1. If $g_x(n) = O(f_x(n))$, prove that $\sum_{i=1}^{x} g_i(n) = O(\sum_{i=1}^{x} f_i(n))$ for $x \geq 2$.

   B/C: When $x = 2$,

   $g_1(n) \leq c_1 \cdot f_1(n)$, for $n \geq n_1$ and $g_2(n) \leq c_2 \cdot f_2(n)$, for $n \geq n_2$

   If $\sum_{i=1}^{2} g_i(n) = O(\sum_{i=1}^{2} f_i(n))$, then $g_1(n) + g_2(n) \leq c_1 \cdot f_1(n) + c_2 \cdot f_2(n)$.

   Suppose there is some c_3 and n_3 such that $c_3 = \max(c_1, c_2)$ and $n_3 = \max(n_1, n_2)$.

   Then, the above is equivalent to:
   $$g_1(n) + g_2(n) \leq c_3 \cdot f_1(n) + c_3 \cdot f_2(n)$$
   $$g_1(n) + g_2(n) \leq c_3 \cdot (f_1(n) + f_2(n))$$
   $$g_1(n) + g_2(n) = O(\left(f_1(n) + f_2(n)\right)\text{, for } c = c_3 \text{ and } n \geq n_3.$$

   I/S: Suppose that $\sum_{i=1}^{k} g_i(n) = O(\sum_{i=1}^{k} f_i(n))$, for some $k \geq 2$.

   $\sum_{i=1}^{k+1} g_i(n) = \sum_{i=1}^{k} g_i(n) + g_{k+1}(n)$

   $\sum_{i=1}^{k+1} g_i(n) \leq \sum_{i=1}^{k} g_i(n) + c_{k+1} \cdot f_{k+1}(n)$, for all $n \geq n_{k+1}$

   $\sum_{i=1}^{k+1} g_i(n) \leq c_k \cdot \sum_{i=1}^{k} f_i(n) + c_{k+1} \cdot f_{k+1}(n)$, for all $n_{max} = \max(n_k, n_{k+1})$

   Suppose there is some c_{max} such that $c_{max} = \max(c_k, c_{k+1})$.

   Then $\sum_{i=1}^{k+1} g_i(n) \leq c_{max} \cdot \sum_{i=1}^{k} f_i(n) + c_{max} \cdot f_{k+1}(n)$, which is equivalent to

   $\sum_{i=1}^{k+1} g_i(n) \leq c_{max} \cdot \sum_{i=1}^{k+1} f_i(n)$, for all $n \geq n_{max}$

2. Suppose $f(n) = 561 \cdot n \cdot \lg(n) + 17.9 \cdot n\sqrt{n} + 1024$, $g(n) = \Theta(f(n))$, and $h(n) = O\left(f(n)\right)$. We may assume that $\ln n = O(\sqrt{n})$ and $n\sqrt{n} = \Omega(1)$.

   $$f(n) = 561 \cdot n \cdot \lg(n) + 17.9 \cdot n \cdot \sqrt{n} + 1024$$
   $$f(n) = \Theta\left(561 \cdot n \cdot \lg(n) + 17.9 \cdot n \cdot \sqrt{n} + 1024\right)$$
   $$f(n) = \Theta(561 \cdot n \cdot \lg(n)) + \Theta\left(17.9 \cdot n \cdot \sqrt{n}\right) + \Theta(1024)$$
   $$f(n) = \Theta(1 \cdot n \cdot \lg(n)) + \Theta\left(1 \cdot n \cdot \sqrt{n}\right) + \Theta(1)$$
   $$f(n) = \Theta(n \cdot \lg(n)) + \Theta\left(n \cdot \sqrt{n}\right) + \Theta(1)$$
   $$f(n) = \Theta(n) \cdot \Theta(\lg(n)) + \Theta\left(n\right) \cdot \Theta(\sqrt{n}) + \Theta(1)$$
   $$f(n) = \Theta(n) \cdot (\Theta(\lg(n)) + \Theta(\sqrt{n})) + \Theta(1)$$
   $$f(n) = \Theta(n) \cdot \Theta(\sqrt{n}) + \Theta(1)$$
   $$f(n) = \Theta(n\sqrt{n} + 1)$$
   $$f(n) = \Theta(n\sqrt{n})$$

   By the anti-symmetry property, $g(n) = \Theta\left(f(n)\right) = O\left(f(n)\right)$ and $f(n) = \Theta\left(n\sqrt{n}\right) = O\left(n\sqrt{n}\right)$.

   Since $h(n) = O\left(f(n)\right)$, then $g(n) \cdot h(n) = O(O\left(n\sqrt{n}\right)) \cdot O(O\left(n\sqrt{n}\right))$ which, by the transitivity property, is equal to $O\left(n\sqrt{n}\right) \cdot O\left(n\sqrt{n}\right)$. Finally, by the envelopment property of multiplication, $g(n) \cdot h(n) = O\left(n\sqrt{n} \cdot n\sqrt{n}\right) = O(n \cdot n \cdot n) = O(n^3)$.