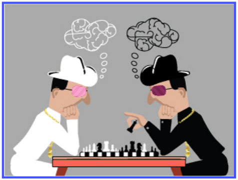
	A	B
A	-10 \ -10	-25 \ 0
B	0 \ -25	-20 \ -20



Game Theory: Solution Concepts in Strategic Games

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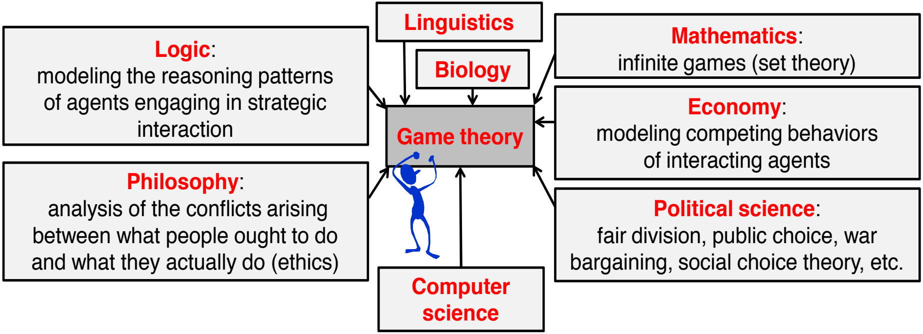
Game Theory in Different Fields

- Study of mathematical models to analyze *strategic interactions* (conflict and cooperation) between rational agents
- A **game** is a formal description of a strategic situation
- It typically involves several **players**; a game with only one player is usually called a *decision problem*



Game Theory vs. Computer Science

- **Algorithms** for predicting what the outcome might be
- Analysis of **complexity** of computing the equilibria of a game
- **Artificial Intelligence**: studying interaction between the software agents in a multiagent system

P. Stone et al. **Articial Intelligence and Life in 2030**. One Hundred Year Study on Artificial Intelligence. *Stanford*, 2016

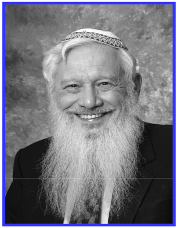
- large-scale machine learning
- reinforcement learning
- deep learning
- robotics
- computer vision
- natural language processing
- internet of things
- collaborative systems
- neuromorphic computing
- crowdsourcing and human computation
- **algorithmic game theory and computational social choice**

Hot topics in AI

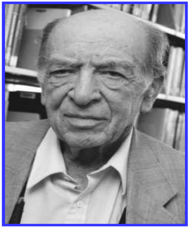
Famous Nobel Prize Winners

Herbert Simon
1978

William Vickrey
1996

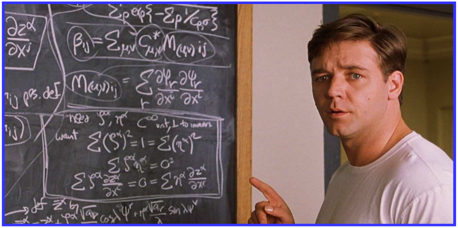


Robert Aumann
2005



Leonid Hurwicz
2007

John Nash
1994

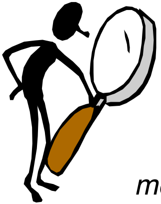


A Beautiful Mind, directed by Ron Howard
2001

Example: Split or Steal (1)

The split-or-steal game in the British television show "**Golden Balls**", particularly the one aired on 14 March 2008, is a good example:
http://youtu.be/p3Uos2fzIJ0

- The normal form of this **strategic** and **non-cooperative** game
- This is a one-shot game with **perfect information**



players

actions

A \ B	**Split**	**Steal**
Split	50k \ 50k	0 \ 100k
Steal	100k \ 0	0 \ 0

outcomes (utilities)

money, satisfaction, chances for transferring genes, etc.

Example: Split or Steal (2)

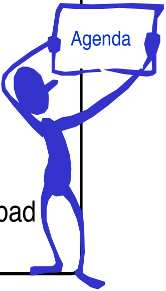
- The normal form of this **strategic** and **non-cooperative** game
- This is a **one-shot** game with **perfect information**

A \ B	**Split**	**Steal**
Split	50k \ 50k	0 \ 100k
Steal	100k \ 0	0 \ 0

- Other games (like chess) can also be modeled using the **extensive** form (as a ”game tree")
- In a **coalitional (cooperative)** game, we might instead ask players to find a split that fairly reflects individual contributions or form coalitions to attain their common goals
- When the information is **incomplete**, the players do not have perfect knowledge of actions and/or utilities

Plan for Today: Strategic Games in Normal Form

- Examples for and formal definition of **normal-form games**
- A definition of **stability** of an outcome (rational for all individuals)
- A definition of **efficiency** of an outcome (good for the group)
- **Pure and mixed Nash equilibria**
- How to compute the Nash equilibria for small games
- The existence of Nash equilibria for arbitrary games
- Equilibrium in **dominant strategies**: do what's definitely good
- **Elimination of dominated strategies**: don't do what's definitely bad
- **Correlated equilibrium**: follow some external recommendation



K. Leyton-Brown and Y. Shoham. Essentials of Game Theory: A Concise, Multi-disciplinary Introduction. *Morgan & Claypool Publishers*, 2008

Prisoner's Dilemma (1)

*Two hardened criminals, **Oliwia** and **Mateusz**, got caught by police and are being interrogated in separate cells. The police only have evidence for some of their minor crimes. Each is facing this dilemma:*

- If we cooperate (**C**) and don't talk, then we each get 10 years for the minor crimes (*utility* = -10)
- If I cooperate (**C**) but my partner defects (**D**) and talks, then I get 25 years (*utility* = -25)
- If my partner cooperates (**C**) but I defect (**D**), then I go free (as crown witness) (*utility* = 0)
- If we both defect (**D**), then we share the blame and get 20 years each (*utility* = -20)

players

actions

O \ M	**C**	**D**
C	-10 \ -10	-25 \ 0
D	0 \ -25	-20 \ -20

outcomes (utilities)

Prisoner's Dilemma (2)

What would you do? Why?

- ***Oliwia***: cooperate (**C**) or defect (**D**)
- ***Mateusz***: cooperate (**C**) or defect (**D**)

O \ M	**C**	**D**
C	-10 \ -10	-25 \ 0
D	0 \ -25	-20 \ -20

Prisoner's Dilemma (3)

assume **O** and ***M*** *cooperate* (***CC***):
for both of them, it would be better
to unilaterally defect (**D**)
they would get 0 rather than -10

Focus on:

- **stability** of an outcome (rational for all individuals)

What would you do? Why?

O \ M	C	D
C	-10 \ -10	-25 \ 0
D	0 \ -25	-20 \ -20

assume **O** defects and ***M*** *cooperates* (***DC***):
for ***M*** it would be better to unilaterally
change his assigned strategy (**D**)
he would get -20 rather than -25
analogously for ***CD***

assume **O** and ***M*** *defect* (***DD***):
for both of them there is no reason to
unilaterally change their assigned strategies
they would get -25 rather than -20

Prisoner's Dilemma - Real-World Relevance

Variants of the Prisoner's Dilemma commonly occur in real life (often with more than two players):

- **countries agreeing to caps on greenhouse gas emissions**
- firms cooperating by not aggressively competing on price
- **NATO and the Warsaw Pact** had the choice to arm or disarm
- network users claiming only limited bandwidth
- doping / **cheating in sport**


Strategic Games in Normal Form

A normal-form game is a tuple **<N, A, u>** where:

- $N = \{1, \ldots, n\}$ is a **finite set of players** (or agents)
- $A = A_1 \times \ldots \times A_n$ is a **finite set of action profiles** $a = (a_1, \ldots, a_n)$, with A_i being the **set of actions** available to player $i \in N$
- $u = (u_1, \ldots, u_n)$ is a **profile of utility functions** $u_i : A \to \mathbb{R}$

- Every player i chooses an action, say a_i, giving rise to the profile a
- Actions are played simultaneously
- Player i then receives payoff $u_i(a)$

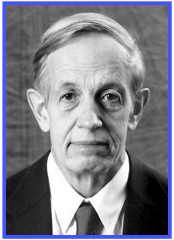
O \ M	C	D
C	-10 \ -10	-25 \ 0
D	0 \ -25	-20 \ -20

- $N = \{O, M\}$
- $A_O = \{C,D\}$, $A_M = \{C,D\}$, $A = A_O \times A_M$
- *e.g.*, $a = (C,C)$
- *e.g.*, $u_O(C,C) = -10$, $u_O(C,D) = -25$, $u_O(D,C) = 0$, $u_O(D,D) = -20$

Nash Equilibria in Pure Strategies: rational for individuals

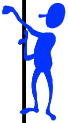
- First we restrict attention to **pure strategies**: strategy = action
- Later we will allow players to randomize over actions

- Notation: (a_i', a_{-i}) is short for $\boldsymbol{a} = (a_1, \ldots, a_{i-1}, a_i', a_{i+1}, \ldots, a_n)$
- $\boldsymbol{a} = (C,C)$; $(D, a_{-O}) = (D,C)$; $(D, a_{-M}) = (C,D)$



John Nash

- We say that $a_i^* \in A_i$ is the **best response** for player i to the (partial) action profile a_{-i}, if $u_i(a_i^*, a_{-i}) \geq u_i(a_i', a_{-i})$ for all $a_i' \in A_i$
- We say that action profile $\boldsymbol{a} = (a_1, \ldots, a_n)$ is a **pure Nash equilibrium**, if a_i is a best response to a_{-i} for every agent $i \in N$



Remark: pure Nash equilibria are **stable outcomes**: *no player has an incentive to unilaterally deviate from her assigned strategy*

Exercise: How Many Pure Nash Equilibria?

	L	R
T	2 \ 2	2 \ 1
B	1 \ 3	3 \ 2

	L	R
T	2 \ 2	2 \ 2
B	2 \ 2	2 \ 2

	L	R
T	1 \ 2	2 \ 1
B	2 \ 1	1 \ 2

- There might be multiple pure Nash equilibria

- There might not be any pure Nash equilibria

Pareto Efficiency: good for the group

Vilfredo Pareto

- Action profile ***a*** **Pareto-dominates** profile ***a'***, if $u_i(a) \geq u_i(a')$ for all players $i \in N$ and this inequality is strict in at least one case
- Action profile ***a*** is called **Pareto efficient** if it is not Pareto-dominated by any other profile, i.e., if you cannot improve things for one player without harming any of the others

	C	D
C	-10 \ -10	-25 \ 0
D	0 \ -25	-20 \ -20

(C,C) **dominates** (D,D)
(C,C) is **Pareto efficient**
(D,D) is not Pareto efficient
The Prisoner's Dilemma illustrates a conflict between ***stability*** *(both players defect) and* ***efficiency*** *(both players cooperate)*

A (pure) **coordination game** is a normal-form game **<*N*, *A*, *u*>** with $u_i(a) = u_j(a)$ for all players $i, j \in N$ and all action profiles $a \in A$

Example:
A world with just two drivers.
Which side of the road to use?

	L	R
L	1 \ 1	0 \ 0
R	0 \ 0	1 \ 1

Remark: For this game, all Nash equilibria are Pareto efficient

Exercise: Is this the case for all coordination games?

A **zero-sum game** is a two-player normal-form game $\boldsymbol{<N, A, u>}$ with $\boldsymbol{u_1(a) + u_2(a) = 0}$ for all action profiles $\boldsymbol{a \in A}$

				
Example: Rock		0 \ 0	-1 \ 1	1 \ -1
Paper		1 \ -1	0 \ 0	-1 \ 1
Scissors		-1 \ 1	1 \ -1	0 \ 0

What are the pure NE of this game?
Intuitively, how should you play?

Mixed Strategies

- So far, the space of strategies available to player ***i*** has simply been her set of actions $\boldsymbol{A}_i$ (pure strategy = action)
- We now generalize and allow player ***i*** to **play any action** *in* $\boldsymbol{A}_i$ **with a certain probability**

- For any finite set $\boldsymbol{X}$, let $\Pi(\boldsymbol{X}) = \{\, p : \boldsymbol{X} \rightarrow [0,1] \mid \Sigma_{x \in \boldsymbol{X}}\, p(x) = 1 \,\}$ be the set of all probability distributions over $\boldsymbol{X}$
- A **mixed strategy** $\boldsymbol{s}_i$ for player ***i*** is a **probability distribution** in $\Pi(\boldsymbol{A}_i)$
- The set of all her mixed strategies is $\boldsymbol{S}_i = \Pi(\boldsymbol{A}_i)$
- A mixed-strategy profile $\boldsymbol{s} = (\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n)$ is an element of $\boldsymbol{S}_1 \times \ldots \times \boldsymbol{S}_n$

actions conducted with a certain pre-defined probablity

x	x_1	x_2	x_3	x_4
$p(x)$	0.50	0.25	0.20	0.05

Example: Battle of the Sexes

Traditionally minded **Oliwia** and **Mateusz** are planning a social activity. Worst of all would be not to agree on joint activity, but if they do manage, **Mateusz** prefers auto (**A**) racing, and **Oliwia** really prefers ballet (**B**).

O \ M	**A**	**B**
A	2 \ 4	0 \ 0
B	0 \ 0	8 \ 3

Suppose **Oliwia** chooses to go to the ballet (**B**) with 75% probability

$\boldsymbol{s_1 = (1/4, 3/4)} \quad \boldsymbol{s_2 = (1, 0)}$

Mateusz chooses to go to the races (**A**) with certainty (pure strategy)

$\boldsymbol{s = (s_1, s_2) = ((1/4, 3/4), (1, 0))}$

The **support of strategy** s_i is the set of actions $\{ a_i \in A_i \mid s_i(a_i) > 0 \}$

- A mixed strategy s_i is pure iff its support is a singleton
- A mixed strategy s_i is truly mixed if it is not pure
- A mixed strategy s_i is fully mixed if its support is the full set A_i

- A mixed-strategy profile $\boldsymbol{s} = (\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n)$ is an element of $\boldsymbol{S}_1 \times \ldots \times \boldsymbol{S}_n$
- The **expected utility** of player $\boldsymbol{i}$ for the mixed-strategy profile $\boldsymbol{s}$ is:

$$u_i(s) = \sum_{a \in \boldsymbol{A}} \left[u_i(a) \cdot \prod_{j \in \boldsymbol{N}} s_j(a_j) \right]$$

O \ **M**	**A** (1)	**B** (0)
A (1/4)	2 \ 4	0 \ 0
B (3/4)	0 \ 0	8 \ 3

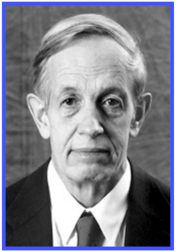
$\boldsymbol{s}_1 = \mathbf{(1/4, 3/4)}$ $\boldsymbol{s}_2 = \mathbf{(1, 0)}$

$\boldsymbol{s} = (\boldsymbol{s}_1, \boldsymbol{s}_2) = \mathbf{((1/4, 3/4), (1, 0))}$

Thus: $u_1(s) = 2 \cdot (1/4 \cdot 1) + 0 \cdot (1/4 \cdot 0) + 0 \cdot (3/4 \cdot 1) + 8 \cdot (3/4 \cdot 0) = 1/2$

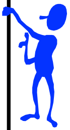
$u_2(s) = 4 \cdot (1/4 \cdot 1) + 0 \cdot (1/4 \cdot 0) + 0 \cdot (3/4 \cdot 1) + 3 \cdot (3/4 \cdot 0) = 1$

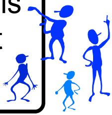
Consider a game $\langle \boldsymbol{N}, \boldsymbol{A}, \boldsymbol{u} \rangle$ with associated (mixed) strategies $\boldsymbol{s}_i \in \boldsymbol{S}_i$



John Nash

- We say that $s_i^* \in \boldsymbol{S}_i$ is the best response for player $\boldsymbol{i}$ to the (partial) strategy profile $\boldsymbol{s}_{-i}$, if $\boldsymbol{u}_i(s_i^*, \boldsymbol{s}_{-i}) \geq \boldsymbol{u}_i(s_i', \boldsymbol{s}_{-i})$ for all $s_i' \in \boldsymbol{S}_i$
- We say that action profile $\boldsymbol{s} = (\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n)$ is a **mixed Nash equilibrium**, if $\boldsymbol{s}_i$ is a best response to $\boldsymbol{s}_{-i}$ for every player $\boldsymbol{i} \in \boldsymbol{N}$





Mixed Nash equilibria are **stable outcomes**:
no player has an incentive to unilaterally change her strategy
Remark: Note how this definition mirrors that of pure Nash equilibria

Computing Nash Equilibria (1)

Recall:
A world with just two drivers
Which side of the road to use?

	L	**R**
L	1 \ 1	0 \ 0
R	0 \ 0	1 \ 1

For this game, it is easy to guess what the Nash equilibria are:

- Pure NE: both pick left (**L**) with certainty: ((1,0), (1,0))
- Pure NE: both pick right (**R**) with certainty: ((0,1), (0,1))
- Both choose fifty-fifty: ((1/2,1/2), (1/2,1/2))

Remark:

- There is no other NE: suppose I pick (1/2+ε, 1/2-ε), e.g., (0.51, 0.49)
- Then your best response is (1,0), because it is better to pick **L** than **R** (expected utility 0.51 vs. 0.49), to which my best response is (1,0)

Computing Nash Equilibria (2)

Suppose we have guessed (correctly) that this game has exactly one NE (**s_1**, **s_2**) and that it is fully mixed. How to compute it?

	L (q)	**R** (1-q)
T (p)	6 \ 4	7 \ 5
B (1-p)	3 \ 2	8 \ 1

Let s_1 = (p,1-p) and s_2 = (q,1-q). If you use a mixed strategy, you must be indifferent between your two actions.

Thus: Player 1 is indifferent: $6 \cdot q + 7 \cdot (1\text{-}q) = 3 \cdot q + 8 \cdot (1\text{-}q) \to q = 1/4$

Player 2 is indifferent: $4 \cdot p + 2 \cdot (1\text{-}p) = 5 \cdot p + 1 \cdot (1\text{-}p) \to p = 1/2$

Mixed Nash Equilibrium: (s_1, s_2) = ((1/2, 1/2), (1/4, 3/4))

Exercise: Game of Chicken

*To establish their relative levels of bravery, **Oliwia** and **Mateusz** race their cars on a collision course straight towards each other at full speed. Each can swerve (**S**) or go straight (**G**).*

- If both **go straight** (***G***), they die
- If both **swerve** (***S***), nothing happens
- Otherwise, whoever swerves faces humiliation, while the other one wins

	S	**G**
S	0 \ 0	-5 \ 10
G	10 \ -5	-20 \ -20

What are the Nash equilibria of this game?

Computing Nash Equilibria (3)

O \ M	S (q)	G ($1-q$)
S (p)	0 \ 0	-5 \ 10
G ($1-p$)	10 \ -5	-20 \ -20

$s_O = (p, 1-p)$
$s_M = (q, 1-q)$
best response of player O depends on q

Player **O** is indifferent if $u_O(S,q) = u_O(G,q)$:
$0 \cdot q + -5 \cdot (1-q) = 10 \cdot q + -20 \cdot (1-q) \rightarrow q = 3/5$

$p \in best_O(q) = [0,1]$
if $u_O(S,q) = u_O(G,q)$, i.e., if $q = 3/5$

Player **O** would play **S** if $u_O(S,q) > u_O(G,q)$:
$0 \cdot q + -5 \cdot (1-q) > 10 \cdot q + -20 \cdot (1-q) \rightarrow q < 3/5$

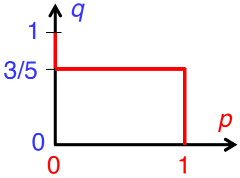
$p \in best_O(q) = 1$
if $u_O(S,q) > u_O(G,q)$, i.e., if $q < 3/5$

Player **O** would play **G** if $u_O(S,q) < u_O(G,q)$:
$0 \cdot q + -5 \cdot (1-q) < 10 \cdot q + -20 \cdot (1-q) \rightarrow q > 3/5$

$p \in best_O(q) = 0$
if $u_O(S,q) < u_O(G,q)$, i.e., if $q > 3/5$

Summary: best response of player **O** depends on q

$$p \in best_O(q) = \begin{cases} 1 & \text{if } q < 3/5 \\ [0,1] & \text{if } q = 3/5 \\ 0 & \text{if } q > 3/5 \end{cases} \Rightarrow$$



Computing Nash Equilibria (4)

O \ M	S (q)	G ($1-q$)
S (p)	0 \ 0	-5 \ 10
G ($1-p$)	10 \ -5	-20 \ -20

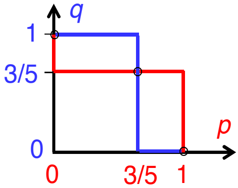
$s_O = (p, 1-p)$
$s_M = (q, 1-q)$
best response of player ***M*** *depends on* ***p***

Player ***M*** is indifferent if $\boldsymbol{u_M(S,p) = u_M(G,p)}$:
$0 \cdot p + -5 \cdot (1-p) = 10 \cdot p + -20 \cdot (1-p) \rightarrow \boldsymbol{p} = 3/5$

$q \in \text{best}_2(p) = [0,1]$
if $\boldsymbol{u_M(S,p) = u_M(G,p)}$, i.e., if $\boldsymbol{p} = 3/5$

Summary: best response of *player* ***M*** depends on ***p***

$$q \in \text{best}_M(\boldsymbol{p}) = \begin{cases} 1 & \text{if } \boldsymbol{p} < 3/5 \\ [0,1] & \text{if } \boldsymbol{p} = 3/5 \\ 0 & \text{if } \boldsymbol{p} > 3/5 \end{cases}$$



Analyze all intersections of best responses curves:

Pure NE: $(s_1, s_2) = ((0,1), (1,0))$
Pure NE: $(s_1, s_2) = ((1,0), (0,1))$
Mixed NE: $(s_1, s_2) = ((3/5, 2/5), (3/5, 2/5))$

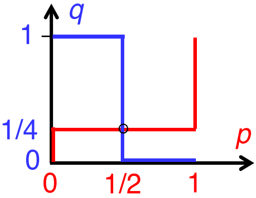
Computing Nash Equilibria (5)

Player 1 is indifferent: $6 \cdot q + 7 \cdot (1-q) = 3 \cdot q + 8 \cdot (1-q) \rightarrow q = 1/4$
Player 2 is indifferent: $4 \cdot p + 2 \cdot (1-p) = 5 \cdot p + 1 \cdot (1-p) \rightarrow p = 1/2$

	L (q)	**R** (1-q)
T (p)	6 \ 4	7 \ 5
B (1-p)	3 \ 2	8 \ 1

$$p \in \text{best}_1(q) = \begin{cases} 0 & \text{if } q < 1/4 \\ [0,1] & \text{if } q = 1/4 \\ 1 & \text{if } q > 1/4 \end{cases}$$

$$q \in \text{best}_2(p) = \begin{cases} 1 & \text{if } p < 1/2 \\ [0,1] & \text{if } p = 1/2 \\ 0 & \text{if } p > 1/2 \end{cases}$$



Analyze all intersections of best response curves:

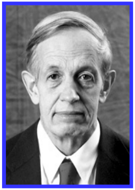
Mixed NE: $(s_1, s_2) = ((1/2, 1/2), (1/4, 3/4))$

Nash's Theorem

Recall that some games do not have pure Nash equilibria. Good news:

Theorem (Nash, 1951) Every (finite) normal-form game has at least one (truly mixed or pure) Nash equilibrium

J.F. Nash. Non-Cooperative Games.
Annals of Mathematics, 54(2):286-295, 1951.



Both the design of algorithms for computing Nash equilibria and the analysis of the computational complexity of this task are important research topics in algorithmic game theory

Regarding the complexity:

- Finding a NE is in NP: if you guess a NE, I can easily verify
- But likely not NP-hard: guaranteed existence would be atypical
- Complete for PPAD (”polynomial parity argument for directed graphs"), which lies ”between" P and NP. Believed to be intractable

Strictly Dominant Strategy

- **Pure and mixed Nash equilibria are examples of solution concepts**: formal models to predict what might be the outcome of a game
- *Have we maybe missed the most obvious solution concept?*

- You should play the action $a_i^* \in A_i$ that gives you a better payoff than any other action $a_i' \in A_i$, whatever the others do (such as playing s_{-i}):

$$u_i(a_i^*, s_{-i}) > u_i(a_i', s_{-i}) \text{ for all } a_i' \in A_i \text{ and for all } s_{-i} \in S_{-i}$$

- Action a_i^* is called a **strictly dominant strategy** for player i
- Profile $a^* \in A$ is called **equilibrium in strictly dominant strategies** if, for every player $i \in N$, action a_i^* is a strictly dominant strategy

Downside: This does not always exist (in fact, it usually does not!)

Example: Strictly Dominant Strategy

*Analysis for player **M***:

*assume **O** cooperates (**C**):*
being ***M***, it is better to defect (**D**) than cooperate (**C**) (0 > -10)

Analysis for player **O** :

*assume **M** cooperates (**C**):*
being **O**, it is better to defect (**D**) than cooperate (**C**) (0 > -10)

O \ M	C	D
C	-10 \ -10	-25 \ 0
D	0 \ -25	-20 \ -20

*Analysis for player **M***:

*assume **O** defects (**D**):*
being ***M***, it is better to defect (**D**) than cooperate (**C**) (-20 > -25)

Analysis for player **O** :

*assume **M** defects (**D**):*
being **O**, it is better to defect (**D**) than cooperate (**C**) (-20 > -25)

- ***D*** is a strictly dominant strategy for *player* **O**
- ***D*** is a strictly dominant strategy for *player* **M**
- (***D***,***D***) is an equilibrium in strictly dominant strategies

Elimination of Dominated Strategies

- Action $a_i \in S_i$ is **strictly dominated** by strategy $s_i^* \in S_i$ if:

$$u_i(s_i^*, a_{-i}) > u_i(a_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}$$

- If we assume i is rational, action a_i can be eliminated

This induces a **solution concept**:

all mixed-strategy profiles of the reduced game that survive ***iterated elimination of strictly dominated strategies*** (*IESDS*)

Example (where the dominating strategies happen to be pure):

	L	R
T	4 \ 4	1 \ 6
B	6 \ 1	2 \ 2

$\Rightarrow$

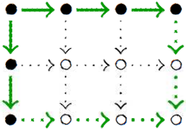
	R
T	1 \ 6
B	2 \ 2

$\Rightarrow$

	R
B	2 \ 2

Order Independence of IESDS

- Suppose $A_i \cap A_j = \varnothing$
- Then we can think of the reduced game G^t after t eliminations simply as the subset of $A_1 \cup \ldots \cup A_n$ that survived



Theorem (Gilboa et al., 1990) Any order of eliminating strictly dominated strategies leads to the same reduced game

I. Gilboa, E. Kalai, and E. Zemel. On the Order of Eliminating Dominated Strategies. *Operations Research Letters*, 9(2):85-89, 1990.

Example (where the dominating strategies happen to be pure):

	L	R
T	4 \ 4	1 \ 6
B	6 \ 1	2 \ 2

⇒

	L	R
B	6 \ 1	2 \ 2

⇒

	R
B	2 \ 2

Iterated Elimination of Strictly Dominated Strategies

R is dominated by **L** and **C**

	L	**C**	**R**
T	3 \ 1	0 \ 1	0 \ **0**
M	1 \ 1	1 \ 1	5 \ **0**
B	0 \ 1	4 \ 1	0 \ **0**

⇒

	L	**C**
T	3 \ 1	0 \ 1
M	**1** \ 1	**1** \ 1
B	0 \ 1	4 \ 1

M is dominated neither by **T** nor **B**, but …
… **M is dominated by the mixed strategy**
that selects **T** nor **B** with equal probability

⇓

	L	**C**
T	3 \ 1	0 \ 1
B	0 \ 1	4 \ 1

⇑

maximally reduced game

Solution concept

the set of all strategy profiles that assign zero probability to playing any action that would be removed through iterated removal of strictly dominated strategies

Remark: *much weaker than Nash equilibrium*

Idea: Recommend Good Strategies

Consider the following variant of the Battle of the Sexes (previously, we had discussed a variant with different payffos)

O \ M	**A**	**B**
A	2 \ 1	0 \ 0
B	0 \ 0	1 \ 2

Nash equilibria:

- pure AA: utility = 2 & 1
- pure BB: utility = 1 & 2
- mixed ((2/3, 1/3), (1/3, 2/3)): expected utility = 2/3 & 2/3
  → *either unfair or low payoffs*

Ask **Oliwia** and **Mateusz** to toss a fair coin and to *pick both* **A** in case of heads and **B** otherwise. They don't have to, but if they do:

$$\textit{expected utility} = 1/2 \cdot 2 + 1/2 \cdot 1 = 3/2 \text{ for } \textbf{Oliwia}$$
$$1/2 \cdot 1 + 1/2 \cdot 2 = 3/2 \text{ for } \textbf{Mateusz}$$

- A random public event occurs
- **Each player *i* receives private signal x_i**
- Modeled as random variables $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \in \boldsymbol{D}_1 \times \ldots \times \boldsymbol{D}_n = \boldsymbol{D}$ with joint probability distribution $\boldsymbol{\pi}$ (so the $\boldsymbol{x}_i$ can be correlated)

- Player *i* uses function $\boldsymbol{\sigma}_i : \boldsymbol{D}_i \to \boldsymbol{A}_i$ to translate signals to actions
- A **correlated equilibrium** is a tuple $\langle \boldsymbol{x}, \boldsymbol{\pi}, \boldsymbol{\sigma} \rangle$ with $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_n)$, such that, for all $i \in N$ and all alternative choices $\sigma_i' : \boldsymbol{D}_i \to \boldsymbol{A}_i$, we get:

$$\sum_{d \in \boldsymbol{D}} \pi(d) \cdot u_1(\sigma_1(d_1), \ldots, \sigma_n(d_n)) \geq$$

$$\geq \sum_{d \in \boldsymbol{D}} \pi(d) \cdot u_1(\sigma_1(d_1), \ldots, \sigma_{i-1}(d_{i-1}), \sigma_i'(d_i), \sigma_{i+1}(d_{i+1}), \ldots, \sigma_n(d_n))$$

Interpretation: Player *i* controls whether to play $\boldsymbol{\sigma}_i$ or σ_i', but has to choose before nature draws $d \in \boldsymbol{D}$ from $\boldsymbol{\pi}$. She knows $\boldsymbol{\sigma}_{-i}$ and $\boldsymbol{\pi}$.

Example: Approaching an Intersection

Oliwia and **Mateusz** reach an intersection in their cars
and each of them has to decide whether to drive on (**D**) or stop (**S**)

O \ M	**D**	**S**
D	-10 \ -10	3 \ 0
S	0 \ 3	-2 \ -2

Nash equilibria:

- pure DS: utility = 3 & 0
- pure SD: utility = 0 & 3
- mixed ((1/3, 2/3), (1/3, 2/3)):
  EU = -4/3 & -4/3
  → *the only fair NE is pretty bad!*

Could instead use this "randomised device" to get CE:

- $D_i = \{red, green\}$ for both players i
- $\pi(red, green) = \pi(green, red) = 1/2$
  $\pi(red, red) = \pi(green, green) = 0$
- recommend to each player to use $\sigma_i : a_i \rightarrow \begin{cases} \textit{drive}\ (\mathbf{D}) \text{ if } x_i = \textit{green} \\ \textit{stop}\ (\mathbf{S}) \text{ if } x_i = \textit{red} \end{cases}$
- *expected utility* = 0·-10 + 1/2·0 + 1/2·3 + 0·-2 = 3/2 for **Oliwia**
  0·-10 + 1/2·3 + 1/2·0 + 0·-2 = 3/2 for **Mateusz**

Correlated Equilibria and Nash Equilibria

Theorem (Aumann, 1974) For **every Nash equilibrium, there exists a correlated equilibrium** inducing the same distribution over outcomes

Corollary Every normal-form game has a correlated equilibrium

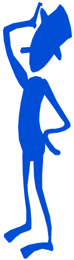
R.J. Aumann. Subjectivity and Correlation in Randomized Strategies. *Journal of Mathematical Economics*, 1(1):67-96, 1974

Proof: Let $\boldsymbol{s} = (\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n)$ be an arbitrary Nash equilibrium

Define a tuple $\boldsymbol{<x, \pi, \sigma>}$ as follows:

- let domain of each $\boldsymbol{x}_i$ be $\boldsymbol{D}_i := \boldsymbol{A}_i$
- fix $\boldsymbol{\pi}$ so that $\boldsymbol{\pi}(\boldsymbol{a}) = \prod_{i \in \boldsymbol{N}} s_i(a_i)$
- let each $\boldsymbol{\sigma}_i : \boldsymbol{A}_i \rightarrow \boldsymbol{A}_i$ be the identity function (***i*** accepts recommendation)

Then $\boldsymbol{<x, \pi, \sigma>}$ is the kind of correlated equilibrium we want



Summary

We have reviewed several **solution concepts** for normal-form games

- equilibrium in dominant strategies: great if it exists
- **Nash's Theorem**: every normal-form game has a (mixed) Nash equilibrium
- correlated equilibrium: accept external advice
- IESDS: iterated elimination of strictly dominated strategies

***Inclusions between sets of strategy profiles** that are solutions for a given game according to certain solution concepts*

$$\underbrace{\text{Dom} \subseteq \text{PureNash}}_{\textit{might be empty}} \subseteq \underbrace{\text{Nash} \subseteq \text{CorrEq} \subseteq \text{IESD}}_{\textit{always non-empty}}$$