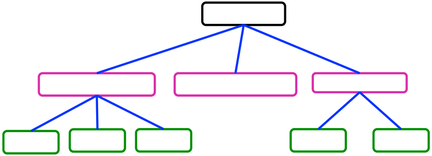

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Analytical Hierarchy Process and the Choquet integral

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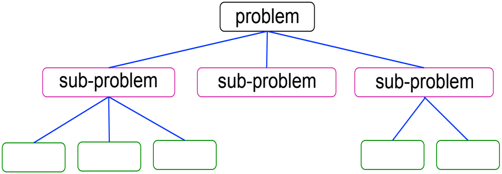
Analytic Hierarchy Process

- Proposed by Thomas Saaty (1977, 1980)
- **Multiple criteria choice and ranking problems**
- Undoubtedly, **the most popular and widely used MCDA method**
- Deals with numerical values (priorities) at all of its stages
  - Recommended when the DMs are unable to construct value functions
- Technically valid and practically useful, though subject to well-grounded and diverse criticism

Thomas L. Saaty

- AHP is based on the motto **divide and conquer**
- Break down multiple criteria problems and solve one 'sub-problem' at a time
- Permits **hierarchical decomposition** of a complex decision problem

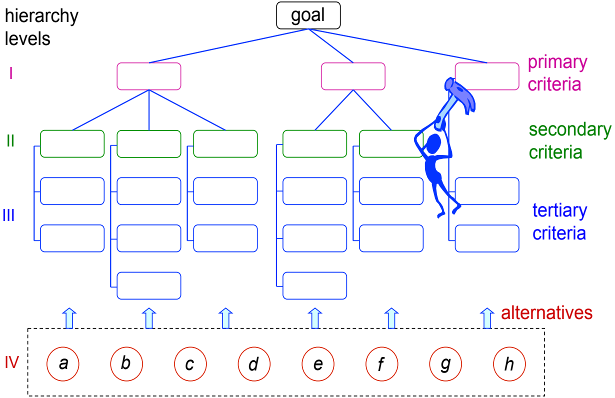


T. Saaty, The Analytic Hierarchy Process : planning, priority setting, resource allocation, New York, McGraw, 1980

Hierarchical Problem Structuring

The problem is structured according to a **hierarchy**

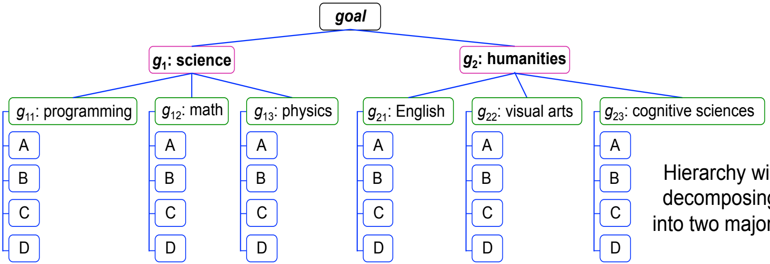
- The **top** element is the **goal** of the decision
- The second level stands for the primary criteria
- The **lowest** level = the **alternatives**
- More levels can be added, representing sub-criteria, sub-sub-criteria, etc.



The breakdown is done in two phases of the decision process:

- **problem structuring** (representing the complex problems and relating them to the goal)
- **elicitation of preference information** (representing preferences, quantifying the importance of various elements, and passing information)

Illustrative Study



Hierarchy with four levels, decomposing the problem into two major sub-problems

Problem:

- Select a student to participate in a research project at the technical university
- **Four candidates**: Ana, Brooke, Caden, and Demi
- Small research team requiring balanced and diverse competencies
- Complete ranking with scores

Criteria:

Two macro subjects: $\boldsymbol{g_1}$ **:** (exact) **science** and $\boldsymbol{g_2}$ **: humanities**, each consisting of three sub-criteria reflecting capacities and skills relevant for the project

- g_{11} : programming, g_{12} : math, g_{13} : physics
- g_{21} : English, g_{22} : visual arts, g_{23} : cognitive sciences

No precise information on the performances (except for English), but the DM knows the candidates and can assess them on each criterion, e.g., by comparing them pairwise

AHP - Major Steps

Set up **decision hierarchy**

⇩

Make systematical **pairwise comparisons of hierarchy elements** (criteria, sub-criteria, and alternatives) with respect to their impact on the element above them

- Human judgments about the relative meaning and importance
- The use of concrete data on performances is possible

⇩

Transform comparisons into **priorities (weights)** and check the consistency

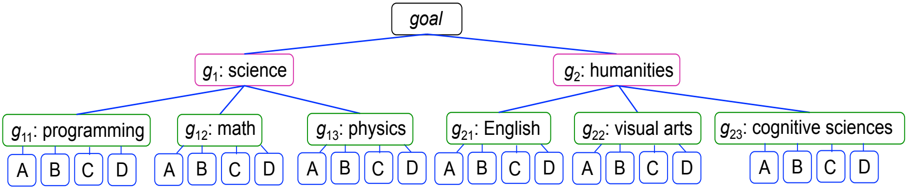
- Numerical values that can be processed and compared over the entire problem
- Diverse elements can be compared in a consistent way

⇩

Use priorities to obtain **comprehensive scores** for alternatives

- Relative abilities of alternatives to achieve the decision goal

Prioritization of Hierarchy Elements



Each lower level is prioritized according to its immediate upper level

The appropriate question to ask with regard to prioritization depends on the context:

- "Which macro subject is more important for choosing the best student for the project?"
- "Which subject is more important for the competencies and skills in exact sciences?"
- "Which student is preferable to fulfill the given competency or skill and to what extent?"
- For the considered problem, nine (6+2+1 = 9) different prioritizations are required, including six prioritizations of alternatives with regard to each elementary criterion

Preference information for elements at each level of the hierarchy: pairwise comparison of all elements i, j at the level of hierarchy h with a common predecessor at level h-1

- The psychologists argue that it is easier and more accurate to express a preference between only two alternatives than simultaneously among all the alternatives

Pairwise Comparison of Hierarchy Elements

The **pairwise comparison** between elements *i* and *j* can be carried out using:

- A **number a_{ij}** expressing the intensity of preference of element *i* over element *j*
- **Verbal appreciation**, more familiar to our daily life that is arbitrarily transformed into a numerical judgment
- **Saaty's scale of preference intensity**

Number	Verbal judgment
1	Equal importance
3	Moderate (weakly more important)
5	Strong (much more important)
7	Very strong (very much more important)
9	Extreme (absolutely more important)
2, 4, 6, 8	Intermediate values (2 – equal to moderate)

Psychologists suggest that:

- a smaller scale, say 1–5, would not give the same level of detail in a data set
- the DM can be lost on a larger scale (e.g., on a 1-100 scale, how to distinguish between 62 and 63?)

Consistent Pairwise Comparison Matrix

- Numbers a_{ij} compose a square matrix of pairwise comparisons
- a_{ij} is supposed to be a ratio of the priorities (weights) of elements i, j

$$a_{ij} = \frac{w_i}{w_j}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

g_1	g_{11}	g_{12}	g_{13}
g_{11}	1	**2**	**6**
g_{12}	1/2	1	**3**
g_{13}	1/6	1/3	1

Example for sub-criteria of **exact sciences**:

- Programming is two times more important than math
- Programming is six times more important than physics
- Math is three times more important than physics

Consistency condition of pairwise comparisons (CCPC):

$$a_{ij} = \frac{1}{a_{ji}}, \quad i, j = 1, \dots, n$$

- Imposed by AHP; 1s on the main diagonal; the matrix is reciprocal (the upper triangle is the reverse of the lower triangle)
- The number of necessary comparisons for each comparison matrix is: $n \cdot (n-1)/2$

Cardinal consistency condition (CCC): $a_{ik} \times a_{kj} = a_{ij}, \quad i, j, k = 1, \dots, n$

$$a_{12} \times a_{23} = a_{13}$$

$$2 \times 3 = 6$$

- Satisfied for the example matrix above
- Not imposed by AHP, though the level of inconsistency is verified (*later*)
- The advantage of precision requires more effort

Inconsistent Pairwise Comparison Matrix

When several successive pairwise comparisons are presented, they may contradict each other

- When the matrix is complete, a **consistency check should be performed**
- Human nature is often inconsistent
- The reasons could be, e.g., vaguely defined problems, lack of sufficient information (known as bounded rationality), uncertain information, lack of concentration, or (an overwhelming) number of comparisons to be performed

g_2	g_{21}	g_{22}	g_{23}
g_{21}	1	**2**	**5**
g_{22}	1/2	1	**3**
g_{23}	1/5	1/3	1

Example for sub-criteria of **humanities** (*involving inconsistency*):

- English is two times more important than visual arts
- Visual arts are three times more important than cognitive sciences
- English is **five** times more important than cognitive science (should be six)

To allow the inconsistent reality, AHP allows up to a 10% inconsistency compared to the average inconsistency of randomly filled matrices (more details to come)

Pairwise Comparison Matrix vs. Priorities

- The **pairwise comparisons matrix** is used to derive **priorities of hierarchy elements**
- A priority $\boldsymbol{w_i}$ is a score that ranks the importance of the alternative, criterion, or sub-criterion in a given group of elements

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} w_1/w_1 & w_1/w_2 & \dots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \dots & w_2/w_n \\ \dots & \dots & \dots & \dots \\ w_n/w_1 & w_n/w_2 & \dots & w_n/w_n \end{bmatrix}$$

- Priorities are numbers associated with the hierarchy nodes, representing the relative weights of the nodes in any group
- The priorities for elements with the same predecessor are automatically normalized to sum to 1

$$\begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{bmatrix}$$

Intermediate results for our problem: macro-criteria, sub-criteria, and local alternative priorities

- E.g., macro-criteria priorities capture the importance of macro criteria with respect to the top goal

Global alternative priorities (to be computed) rank alternatives with respect to all criteria and consequently the overall goal

***FOCUS*: How to determine *w* based on *A*?**

Computing Priorities for Consistent Matrices

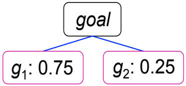
Let us consider a pairwise comparison matrix concerning a pair of macro criteria g_1 and g_2

G	g_1	g_2
g_1	1	3
g_2	1/3	1

If **g_1 is 3 times more important than g_2**, then the priority w_1 of g_1 needs to be 3 times greater than the priority w_2 of g_2

- We opt for values normalized, to sum up to 1

	w	**w'**
g_1	3	**0.75**
g_2	1	**0.25**

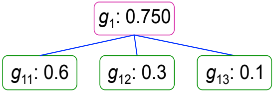


Let us consider a pairwise comparison matrix concerning sub-criteria g_{11}, g_{12}, and g_{13} of science (g_1)

g_1	g_{11}	g_{12}	g_{13}
g_{11}	1	2	6
g_{12}	1/2	1	3
g_{13}	1/6	1/3	1
Sum	10/6	10/3	10/1
Inverse	6/10	3/10	1/10

If $a_{ij}=w_i/w_j$, and **both consistency conditions are satisfied**, then: $\boxed{A\,\mathbf{w} = n\,\mathbf{w}}$

$$\begin{bmatrix} w_1/w_1 & w_1/w_2 & \dots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \dots & w_2/w_n \\ \dots & \dots & \dots & \dots \\ w_n/w_1 & w_n/w_2 & \dots & w_n/w_n \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{bmatrix} = \begin{bmatrix} n\,w_1 \\ n\,w_2 \\ \dots \\ n\,w_n \end{bmatrix}$$



	w	**w'**
g_{11}	6	**0.6**
g_{12}	3	**0.3**
g_{13}	1	**0.1**

Sum up elements in each column j of A (j=1,…,n) and inverse the resulting sums for each j:

$$\boxed{col_j = \sum_{i=1}^{n} w_i \big/ w_j}$$

$$\boxed{w'_j = \frac{1}{col_j} = \frac{w_j}{\sum_{i=1}^{n} w_i}}$$

Computing Priorities for Inconsistent Matrices

If $a_{ij}=w_i/w_j$, and both consistency conditions are satisfied, then: $\boxed{A\mathbf{w} = n\mathbf{w}}$

- For an inconsistent matrix, this relation is no longer valid
- A number of conversion methods of a matrix into priorities are possible, but AHP uses a mathematical approach based on eigenvalues and eigenvectors

g_2	g_{21}	g_{22}	g_{23}
g_{21}	1	2	5
g_{22}	1/2	1	3
g_{23}	1/5	1/3	1

$\Downarrow$

$$\boxed{A\mathbf{w} = \lambda\mathbf{w}}$$

For matrix A satisfying the consistency condition of pairwise comparisons (CCPC), but **not necessarily satisfying cardinal consistency condition** (CCC):

- The dimension n is replaced by the unknown λ
- The calculation of λ and $\boldsymbol{w}$ is called, in linear algebra, an **eigenvalue problem**
- Any value λ satisfying this equation is called an **eigenvalue** and $\boldsymbol{w}$ is its associated **eigenvector**
- According to the Perron theorem, a positive matrix has a positive eigenvalue

The non-trivial eigenvalue is the maximum eigenvalue λ_{max} of matrix A ($\lambda_{max} \geq n$)

$$\boxed{A\mathbf{w} = \lambda_{\max}\mathbf{w}}$$

- AHP looks for the **principal eigenvector of *A*** (eigenvector corresponding to λ_{max})

In case the matrix is consistent, $\boldsymbol{w}$ is an eigenvector of A with an eigenvalue equal to n ($\lambda_{max} = n$)

- Otherwise, the difference ($\lambda_{max} - n$) is a measure of the inconsistency

Approximating Principal Eigenvector (1)

Step 1. Sum up elements in each column j of A (j=1,…,n):

$$col_j = \sum_{i=1}^{n} a_{ij}$$ $$\left(col_j = \sum_{i=1}^{n} w_i / w_j \right)$$

***Step* 2.** Build normalized matrix $A' = [a'_{ij}]$, i,j=1,…,n:

$$a'_{ij} = a_{ij} / col_j$$ $$\left(a'_{ij} = w_i / \sum_{i=1}^{n} w_i \right)$$

***Step* 3.** Calculate approximate, normalized weight w'_i as an arithmetic mean of the row elements of the normalized matrix A' (i=1,…,n):

$$w'_i = \frac{1}{n} \sum_{j=1}^{n} a'_{ij} = \frac{1}{n} \sum_{j=1}^{n} \frac{a_{ij}}{\sum_{i=1}^{n} a_{ij}}$$ $$\left(w'_i = \frac{1}{n} \times \frac{n \times w_i}{\sum_{i=1}^{n} w_i} = \frac{w_i}{\sum_{i=1}^{n} w_i} \right)$$

a_{ij}	g_{21}	g_{22}	g_{23}
g_{21}	1	2	5
g_{22}	1/2	1	3
g_{23}	1/5	1/3	1
col_j	17/10	10/3	9/1

Normalized matrix *A'*

a'_{ij}	g_{21}	g_{22}	g_{23}
g_{21}	10/17	6/10	5/9
g_{22}	5/17	3/10	3/9
g_{23}	2/17	1/10	1/9

w'	w	w'
g_{21}	10/17 + 6/10 + 5/9 = 1.744	1/3·1.744 = 0.581
g_{22}	5/17 + 3/10 + 3/9 = 0.927	1/3·0.927 = 0.309
g_{23}	2/17 + 1/10 + 1/9 = 0.329	1/3·0.329 = 0.110

*To find the ranking of priorities (eigenvector) – **short description**:*
Normalize the column entries by dividing each entry by the sum of the column.
Take the overall row averages.

Approximating Principal Eigenvector (2)

In order to calculate the eigenvector associated with the maximum eigenvalue, use an **iterative power method**:

***Step* 1**. The (pairwise comparison) matrix is squared: $A_{t+1} = A_t \cdot A_t$

***Step* 2**. The row sums are then calculated and normalized, to sum up to the unity. This is the first **approximation of the eigenvector**.

***Step* 3**. Using the matrix A_{t+1}, *steps* 1 and 2 are repeated until the difference between these sums in two consecutive approximations of the eigenvector is smaller than a pre-defined accuracy threshold

A_1	g_{21}	g_{22}	g_{23}
g_{21}	1	2	5
g_{22}	1/2	1	3
g_{23}	1/5	1/3	1

$A_2 = A_1 \cdot A_1$

A_2	g_{21}	g_{22}	g_{23}		Row sum	*w*'
g_{21}	3	5.67	16		**24.67**	**0.582**
g_{22}	1.60	3	8.50		**13.10**	**0.309**
g_{23}	0.57	1.07	3		**4.63**	**0.109**

$A_3 = A_2 \cdot A_2$

A_3	g_{21}	g_{22}	g_{23}		Row sum	*w*'
g_{21}	27.13	51.07	144.17		**222.37**	**0.582**
g_{22}	14.42	27.13	76.60		**118.15**	**0.309**
g_{23}	5.11	9.61	27.13		**41.85**	**0.109**

Interpretation of Priorities

g_2	g_{21}	g_{22}	g_{23}
g_{21}	1	**2**	**5**
g_{22}	1/2	1	**3**
g_{23}	1/5	1/3	1

priorites

	w'
g_{21}	0.581
g_{22}	0.309
g_{23}	0.110

The priorities with both methods differ marginally

Arithmetic mean of the normalized matrix

Iterative power method

	w'
g_{21}	0.582
g_{22}	0.309
g_{23}	0.110

There exist **accurate methods** for computing the principal eigenvector of matrix A

	w	w'
g_{21}	5.313	**0.582**
g_{22}	2.823	**0.309**
g_{23}	1.000	**0.109**
λ	3.004	

Pairwise comparisons based on w'

e.g., $a'_{12} = w'_1 / w'_2 =$ **= 0.582 / 0.309 = 1.88**

A	g_{21}	g_{22}	g_{23}
g_{21}	1	**1.88**	**5.31**
g_{22}	0.53	1	**2.82**
g_{23}	0.19	0.35	1

Intuitively, the consistency is high: $\lambda_{max} = 3.004$ ($n = 3$)

Priorities included in the hierarchy

g_2: 0.250

g_{21}: 0.582 g_{22}: 0.309 g_{13}: 0.109

Aims vs. mathematical reality:

- g_{21} is **2** times more important than g_{22} – we have **1.88**
- g_{22} is **3** times more important than g_{23} – we have **2.82**
- g_{21} is **5** times more important than g_{23} – we have **5.31**

Preference Information in AHP

For programming (inconsistent)

g_{11}	*A*	*B*	*C*	D
A	1	2	5	1
B	1/2	1	3	2
C	1/5	1/3	1	1/4
D	1	1/2	4	1

g_{11}	*w*	*w'*
A	1.473	0.379
B	1.129	0.290
C	0.289	0.074
D	1.000	0.257
λ	4.191	

For math (inconsistent)

g_{12}	*A*	*B*	*C*	D
A	1	2	1/2	2
B	1/2	1	1/3	3
C	2	3	1	4
D	1/2	1/3	1/4	1

g_{12}	*w*	*w'*
A	2.610	0.254
B	1.886	0.184
C	4.762	0.464
D	1.000	0.097
λ	4.124	

For physics (inconsistent)

g_{13}	*A*	*B*	*C*	D
A	1	4	5	4
B	1/4	1	3	1/2
C	1/5	1/3	1	1/3
D	1/4	2	3	1

g_{13}	*w*	*w'*
A	2.724	0.569
B	0.709	0.148
C	0.353	0.074
D	1.000	0.209
λ	4.158	

For visual arts (consistent)

g_{22}	*A*	*B*	*C*	D
A	1	1/3	1	3
B	3	1	3	9
C	1	1/3	1	3
D	1/3	1/9	1/3	1

g_{22}	*w*	*w'*
A	3	0.188
B	9	0.563
C	3	0.188
D	1	0.063
λ	4	

The preference elicitation mode and the way of computing the priorities is the same:

- irrespective of the hierarchy level
- irrespective of whether we deal with criteria, sub-criteria, or alternatives

For cognitive sciences (inconsistent)

g_{23}	*A*	*B*	*C*	D
A	1	1/7	1/3	1/5
B	7	1	5	3
C	3	1/5	1	1/3
D	5	1/3	3	1

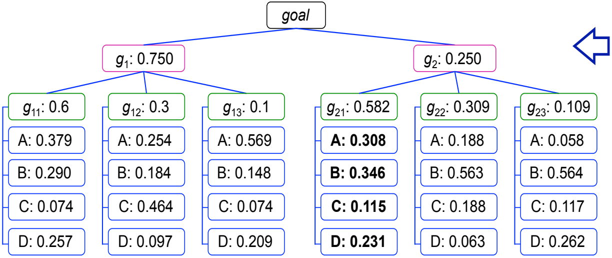
g_{23}	*w*	*w'*
A	0.220	0.058
B	2.155	0.564
C	0.448	0.117
D	1.000	0.262
λ	4.117	

Considering Qualitative Information in AHP

For some elementary criteria, **precise quantitative information is available**

- Assume that for English (g_{21}), we have results of the level test
- Ratios between such objective data can be used to determine the priorities
- *Pairwise comparison judgments may still be used in some cases*
- When the criterion is to be minimized, the score should be inverted (x becomes $1/x$)

	g_{21}	w'
A	80	**0.308**
B	90	**0.346**
C	30	**0.115**
D	60	**0.231**
Sum	260	1



Complete hierarchy filled with the priorities

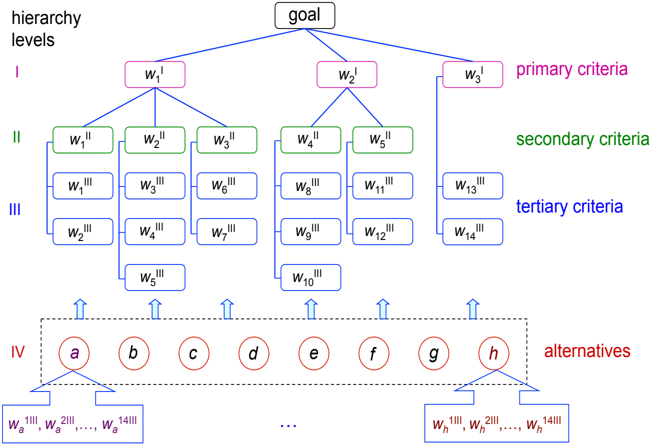
- The priorities for a group of elements with the common predecessor sum up to 1
- The priorities of alternatives differ on various criteria
- Need for aggregation

Computing Comprehensive Score in AHP (1)

Each alternative *a* gets a score (value) equal to the sum of products of priorities (weights) of elements of the hierarchy on **all paths from the alternative to the goal**

$$\textbf{Score}(a) = w_1^{I} w_1^{II} w_1^{III} w_a^{1III} + w_1^{I} w_1^{II} w_2^{III} w_a^{2III} + \ldots + w_2^{I} w_4^{II} w_8^{III} w_a^{8III} + \ldots + w_3^{I} w_{14}^{III} w_a^{14III}$$

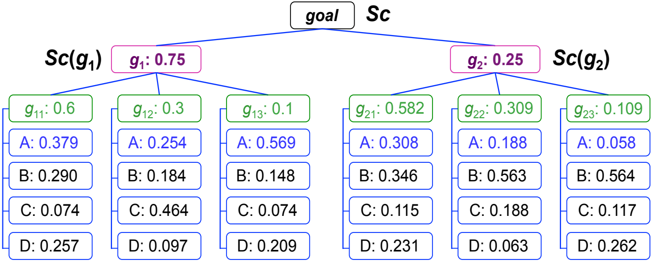
$$\textbf{Score}(h) = w_1^{I} w_1^{II} w_1^{III} w_h^{1III} + w_1^{I} w_1^{II} w_2^{III} w_h^{2III} + \ldots + w_2^{I} w_4^{II} w_8^{III} w_h^{8III} + \ldots + w_3^{I} w_{14}^{III} w_h^{14III}$$



Computing Comprehensive Score in AHP (2)

The aggregation of the local, sub-criteria, and macro criteria prioritizations leads to **global prioritizations**

$$\boldsymbol{Score(A)} = 0.379\cdot0.6\cdot0.75 + 0.254\cdot0.3\cdot0.75 + 0.569\cdot0.1\cdot0.75 + 0.308\cdot0582\cdot0.25 + 0.188\cdot0.309\cdot0.25 + 0.058\cdot0.109\cdot0.25 = \mathbf{0.331}$$



Scores can also be computed for any sub-hierarchy, e.g., g_1 and g_2

RANKING

A → ***B*** → ***D*** → ***C***

	Sc	$Sc(g_1)$	$Sc(g_2)$
A	**0.331**	0.360	0.243
B	**0.292**	0.244	0.437
C	**0.178**	0.191	0.138
D	**0.199**	0.204	0.182
Sum	**1**	1	1

Along with the priorities, AHP also yields **inconsistency indices** *CI* and *CR*

- Designed to alert the DM to any inconsistencies in the comparisons

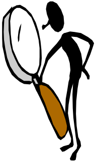
Consistency index (*CI*) says to what degree the judgments of the DM satisfy the cardinal consistency condition

Relative consistency index (*CR*) is the ratio of *CI* and *RI*

- *RI* is a random index defined as an average value of consistency index *CI* for an $n \times n$ matrix *A* filled randomly over diagonal
- *CR* measures how consistent the judgments have been relative to large samples of purely random judgments

$$CI = \frac{\lambda_{max} - n}{n - 1}$$

$$CR = \frac{CI}{RI}$$



n	1	2	3	4	5	6	7	8	9	10	11
RI	0	0	0.58	0.9	1.12	1.24	1.32	1.41	1.45	1.49	1.51

- ***CR* ≤ 0.1 is considered satisfactory** = an inconsistency of 10% or less implies that the adjustment is none or small as compared to the actual values of the eigenvector entries
- **CR > 0.1 = the judgments are untrustworthy** because they are too close to randomness and the preference elicitation is valueless or must be repeated

Inconsistency Verification (2)

$$CI = \frac{\lambda_{max} - n}{n - 1}$$

n	1	2	3	4	5	6	7	8	9	10	11
RI	0	0	0.58	0.9	1.12	1.24	1.32	1.41	1.45	1.49	1.51

$$CR = \frac{CI}{RI}$$

For visual arts

g_{22}	A	B	C	D
A	1	1/3	1	3
B	3	1	3	9
C	1	1/3	1	3
D	1/3	1/9	1/3	1

$n = 4$

g_{22}	w	w'
A	3	0.188
B	9	0.563
C	3	0.188
D	1	0.063
λ_{max}	4	

CI = (4 – 4) / (4 – 1) = 0

CR = CI / 0.9 = 0 / 0.9 = 0 ≤ 0.1

Full consistency

For programming

g_{11}	A	B	C	D
A	1	2	5	1
B	1/2	1	3	2
C	1/5	1/3	1	1/4
D	1	1/2	4	1

$n = 4$

g_{11}	w	w'
A	1.473	0.379
B	1.129	0.290
C	0.289	0.074
D	1.000	0.257
λ_{max}	4.191	

Satisfactory consistency

CI = (4.191 – 4) / (4 – 1) = 0.064

CR = CI / 0.9 = 0.064 / 0.9 = 0.071 ≤ 0.1

For some unknown criterion

$g_?$	A	B	C	D
A	1	1/3	1	3
B	3	1	3	9
C	1	1/3	1	3
D	1/3	1/9	1/3	1

$g_?$	w	w'
A	2.241	0.350
B	1.232	0.192
C	1.936	0.302
D	1.000	0.156
λ_{max}	7.503	

CI = (7.503 – 4) / (4 – 1) = 1.17

CR = CI / 0.9 = 1.17 / 0.9 = 1.30 > 0.1

The judgments are more random than an average randomly filled matrix of size 4×4

Unacceptable!

AHP - Summary

AHP

Input data: a finite set of alternatives $A=\{a, b, c, \ldots,\}$
hierarchy of criteria $G=\{g_1^{I},\ldots,g_p^{I}, g_1^{II},\ldots,g_s^{II}, g_1^{III},\ldots,g_t^{III},\ldots\}$

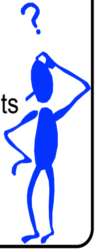
- Criteria, sub-criteria, and alternatives are called elements of the hierarchy

Preference information for elements at each level of the hierarchy:

- pairwise comparisons of all elements *i*, *j* with a common predecessor

Preference model: weights w_i (called priorities) of each element *i* at each level of the hierarchy

- Each alternative gets a score (value) equal to the sum of products of weights of elements of the hierarchy on the paths from the alternative to the goal



Choice: selection of the most preferred alternative

Ranking: putting the alternatives in order from the most to the least desirable

AHP - Drawbacks

- AHP requires **pairwise comparisons** of all elements linked by the same parent element: $n(n–1)/2$ **may easily run into hundreds**
  - Difficult to use when the number of criteria or alternatives is high (n > 7)
- **Arbitrary conversion** from verbal to numeric scale (ordinal is not cardinal!)
- Problems of 1 to 9 scale (not sufficient for representing DMs' value systems)
  - Easily run into inconsistency; the need for clusters that are compared directly and a pivot serving as a link between clusters (conversion rate)
- Comparison of ordinal evaluations on a ratio scale is meaningless: e.g., how many times „good" is better than „sufficient" ?
- The result of each pairwise comparison is expressed on a ratio scale
  - The **absolute zero is unstable** from one comparison to another

- E.g., compare two cars *i,j* w.r.t. maximal speed criterion (V_{max}); if a_{ij}=3, and $V_{max}(i)$=200 km/h, $V_{max}(j)$=170 km/h, than this means that:

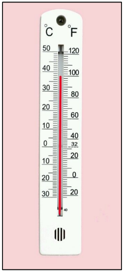
$$(200 - x) / (170 – x) = 3$$

where, assuming a linear transformation of the scale of V_{max} into the scale of preference, x is an absolute zero on the preference scale (x = 155 km/h)

From above, If $V_{max}(k)$=250 km/h, $V_{max}(j)$=170 km/h and a_{kj}=5, then:

$$(250 - x') / (170 - x') = 5 \quad \text{and} \quad x'=150 \text{ km/h}$$

8	1	6
3	5	7
4	9	2



Violating Condition of Order Preservation

The eigenvalue method of deriving the priority vector may not satisfy the "**Condition of Order Preservation**"

Def. The priority vector ***w*** satisfies the condition of order preservation (COP) if for all elements *i*,*j*,*k*,*h* being compared, the weights w_i,w_j, w_k,w_h respect the equivalence:

$$\boldsymbol{w_i / w_j > w_k / w_h \Leftrightarrow a_{ij} > a_{kh}}$$

where a_{ij} (i,j = 1,…,n) are the judgments given by the DM in the pairwise comparison matrix *A*

Example violation of condition of order preservation

Pairwise comparison matrix

	A	***B***	***C***	***D***
A	1	2	3	9
B	1/2	1	3/2	8
C	1/3	2/3	1	5/2
D	1/9	1/8	2/5	1

Priorities

	w
A	0.497
B	0.292
C	0.159
D	0.051

e.g.:

0.497 / 0.292 = 1.70

0.292 / 0.159 = 1.84

Pairwise comparison matrix using priorities

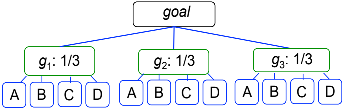
	A	***B***	***C***	***D***
A	1	1.70	3.13	9.75
B	0.59	1	1.84	5.73
C	0.32	0.54	1	3.12
D	0.10	0.17	0.32	1

w does not satisfy COP, because $\boldsymbol{w_A/w_B < w_B/w_C}$, but originally $\boldsymbol{a_{AB} > a_{BC}}$, while it satisfies the cardinal consistency condition $w_1/w_2 \times w_2/w_3 = w_1/w_3$

Rank Reversal Phenomenon

- **Rank reversal** is a change in the rank ordering of alternatives when, e.g., the set of alternatives is modified (*there exist various types of ranking reversals*)
- Addition of a copy of an alternative may change the rank order of the terminal scores, even if criteria and criterion weights remain the same

- Initially, there are 3 alternatives *A, B, C* and 3 criteria with equal weights (1/3)
- Later, *D* will be added (*D* is a copy of *B*)
- The cardinal consistency condition holds



weight w_1 = 1/3				
g_1	***A***	***B***	***C***	***D***
A	1	1/9	1	1/9
B	9	1	9	1
C	1	1/9	1	1/9
D	9	1	9	1

weight w_2 = 1/3				
g_2	***A***	***B***	***C***	***D***
A	1	9	9	9
B	1/9	1	1	1
C	1/9	1	1	1
D	1/9	1	1	1

weight w_3 = 1/3				
g_3	***A***	***B***	***C***	***D***
A	1	8/9	8	8/9
B	9/8	1	9	1
C	1/8	1/9	1	1/9
D	9/8	1	9	1

The normalized scores for *A*, *B*, *C* are:

B	***A***	***C***
0.47	0.45	0.08

When **a copy *D* of *B*** is added, the scores become:

A	***B***	***D***	***C***
0.37	0.29	0.29	0.06

The order of *A* and *B* has been reversed

Consequence: difficult to add or take out a new criterion or alternative (*the ranking may change*)

AHP - Main Advantages

Formal structuring of the decision problem

- Complex problems to be decomposed into sets of simpler sub-problems (judgments)
- Provides a documented rationale for the choice of a particular alternative

The simplicity of pairwise comparisons

- Only two criteria or alternatives have to be considered at any one time so that the DM's task is simplified
- Verbal comparisons are likely to be preferred by DMs who have difficulty in judging numerically

Applicable when it is challenging to formulate criteria evaluations

- It allows qualitative and quantitative evaluations

Redundancy allows consistency to be checked

- AHP requires more comparisons to be made by the DM than are needed to establish a set of weights
- If the DM indicates that A is twice as important as B, and B, in turn, is three times as important as C, then it can be inferred that A is six times more important than C
- By also asking the DM to compare A with C, it is possible to check the consistency of the judgments

Numerical values (priorities) at the local, intermediate, and global levels

Often used to determine only the criteria weights

AHP for Group Decision Making

One can use **additional levels of the hierarchy** to model:

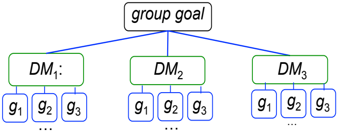
- **Multiple Decision Makers** with different importance

If the DMs do not have equal weights, their priorities must be determined

- Reflect the expertise of DMs or their impact on the decision
- Allocated by supra DM or by a participatory approach

Group decision making with AHP: overall priorities are determined by the weighted averages of the priorities obtained from members of the group

- Understand the conflicting ideas in the organization and try to reach a consensus
- Reduce dominance by a strong member of the group



Pairwise comparisons by supra DM

	DM_1	DM_2	DM_3
DM_1	1	2	5
DM_2	1/2	1	3
DM_3	1/5	1/4	1

*In the same spirit, AHP can account for **multiple scenarios** of uncertainty with different credibilities*

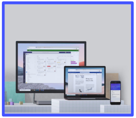
Example Applications of AHP

The **applications** of AHP to complex problems have **numbered in thousands**

AHP has been very widely applied in areas such as economics and planning, budget and resource allocations, priority setting, energy policy, material handling and purchasing, project selection, and forecasting

- Quantifying the overall quality of software systems (*Microsoft*)
- Choose the entertainment system for its entire fleet (*British Airways*)
- Establish priorities for criteria that improve customer satisfaction (*Ford Motor Company*)
- Allocate close to a billion dollars to the research projects (*Xerox*)
- Allocate resources to diverse activities (*US Department of Defence*)
- Determine the best relocation site for the earthquake-devastated Turkish city Adapazari

The wide range of applications of the AHP is evidence of its versatility



Weighted Sum Model

A weighted sum model: a comprehensive value is a weighted sum of original performances

$$WS(a) = \sum_{i=1,\dots,n} w_i \cdot g_i(a) = w_1 \cdot g_1(a) + \dots + w_n \cdot g_n(a)$$

weighted sum	weight associated with criterion g_i	performance of alternative *a* on g_i

- Also called simple additive weighting (SAW) or weighted linear combination (WLC)
- **It is the best known and simplest MCDA method**
- Claimed to be the most intuitive model due to differentiating the impact of individual criteria and not requiring the transformation of original performances

Easy exploitation of the preference relation induced by **WS** in the set of alternatives *A*:

- Preference relation (*P*, >): $a > b \Leftrightarrow WS(a) > WS(b)$
- Indifference relation (*I*, ~): $a \sim b \Leftrightarrow WS(a) = WS(b)$

- **Four students**: Ana, Brooke, Caden, and Demi
- **Three criteria** of gain type:
  - g_1 – math
  - g_2 – physics
  - g_3 – literature

	w_1	w_2	w_3	**Example computations of the weighted sums**	
Weight	**3/8**	**3/8**	**2/8**		
Alt.	g_1	g_2	g_3	**WS**	**Rank**
A	18	16	14	3/8·18 + 3/8·16+ 2/8·14 = 16.25	1
B	18	14	16	3/8·18 + 3/8·14+ 2/8·16 = 16.00	2
C	14	16	14	3/8·14 + 3/8·16+ 2/8·14 = 14.75	3
D	14	14	16	3/8·14 + 3/8·14+ 2/8·16 = 14.50	4

Weights as Trade-off Coefficients

- The preference information = weights w_j
- Weights in a weighted sum model are interpreted as trade-offs coefficients (not importances)
- The weighted sum allows trade-off (compensation) between criteria

Focus on three criteria: $WS(a) = WS(g_1(a), g_2(a), g_3(a)) = w_1 \cdot g_1(a) + w_2 \cdot g_2(a) + w_3 \cdot g_3(a)$

What change (x) on criterion g_3 can compensate a change by 1 unit on criterion g_2?

$$WS(g_1(a), g_2(a), g_3(a)) = WS(g_1(a), g_2(a) + 1, g_3(a) - x)$$

$$w_1 \cdot g_1(a) + w_2 \cdot g_2(a) + w_3 \cdot g_3(a) = w_1 \cdot g_1(a) + w_2 \cdot [g_2(a) + 1] + w_3 \cdot [g_3(a) - x]$$

$$w_2 \cdot g_2(a) + w_3 \cdot g_3(a) = w_2 \cdot g_2(a) + w_3 \cdot g_3(a) + w_2 - w_3 \cdot x$$

$x = w_2 / w_3$ (for example, $x = 3/8 / 2/8 = 3/2$)

- The **ratio between weights** answers the questions on the trade-offs (changes required on one criterion to compensate a change by 1 on another criterion)
- **Weights depend on the performance units**! They are scaling coefficients controlling the contribution in the weighted sum (comprehensive score).
- Not easy to elicit! Greater weight does not mean greater importance!

Weights as Scaling Constants

	w_1	w_2	w_3	
Weight	**3/8**	**3/8**	**2/8**	
Alt.	g_1	g_2	g_3	**WS**
A	18	16	140	47.25
B	18	14	160	52.00
C	14	16	140	46.25
D	14	14	160	50.50

Three criteria of gain type: g_1 – math, g_2 – physics, g_3 – literature

- Assume the scale of g_1 and g_2 is between 0 and 20
- The scale of g_3 is modified to the range between 0 and 200 (the performances have been multiplied by 10 w.r.t. the previous scenario)
- Assume we use the olds weights (w_1 = 3/8, w_2 = 3/8, w_3 = 2/8)
- ***B*** and ***D*** are exceptionally good comprehensively only because they score favorably on g_3, which has the most extensive scale of performances
- *This is an undesired effect*!

With the previous performances scale of criterion g_3, a change by 1 unit on criterion g_2 was compensated with a change **x** = 3/2 = 1.5 on criterion g_3

x = w_2 / w_3 = 3/8 / 2/8 = 3/2

- To ensure the same trade-off effect with the new performance scale of g_3, we need to adjust the weights
- A change by 1 unit on criterion g_2 should be compensated with a change **x'** = 15 on criterion g_3
- Therefore, while fixing w_2 = 3/8, the weight w_3 of criterion g_3 should be: w_3 = w_2 / **x'** = 3/8 / 15 = 2/80
- Since the performances on g_3 were multiplied by 10, the weight w_3 of g_3 should be divided by 10

	w_1	w_2	w_3	
Weight	**3/8**	**3/8**	**2/80**	
Alt.	g_1	g_2	g_3	**WS**
A	18	16	140	16.25
B	18	14	160	16.00
C	14	16	140	14.75
D	14	14	160	14.50

To ease the preference elicitation, one often assumes that all performances are expressed in exactly the same unit

The Meaning of Weights

- **Weights** can have **different interpretations,** which is determined by the model in which they are incorporated
- Values of preference parameters are meaningless as long as the preference model in which they are used is not specified

WEIGHTS

	w_1	w_2	w_3
Weight	**3/8**	**3/8**	**2/8**
Alt.	g_1	g_2	g_3
E	18	14	14
F	14	16	16

Consider two alternatives: ***E*** = (18, 14, 14) and ***F*** = (14, 16, 16)

Consider the **weighed sum model**:

- $WS(\boldsymbol{E}) = 3/8 \cdot 18 + 3/8 \cdot 14 + 2/8 \cdot 14 = 15.5$
- $WS(\boldsymbol{F}) = 3/8 \cdot 14 + 3/8 \cdot 16 + 2/8 \cdot 16 = 15.25$
- ***Effect:*** ***E*** is preferred (**>**) to ***F***

Consider the **Condorcet aggregation**: ***a*** is preferred (**>**) to ***b*** *iff* $\Sigma_{j:\, g_j(\boldsymbol{a}) \geq g_j(\boldsymbol{b})}\, w_j > \Sigma_{j:\, g_j(\boldsymbol{b}) \geq g_j(\boldsymbol{a})}\, w_j$

- the sum of weights of criteria supporting that ***a*** is at least as good as ***b*** is greater than the sum of weights of criteria supporting that ***b*** is at least as good as ***a***

- ***Effect:*** ***F*** is preferred (**>**) to ***E*** because $w_2 + w_3 = 5/8 > w_1 = 3/8$

Depending on the model with the same parameter (weight) values, the results can be completely different

Preference Independence of Criteria

- The weights and thus the trade-offs in the weighted sum model are **constant** for the whole range variation of criteria values (performances)
- The weighted sum and, more generally, an additive value function requires that criteria are **independent in the sense of preference**
- Intuitively, this means that the contribution each alternative $\boldsymbol{a}$ gets from criterion $\boldsymbol{g_i}$, i.e., $\boldsymbol{u_i(a) = w_i \cdot g_i(a)}$ does not change with a change of the performance on some other criterion $\boldsymbol{g_j(a)}$, $j=1,\ldots,n$, $j \neq i$

	w_1	w_2	w_3	Results	
Weight	**3/8**	**3/8**	**2/8**		
Alt.	g_1	g_2	g_3	**WS**	**Rank**
A	18	16	14	16.25	1
B	18	14	16	16.00	2
C	14	16	14	14.75	3
D	14	14	16	14.50	4

Assume we agree with ***C*** > ***D***, but not with ***A*** > ***B***

- ***A*** has good scores on math and physics, which is natural, but a student good at math or physics (such as ***B***) are usually not good at literature, so they should get some bonus

The weighted sum model is not able to represent such preferences: ***B*** > ***A*** and ***C*** > ***D***

- Due to the "**preference independence**" condition, WSM requires that if ***B*** > ***A*** then ***D*** > ***C*** and if ***C*** > ***D*** then ***A*** > ***B***

$\boldsymbol{B} > \boldsymbol{A}$ *iff* $WS(\boldsymbol{B}) = 18 \cdot w_1 + 14 \cdot w_2 + 16 \cdot w_3 > WS(\boldsymbol{A}) = 18 \cdot w_1 + 16 \cdot w_2 + 14 \cdot w_3$ *iff* $14 \cdot w_2 + 16 \cdot w_3 > 16 \cdot w_2 + 14 \cdot w_3$

$\boldsymbol{C} > \boldsymbol{D}$ *iff* $WS(\boldsymbol{C}) = 14 \cdot w_1 + 16 \cdot w_2 + 14 \cdot w_3 > WS(\boldsymbol{D}) = 14 \cdot w_1 + 14 \cdot w_2 + 16 \cdot w_3$ *iff* $14 \cdot w_2 + 16 \cdot w_3 < 16 \cdot w_2 + 14 \cdot w_3$

Need for Interactions Between Criteria

	w_1	w_2	w_3	Results	
Weight	**3/8**	**3/8**	**2/8**		
Alt.	g_1	g_2	g_3	**WS**	**Rank**
A	18	16	14	16.25	1
B	18	14	16	16.00	2
C	14	16	14	14.75	3
D	14	14	16	14.50	4

- The performance of 14 on math – g_1 (see ***C*** and ***D***) represents a different level than the performance of 18 on g_1 (see ***A*** and ***B***)
- When considering students with low math skills (14), we may prefer somebody better in terms of physics – g_2: ***C*** > ***D***
- When considering students with high math skills (18), we may prefer somebody better in terms of literature – g_3 (with a more balanced profile): ***B*** > ***A***
- We wish to represent **interactions between criteria!**

- There is a redundancy (**negative synergy**) between mathematics and physics: risk of over-evaluation of students being good in math and physics because students good in math are usually good in physics
- **Redundancy (weakening effect)** for math (g_1) and physics (g_2) – the weight that should be assigned to g_1 and g_2 jointly should be lower than the sum of individual weights of g_1 and g_2

$$\mu(\{g_1, g_2\}) < \mu(\{g_1\}) + \mu(\{g_2\})$$

- There is a complementary (**positive synergy**) between math or physics and literature, because we wish to give a bonus to students who, besides math or physics, are also good at literature
- **Complementarity (strengthening effect)** for math (g_1) and literature (g_3) – the weight that should be assigned to g_1 and g_3 jointly should be greater than the sum of individual weights of g_1 and g_3 (the same for g_2 and g_3)

$$\mu(\{g_1, g_3\}) > \mu(\{g_1\}) + \mu(\{g_3\}) \quad \text{and} \quad \mu(\{g_2, g_3\}) > \mu(\{g_2\}) + \mu(\{g_3\})$$

***We need to represent the weights of subsets of criteria rather than only the weights of individual criteria*!**

Instead of weights w_i for each criterion $g_j \in G$ in a weighted sum: $\mu(F)$ represents a **joint weight of criteria** from a subset $F \subseteq G$

$\mu : 2^G \to [0,1]$ is an non-additive measure (***capacity***) defined over all subsets of criteria, taking values in the range [0,1]

- **Normalization**: $\mu(\emptyset)=0$, $\mu(G)=1$
- **Monotonicity**: for $F' \subset F \subseteq G$, $\mu(F') \leq \mu(F)$

In general, it is admissible that $\mu(F' \cup F'') \neq \mu(F') + \mu(F'')$

- **Positive interaction** (synergy): $\mu(F' \cup F'') > \mu(F') + \mu(F'')$
- **Negative interaction** (redundancy): $\mu(F' \cup F'') < \mu(F') + \mu(F'')$

Example capacities:

- $\mu(\emptyset)=0$

For individual criteria:

- $\mu(\{g_1\}) = \mu(\{g_2\}) = 0.45$, $\mu(\{g_3\}) = 0.3$

For criteria pairs:

- $\mu(\{g_1,g_2\}) = 0.5$, $\mu(\{g_1,g_3\}) = \mu(\{g_2,g_3\})=0.85$

For all criteria:

- $\mu(\{g_1,g_2,g_3\}) = 1$

$\Downarrow$

Positive interaction (synergy)::

- $\mu(\{g_1\}) + \mu(\{g_3\}) = 0.45 + 0.3 = 0.75$
- $\mu(\{g_1\}) + \mu(\{g_3\}) < \mu(\{g_1,g_3\}) = 0.85$
- $\mu(\{g_2\}) + \mu(\{g_3\}) = 0.75 < \mu(\{g_2,g_3\}) = 0.85$

Negative interaction (redundancy):

- $\mu(\{g_1\}) + \mu(\{g_2\}) = 0.45 + 0.45 = 0.9$
- $\mu(\{g_1\}) + \mu(\{g_2\}) > \mu(\{g_1,g_2\}) = 0.5$

- Weights (capacities) μ permit to take into account the **interactions between criteria**
- Weights for the subsets of criteria are incorporated in the **Choquet integral**

A Different Look at the Weighted Sum Model

	w_1	w_2	w_3
Weight	**3/8**	**3/8**	**2/8**
Alt.	g_1	g_2	g_3
$\boldsymbol{A}$	18	16	14

$WS(\boldsymbol{A}) = w_1 \cdot g_1(\boldsymbol{A}) + w_2 \cdot g_2(\boldsymbol{A}) + w_3 \cdot g_3(\boldsymbol{A}) = 18 \cdot w_1 + 16 \cdot w_2 + 14 \cdot w_3 =$

$= (14 + 4) \cdot w_1 + (14 + 2) \cdot w_2 + 14 \cdot w_3 = 14 \cdot (w_1 + w_2 + w_3) + 4 \cdot w_1 + 2 \cdot w_2 =$

$= 14 \cdot (w_1 + w_2 + w_3) + (2 + 2) \cdot w_1 + 2 \cdot w_2 = 14 \cdot (w_1 + w_2 + w_3) + 2(w_1 + w_2) + 2 \cdot w_1 =$

$= 14 \cdot (w_1 + w_2 + w_3) + 2(w_1 + w_2) + 2 \cdot w_1 =$

$= (14 – 0) \cdot (w_1 + w_2 + w_3) + (16 – 14) \cdot (w_1 + w_2) + (18 – 16) \cdot w_1 =$

$= (g_3(\boldsymbol{A}) – 0) \cdot (w_1 + w_2 + w_3) + (g_2(\boldsymbol{A}) – g_3(\boldsymbol{A})) \cdot (w_1 + w_2) + (g_1(\boldsymbol{A}) – g_2(\boldsymbol{A})) \cdot w_1$

A weighted sum model: $WS(a) = \sum_{i = 1,\dots,n} w_i \cdot g_i(a)$ can be expressed as:

$$WS(a) = \sum_{i = 1,\dots,n} [g_{(i)}(a) - g_{(i-1)}(a)] \cdot \sum_{j = i,\dots,n} w_{(j)}$$

where $_{(\cdot)}$ is an index permutation $\{1,\dots,n\}$ such that the performances are ordered in the non-decreasing order: $0 = g_{(0)}(a) \leq g_{(1)}(a) \leq g_{(2)}(a) \leq \dots \leq g_{(n)}(a)$ (0 is added as the worst, artificial performance)

- A weighted sum of differences between consecutive performances in the non-decreasing order with weights defined as the sum of weights of criteria on which a given performance level is attained

Assume that $\mu(\{(i),\dots,(n)\})$ is a **weight of the subset of criteria** $(\{(i),\dots,(n)\})$ defined as the sum $\sum_{j = i,\dots,n} w_{(j)}$ of weights of the elementary criteria contained in the subset, i.e.:

$$\mu(\{(i),\dots,(n)\}) = \sum_{j = i,\dots,n} w_{(j)}$$

Then, $WS(a) = \sum_{i = 1,\dots,n} [g_{(i)}(a) - g_{(i-1)}(a)] \cdot \mu(\{(i),\dots,(n)\})$

Weighted Sum Model - Example (1)

New perspective on the weighted sum: $WS(a) = \sum_{i=1,\dots,n} [g_{(i)}(a) - g_{(i-1)}(a)] \cdot \mu(\{(i),\dots,(n)\})$
where $\mu(\{(i),\dots,(n)\}) = \sum_{j=i,\dots,n} w_{(j)}$

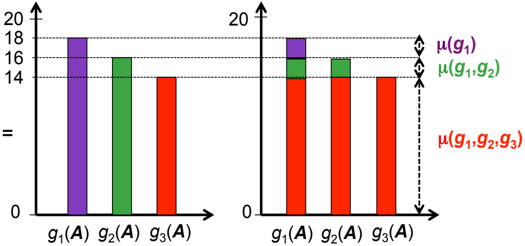
	w_1	w_2	w_3
Weight	**3/8**	**3/8**	**2/8**
Alt.	g_1	g_2	g_3
A	18	16	14

The ordered performances of aternative A:
$0 = g_{(0)}(A) \le g_{(1)}(A) = g_3(A) = 14 \le g_{(2)}(A) = g_2(A) = 16 \le g_{(3)}(A) = g_1(A) = 18$

Assuming that $\mu(\{g_1\}) = 3/8$, $\mu(\{g_2\}) = 3/8$, and $\mu(\{g_3\}) = 2/8$, we have:

- $\mu(\{g_1,g_2\}) = 3/8 + 3/8 = 6/8$; $\mu(\{g_1,g_3\}) = 3/8 + 2/8 = 5/8$; $\mu(\{g_2,g_3\}) = 3/8 + 2/8 = 5/8$
- $\mu(\{g_1,g_2,g_3\}) = 3/8 + 3/8 + 2/8 = 1$

$$WS(A) = [14-0]\cdot\mu(\{g_1,g_2,g_3\}) + [16-14]\cdot\mu(\{g_1,g_2\}) + [18-16]\cdot\mu(\{g_1\}) =$$
$$= 14\cdot\mu(\{g_1,g_2,g_3\}) + 2\cdot\mu(\{g_1,g_2\}) + 2\cdot\mu(\{g_1\}) =$$
$$= 14\cdot 1 + 2\cdot 6/8 + 2\cdot 3/8 = 16.25$$



Weighted Sum Model - Example (2)

New perspective on the weighted sum: $WS(a) = \sum_{i=1,\ldots,n} [g_{(i)}(a) - g_{(i-1)}(a)]\cdot\mu(\{(i),\ldots,(n)\})$

where $\mu(\{(i),\ldots,(n)\}) = \sum_{j=i,\ldots,n} w_{(j)}$

	w_1	w_2	w_3
Weight	3/8	3/8	2/8
Alt.	g_1	g_2	g_3
B	18	14	16

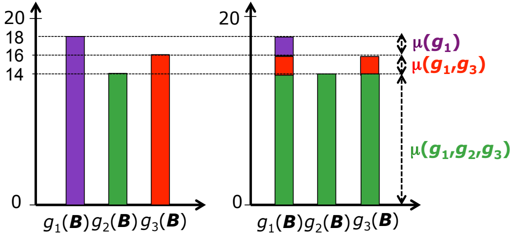
The ordered performances of alternative B:

$0 = g_{(0)}(B) \leq g_{(1)}(B) = g_2(B) = 14 \leq g_{(2)}(B) = g_3(B) = 16 \leq g_{(3)}(B) = g_1(B) = 18$

Assuming that $\mu(\{g_1\}) = 3/8$, $\mu(\{g_2\}) = 3/8$, and $\mu(\{g_3\}) = 2/8$, we have:

- $\mu(\{g_1,g_2\}) = 3/8 + 3/8 = 6/8$; $\mu(\{g_1,g_3\}) = 3/8 + 2/8 = 5/8$;
  $\mu(\{g_2,g_3\}) = 3/8 + 2/8 = 5/8$
- $\mu(\{g_1,g_2,g_3\}) = 3/8 + 3/8 + 2/8 = 1$

$$WS(B) = [14-0]\cdot\mu(\{g_1,g_2,g_3\}) + [16-14]\cdot\mu(\{g_1,g_3\}) + [18-16]\cdot\mu(\{g_1\}) =$$
$$= 14\cdot\mu(\{g_1,g_2,g_3\}) + 2\cdot\mu(\{g_1,g_3\}) + 2\cdot\mu(\{g_1\}) =$$
$$= 14\cdot 1 + 2\cdot 5/8 + 2\cdot 3/8 = 16.0$$



Is it really necessary to impose $\mu(\{(i),\ldots,(n)\}) = \sum_{j=i,\ldots,n} w_{(j)}$?

NO!

The Choquet Integral

The **Choquet integral** can be defned in the same way as the revised weighted sum:

$$Ch(a) = \sum_{i=1,\dots,n} [g_{(i)}(a) - g_{(i-1)}(a)] \cdot \mu(\{(i),\dots,(n)\})$$

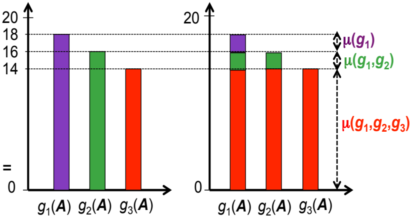
where $_{(\cdot)}$ is an index permutation $\{1,\dots, n\}$ such that the performances are ordered in the non-decreasing order:
$0 = g_{(0)}(a) \leq g_{(1)}(a) \leq g_{(2)}(a) \leq \dots \leq g_{(n)}(a)$ (0 is added as the worst, artifcial performance)

- without imposing the condition $\mu(\{(i),\dots,(n)\}) = \sum_{j=i,\dots,n} w_{(j)}$
- the capacities need to satisfy the previously defined normalization and monotonicity constraints, but can represent positive ($\mu(\{(i),\dots,(n)\}) > \sum_{j=i,\dots,n} w_{(j)}$) or negative ($\mu(\{(i),\dots,(n)\}) < \sum_{j=i,\dots,n} w_{(j)}$) interactions

Alt.	g_1	g_2	g_3
A	18	16	14

Example capacities (defined previously):
$\mu(\emptyset)=0$, $\mu(\{g_1,g_2,g_3\}) = 1$
$\mu(\{g_1\}) = \mu(\{g_2\}) = 0.45$, $\mu(\{g_3\}) = 0.3$
$\mu(\{g_1,g_2\}) = 0.5$, $\mu(\{g_1,g_3\}) = \mu(\{g_2,g_3\})=0.85$

$Ch(A) = [14-0]\cdot\mu(\{g_1,g_2,g_3\}) + [16-14]\cdot\mu(\{g_1,g_2\}) + [18-16]\cdot\mu(\{g_1\}) =$
$= 14\cdot\mu(\{g_1,g_2,g_3\}) + 2\cdot\mu(\{g_1,g_2\}) + 2\cdot\mu(\{g_1\}) =$
$= 14\cdot 1 + 2\cdot 0.5 + 2\cdot 0.45 = 15.9$



The Choquet Integral - Example

The Choquet integral $Ch(a) = \sum_{i=1,\ldots,n} [g_{(i)}(a) - g_{(i-1)}(a)] \cdot \mu(\{(i),\ldots,(n)\})$

Example capacities (defined previously):

$\mu(\emptyset)=0$, $\mu(\{g_1,g_2,g_3\}) = 1$

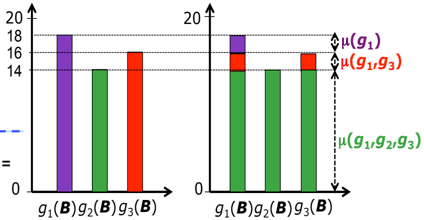
$\mu(\{g_1\}) = \mu(\{g_2\}) = 0.45$, $\mu(\{g_3\}) = 0.3$

$\mu(\{g_1,g_2\}) = 0.5$, $\mu(\{g_1,g_3\}) = \mu(\{g_2,g_3\}) = 0.85$

$Ch(B) = [14 - 0] \cdot \mu(\{g_1,g_2,g_3\}) + [16 - 14] \cdot \mu(\{g_1,g_3\}) + [18 - 16] \cdot \mu(\{g_1\}) =$

$= 14 \cdot \mu(\{g_1,g_2,g_3\}) + 2 \cdot \mu(\{g_1,g_3\}) + 2 \cdot \mu(\{g_1\}) =$

$= 14 \cdot 1 + 2 \cdot 0.85 + 2 \cdot 0.45 = 16.6$



- The Choquet integral is able to represent preferences:

  B* > *A* and *C* > *D

  by considering positive and negative interactions between criteria, and not adhering to the condition of preference independence
- *Drawback*: all criteria must have **the same cardinal evaluation scale**

Alt.	g_1	g_2	g_3	*Ch*	Rank
A	18	16	14	15.9	2
B	18	14	16	16.6	1
C	14	16	14	14.9	3
D	14	14	16	14.6	4

2-additive Choquet Integral

- The Choquet integral admitting interactions between any subsets of criteria is hard to interpret
- It is also difficult to understand the impact of various subsets on the integral's values for different alternatives (their intervention depends on the order of performances of a particular alternative)
- In practice, one often reduces the interactions to pairs of criteria (**2-additive Choquet integral**)

Let us consider alternative a evaluated in terms of only two criteria ($n = 2$): $a = [g_1(a), g_2(a)]$

- *If* $g_1(a) \leq g_2(a)$, then $Ch(a) = g_1(a) \cdot \mu(\{1,2\}) + [g_2(a) - g_1(a)] \cdot \mu(\{2\})$
- *If* $g_1(a) > g_2(a)$, then $Ch(a) = g_2(a) \cdot \mu(\{1,2\}) + [g_1(a) - g_2(a)] \cdot \mu(\{1\})$

In general: $Ch(a) = g_1(a) \cdot \mu(\{1\}) + g_2(a) \cdot \mu(\{2\}) + \min\{g_1(a), g_2(a)\} \cdot [\mu(\{1,2\}) - \mu(\{1\}) - \mu(\{2\})]$

e.g., *If* $g_1(a) \leq g_2(a)$, then $Ch(a) = g_1(a) \cdot \mu(\{1\}) + g_2(a) \cdot \mu(\{2\}) + g_1(a) \cdot [\mu(\{1,2\}) - \mu(\{1\}) - \mu(\{2\})] =$

$$= g_1(a) \cdot [\mu(\{1,2\})] + [g_2(a) - g_1(a)] \cdot \mu(\{2\})$$

- If $\mu(\{1,2\}) - \mu(\{1\}) - \mu(\{2\}) < 0$ (i.e., $\mu(\{1,2\}) < \mu(\{1\}) + \mu(\{2\})$), then the interaction is negative
- If $\mu(\{1,2\}) - \mu(\{1\}) - \mu(\{2\}) > 0$ (i.e., $\mu(\{1,2\}) > \mu(\{1\}) + \mu(\{2\})$), then the interaction is positive

Let us formulate the Choquet integral as:

$$Ch(a) = m(\{1\}) \cdot g_1(a) + m(\{2\}) \cdot g_2(a) + m(\{1,2\}) \cdot \min\{g_1(a), g_2(a)\}$$

where $m(\{1\})$, $m(\{2\})$ are weights of individual criteria and $m(\{1,2\})$ is the interaction coefficient

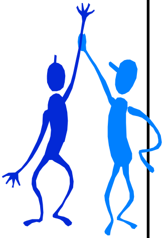
- $m(\{1\}) = \mu(\{1\})$, $m(\{2\}) = \mu(\{2\})$, $m(\{1,2\}) = \mu(\{1,2\}) - \mu(\{1\}) - \mu(\{2\})$
- *We change symbols from* μ *to* m to *avoid confusion between different notations*

Möbius Representation of the 2-additive Choquet Integral

- The Choquet integral for problems involving two criteria (n = 2):

$$\boldsymbol{Ch(a)} = m(\{1\})\cdot g_1(a) + m(\{2\})\cdot g_2(a) + m(\{1,2\})\cdot \min\{g_1(a),g_2(a)\} =$$
$$= \sum_{i=1,2} m(\{i\})\cdot g_i(a) + m(\{1,2\})\cdot \min\{g_1(a),g_2(a)\}$$

- The weights of individual criteria need to be non-negative, but the interaction coefficients are not restricted in this way (they can be negative, positive, or equal to 0)
- The above model can be generalized to the **Möbius representation of the 2-additive Choquet integral**:



$$\boldsymbol{Ch(a)} = \sum_{i=1,\ldots,n} m(\{i\})\cdot g_i(a) + \sum_{\{i,j\}\subseteq G} m(\{i,j\})\cdot \min\{g_i(a),g_j(a)\}$$

where:

- **normalization**: $m(\emptyset) = 0$ and $\sum_{i=1,\ldots,n} m(\{i\}) + \sum_{\{i,j\}\subseteq G} m(\{i,j\}) = 1$
- **monotonicity**: $m(\{i\}) \geq 0$, for $i = 1,\ldots,n$

$$m(\{i\}) + \sum_{j\in F} m(\{i,j\}) \geq 0, \text{ for } i = 1,\ldots,n \text{ and } F \subseteq G \setminus \{i\},\ G \neq \emptyset$$

- The Möbius representation clearly exhibits the contributions of both individual criteria and pairs of criteria that interact **positively ($m(\{i,j\}) > 0$)** or **negatively ($m(\{i,j\}) < 0$)**

2-additive Choquet Integral - Example

The **Möbius representation** of the 2-additive Choquet integral:

$$\boldsymbol{Ch}(\boldsymbol{a}) = \sum_{i=1,\dots,n} m(\{i\})\cdot g_i(a) + \sum_{\{i,j\}\subseteq G} m(\{i,j\})\cdot \min\{g_i(a), g_j(a)\}$$

Example weights satisfying normalization and monotonicity constraints:

$m(\{\boldsymbol{g_1}\}) = m(\{\boldsymbol{g_2}\}) = 0.45$, $m(\{\boldsymbol{g_3}\}) = 0.3$

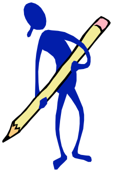
$m(\{\boldsymbol{g_1},\boldsymbol{g_2}\}) = -0.4$ (negative interaction)

$m(\{\boldsymbol{g_1},\boldsymbol{g_3}\}) = m(\{\boldsymbol{g_2},\boldsymbol{g_3}\}) = 0.1$ (positive interaction)

Alt.	g_1	g_2	g_3	*Ch*	Rank
A	18	16	14	**15.9**	2
B	18	14	16	**16.6**	1
C	14	16	14	14.9	3
D	14	14	16	14.6	4

$$Ch(\boldsymbol{A}) = 18\cdot m(\{\boldsymbol{g_1}\}) + 16\cdot m(\{\boldsymbol{g_2}\}) + 14\cdot m(\{\boldsymbol{g_3}\}) +$$
$$+ \min\{18,16\}\cdot m(\{\boldsymbol{g_1},\boldsymbol{g_2}\}) + \min\{18,14\}\cdot m(\{\boldsymbol{g_1},\boldsymbol{g_3}\}) + \min\{16,14\}\cdot m(\{\boldsymbol{g_2},\boldsymbol{g_3}\}) =$$
$$= 18\cdot 0.45 + 16\cdot 0.45 + 14\cdot 0.3 + 16\cdot(-0.4) + 14\cdot 0.1 + 14\cdot 0.1 = \mathbf{15.9}$$

$$Ch(\boldsymbol{B}) = 18\cdot m(\{\boldsymbol{g_1}\}) + 14\cdot m(\{\boldsymbol{g_2}\}) + 16\cdot m(\{\boldsymbol{g_3}\}) +$$
$$+ \min\{18,14\}\cdot m(\{\boldsymbol{g_1},\boldsymbol{g_2}\}) + \min\{18,16\}\cdot m(\{\boldsymbol{g_1},\boldsymbol{g_3}\}) + \min\{14,16\}\cdot m(\{\boldsymbol{g_2},\boldsymbol{g_3}\}) =$$
$$= 18\cdot 0.45 + 14\cdot 0.45 + 16\cdot 0.3 + 14\cdot(-0.4) + 16\cdot 0.1 + 14\cdot 0.1 = \mathbf{16.6}$$



- In many real decision problems, it suffices to consider 2-additive measures
  - In this case, positive and negative interactions between couples of criteria are modeled
- From the point of view of MCDA, the use of 2-additive measures is justified by observing that the information on the importance of the single criteria and the interactions between couples of criteria are noteworthy
- It could be not easy for the DM to provide information on the interactions among three or more criteria during the decision procedure