Big-Oh Challenge Problem Sample Solution

Due 09/13/16

September 6, 2016

In Homework 3, we proved that $(n!)^2 = O((2n)!)$. While these two functions are closely related, $(n!)^2 \neq \Theta((2n)!)$; however, there *is* a function $f(n)$ such that $f(n)(n!)^2 = \Theta((2n)!)$. Your challenge—should you choose to accept it—is to find such a function and use the formal definition of Big-Theta to prove that $f(n)(n!)^2 = \Theta((2n)!)$.

Hint: review your (or my) proof of problem 2 on Homework 3, and consider how much bigger $(n!)^2$ needs to get in order to "keep up" with $(2n)!$. In other words, what do you need to multiply $(n!)^2$ and $(2n)!$ by every time you go from $n = k$ to $n = k + 1$ and how do these compare (approximately)? Once you have an idea of what $f(n)$ is, evaluating $f(n)(n!)^2$ and $(2n)!$ for a few values of n should give you an idea of whether they are growing at the same rate.

Let $f(n) = 4^n$. We show that $f(n)(n!)^2 = \Theta((2n)!)$ by proving that $(2n)! \leq 4^n(n!)^2 \leq 2(2n)!$ for all $n \geq 1$ by induction.

Proof. (*Base case*) When $n = 1$, $(2n)! = 2! = 2$, and $4^n(n!)^2 = 4^1(1!)^2 = 4$, so $(2n)! \leq 4^n(n!)^2 \leq 2(2n)!$ when $n = 1$.

(*Inductive step*) Suppose that $(2k)! \leq 4^k(k!)^2 \leq 2(2k)!$ for some $k \geq 1$, and consider $n = k + 1$. First, we prove that $(2(k+1))! \leq 4^{k+1}((k+1)!)^2$:

$$\begin{aligned}
(2(k+1))! &= (2k+2)! \\
&= (2k+2)(2k+1)(2k)! \\
&\leq (2k+2)(2k+1)4^k(k!)^2 \\
&\leq (2k+2)^2 4^k(k!)^2 \\
&= 2^2(k+1)^2 4^k(k!)^2 \\
&= 4(k+1)^2 4^k(k!)^2 \\
&= 4^{k+1}((k+1)!)^2
\end{aligned}$$

Now, we prove that $4^{k+1}((k+1)!)^2 \leq 2(2(k+1))!$:

$$
\begin{aligned}
4^{k+1}((k+1)!)^2 &= 4(k+1)^2 4^k (k!)^2 \\
&\le 4(k+1)^2(2k)! \\
&\le 4(k+1)(2k+1)(2k)! \\
&= 2(2k+2)(2k+1)(2k)! \\
&= 2(2k+2)! \\
&= 2(2(k+1))!
\end{aligned}
$$

Since $(2(k+1))! \le 4^{k+1}((k+1)!)^2$ and $4^{k+1}((k+1)!)^2 \le 2(2(k+1)!)$, $(2(k+1))! \le 4^{k+1}((k+1)!)^2 \le 2(2(k+1))!$. Therefore, $(2n)! \le 4^n(n!)^2 \le 2(2n)$ for all $n \ge 1$, by induction.

As such, there exist positive constants c_1, c_2, and n_0 (namely $c_1 = 1$, $c_2 = 2$, and $n_0 = 1$) such that $c_1(2n)! \le 4^n(n!)^2 \le c_2(2n)!$ for all $n \ge n_0$, so $4^n(n!)^2 = \Theta((2n)!)$. □

Why does $f(n) = 4^n$ work?

There are multiple ways to arrive at $f(n) = 4^n$ as a solution to this problem. Per the problem hint, if we compare $n!^2$ and $(2n)!$ as they go from $n = k$ to $n = k+1$, we see that $(k+1)!^2 = (k+1)^2k!^2$, while $(2k+2)! = (2k+2)(2k+1)(2k)!$. As $2k+2$ and $2k+1$ are both roughly twice as large as $k+1$, $(2n)!$ is growing roughly 4 times as fast as $n!^2$ every time n increments. Since $f(n) = 4^n$ is the function that increases by a factor of 4 every time n increments, it approximates the ratio between these functions.