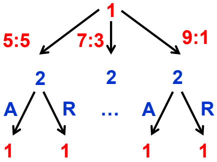

Game Theory:
Congestion and Extensive Games

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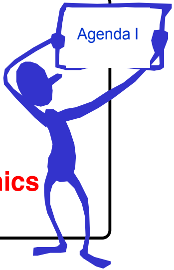
Congestion Games - Agenda

Every normal-form game has a Nash equilibrium,
although not necessarily one that is pure. Pure equilibria are nicer ☺

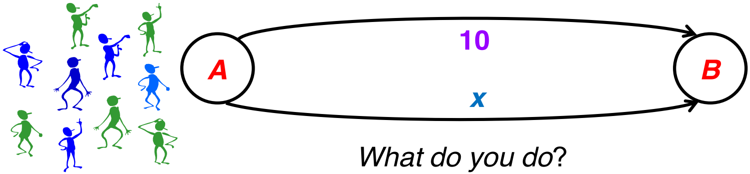
A family of games of practical interest where we can guarantee the existence of pure Nash equilibria

- **Congestion games**: example and definition
- **Potential games**: tool to analyze congestion games
- Existence of pure Nash equilibria for both types of games
- Finding those equilibria by means of **better-response dynamics**
- (Briefly) price of anarchy: quality guarantees for equilibria



Y. Shoham and K. Leyton-Brown. Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations. *Cambridge University Press*, 2012

10 *people need to get from* ***A*** *to* ***B****. Everyone can choose between the* ***top*** *and the* ***bottom route****. Via the top route, the trip takes* ***10 mins****. Via the bottom route, it depends on* ***the number of fellow travelers****: it takes as many minutes* ***x*** *as there are people using this route.*



What do you do?

What are the pure Nash equilibria?

Example: The El Farol Bar Problem

100 *people consider visiting the* ***El Farol Bar*** *on a Monday night. They all have identical preferences*:

- If 60 or more people show up, it's nicer to be at home
- If fewer than 60 people show up, it's nicer to be at the bar

Now what? What are the pure Nash equilibria of this game?

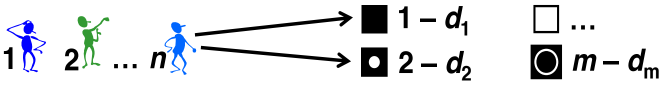
Congestion Games - Notation

A normal-form game is a tuple $\langle N, R, A, d \rangle$ where:

- $N = \{1, \dots, n\}$ is a **finite set of players** (or agents)
- $R = \{1, \dots, m\}$ is a **finite set of resources**
- $A = A_1 \times \dots \times A_n$ is a **finite set of action profiles** $a = (a_1, \dots, a_n)$, with $A_i \subseteq 2^R$ being the **set of actions** available to player $i \in N$
- $d = (d_1, \dots, d_m)$ is a vector of **delay (cost) functions** $d_r : N \rightarrow \mathbb{R}_{\geq 0}$, each of which is required to be non-decreasing

- Every player i chooses a subset of resources to use (that is action)
- Each d_r is associated with a resource (not with a player)

Simplification of games via constraints on the effects that a single player's action can have on other players' utilities



Congestion Games - Utility Function

Let $\boldsymbol{n_r^a} = \#\{\boldsymbol{i \in N : r \in a_i}\}$ be the numbers of players claiming $\boldsymbol{r}$ in $\boldsymbol{a}$.
The ***cost*** incurred by player $\boldsymbol{i}$ is the sum of the delays she experiences due to the congestion of the resources she picks. Her ***utility*** then is:

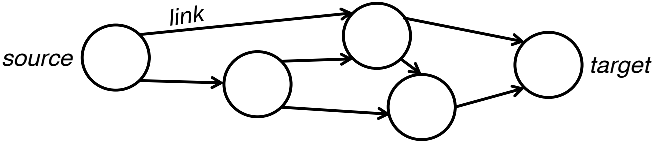
$$u_i(\boldsymbol{a}) = -\, cost_i(\boldsymbol{a}) = -\sum_{r \in a_i} d_r(n_r^a)$$

- **The same utility function for all players**
- ***Anonimity***: you can about how many others use a resource, but not about which other do so

$u_1(\boldsymbol{a})$ $u_2(\boldsymbol{a})$ ... $u_n(\boldsymbol{a})$

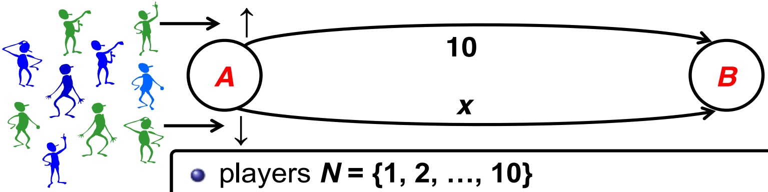
- Imagine a ***computer network*** in which several users want to send a message from one node to another at the same time
- Each ***link is a resource***, and an action for a user is to select a path of links connecting the source and the target
- The ***cost*** function = the latency on link as a function of its congestion

Example:
computer network



Formally Modelling the Examples (1)

10 *people need to get from* ***A*** *to* ***B****. Everyone can choose between the* ***top*** *and the* ***bottom route****. Via the top route, the trip takes 10 mins. Via the bottom route, it depends on the number of fellow travellers: it takes as many minutes x as there are people using this route.*



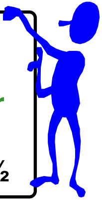
- players $\boldsymbol{N} = \{1, 2, \dots, 10\}$
- resources $\boldsymbol{R} = \{\uparrow, \downarrow\}$
- action spaces $\boldsymbol{A_i} = \{\{\uparrow\}, \{\downarrow\}\}$ representing the two routes
- delay functions $\boldsymbol{d_\uparrow} = \mathbf{x} \to \mathbf{10}$ and $\boldsymbol{d_\downarrow} = \mathbf{x} \to \mathbf{x}$

Formally Modelling the Examples (2)

100 *people consider visiting the* ***El Farol Bar*** *on a Monday night. They all have identical preferences:*

- If 60 or more people show up, it's nicer to be at home
- If fewer than 60 people show up, it's nicer to be at the bar

- players $\boldsymbol{N} = \{1, 2, \ldots, 100\}$
- resources $\boldsymbol{R} = \{ ☺, ⌂_1, ⌂_2, \ldots, ⌂_{100} \}$
- action spaces $\boldsymbol{A}_i = \{ ☺, ⌂_i \}$ representing the **bar** and player *i*'s home
- delay functions $\boldsymbol{d}_{☺} = \boldsymbol{x} \rightarrow \boldsymbol{1}_{x \geq 60}$ and $\boldsymbol{d}_{⌂_i} = \boldsymbol{x} \rightarrow ½$



Remark: *Neither example makes full use of the power of the model, as every player only ever claims a single resource*

Variations and Existence of Pure Nash Equilibria

Our model of congestion games has certain restrictions:

- Utility functions are ***additive*** (no synergies between resources)
- Delay functions are ***not player-specific*** (no individual tastes)

Some careful relaxations of these assumptions have been considered in the literature, but we are not going to do so here.

Theorem (Rosenthal, 1973) Every **congestion game** has at least one **pure Nash equilibrium**

R.W. Rosenthal. A Class of Games Possessing Pure-Strategy Nash Equilibria. *International Journal of Game Theory*, 2(1):65-67, 1973.

Nice, because mixed-strategy equilibria are open to criticism and if we want to compute a sample equilibrium, it is relatively easy

A normal-form game $\langle N, A, u \rangle$ is a **potential game** if there exists a function $P : A \to \mathbf{R}$ such that, for all $i \in N$, $a \in A$, and $a_i' \in A_i$:

$$u_i(a) - u_i(a_i', a_{-i}) = P(a) - P(a_i', a_{-i})$$

The game underlying the Prisonner's Dilemma is a ***potential game****, because we can define a function* ***P*** *from action profiles (matrix cells) to the reals that correctly tracks any* ***unilateral deviation****:*

O \ M	**C**	**D**
C	-10 \ -10	-25 \ 0
D	0 \ -25	-20 \ -20

$P(C,C) = 50$
$P(C,D) = 60$
$P(D,C) = 60$
$P(D,D) = 65$

For example, if ***Mateusz*** *deviates from* **(C,C)** *to* **(C,D)**, *his utility will increase by 10, and indeed:* $P(C,D) - P(C,C) = 60 - 50 = 10$

Example: Matching Pennies

*Each player gets a penny and secretly displays either **H**eads or **T**ails. **Oliwia** wins if the two pennies agree; **Mateusz** wins if they don't.*

O \ M	H	T
H	1 \ -1	-1 \ 1
T	-1 \ 1	1 \ -1

***P*(H,H)** = 5
***P*(H,T)** = 7
***P*(T,H)** = 3
***P*(T,T)** = ? (1 and 9)

Exercise: Show that this is **not** a potential game

Existence of Pure Nash Equilibria

A game $\langle N, A, u\rangle$ is called a **potential game** if there exists a function $P : A \rightarrow \mathbf{R}$ such that, for all $i \in N$, $a \in A$, and $a_i' \in A_i$:

$$u_i(a) - u_i(a_i', a_{-i}) = P(a) - P(a_i', a_{-i})$$

Theorem (Monderer and Shapley, 1996)

Every **potential game** has at least one **pure Nash equilibrium**

D. Monderer and L.S. Shapley. Potential Games. *Games and Economic Behavior*, 14(1):124-143, 1996.

Proof: Take (one of) the action profile(s) a for which P is maximal. By definition, no player can benefit by deviating using a pure strategy. Thus, also not using a mixed strategy. Hence, a must be a pure NE.

O \ M	C	D
C	-10 \ -10	-25 \ 0
D	0 \ -25	-20 \ -20

$P(C,C) = 50$
$P(C,D) = 60$
$P(D,C) = 60$
$P(D,D) = 65$

We still need to prove that also every congestion game has a pure NE. We are done, if we can prove the following lemma:

Lemma: Every **congestion game** is a **potential game**.

Proof: Take any congestion game $\langle \boldsymbol{N}, \boldsymbol{R}, \boldsymbol{A}, \boldsymbol{d} \rangle$.

$$u_i(\boldsymbol{a}) = -\sum_{r \in a_i} d_r(\boldsymbol{n}_r^{\mathbf{a}}), \text{ where } \boldsymbol{n}_r^{\mathbf{a}} = \#\{\boldsymbol{i} \in \boldsymbol{N} : \boldsymbol{r} \in \boldsymbol{a}_i\}$$

Now define function $\boldsymbol{P}$ as follows:

$$P(\boldsymbol{a}) = -\sum_{r \in R} \sum_{k=1,\dots,n_r^a} d_r(\boldsymbol{k}), \text{ for all } \boldsymbol{a} \in \boldsymbol{A}$$

It is easy to verify that: $\boldsymbol{u_i(a) - u_i(a_i', a_{-i}) = P(a) - P(a_i', a_{-i})}$.

Thus, $\boldsymbol{P}$ is a potential for our congestion game.

Intuition: $d_r(\boldsymbol{k})$ is the cost of the $\boldsymbol{k}$-th player arriving at resource $\boldsymbol{r}$

Better-Response Dynamics

We ***start*** in some action profile $\boldsymbol{a}^0$. Then, ***at every step***, some player ***i unilaterally deviates*** to achieve an ***outcome*** that is ***better*** for her:

- $a_i^k \in A_i$ such that - $u_i(a_i^k, a_{-i}^{k-1}) > u_i(a_i^{k-1})$
- $a_{i'}^k = a_{i'}^{k-1}$ for all other players $i \in N \setminus \{i\}$

This leads to a sequence $\boldsymbol{a}^0 \to \boldsymbol{a}^1 \to \boldsymbol{a}^2 \to \boldsymbol{a}^3 \to \ldots$

	A	B
A	3 \ 3	0 \ 4
B	4 \ 0	2 \ 2

$\boldsymbol{a}^0 = (\mathbf{A}, \mathbf{A}) \to$
$\to \boldsymbol{a}^1 = (\mathbf{A}, \mathbf{B}) \to$
$\to$ **… ?**

A game has the **finite improvement property** (FIP) if it does not permit an infinite sequence of better responses of this kind.

Observation: If a profile ***a*** does not admit a ***better response***, then ***a*** is a ***pure Nash equilibrium***. The converse is also true.

Observation: Every game with the ***FIP*** has a ***pure Nash equilibrium***. The converse is not true (*see next slide*).

Exercise: Better-Response Dynamics

For the games below, all pure Nash equilibria are shown in boldface:

	C	D
C	-10 \ -10	-25 \ 0
D	0 \ -25	**-20 \ -20**

	L	C	R
T	1 \ -1	-1 \ 1	-5 \ -5
M	-1 \ 1	1 \ -1	-5 \ -5
B	-5 \ -5	-5 \ -5	**5 \ 5**

	H	T
H	1 \ -1	-1 \ 1
T	-1 \ 1	1 \ -1

Suppose we start in the upper lefthand cell
and players keep playing better (or best) responses.
What will happen?

On some games, the myopic best response algorithm can fail to terminate, but for the congestion and potential games…

Finite Improvement Property

Potential and congestion games not only all have pure Nash equilibria, but it also is natural to believe players will actually find them ...

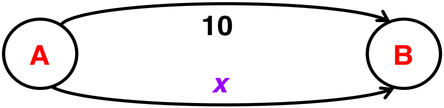
Theorem (Moderer and Shapley, 1996) Every **potential game** has the ***FIP***. Thus, also every ***congestion game*** has the ***FIP***.

D. Monderer and L.S. Shapley. Potential Games. *Games and Economic Behavior*, 14(1):124-143, 1996.

Proof: By definition of the potential P, we get $P(a^k)$ - $P(a^{k-1})$ for any two consecutive action profiles in a better-response sequence. The claim then follows from finiteness.

So: in a congestion game, the natural better-response dynamics will always lead us to a pure NE. Nice. *But*: how good is that equilibrium?

Recall our **traffic congestion example**:



10 people overall
top delay = 10 minutes
bottom delay = # on route

If $x \leq 10$ players use bottom route, social welfare (sum of utilities) is:

$$sw(x) = -[x \cdot x + (10 - x) \cdot 10] = -[x^2 - 10 \cdot x + 100]$$

- This function is maximal for $x = 5$ and minimal for $x = 0$ and $x = 10$
- In equilibrium, 9 or 10 people will use the bottom route (10 is worse)

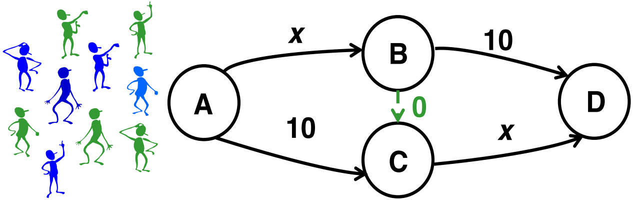
The so-called **Price of Anarchy (PoA)** of this game is:

$$sw(10)/sw(5) = -100/-75 = 4/3$$

Thus: not perfect, but not too bad either (for this example).

Intuitively, **Price of Anarchy** is the proportion of additional social cost that is incurred because of players' self-interested behavior

Something to think about. 10 people have to get from A to D:



If we consider the **delay-free link** from ***B*** to ***C***, this happens:

- In the worst equilibrium, everyone will take the route ***A-B-C-D*** and take 20 minutes! (Other equilibria are only slightly better.)

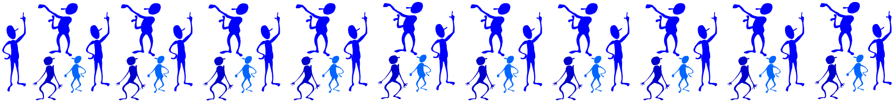
If the **delay-free link** from ***B*** to ***C*** is ***not*** present:

- In equilibrium, 5 people will use the top route ***A-B-D*** and 5 people the bottom route ***A-C-D***. Everyone will take 15 minutes.

Congestion Games - Summary

We have analyzed a specific class of games, the **congestion games**:

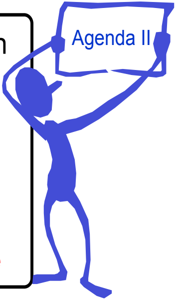
- Natural model for ***applications***
- ***Rosenthal's Theorem***: every congestion game has a pure NE
- Analysis via the more general ***potential games***
- Finite improvement property: can find NE via better responses
- **Price of anarchy**: how much worse if we don't impose outcome?
- Beware of counter-intuitive effects (**Braess' paradox**)



Extensive Games - Agenda

Extensive games model individual **actions being played in sequence** (i.e., we do not assume players act simultaneously)

- Focus on modeling **extensive games** of perfect information (the temporal component made explicit)
- **Translation** from the extensive into the normal form
- **Zermelo's Theorem**: existence of pure Nash equilibria
- New solution concept: **subgame-perfect equilibria** (explicitly refer to the sequence in which players act)
- Famous examples: **ultimatum game** and **centipede game**

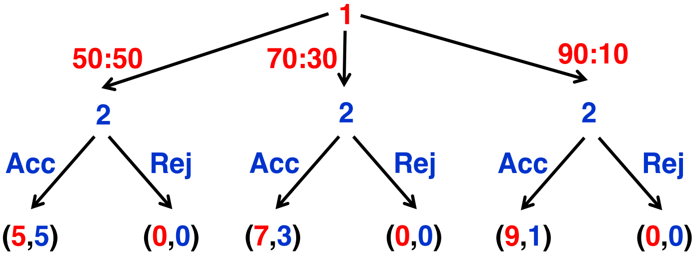


K. Leyton-Brown and Y. Shoham. Essentials of Game Theory: A Concise, Multi-disciplinary Introduction. *Morgan & Claypool Publishers*, 2008

Extensive Games - Example

Tree in the sense of graph theory, in which:

- **Node** represents the choice of one of the players
- **Edge** represents a possible action
- **Leaf** represents final outcomes



Player 1 chooses a division of a given amount of money.
Player 2 accepts this division or rejects it (in which case both get nothing).

Strategic Games in Extensive Form

An extensive-form game is a tuple $\langle N, A, H, Z, \underline{i}, \underline{A}, \sigma, u \rangle$ where:

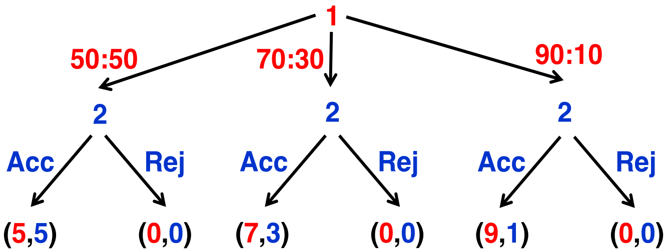
- $N = \{1, \ldots, n\}$ is a **finite set of players**
- A is a **single set of actions**
- H is a set of **choice nodes** (non-leaf nodes of the tree)
- Z is a set of **outcome nodes** (leaf nodes of the tree)
- $\underline{i} : H \rightarrow N$ is the **turn function**, fixing whose turn it is when
- $\underline{A} : H \rightarrow 2^A$ is the **action function**, fixing the playable action
- $\sigma : H \times A \rightarrow H \cup Z$ is the (injective) **successor function**
- $u = (u_1, \ldots, u_n)$ is a **profile of utility functions** $u_i : Z \rightarrow \mathbf{R}$

Must be finite. Must have exactly one root $h_0 \in H$ s.t. $h_0 = \sigma(h,a)$ for all $h \in H$ and $a \in A$. Must have $\underline{A}(h) \neq \emptyset$ for all nodes $h \in H$.

Remark: Requiring σ to be **injective** ensures every node has (at most) one parent (so the descendants of h_0 really form a tree).

Example: Ultimatum Game

***Player* 1** chooses a division of a given amount of money.
***Player* 2** accepts this division or rejects it (in which case both get nothing).



What strategy would you adopt as ***Player* 1**? And as ***Player* 2**?

<u>*Exercise*</u>: Describe this game using our formal definition of extensive games. Note that the picture is missing names for nodes in $H \cup Z$.

$N = \{1,2\}$, A, H, Z, $\underline{i} : H \to N$ - **turn function**, $\underline{A} : H \to 2^A$ - **action function**, $\sigma : H \times A \to H \cup Z$ - **successor function**, $\underline{u_i} : Z \to \mathbb{R}$ - **utility function**

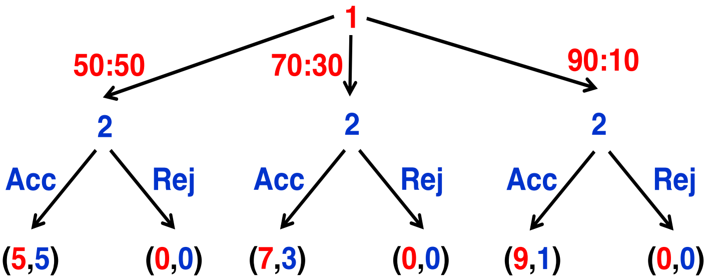
Pure Startegies

- Notation: Write $\boldsymbol{H_i} := \{\boldsymbol{h} \in \boldsymbol{H} \mid \underline{\boldsymbol{i}}(\boldsymbol{h}) = \boldsymbol{i}\}$ for the set of choice nodes in which it is player ***i***'s turn to choose an action.
- A **pure strategy** for player ***i*** maps nodes $\boldsymbol{h} \in \boldsymbol{H_i}$ to actions in $\underline{\boldsymbol{A}}(\boldsymbol{h})$
  Thus, it is a function $\boldsymbol{\alpha_i} : \boldsymbol{H_i} \rightarrow \boldsymbol{A}$ that respects $\boldsymbol{\alpha_i(h) \in \underline{A}(h)}$
  (a complete specification of which deterministic action to take at every node "belonging" to player ***i***)
  Remark: A strategy describes what to do for every choice node where it would be your turn, even those you may never actually reach.

Given a profile $\boldsymbol{\alpha} = (\boldsymbol{\alpha_1}, \ldots, \boldsymbol{\alpha_n})$ of pure strategies, the **outcome** of the game is the outcome node computed by this program:

```
h ← h_0                      // start at the root node
while h∉Z do                 // player i, whose turn it is, conducts
    h ← σ(h, α_{i(h)}(h))       an action (moving from one node to another)
return h                     // terminate once reaching some leaf
```

Pure Strategies - Example



The **pure strategies** of the players:

- $A_1^* = \{50{:}50, 70{:}30, 90{:}10\}$
- $A_2^* = \{$Acc-Acc-Acc, Acc- Acc-Rej, Acc-Rej-Acc, Acc-Rej-Rej, Rej-Acc-Acc, Rej-Acc-Rej, Rej-Rej-Acc, Rej-Rej-Rej$\}$

Translation to Normal Form

Every extensive-form game can be translated into a normal-form game

We can translate $<N, A, H, Z, \underline{i}, \underline{A}, \sigma, u>$ to normal-form game $<N^*, A^*, u^*>$:

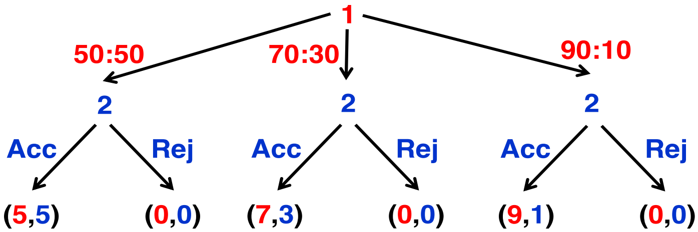
- $N = N^*$ – the same set of players
- $A^* = A_1^* \times \ldots \times A_n^*$ with $A_i^* = \{\alpha_i^* : H_i \rightarrow A \mid \alpha_i(h) \in \underline{A}(h)\}$, i.e., the set of action profiles in the normal-form game is the set of pure-strategy profiles in the extensive game
- $u^* = (u_1^*, \ldots, u_n^*)$ with $u_i^*: \alpha = (\alpha_1, \ldots, \alpha_n) \rightarrow u_i(\text{out}(\alpha))$, where $\text{out}(\alpha)$ is the outcome of the extensive game under pure-strategy profile α

Thus, the full machinery developed for normal-form games (such as mixed strategies, Nash equilibria, other solution concepts) is available.

So why use the extensive form at all?

- Because it (often) is a more ***compact*** (exponentially smaller) as well as ***intuitive*** form of representation (much more natural to reason about)

Exercise: Translation to Normal Form



Convertion to the normal form image of the game:

- Strategy spaces pf the two games are the same
- The pure (and mixed) Nash equilibria are the same
- The transformation can result in a certain redundancy in the normal form and exponential blowup of the game representation

	Acc-Acc-Acc	Acc-Acc-Rej	Acc-Rej-Acc	...
50:50	5 \ 5	5 \ 5	5 \ 5	...
70:30	7 \ 3	7 \ 3	0 \ 0	...
90:10	9 \ 1	0 \ 0	9 \ 1	...

Translation From Normal Form

Can we also translate from **normal-form** to **extensive-form** games? **No!**

	C	D
C	-10 \ -10	-25 \ 0
D	0 \ -25	**-20 \ -20**

At least not in all cases.
The perfect information game cannot model simultaneity.
So the **normal form is more general**.

Existence of Pure Nash Equilibria

Theorem (Zermelo, 1913) Every (finite) **extensive-form game** has at least one ***pure Nash equilibrium***.

E. Zermelo. Uber eine Anwendung der Mengenlehre auf die Theorie des **Schachspiels**. *Proc. 5th International Congress of Mathematicians*, 1913

- Players take turns and everyone gets to see eveything that happened thus far
- It is never necessary to introduce randomness into action selection to find an equilibrium

Existence of Pure Nash Equilibria - Backward Induction

Theorem (Zermelo, 1913) Every (finite) **extensive-form game** has at least one ***pure Nash equilibrium***.

E. Zermelo. Uber eine Anwendung der Mengenlehre auf die Theorie des **Schachspiels**. *Proc. 5th International Congress of Mathematicians*, 1913

Proof: Work your way up, from the "lowest" choice nodes to the root. Label each $h \in H$ with an action $a^* \in \underline{A}(h)$ and a vector $(u_1^h, \ldots, u_n^h)$:

- Find (one of) the best action(s) for the selected player $i^* = \underline{i}(h)$:

$$a^* \in argmax_{a \in \underline{A}(h)}\, u_{i^*}^{\sigma(h,a)}$$

- Compute the utility labels u_i^h for node h for all agents $i \in N$:

$$u_i^h = u_i^{\sigma(h,a^*)} \text{ (where } u_i^z = u_i(z) \text{ for any } z \in Z\text{)}$$

This process is well-defined and terminates. And by construction, the resulting assignment $\{h \rightarrow a^*\}$ of nodes to pure strategies is a NE. This method for solving a game is called **backward induction**.

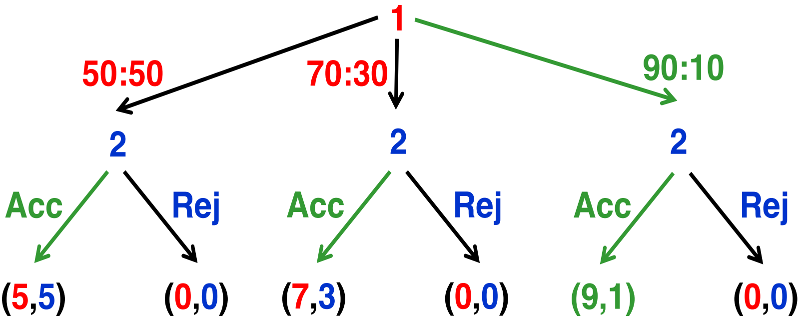
Historical Note: Relevance to Chess

Of course, Zermelo did not phrase his result quite like that: extensive games and Nash equilibria were introduced much later than 1913.

The title of Zermelo's paper mentions chess (das Schachspiel):

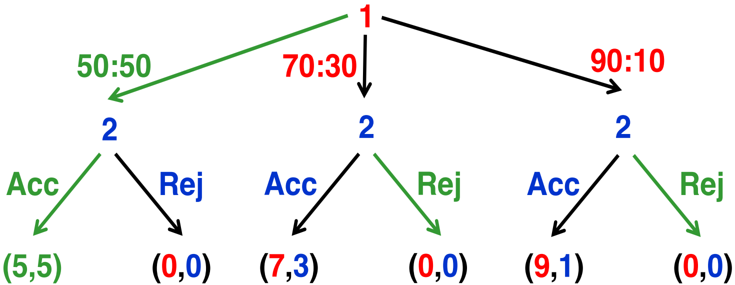
- Using essentially the same argument we have (**backward induction**) it is easy to see that chess must be determined: either White has a **winning strategy**, or Black has, or both players can force a draw
- Of course, the existence of such a strategy does not mean that anyone knows what it actually looks like (the game tree is too big)
- Still, the basic idea of backward induction is at the bottom of any chess-playing program (and the same is true for similar games)

Here is the (only!) Nash equilibrium (**90:10**, **Acc-Acc-Acc**) you will find by applying **backward induction** to the Ultimatum Game:



Exercise: Is this the only pure Nash equilibrium for this game?

There are several other Nash equilibria, such as (**50:50**, **Acc-Rej-Rej**):



Indeed, no player has an incentive to unilaterally change her strategy.
Nevertheless, this does not seem a reasonable solution for the game:

- **Player 2**'s threats to reject are not credible

Example: In the hypothetical situation where the righthand subgame is reached, to reject (**Rej**) would be a *strictly dominated strategy* for **Player 2**

Subgame-Perfect Equilibrium

- Every internal node $h \in H$ induces a **subgame** in the natural manner
- A strategy profile s is a **subgame-perfect equilibrium** of an extensive game G_0 if, for every (not necessarily proper) subgame G of G_0, the restriction of s to G is a Nash equilibrium.

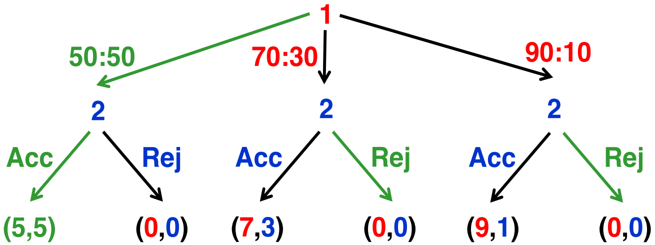
Theorem (Selten, 1965) Every (finite) **extensive-form game** has at least one ***subgame-perfect equilibrium***

R. Selten. Spieltheoretische Behandlung eines Oligopolmodells mit Nachfragetragheit. Zeitschrift fur die Gesamte Staatswissenschaft, 121(2):301-324, 1965

Proof: This is what we showed when we proved Zermelo's Theorem.

Remark: Selten (1965) introduced the concept of SPE for a more specific family of games and did not quite state the theorem above, but these ideas are clearly implicit in that paper.

Example: Subgame-Perfect Equilibrium



- The Selten's theorem rules out "non-credible threats"
- The Nash equilibrium (**50:50**, **Acc-Rej-Rej**) is not subgame perfect

- Consider the middle node of **Player 2**'s choice
- The unique Nash equilibrium of this (trivial) game is for Player 2 to play **Acc**
- Thus, the action **Rej** is not optimal in this subgame and cannot be part of a subgame-perfect equilibrium of the larger game

Backward induction guarantees to find a subgame-perfect equilibrium
(rather than a Nash equilibrium that involves non-credible threats)

Example: Centipede Game

The concept of backward induction is not without controversy

We start in the choice node on the left. Players alternate in making decisions. At each step, the player whose turn it is can choose between going down (i.e., ending the game) and going right (continuing the game except for the last node where going right also ends the game)

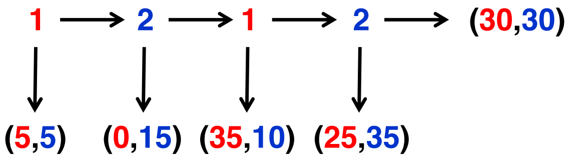
$$
\begin{array}{ccccccccc}
\mathbf{1} & \longrightarrow & \mathbf{2} & \longrightarrow & \mathbf{1} & \longrightarrow & \mathbf{2} & \longrightarrow & \mathbf{(30,30)} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbf{(5,5)} & & \mathbf{(0,15)} & & \mathbf{(35,10)} & & \mathbf{(25,35)} & &
\end{array}
$$

Strategies:

- down-down
- down-right
- right-down
- right-right

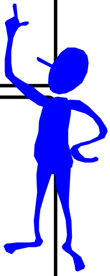
What strategy would you adopt as ***Player 1***?
And as ***Player 2***?

Centipede Game - Discussion (1)



Strategies:
- down-down
- down-right
- right-down
- right-right

The only **subgame-perfect equilibrium** in this game is to always choose to **go down** ↓



Consider the last node:
- The best choice for the player is to go down (35 vs. 30)

Since it is the case, going down is the best choice for the other player in the previous choice point

- By induction the same argument holds for all choice points

- It appears that humans rarely play their SPE strategies
- Even when they do, this can result in **counterintuitive effects**

- Suppose you play your SPE strategy, but your opponent doesn't
- Then you are committed to continuing to play a strategy that you devised on the basis of an assumption (full rationality of your opponent) that just turned out to be wrong…
- How should you amend your beliefs and course of action based on the measure-zero event?
- There exist different accounts of this situation based on what is common knowledge and how one revises beliefs in the face of measure-zero events

This has been an **introduction to extensive games**, where we model the sequential nature of most real games:

- Definition of the formal model
- Pure strategies as functions from choice nodes to actions
- **Translation into normal form is always possible**
- Translation from normal form into extensive form is not
- Noncredible threats call for new solution concept: SPE
- **Subgame-perfect equilibrium** = NE in every subgame
- **Backward induction** shows: SPE and NE always exist