MACIEJ MEDYK Z-15197421

*COT6405 Analysis of Algorithms*
*Spring 16*

# Assignment 1

Posted on Jan 26, due on Feb 9

Maximum total 70 points

10 points

1. Find a $\Theta$ notation in terms of $n$ for the number of times the statement x = x + 1 is executed in the segment:

**for** i = 1 to $n$
  **for** j = 1 to i
    **for** k = 1 to i
      x = x + 1

$\boxed{RT = \Theta(n^3)}$

10 points

2. Arrange the following functions in ascending order of growth rate. That is, if function g(n) immediately follows function f(n) in your list, then it should be the case that f(n) = O(g(n)).

$$f_1(n) = \left(\frac{1}{3}\right)^n + \boxed{(\log_3 n)^{100}}$$

$$f_2(n) = \boxed{n^2(\log_2 n)^3}$$

$$f_3(n) = \boxed{\sqrt{8n^3}}$$

$$f_4(n) = \left(\frac{1}{5}\right)^n \boxed{+ n!}$$

$$f_5(n) = 50n - 100 \boxed{+ n^2 \log_2 n}$$

$$f_6(n) = \boxed{4^{\log_2 n} - 1000}$$

FUNCTIONS IN ASCENDING ORDER OF GROWTH

$\boxed{F1, F3, F6, F5, F2, F4}$

20 points

3. Use the formal definitions to show that:
   - $2n^3 - 25n = \Omega(n^2)$. Find some positive values c and $n_0$. $\rightarrow$ $\boxed{C = 1 \quad N_0 = 5}$
   - $5n^2 - 8 = o(n^3)$. Find $n_0$.

$N_0 = \text{MAX}\left(2\sqrt{\frac{2}{5}}, \frac{5}{c}\right)$

*(continued next page)*

20 points

4. Solve the following recurrences:

$T(n) = \Theta(3^n)$

   a. $T(n) = 3T(n-1) + 2$ for $n > 1$, $T(1) = 4$.
      Solve using backward substitution.

   b. $T(n) = 2T(\sqrt[3]{n^2}) + (\log_2 n)^3$. Use the change of variable $m = \log_2 n$

$T(n) = \Theta(\lg^3 n)$

10 points

5. There are n bacteria and 1 virus in a Petri dish. Within the first minute, the virus kills one bacterium and produces another copy of itself, and all of the remaining bacteria reproduce, making 2 viruses and 2(n–1) bacteria. In the second minute, each of the viruses kills a bacterium and produces a new copy of itself, resulting in 4 viruses and 2(2(n–1) –2) = 4n – 8 bacteria; again, the remaining bacteria reproduce. This process continues every minute.
   - Derive formulas to compute the number of viruses and the number of bacteria in the k[th] minute. These formulas are functions of k and n.
   - Will the viruses eventually kill all bacteria? If so, how many steps (minutes) will it take?

FORMULA TO COMPUTE NUMBERS FOR

VIRUSES $\Rightarrow$ $2^k$

BACTERIA $\Rightarrow$ $2^k(n-k)$

THE VIRUS WILL KILL ALL THE BACTERIA

IN N NUMBER OF MINUTES.

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1

for i=1 to n $\Theta(n)$
 for j=1 to i $\Theta(n \cdot n)$
  for k=1 to i $\Theta(n \cdot n \cdot n)$

$\boxed{\Theta(n^3)}$

2

$f_1 = \left(\frac{1}{3}\right)^n + \boxed{(\log_3 n)^{100}}$

$f_2 = \boxed{n^2 (\log_2 n)^3}$

$f_3 = \sqrt{8n^3} = \boxed{\sqrt{8}\, n^{\frac{3}{2}}}$

$f_4 = \left(\frac{1}{5}\right)^n + \boxed{n!}$

$f_5 = 50n - 100 + \boxed{n^2 \log_2 n}$

$f_6 = 4^{\log_2 n} - 1000 = \boxed{n^2} - 1000$

FUNCTIONS IN ASCENDING ORDER

$f_1, f_3, f_6, f_5, f_2, f_4$

3

a

$2n^3 - 25n = \Omega(n^2)$

$0 \leqslant cn^2 \leqslant 2n^3 - 25n$

$cn^2 \geqslant 0$ ☑

$2n^3 - cn^2 - 25n \geqslant 0$ $\boxed{c=1}$

$n^3 - n^2 + n^3 - 25n \geqslant 0$

$n^2(n-1) + n(n^2-25) \geqslant 0$

$\boxed{n \geqslant 1}$ $\boxed{n \geqslant 5}$

FINAL ANSWER

$c = 1 \quad n_0 = 5$

(3)

(B)

$$5n^2 - 8 = O(n^3)$$

$$0 \leq 5n^2 - 8 < cn^3$$

$$5n^2 - 8 \geq 0 \qquad cn^3 - 5n^2 + 8 > 0$$

$$5n^2 \geq 8 \qquad cn^2\left(n - \tfrac{5}{c}\right) + 8 > 0$$

$$n \geq \sqrt{\tfrac{8}{5}} \qquad n - \tfrac{5}{c} \geq 0$$

$$n \geq 2\sqrt{\tfrac{2}{5}} \qquad n \geq \tfrac{5}{c}$$

$$n_0 = \max\left(2\sqrt{\tfrac{2}{5}}, \tfrac{5}{c}\right)$$

(4)

(A)

$$T(n) = 3T(n-1) + 2$$
$$T(n-1) = 3T(n-2) + 2$$
$$T(n-2) = 3T(n-3) + 2$$
$$T(n-3) = 3T(n-4) + 2$$

$$T(n) = 3(3(3(3T(n-4)+2)+2)+2)+2$$

$$T(n) = 3^4 T(n-4) + 3^3 \cdot 2 + 3^2 \cdot 2 + 3^1 \cdot 2 + 3^0 \cdot 2$$

$$T(n) = 3^k T(n-k) + 3^{k-1} \cdot 2 + 3^{k-2} \cdot 2 + 3^{k-3} \cdot 2 + \dots + 3^1 \cdot 2 + 3^0 \cdot 2$$

$$T(n) = 3^k T(n-k) + 2 \cdot \sum_{i=0}^{k-1} 3^i = 3^k T(n-k) + \cancel{2}\, \frac{3^{k-1+1} - 1}{\cancel{2}} =$$

$3^k - 1$

$$T(n) = 3^k \cdot (T(n-k) + 1) - 1$$

let $k = n-1$

$$T(n) = 3^{n-1}\left(T(n-(n-1)+1)\right) - 1 = 3^{n-1}(T(1)+1) - 1$$

$$T(n) = 3^{n-1}(4+1) - 1 = 3^{n-1}(5) - 1$$

$$T(n) = \Theta(3^n)$$

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$$T(n) = 2T(\sqrt[3]{n^2}) + (\lg n)^3 \quad \sqrt[3]{n^2} \to n^{\frac{2}{3}}$$

$m = \lg n$

let $n = 2^m$

$$T(2^m) = 2T(2^{\frac{2m}{3}}) + m^3$$

$$S(m) = T(2^m)$$

$$S(m) = 2S\left(\frac{2m}{3}\right) + m^3$$

$a = 2$

$b = \frac{3}{2}$

$m^3$ vs $m^{\log_{\frac{3}{2}} 2}$

$m^3$ vs $m^{1.71}$ $\quad \varepsilon = 0.29$

$m^3$ vs $m^{1.71+0.29}$

CASE 3 MASTER THEOREM $\boxed{T(2^m) = \Theta(m^3)}$

$\boxed{T(n) = \Theta(\lg^3 n)}$

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$$2\left(\frac{2m}{3}\right)^3 \leq cm^3$$

$$\frac{16\cancel{m^3}}{27} \leq c\cancel{m^3} \Rightarrow \boxed{c \geq \frac{16}{27}}$$

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⑤

| | BACTERIA $2^k(n-k)$ | VIRUS $2^k$ |
|---|---|---|
| $k=0$ | $2^0(n-0) = \boxed{n}$ | $\boxed{1}$ |
| $k=1$ | $\boxed{2(n-1)}$ | $\boxed{2}$ |
| $k=2$ | $4(n-2) = \boxed{4n-8}$ | $\boxed{4}$ |
| $k=3$ | $\boxed{8n-24}$ | $\boxed{8}$ |
| $k=4$ | $\boxed{16n-64}$ | $\boxed{16}$ |

$$\cancel{2^k}(n-k) < \cancel{2^k}1$$

$$n - k < 1$$

$$\boxed{n < k+1} \Rightarrow \boxed{k > n-1}$$

YES THE VIRUS WILL KILL THE BACTERIA IN N NUMBER OF TICKS THEREFORE N MINUTES

EXAMPLE $N = 5$

| K | BACTERIA | | VIRUS |
|---|---|---|---|
| 0 | 5 | > | 1 |
| 1 | 8 | > | 2 |
| 2 | 12 | > | 4 |
| 3 | 16 | > | 8 |
| 4 | 16 | = | 16 |
| 5 | 0 | < | 32 |

BACTERIA DIED IN 5 TICKS