# Derivation of Nonlinear Risk Sensitive Control With Measurement Uncertainty

October 2020

**Abstract**

Demonstrate how to derive risk sensitive optimal control solver that takes into account the measurement uncertainty.

## 1 Linearized dynamics and Deviations

Starting from a nonlinear dynamical system in discrete tine

$$x_{t+1} = f\left(x_t, u_t\right) \tag{1}$$

then write in terms of deviation along a nominal trajectory $\{x_t^n, u_t^n\}$ starting from the nonlinear dynamics model,

$$\delta x_t = x_t - x_t^n \tag{2}$$
$$x_{t+1}^n = x_t^n + \Delta t.f\left(x_t^n, u_t^n\right) \tag{3}$$
$$x_{t+1} = x_t + \Delta t\left[f\left(x_t^n, u_t^n\right) + \frac{\partial f}{\partial x}\left(x_t - x_t^n\right) + \frac{\partial f}{\partial u}\left(u_t - u_t^n\right)\right] \tag{4}$$
$$\delta x_{t+1} = \left[I + \Delta t\frac{\partial f}{\partial x}\right]\delta x_t + \Delta t\frac{\partial f}{\partial u}\delta u_t \tag{5}$$

Then we can approximate the system dynamics and cost function as

$$\delta x_{t+1} = A_t\delta x_t + B_t\delta u_t + C_t\omega_t \tag{6}$$
$$\mathcal{J}^* = \min_{u_t}\ \mathbb{E}\left[\exp\left(\sigma\left[l_T\left(\delta x_T\right) + \sum_0^{T-1} l_t\left(\delta x_t, \delta u_t\right)\right]\right)\right] \tag{7}$$

keep in mind that the cost is also approximated as a quadratic

$$l_t\left(\delta x_t, \delta u_t\right) = \frac{1}{2}\delta x_t^T Q_t\delta x_t + \delta x_t^T q_t + \bar{q}_t + \frac{1}{2}\delta u_t^T R_t\delta u_t + \delta u_t^T r_t + \bar{r}_t + \delta x_t^T P_t\delta u_t \tag{8}$$

## 2 Problem Definition

start with system dynamics and measurements

$$\delta x_{t+1} = A_t\delta x_t + B_t\delta u_t + C_t\omega_t \tag{9}$$
$$\delta y_t = D_t\delta x_t + E_t\gamma_t \tag{10}$$

How do we design an Extended Kalman Filter for this to get an estimate $\delta\hat{x}_{t+1}$ of $\delta x_{t+1}$? Any choice of filter that returns locally linear gains as described below is ok. Assume a simple EKF of the form

$$\delta\hat{x}_{t+1} = A_t\delta\hat{x}_t + B_t\delta u_t + K_tD_t(\delta x_t - \delta\hat{x}_t) + K_tE_t\gamma_t \tag{11}$$

then following the idea of Ponton [4], we can create a new system with double the states

$$\underbrace{\begin{bmatrix}\delta x_{t+1}\\ \delta \hat{x}_{t+1}\end{bmatrix}}_{\delta\tilde{x}_{t+1}} = \underbrace{\begin{bmatrix}A_t\delta x_t + B_t\delta u_t\\ A_t\delta\hat{x}_t + B_t\delta u_t + K_tD_t(\delta x_t - \delta\hat{x}_t)\end{bmatrix}}_{\tilde{f}(\delta\tilde{x}_t,\delta u_t)} + \begin{bmatrix}C_t & 0\\ 0 & K_tE_t\end{bmatrix}\begin{bmatrix}w_t\\ \gamma_t\end{bmatrix} \tag{12}$$

this defines an augmented state vector $\tilde{x}_t$ where

$$\tilde{x}_t = \begin{bmatrix}x_t\\ \hat{x}_t\end{bmatrix} \tag{13}$$

let's do some rearranging

$$\begin{bmatrix}\delta x_{t+1}\\ \delta \hat{x}_{t+1}\end{bmatrix} = \underbrace{\begin{bmatrix}A_t & 0\\ K_tD_t & A_t - K_tD_t\end{bmatrix}}_{\tilde{A}_t}\begin{bmatrix}\delta x_t\\ \delta\hat{x}_t\end{bmatrix} + \underbrace{\begin{bmatrix}B_t\\ B_t\end{bmatrix}}_{\tilde{B}_t}\delta u_t + \underbrace{\begin{bmatrix}C_t & 0\\ 0 & K_tE_t\end{bmatrix}}_{\tilde{C}_t}\underbrace{\begin{bmatrix}w_t\\ \gamma_t\end{bmatrix}}_{\xi_t} \tag{14}$$

now the new system dynamics looks like,

$$\delta\tilde{x}_{t+1} = \tilde{A}_t\delta\tilde{x}_t + \tilde{B}_t\delta u_t + \tilde{C}_t\xi_t \tag{15}$$

From Jacobson [1] the value function is quadratic in the state deviations

$$V\left(t,\delta\tilde{x}_t\right) = F_t \exp\left\{\frac{\sigma}{2}\delta\tilde{x}_t^T\underbrace{\begin{bmatrix}S_t^{xx} & S_t^{x\hat{x}}\\ S_t^{\hat{x}x} & S_t^{\hat{x}\hat{x}}\end{bmatrix}}_{\tilde{S}_t}\delta\tilde{x}_t + \sigma\delta\tilde{x}_t^T\underbrace{\begin{bmatrix}s_t^x\\ s_t^{\hat{x}}\end{bmatrix}}_{\tilde{s}_t}\right\} \tag{16}$$

the immediate cost is only a function of $x_t$ and the recursive principle of optimality can be written as

$$V\left(t,\delta\tilde{x}_t\right) = \min_{u_t}\left\{\exp\left(\sigma l_t\left(\delta x_t,\delta u_t\right)\right)\mathbb{E}\left[V\left(t+1,\delta\tilde{x}_{t+1}\right)\right]\right\} \tag{17}$$

# 3 Expectation of the Value Function

How does the expectation $\mathbb{E}\left[V\left(t+1,\delta\tilde{x}_{t+1}\right)\right]$ look like ? This follows the steps of Jacobson [1], recall the definition of expectation

$$\mathbb{E}\left[f(\xi_t)\right] = \int_{-\infty}^{+\infty} pdf(\xi_t)f(\xi_t)d\xi_t \tag{18}$$

if $\xi_t \sim \mathcal{N}\left(0,\Xi_t\right)$ then $pdf(\xi_t)$ is

$$pdf(\xi_t) = \frac{1}{\sqrt{det(2\pi\Xi)}}\exp\left\{-\frac{1}{2}\xi_t^T\Xi_t^{-1}\xi_t\right\} \tag{19}$$

we proceed with expanding the value function to compute the integral

$$V\left(t+1,\delta\tilde{x}_{t+1}\right) = F_{t+1}\exp\left\{\frac{\sigma}{2}\delta\tilde{x}_{t+1}^T\tilde{S}_{t+1}\delta\tilde{x}_{t+1} + \sigma\delta\tilde{x}_{t+1}^T\tilde{s}_{t+1}\right\} \tag{20}$$

ok, let's expand the argument of the exponential

$$\begin{aligned}
\frac{\sigma}{2}\delta\tilde{x}_{t+1}^T\tilde{S}_{t+1}\delta\tilde{x}_{t+1} =& \frac{\sigma}{2}\left(\tilde{A}_t\delta\tilde{x}_t + \tilde{B}_t\delta u_t + \tilde{C}_t\xi_t\right)^T \tilde{S}_{t+1}\left(\tilde{A}_t\delta\tilde{x}_t + \tilde{B}_t\delta u_t + \tilde{C}_t\xi_t\right) \\
=& \frac{\sigma}{2}\left(\delta\tilde{x}_t^T\tilde{A}_t^T\tilde{S}_{t+1}\tilde{A}_t\delta\tilde{x}_t + \delta\tilde{x}_t^T\tilde{A}_t^T\tilde{S}_{t+1}\tilde{B}_t\delta u_t + \delta\tilde{x}_t^T\tilde{A}_t^T\tilde{S}_{t+1}\tilde{C}_t\xi_t\right) \\
&+ \frac{\sigma}{2}\left(\delta u_t^T\tilde{B}_t^T\tilde{S}_{t+1}\tilde{A}_t\delta\tilde{x}_t + \delta u_t^T\tilde{B}_t^T\tilde{S}_{t+1}\tilde{B}_t\delta u_t + \delta u_t^T\tilde{B}_t^T\tilde{S}_{t+1}\tilde{C}_t\xi_t\right) \\
&+ \frac{\sigma}{2}\left(\xi_t^T\tilde{C}_t^T\tilde{S}_{t+1}\tilde{A}_t\delta\tilde{x}_t + \xi_t^T\tilde{C}_t^T\tilde{S}_{t+1}\tilde{B}_t\delta u_t + \xi_t^T\tilde{C}_t^T\tilde{S}_{t+1}\tilde{C}_t\xi_t\right) \qquad (21)
\end{aligned}$$

$$\begin{aligned}
\sigma\tilde{x}_{t+1}^T\tilde{s}_{t+1} =& \sigma\left(\tilde{A}_t\delta\tilde{x}_t + \tilde{B}_t\delta u_t + \tilde{C}_t\xi_t\right)^T \tilde{s}_{t+1} \\
=& \sigma\delta\tilde{x}_t^T\tilde{A}_t^T\tilde{s}_{t+1} + \sigma\delta u_t^T\tilde{B}_t^T\tilde{s}_{t+1} + \sigma\xi_t^T\tilde{C}_t^T\tilde{s}_{t+1} \qquad (22)
\end{aligned}$$

let $M_t$ be the group of terms not containing $\xi_t$ and $N_t$ be the group of terms containing $\xi_t$, keep in mind both are scalars, then

$$\begin{aligned}
M_t =& \frac{\sigma}{2}\delta\tilde{x}_t^T\tilde{A}_t^T\tilde{S}_{t+1}\tilde{A}_t\delta\tilde{x}_t + \frac{\sigma}{2}\delta u_t^T\tilde{B}_t^T\tilde{S}_{t+1}\tilde{B}_t\delta u_t + \sigma\delta\tilde{x}_t^T\tilde{A}_t^T\tilde{S}_{t+1}\tilde{B}_t\delta u_t \\
&+ \sigma\delta\tilde{x}_t^T\tilde{A}_t^T\tilde{s}_{t+1} + \sigma\delta u_t^T\tilde{B}_t^T\tilde{s}_{t+1} \qquad (23)
\end{aligned}$$

$$N_t = \frac{\sigma}{2}\xi_t^T\tilde{C}_t^T\tilde{S}_{t+1}\tilde{C}_t\xi_t + \sigma\xi_t^T\left(\tilde{C}_t^T\tilde{S}_{t+1}\tilde{A}_t\delta\tilde{x}_t + \tilde{C}_t^T\tilde{S}_{t+1}\tilde{B}_t\delta u_t + \tilde{C}_t^T\tilde{s}_{t+1}\right) \qquad (24)$$

now, define the vector $Z_t$ as

$$Z_t = \tilde{C}_t^T\tilde{S}_{t+1}\tilde{A}_t\delta\tilde{x}_t + \tilde{C}_t^T\tilde{S}_{t+1}\tilde{B}_t\delta u_t + \tilde{C}_t^T\tilde{s}_{t+1} \qquad (25)$$

Then $N_t$ becomes

$$N_t = \frac{\sigma}{2}\xi_t^T\tilde{C}_t^T\tilde{S}_{t+1}\tilde{C}_t\xi_t + \sigma\xi_t^T Z_t \qquad (26)$$

Then the overall expectation of the value function at the next time step can be written as

$$\mathbb{E}\left[V\left(t+1, \delta\tilde{x}_{t+1}\right)\right] = \frac{1}{\sqrt{det(2\pi\Xi)}}\int_{-\infty}^{+\infty}\exp\left\{-\frac{1}{2}\xi_t^T\Xi_t^{-1}\xi_t + M_t + N_t\right\}d\xi_t \qquad (27)$$

we want to write $N_t - \frac{1}{2}\xi_t^T\Xi_t^{-1}\xi_t$ as a perfect square

$$\begin{aligned}
N_t - \frac{1}{2}\xi_t^T\Xi_t^{-1}\xi_t =& -\frac{1}{2}\left(\xi_t - \bar{\xi}_t\right)^T W_t^{-1}\left(\xi_t - \bar{\xi}_t\right) \qquad (28) \\
=& -\frac{1}{2}\xi_t^T W_t^{-1}\xi_t - \frac{1}{2}\bar{\xi}_t^T W_t^{-1}\bar{\xi}_t + \xi_t^T W_t^{-1}\bar{\xi}_t \qquad (29)
\end{aligned}$$

just by comparison we can see that

$$\begin{aligned}
W_t^{-1} &= \Xi - \sigma\tilde{C}_t^T\tilde{S}_{t+1}\tilde{C}_t \qquad (30) \\
\xi_t^T W_t^{-1}\bar{\xi}_t &= \sigma\xi_t^T Z_t \qquad (31) \\
W_t^{-1}\bar{\xi}_t &= \sigma Z_t \qquad (32) \\
\bar{\xi}_t &= \sigma W_t Z_t \qquad (33) \\
\frac{1}{2}\bar{\xi}_t^T W_t^{-1}\bar{\xi}_t &= \frac{\sigma^2}{2} Z_t^T W_t Z_t \qquad (34)
\end{aligned}$$

then, the arguments of the exponential inside the integral become

$$M_t + \frac{\sigma^2}{2}\ Z_t^T W_t Z_t - \frac{1}{2}\left(\xi_t - \bar{\xi}_t\right)^T W_t^{-1}\left(\xi_t - \bar{\xi}_t\right) \tag{35}$$

where

$$\begin{aligned} Z_t =& \tilde{C}_t^T \tilde{S}_{t+1}\tilde{A}_t \delta\tilde{x}_t + \tilde{C}_t^T \tilde{S}_{t+1}\tilde{B}_t \delta u_t + \tilde{C}_t^T \tilde{s}_{t+1} &(36)\\ W_t =& \left(\Xi - \sigma \tilde{C}_t^T \tilde{S}_{t+1}\tilde{C}_t\right)^{-1} &(37)\\ \bar{\xi}_t =& \sigma W_t Z_t &(38)\end{aligned}$$

the arguments of the exponential are scalar, so we can break it up

$$\mathbb{E}\left[V\left(t+1, \delta\tilde{x}_{t+1}\right)\right] = \frac{1}{\sqrt{det(2\pi\Xi)}}\exp\left\{M_t + \frac{\sigma^2}{2}\ Z_t^T W_t Z_t\right\}\int_{-\infty}^{+\infty}\exp\left\{-\frac{1}{2}\left(\xi_t - \bar{\xi}_t\right)^T W_t^{-1}\left(\xi_t - \bar{\xi}_t\right)\right\}d\xi_t \tag{39}$$

but we know that

$$\int_{-\infty}^{+\infty}\frac{1}{\sqrt{det(2\pi W_t)}}\exp\left\{-\frac{1}{2}\left(\xi_t - \bar{\xi}_t\right)^T W_t^{-1}\left(\xi_t - \bar{\xi}_t\right)\right\}d\xi_t = 1 \tag{40}$$

then

$$\mathbb{E}\left[V\left(t+1, \delta\tilde{x}_{t+1}\right)\right] = \frac{\sqrt{det(2\pi W_t)}}{\sqrt{det(2\pi\Xi)}}\exp\left\{M_t + \frac{\sigma^2}{2}\ Z_t^T W_t Z_t\right\} \tag{41}$$

the noise contribution to the cost is the last term described as $\frac{\sigma^2}{2}\ Z_t^T W_t Z_t$
Bellman Principle for the risk sensitive problem becomes

$$V\left(t, \delta\tilde{x}_t\right) = \min_{\delta u_t}\left\{\frac{\sqrt{det(2\pi W_t)}}{\sqrt{det(2\pi\Xi)}}\exp\left\{\sigma l_t\left(\delta x_t, \delta u_t\right) + M_t + \frac{\sigma^2}{2}\ Z_t^T W_t Z_t\right\}\right\} \tag{42}$$

# 4 Control Minimizing the Value Function

it is sufficient to minimize the argument of the exponential, so collect the terms containing $\delta u_t$, for this we have to expand, collect etc. Recall that

$$\begin{aligned} \sigma l_t\left(\delta x_t, \delta u_t\right) =& \frac{\sigma}{2}\delta x_t^T Q_t \delta x_t + \sigma\delta x_t^T q_t + \sigma\bar{q}_t + \frac{\sigma}{2}\delta u_t^T R_t \delta u_t + \sigma\delta u_t^T r_t + \sigma\bar{r}_t + \sigma\delta x_t^T P_t \delta u_t &(43)\\ M_t =& \frac{\sigma}{2}\delta\tilde{x}_t^T \tilde{A}_t^T \tilde{S}_{t+1}\tilde{A}_t\delta\tilde{x}_t + \frac{\sigma}{2}\delta u_t^T \tilde{B}_t^T \tilde{S}_{t+1}\tilde{B}_t \delta u_t + \sigma\delta\tilde{x}_t^T \tilde{A}_t^T \tilde{S}_{t+1}\tilde{B}_t \delta u_t \\ &+ \sigma\delta\tilde{x}_t^T \tilde{A}_t^T \tilde{s}_{t+1} + \sigma\delta u_t^T \tilde{B}_t^T \tilde{s}_{t+1} &(44)\end{aligned}$$

now let's expand the noise contribution

$$\begin{aligned} \frac{\sigma^2}{2}\ Z_t^T W_t Z_t =& \frac{\sigma^2}{2}\left(\tilde{C}_t^T \tilde{S}_{t+1}\tilde{A}_t\delta\tilde{x}_t + \tilde{C}_t^T \tilde{S}_{t+1}\tilde{B}_t\delta u_t + \tilde{C}_t^T \tilde{s}_{t+1}\right)^T W_t\left(\tilde{C}_t^T \tilde{S}_{t+1}\tilde{A}_t\delta\tilde{x}_t + \tilde{C}_t^T \tilde{S}_{t+1}\tilde{B}_t\delta u_t + \tilde{C}_t^T \tilde{s}_{t+1}\right)\\ =& \frac{\sigma^2}{2}\left(\delta\tilde{x}_t^T \tilde{A}_t^T \tilde{S}_{t+1}\tilde{C}_t W_t \tilde{C}_t^T \tilde{S}_{t+1}\tilde{A}_t\delta\tilde{x}_t + \delta\tilde{x}_t^T \tilde{A}_t^T \tilde{S}_{t+1}\tilde{C}_t W_t \tilde{C}_t^T \tilde{S}_{t+1}\tilde{B}_t\delta u_t + \delta\tilde{x}_t^T \tilde{A}_t^T \tilde{S}_{t+1}\tilde{C}_t W_t \tilde{C}_t^T \tilde{s}_{t+1}\right)\\ &+ \frac{\sigma^2}{2}\left(\delta u_t^T \tilde{B}_t^T \tilde{S}_{t+1}\tilde{C}_t W_t \tilde{C}_t^T \tilde{S}_{t+1}\tilde{A}_t\delta\tilde{x}_t + \delta u_t^T \tilde{B}_t^T \tilde{S}_{t+1}\tilde{C}_t W_t \tilde{C}_t^T \tilde{S}_{t+1}\tilde{B}_t\delta u_t + \delta u_t^T \tilde{B}_t^T \tilde{S}_{t+1}\tilde{C}_t W_t \tilde{C}_t^T \tilde{s}_{t+1}\right)\\ &+ \frac{\sigma^2}{2}\left(\tilde{s}_{t+1}^T \tilde{C}_t W_t \tilde{C}_t^T \tilde{S}_{t+1}\tilde{A}_t\delta\tilde{x}_t + \tilde{s}_{t+1}^T \tilde{C}_t W_t \tilde{C}_t^T \tilde{S}_{t+1}\tilde{B}_t\delta u_t + \tilde{s}_{t+1}^T \tilde{C}_t W_t \tilde{C}_t^T \tilde{s}_{t+1}\right) &(45)\end{aligned}$$

then the optimal solution is

$$\delta u_t^* = \arg\min_{\delta u_t} \left\{ \frac{\sigma}{2}\delta u_t^T R_t \delta u_t + \sigma \delta u_t^T r_t + \sigma \delta x_t^T P_t \delta u_t + \frac{\sigma}{2}\delta u_t^T \tilde{B}_t^T \tilde{S}_{t+1}\tilde{B}_t \delta u_t + \sigma \delta \tilde{x}_t^T \tilde{A}_t^T \tilde{S}_{t+1}\tilde{B}_t \delta u_t \right.$$
$$+ \sigma \delta u_t^T \tilde{B}_t^T \tilde{s}_{t+1} + \sigma^2 \delta u_t^T \tilde{B}_t^T \tilde{S}_{t+1}\tilde{C}_t W_t \tilde{C}_t^T \tilde{S}_{t+1}\tilde{A}_t \delta \tilde{x}_t + \frac{\sigma^2}{2}\delta u_t^T \tilde{B}_t^T \tilde{S}_{t+1}\tilde{C}_t W_t \tilde{C}_t^T \tilde{S}_{t+1}\tilde{B}_t \delta u_t$$
$$\left. + \sigma^2 \delta u_t^T \tilde{B}_t^T \tilde{S}_{t+1}\tilde{C}_t W_t \tilde{C}_t^T \tilde{s}_{t+1} \right\} \tag{46}$$

group the terms into quadratic form in the control, watch out, some terms only multiply $\delta x_t$

$$\delta u_t^* = \arg\min_{\delta u_t} \left\{ \sigma \left( \frac{1}{2}\delta u_t^T G_t \delta u_t + \delta u_t^T g_t + \delta u_t^T H_t \delta \tilde{x}_t \right) \right\} \tag{47}$$

now define

$$\tilde{P}_t = \begin{bmatrix} P_t \\ 0 \end{bmatrix} \quad \tilde{Q}_t = \begin{bmatrix} Q_t & 0 \\ 0 & 0 \end{bmatrix} \quad \tilde{q}_t = \begin{bmatrix} q_t \\ 0 \end{bmatrix} \tag{48}$$

then we can write the terms of the current time cost as

$$\sigma \delta x_t^T P_t \delta u_t = \sigma \delta u_t^T \tilde{P}_t^T \delta \tilde{x}_t \tag{49}$$
$$\frac{\sigma}{2}\delta x_t^T Q_t \delta x_t + \sigma \delta x_t^T q_t = \frac{\sigma}{2}\delta \tilde{x}_t^T \tilde{Q}_t \delta \tilde{x}_t + \sigma \delta \tilde{x}_t^T \tilde{q}_t \tag{50}$$

so the control terms become

$$G_t = R_t + \tilde{B}_t^T \tilde{S}_{t+1}\tilde{B}_t + \sigma \tilde{B}_t^T \tilde{S}_{t+1}\tilde{C}_t W_t \tilde{C}_t^T \tilde{S}_{t+1}\tilde{B}_t \tag{51}$$
$$g_t = r_t + \tilde{B}_t^T \tilde{s}_{t+1} + \sigma \tilde{B}_t^T \tilde{S}_{t+1}\tilde{C}_t W_t \tilde{C}_t^T \tilde{s}_{t+1} \tag{52}$$
$$H_t = \tilde{B}_t^T \tilde{S}_{t+1}\tilde{A}_t + \sigma \tilde{B}_t^T \tilde{S}_{t+1}\tilde{C}_t W_t \tilde{C}_t^T \tilde{S}_{t+1}\tilde{A}_t + \tilde{P}_t^T \tag{53}$$

clearly $G_t$ is a symmetric $m \times m$ matrix, it is positive definite if $W_t$ is positive definite, then the derivative of the quadratic in $\delta u_t$ can be computed and set to zero to obtain the minimizer, hence

keep in mind that $H_t = \begin{bmatrix} H_t^x & H_t^{\hat{x}} \end{bmatrix}$

$$\delta u_t^* = -G_t^{-1} g_t - G_t^{-1} H_t \delta \tilde{x}_t = -G_t^{-1} g_t - G_t^{-1} H_t^x \delta x_t - G_t^{-1} H_t^{\hat{x}} \delta \hat{x}_t \tag{54}$$

## 4.1 the expectation of $\delta x_t$

since $\delta x_t$ is the actual state, a vector that we don't know, then following the assumption from [2], that is

$$\mathbb{E}_{\delta x_t | \delta \hat{x}_t} [\delta u_t^*] = -G_t^{-1} g_t - G_t^{-1} \left( H_t^x + H_t^{\hat{x}} \right) \delta \hat{x}_t \tag{55}$$

or better to simplify the comparison, write it as a function of $\delta \tilde{x}_t$

$$\mathbb{E}_{\delta x_t | \delta \hat{x}_t} [\delta u_t^*] = \underbrace{-G_t^{-1} g_t}_{\tilde{h}_t} + \underbrace{\begin{bmatrix} 0 & -G_t^{-1} \left( H_t^x + H_t^{\hat{x}} \right) \end{bmatrix}}_{\tilde{H}_t} \delta \tilde{x}_t = \tilde{h}_t + \tilde{H}_t \delta \tilde{x}_t \tag{56}$$

now we can plug back into the value function and obtain the recursions, we need to break the $\delta \tilde{x}_t$ and match the segments.

## 5 The Value Function Recursions

let's start by the contribution of $\sigma l_t(\delta x_t, \delta u_t)$ to the value function

$$\begin{aligned}
\sigma l_t(\delta x_t, \delta u_t) =& \frac{\sigma}{2}\delta\tilde{x}_t^T\tilde{Q}_t\delta\tilde{x}_t + \sigma\delta\tilde{x}_t^T\tilde{q}_t + \sigma\bar{q}_t + \frac{\sigma}{2}\delta u_t^T R_t\delta u_t + \sigma\delta u_t^T r_t + \sigma\bar{r}_t + \sigma\delta\tilde{x}_t^T\tilde{P}_t\delta u_t \\
=& \frac{\sigma}{2}\delta\tilde{x}_t^T\tilde{Q}_t\delta\tilde{x}_t + \sigma\delta\tilde{x}_t^T\tilde{q}_t + \sigma\bar{q}_t + \frac{\sigma}{2}\left(\tilde{h}_t + \tilde{H}_t\delta\tilde{x}_t\right)^T R_t\left(\tilde{h}_t + \tilde{H}_t\delta\tilde{x}_t\right) \\
&+\sigma\left(\tilde{h}_t + \tilde{H}_t\delta\tilde{x}_t\right)^T r_t + \sigma\bar{r}_t + \sigma\delta\tilde{x}_t^T\tilde{P}_t\left(\tilde{h}_t + \tilde{H}_t\delta\tilde{x}_t\right) \\
=& \frac{\sigma}{2}\delta\tilde{x}_t^T\left(\tilde{Q}_t + \tilde{H}_t^T R_t\tilde{H}_t + 2\tilde{P}_t\tilde{H}_t\right)\delta\tilde{x}_t + \sigma\delta\tilde{x}_t^T\left(\tilde{q}_t + \tilde{H}_t^T R_t\tilde{h}_t + \tilde{H}_t^T r_t + \tilde{P}_t\tilde{h}_t\right) \\
&+\sigma\bar{q}_t + \sigma\bar{r}_t + \frac{\sigma}{2}\tilde{h}_t^T R_t\tilde{h}_t + \sigma\tilde{h}_t^T r_t
\end{aligned} \tag{57}$$

the second term

$$\begin{aligned}
M_t =& \frac{\sigma}{2}\delta\tilde{x}_t^T\tilde{A}_t^T\tilde{S}_{t+1}\tilde{A}_t\delta\tilde{x}_t + \frac{\sigma}{2}\delta u_t^T\tilde{B}_t^T\tilde{S}_{t+1}\tilde{B}_t\delta u_t + \sigma\delta\tilde{x}_t^T\tilde{A}_t^T\tilde{S}_{t+1}\tilde{B}_t\delta u_t + \sigma\delta\tilde{x}_t^T\tilde{A}_t^T\tilde{s}_{t+1} + \sigma\delta u_t^T\tilde{B}_t^T\tilde{s}_{t+1} \\
=& \frac{\sigma}{2}\delta\tilde{x}_t^T\tilde{A}_t^T\tilde{S}_{t+1}\tilde{A}_t\delta\tilde{x}_t + \frac{\sigma}{2}\left(\tilde{h}_t + \tilde{H}_t\delta\tilde{x}_t\right)^T\tilde{B}_t^T\tilde{S}_{t+1}\tilde{B}_t\left(\tilde{h}_t + \tilde{H}_t\delta\tilde{x}_t\right) + \sigma\delta\tilde{x}_t^T\tilde{A}_t^T\tilde{S}_{t+1}\tilde{B}_t\left(\tilde{h}_t + \tilde{H}_t\delta\tilde{x}_t\right) \\
&+\sigma\delta\tilde{x}_t^T\tilde{A}_t^T\tilde{s}_{t+1} + \sigma\left(\tilde{h}_t + \tilde{H}_t\delta\tilde{x}_t\right)^T\tilde{B}_t^T\tilde{s}_{t+1} \\
=& \frac{\sigma}{2}\delta\tilde{x}_t^T\left(\tilde{A}_t^T\tilde{S}_{t+1}\tilde{A}_t + \tilde{H}_t^T\tilde{B}_t^T\tilde{S}_{t+1}\tilde{B}_t\tilde{H}_t + 2\tilde{A}_t^T\tilde{S}_{t+1}\tilde{B}_t\tilde{H}_t\right)\delta\tilde{x}_t \\
&+\sigma\delta\tilde{x}_t^T\left(\tilde{H}_t^T\tilde{B}_t^T\tilde{S}_{t+1}\tilde{B}_t\tilde{h}_t + \tilde{A}_t^T\tilde{S}_{t+1}\tilde{B}_t\tilde{h}_t + \tilde{A}_t^T\tilde{s}_{t+1} + \tilde{H}_t^T\tilde{B}_t^T\tilde{s}_{t+1}\right) + \frac{\sigma}{2}\tilde{h}_t^T\tilde{B}_t^T\tilde{S}_{t+1}\tilde{B}_t\tilde{h}_t + \sigma\tilde{h}_t^T\tilde{B}_t^T\tilde{s}_{t+1}
\end{aligned} \tag{58}$$

the last term is a bit longer,

$$\begin{aligned}
\frac{\sigma^2}{2}Z_t^T W_t Z_t =& \frac{\sigma^2}{2}\delta\tilde{x}_t^T\tilde{A}_t^T\tilde{S}_{t+1}\tilde{C}_t W_t\tilde{C}_t^T\tilde{S}_{t+1}\tilde{A}_t\delta\tilde{x}_t + \sigma^2\delta\tilde{x}_t^T\tilde{A}_t^T\tilde{S}_{t+1}\tilde{C}_t W_t\tilde{C}_t^T\tilde{S}_{t+1}\tilde{B}_t\delta u_t \\
&+\sigma^2\delta\tilde{x}_t^T\tilde{A}_t^T\tilde{S}_{t+1}\tilde{C}_t W_t\tilde{C}_t^T\tilde{s}_{t+1} + \frac{\sigma^2}{2}\delta u_t^T\tilde{B}_t^T\tilde{S}_{t+1}\tilde{C}_t W_t\tilde{C}_t^T\tilde{S}_{t+1}\tilde{B}_t\delta u_t \\
&+\sigma^2\delta u_t^T\tilde{B}_t^T\tilde{S}_{t+1}\tilde{C}_t W_t\tilde{C}_t^T\tilde{s}_{t+1} + \frac{\sigma^2}{2}\tilde{s}_{t+1}^T\tilde{C}_t W_t\tilde{C}_t^T\tilde{s}_{t+1} \\
=& \frac{\sigma^2}{2}\delta\tilde{x}_t^T\tilde{A}_t^T\tilde{S}_{t+1}\tilde{C}_t W_t\tilde{C}_t^T\tilde{S}_{t+1}\tilde{A}_t\delta\tilde{x}_t + \sigma^2\delta\tilde{x}_t^T\tilde{A}_t^T\tilde{S}_{t+1}\tilde{C}_t W_t\tilde{C}_t^T\tilde{S}_{t+1}\tilde{B}_t\left(\tilde{h}_t + \tilde{H}_t\delta\tilde{x}_t\right) \\
&+\sigma^2\delta\tilde{x}_t^T\tilde{A}_t^T\tilde{S}_{t+1}\tilde{C}_t W_t\tilde{C}_t^T\tilde{s}_{t+1} + \frac{\sigma^2}{2}\left(\tilde{h}_t + \tilde{H}_t\delta\tilde{x}_t\right)^T\tilde{B}_t^T\tilde{S}_{t+1}\tilde{C}_t W_t\tilde{C}_t^T\tilde{S}_{t+1}\tilde{B}_t\left(\tilde{h}_t + \tilde{H}_t\delta\tilde{x}_t\right) \\
&+\sigma^2\left(\tilde{h}_t + \tilde{H}_t\delta\tilde{x}_t\right)^T\tilde{B}_t^T\tilde{S}_{t+1}\tilde{C}_t W_t\tilde{C}_t^T\tilde{s}_{t+1} + \frac{\sigma^2}{2}\tilde{s}_{t+1}^T\tilde{C}_t W_t\tilde{C}_t^T\tilde{s}_{t+1} \\
=& \frac{\sigma^2}{2}\delta\tilde{x}_t^T\left(\tilde{A}_t^T\tilde{S}_{t+1}\tilde{C}_t W_t\tilde{C}_t^T\tilde{S}_{t+1}\tilde{A}_t + 2\tilde{A}_t^T\tilde{S}_{t+1}\tilde{C}_t W_t\tilde{C}_t^T\tilde{S}_{t+1}\tilde{B}_t\tilde{H}_t + \tilde{H}_t^T\tilde{B}_t^T\tilde{S}_{t+1}\tilde{C}_t W_t\tilde{C}_t^T\tilde{S}_{t+1}\tilde{B}_t\tilde{H}_t\right)\delta\tilde{x}_t \\
&+\sigma^2\delta\tilde{x}_t^T(\tilde{A}_t^T\tilde{S}_{t+1}\tilde{C}_t W_t\tilde{C}_t^T\tilde{s}_{t+1} + 2\tilde{A}_t^T\tilde{S}_{t+1}\tilde{C}_t W_t\tilde{C}_t^T\tilde{S}_{t+1}\tilde{B}_t\tilde{h}_t \\
&+\tilde{H}_t^T\tilde{B}_t^T\tilde{S}_{t+1}\tilde{C}_t W_t\tilde{C}_t^T\tilde{S}_{t+1}\tilde{B}_t\tilde{h}_t + \tilde{H}_t^T\tilde{B}_t^T\tilde{S}_{t+1}\tilde{C}_t W_t\tilde{C}_t^T\tilde{s}_{t+1}) \\
&+\frac{\sigma^2}{2}\tilde{h}_t^T\tilde{B}_t^T\tilde{S}_{t+1}\tilde{C}_t W_t\tilde{C}_t^T\tilde{S}_{t+1}\tilde{B}_t\tilde{h}_t + \frac{\sigma^2}{2}\tilde{s}_{t+1}^T\tilde{C}_t W_t\tilde{C}_t^T\tilde{s}_{t+1} + \sigma^2\tilde{h}_t^T\tilde{B}_t^T\tilde{S}_{t+1}\tilde{C}_t W_t\tilde{C}_t^T\tilde{s}_{t+1}
\end{aligned} \tag{59}$$

now separate and collect what's quadratic in $\delta\tilde{x}$ and what is linear to obtain the recursive value function approximations

$$\begin{aligned}
\tilde{S}_t =& \tilde{Q}_t + \tilde{H}_t^T R_t\tilde{H}_t + 2\tilde{P}_t\tilde{H}_t + \tilde{A}_t^T\tilde{S}_{t+1}\tilde{A}_t + \tilde{H}_t^T\tilde{B}_t^T\tilde{S}_{t+1}\tilde{B}_t\tilde{H}_t + 2\tilde{A}_t^T\tilde{S}_{t+1}\tilde{B}_t\tilde{H}_t \\
&+\sigma\left(\tilde{A}_t^T\tilde{S}_{t+1}\tilde{C}_t W_t\tilde{C}_t^T\tilde{S}_{t+1}\tilde{A}_t + 2\tilde{A}_t^T\tilde{S}_{t+1}\tilde{C}_t W_t\tilde{C}_t^T\tilde{S}_{t+1}\tilde{B}_t\tilde{H}_t + \tilde{H}_t^T\tilde{B}_t^T\tilde{S}_{t+1}\tilde{C}_t W_t\tilde{C}_t^T\tilde{S}_{t+1}\tilde{B}_t\tilde{H}_t\right)
\end{aligned} \tag{60}$$

$$
\begin{aligned}
\tilde{s}_t =& \tilde{q}_t + \tilde{H}_t^T R_t \tilde{h}_t + \tilde{H}_t^T r_t + \tilde{P}_t \tilde{h}_t + \tilde{H}_t^T \tilde{B}_t^T \tilde{S}_{t+1} \tilde{B}_t \tilde{h}_t + \tilde{A}_t^T \tilde{S}_{t+1} \tilde{B}_t \tilde{h}_t + \tilde{A}_t^T \tilde{s}_{t+1} + \tilde{H}_t^T \tilde{B}_t^T \tilde{s}_{t+1} \\
& + \sigma \Big( \tilde{A}_t^T \tilde{S}_{t+1} \tilde{C}_t W_t \tilde{C}_t^T \tilde{s}_{t+1} + 2 \tilde{A}_t^T \tilde{S}_{t+1} \tilde{C}_t W_t \tilde{C}_t^T \tilde{S}_{t+1} \tilde{B}_t \tilde{h}_t + \tilde{H}_t^T \tilde{B}_t^T \tilde{S}_{t+1} \tilde{C}_t W_t \tilde{C}_t^T \tilde{S}_{t+1} \tilde{B}_t \tilde{h}_t \\
& + \tilde{H}_t^T \tilde{B}_t^T \tilde{S}_{t+1} \tilde{C}_t W_t \tilde{C}_t^T \tilde{s}_{t+1} \Big)
\end{aligned} \tag{61}
$$

$$
\begin{aligned}
F_t = \frac{\sqrt{det(2\pi W_t)}}{\sqrt{det(2\pi \Xi)}} \exp \Bigg\{ & \sigma \bar{q}_t + \sigma \bar{r}_t + \frac{\sigma}{2} \tilde{h}_t^T R_t \tilde{h}_t + \sigma \tilde{h}_t^T r_t + \frac{\sigma}{2} \tilde{h}_t^T \tilde{B}_t^T \tilde{S}_{t+1} \tilde{B}_t \tilde{h}_t + \sigma \tilde{h}_t^T \tilde{B}_t^T \tilde{s}_{t+1} \\
& + \frac{\sigma^2}{2} \tilde{h}_t^T \tilde{B}_t^T \tilde{S}_{t+1} \tilde{C}_t W_t \tilde{C}_t^T \tilde{S}_{t+1} \tilde{B}_t \tilde{h}_t + \frac{\sigma^2}{2} \tilde{s}_{t+1}^T \tilde{C}_t W_t \tilde{C}_t^T \tilde{s}_{t+1} + \sigma^2 \tilde{h}_t^T \tilde{B}_t^T \tilde{S}_{t+1} \tilde{C}_t W_t \tilde{C}_t^T \tilde{s}_{t+1} \Bigg\}
\end{aligned} \tag{62}
$$

# 6 Few Comments on Solving Problems With Measurement Uncertainty

Given an initially feasible trajectory in the states and controls, the linear approximation of the dynamics and quadratic approximation of the cost are computed, just like any well known iLQR variation [3]. The first difference is though, to also compute the Kalman Filter gains along this trajectory.

For the backward pass, the filter gains are fixed and considered to be fixed parts of the A,B,C matrix of the extended dynamics, and the optimal control problem is solved recursively using the equations derived in this document.

This formulation returns an optimal control that is linear in the extended state vector i.e. $\tilde{x} = \left[x, \hat{x}\right]$, however, we don't actually know $x$, so we take the expectation of $x$ conditioned on $\hat{x}$ denoted $\mathbb{E}_{\delta x_t | \delta \hat{x}_t}\left[\delta u_t^*\right]$ in (55).

# References


[1] David Jacobson. Optimal stochastic linear systems with exponential performance criteria and their relation to deterministic differential games. *IEEE Transactions on Automatic control*, 18(2):124–131, 1973.

[2] Weiwei Li and Emanuel Todorov. Iterative linearization methods for approximately optimal control and estimation of non-linear stochastic system. *International Journal of Control*, 80(9):1439–1453, 2007.

[3] Carlos Mastalli, Rohan Budhiraja, Wolfgang Merkt, Guilhem Saurel, Bilal Hammoud, Maximilien Naveau, Justin Carpentier, Ludovic Righetti, Sethu Vijayakumar, and Nicolas Mansard. Crocoddyl: An efficient and versatile framework for multi-contact optimal control. In *2020 IEEE International Conference on Robotics and Automation (ICRA)*, pages 2536–2542. IEEE, 2020.

[4] Brahayam Pontón, Stefan Schaal, and Ludovic Righetti. On the effects of measurement uncertainty in optimal control of contact interactions. In *Algorithmic Foundations of Robotics XII*, pages 784–799. Springer, 2020.