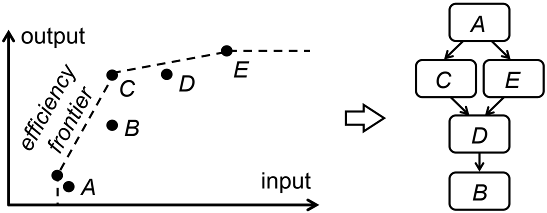


# Data Envelopment Analysis: Rankings of Decision Making Units

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# Key concepts in DEA

Measures relative **efficiency** of Decision Making Units

Handles complex relation between **multiple inputs and multiple outputs**

**DEA**

**Performance** analysis and measurement to set a **benchmark**

*input* 1
*input* 2
*input* 3
*input* 4

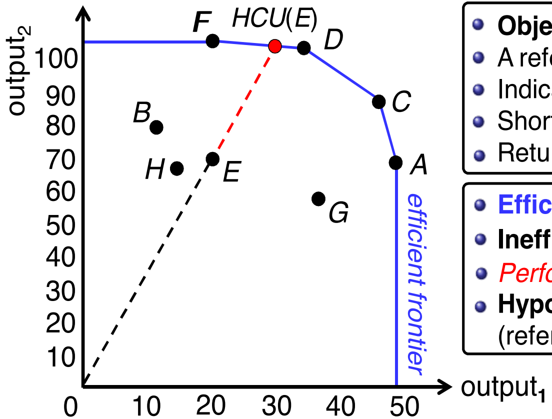
Decision Making Unit (DMU)

*output* 1
*output* 2
*output* 3

- Medical doctors (**A** to H) work at a hospital for the same **160 hours per month** (equal input), performing **exams** and **surgeries** (two outputs)
- Which ones are the most "efficient"?

| Doctor | DMU | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|---|
| Hours | **Input$_1$** | 160 | 160 | 160 | 160 | 160 | 160 | 160 | 160 |
| Exams | **Output$_1$** | 48 | 12 | 45 | 31 | 20 | 20 | 36 | 15 |
| Surgeries | **Output$_2$** | 68 | 80 | 87 | 100 | 70 | 105 | 53 | 65 |

# Results and Advantages of DEA




- **Objective measures of efficiency**
- A reference set of comparable units
- Indicators of excess use of inputs
- Shortfalls in the production of outputs
- Returns to scale measure

- **Efficient doctors**: **F, D, C** and **A**
- **Inefficient doctors**: **B, H, E** and **G**
- *Performance gap*
- **Hypothetical comparison unit** (reference set for *E* is {*F*,*D*})

- Performance evaluation and measurement
- Benchmarking to identify ”best practice” units
- ”Data mining” to generate hypotheses about the drivers of efficiency

# Model for Ratio-based Efficiency Analysis

Based on **linear algebra** and related to **linear programming** concepts

**Data**

- $K$ operating DMUs, $k = 1, \ldots, K$
- $M$ inputs ($m = 1, \ldots, M$)
- $x_{mk}$ – level of **input**$_m$ from **DMU**$_k$
- $v_m$ – weight of **input**$_m$
- $N$ outputs ($n = 1, \ldots, N$)
- $y_{nk}$ – level of **output**$_n$ from **DMU**$_k$
- $u_n$ – weight of **output**$_n$

$$\max \sum_{n=1,\ldots,N} u_n \cdot y_{nk}$$

$$\text{s.t.} \quad \sum_{m=1,\ldots,M} v_m \cdot x_{mk} = 1$$

for $j = 1, \ldots, K$

$$\sum_{n=1,\ldots,N} u_n \cdot y_{nj} \le \sum_{m=1,\ldots,M} v_m \cdot x_{mj}$$

$$v_m \ge 0, \; m = 1, \ldots, M$$
$$u_n \ge 0, \; n = 1, \ldots, N$$

- Ratio-based efficiency model

$$E_k = \frac{\sum_{n=1,\ldots,N} u_n \cdot y_{nk}}{\sum_{m=1,\ldots,M} v_m \cdot x_{mk}}$$

- There exist other efficiency models in DEA

- Production possibilites: **CRS/CCR** (conical combinations) or VRS/BCC (convex combinations)
- Orientation: **input** or output
- Perspective: units' combinations or **efficiencies**

Today's focus, but all ideas can be generalized

# Understanding Efficiency Scores

| DMU | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| $Input_1$ | 160 | 160 | 160 | 160 | 160 | 160 | 160 | 160 |
| $Output_1$ | 48 | 12 | 45 | 31 | 20 | 20 | 36 | 15 |
| $Output_2$ | 68 | 80 | 87 | 100 | 70 | 105 | 53 | 65 |

**Analysis for A**

$$\begin{aligned}
&\max\ 48\cdot u_1 + 68\cdot u_2\\
&\text{s.t.}\quad 160\cdot v_1 = 1\\
&\quad 48\cdot u_1 + 68\cdot u_2 \le 160\cdot v_1\\
&\quad 12\cdot u_1 + 80\cdot u_2 \le 160\cdot v_1\\
&\quad 45\cdot u_1 + 87\cdot u_2 \le 160\cdot v_1\\
&\quad \cdots\\
&\quad 15\cdot u_1 + 65\cdot u_2 \le 160\cdot v_1\\
&\quad v_1,\ u_1,\ u_2 \ge 0
\end{aligned}$$

**Analysis for B**

$$\begin{aligned}
&\max\ 12\cdot u_1 + 80\cdot u_2\\
&\text{s.t.}\quad 160\cdot v_1 = 1\\
&\quad 48\cdot u_1 + 68\cdot u_2 \le 160\cdot v_1\\
&\quad 12\cdot u_1 + 80\cdot u_2 \le 160\cdot v_1\\
&\quad 45\cdot u_1 + 87\cdot u_2 \le 160\cdot v_1\\
&\quad \cdots\\
&\quad 15\cdot u_1 + 65\cdot u_2 \le 160\cdot v_1\\
&\quad v_1,\ u_1,\ u_2 \ge 0
\end{aligned}$$

$v_1 = 0.00625$
$u_1 = 0.017025089$
$u_2 = 0.002688172$
$E_A = 1.000 \Rightarrow$ **A** is efficient

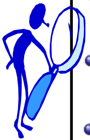
$v_1 = 0.00625$
$u_1 = 0.0$
$u_2 = 0.009523809$
$E_B = 0.762 \Rightarrow$ **B** is inefficient

# Weaknesses of DEA

**Data Envelopment Analysis** is a method to **assign a score** to each DMU, in order to measure its efficiency:

- **score = 1: DMU is efficient**
- **score < 1: DMU is inefficient**

| DMU | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| **$\text{Input}_1$** | 160 | 160 | 160 | 160 | 160 | 160 | 160 | 160 |
| **$\text{Output}_1$** | 48 | 12 | 45 | 31 | 20 | 20 | 36 | 15 |
| **$\text{Output}_2$** | 68 | 80 | 87 | 100 | 70 | 105 | 53 | 65 |
| **Efficiency** | 1.0 | 0.762 | 1.0 | 1.0 | 0.693 | 1.0 | 0.755 | 0.629 |



- **Poor discriminative power** (many DMUs may be efficient)
- Does not perform well with too many inputs/outputs (DMUs tend to be ”all optimal”)
- Enough DMUs in relation to inputs/outputs for building an efficient frontier: $K > 2(M + N)$
- DEA scores reflect DMUs **performance only for weights that are most favorable** to it

# Rankings of Decision Making Units

Poor discriminative power for the efficient units

With too many DMUs, they become too specialized

**DEA weaknesses**

With too many inputs and outputs, many DMUs prove to be efficient

Extremely limited set of weight vectors is taken into account

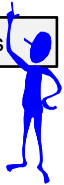
Super-efficiency

Cross-efficiency

**Making DMUs more comparable**

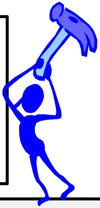
Weight constraints

Robustness analysis with Monte Carlo simulation



# Super-efficiency

- Allows calculating efficiency improvements for efficient units
- **Helps determine how much more efficient an efficient DMU is relative to other DMUs**
- Does help to rank efficient DMUs



$$\boldsymbol{max} \sum_{n=1,\ldots,N} \boldsymbol{u_n} \cdot \boldsymbol{y_{nk}}$$

$$\text{s.t.} \quad \sum_{m=1,\ldots,M} \boldsymbol{v_m} \cdot \boldsymbol{x_{mk}} = 1$$

$$\text{for } \boldsymbol{j} = 1, \ldots, \boldsymbol{K},\ \boldsymbol{j \neq k}$$

$$\sum_{n=1,\ldots,N} \boldsymbol{u_n} \cdot \boldsymbol{y_{nj}} \leq \sum_{m=1,\ldots,M} \boldsymbol{v_m} \cdot \boldsymbol{x_{mj}}$$

$$\boldsymbol{v_m} \geq 0,\ \boldsymbol{m} = 1, \ldots, M$$

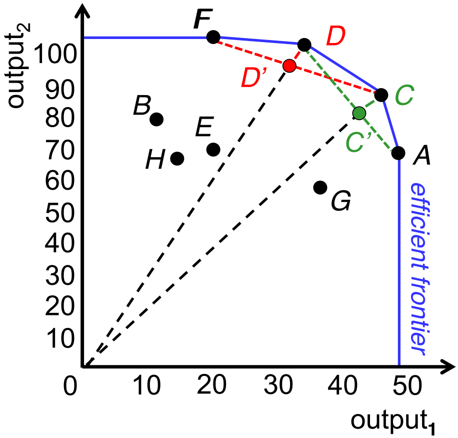
$$\boldsymbol{u_n} \geq 0,\ \boldsymbol{n} = 1, \ldots, N$$

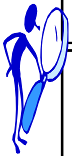
Super-efficiency model for $DMU_k$

$DMU_k$ (under consideration) is removed from the constraint set, thereby allowing its efficiency to exceed the value of 1

P. Andersen, N. Petersen, A procedure for ranking efficient units in DEA. *Management Science*, 39,1261-1264, 1994

# Super-efficiency – Graphical Analysis



- ***D*** evaluated relative to the frontier defined by *F*-***C***-*A*
- Super-efficiency defined by the ratio **OD/OD'**

- ***C*** evaluated relative to the frontier defined by *F*-***D***-*A*
- Super-efficiency defined by the ratio **OC/OC'**

- By visual inspection ***C*** is slightly more super-efficient than ***D***

- *B*, *E*, *G*, and *H* are evaluated relative to the frontier defined by *F*-***C***-***D***-*A*

# Super-efficiency vs. Efficiency

| DMU | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| $\text{Input}_1$ | 160 | 160 | 160 | 160 | 160 | 160 | 160 | 160 |
| $\text{Output}_1$ | 48 | 12 | 45 | 31 | 20 | 20 | 36 | 15 |
| $\text{Output}_2$ | 68 | 80 | 87 | 100 | 70 | 105 | 53 | 65 |
| Efficiency | 1.0 | 0.762 | 1.0 | 1.0 | 0.693 | 1.0 | 0.755 | 0.629 |
| Super-eff | 1.067 | 0.762 | 1.084 | 1.024 | 0.693 | 1.050 | 0.755 | 0.629 |

**Analysis for A**

$$\max\ 48\cdot u_1 + 68\cdot u_2$$

$$\text{s.t.}\quad 160\cdot v_1 = 1$$

$$\cancel{48\cdot u_1 + 68\cdot u_2 \le 160\cdot v_1}$$

$$12\cdot u_1 + 80\cdot u_2 \le 160\cdot v_1$$

$$45\cdot u_1 + 87\cdot u_2 \le 160\cdot v_1$$

$$\ldots$$

$$15\cdot u_1 + 65\cdot u_2 \le 160\cdot v_1$$

$$v_1,\ u_1,\ u_2 \ge 0$$

- *For efficient DMUs*, super-efficiencies are not less than 1 (*they can be* 1)
- *For inefficient DMUs*, super-efficiencies are equal to the respective efficiencies
- Helps to rank efficient DMUs (compare *C* against *D*)

*C* → *A* → *F* → *D*

# Cross-efficiency

- Efficiencies are based on the weights which are most favorable to the DMU being evaluated
- It may be of interest to know **how the DMU performs when using other weights** as well
- The **cross-efficiency** score represents how the DMU performs when evaluated with the optimal weights for all DMUs

| DMU | 1 | 2 | … | K |
|---|---|---|---|---|
| 1 | $E_{11}$ | $E_{12}$ | … | $E_{1K}$ |
| 2 | $E_{21}$ | $E_{22}$ | … | $E_{2K}$ |
| … | … | … | … | … |
| K | $E_{K1}$ | $E_{K2}$ | … | $E_{KK}$ |

efficiency of $DMU_2$ when evaluated with the optimal weights for $DMU_1$

efficiency of $DMU_K$ when evaluated with the optimal weights for $DMU_2$

**Cross-efficiency** for $\mathrm{DMU}_k$ is the average of efficiencies attained for the weights optimal for other DMUs:

$$\mathbf{CRE}_k = \frac{1}{K} \sum_{i=1,\ldots,K} E_{ik}$$

# Deriving Weights for Cross-efficiency

- Multiple optima (weight vectors for which the efficiency is maximal) are possible
- Selection of the weight vector based on either aggressive (***min***) or benevolent (***max***) formulation (optimization of the efficiency of a composite DMU)
- The inputs and outputs of a **composite DMU** are sums of, respectively, inputs ($\sum_{j=1,\ldots,K,\, j\neq k} x_{mj}$) or outputs ($\sum_{j=1,\ldots,K,\, j\neq k} y_{nj}$) of DMUs other than $DMU_k$

$$\textit{min} \text{ or } \textit{max} \sum_{n=1,\ldots,N} u_n \cdot \sum_{j=1,\ldots,K,\, j\neq k} y_{nj}$$

$$\text{s.t.} \quad \sum_{m=1,\ldots,M} v_m \cdot \sum_{j=1,\ldots,K,\, j\neq k} x_{mj} = 1$$

for $j = 1, \ldots, K,\ j \neq k$

$$\sum_{n=1,\ldots,N} u_n \cdot y_{nj} \leq \sum_{m=1,\ldots,M} v_m \cdot x_{mj}$$

$$\sum_{n=1,\ldots,N} u_n \cdot y_{nk} = E_k \cdot \sum_{m=1,\ldots,M} v_m \cdot x_{mk}$$

$v_m \geq 0,\ m = 1, \ldots, M$

$u_n \geq 0,\ n = 1, \ldots, N$

**Deriving optimal weights for $DMU_k$**

optimize efficiency of a composite DMU

assume the efficiencies of DMUs other than $DMU_k$ are not greater than 1

assume the efficiency of $DMU_k$ is maximal (i.e., equal to $E_k$)

# Cross-efficiency – Computational Aspects (1)

| DMU | A | B | C | D | E | F | G | H | Composite: sum B-H |
|---|---|---|---|---|---|---|---|---|---|
| $\text{Input}_1$ | 160 | 160 | 160 | 160 | 160 | 160 | 160 | 160 | 160 + … + 160 = 1120 |
| $\text{Output}_1$ | 48 | 12 | 45 | 31 | 20 | 20 | 36 | 15 | 12 + … + 15 =179 |
| $\text{Output}_2$ | 68 | 80 | 87 | 100 | 70 | 105 | 53 | 65 | 80 + … + 65 = 560 |
| Efficiency | 1.0 | 0.762 | **1.0** | **1.0** | 0.693 | **1.0** | 0.755 | 0.629 | |

**Analysis for A**

$$
\begin{aligned}
\min\ & 179\cdot u_1 + 560\cdot u_2 \ (\textit{aggresive})\\
\text{s.t. } & 1120\cdot v_1 = 1\\
& 12\cdot u_1 + 80\cdot u_2 \le 160\cdot v_1\\
& 45\cdot u_1 + 87\cdot u_2 \le 160\cdot v_1\\
& \dots\\
& 15\cdot u_1 + 65\cdot u_2 \le 160\cdot v_1\\
& 48\cdot u_1 + 68\cdot u_2 = 1.0\cdot 160\cdot v_1\\
& v_1, u_1, u_2 \ge 0
\end{aligned}
$$

$v_1 = 0.000892857$
$u_1 = 0.002976190$
$u_2 = 0.0$

optimal weights for *A*

| DMU | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| $E_{Ak}$ | 1.0 | 0.250 | 0.937 | 0.646 | 0.417 | 0.417 | 0.750 | 0.313 |

efficiency of *B* when evaluated with optimal weights for *A*

J. Doyle, R. Green, Efficiency and cross-efficiency in DEA: Derivations, meanings and uses. *Journal of the Operational Research Society*, 45, 567-578,1994

# Cross-efficiency – Computational Aspects (2)

| DMU | $E_{kA}$ | $E_{kB}$ | $E_{kC}$ | $E_{kD}$ | $E_{kE}$ | $E_{kF}$ | $E_{kG}$ | $E_{kH}$ |
|---|---|---|---|---|---|---|---|---|
| $E_{Ak}$ | 1.0 | 0.250 | 0.937 | 0.646 | 0.417 | 0.417 | 0.750 | 0.313 |
| $E_{Bk}$ | 0.648 | 0.762 | 0.829 | 0.952 | 0.667 | 1.000 | 0.505 | 0.619 |
| $E_{Ck}$ | 1.0 | 0.419 | 1.0 | 0.797 | 0.529 | 0.623 | 0.755 | 0.430 |
| $E_{Dk}$ | 0.787 | 0.749 | 0.942 | 1.0 | 0.693 | 1.0 | 0.608 | 0.629 |
| $E_{Ek}$ | 0.787 | 0.749 | 0.942 | 1.000 | 0.693 | 1.0 | 0.608 | 0.629 |
| $E_{Fk}$ | 0.648 | 0.762 | 0.829 | 0.952 | 0.667 | 1.0 | 0.505 | 0.619 |
| $E_{Gk}$ | 1.0 | 0.419 | 1.0 | 0.797 | 0.529 | 0.623 | 0.755 | 0.430 |
| $E_{Hk}$ | 0.787 | 0.749 | 0.942 | 1.0 | 0.693 | 1.0 | 0.608 | 0.629 |
| $CRE_k$ | 0.832 | 0.607 | 0.928 | 0.893 | 0.611 | 0.833 | 0.637 | 0.537 |

$$CRE_A = 1/8 \cdot (1.0 + 0.648 + 1.0 + \ldots + 0.787) = \mathbf{0.832}$$

A DMU with a high cross-efficiency can be considered as a **good overall performer**; others are more "**niche**" DMUs

C → D → F → A → G → E → B → H

# Preference Information – Weight Constraints

- DMUs may attain their efficiency scores for "extreme" weights in conventional DEA models
- **Preference information** can be captured through preference statements about the relative values of input and/or output units
- Statement impose **constraints on the input/output weights**

| Doctor | DMU | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|---|
| Hours | **$\text{Input}_1$** | 160 | 160 | 160 | 160 | 160 | 160 | 160 | 160 |
| Exams | **$\text{Output}_1$** | 48 | 12 | 45 | 31 | 20 | 20 | 36 | 15 |
| Surgeries | **$\text{Output}_2$** | 68 | 80 | 87 | 100 | 70 | 105 | 53 | 65 |

*A single surgery is at least as valuable 2 examinations,* ⇨ $u_2 \geq 2u_1$
*but not more valuable than 4 examinations* $u_2 \leq 4u_1$

- **Only relative weights matter**
- Several elicitation procedures can be employed (see MCDA)

V. Podinovski, The explicit role of weight bounds in models of data envelopment analysis. *Journal of the Operational Research Society*, 56, 1408-1418, 2005

# Efficiency Analysis with Weight Constraints

**Feasible sets defined by preference information (constraints)**

- *space of feasible input weights*
  $S_v = \{v=(v_1,\dots,v_M) \neq 0 \mid v \geq 0, A_v v \leq 0\}$
- *space of feasible output weights*
  $S_u = \{u=(u_1,\dots,u_N) \neq 0 \mid u \geq 0, A_u u \leq 0\}$

$$\max \sum_{n=1,\dots,N} u_n \cdot y_{nk}$$

$$\text{s.t.} \quad \sum_{m=1,\dots,M} v_m \cdot x_{mk} = 1$$

for $j = 1, \dots, K$

$$\sum_{n=1,\dots,N} u_n \cdot y_{nj} \leq \sum_{m=1,\dots,M} v_m \cdot x_{mj}$$

$v=(v_1,\dots,v_M) \in S_v$

$u=(u_1,\dots,u_N) \in S_u$

| DMU | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| $\text{Input}_1$ | 160 | 160 | 160 | 160 | 160 | 160 | 160 | 160 |
| $\text{Output}_1$ | 48 | 12 | 45 | 31 | 20 | 20 | 36 | 15 |
| $\text{Output}_2$ | 68 | 80 | 87 | 100 | 70 | 105 | 53 | 65 |

*weight constraints*

$u_2 \geq 2u_1$

$u_2 \leq 4u_1$

**Analysis for A**

$$\max\ 48 \cdot u_1 + 68 \cdot u_2$$

$$\text{s.t.} \quad 160 \cdot v_1 = 1$$

$$48 \cdot u_1 + 68 \cdot u_2 \leq 160 \cdot v_1$$

$$12 \cdot u_1 + 80 \cdot u_2 \leq 160 \cdot v_1$$

$$\dots$$

$$15 \cdot u_1 + 65 \cdot u_2 \leq 160 \cdot v_1$$

$$4u_1 \geq u_2 \geq 2u_1$$

$$v_1,\ u_1,\ u_2 \geq 0$$

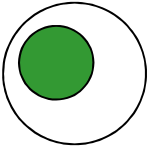
*With weight constraints:* $v_1 = 0.00625$
$u_1 = 0.004329004$, $u_2 = 0.008658008$
$E_A = 0.797$ ⇨ **A** is inefficient

*Without weight constraints:* $v_1 = 0.00625$
$u_1 = 0.017025089$, $u_2 = 0.002688172$
$u_1$ and $u_2$ do not satisfy the constraints
$E_A = 1.000$ ⇨ **A** was efficient

# Efficiency Scores With and Without Weight Constraints

| DMU | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| $Input_1$ | 160 | 160 | 160 | 160 | 160 | 160 | 160 | 160 |
| $Output_1$ | 48 | 12 | 45 | 31 | 20 | 20 | 36 | 15 |
| $Output_2$ | 68 | 80 | 87 | 100 | 70 | 105 | 53 | 65 |
| **Efficiency** (no constr.) | **1.0** | 0.762 | **1.0** | **1.0** | 0.693 | **1.0** | 0.755 | 0.629 |
| **Efficiency** (with constr.) | 0.797 | 0.755 | 0.948 | **1.0** | 0.693 | **1.0** | 0.615 | 0.629 |

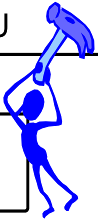
- **Preference information contracts the space of feasible inputs/output weights**
- The introduction of weight information often leads to lower (but never higher) efficiency scores



*When considering preference information, A and C become inefficient* (better discrimination among DMUs)

# Need for Robustness Analysis

- Analysis of **most favourable weights** (not unique)
- Extremely **small share of weights** is analyzed (others neglected while being equally desirable)
- Linear programming techniques **require nromalization of weights** for each DMU individually (meaningful comparison of weights across different DMUs is difficult)
- Assumptions about **returns to scale** may be difficult to formulate
- Results **sensitive** to removal or inclusion of a single DMU



Addressing all aforementioned concerns comprehensively requires incorporation of **robustness analysis**

# Monte Carlo Simulation

- Accounting for **uncertainties** observed in real-world problems
- **Robust** = true for all or the most plausible combinations of parameter values / feasible scenarios
- In the context of DEA, the uncertainty is related to the selection of feasible input/output weights

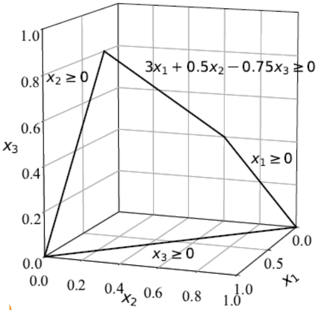
**Monte Carlo simulation**

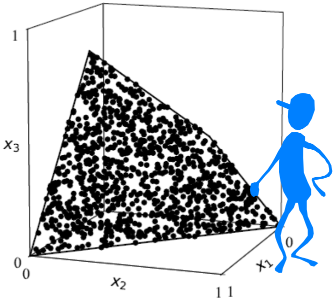
- Broad class of computational algorithms that rely on **repeated random sampling** to obtain numerical results
- Use randomness to solve problems that might be deterministic in principle

**Monte Carlo methods used in**: **numerical analysis** (calculating the numerical value of a definite integral), **generating draws from a prob. distribution**, and **optimization** (selection of the best option)

# Hit-And-Run Algorithm

- **Markov Chain Monte Carlo** (series of depending samples)
- Mixes rapidly from any interior point
- Provides block samples from the **uniform distribution**
- Avoids the rejection step (improved acceptance rate)




https://github.com/gertvv/hitandrun

https://github.com/kciomek/polyrun/

# Monte Carlo Simulation in Efficiency Analysis

- Apply Monte Carlo to derive **a representative sample of all feasible input/output weights** from the uniform distribution
- Check the results for each weight vector and **summarize how probable are particular outcomes** / what is their distribution in view of all samples
- No assumptions with respect to the production possiblities beyond the set of DMUs under consideration
- Three perspectives for robustness analysis:
  - Efficiency scores
  - Pairwise efficiency preference relation
  - Efficiency ranks
- **Stochastic indices** can be estimated through simulation up to a pre-defined accuracy (with 10,000 samples – 0.01 with 95% confid.)
- Results derived from pairwise comparisons are less sensitive to outliers

M. Kadziński, A. Labijak, M. Napieraj, Integrated framework for robustness analysis using ratio-based efficiency model with application to evaluation of Polish airports, *Omega*, 67, 1-18, 2017

# Stochastic Analysis – Focus on Ranks

- Efficiency rank acceptability indices ERAI($DMU_k$) for each $DMU_k$ and rank $r$

  **ERAI($DMU_k$,$r$)** = share of feasible weights for which **$DMU_k$** attains rank **$r$** =

  $$= \int_{(v,u)\in(S_v,S_u)} m((v,u),k,r)\, d(v,u),$$

  *where* $m((v,u),k,r) = 1$, if $rank(DMU_k,(v,u)) = r$,

  $m((v,u),k,r) = 0$, otherwise

- Based on pairwise one-on-one comparisons (more stable)
- For each $DMU_k$: $\sum_{r=1,\ldots,K}$ ERAI($DMU_k$,r) = 1
- Estimate of an expected rank for $DMU_k$: $ER_k$ = average rank across all samples

*for 11% of feasible weights, A attains the greatest efficiency* (*i.e., the first rank*)

| DMU | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $ER_k$ |
|---|---|---|---|---|---|---|---|---|---|
| **A** | 0.11 | 0.15 | 0.08 | 0.39 | 0.23 | 0.04 | 0.0 | 0.0 | 3.60 |
| **B** | 0.0 | 0.0 | 0.0 | 0.27 | 0.33 | 0.03 | 0.23 | 0.14 | 5.64 |
| **C** | 0.35 | 0.22 | 0.43 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 2.08 |
| **…** | … | … | … | … | … | … | | | … |

- Pairwse efficiency outranking index PEOI($DMU_k$,$DMU_l$) for each pair ($DMU_k$,$DMU_l$)

  **PEOI($DMU_k$,$DMU_l$)** = share of feasible weights for which $DMU_k$ attains efficiency not worse than $DMU_l$

- Reflects the outcome of a pairwise comparison (not influenced by other DMUs)

*inefficient B can be better than efficient A*

*significant advantage*

| DMU | A | B | C | … |
|---|---|---|---|---|
| A | 1.0 | 0.72 | 0.10 | … |
| B | 0.28 | 1.0 | 0.0 | … |
| C | 0.90 | 1.0 | 1.0 | … |
| … | … | … | … | … |

*robust relation*

# Stochastic Analysis – Focus on Efficiency Scores

- Efficiency distribution:
  Efficiency acceptability interval index **EAII($DMU_k$,$b_i$)**

  *efficiency buckets* $b_i$

| [0.0,0.1] | (0.1,0.2] | (0.2,0.3] | … | (0.9,1.0] |
|---|---|---|---|---|

  **EAII($DMU_k$,$b_i$)** = share of feasible weights for which $DMU_k$ attains efficiciency in the interval $b_i$

- Estimate of an expected efficiency for $DMU_k$:
  $EE_k$ = average efficiency score across all samples

| DMU | [0.0,0.1] | … | (0.6,0.7] | (0.7,0.8] | (0.8,0.9] | (0.9,1] | $EE_k$ |
|---|---|---|---|---|---|---|---|
| **A** | 0.0 | … | 0.18 | 0.20 | 0.26 | 0.36 | 0.840 |
| **B** | 0.0 | … | 0.13 | 0.56 | 0.0 | 0.0 | 0.636 |
| **C** | 0.0 | … | 0.0 | 0.0 | **0.24** | **0.76** | **0.945** |
| **…** | … | … | … | … | … | … | … |

# Data Envelopment Analysis – Summary

We have introduced **basic and more advanced methods for DEA**:

- Estimation of production frontier in production theory in economics
- Empirically measure the efficiency of Decision Making Units
- Benchmarking in operations management (best-practice frontier)
- **Thousands of applications** in health care, banking, agriculture, transportation, education, energy, manufacturing, sport, …
- "Proof" that **linear programming is extremely useful**, bringing lots of money to those who can use it properly and interpret the results
- Ideas used in efficiency analysis and ranking of DMUs can be (and actually are) adopted in other fields of operations research