Homework 4 sample solution

Due 09/09/16

September 6, 2016

Use the *formal definitions* of Big-Oh, Big-Omega, and Big-Theta to prove the following.

1. Prove that $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

   Answer:

   Proof. ($f(n) = \Theta(g(n)) \rightarrow f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$) If $f(n) = \Theta(g(n))$, then there exist positive constants c_1, c_2, and n_0 such that $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0$. Since $f(n) \geq c_1 g(n)$ for all $n \geq n_0$, $f(n) = \Omega(g(n))$. Similarly, since $f(n) \leq c_2 g(n)$ for all $n \geq n_0$, $f(n) = O(g(n))$.

   ($f(n) = O(g(n))$ and $f(n) = \Omega(g(n)) \rightarrow f(n) = \Theta(g(n))$) If $f(n) = O(g(n))$, then there exist positive constants c_1 and n_1 such that $f(n) \leq c_1 g(n)$ for all $n \geq n_1$, and since $f(n) = \Omega(g(n))$, there exist positive constants c_2 and n_2 such that $f(n) \geq c_2 g(n)$ for all $n \geq n_2$. Note that both of these inequalities hold for $n \geq \max\{n_1, n_2\}$. Hence, there exist positive constants c_1, c_2, and n_0, namely $n_0 = \max\{n_1, n_2\}$, such that $c_2 g(n) \leq f(n) \leq c_1 g(n)$ for all $n \geq n_0$, so $f(n) = \Theta(g(n))$. $\square$

2. Prove that if $f_1(n) = \Omega(g_1(n))$ and $f_2(n) = \Omega(g_2(n))$, then $f_1(n) + f_2(n) = \Omega(g_1(n) + g_2(n))$.

   Answer:

   Proof. If $f_1(n) = \Omega(g_1(n))$, then there exist positive constants c_1 and n_1 such that $f_1(n) \geq c_1 g_1(n)$ for all $n \geq n_1$, and if $f_2(n) = \Omega(g_2(n))$, then there exist positive constants c_2 and n_2 such that $f_2(n) \geq c_2 g_2(n)$ for all $n \geq n_2$. Note that both of these inequalities will be true for all $n \geq \max\{n_1, n_2\}$. Adding these inequalities yields $f_1(n) + f_2(n) \geq c_1 g_1(n) + c_2 g_2(n)$, which is true for all $n \geq \max\{n_1, n_2\}$. If we let $c_3 = \min c_1, c_2$, $f_1(n) + f_2(n) \geq c_1 g_1(n) + c_2 g_2(n) \geq c_3 g_1(n) + c_3 g_2(n) = c_3(g_1(n) + g_2(n))$. Thus, there exist positive coefficients c_3 and n_3, namely $c_3 = \max c_1, c_2$ and $n_3 = \max\{n_1, n_2\}$, such that $f_1(n) + f_2(n) \geq c_3(g_1(n) + g_2(n))$ for all $n \geq n_3$, so $f_1(n) + f_2(n) = \Omega(g_1(n) + g_2(n))$. $\square$