Homework 5 sample solution

Due 09/13/16

September 8, 2016

1. Use the formal definition of Big-Oh to prove the extension of the Envelopment Property of Addition to more than two functions; that is, if $f_1(n)$, $f_2(n)$, ..., $f_x(n)$, and $g_1(n)$, ..., $g_x(n)$ are functions of n such that $g_1(n) = O(f_1(n))$, $g_2(n) = O(f_2(n))$, etc., then

   $$\sum_{i=1}^{x} g_i(n) = O\left(\sum_{i=1}^{x} f_i(n)\right),$$

   for all $x \geq 2$.

   Answer:

   Proof. We prove the claim directly, by using an arbitrary value of $x \geq 2$. Note that the claim could also be proven by induction, using a similar argument.

   Suppose that $f_1(n)$, ..., $f_x(n)$ and $g_1(n)$, ..., $g_x(n)$ are functions such that $g_i(n) = O(f_i(n))$, for all $g_i(n)$. By the definition of Big-Oh, there exist positive constants c_1, n_1, c_2, n_2, ..., c_x, and n_x such that $g_1(n) \leq c_1 f_1(n)$ for all $n \geq n_1$, $g_2(n) \leq c_2 f_2(n)$ for all $n \geq n_2$, ..., and $g_x(n) \leq c_x f_x(n)$ for all $n \geq n_x$. If we let $c_0 = \max\{c_1, c_2, \ldots, c_x\}$ and $n_0 = \max\{n_1, n_2, \ldots, n_x\}$, we see that $g_1(n) \leq c_1 f_1(n)$, $g_2(n) \leq c_2 f_2(n)$, ..., $g_x(n) \leq c_x f_x(n)$ for all $n \geq n_0$. Hence:

   $$\begin{aligned}
   \sum_{i=1}^{x} g_i(n) &= g_1(n) + g_2(n) + \ldots + g_x(n) \\
   &\leq c_1 f_1(n) + c_2 f_2(n) + \ldots + c_x f_x(n) && \forall n \geq n_0 \\
   &\leq c_0 f_1(n) + c_0 f_2(n) + \ldots + c_0 f_x(n) && \forall n \geq n_0 \\
   &= c_0(f_1(n) + f_2(n) + \ldots + f_x(n)) && \forall n \geq n_0 \\
   &= c_0 \left(\sum_{i=1}^{x} f_i(n)\right) && \forall n \geq n_0
   \end{aligned}$$

   Since c_0 and n_0 are the maxima of positive values, they must themselves be positive, so there exist positive constants $_0$ and n_0 such that $\sum_{i=1}^{x} g_i(n) \leq$

$c_0(\sum_{i=1}^{x} f_i(n))$ for all $n \geq n_0$. Thus, $\sum_{i=1}^{x} g_i(n) = O(\sum_{i=1}^{x} f_i(n))$ by the formal definition of Big-Oh. $\square$

2. Use the properties of Big-Theta presented in class (*not the formal definition*) to prove that if $f(n) = 561n\lg(n) + 17.9n\sqrt{n} + 1024$, $g(n) = \Theta(f(n))$, and $h(n) = O(f(n))$, then $g(n)h(n) = O(n^3)$. You may assume that $\ln n = O(\sqrt{n})$ and $n\sqrt{n} = \Omega(1)$. *Hint*: start by proving that $f(n) = \Theta(n\sqrt{n})$.

Answer:

Proof. First, note that since $\lg n = O(\sqrt{n})$ and $n = O(n)$ (by the Reflexive Property), $n \lg n = O(n\sqrt{n})$. Also, since $n\sqrt{n} = \Omega(1)$, $1 = O(n\sqrt{n})$. Thus:

$$
\begin{aligned}
f(n) &= 561n\lg(n) + 17.9n\sqrt{n} + 1024 \\
&= \Theta(561n\lg(n)) + \Theta(17.9n\sqrt{n}) + \Theta(1024) && \text{Reflexive} \\
&= \Theta(n\lg(n)) + \Theta(n\sqrt{n}) + \Theta(1) && \text{Constant coefficients} \\
&= \Theta(n\lg(n)) + \Theta(n\sqrt{n} + 1) && \text{Envelopment (addition)} \\
&= \Theta(n\lg(n)) + \Theta(n\sqrt{n}) && \text{Greatest term: } 1 = O(n\sqrt{n})) \\
&= \Theta(n\lg(n) + n\sqrt{n}) && \text{Envelopment (addition)} \\
&= \Theta(n\sqrt{n}) && \text{Greatest term: } n\lg(n) = O(n\sqrt{n})
\end{aligned}
$$

As such, $f(n) = \Theta(n\sqrt{n})$. So, $g(n) = \Theta(n\sqrt{n})$ by Transitivity. Since $f(n)$ and $g(n)$ are $\Theta(n\sqrt{n})$, they are both $O(n\sqrt{n})$. Hence, $h(n) = O(n\sqrt{n})$, by Transitivity. Thus:

$$
\begin{aligned}
g(n)h(n) &= O(n\sqrt{n})O(n\sqrt{n}) \\
&= O(n\sqrt{n}(n\sqrt{n})) && \text{Envelopment (multiplication)} \\
&= O(n^3)
\end{aligned}
$$

$\square$