

Locality in Online Algorithms

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Abstract

Online algorithms make decisions based on past inputs. In general, the decision may depend on the entire history of inputs. If many computers run the same online algorithm with the same input stream but are started at different times, they do not necessarily make consistent decisions.

In this work we introduce *time-local online algorithms*. These are online algorithms where the output at a given time only depends on $T = O(1)$ latest inputs. The use of (deterministic) time-local algorithms in a distributed setting automatically leads to globally consistent decisions.

Our key observation is that time-local online algorithms (in which the output at a given time only depends on local inputs in the temporal dimension) are closely connected to *local distributed graph algorithms* (in which the output of a given node only depends on local inputs in the spatial dimension). This makes it possible to interpret prior work on distributed graph algorithms from the perspective of online algorithms.

We describe an *algorithm synthesis method* that one can use to design optimal time-local online algorithms for small values of T . We demonstrate the power of the technique in the context of a variant of the *online file migration problem*, and show that e.g. for two nodes and unit migration costs there exists a 3-competitive time-local algorithm with horizon $T = 4$, while no deterministic online algorithm (in the classic sense) can do better. We also derive upper and lower bounds for a more general version of the problem; we show that there is a 6-competitive deterministic time-local algorithm and a 2.62-competitive randomized time-local algorithm for any migration cost $\alpha \geq 1$.

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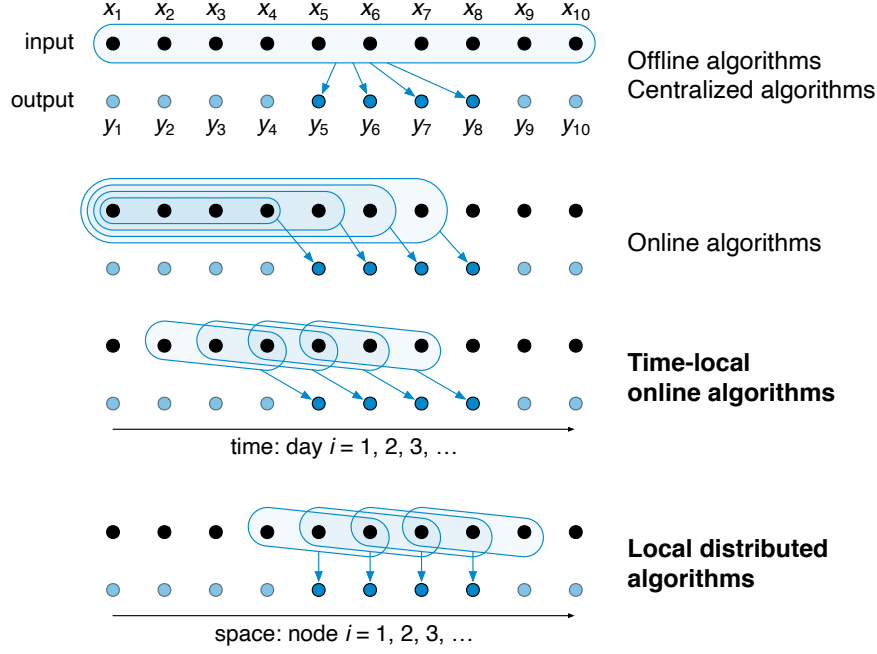
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1 Introduction

Online algorithms [13] make decisions based on past inputs, with the goal of being competitive against an algorithm that sees also future inputs. In this work, we introduce *time-local online algorithms*; these are online algorithms in which the output at any given time is a function of only T latest inputs (instead of the full history of past inputs).

Our main observation is that time-local online algorithms are closely connected to *local distributed graph algorithms*: distributed algorithms make decisions based on the *local*



■ **Figure 1** Local decision-making in time vs. space dimensions.

information in the spatial dimension, while time-local online algorithms make decisions based on the *local information in the temporal dimension*; see Figure 1. We formalize this connection, and show how we can directly use the tools developed to study distributed approximability of graph optimization problems to prove upper and lower bounds on the competitive ratio achieved with time-local online algorithms. Moreover, we show how to use *computational techniques* to synthesize optimal time-local algorithms.

1.1 Motivation

Time-local online algorithms have some attractive properties. Let us give two examples.

Fault-Tolerant Distributed Decision. Consider a setting in which many geographically distributed computers need to make *consistent* decisions. All computers can observe the same input stream, and each day each of them has to announce its own decision.

If all computers are started at the same time, we can take any deterministic online algorithm and let each computer run its own copy of the algorithm. However, this approach does not *tolerate failures*: if a computer crashes and is restarted, the local state of the algorithm is lost, and as the decisions may depend in general on the full history of inputs, it will no longer make consistent decisions with the others.

One solution is to run a *consensus* protocol [38], with which the computers can ensure that they all agree on the same decision (or on the same local state). However, this introduces overhead, and is only applicable if we can assume that not too many computers are faulty; for example, with Byzantine failures consensus is possible only if fewer than $1/3$ of the computers are faulty [38].

Deterministic time-local online algorithms provide automatically the guarantee that all computers will make consistent decisions. The system will tolerate an arbitrary number of failures and ensure that the computers will also recover from transient faults, i.e., it is

self-stabilizing [24, 25]: in T steps since the latest failure, all computers will deterministically make consistent decisions, without any communication.

Random Access to the Decision History. A second benefit of time-local online algorithms is that they make it possible to efficiently access any past decision, with zero additional storage beyond the storage of the input stream. To look up a decision at any given time i , it is enough to look up the inputs at the T previous time points and apply the deterministic time-local algorithm. With classic online algorithms one would have to either store the decision, store the local state, or re-run the entire algorithm up to point i .

1.2 Online Problems and Algorithms

Online Problems as Request-Answer Games. Online problems are often formalized as request-answer games [13, Ch. 7] that consist of an input set X (requests), an output set Y (answers), and an infinite sequence $(f_n)_{n \geq 1}$ of cost functions

$$f_n: X^n \times Y^n \rightarrow \mathbb{R} \cup \{\infty\} \text{ for each } n \in \mathbb{N}.$$

The *optimal offline cost* of an input sequence $\mathbf{x} \in X^n$ is $\text{OPT}(\mathbf{x}) = \min\{f_n(\mathbf{x}, \mathbf{y}) : \mathbf{y} \in Y^n\}$.

Classic Online Algorithms. An online algorithm A in the classic sense, i.e., an algorithm that has access to all past inputs, can be defined as a sequence $(A_i)_{i \geq 1}$ of functions $A_i: X^{i-1} \rightarrow Y$. The output $\mathbf{y} = A(\mathbf{x})$ of the algorithm on input $\mathbf{x} \in X^n$ is given by

$$y_i = A_i(x_1, \dots, x_{i-1}) \text{ for each } 1 \leq i \leq n.$$

The quality of an online algorithm is measured by comparing the cost of its output against the optimal offline cost. An algorithm is said to be c -competitive (have a *competitive ratio* c) if on any input sequence $\mathbf{x} \in X^n$ its output $\mathbf{y} = A(\mathbf{x})$ satisfies $f_n(\mathbf{x}, \mathbf{y}) \leq c \cdot \text{OPT}(\mathbf{x}) + d$ for a fixed constant d . We say that an algorithm is *strictly c -competitive* if additionally $d = 0$.

Time-Local Algorithms. We now start with a simplified definition of (deterministic) time-local algorithms; the general definitions are given later. Let $T \in \mathbb{N}$ be a constant. A time-local algorithm that has access to T latest inputs is given by a sequence of maps $(A_i)_{i \geq 1}$ of the form $A_i: (X \cup \{\perp\})^T \rightarrow Y$, where $\perp \notin X$. The output of the algorithm is given by

$$y_i = A_i(x_{i-T}, \dots, x_{i-1}) \text{ for each } 1 \leq i \leq n,$$

where we let $x_i = \perp$ be placeholder values for $i < 1$. We say that the algorithm is *unclocked* if all maps A_i are equal and otherwise it is *clocked*. That is, in the latter case the i th decision y_i made by the algorithm may depend on the current time step i . On the other hand, unclocked algorithms are given by a single map $A: X^T \rightarrow Y$ and the output does not depend on the number of steps taken so far on long inputs.

The Online File Migration. As our running example, we consider a simple variant of the *online file migration problem*: we are given a network, modeled as an undirected graph with two nodes, and an indivisible shared resource, a *file*, initially stored at one of the nodes. Requests to access the file arrive from nodes of the network over time, and the serving cost of a request is 0 if the file is collocated with the request, and 1 otherwise. After serving a request, we may decide to *migrate* the file to a different node of the network, paying α

units of migration cost for some parameter $\alpha \geq 0$ (usually $\alpha \geq 1$). Our goal is to minimize the total cost of accesses and migrations. In the classic setting, for $\alpha \geq 1$, there exists a 3-competitive deterministic algorithm for the problem [14], and no deterministic algorithm can do better [11]. However, randomized algorithms can beat this bound: for $\alpha \geq 1$, there exists a $(1 + \phi)$ -competitive randomized algorithm against the oblivious adversary [44], where $\phi \approx 1.62$ is the golden ratio.

1.3 Contributions

In this work, we initiate the study of temporal locality in online algorithms. We give a series of results and techniques that illustrate different aspects of time-local algorithms. All technical details and further results are provided in the subsequent sections.

Introducing Time-Local Online Algorithms (Section 3). We formalize the new notion of temporal locality in online algorithms. We investigate the power of time-local online algorithms by focusing on two basic models, unlocked and clocked. We formally introduce the deterministic variants of these models in Section 3, and randomized variants of the models in Section 6. We show that clocked algorithms are often strictly more powerful than unlocked algorithms, but more surprisingly, unlocked time-local algorithms can sometimes be as competitive as classic (non-local) online algorithms.

Transferring Results from Distributed Computing (Section 4). We identify connections between different models studied in *distributed graph algorithms* and different variants of *time-local online algorithms*. We exploit this connection, and show how to lift many results from theory of distributed computing to establish *impossibility results* for time-local algorithms essentially for free. For example, it turns out that if an online problem has a component that is equivalent to a symmetry-breaking task or to a nontrivial distributed optimization problem, such as in online load balancing (Example 3.2), it follows that deterministic unlocked time-local algorithms cannot perform well, but randomization helps.

On the other hand, this suggests that problems where symmetry-breaking is not a critical component, such as in the online file migration problem (Example 3.1), deterministic unlocked time-local algorithms can perform well. This heuristic argument is corroborated by our case study on online file migration: the problem admits competitive time-local algorithms.

The Power of Clocked Algorithms (Section 5). Second, we establish that deterministic *clocked* time-local algorithms are powerful. For the family of *bounded monotone minimization games*, we can turn any classic deterministic online algorithm into a deterministic clocked time-local algorithm with only a small increase in the competitive ratio.

► **Theorem 1.1.** *Let \mathcal{F} be a bounded monotone minimization game. If there exists an online algorithm A with competitive ratio c for \mathcal{F} , then for any constant $\varepsilon > 0$ there exists some constant T and a clocked T -time-local algorithm B with competitive ratio $(1 + \varepsilon)c$ for \mathcal{F} .*

We also show that there are problems outside this family that do not admit competitive clocked algorithms: for the classic *online caching problem* [42], no deterministic clocked time-local algorithm can achieve a finite competitive ratio, but competitive classic online algorithms exist.

Randomization in Time-Local Algorithms (Section 6). In the classic online setting, randomized algorithms have two equivalent characterizations:

1. at the start, randomly sample a deterministic algorithm from the set of all algorithms, or
2. in each step, make a random decision based on e.g. a sequence of random coin flips.

The former corresponds to *mixed strategies*, where we sample all random bits used by the algorithm before seeing any of the input, whereas the latter corresponds to *behavioral strategies*, where the algorithm generates random bits along the way as it needs them. We observe that these two types of randomized algorithms differ for time-local algorithms, we illustrate differences between them by showing both types of algorithms, and give a lower bound that holds for both types (see below).

Automated Synthesis of Time-Local Algorithms (Section 7). Again by leveraging the connection to local graph algorithms, we describe and implement a novel *algorithm synthesis method* that allows us to automate the design of *optimal* unlocked time-local algorithms (see Section 7). Specifically, the synthesis task can be formulated as a certain weighted optimization problem in dual de Bruijn graphs.

We formalize and implement a technique for the automated design of time-local algorithms for *local optimization problems* (defined in this work). This technique allows us to automatically obtain *tight* upper and lower bounds for unlocked time-local algorithms. First, we prove the following result.

► **Theorem 1.2** (informal). *Let Π be a local optimization problem and A an unlocked time-local algorithm with horizon T . Then there is a finite, dual-weighted graph $G = G(\Pi, A)$ such that the competitive ratio of A is determined by the cycle with the heaviest weight ratio in G .*

Recall that each unlocked time-local algorithm with horizon T is given by some map $A: X^T \rightarrow Y$. For local optimization problems with finite input set X and output set Y , we can iterate through all of the $|Y|^{|X|^T}$ maps to find an optimal algorithm for any given T .

We illustrate the usefulness of this technique by synthesizing several optimal deterministic time-local algorithms for the online file migration problem. For example, we show that for unit costs ($\alpha = 1$) there exists a 3-competitive time-local algorithm with $T = 4$, which is the best competitive ratio achieved by *any* deterministic online algorithm [11]. Moreover, we also synthesize *randomized* time-local algorithms with strictly better (expected) competitive ratio than the optimal deterministic algorithms under the oblivious adversary. See Figure 2 for a summary.

Online File Migration: A Case Study (Section 8). Finally, we investigate analytically how the length T of the visible input horizon influences the quality of solutions given by time-local online algorithms. For the online file migration problem, we show the following:

► **Theorem 1.3** (informal; see Theorem 8.1). *For any $\alpha \geq T$, there is no randomized clocked time-local algorithm that achieves a competitive ratio better than $2\alpha/T$.*

► **Theorem 1.4** (informal; see Theorem 8.2). *For any $\alpha \geq 1$, there is a randomized clocked time-local algorithm that is $(1 + \phi)$ -competitive, where $\phi \approx 1.62$ is the golden ratio, for some $T = \Theta(\alpha)$. In general, it achieves a competitive ratio of $\max\{2 + 2\alpha/T, 1 + (T + 1)/(2\alpha)\}$.*

► **Theorem 1.5** (informal; see Corollary 8.13). *For any $\alpha \geq 1$, there is a deterministic unlocked time-local algorithm that is 6-competitive for $T \geq 6\alpha$. Moreover, for $1 \leq T < 6\alpha$ the algorithm is $(4 + 12\alpha/T)$ -competitive.*

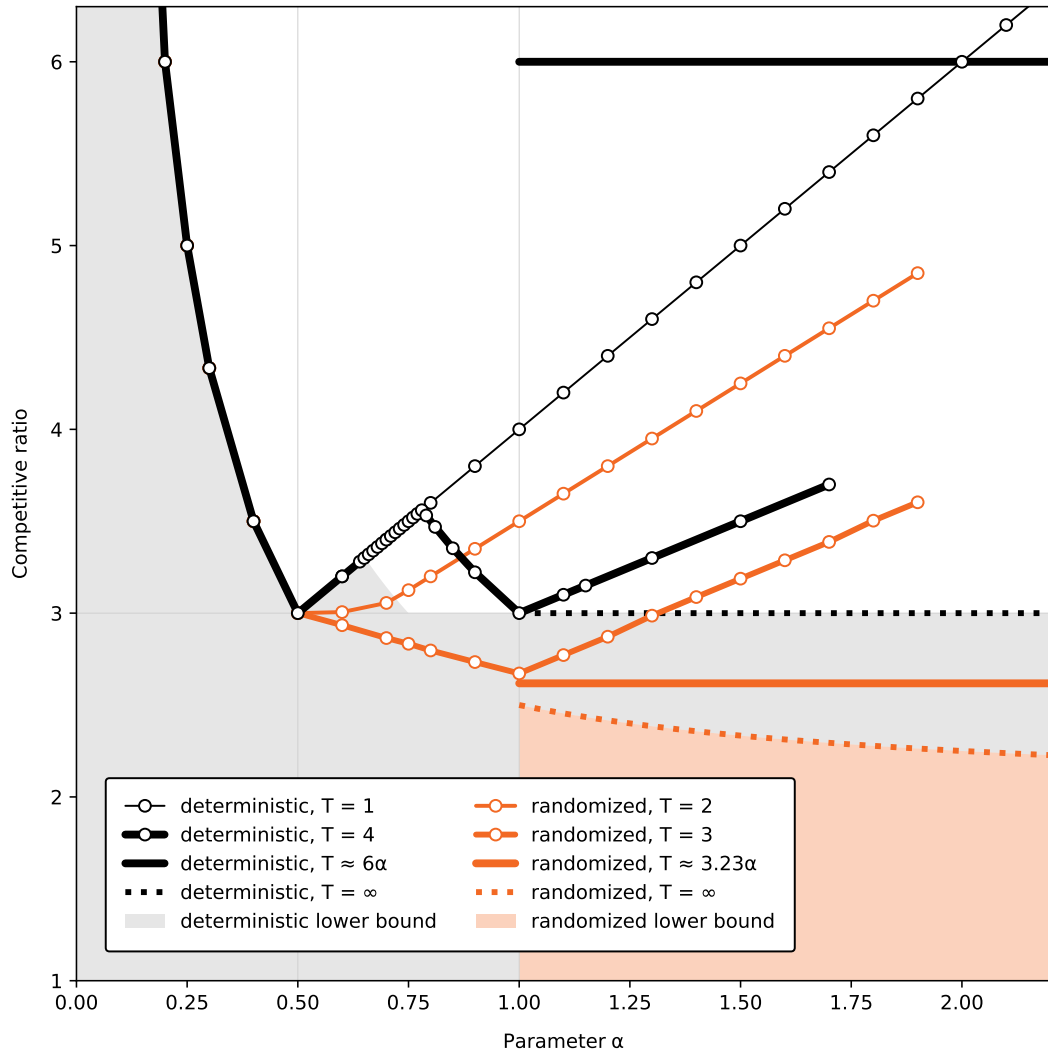


Figure 2 Upper and lower bounds for the online file migration problem. The visualization includes the upper bounds from Table 2 for small values of T , as well as the upper bounds from Corollaries 8.3 and 8.13, and the lower bound from Corollary 8.16.

The above deterministic algorithm uses a simple sliding window rule. We say that the last T inputs contain a b -window for $b \in \{0, 1\}$ if there is a subsequence of requests in which the number of b -requests is at least twice the number of $(1 - b)$ -requests. The algorithm is:

1. Output $b \in \{0, 1\}$ if the most recent window in the last T inputs is a b -window.
2. Otherwise, output 0.

However, despite the deceptive simplicity of the algorithm, its analysis is surprisingly challenging and illuminates aspects involved in the competitive analysis of time-local algorithms.

The technical results in the case study illustrate how the style of arguments and techniques used in the analysis of time-local online algorithms differs from the classic online algorithms setting. The analytical and synthesized algorithms are summarized and compared against classic online algorithms in Figure 2.

2 Related Work

Restricted Models of Online Algorithms. To our best knowledge, *temporal locality* of online algorithms has not been systematically studied. However, other restricted forms of online algorithms have received some attention. For example, Chrobak [19] introduced the notion of *memoryless* online algorithms: the answer to the current request can only depend on the current configuration instead of being an arbitrary function of the entire past history as in general online algorithms. In particular, memoryless online algorithms can be synthesized using a fixed point approach. However, memoryless online algorithms differ from time-local algorithms, as memoryless algorithms have access to the previous configuration of the algorithm instead of last T inputs.

Ben-David et al. [8] investigated local online problems within the request-answer game framework of online algorithms. However, their notion of locality applies to the *cost functions* defining the online problem instead of algorithms solving them: the cost of a solution cannot depend too much on past inputs. In this work, to avoid confusion, we call these games *bounded monotone minimization games*, and show that if a problem in this class admits a competitive online algorithm, then any such algorithm can be converted into a competitive clocked time-local algorithm. Moreover, we introduce an alternative characterization of local games, *local optimization games*, which are different from bounded monotone minimization games, but they have a natural interpretation as distributed optimization problems on paths. We give a general synthesis method to automate the design of optimal time-local algorithms for local optimization problems.

Synthesis of Online Algorithms. As mentioned, already the early work on online algorithms considered synthesis in the context of memoryless algorithms [19]. Computer-aided design techniques have also been used to design optimal online algorithms for specific problems. Coppersmith et al. [22] studied the design and analysis of randomized online algorithms for k -server problems, metrical task systems and a class of graph games; they show that algorithm synthesis is equivalent to the synthesis of random walks on graphs. For a variant of the *online knapsack* problem [31], Horiyama, Iwama and Kawahara [29] obtained an optimal algorithm by using a problem-specific finite automaton and solving a set of inequalities for each of its states. More recently, the synthesis of optimal algorithms for preemptive variants of *online scheduling* [7] was reduced to a two-player graph game [18, 37].

Synthesis of Local Algorithms. Synthesis of distributed graph algorithms has a long history, mostly focusing on so-called locally checkable labeling (LCL) problems in the LOCAL

model of distributed computing [4, 5, 15, 17, 28, 40]. In their foundational work, Naor and Stockmeyer [36] showed that it is undecidable to determine whether an LCL problem admits a local algorithm in general, but it is decidable for unlabeled directed paths and cycles. Balliu et al. [4] showed that determining the distributed round complexity of LCL problems on paths and cycles with inputs is decidable, but PSPACE-hard. Moreover, when restricted to LCL problems with binary inputs, there is a simple synthesis procedure [5]. Recently, Chang et al. [17] showed that synthesis in unlabeled paths, cycles and rooted trees can be done efficiently.

Beyond decidability results, synthesis has also been applied in practice to obtain optimal local algorithms. Rybicki and Suomela [40] showed how to synthesize optimal distributed coloring algorithms on directed paths and cycles. Brandt et al. [15] gave a technique for synthesizing efficient distributed algorithms in 2-dimensional toroidal grids, but showed that in general determining the complexity of an LCL problem is undecidable in grids. Similarly to our work, Hirvonen et al. [28] considered the synthesis of *optimization problems*. They gave a method for synthesizing randomized algorithms for the max cut problem in triangle-free regular graphs.

Our work identifies the connection between temporal locality online algorithms and spatial locality in distributed algorithms. As we show in this work, time-local online algorithms can be seen as local graph algorithms on directed paths. However, formally the computational power of the LOCAL model resides between unclocked and clocked time-local models we study in this work. Thus, decidability results and synthesis techniques do not directly carry over to the time-local online algorithms setting.

3 Models and Formalism

3.1 Local Optimization Problems

We now introduce the family of local online problems that we study in this work. The definition is somewhat technical, but the basic idea is simple: at each time step i , the cost (or utility) of our decision y_i is defined to be some function of the current input x_i and up to $r = O(1)$ previous inputs and outputs.

This formalism has several attractive features. First, it is flexible enough to e.g. define online problems in which we reward correct decisions (e.g. whenever we predict correctly $y_i = x_i$, we get some profit), we penalize costly moves (e.g. whenever we change our mind and switch to a new output $y_i \neq y_{i-1}$, we get some penalty), and we prevent invalid choices (e.g. by defining infinite penalties for decisions that are not compatible with the previous inputs and/or previous decisions). Second, this formalism can capture problems that are relevant in distributed graph algorithms (e.g. x_i represents the weight of node i along a path, y_i indicates which nodes are selected, and we pay x_i whenever we select a node). Finally, this family of problems is amenable to automated algorithm synthesis, as we will later see.

We will now present the formal definition, and then give several examples of different kinds of problems, both from the areas of online and distributed graph algorithms.

Formalism. A *local optimization problem* is a tuple $\Pi = (X, Y, r, v, \text{aggr}, \text{obj})$, where

- X is the *set of inputs*,
- Y is the *set of outputs*,
- $r \in \mathbb{N}$ is the *horizon*,
- $v: X^{r+1} \times Y^{r+1} \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ is the *local cost function*,
- $\text{aggr} \in \{\text{sum}, \text{min}, \text{max}\}$ is the *aggregation function*,

■ $\text{obj} \in \{\min, \max\}$ is the *objective*.

The input for the problem Π is a sequence $\mathbf{x} = (x_1, x_2, \dots, x_n) \in X^n$ and a solution is a sequence $\mathbf{y} = (y_1, y_2, \dots, y_n) \in Y^n$. For convenience, we will use placeholder values $x_i = y_i = \perp$ for $i < 1$ and $i > n$. With each index, we associate a *value* $u_i(\mathbf{x}, \mathbf{y})$ defined as

$$u_i(\mathbf{x}, \mathbf{y}) = v(x_{i-r}, \dots, x_i, y_{i-r}, \dots, y_i).$$

Finally, we apply the aggregation function aggr to values u_i to determine the value $u(\mathbf{x}, \mathbf{y})$ of the solution. That is, if the aggregation function is sum, the cost function is given by

$$f_n(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n u_i(\mathbf{x}, \mathbf{y}).$$

For example, if the objective is min, the task in Π is to find a solution \mathbf{y} that minimizes $u(\mathbf{x}, \mathbf{y})$ for a given input \mathbf{x} , and so on. Note that X, Y and $(f_n)_{n \geq 1}$ define a request-answer game.

Shorthand Notation. In general, the local cost function v is a function with $2(r+1)$ arguments. However, it is often more convenient to represent v as a function that takes one matrix with two rows and $r+1$ columns and use “.” to denote irrelevant parameters, e.g.

$$v\left(\begin{smallmatrix} \cdot & \cdot & c \\ d & e & \cdot \end{smallmatrix}\right) = \gamma$$

is equivalent to saying that $v(a, b, c, d, e, f) = \gamma$ for all $a, b \in X$ and $f \in Y$.

Examples of Online Problems. Let us first see how to encode typical online problems in our formalism. We start with a highly simplified version of the *online file migration problem*, a.k.a. *online page migration* [11]. This will be the running example that we use throughout this work.

► **Example 3.1** (online file migration). We are given a network consisting of two nodes, and an indivisible shared resource, a *file*, initially stored at one of the nodes. Requests to access the file arrive from nodes of the network over time, and the serving cost of a request is the distance from the requesting node to the file, i.e., 0 if the file is collocated with the request, and 1 otherwise. After serving a request, we may decide to *migrate* the file to a different node of the network, paying α units of migration cost for some parameter $\alpha \geq 0$.

Let us express the online file migration problem introduced earlier using the above formalism. The problem is modeled so that input $x_i \in X = \{0, 1\}$ represents access to the file at time i from the node x_i of the network, and output $y_i \in Y = \{0, 1\}$ represents the location of the file at time i . We choose the horizon $r = 1$, aggregation function “sum”, and objective “min”, and define the local cost function as

$$\begin{aligned} v\left(\begin{smallmatrix} \cdot & 0 \\ 0 & 0 \end{smallmatrix}\right) &= 0, & v\left(\begin{smallmatrix} \cdot & 0 \\ 1 & 1 \end{smallmatrix}\right) &= 1, & v\left(\begin{smallmatrix} \cdot & 0 \\ 1 & 0 \end{smallmatrix}\right) &= \alpha, & v\left(\begin{smallmatrix} \cdot & 0 \\ 0 & 1 \end{smallmatrix}\right) &= 1 + \alpha, \\ v\left(\begin{smallmatrix} \cdot & 1 \\ 0 & 1 \end{smallmatrix}\right) &= 0, & v\left(\begin{smallmatrix} \cdot & 1 \\ 1 & 0 \end{smallmatrix}\right) &= 1, & v\left(\begin{smallmatrix} \cdot & 1 \\ 0 & 0 \end{smallmatrix}\right) &= \alpha, & v\left(\begin{smallmatrix} \cdot & 1 \\ 1 & 1 \end{smallmatrix}\right) &= 1 + \alpha. \end{aligned}$$

Recall that $\alpha > 0$ is the cost of migrating the file. Intuitively, the four columns represent local access, remote access, local and remote access after reconfiguration.

Let us now look at a problem of a different flavor, a variant of load balancing [3].

► **Example 3.2** (online load balancing). Each day i a job arrives; the job has a duration $x_i \in X = \{1, 2, \dots, \ell\}$. We need to choose a machine $y_i \in Y$ that will process the job. If, e.g., $x_i = 3$, then machine y_i will process job i during days $i, i + 1$, and $i + 2$. The *load* of a machine is the number of concurrent jobs that it is processing at a given day, and our task is to minimize the maximum load of any machine at any point of time.

In this case we can choose the horizon $r = \ell - 1$, aggregation function “max”, and objective “min”, and define the local cost function as follows:

$$v(x_{i-r}, \dots, x_i, y_{i-r}, \dots, y_i) = \max_{y \in Y} |\{j \in X : x_{i-j+1} \geq j \text{ and } y_{i-j+1} = y\}|.$$

That is, we count the number of jobs that were assigned to each machine $y \in Y$ on days $i - r, \dots, i$ and that are long enough so that they are still being processed during day i . For example, if $X = Y = \{1, 2\}$, this is equivalent to

$$v\left(\begin{smallmatrix} 1 \\ \vdots \end{smallmatrix}\right) = 1, \quad v\left(\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right) = 2, \quad v\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right) = 2, \quad v\left(\begin{smallmatrix} 2 \\ 1 \\ 2 \end{smallmatrix}\right) = 1, \quad v\left(\begin{smallmatrix} 2 \\ 2 \\ 1 \end{smallmatrix}\right) = 1.$$

Encoding Online Problems: Answers vs. Configurations. The same online problem can often be formalized and encoded in several different ways. In general, the output of an algorithm in each step can naturally be defined either as

1. the new *configuration* of the system (e.g. “machine 1 is executing two jobs with remaining durations a and b , and machine 2 is executing one job with remaining duration c ”), or
2. the *action* chosen (a transition from the previous configuration to the next one e.g. “enqueue the incoming job on machine 2”).

In the classic online setting, where algorithms have access to the entire past input sequence, this distinction tends to matter little (actions can be computed from the full sequence of configurations, and vice versa); one simply uses the most convenient formulation.

However, in restricted models of computation, such as time-local online algorithms, the situation is different. The choice of encoding—e.g. whether the algorithm outputs the actions it makes or the current configuration—may dramatically impact the solvability of a problem. We discuss this question in more detail in Section 8.4. Unless otherwise specified, we assume that algorithms for online problems output the current configuration in each step.

Examples of Graph Problems on Paths. In this paper, we uncover and exploit connections between time-local online algorithms and distributed graph algorithms on paths. We have seen that the formalism that we use is expressive enough to capture typical online problems; we now express some classic graph optimization problems studied in distributed computing. Let us now see how to express some classic graph optimization problems that have been studied in the theory of distributed computing.

We interpret each index i as a node in a path, where nodes i and $i + 1$ are connected by an edge. Input x_i is the *weight* of node i , and output y_i encodes a subset of nodes $S \subseteq \{1, 2, \dots, n\}$, with the interpretation that $i \in S$ whenever $y_i = 1$. Hence $X = \mathbb{R}_{\geq 0}$ and $Y = \{0, 1\}$.

► **Example 3.3** (maximum-weight independent set). We can capture a problem equivalent to the classic maximum-weight independent set as follows: we choose the horizon $r = 1$, aggregation function “sum”, and objective “max”, and define the local cost function as follows:

$$v\left(\begin{smallmatrix} \cdot \\ \cdot \end{smallmatrix}\right) = 0, \quad v\left(\begin{smallmatrix} \cdot \\ 0 \end{smallmatrix}\right) = \alpha, \quad v\left(\begin{smallmatrix} \cdot \\ 1 \end{smallmatrix}\right) = -\infty.$$

That is, a node of weight α is worth α units if we select it. The last case ensures that the solution represents a valid independent set (no two nodes selected next to each other).

► **Example 3.4** (minimum-weight dominating set). To represent minimum-weight dominating sets, we choose $r = 2$, $\text{aggr} = \text{sum}$, and $\text{obj} = \text{min}$. We define the local cost function as follows:

$$v\left(\begin{smallmatrix} \cdot & \alpha & \cdot \\ & 1 & \end{smallmatrix}\right) = \alpha, \quad v\left(\begin{smallmatrix} \cdot & \cdot & \cdot \\ & 0 & 1 \end{smallmatrix}\right) = 0, \quad v\left(\begin{smallmatrix} \cdot & 1 & \cdot \\ & 1 & \cdot \end{smallmatrix}\right) = 0, \quad v\left(\begin{smallmatrix} \cdot & \cdot & \cdot \\ 0 & 0 & 0 \end{smallmatrix}\right) = +\infty.$$

Here if we select a node of cost α , we pay α units. Nodes that are not selected but that are correctly dominated by a neighbor are free. We ensure correct domination by assigning an infinite cost to unhappy nodes.

Technically, when we select a node i , we will pay for it at time $i + 1$, not at time i , but this is fine, as we will in any case sum over all nodes (and ignore constantly many nodes near the boundaries).

3.2 Local Algorithms in Time and Space

So far we have introduced the formalism we use to define computational problems. Let us now define the model of computing. For convenience, we will extend the definition of inputs to include a placeholder value \perp and let $x_i = \perp$ for $i < 1$ and for $i > n$. The key models of computing that we study are all captured by the following definition:

► **Definition 3.5** (local algorithm). An $[a, b]$ -local algorithm is a sequence $(A_i)_{i \geq 1}$ of functions of the form $A_i: X^{a+1+b} \rightarrow Y$. The output \mathbf{y} of an algorithm A for input $\mathbf{x} \in X^n$, in notation $\mathbf{y} = A(\mathbf{x})$, is defined as follows:

$$y_i = A_i(x_{i-a}, \dots, x_{i+b}) \text{ for each } i = 1, \dots, n.$$

If $A_i = A_j$ for all $i, j \in \mathbb{N}$, then the algorithm A is *unclocked*. Otherwise, the algorithm is *clocked*.

Note that *unclocked* time-local algorithms as defined above are unaware of the current time step i ; they make the same deterministic decision every time for the same (local) input pattern. We can quantify the cost of not being aware of the current time step, by comparing unclocked algorithms against the stronger model of *clocked* algorithms, which can make different decisions based on the current time step i .

Classic Models of Online and Distributed Algorithms. Using the notion of *unclocked* time-local algorithms, we can characterize algorithms studied in prior work as follows; see also Figure 1. In what follows, T is a constant independent of the length n of input:

- **$[\infty, \infty]$ -local:** These are algorithms with access to the full input. In the context of online algorithms, these are usually known as *offline algorithms*, while in the context of distributed computing, these are usually known as *centralized algorithms*.
- **$[\infty, -1]$ -local:** These are *online algorithms* in the usual sense. The output for a time step i is chosen based on inputs for all previous time steps up to the time step $i - 1$. This is an appropriate definition for the online file migration problem (Example 3.1): we need to decide where to move the file before we see the next request.
- **$[\infty, 0]$ -local:** These are online algorithms with one unit of *lookahead*. The output for a time step i is chosen based on inputs up to the time step i . This is an appropriate definition for the online load balancing problem (Example 3.2): we can choose the machine once we see the parameters of the new job.

■ **Table 1** Correspondence between time-local online algorithms and distributed graph algorithms.

	Time-local online algorithms	Local distributed graph algorithms on directed paths
Weakest	unclocked $[T, T]$ -local	T rounds in the PN model [1, 2, 45]
	N/A	T rounds in the LOCAL model [33, 39]
	clocked $[T, T]$ -local	T rounds in the numbered LOCAL model
Strongest	N/A	T rounds in the supported LOCAL model [26, 41]

- **$[T, T]$ -local:** These can be interpreted as T -round *distributed algorithms* in directed paths in the port-numbering model. In the port-numbering model, in T synchronous communication rounds, each node can gather full information about the inputs of all nodes within distance T from it, and nothing else. This is a setting in which it is interesting to study graph problems such as the maximum-weight independent set (Example 3.3) and the minimum-weight dominating set (Example 3.4).

New Models: Time-Local Online Algorithms. Now we are ready to introduce the main objects of study for the present work:

- **unclocked $[T, -1]$ -local:** These are *time-local algorithms* with horizon T , i.e., online algorithms that make decisions based on only T latest inputs.
- **unclocked $[T, 0]$ -local:** These are time-local algorithms with one unit of *lookahead*.
- **clocked $[T, T]$ -local:** As we will see later, these algorithms are equivalent to T -round distributed algorithms a restricted variant of the *supported* LOCAL model [26, 41].
- **clocked $[T, -1]$ -local:** These are *clocked time-local algorithms* that make decisions based on only T latest inputs, but the decision may depend on the current time step i .

We note that there is nothing fundamental about the constants -1 and 0 that appear above; they are merely constants that usually make most sense in applications. Hence one can perfectly well study, say, $[10, 7]$ -local algorithms, and interpret them either as distributed algorithms that make decisions based on an asymmetric local neighborhood, or interpret them as time-local algorithms that can postpone decisions and choose y_i only after seeing inputs up to $i + 7$.

4 Time-Local Online Algorithms vs. Local Graph Algorithms

In this section, we discuss the connection between time-local online algorithms and local distributed graph algorithms on paths. Although the former deal with locality in the *temporal* dimension and the latter in *spatial* dimension, we will see that these two worlds are closely connected. In particular, we show how to transfer results from distributed computing to the time-local online setting.

We focus on two standard models with very different computational power: the anonymous port-numbering model (a weak model) and the supported LOCAL model (a strong model). In the deterministic setting, the correspondence between these models and time-local online algorithms is summarized in Table 1.

4.1 Distributed Graph Algorithms

Let $G = (V, E)$ be a graph that represents the communication topology of a distributed system consisting of n nodes $V = \{v_1, \dots, v_n\}$. Each node $v_i \in V$ corresponds to a processor and the edges denote direct communication links between processors, i.e., any pair of nodes connected by an edge can directly communicate with each other. In this work, G will always be a path of length n with the set of edges given by $E = \{\{v_i, v_{i+1}\} : 1 \leq i < n\}$.

Synchronous distributed computation. We start with the basic synchronous message-passing model of computation. Let X and Y be the set of input and output labels, respectively. The input is the vector $\mathbf{x} = (x_1, \dots, x_n) \in X^n$, where x_i is the local input of node v_i . Initially, each node v_i only knows its local input $x_i \in X$.

The computation proceeds in synchronous rounds, where in each round $t = 1, 2, \dots$, all nodes in parallel perform the following in lock-step:

1. send messages to their neighbors,
2. receive messages from their neighbors, and
3. update their local state.

An algorithm has running time T if at the end of round T , each node v_i halts and declares its own local output value y_i . The output of the algorithm is the vector $\mathbf{y} = (y_1, \dots, y_n) \in Y^n$.

Note that—since there is no restriction on message sizes—every T -round algorithm can be represented as a simple full-information algorithm: In every round, each node broadcasts all the information it currently has, i.e., its own local input and inputs it has received from others, to all of its neighbors. After executing this algorithm for T rounds, this algorithm has obtained all the information *any* T -round algorithm can. Thus, every T -round algorithm can be represented as map from radius- T neighborhoods to output values.

4.2 Distributed Algorithms vs. Time-Local Online Algorithms

The distributed computing literature has extensively studied the computational power of different variants of the above basic model of graph algorithms. The variants are obtained by considering different types of *symmetry-breaking information*: in addition to the problem specific local input $x_i \in X$, each node v_i also receives some input z_i that encodes additional model-dependent symmetry-breaking information.

We will now discuss four such models in increasing order of computational power. The correspondence between these models and time-local online algorithms is summarized by Table 1.

The port-numbering model PN on directed paths. In the PN model [1, 2, 45] all nodes are anonymous, but the edges of G are consistently oriented from v_i towards v_{i+1} for all $1 \leq i < n$. The nodes know their degree and can distinguish between the incoming and outgoing edges. The orientation only serves as symmetry-breaking information; the communication links are bidirectional.

Any deterministic algorithm in this model corresponds to a map $A: X^{2T+1} \rightarrow Y$ such that the output of node v_i for $1 \leq i \leq n$ is

$$y_i = A(x_{i-T}, \dots, x_i, \dots, x_{i+T}),$$

where we let $x_j = \perp$ for any $j < 0$ or $j > n$ (the \perp values are used in the scenarios where nodes near the endpoints of the path observe these endpoints). Note that this is exactly the definition of an unlocked $[T, T]$ -local algorithms (Definition 3.5).

The LOCAL model on directed paths. In the LOCAL model [33, 39] each node receives the same information as in the port-numbering model PN, but in addition, each node v_i is also given a unique identifier $ID(v_i)$ from the set $\{1, \dots, n^c\}$ for some constant $c \geq 1$; the nodes do not know n . Lower bounds for this model also hold in the weaker PN model.

The numbered LOCAL model on directed paths. The numbered LOCAL model further assumes that the unique identifiers have a specific, ordered structure: node v_i is given the identifier $ID(v_i) = i$ as local input in addition to the problem specific input $x_i \in X$. That is, each node knows its distance from the start of the path. Any deterministic algorithm in this model corresponds to a map $A: X^{2T+1} \times \mathbb{N} \rightarrow Y$ such that the output of node v_i for $1 \leq i \leq n$ is

$$y_i = A(x_{i-T}, \dots, x_i, \dots, x_{i+T}, i) = A_i(x_{i-T}, \dots, x_i, \dots, x_{i+T}).$$

Observe that this coincides with *clocked* $[T, T]$ -local algorithms (Definition 3.5).

This model is *not* something that to our knowledge has been studied in the distributed computing literature; the name “numbered LOCAL model” is introduced here. However, it is very close to another model, so-called supported LOCAL model, which has been studied in the literature.

The supported LOCAL model on directed paths. The supported LOCAL model [26, 41] is the same as the numbered LOCAL model, but each node is also given the length n of the path as local input. This would correspond to *clocked* $[T, T]$ -local algorithms that also know the length of the input in advance but do not see the full input. This is the most powerful model, and hence, all impossibility results in this model also hold for all the previous models.

4.3 Transferring Results From Distributed Computing

4.3.1 Symmetry-Breaking Tasks in Distributed Computing

One of the key challenges in distributed graph algorithms is local symmetry breaking: two adjacent nodes in a graph (here: two consecutive nodes along the path) have got isomorphic local neighborhoods but are expected to produce different outputs.

In distributed computing, a canonical example is the *vertex coloring problem*. Consider, for example, the task of finding a proper coloring with k colors. This is trivial in the supported and numbered models (node number i can simply output e.g. $i \bmod 2$ to produce a proper 2-coloring). However, the case of the PN model and the LOCAL model is a lot more interesting.

One can use simple arguments based on *local indistinguishability* [12, 45] to argue that such tasks are not solvable in $o(n)$ rounds in the PN model. In brief, if two nodes have identical radius- T neighborhoods, then they will produce the same output in any deterministic PN-algorithm that runs in T rounds. For example, it immediately follows k -coloring for any k requires $\Omega(n)$ rounds in the deterministic PN model.

Yet another idea one can exploit in the analysis of symmetry-breaking tasks is *rigidity* (or, put otherwise, the lack of *flexibility*); see e.g. [15, 17]. For example, 2-coloring is a rigid problem: once the output of one node is fixed, all other nodes have fixed their outputs. Informally, two nodes arbitrarily far from each other need to be able to coordinate their decisions—or otherwise there is at least one node between them that produces the wrong output. This idea can be used to quickly show that e.g. 2-coloring in the LOCAL model

requires also $\Omega(n)$ rounds, and this holds even if we consider *randomized* algorithms (say, Monte Carlo algorithms that are supposed to work w.h.p.).

This leaves us with the case of symmetry-breaking tasks that *are* flexible. A canonical example is the 3-coloring problem. Informally, one can fix the colors of any two nodes (sufficiently far from each other), and it is always possible to complete the coloring between them. While the 3-coloring problem requires $\Omega(n)$ rounds in the deterministic PN model, it is a problem that can be solved much faster in the deterministic LOCAL model and also in the randomized PN model: the Cole–Vishkin technique [21] can be used to do it in only $O(\log^* n)$ rounds. However, what is important for us in this work is that this is also known to be tight [33, 35]: 3-coloring is not possible in $o(\log^* n)$ rounds, not even if we use both unique identifiers and randomness.

Moreover, the same holds for *all* problems in which the task is to label a path with some labels from a constant-sized set Y , and arbitrarily long sequences of the same label are forbidden: no such problem can be solved in constant time in the PN or LOCAL model, not even if one has got access to randomness [16, 33, 35, 36, 43].

We will soon see what all of this implies for us, but let us discuss one technicality first: symmetric vs. asymmetric horizons.

4.3.2 Symmetric vs. Asymmetric Horizons

While the standard models in distributed computing correspond to *symmetric* horizons ($[T, T]$ -local algorithms) and the study of online algorithms is typically interested in *asymmetric* horizons (e.g. $[T, -1]$ -local algorithms), in many cases this distinction is inconsequential when one considers symmetry-breaking tasks.

Consider, for example, the vertex coloring problem Π . Assume one is given an $[a, b]$ -local algorithm A for solving Π . Now for any constant c one can construct an $[a + c, b - c]$ -local algorithm A' that solves the same problem. In essence, node x_i in algorithm A' simply outputs $A(x_{i-a-c}, \dots, x_{i+b-c})$. Now if one compares the outputs of A' and A , we produce the same sequence of colors but shifted by c steps. This is the standard trick one uses to convert algorithms for directed paths into algorithms for rooted trees and vice versa; see e.g. [17, 40]. The only caveat is that we need to worry about what to do near the boundaries, but for our purposes the very first and the very last outputs are usually inconsequential (can be handled by an ad hoc rule, or simply ignored thanks to the additive constant in the definition of the competitive ratio).

Hence, in essence everything that we know about symmetry-breaking tasks in the context of $[T, T]$ -local algorithms can be easily translated into equivalent results for $[T', -1]$ -local algorithms for $T' = 2T + 1$, and vice versa.

4.3.3 Distributed Optimization and Approximation

So far we have discussed distributed graph problems in which the task is to find *any* feasible solution subject to some local constraints. However, especially in the context of online algorithms, we are usually interested in finding *good* solutions. Typical examples are problems such as the task of finding the minimum dominating set problem and the maximum independent set problem.

These are not, strictly speaking, symmetry-breaking tasks. Nevertheless, it turns out to be useful to look at also such tasks through the lens of symmetry breaking. In brief, the following picture emerges [23, 27, 36, 43]:

- Deterministic $O(1)$ -round LOCAL-model algorithms are not any more powerful than deterministic $O(1)$ -round PN-model algorithms.
- Randomized $O(1)$ -round algorithms are strictly stronger than deterministic $O(1)$ -round algorithms.

For example, if we look at the *minimum dominating set problem* in unweighted paths, the only possible deterministic $O(1)$ -round PN-algorithm produces a constant output: all nodes (except possibly some nodes near the boundaries) are part of the solution. Deterministic $O(1)$ -round LOCAL-algorithm can *try* to do something much more clever, with the help of unique identifiers, but a Ramsey-type argument [23, 27, 36] shows that it is futile: there always exists an adversarial assignment of unique identifiers such that the algorithm produces a near-constant output for all but ϵn many nodes, for an arbitrarily small $\epsilon > 0$. However, randomized algorithms can do much better (at least on average); to give a simple example, consider an algorithm that first takes each node with some fixed probability $0 < p < 1$, and then adds the nodes that were not yet dominated. Finally, in the numbered and supported models one can obviously do much better, even deterministically (simply pick every third node).

This is now enough background on the most relevant results related to T -round algorithms in deterministic and randomized PN and LOCAL models.

4.3.4 Consequences: Time-Local Solvability

It turns out to be highly beneficial to try to classify online problems in the above terms: whether there is a component that is equivalent to a symmetry-breaking task or to a nontrivial distributed optimization problem. This is easiest to explain through examples:

Online file migration (Example 3.1). This problem is trivial to solve for a constant input; the same also holds for any input sequence that is strictly periodic. Indeed, if the adversary gives a long sequence of constant inputs (or follows a fixed periodic pattern), it only helps us. Hence none of the above obstacles are in our way; interesting inputs are sequences that already break symmetry locally. Furthermore, as we also know that this is a well-known online problem solvable with the full history, we would expect that there is also an unclocked time-local algorithm for solving the task, with a nontrivial competitive ratio. While this is a *heuristic* argument (based on the *lack* of specific obstacles), we will see in Section 8 that the argument works very well in this case.

Online load balancing (Example 3.2). This problem is fundamentally different from the file migration problem. Let us assume that the algorithm needs to output the action (on which machine to schedule the current job). Consider an input sequence that consists of the constant value 2. In such a case, there is an optimal solution that alternately assigns the 2-unit jobs to the two machines, ensuring that the load of any machine at any time is exactly 1. But this means that an optimal algorithm has to turn the constant input $2, 2, 2, \dots$ into a strictly alternating sequence like $1, 2, 1, 2, \dots$. Any deviation from it will result at least momentarily in a load of 2. Hence in an *optimal* solution we need to *at least* solve the 2-coloring problem within each segment of such constant inputs. As we discussed, this is not possible in the PN or LOCAL model in $O(1)$ rounds, not even with the help of randomness; it follows that there certainly is no optimal unclocked time-local online algorithm, with any constant horizon T . Optimal solutions have to resort to the clock.

However, this does not prevent us from solving the problem with a finite competitive ratio. Indeed, even the trivial solution that outputs always 1 will result in a maximum load

that is at most 2 times as high as optimal.

Furthermore, if we were not interested in the *maximum* load but the *average* load, we arrive at a task that is, in essence, a distributed optimization problem. Unclocked *randomized* time-local algorithms may then have an advantage over unclocked *deterministic* time-local algorithms, and indeed this turns out to be the case here: simply choosing the machine at random is already better on average than assigning all tasks to the same machine.

4.3.5 Consequences: Time-Local Models

On a more general level, the above discussion also leads to the following observation: the definition of *unclocked* time-local algorithms is *robust*. Now it coincides with the PN model, but even if one tried to strengthen it so that its expressive power was closer to the LOCAL model, very little would change in terms of the results.

Conversely, if one weakened the *clocked* model so that e.g. the clock values are not increasing by one but they are only a sequence of monotone, polynomially-bounded time stamps, we would arrive at a model very similar to the LOCAL model, and as we have seen above, time-local algorithms in such a model cannot solve symmetry-breaking tasks any better than in the unclocked model. Hence in order to capture the idea of a model that is strictly more powerful than the unclocked model, it is not sufficient to have a definition in which the clock values are merely monotone and polynomially bounded, but one has to further require e.g. that the clock values increase at each step at most by a constant. (Such a model with constant-bounded clock increments would indeed be a meaningful alternative, and it would fall in its expressive power strictly between our unclocked and clocked models. It would be strong enough to solve 3-coloring but not strong enough to solve 2-coloring in a time-local fashion. We do not explore this variant further, but it may be an interesting topic for further research, especially when comparing its power with randomized PN algorithms.)

5 Clocked Time-Local Algorithms

In this section, we examine the power of clocked time-local algorithms. First, we show that clocked time-local algorithms can be powerful, despite their limited access to the input: for many problems, competitive classic online algorithms can be converted into competitive clocked time-local algorithms. Second, we complement the first result by giving an example of a classic online problem that does admit a competitive clocked time-local algorithm, despite having competitive classic online algorithms.

5.1 Clocked Time-Local Algorithms from Full-History Algorithms

We now show that for a large class of online problems the following result holds: if the problem admits a *deterministic* classic online algorithm with competitive ratio c , then for any given constant $\varepsilon > 0$ there exists a deterministic clocked time-local algorithm with a competitive ratio of at most $(1 + \varepsilon)c$ for some constant horizon T .

The proof follows a similar structure as the constructive derandomization proof of Ben-David et al. [8, Section 4] for classic online algorithms: we chop the input sequence into short segments and show that under certain assumptions both the offline and competitive online algorithms pay roughly the same cost. However, some care is needed to adapt the proof strategy, as in the case of time-local algorithms, we can only use constant-size segments.

Bounded Minimization Games. We now define the class of request-answer games for which we prove our result. A (minimization) game is *monotone* if for all $n \in \mathbb{N}$

$$f_{n+1}(\mathbf{x} \cdot x, \mathbf{y} \cdot y) \geq f_n(\mathbf{x}, \mathbf{y}) \text{ for all } \mathbf{x} \in X^n, \mathbf{y} \in Y^n, x \in X, y \in Y.$$

That is, the cost cannot decrease when extending the input-output sequence. We say that a monotone game has *bounded delay* if for every $h \in \mathbb{R}$ the set

$$L(h) = \{\mathbf{x} \in \bigcup_{n=1} X^n : \text{OPT}(\mathbf{x}) \leq h\}$$

is finite (sometimes this property is called locality [8]). That is, there cannot be arbitrarily long sequences of a fixed cost: eventually the cost of any sequence must increase. Finally, the diameter of the game is

$$D = \sup \left\{ |f(\mathbf{x} \cdot \mathbf{x}', \mathbf{y} \cdot \mathbf{y}') - f(\mathbf{x}, \mathbf{y}) - f(\mathbf{x}', \mathbf{y}')| : (\mathbf{x}, \mathbf{y}), (\mathbf{x}', \mathbf{y}') \in \bigcup_{n \geq 0} X^n \times Y^n \right\}.$$

We define that a *bounded monotone minimization game* is a monotone minimization game that has bounded delay, finite diameter, and finite input set X . Note that bounded monotone minimization games are not necessarily local optimization problems as defined in Section 3.1. The latter are monotone games with finite diameter, but they do not necessarily have bounded delay. The following result holds for deterministic algorithms:

► **Theorem 5.1.** *Let \mathcal{F} be a bounded monotone minimization game. If there exists an online algorithm A with competitive ratio c for \mathcal{F} , then for any constant $\varepsilon > 0$ there exists some constant T and a clocked T -time-local algorithm B with competitive ratio $(1 + \varepsilon)c$ for \mathcal{F} .*

Proof. Since A has competitive ratio c , then there exists some constant d such that for every input \mathbf{x} the output $\mathbf{y} = A(\mathbf{x})$ satisfies $f(\mathbf{x}, \mathbf{y}) \leq c \cdot \text{OPT}(\mathbf{x}) + d$. Let D be the diameter of the game and fix

$$\delta = \frac{2\varepsilon}{3} \quad \text{and} \quad H = \left(\frac{2 + \delta}{\delta} \right) \cdot \max\{d, D\}.$$

Since the game has bounded delay, we have that

$$L(H) = \{\mathbf{x} : \text{OPT}(\mathbf{x}) \leq H\} \quad \text{and} \quad T = \max\{k + 1 : (x_1, \dots, x_k) \in L(H)\}$$

are finite. Note that T is independent of n , as it only depends on H . Observe that since the cost functions are monotone, for all $n \geq T$ any input sequence $\mathbf{x} \in X^n$ satisfies $\text{OPT}(\mathbf{x}) \geq H$.

We can now construct the *clocked* time-local algorithm that only sees the T latest inputs and the total number of requests served so far. Let $A = (A_i)_{i \geq 1}$ be the classic online algorithm. The clocked time-local algorithm B is given by sequence $(B_j)_{j \geq 1}$, where

$$B_{Tk+i}(z_1, \dots, z_T) = A_i(z_1, \dots, z_i) \text{ for } 1 \leq i \leq T \text{ and } k \geq 0.$$

That is, the clocked time local algorithm B simulates the classic online algorithm A by resetting it every time T inputs have been served since the last reset.

We now analyze the clocked time local algorithm B . For any $n \in \mathbb{N}$, let $\mathbf{x} \in X^n$ be some input sequence and $\mathbf{y} \in Y^n$ be the output of B on the input sequence \mathbf{x} . Let $\mathbf{x}(1), \dots, \mathbf{x}(k)$ be the subsequences of \mathbf{x} , where $\mathbf{x}(1)$ denote the first T inputs, $\mathbf{x}(2)$, denote the next T inputs, and so on. Define the shorthand $C(i) = \text{OPT}(\mathbf{x}(i))$ for each $1 \leq i \leq k$. Note that

$C(i) \geq H$ for each $1 \leq i < k$. The last subsequence $\mathbf{x}(k)$ may consist of fewer than T inputs, so we have no lower bound for $C(k)$. For $1 \leq i < k$, we get that

$$C(i) - D \geq \left(\frac{2}{2+\delta}\right) \cdot C(i) \quad \text{and} \quad D + d \leq \left(\frac{2\delta}{2+\delta}\right) \cdot C(i)$$

by applying the fact that $C(i) \geq H$ and the definition of H .

By repeatedly applying the definition of diameter, we get that the optimum offline solution is lower bounded by

$$\text{OPT}(\mathbf{x}) \geq C(1) + \sum_{i=2}^k (C(i) - D) \geq \sum_{i=1}^k (C(i) - D) \geq \left(\frac{2}{2+\delta}\right) \sum_{i=1}^k C(i).$$

Since A has competitive ratio c , the output of B has cost

$$f(\mathbf{x}, B(\mathbf{x})) \leq c \cdot C(1) + d + \sum_{i=2}^k (c \cdot C(i) + d + D) \leq \left(c + \frac{2\delta}{2+\delta}\right) \sum_{i=1}^k C(i) + d.$$

Now using the lower bound on $\text{OPT}(\mathbf{x})$ and the definition of δ , we get that the output of B has cost bounded by

$$\begin{aligned} f(\mathbf{x}, B(\mathbf{x})) &\leq \left(c + \frac{2\delta}{2+\delta}\right) \sum_{i=1}^k C(i) + d \\ &\leq \left(c + \frac{2\delta}{2+\delta}\right) \cdot \left(\frac{2+\delta}{2}\right) \text{OPT}(\mathbf{x}) + d \\ &= c \left(1 + \frac{\delta}{2}\right) + \delta + d = (1 + \varepsilon)c + d. \end{aligned} \quad \blacktriangleleft$$

5.2 The Limitations of Clocked Time-Local Algorithms

If the assumptions of Theorem 5.1 are not satisfied, how badly can clocked time-local algorithms perform? We now give an example of an online problem for which no time-local algorithm is competitive, despite the existence of competitive classic online algorithms: the *online caching problem* [42].

In this problem, the input set $X = \{1, \dots, m\}$ coincides with the set of elements that can be stored in the cache of size k , and the set $Y = \{A \subseteq X : |A| \leq k\}$ of outputs corresponds to the set of files currently stored in the cache. The result holds already for universe of size $m = 3$ and cache of size $k = 2$.

► **Theorem 5.2.** *There is no deterministic clocked time-local algorithm for online caching with cache size $k = 2$ that has finite competitive ratio.*

Proof. Consider the caching problem with the cache size $k = 2$ and a universe of $m = 3$ elements $X = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$. Let A be any clocked time-local algorithm with horizon $T = O(1)$. We say that A is *decisive* if on the infinite sequence \mathbf{a}^* there is some t such that $y_{t'} = y_t$ for all $t' > t$. Otherwise, A is *indecisive*; note that any indecisive deterministic algorithm A must be a clocked algorithm.

We distinguish two cases. First, suppose that A is indecisive and consider an input sequence family $\mathcal{I} := \{\mathbf{a}^L : L \in \mathbb{N}\}$. Intuitively, an indecisive algorithm changes its output infinitely many times on the infinite sequence consisting only of requests to \mathbf{a} . Clearly, the optimal offline solution on any input $\mathbf{x} \in \mathcal{I}$ is constant. However, since A is indecisive, A may

incur an arbitrarily large cost on requests from this family, due to an arbitrary number of changes in the output, each causing a cache reconfiguration. Thus, any indecisive algorithm has an unbounded competitive ratio.

Second, suppose that A is decisive. A crucial observation is that for a fixed deterministic time-local algorithm, the following holds: if two inputs share a subsequence a^T at a time step τ , then A produces the same output at τ on both. Therefore, a decisive algorithm eventually settles on a fixed output on a^* , and consequently it also eventually settles for a fixed output when its visible horizon is a^T on any input from \mathcal{I}' . Formally, there exists a time τ_0 so that A after time τ_0 always outputs a fixed configuration when faced with T many requests to \mathbf{a} . We refer to this configuration as the *default* configuration, and w.l.o.g. we assume that the default configuration is $\{\mathbf{a}, \mathbf{b}\}$.

Consider an input sequence family $\mathcal{I}' := \{(a^T c)^L : L \in \mathbb{N}\}$. Starting from τ_0 , A outputs the default configuration $\{\mathbf{a}, \mathbf{b}\}$ after seeing any sequence of T requests to \mathbf{a} . Hence, after τ_0 , each request to \mathbf{c} incurs the cost at least 1, and this accumulated cost can be arbitrarily large (it grows with the length of the sequence). The optimal solution for all inputs in \mathcal{I}' is to move to the configuration $\{\mathbf{a}, \mathbf{c}\}$ at the beginning, where all requests are free, hence the optimal offline solution incurs a constant cost. Thus, A is not finitely competitive on \mathcal{I}' . ◀

We note that the above result relies upon a specific, yet natural, choice of the request and answer set in our formulation, coinciding with the configuration set. However, other encodings may admit competitive time-local algorithms. For a discussion on the choice of the request and answer set and their influence on the competitive ratio, see Section 8.4.

We note that similar arguments can establish the hardness of a wide range of metrical task systems, and many k -server problem variants. Moreover, the theorem can be strengthened to show intractability for a wider class of randomized clocked time-local online algorithms; the reasoning is equivalent to the proof Theorem 8.1.

6 Power of Randomness

In this section, we define randomized time-local algorithms. In the classic online setting, there are two equivalent ways of describing randomized algorithms:

- at the start, randomly sample an algorithm from a set of deterministic algorithms, or
- at each step, make a random decision based on coin flips.

The former corresponds to *mixed strategies*, where we sample all random bits used by the algorithm before seeing any of the input, whereas the latter corresponds to *behavioral strategies*, where the algorithm generates random bits along the way as it needs them.

Mixed vs. Behavioral Strategies in Time-Local Algorithms. The above two characterizations are equivalent in classic online algorithms [13]: to simulate a behavioral strategy with a mixed strategy, we can generate an infinite sequence $(r_i)_{i \geq 1}$ of random bit strings in advance and use the random bits given by r_i in step i . Conversely, we can choose to flip coins only at the beginning and store the outcomes in memory and refer to them consistently at later steps.

In contrast, for time-local algorithms, the behavioral and mixed strategies differ in a way we can exploit randomness, and each type of strategy brings distinct advantages. If we use a behavioral strategy, at each step the algorithm can make coin flips that are independent of the previous coin flips. This enables algorithmic strategies that can e.g. break ties in an independent manner in successive steps. If we use a mixed strategy, we commit to a randomly chosen (consistent) strategy: the initial random choice influences all outputs. In a sense,

in time-local algorithms, behavioral strategies correspond to *private randomness* available at each step i , whereas mixed strategies correspond to *shared randomness* across the whole sequence. Interestingly, in the time-local setting, it is also natural to consider a *combination of both*: we choose a behavioral time-local strategy at random.

With this in mind, we arrive at three natural definitions of randomized time-local algorithms:

- *behavioral strategy* time-local algorithms,
- *mixed strategy* time-local algorithms, and
- *general strategy* time-local algorithms that use a combination of both.

We now give formal definitions for each class of randomized time-local algorithms.

► **Definition 6.1** (behavioral local algorithms). *A behavioral $[a, b]$ -local algorithm is given by the sequence of maps $(A_i)_{i \geq 1}$ of the form $A_i: X^{a+b} \times [0, 1) \rightarrow Y$, where the output is given by*

$$y_i = A_i(x_{i-a}, \dots, x_{i+b}, r_i),$$

where $(r_i)_{i \geq 1}$ is a sequence of i.i.d. real values sampled uniformly from the unit range. If $A_i = A_j$ for all i, j , then the algorithm is unclocked. Otherwise, it is clocked.

► **Definition 6.2** (mixed local algorithms). *Let \mathcal{D} be a nonempty set of (deterministic) $[a, b]$ -local algorithms. A mixed $[a, b]$ -local algorithm over \mathcal{D} is a probability measure $A: \mathcal{D} \rightarrow [0, 1]$ over \mathcal{D} . The output of A on input \mathbf{x} is the random vector $\mathbf{y} = P(\mathbf{x})$, where P is a deterministic time-local algorithm sampled from \mathcal{D} according to A . If \mathcal{D} is a subset of all unclocked $[a, b]$ -local algorithms, then A is unclocked. If \mathcal{D} is a subset of all clocked $[a, b]$ -local algorithms, then A is clocked.*

► **Definition 6.3** (general randomized local algorithms). *A general randomized unclocked $[a, b]$ -local algorithm is a mixed $[a, b]$ -local algorithm over the set of unclocked behavioral $[a, b]$ -local algorithms.*

Observe that in the case of *clocked* algorithms, general randomized time-local algorithms coincide with mixed clocked time-local algorithms, as the former can be simulated by the latter. To this end, we can generate an infinite sequence $(r_i)_{i \geq 1}$ of random bit strings in advance and store them in functions A_i of the deterministic clocked $[a, b]$ -local algorithms, and use the random bits given by r_i in step i . However, this is not the case with unclocked algorithms: as time-local algorithms do not have memory to store past random outcomes, it is impossible to directly simulate mixed time-local algorithms by a behavioral time-local algorithm that flips coins only at the beginning.

Adversaries and the Expected Competitive Ratio. We naturally extend the notion of competitiveness to randomized algorithms. For randomized algorithms, the answer sequence and the cost of an algorithm is a random variable. We will abuse the notation slightly to let $\mathbf{y} = A(\mathbf{x})$ denote the random output generated by a randomized algorithm A on input \mathbf{x} .

We say that a randomized online algorithm A for a game defined with cost functions $(f_n)_{n \geq 1}$ is *c-competitive* if

$$\mathbb{E}[f_n(\mathbf{x}, A(\mathbf{x}))] \leq c \cdot \text{OPT}(\mathbf{x}) + d$$

for any input sequence \mathbf{x} and a fixed constant d . The input sequence and the benchmark solution OPT is generated by an adversary. We distinguish between the notion of competitiveness against various adversaries, having different knowledge about A and different knowledge while

producing the solution OPT . Competitive ratios for a given problem may vary depending on the power of the adversary.

An *oblivious offline adversary* must produce an input sequence in advance, merely knowing the description of the algorithm it competes against (in particular, it may have access to probability distributions that the algorithm uses, but not the random outcomes), and pays an optimal offline cost for the sequence. An *adaptive online adversary* produces an input sequence based on the actions of the algorithm, and serves this request sequence online. An *adaptive offline adversary* produces an input sequence based on the actions of the algorithm, and pays an optimal offline cost for the sequence. For a comprehensive overview of adversary types, see [13].

Later in this paper, we present time-local algorithms that are competitive against the oblivious offline (cf. Section 8.2) and the adaptive online adversary (cf. Section 8.4). We note an interesting question regarding the adaptive offline adversary in the time-local setting. A well-known result in classic online algorithms states that if there exists a c -competitive randomized algorithm against it, then there exists a deterministic c -competitive algorithm, for any c [8]. Does the existence of a competitive randomized time-local algorithm against the adaptive offline adversary imply the existence of any competitive deterministic *time-local* algorithm?

7 Automated Algorithm Synthesis

In this section, we describe a technique for automated design of time-local algorithms for local optimization problems. This technique allows us to automatically obtain both upper and lower bounds for unclocked time-local algorithms. In particular, for deterministic algorithms, we can synthesize *optimal* algorithms. We also discuss how to extend our approach to randomized algorithms. As our case study problem, we use the simplified variant of online file migration.

7.1 Overview of the Approach

We now assume that the input and output sets X and Y are finite. Recall that an unclocked time-local algorithm that has access to last T inputs is given by a map $A: X^T \rightarrow Y$. The synthesis task is as follows: given the length $T \in \mathbb{N}$ of the input horizon, find a map A that minimizes the competitive ratio. For simplicity of presentation, we will ignore short instances of length $n < T$, as short input sequences do not influence the competitive ratio.

The Synthesis Method. The high-level idea of our synthesis approach is simple:

1. Iterate through all of the algorithm candidates in the set $\mathcal{A} = \{X^T \rightarrow Y\}$.
2. Compute the competitive ratio $c(A)$ for each algorithm $A \in \mathcal{A}$.
3. Choose the algorithm A that minimizes the competitive ratio.

Given that the input and output sets X and Y are finite, the set \mathcal{A} of algorithms is also finite: there are exactly $|Y|^{|X|^T}$ algorithms we need to check.

Evaluating the Competitive Ratio. Obviously, the challenging part is implementing the second step, i.e., computing the competitive ratio of a given algorithm A . A priori it may seem that we would need to consider infinitely many input strings in order to determine the competitive ratio of the algorithm. However, for any local optimization problem Π with finite input and output sets, it turns out that we can capture the competitive ratio by analyzing a finite combinatorial object.

We show that for any time-local algorithm A , we can construct a (finite) weighted, directed graph $G(\Pi, A)$ that captures the costs of output sequences as walks in $G(\Pi, A)$. The cost of any unclocked time-local algorithm on adversarial input sequences can be obtained by evaluating the weight of all cycles defined in this graph $G(\Pi, A)$.

7.2 Evaluating the Competitive Ratio of an Algorithm

We will now describe how to construct the graph $G(\Pi, A)$ for a given local optimization problem Π and an unclocked local algorithm A . For the sake of simplicity, we only consider the sum aggregation function; the construction for min aggregation is defined analogously. Let $r \in \mathbb{N}$ be the horizon of the local optimization problem Π , v the valuation function of Π , and $A: X^T \rightarrow Y$ be the time-local algorithm. To avoid unnecessary notational clutter, we describe the construction for $r = 1$; however, the construction is straightforward to generalize.

The Dual de Bruijn Graph. We construct a directed graph $G = (V, E)$ on the set of vertices $V = X^T \times Y$. For any $\mathbf{x} = (x_1, \dots, x_k)$, we define $s(\mathbf{x}, a) = (x_2, \dots, x_k, a)$ to be the successor of \mathbf{x} on a . For each vertex $(\mathbf{a}, y) \in V$, there is a directed edge towards the vertex $(\mathbf{a}', y') \in V$, where for all $y' \in Y$, $\mathbf{a}' = s(\mathbf{a}, x)$ and $x \in X$. Note that there are self-loops in this graph.

The idea is that for any sufficiently long input $n \geq T$, an input sequence $\mathbf{x} \in X^n$ and an output sequence $\mathbf{y} \in Y^n$ define a walk $\rho(\mathbf{x}, \mathbf{y})$ in the graph G . After the time step $i \geq T$, we are at vertex $(x_{i-T-1}, \dots, x_i, y_i) \in V$ and the next vertex is given by $(x_{i-T}, \dots, x_{i+1}, y_{i+1}) \in V$. In particular, from any walk ρ we can obtain the following sequences:

- an input sequence $\mathbf{x}(\rho) = (x_1, \dots, x_n) \in X^n$,
- some (possibly optimal) solution $\mathbf{y}^*(\rho) = (y_1, \dots, y_n)$ for $\mathbf{x}(\rho)$, and
- the output $y(\rho) = A(\mathbf{x}(\rho))$ given by the algorithm on $\mathbf{x}(\rho)$.

Vice versa, any pair of input \mathbf{x} and output \mathbf{y}^* sequences defines a walk $\rho(\mathbf{x}, \mathbf{y}^*)$ in G .

Assigning the Costs. For each edge $e \in E$ in the graph, we assign *two* costs for the edge: the first describes the cost paid by some (possibly optimal) output, and the second, the cost paid by the algorithm A . Recall that for a local optimization problem Π , the costs are given by the valuation function $v: X^{r+1} \times Y^{r+1} \rightarrow \mathbb{R} \cup \{\infty\}$. Consider an edge $e = ((\mathbf{a}, \mathbf{b}), (\mathbf{a}', \mathbf{b}')) \in E$, where $\mathbf{a}' = (a_2, \dots, a_{T+r-1}, x)$ and $\mathbf{b}' = (b_2, \dots, b_r, y)$ for some $x \in X$ and $y \in Y$.

We now define the *adversary cost* $w(e)$ and *algorithm cost* $q(e)$ of the edge e . For the sake of simplicity, we now only describe the simpler case of $r = 1$; the costs generalize to general $r > 1$ by applying the definitions of local cost functions given in Section 3.1. We define

$$\begin{aligned} w(e) &= v(a_T, x, b'_1, b'_2) \text{ is the cost paid on some output } b'_r = y \text{ on input } x, \\ q(e) &= v(a_T, x, b'_1, A(\mathbf{a})) \text{ is the cost paid by the algorithm on input } x, \end{aligned}$$

where $v: X \times Y \rightarrow \mathbb{R} \cup \{\infty\}$ is the valuation function of the problem Π .

The Cost Ratio of a Walk. Finally, for any walk $\rho = (v_1, \dots, v_k)$ in G , we define

$$w(\rho) = \sum_{i=1}^{k-1} w(v_i, v_{i+1}), \quad q(\rho) = \sum_{i=1}^{k-1} q(v_i, v_{i+1}).$$

Here $w(\rho)$ and $q(\rho)$ define the *total* adversary and algorithm costs for the walk ρ . The *cost ratio* of a walk ρ is defined as

$$r(\rho) = \begin{cases} q(\rho)/w(\rho) & \text{if } w(\rho) > 0 \\ 1 & \text{if } q(\rho) = w(\rho) = 0 \\ \infty & \text{otherwise.} \end{cases}$$

That is, on input $\mathbf{x}(\rho)$ the algorithm A will pay a cost of $q(\rho) + O(1)$, whereas the optimum solution has cost at most $w(\rho) + O(1)$; there is a constant overhead on the costs since we ignore the costs incurred during the first $T - 1 + r = O(1)$ inputs.

Bounding the Competitive Ratio. We now show that we can compute the competitive ratio of the algorithm A using the graph $G = G(\Pi, A)$. We say a walk $\rho = (v_1, \dots, v_k) \in V^k$ is *closed* if its starts and ends in the same vertex $v_1 = v_k$. A *directed cycle* is a closed walk that is non-repeating, i.e., $v_i \neq v_j$ for all $1 \leq i \leq j < k$.

► **Theorem 7.1.** *The competitive ratio of algorithm A is $\max\{r(\rho) : \rho \text{ is a directed cycle of } G\}$.*

Figure 3 gives an example of the dual de Bruijn graph for the online file migration problem (Example 3.1) and an algorithm with local horizon $T = 2$.

To prove the above theorem, we introduce three lemmas and the following definitions. A *closed extension* of a walk ρ is a closed walk ρ' that contains ρ as a prefix. A *subwalk* ρ' of a walk $\rho = (v_1, \dots, v_k)$ is a subsequence (v_i, \dots, v_j) for some $1 \leq i \leq j \leq k$. A *decomposition* of ρ into L subwalks is a sequence of subwalks ρ_1, \dots, ρ_L of ρ such that their concatenation $\rho = \rho_1 \cdots \rho_L$.

► **Lemma 7.2.** *Let ρ be a walk in G . For any decomposition of ρ into L subwalks $\rho_1 \cdots \rho_L$, there exists some $1 \leq i \leq L$ such that $r(\rho_i) \geq r(\rho)$.*

Proof. Let π be a permutation on $\{1, \dots, L\}$ and $\tau_i = \rho_{\pi(i)}$ such that

$$r(\tau_1) \leq r(\tau_2) \leq \dots \leq r(\tau_L).$$

Moreover, for $1 \leq i \leq L$ we define $r(\tau_i) = q_i/w_i$, where $q_i = q(\tau_i)$ and $w_i = w(\tau_i)$. We use the shorthands $Q(i) = \sum_{j=1}^i q_j$ and $W(i) = \sum_{j=1}^i w_j$. Note that for the aggregate cost ratio for a local optimization problem using the sum as its aggregation function gives that

$$r(\rho) = \frac{Q(L)}{W(L)} = \frac{\sum_{j=1}^L q_j}{\sum_{j=1}^L w_j}.$$

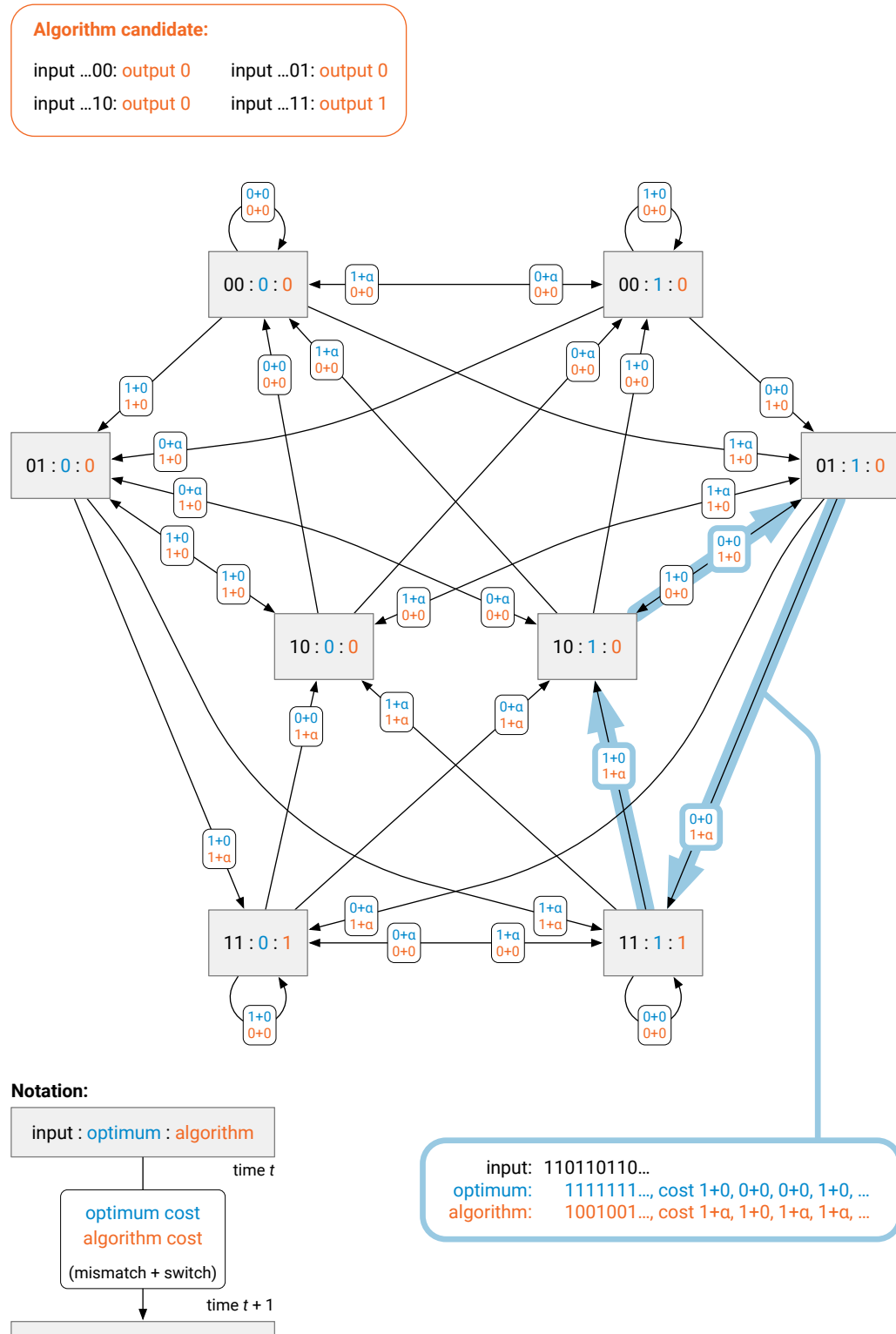
We now show by induction that for all $1 \leq i \leq L$ we have that

$$r(\tau_i) = \frac{q_i}{w_i} \geq \frac{Q(i)}{W(i)}.$$

Observe that this implies that $r(\rho_{\pi(L)}) = r(\tau_L) \geq Q(L)/W(L) = r(\rho)$.

The base case $i = 1$ is vacuous. For the inductive step, assume that the claim holds for some $1 \leq i < L$. For the sake of contradiction, assume that claim does not hold for $i + 1$, i.e.,

$$r(\tau_{i+1}) = \frac{q_{i+1}}{w_{i+1}} < \frac{Q(i+1)}{W(i+1)}.$$



■ **Figure 3** Dual de Bruijn graph for $T = 2$. The highlighted cycle shows how an adversary can force the candidate algorithm to pay $3 + 2\alpha$ when optimum pays only 1; hence this specific time-local algorithm cannot be better than $(3 + 2\alpha)$ -competitive.

By rearranging the terms, we get

$$\frac{Q(i+1)w_{i+1} - W(i+1)q_{i+1}}{W(i+1)} > 0,$$

which in turn implies that $Q(i+1) \cdot w_{i+1} > W(i+1) \cdot q_{i+1}$ holds. Now observing that

$$Q(i)w_{i+1} + q_{i+1}w_{i+1} = Q(i+1)w_{i+1} > W(i+1) \cdot q_{i+1} = W(i)q_{i+1} + w_{i+1}q_{i+1},$$

we get that

$$r(\tau_{i+1}) = \frac{q_{i+1}}{w_{i+1}} < \frac{Q(i)}{W(i)} \leq \frac{q_i}{w_i} = r(\tau_i),$$

where the second inequality follows from the induction assumption. However, this contradicts the fact that τ_1, \dots, τ_L were ordered according to increasing cost ratio. \blacktriangleleft

► **Lemma 7.3.** *Let ρ be a directed cycle in G . The competitive ratio of A is at least $r(\rho)$.*

Proof. Recall that the cycle defines an input sequence $\mathbf{x} = \mathbf{x}(\rho)$. By definition, the algorithm has cost at least $q(\rho)$ on this input sequence, whereas the optimum solution has cost at most $w(\rho) + d$ for some constant d . Thus, the algorithm has a cost of at least $q(\rho) \geq r(\rho)(w(\rho) + d) \geq r(\rho) \cdot \text{OPT}(\mathbf{x}) + O(1)$. \blacktriangleleft

► **Lemma 7.4.** *If the competitive ratio of A is greater than $c + \varepsilon$ for some $\varepsilon > 0$, then there exists a directed cycle ρ in G with cost ratio $r(\rho) > c$.*

Proof. For any given walk ρ in G , let $\hat{\rho}$ be the shortest closed extension of ρ that minimizes the cost of $\mathbf{y}^*(\hat{\rho})$. Note there may be multiple shortest closed extensions, so we pick the one with the cheapest adversarial cost. We let $\hat{\rho} \setminus \rho$ denote the suffix of $\hat{\rho}$ that satisfies $\hat{\rho} = \rho \cdot (\hat{\rho} \setminus \rho)$. Define

$$\delta = \max\{w(\hat{\rho} \setminus \rho) : \rho \text{ is a walk in } G\}.$$

Note that δ is a constant, since $\hat{\rho}$ is a minimal closed extension of ρ and G is finite.

Let \mathbf{x} be an input sequence and \mathbf{y}^* an optimal output sequence. For the walk $\rho = \rho(\mathbf{x}, \mathbf{y}^*)$, we have that

$$r(\hat{\rho}) = \frac{q(\hat{\rho})}{w(\hat{\rho})} \geq \frac{q(\rho) + q(\hat{\rho} \setminus \rho)}{w(\rho) + w(\hat{\rho} \setminus \rho)} \geq \frac{q(\rho) + \delta}{w(\rho) + \delta},$$

since $w(\rho) \leq q(\rho)$, as the cost of the algorithm is never less than the cost of the optimal solution \mathbf{y}^* . Asymptotically, as the length of the walk goes to infinity, we have that $r(\hat{\rho}) = r(\rho) - o(1)$. In particular, for any constant $\varepsilon_0 > 0$ we can find n_0 such that all input sequences \mathbf{x} of length $n \geq n_0$, the walk $\rho = \rho(\mathbf{x}, \mathbf{y}^*)$ given by \mathbf{x} and the optimal output sequence \mathbf{y}^* , satisfies

$$r(\hat{\rho}) \geq r(\rho) - \varepsilon_0.$$

By assumption A had a competitive ratio of at least $c + \varepsilon$. We can pick a sufficiently long input sequence \mathbf{x} and an optimal solution \mathbf{y}^* such that $\rho = \rho(\mathbf{x}, \mathbf{y}^*)$ satisfies

$$r(\hat{\rho}) \geq r(\rho) - \varepsilon_0 \geq \frac{f_n(\mathbf{x}, A(\mathbf{x}))}{\text{OPT}(\mathbf{x})} - \varepsilon' - \varepsilon_0 \geq c - \varepsilon' - \varepsilon_0 + \varepsilon > c,$$

where $f_n(\mathbf{x}, A(\mathbf{x}))$ denotes the cost of the algorithm A on input \mathbf{x} and ε_0 and ε' are appropriately chosen constants. Thus, we have now obtained a closed walk $\hat{\rho}$ with $r(\hat{\rho}) \geq c$. Since we can decompose $\hat{\rho}$ into a sequence $\hat{\rho}_1, \dots, \hat{\rho}_K$ of directed cycles, by applying Lemma 7.2 we get that some directed cycle $\hat{\rho}_i$ satisfies $r(\hat{\rho}_i) > c$, as claimed. \blacktriangleleft

Proof of Theorem 7.1. The above two lemmas yield that the competitive ratio of A is

- at least as large as the cost-ratio of some directed cycle in G (Lemma 7.3), and
- at most as large as the cost-ratio of some directed cycle in G (Lemma 7.4).

Thus, the directed cycle with the highest cost-ratio determines the competitive ratio of the algorithm A . Since the graph G is finite, it suffices to check all directed cycles of G to determine the competitive ratio of A .

7.3 Optimizations

We now overview few techniques for optimizing the synthesis for our case study problem of online file migration.

We can reduce the amount of computation needed to find the best algorithm A for a fixed T and α , by eliminating some algorithms. For example, we can often quickly identify some simple property of G that immediately disqualifies an algorithm candidate.

The Role of Self-Loops. If the competitive ratio of A is K , then the cost-ratio of any directed cycle has to be at most K . In particular, the cost-ratio of any directed cycle has to be finite. So we can directly eliminate all cases in which there is a cycle ρ with adversary-cost $w(\rho) = 0$ and a positive algorithm-cost $q(\rho) > 0$. For example, we can apply this reasoning to self-loops in the graph G . If the adversary-cost of a self-loop is zero, then the algorithm-cost of the same loop has to be also zero. It follows that e.g. we must have $A(0, \dots, 0) = 0$ and $A(1, \dots, 1) = 1$ for any algorithm A . In the case of $T = 3$, this reduces the number of algorithms that need to be checked from $2^8 = 256$ to only $2^{2^3-2} = 2^6 = 64$ instead.

Detecting Heavy Cycles. When searching for algorithms with best competitive ratio, it is useful to keep track of the best cost-ratio found so far: when checking a new algorithm candidate A and its corresponding graph G , we can first check small cycles of length at most L to see if any such cycle has cost-ratio larger than the best found cost-ratio for any other algorithm so far. If we encounter a cycle ρ with cost-ratio $r(\rho)$ that is larger or equal than the competitive ratio of some previously considered algorithm A' , then we know that the competitive ratio of A is larger or equal than that of A' . Thus, we can immediately disregard A and move on to check the next possible algorithm candidate.

Indeed, it turns out that in many cases, cycles with large cost-ratio are already found when examining only short cycles. However, if high-cost short cycles are not found, we can always fall back to an exhaustive search that checks all cycles.

7.4 On the Synthesis of Randomized Algorithms

We note that we can extend our approach to the synthesis of randomized algorithms as well. Here, the synthesis bounds the *expected* competitive ratio of the algorithm against an oblivious randomized adversary. We consider the synthesis for randomized *behavioral* algorithms (cf. Section 6). Synthesis of *mixed* algorithms would correspond to finding a good probability distribution over the finite set of algorithms, but we restrict our attention to the behavioral algorithms.

In the deterministic synthesis, we considered maps $A: \{0, 1\}^T \rightarrow \{0, 1\}$, and now we consider maps $A: \{0, 1\}^T \rightarrow [0, 1]$, where $A(\mathbf{a})$ gives the *probability* that A outputs 1 upon

seeing the sequence $\mathbf{a} \in X^T$ of last T inputs. Thus,

$$\begin{aligned} A(\mathbf{a}) &= \Pr[A \text{ outputs } 1 \text{ on input } \mathbf{a} \in X^T] \\ 1 - A(\mathbf{a}) &= \Pr[A \text{ outputs } 0 \text{ on input } \mathbf{a} \in X^T]. \end{aligned}$$

We assign the algorithm cost $q(e)$ for any edge (\mathbf{a}, y) to $(s(\mathbf{a}, x), y')$ as follows:

- On a mismatch, the algorithm pays the cost

$$q_{\text{mismatch}}(e) = \begin{cases} 1 - A(\mathbf{a}) & \text{if } x = 1 \\ A(\mathbf{a}) & \text{otherwise.} \end{cases}$$

- The switching cost is given by

$$q_{\text{switch}}(e) = \alpha \cdot [A(\mathbf{a}) \cdot (1 - A(s(\mathbf{a}, a))) + (1 - A(\mathbf{a})) \cdot A(s(\mathbf{a}, x))].$$

- The total cost is $q(e) = q_{\text{mismatch}}(e) + q_{\text{switch}}(e)$.

We calculate the adversary-cost in the same manner as we do in the deterministic model. That is our adversary always outputs 0 or 1 (but not, e.g. 0.6). Thus, the graph G will have the same structure as in the deterministic case.

Since there are uncountably many possible randomized algorithms A for any T , we instead discretize the probability space into finitely many segments. Thus, we cannot guarantee that we find optimal randomized algorithms. Nevertheless, this method can be used to obtain synthesized algorithms that beat the deterministic algorithms.

7.5 Results

We now give some results for the online file migration problem obtained using the synthesis approach. First, we discuss algorithms with small values of $T = 1, 2, 3$ and then provide observations for $T = 4$ and $T = 5$.

7.5.1 Synthesized Algorithms for $T = 1, 2, 3$

Table 2 summarizes results for $T = 1, 2, 3$ and $0.1 \leq \alpha \leq 1.6$. For deterministic algorithms, we list the competitive ratios of the *optimal* deterministic algorithms for the given values of parameters T and α .

For randomized algorithms, we list the best competitive ratios found by the synthesis method for the given values of T and α . As discussed, the search for randomized algorithms was conducted in a discretized search space, so it is possible that some randomized algorithms with better competitive ratios may have been missed by the search method.

The Power of Randomness. Note that already with $T = 2$ we can obtain algorithms with strictly better competitive ratios when randomness is used. Moreover, with only $T = 3$, we are able to obtain randomized algorithms with competitive ratio < 3 (e.g., when $\alpha = 1.0$). This is strictly better than *any* (non-time-local) deterministic algorithm for $\alpha = 1$. Table 3 gives an example of such an algorithm that achieve competitive ratio of roughly 2.67 for $T = 3$ and $\alpha = 1$.

After checking all cycles in the constructed dual de Bruijn graph, the cycle with the maximum cost-ratio (cost-ratio of about 2.67) happens to be the following:

- Last T inputs: 000, adversary output: 0.
- Last T inputs: 001, adversary output: 0.

- Last T inputs: 011, adversary output: 0.
- Last T inputs: 110, adversary output: 0.
- Last T inputs: 100, adversary output: 0.

7.5.2 The Case of $T = 4$

For $T = 4$ we can obtain better deterministic algorithms than with $T = 3$. Interestingly, we can find several optimal algorithms for the case $\alpha = 1$: even a full-history deterministic online algorithm cannot achieve a better competitive ratio. Table 4 lists all the 3-competitive algorithms that exist for parameter values of $T = 4$, $\alpha = 1$. This shows that even very simple time-local algorithms can perform well compared to classic online algorithms. Table 2 contains some of the results for $T = 4$ and $0.1 \leq \alpha \leq 1.5$.

7.5.3 Negative Results for $T = 5$

Since the number of cycles to be checked increases exponentially in T , we were not able to obtain any positive results for the case of $T = 5$. However, negative results could still be obtained, since verifying for a certain lower bound does not require to check all the cycles for all the algorithms. Instead, it is sufficient to find at least one cycle with a large enough cost-ratio to disregard a certain algorithm and move on to the next one. We were able to obtain the following results:

- **Observation 7.5.** *With parameter values of $\alpha = 1$ and $T = 5$, the best competitive ratio remains 3. That is, for each deterministic algorithm, after the dual de Bruijn graph has been constructed, there is a cycle with a cost-ratio of at least 3.*
- **Observation 7.6.** *No algorithm with ratio < 3.1 exists for parameter values of $T = 5$ and $\alpha = 1.1$.*
- **Observation 7.7.** *No algorithm with ratio < 3.2 exists for parameter values of $T = 5$ and $\alpha = 1.2$.*

8 Case Study: Online File Migration

We study a variant of online file migration (defined in Section 1.2) in the time-local setting. The case study serves three purposes:

- to show an example of a problem that admits competitive algorithms in the time-local setting,
- to highlight the challenges of algorithm design and analysis present in the time-local setting and propose techniques used to deal with them,
- to study the influence of limiting the visible horizon size on degradation of the competitive ratio.

We assume the following encoding of the problem: on each request, a time-local algorithm takes the current visible horizon as input and outputs the next location of the file. For a discussion on alternative encodings of this problem, see Section 8.4. Unless stated explicitly, we assume that the migration cost α is at least 1.

We start by providing insights into techniques which are useful for studying online problems in time-local setting. Next, we present a lower bound showing that the degradation of the competitive ratio is inevitable with the decrease of T , even with access to a global

■ **Table 2** The best competitive ratios for some values of α and T ; see also Figure 2.

α	$T = 1$	$T = 2$		$T = 3$		$T = 4$
	deterministic	deterministic	randomized	randomized	randomized	deterministic
0.1	11	11	11	11	11	11
0.2	6	6	6	6	6	6
0.3	4.333	4.333	4.333	4.333	4.333	4.333
0.4	3.5	3.5	3.5	3.5	3.5	3.5
0.5	3	3	3	3	3	3
0.6	3.2	3.2	3.006	3.2	2.934	3.2
0.7	3.4	3.4	3.055	3.4	2.864	3.4
0.8	3.6	3.6	3.2	3.6	2.797	
0.9	3.8	3.8	3.35	3.8	2.734	3.222
1.0	4	4	3.5	4	2.672	3
1.1	4.2	4.2	3.65	4.2	2.772	3.1
1.2	4.4	4.4	3.8	4.4	2.872	
1.3	4.6	4.6	3.95	4.6	2.986	3.3
1.4	4.8	4.8	4.1	4.8	3.088	
1.5	5	5	4.25	5	3.188	3.5
1.6	5.2	5.2	4.4	5.2	3.288	

■ **Table 3** A randomized algorithm for $T = 3$ and $\alpha = 1$ with expected competitive ratio ≈ 2.67 .

Last T inputs	The probability to output 1
000	0
001	0.3309
010	0.2711
011	1
100	0
101	0.7289
110	0.6691
111	1

■ **Table 4** Three 3-competitive algorithms for $T = 4$, $\alpha = 1$.

Last T inputs	Output		
	A_1	A_2	A_3
...0000	0	0	0
...0001	0	0	0
...0010	0	0	0
...0011	1	1	1
...0100	0	0	0
...0101	0	0	1
...0110	1	1	1
...0111	1	1	1
...1000	0	0	0
...1001	0	0	0
...1010	1	0	1
...1011	1	1	1
...1100	0	0	0
...1101	1	1	1
...1110	1	1	1
...1111	1	1	1

clock and using randomization. Then, we discuss an adaptation of a well-known randomized algorithm to the time-local setting, and design a competitive deterministic algorithm for 2-node networks.

Techniques for designing time-local algorithms. The challenges in designing time-local algorithms come from two sources: (1) the algorithm can make decisions based on the most recent input history only, and (2) the algorithm is unaware of its current configuration. Note that the latter challenge is not present in *memoryless* online algorithms [19]. To tackle these challenges, we highlight a useful technique of *tracking* distinguished subsequences of the input as they recede in the visible horizon. Implementing consistent tracking is simpler in the clocked setting; we study an example where we track requests issued at certain points in time, initially chosen uniformly at random.

Tracking is significantly more challenging to implement in non-clocked setting. Without knowing the temporal position of requests, requests from the same node are indistinguishable. We overcome this limitation by tracking distinguishable subsequences of requests instead of single requests, and we specify it in Section 8.3.

Lower bounds under time-local setting. We are able to reason about a time-local algorithm's performance on different sequences that share the same subsequences of T requests. The algorithm is a function of the last T requests, hence its output on one input is identical to the output on the second. A common lower bound design technique for classic online algorithms involves reasoning about distributions over inputs with Yao's principle. Instead, a simpler reasoning about a time-local algorithm may be sufficient in certain situations. We may argue about the performance of the algorithm on various relevant inputs independently, and come to conclusions about its performance on every input.

8.1 A Lower Bound

In this section, we present a lower bound for time-local algorithms for online file migration that shows an inevitable degradation of the competitive ratio when the visible horizon is limited. The following lower bound assumes the length of the visible horizon is given. Hardness for simpler settings is implied, in particular for non-clocked time-local algorithms. In the subsequent Sections 8.2 and 8.3, we present randomized and deterministic algorithms that asymptotically match this lower bound.

► **Theorem 8.1.** *Fix any randomized T -time-local clocked algorithm A for online file migration for networks with at least 2 nodes. Assume that the file size is α , and $\alpha \geq T$. If A is c -competitive against an oblivious offline adversary, then $c \geq 2\alpha/T$.*

Proof. To state A 's properties, we consider two infinite sequence of requests, 0^* and 1^* . Later we will reason about A 's performance on finite sequences. We say that a deterministic algorithm is *resisting* if for each t there exists a time $t' > t$ such that the algorithm either outputs 1 at t' when faced with 0^* , or outputs 0 at t' when faced with 1^* . A time-local algorithm may be resisting if it has access to a global clock.

Recall that A is a distribution over deterministic clocked algorithms. First, assume that A has a resisting strategy in its support. Consider two input sequence families, $\mathcal{I}_0 := \{0^L : L \in \mathbb{N}\}$ and $\mathcal{I}_1 := \{1^L : L \in \mathbb{N}\}$. Note that A may incur an arbitrarily large cost on requests from either of these families of inputs, say \mathcal{I}_0 , due to an arbitrary number of 0-requests served in the configuration 1. The cost of an optimal offline solution on such sequences is constant: an offline algorithm moves the file at the beginning to the only node that requests the file. We conclude that the competitive ratio of A can be arbitrarily large on inputs from \mathcal{I}_0 or \mathcal{I}_1 .

For the rest of this proof, we assume that A does not have any resisting strategy in its support. A crucial observation is that for a fixed deterministic time-local algorithm, its output on 0^* (resp. 1^*) for any time τ determines its output on other sequences that contain a sequence of T requests to 0 (resp. 1) at time τ . As A does not have a resisting strategy in its support, there exists a time τ_{det} such that after τ_{det} , all strategies in A 's support always output b when faced with T many requests to b , for $b \in \{0, 1\}$.

Consider an input $\sigma = (1^T 0^T)^L$ for some L to be determined. Let σ' be the subsequence of σ starting from the first 0-request that comes after τ_{det} . Fix any optimal offline algorithm OPT for σ . For $\sigma \setminus \sigma'$, we simply claim $A(\sigma \setminus \sigma') \geq \text{OPT}(\sigma \setminus \sigma')$, where $A(\cdot)$ and $\text{OPT}(\cdot)$ denotes the cost of the algorithm A and an optimal offline algorithm, respectively. To analyze σ' , we split it into subsegments of $(1^T 0^T)$ that we refer to as *phases*. As no strategy in A 's support is resisting, and requests from σ' come after τ_{det} , A 's behavior on σ' is determined: it must output b faced with b -uniform sequence, for $b \in \{0, 1\}$. Hence, in each phase A incurs cost 2α . On the other hand, in each phase OPT incurs cost at least T — recall that $T \leq \alpha$, thus either it migrated during the phase and paid $\alpha \geq T$ already, or did not migrate and paid for either 0-requests or 1-requests. Summing up the above observations and assuming σ' has length $2T \cdot L'$, we obtain

$$\frac{A(\sigma)}{\text{OPT}(\sigma)} = \frac{A(\sigma \setminus \sigma') + A(\sigma')}{\text{OPT}(\sigma \setminus \sigma') + \text{OPT}(\sigma')} \geq \frac{\text{OPT}(\sigma \setminus \sigma') + 2\alpha \cdot L'}{\text{OPT}(\sigma \setminus \sigma') + T \cdot L'}.$$

By choosing a long enough sequence σ (and consequently a large enough L'), the competitive ratio can be arbitrarily close to $2\alpha/T$. ◀

Note that the result presented in this section implies the lower bound for online file migration on general networks (not necessarily consisting of two vertices).

8.2 A Randomized Algorithm

In this section we discuss a randomized time-local algorithm for online file migration in general networks. The algorithm is an adaptation of an elegant randomized algorithm by Westbrook [44] as a function of the local time horizon and the global clock. We discuss the competitive ratio tradeoffs related to limited view of past requests, and we show that these tradeoffs are asymptotically optimal (cf. Section 8.1).

We assume that the feasible set of answers of any algorithm is coincident with the set of nodes of the network. For a discussion on possible extensions of such setting, see Section 8.4.

Algorithm Mixed Resetting. Fix any network N , size of the file α and the size of visible T -horizon. We describe a set of deterministic strategies, each parameterized by an integer $k \in [1, T]$. The Mixed Resetting algorithm chooses uniformly at random one of these strategies at the beginning.

Now we describe the deterministic strategy for a fixed k . In the classic setting with stateful algorithms, the algorithm may be described as follows. It maintains a counter, initially set to k . Each time it encounters a request, the counter decreases. When the counter drops to 0, the algorithm moves the file to the node where the request comes from at the moment, and resets the counter to T . While the counter is positive, the algorithm does not change the file placement.

This deterministic strategy can be implemented in the clocked time-local setting with the last T requests visible. Let τ be the current value of the clock, and let p, q be integers such that $\tau = T \cdot p + q$ and $q < T$. The algorithm's output at τ is x_{T-q} , the node that requested a file with the index $T - q$ in the current visible T -horizon.

► **Theorem 8.2.** *The competitive ratio of Mixed Resetting against the oblivious offline adversary is $\max\{2 + \frac{2\alpha}{T}, 1 + \frac{T+1}{2\alpha}\}$.*

► **Corollary 8.3.** *The competitive ratio of Mixed Resetting against the oblivious offline adversary is $1 + \phi \approx 2.62$ for $T = \alpha + \frac{1}{2}\sqrt{20\alpha^2 - 4\alpha + 1} - 1$, where ϕ is the Golden Ratio.*

An elegant proof of this theorem was given by Westbrook [44], using a potential function argument. This yields the best currently known algorithm in general networks against the oblivious adversary; the result is not tight, and the best known lower bound against the oblivious adversary is $2 + 1/(2\alpha)$ [20].

For larger T the competitive ratio would increase, thus in such case we simply truncate the visible horizon to its optimal value. Note that the ratio of Mixed Resetting is $O(\alpha/T)$, and it asymptotically matches the lower bound of $\Omega(\alpha/T)$ from Theorem 8.1. The result in this section is more general: the network may be arbitrary (more than two vertices), and the competitive ratio of the time-local algorithm remains unchanged and matches the lower bound even in arbitrary networks.

8.3 A Deterministic Algorithm

In this section, we introduce a constant competitive time-local algorithm, the *Sliding Window Algorithm* (ALG for short), for the online file migration problem restricted to two nodes, identified by 0 and 1. The algorithm takes the last T requests as input, and it outputs a value in $\{0, 1\}$, the (new) location of the file. For each request 0 or 1 (the node requesting the file), ALG pays a unit cost if its last output does not match the request (i.e., if the file is not located at the node). In such cases, we say ALG incurs a *mismatch*. After serving the

request, ALG may choose to migrate the file to the other node (by switching the output) at the cost α , and in such case we say ALG *flips* its output.

The algorithm scans the visible horizon looking for a distinguished subsequence of requests, called a *relevant window*. It decides its output based on the existence of any relevant window, and an invariable property of the relevant window (if any detected). After a window enters the visible horizon, ALG maintains its latest output as long as (i) the window is contained in the visible horizon (while *sliding*), and (ii) it is not succeeded by a more recent relevant window. ALG may flip its output once either of (i) or (ii) is no longer the case. Intuitively, a relevant window serves as a short-lived memory, enabling ALG to maintain the same output for as long as the window is contained in the visible horizon.

Sliding Window Algorithm. We define a b -window for $b \in \{0, 1\}$ as a subsequence of requests in which the number of b -requests is at least twice the number of $\neg b$ -requests, where $\neg b = 1 - b$. ALG outputs 1 only if the most recent b -window in the visible horizon is a 1-window. Therefore, it outputs 1 as long as the visible horizon contains a 1-window that is not succeeded by a (more recent) 0-window. The 1-window slides further to the past as new requests arrive, until it is no longer contained in the visible horizon. At this time, ALG flips back to 0 (as the default output) unless there is a more recent 1-window in the visible horizon.

We describe the sliding window algorithm formally as follows. Given any $T \geq 6$, let $\lambda := \min\{\lceil T/6 \rceil, \alpha\} \geq 1$ be a parameter. We denote a b -window of length 3λ that enters the visible horizon at time t by $\mathbf{W}_t := \sigma(t - 3\lambda, t]$. At any time t , the algorithm ALG takes the visible horizon (the past T requests) as input and outputs either 0 or 1 according to the following rules.

Rule 1. ALG outputs $b \in \{0, 1\}$ if the most recent window in the visible horizon is a b -window.

Rule 2. ALG outputs 0 if the visible horizon contains no b -window for $b \in \{0, 1\}$.

Note that whenever ALG flips to 0, it is either because the visible horizon contains a 0-window (Rule 1), or because there is no b -window in the visible horizon (Rule 2).

8.3.1 Analysis

Our analysis is based on the observation that ALG neither flips too frequently, incurring excessive reconfiguration cost, nor too conservatively, incurring excessive mismatches. To this end, we focus our attention on individual subsequences between two consecutive flips to 1. We show that any of these subsequences contains sufficiently many 0 and 1-request, so that an optimal offline algorithm incurs a cost within a factor $O(\alpha/T)$ of ALG's cost.

Preliminaries. We introduce auxiliary definitions and notations that we use in our analysis. ALG starts serving requests by outputting 0 and flips for the first time at t_{first} , which is a flip to 1. Let (l_i, r_i) denote the i th pair of time indexes after t_{first} , s.t. ALG flips to 0 at l_i and to 1 at r_i , and let m denote the number of these pairs. For the case $m = 0$, we let $r_0 := t_{\text{first}}$. After the last pair, ALG may perform a last flip to 0 at a time denoted by t_{last} . We denote the set of times at which ALG flips its output by $F := \{t_{\text{first}}, l_1, r_1, \dots, l_m, r_m, t_{\text{last}}\}$. We partition the i th phase into two parts as $\mathbf{P}_i = \mathbf{L}_i \mathbf{R}_i$, where $\mathbf{L}_i := \sigma(r_{i-1}, l_i]$ is the *left* part and $\mathbf{R}_i := \sigma(l_i, r_i]$ is the *right* part.

A *(sub)segment* of σ between times i and $j > i$ is a contiguous subsequence of σ specified by $\sigma(i, j]$. We denote the concatenation of any two consecutive segments \mathbf{S}_1 and \mathbf{S}_2 by $\mathbf{S}_1 \mathbf{S}_2$.

We say a segment \mathbf{S} is *short* if $|\mathbf{S}| < 3\lambda$, otherwise $|\mathbf{S}| \geq 3\lambda$ and it is *long*. For $b \in \{0, 1\}$, we denote the number of b -requests in a segment \mathbf{S} by $n_b(\mathbf{S})$.

We compare the cost of our algorithm to the cost of an optimal offline algorithm denoted by OPT . The total cost incurred by ALG and OPT while serving a segment \mathbf{S} is denoted by $\text{ALG}(\mathbf{S})$ and $\text{OPT}(\mathbf{S})$ respectively. We denote the cost of mismatches to ALG for a segment \mathbf{S} by $\text{mis}(\mathbf{S})$. Therefore, $\text{ALG}(\mathbf{W}_t) = \text{mis}(\mathbf{W}_t) + \alpha \leq 2\lambda + \alpha$ for $t \in F$.

Charging Scheme. We partition the input σ into *phases*, separated by flips to 1. Precisely $\sigma = \mathbf{P}_{\text{first}} \mathbf{P}_1 \dots \mathbf{P}_k \mathbf{P}_{\text{last}}$, where $\mathbf{P}_{\text{first}} := \sigma(0, t_{\text{first}}]$ is the subsequence until the first flip, $\mathbf{P}_{\text{last}} := \sigma(r_m, |\sigma|]$ is the subsequence from the last flip to the last request, and each $\mathbf{P}_i := \sigma(r_{i-1}, r_i]$, $1 \leq i \leq m$. In a series of lemmas, we analyze the total cost to OPT and eventually the competitive ratio for each part separately. Then, we aggregate all individual ratios into one competitive ratio in Theorem 8.12.

For any phase \mathbf{P}_i , Lemma 8.4 lower-bounds the number of 0-requests in \mathbf{W}_{l_i} and shows the window is contained in \mathbf{P}_i , given the flip to 0 occurs by Rule 2.

► **Lemma 8.4.** *For any phase \mathbf{P}_i , if the flip to 0 at l_i occurs by Rule 2 then $\mathbf{P}_{i-1} \cap \mathbf{W}_{l_i} = \emptyset$ and $n_0(\mathbf{L}_i) > \lambda$.*

Proof. By definition, ALG flips to 1 at $r_{i-1} < l_i$ in the phase \mathbf{P}_{i-1} . Since $T \geq 6\lambda$ (by definition), the segment $\mathbf{W}_{r_{i-1}}$ is contained in the visible horizon at each time $r_{i-1}, \dots, r_{i-1} + 3\lambda$. Thus, ALG outputs 1 at every time step $r_{i-1}, \dots, r_{i-1} + 3\lambda$. Therefore, the flip to 0 by Rule 2 occurs earliest at $r_{i-1} + 3\lambda + 1$, that is, at $l_i \geq r_{i-1} + 3\lambda + 1$. Hence,

$$|\mathbf{L}_i| = l_i - r_{i-1} \geq 3\lambda + 1 = |\mathbf{W}_{l_i}| + 1.$$

That is, \mathbf{W}_{l_i} is contained in \mathbf{L}_i , which implies $\mathbf{P}_{i-1} \cap \mathbf{W}_{l_i} = \emptyset$. Moreover, we have

$$n_1(\mathbf{W}_{l_i}) \leq 2\lambda - 1,$$

as otherwise ALG would output 1 at l_i , contradicting our assumption. Thus,

$$n_0(\mathbf{L}_i) \geq n_0(\mathbf{W}_{l_i}) = |\mathbf{W}_{l_i}| - n_1(\mathbf{W}_{l_i}) \leq 3\lambda - (2\lambda - 1) \geq \lambda + 1 > \lambda. \quad \blacktriangleleft$$

Lemma 8.4 implies that if the flip to 0 in \mathbf{P}_i occurs by Rule 2 then both windows \mathbf{W}_{l_i} and \mathbf{W}_{r_i} are contained in \mathbf{P}_i , i.e., they do not overlap \mathbf{P}_{i-1} .

► **Corollary 8.5.** *For any phase \mathbf{P}_i , if the flip to 0 at l_i occurs by Rule 2 then both windows \mathbf{W}_{l_i} and \mathbf{W}_{r_i} are contained in \mathbf{P}_i .*

The next lemma bounds the number of 0-requests in \mathbf{L}_i and the number of 1-requests in \mathbf{R}_i , given the flip at l_i occurs by Rule 1.

► **Lemma 8.6.** *For any phase \mathbf{P}_i , if the flip at l_i occurs by Rule 1 then $n_0(\mathbf{L}_i) \geq \lambda$ and $n_1(\mathbf{R}_i) \geq \lambda$.*

Proof. A flip to 1 always occurs by Rule 1 and therefore \mathbf{W}_{r_i} contains exactly 2λ many 1-requests. Assume for contradiction that $x := n_1(\mathbf{W}_{r_i} \setminus \mathbf{W}_{l_i}) < \lambda$. Then the remaining $2\lambda - x$ many 1-requests must be in \mathbf{W}_{l_i} which implies $n_1(\mathbf{W}_{l_i}) \geq 2\lambda - x > \lambda$. Since the flip to 0 is assumed to occur by Rule 1, \mathbf{W}_{l_i} must contain exactly 2λ many 0-requests. However, the number of 0-requests in \mathbf{W}_{l_i} is at most $3\lambda - n_1(\mathbf{W}_{l_i}) \leq 3\lambda - (2\lambda - x) < 2\lambda$, which implies either there is no flip to 0 at l_i , or the flip to 0 occurs by Rule 2, contradicting our assumption. Therefore, $n_1(\mathbf{R}_i) \geq n_1(\mathbf{W}_{r_i} \setminus \mathbf{W}_{l_i}) \geq \lambda$.

The flip to 0 at l_i follows the flip to 1 in the previous phase \mathbf{P}_{i-1} , which occurs by Rule 1. Then, following a similar argument as used for the flip at r_i (involving $\mathbf{W}_{r_{i-1}}$), we conclude $n_0(\mathbf{L}_i) \geq n_0(\mathbf{W}_{l_i} \setminus \mathbf{P}_{i-1}) \geq \lambda$. \blacktriangleleft

A segment \mathbf{S} is a *block* segment if $|\mathbf{S}| = 3\lambda$. The following lemma lower-bounds costs to OPT for any block segment in which ALG never flips, or it flips only at the end of the block, i.e., immediately *after* serving the last request in the block.

► **Lemma 8.7.** *For any block segment \mathbf{S} where ALG pays the mismatch cost $\text{mis}(\mathbf{S})$, if ALG does not flip in \mathbf{S} then $\text{OPT}(\mathbf{S}) \geq \min\{n_0(\mathbf{S}), n_1(\mathbf{S}), \alpha\} \geq \text{mis}(\mathbf{S})/2$. If ALG flips only at end of \mathbf{S} then $\text{OPT}(\mathbf{S}) \geq \lambda$.*

Proof. Assume ALG outputs $b \in \{0, 1\}$ in \mathbf{S} . Then it must hold $\text{mis}(\mathbf{S}) = n_{-b}(\mathbf{S}) \leq 2\lambda - 1$ (otherwise it would flip to $-b$) and hence $n_b(\mathbf{S}) \geq \lambda + 1 \geq n_{-b}(\mathbf{S})/2 = \text{mis}(\mathbf{S})/2$. OPT pays mismatches to b -requests or to $-b$ -requests, or it performs a flip and possibly serves some of them for free. Thus, regardless of OPT's choices in \mathbf{S} ,

$$\text{OPT}(\mathbf{S}) \geq \min\{n_b(\mathbf{S}), n_{-b}(\mathbf{S}), \alpha\} \geq \min\{\text{mis}(\mathbf{S})/2, \text{mis}(\mathbf{S}), \alpha\}$$

and the claim follows since $\text{mis}(\mathbf{S}) \leq 2\lambda - 1$ and thus $\text{mis}(\mathbf{S})/2 < \lambda \leq \alpha$.

Assume that ALG flips to 0 at the end of \mathbf{S} . Then \mathbf{S} contains at most 2λ many 0-requests as otherwise the flip to 0 would occur earlier in \mathbf{S} . Moreover, \mathbf{S} contains at most $2\lambda - 1$ many 1-requests, otherwise ALG would output 1 at the end of \mathbf{S} (by Rule 1). Therefore, \mathbf{S} contains at least λ many of either 0- and 1-requests, that is, $n_b(\mathbf{S}), n_{-b}(\mathbf{S}) \geq \lambda$. Regardless of OPT's actions, we have

$$\text{OPT}(\mathbf{S}) \geq \min\{n_b(\mathbf{S}), n_{-b}(\mathbf{S}), \alpha\} \geq \min\{\lambda, \alpha\} = \lambda.$$

The case where ALG flips to 1 at the end of the segment is analogous (up to swapping 0's and 1's). \blacktriangleleft

Consider any segment \mathbf{U} that can be partitioned into a set of blocks \mathcal{B} . If ALG does not flip in \mathbf{U} then applying Lemma 8.7 to each block separately yields

$$\text{OPT}(\mathbf{U}) = \sum_{\mathbf{B} \in \mathcal{B}} \text{OPT}(\mathbf{B}) \geq \sum_{\mathbf{B} \in \mathcal{B}} \text{mis}(\mathbf{B})/2 = \text{mis}(\mathbf{U})/2.$$

► **Corollary 8.8.** *For any segment \mathbf{U} such that $|\mathbf{U}|$ is a multiple of 3λ , if ALG does not flip in \mathbf{U} then $\text{OPT}(\mathbf{U}) \geq \text{mis}(\mathbf{U})/2$.*

The following lemma lower-bounds costs to OPT for a segment in which OPT does not flip, and ALG flips immediately after serving the segment, i.e., at *the end* of the segment.

► **Lemma 8.9.** *Consider any phase \mathbf{P}_i and a segment $\mathbf{S} \in \{\mathbf{L}_i, \mathbf{R}_i\}$ s.t. $|\mathbf{S}| \geq 3\lambda$. ALG outputs $b \in \{0, 1\}$ for the entire \mathbf{S} and flips to $-b$ at $t \in \{l_i, r_i\}$. Consider the partitioning $\mathbf{S} = \mathbf{U}\mathbf{V}\mathbf{W}_t$, where $|\mathbf{U}|$ is a multiple of 3λ , $|\mathbf{V}| < 3\lambda$. If OPT does not flip in $\mathbf{V}\mathbf{W}_t$ then one of the two cases holds:*

- i) OPT serves $\mathbf{V}\mathbf{W}_t$ in state b and $\text{OPT}(\mathbf{V}\mathbf{W}_t) = \text{mis}(\mathbf{V}\mathbf{W}_t)$
- ii) OPT serves $\mathbf{V}\mathbf{W}_t$ in state $-b$ and $\text{OPT}(\mathbf{V}\mathbf{W}_t) \geq \lambda$.

Proof. We show the claim for the case $b = 1$. The argument for $b = 1$ is analogous subject to swapping 0's and 1's. If OPT serves the entire \mathbf{S} in state $b = 1$ then both OPT and ALG pay mismatches to $-b$'s in the entire \mathbf{S} and $\text{OPT}(\mathbf{S}) = \text{mis}(\mathbf{S})$ which concludes Lemma 8.9.i.

Otherwise, OPT serves the entire \mathbf{S} in state $-b = 0$ and possibly pays no cost for \mathbf{V} . Therefore, $\text{OPT}(\mathbf{S}) \geq \text{OPT}(\mathbf{W}_t) \geq \lambda$, where the last inequality follows from Lemma 8.7, concluding Lemma 8.9.ii. \blacktriangleleft

The following lemma lower-bounds OPT's cost for a segment in which ALG does not flip until the end of the segment.

► **Lemma 8.10.** *Consider any phase P_i and a segment $S \in \{L_i, R_i\}$ s.t. $|S| \geq 3\lambda$. ALG flips to $b \in \{0, 1\}$ at $t \in \{l_i, r_i\}$ after it outputs $\neg b$ for the entire S . Consider the partitioning $S = UVW_t$, where $|U|$ is a multiple of 3λ , $|V| < 3\lambda$. If OPT flips in VW_t then one of the two cases applies:*

- i) OPT flips to b in VW_t and $\text{OPT}(VW_t) \geq \min\{\text{mis}(V), \lambda\} + \alpha$,
- ii) OPT flips to $\neg b$ in VW_t and $\text{OPT}(VW_t) \geq \max\{\text{mis}(VW_t)/2 - \lambda, 0\} + \alpha$.

Proof. We analyze the case where ALG flips to $b = 0$ at t . The case for $b = 1$ follows in a similar way subject to swapping 0's and 1's.

If OPT flips more than once in VW_t then $\text{OPT}(VW_t) \geq 2\alpha$. Since $\min\{\text{mis}(V), \lambda\} \leq \lambda \leq \alpha$, Lemma 8.10.i follows from $\text{OPT}(VW_t) \geq 2\alpha \geq \min\{\text{mis}(V), \lambda\} + \alpha$. Since $\text{mis}(VW_t) < 4\lambda$, we have $\text{mis}(VW_t)/2 - \lambda < \lambda \leq \alpha$. Thus, if OPT flips more than once in VW_t then $\text{OPT}(VW_t) \geq 2\alpha \geq \max\{\text{mis}(VW_t)/2 - \lambda, 0\} + \alpha$ and Lemma 8.10.ii holds. Hence, in the remainder, we assume OPT flips only once in VW_t .

If OPT flips in V then it does not flip in W_t and by Lemma 8.7, $\text{OPT}(W_t) \geq \lambda$. Therefore, $\text{OPT}(VW_t) = \text{OPT}(V) + \text{OPT}(W_t) \geq \alpha + \lambda$. Next, we provide lower bounds for the cost of OPT in VW_t by distinguishing the two cases of OPT's flip, to 0 and to 1.

OPT flips to $b = 0$ in VW_t . If OPT flips in V then

$$\text{OPT}(VW_t) \geq \lambda + \alpha \geq \min\{\text{mis}(V), \lambda\} + \alpha.$$

Otherwise, OPT serves V in state 1, $\text{OPT}(V) = n_0(V) = \text{mis}(V)$, and it flips to 0 in W_t . Therefore,

$$\text{OPT}(VW_t) = \text{OPT}(V) + \text{OPT}(W_t) \geq \text{mis}(V) + \alpha \geq \min\{\text{mis}(V), \lambda\} + \alpha,$$

which concludes Lemma 8.10.i.

OPT flips to $\neg b = 1$ in VW_t . If OPT flips in V then

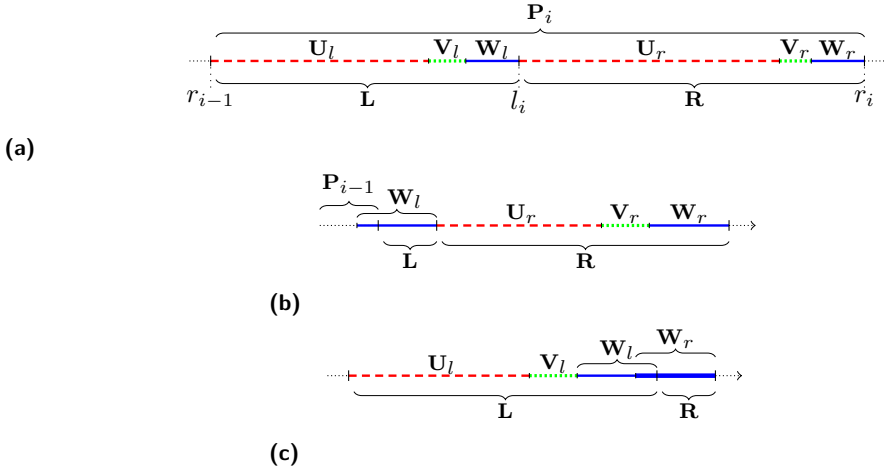
$$\text{OPT}(VW_t) \geq \lambda + \alpha \geq \max\{\text{mis}(VW_t)/2 - \lambda, 0\} + \alpha.$$

Otherwise, OPT serves the entire V in state 0. We lower-bound the cost of mismatches in W_t after OPT flips to 1, using the following observation. If $n_0(V) \geq 1$ then there are at least $\lambda + 1$ many 1-requests between the last 0-request in V and ALG's flip to 0 at t . Assume this is not the case and $n_1(W_t) \leq \lambda$. Therefore $n_0(W_t) = 3\lambda - n_1(W_t) \geq 2\lambda$ and $n_0(V) + n_0(W_t) \geq 2\lambda + 1$, that is, ALG flips to 0 earlier than t in W_t , contradicting our assumption. Therefore, at least $\lambda + 1$ many 1-requests in VW_t occur after the last 0-request in V . Let $\sigma_p = 1$ be the $(\lambda + 1)$ th 1-request in VW_t .

Next, we lower-bound the number of mismatches incurred by OPT after it flips to 1. Either OPT flips to 1 after σ_p , that is, after paying at least $\lambda + 1$ mismatches to 1-requests in VW_t , or it flips to 1 before serving σ_p . In the latter case, OPT pays mismatches to the remaining 0-requests (occurring after σ_p) in W_t . Let x be the number of 0-requests in VW_t after σ_p . Then the number of 0-requests in VW_t and before σ_p is $n_0(VW_t) - x < 2\lambda$, otherwise the flip to 0 would occur earlier than $p < t$, contradicting our assumption.

Summing up all the considered cases, OPT pays at least

$$x > n_0(VW_t) - 2\lambda = \text{mis}(VW_t) - 2\lambda$$



■ **Figure 4** ALG flips to 0 at the end of the segment L and flips to 1 at the end of the segment R . In 4a, both segments L and R are long. As a result, both windows W_l and W_r are contained in the phase. In 4b and 4c, one part is short and the other is long. The window W_l overlaps the phase P_{i-1} in 4b, and it overlaps W_r (in thick blue) in 4c.

mismatches to 0-requests after it flips to 1, and

$$\text{OPT}(\mathbf{VW}_t) \geq \max\{\text{mis}(\mathbf{VW}_t) - 2\lambda, 0\} + \alpha \geq \max\{\text{mis}(\mathbf{VW}_t)/2 - \lambda, 0\} + \alpha,$$

which concludes Lemma 8.10.ii. ◀

► **Lemma 8.11.** *For any phase P_i , we have $\text{ALG}(P_i)/\text{OPT}(P_i) \leq 4 + \frac{2\alpha}{\lambda}$.*

Proof. Recall that in the phase P_i , ALG flips to 0 at $l := l_i$ and later to 1 at $r := r_i$. We bound the costs to OPT and ALG in $L := L_i$ and $R := R_i$ separately, which is followed by the competitive ratio for $P_i = LR$. Thus, $\text{ALG}(P_i) = \text{ALG}(L) + \text{ALG}(R)$ and $\text{OPT}(P_i) = \text{OPT}(L) + \text{OPT}(R)$. If L is long, that is $|L| \geq 3\lambda$, we consider the partitioning $L = U_l V_l W_l$ where $|U_l|$ is a multiple of 3λ and $|V_l| < 3\lambda$. Similarly, $R = U_r V_r W_r$ where $|U_r|$ is a multiple of 3λ and $|V_r| < 3\lambda$.

To obtain our upper bounds, we use the fact $\sum_j a_j / \sum_j b_j \leq \max_j a_j / b_j$ for $a_j, b_j \geq 0$. We bound the costs under four major cases of $|L|$ and $|R|$.

Both parts are short. Since $|L| < 3\lambda = |W_l|$, L is a subsegment of W_l which implies $P_{i-1} \cap W_l \neq \emptyset$. The latter, together with Lemma 8.4 implies that the flip to 0 at l occurs by Rule 1. Then, Lemma 8.6 guarantees at least λ many 0-requests in L and at least λ many 1-requests in R are contained. Therefore, regardless of whether OPT performs a flip in P_i or not, it pays $\text{OPT}(P_i) \geq \min\{\lambda, \alpha\} = \lambda$. Using $\text{mis}(W_l), \text{mis}(W_r) \leq 2\lambda + \alpha$, we obtain

$$\frac{\text{ALG}(P_i)}{\text{OPT}(P_i)} = \frac{\text{ALG}(L) + \text{ALG}(R)}{\text{OPT}(P_i)} \leq \frac{\text{mis}(W_l) + \text{mis}(W_r) + 2\alpha}{\text{OPT}(P_i)} \leq \frac{4\lambda + 2\alpha}{\lambda}. \quad (1)$$

Both parts are long. See Figure 4a for an illustration. Regardless of OPT's actions in L , we have $\text{OPT}(U_l) \geq \text{mis}(U_l)/2$ from Corollary 8.8, and $\text{OPT}(W_t) \geq \lambda$ from Lemma 8.7. Therefore, $\text{OPT}(L) = \text{OPT}(U_l) + \text{OPT}(V_l W_l) \geq \text{mis}(U_l)/2 + \lambda$. Similarly for R , we have $\text{OPT}(R) \geq \text{mis}(U_r)/2 + \lambda$. Thus,

$$\text{OPT}(P_i) = \text{OPT}(L) + \text{OPT}(R) \geq \text{mis}(U_l)/2 + \text{mis}(U_r)/2 + 2\lambda.$$

For ALG's mismatch cost, we have $\text{mis}(\mathbf{V}_l \mathbf{W}_l) = \text{mis}(\mathbf{V}_l) + \text{mis}(\mathbf{W}_l) \leq 2\lambda - 1 + 2\lambda < 4\lambda$ and similarly $\text{mis}(\mathbf{V}_r \mathbf{W}_r) < 4\lambda$. Then, $\text{ALG}(\mathbf{P}_i) = \text{ALG}(\mathbf{L}) + \text{ALG}(\mathbf{R}) = \text{mis}(\mathbf{U}_l) + \text{mis}(\mathbf{V}_l \mathbf{W}_l) + \alpha + \text{mis}(\mathbf{U}_r) + \text{mis}(\mathbf{V}_r \mathbf{W}_r) + \alpha \leq \text{mis}(\mathbf{U}_l) + \text{mis}(\mathbf{U}_r) + 8\lambda + 2\alpha$, and

$$\frac{\text{ALG}(\mathbf{P}_i)}{\text{OPT}(\mathbf{P}_i)} \leq \frac{\text{mis}(\mathbf{U}_l) + \text{mis}(\mathbf{U}_r) + 8\lambda + 2\alpha}{\text{mis}(\mathbf{U}_l)/2 + \text{mis}(\mathbf{U}_r)/2 + 2\lambda} \leq \frac{8\lambda + 2\alpha}{2\lambda} = 4 + \frac{\alpha}{\lambda}. \quad (2)$$

Only the right part is long. Then, \mathbf{L} is a subsegment of \mathbf{W}_l (see Figure 4b) and therefore $\text{ALG}(\mathbf{L}) = \text{mis}(\mathbf{W}_l) + \alpha \leq 2\lambda + \alpha$. For \mathbf{R} , we distinguish several cases.

Case 1.1. OPT enter $\mathbf{V}_r \mathbf{W}_r$ in state 0 and serves it in state 0. In this case, Lemma 8.9.i applies which together with Corollary 8.8 yields $\text{OPT}(\mathbf{R}) = \text{OPT}(\mathbf{U}_r) + \text{mis}(\mathbf{V}_r) + \text{mis}(\mathbf{W}_r) \geq \text{mis}(\mathbf{U}_r)/2 + \text{mis}(\mathbf{V}_r) + \lambda$. Thus, we have, $\text{ALG}(\mathbf{P}_i) = \text{ALG}(\mathbf{L}) + \text{ALG}(\mathbf{R}) = (2\lambda + \alpha) + \text{mis}(\mathbf{U}_r) + \text{mis}(\mathbf{V}_r) + \text{mis}(\mathbf{W}_r) + \alpha \leq \text{mis}(\mathbf{U}_r) + \text{mis}(\mathbf{V}_r) + 4\lambda + 2\alpha$ and thereby

$$\frac{\text{ALG}(\mathbf{P}_i)}{\text{OPT}(\mathbf{P}_i)} \leq \frac{\text{mis}(\mathbf{U}_r) + \text{mis}(\mathbf{V}_r) + 4\lambda + 2\alpha}{\text{mis}(\mathbf{U}_r)/2 + \text{mis}(\mathbf{V}_r) + \lambda} \leq \frac{4\lambda + 2\alpha}{\lambda} = 4 + \frac{2\alpha}{\lambda}. \quad (3)$$

Case 1.2. OPT enters $\mathbf{V}_r \mathbf{W}_r$ in state 0 and flips to 1 in $\mathbf{V}_r \mathbf{W}_r$. In this case, Lemma 8.10.i applies which together with Corollary 8.8 yields $\text{OPT}(\mathbf{R}) = \text{OPT}(\mathbf{U}_r) + \text{OPT}(\mathbf{V}_r \mathbf{W}_r) \geq \text{mis}(\mathbf{U}_r)/2 + \min\{\text{mis}(\mathbf{V}_r), \lambda\} + \alpha$. By distinguishing the two cases $\text{mis}(\mathbf{V}_r) < \lambda$ and $\text{mis}(\mathbf{V}_r) \geq \lambda$, and using $\text{ALG}(\mathbf{P}_i) = \text{ALG}(\mathbf{L}) + \text{ALG}(\mathbf{R}) \leq (2\lambda + \alpha) + \text{mis}(\mathbf{U}_r) + \text{mis}(\mathbf{V}_r) + 2\lambda + \alpha$, we obtain

$$\begin{aligned} \frac{\text{ALG}(\mathbf{P}_i)}{\text{OPT}(\mathbf{P}_i)} &\leq \frac{\text{mis}(\mathbf{U}_r) + \text{mis}(\mathbf{V}_r) + 4\lambda + 2\alpha}{\text{mis}(\mathbf{U}_r)/2 + \min\{\text{mis}(\mathbf{V}_r), \lambda\} + \alpha} \\ &\leq \max\left\{\frac{4\lambda + 2\alpha}{\lambda}, \frac{6\lambda + 2\alpha}{\lambda + \alpha}\right\} = 4 + \frac{2\alpha}{\lambda}. \end{aligned} \quad (4)$$

Case 1.3. OPT enters $\mathbf{V}_r \mathbf{W}_r$ in state 1. If OPT serves it in state 1 then from Lemma 8.7, we have $\text{OPT}(\mathbf{V}_r \mathbf{W}_r) \geq \text{OPT}(\mathbf{W}_r) \geq \lambda$. Else, it flips to 0 and $\text{OPT}(\mathbf{V}_r \mathbf{W}_r) \geq \alpha \geq \lambda$. Therefore, in either case, $\text{OPT}(\mathbf{V}_r \mathbf{W}_r) \geq \lambda$. Since $\mathbf{P}_{i-1} \cap \mathbf{W}_l \neq \emptyset$, Lemma 8.4 implies that the flip to 0 at l occurs by Rule 1. Then, from Lemma 8.6, we have $n_0(\mathbf{L}) = n_0(\mathbf{W}_l \setminus \mathbf{P}_{i-1}) \geq \lambda$.

Next, we lower-bound OPT's cost in \mathbf{P}_i before entering $\mathbf{V}_r \mathbf{W}_r$ (i.e. in $\mathbf{L} \mathbf{U}_r$). Either OPT serves the entire \mathbf{L} in state 1 and $\text{OPT}(\mathbf{L}) = n_0(\mathbf{L}) \geq \lambda$, or it flips in \mathbf{L} at cost $\alpha \geq \lambda$. From Lemma 8.6, we have $\text{OPT}(\mathbf{U}_r) \geq \text{mis}(\mathbf{U}_r)/2$, which yields

$$\text{OPT}(\mathbf{P}_i) \geq \text{OPT}(\mathbf{L}) + \text{OPT}(\mathbf{U}_r) + \text{OPT}(\mathbf{V}_r \mathbf{W}_r) \geq \lambda + \text{mis}(\mathbf{U}_r)/2 + \lambda.$$

Using

$$\text{ALG}(\mathbf{P}_i) = \text{ALG}(\mathbf{L}) + \text{ALG}(\mathbf{R}) \leq (2\lambda + \alpha) + \text{mis}(\mathbf{U}_r) + \text{mis}(\mathbf{V}_r \mathbf{W}_r) + \alpha,$$

we obtain

$$\frac{\text{ALG}(\mathbf{P}_i)}{\text{OPT}(\mathbf{P}_i)} \leq \frac{\text{mis}(\mathbf{U}_r) + 6\lambda + 2\alpha}{\text{mis}(\mathbf{U}_r)/2 + 2\lambda} \leq \frac{6\lambda + 2\alpha}{2\lambda} = 3 + \frac{\alpha}{\lambda}. \quad (5)$$

Only the left part is long. See Figure 4c for an illustration. We distinguish cases of OPT's state when it enters $\mathbf{V}_l \mathbf{W}_l$. Note that ALG's flip to 0 at l possibly occurs by Rule 2.

Case 2.1. OPT enters $\mathbf{V}_l \mathbf{W}_l$ in state 1 and serves it in state 1. This case is symmetric to Case 1.1 and the upper bound (3) holds analogously, after swapping the usage of \mathbf{R} and \mathbf{L} , as well as 0's and 1's.

Case 2.2. OPT enters $\mathbf{V}_r\mathbf{W}_r$ in state 1 and flips to 0 later in this segment. This case is symmetric to Case 1.2 and the upper bound (4) holds analogously, after swapping the usage of \mathbf{R} and \mathbf{L} , as well as 0's and 1's.

Case 2.3. OPT enters $\mathbf{V}_l\mathbf{W}_l$ in state 0. Recall that if the flip to 0 is by Rule 2 then Lemma 8.6 does not apply and possibly $n_1(\mathbf{R}) = n_1(\mathbf{W}_r \setminus \mathbf{W}_l) < \lambda$. However, since the flip to 1 at r is by Rule 1, we have $n_1(\mathbf{W}_r) = 2\lambda$, and since \mathbf{W}_r is a subsegment of $\mathbf{W}_l\mathbf{R}$, we have $n_1(\mathbf{W}_l\mathbf{R}) \geq n_1(\mathbf{W}_r) \geq 2\lambda$. Since $\text{mis}(\mathbf{V}_l\mathbf{W}_l) < 4\lambda$, we have $\text{mis}(\mathbf{V}_l\mathbf{W}_l)/2 - \lambda < \lambda$ and hence

$$\max\{\text{mis}(\mathbf{V}_l\mathbf{W}_l)/2 - \lambda, 0\} + \lambda < 2\lambda \leq n_1(\mathbf{W}_l\mathbf{R}).$$

Either OPT serves the entire segment $\mathbf{V}_l\mathbf{W}_l\mathbf{R}$ in state 0 and

$$\text{OPT}(\mathbf{V}_l\mathbf{W}_l\mathbf{R}) \geq n_1(\mathbf{W}_l\mathbf{R}) \geq 2\lambda \geq \max\{\text{mis}(\mathbf{V}_l\mathbf{W}_l)/2 - \lambda, 0\} + \lambda,$$

or it flips to 1 in this segment. If OPT flips to 1 in \mathbf{R} then by Lemma 8.6,

$$\text{OPT}(\mathbf{V}_l\mathbf{W}_l\mathbf{R}) \geq \text{OPT}(\mathbf{W}_l) + \text{OPT}(\mathbf{R}) \geq \lambda + \alpha \geq 2\lambda \geq \max\{\text{mis}(\mathbf{V}_l\mathbf{W}_l)/2 - \lambda, 0\}.$$

Else, it flips in $\mathbf{V}_l\mathbf{W}_l$; by Lemma 8.10.ii,

$$\text{OPT}(\mathbf{V}_l\mathbf{W}_l) \geq \max\{\text{mis}(\mathbf{V}_l\mathbf{W}_l)/2 - \lambda, 0\} + \lambda.$$

Thus, in any case where OPT enters $\mathbf{V}_l\mathbf{W}_l$ in state 0, we have

$$\text{OPT}(\mathbf{V}_l\mathbf{W}_l) \geq \max\{\text{mis}(\mathbf{V}_l\mathbf{W}_l)/2 - \lambda, 0\} + \lambda.$$

By applying Corollary 8.8 to \mathbf{U}_l , we obtain

$$\text{OPT}(\mathbf{P}_i) \geq \text{OPT}(\mathbf{U}_l) + \text{OPT}(\mathbf{V}_l\mathbf{W}_l\mathbf{R}) \geq \text{mis}(\mathbf{U}_l)/2 + \max\{\text{mis}(\mathbf{V}_l\mathbf{W}_l)/2 - \lambda, 0\} + \lambda. \quad (6)$$

If $\text{mis}(\mathbf{V}_l\mathbf{W}_l) < 2\lambda$ then (6) reduces to $\text{OPT}(\mathbf{P}_i) \geq \text{mis}(\mathbf{U}_r)/2 + \lambda$. Using

$$\text{ALG}(\mathbf{P}_i) \leq \text{mis}(\mathbf{U}_l) + \text{mis}(\mathbf{V}_l\mathbf{W}_l) + \alpha + (2\lambda + \alpha) \leq \text{mis}(\mathbf{U}_l) + 4\lambda + 2\alpha,$$

we obtain

$$\frac{\text{ALG}(\mathbf{P}_i)}{\text{OPT}(\mathbf{P}_i)} \leq \frac{\text{mis}(\mathbf{U}_r) + 4\lambda + 2\alpha}{\text{mis}(\mathbf{U}_r)/2 + \lambda} \leq \frac{4\lambda + 2\alpha}{\lambda} = 4 + \frac{2\alpha}{\lambda}. \quad (7)$$

Else, $\text{mis}(\mathbf{V}_l\mathbf{W}_l) \geq 2\lambda$ holds and (6) reduces to

$$\text{OPT}(\mathbf{V}_l\mathbf{W}_l) \geq (\text{mis}(\mathbf{V}_l\mathbf{W}_l)/2 - \lambda) + \lambda.$$

Let $z := \text{mis}(\mathbf{V}_l\mathbf{W}_l)/2 - \lambda$. Then, $\text{mis}(\mathbf{V}_l\mathbf{W}_l) = 2z + 2\lambda$,

$$\text{ALG}(\mathbf{P}_i) \leq \text{mis}(\mathbf{U}_l) + \text{mis}(\mathbf{V}_l\mathbf{W}_l) + 2\lambda + 2\alpha = \text{mis}(\mathbf{U}_l) + (2z + 2\lambda) + 2\lambda + 2\alpha,$$

and

$$\frac{\text{ALG}(\mathbf{P}_i)}{\text{OPT}(\mathbf{P}_i)} \leq \frac{\text{mis}(\mathbf{U}_r) + 2z + 4\lambda + 2\alpha}{\text{mis}(\mathbf{U}_r)/2 + z + \lambda} \leq \frac{4\lambda + 2\alpha}{\lambda} = 4 + \frac{2\alpha}{\lambda}. \quad (8)$$

From all upper bounds (1)–(8), we conclude $\text{ALG}(\mathbf{P}_i)/\text{OPT}(\mathbf{P}_i) \leq 4 + \frac{2\alpha}{\lambda}$. \blacktriangleleft

► **Theorem 8.12.** *For any input sequence σ , any $T \geq 6$ and $1 \leq \lambda \leq \alpha$, we have*

$$\text{ALG}(\sigma) \leq \left(4 + \frac{2\alpha}{\lambda}\right) \text{OPT}(\sigma) + 6\alpha.$$

Proof. Assume ALG performs at least one flip in σ . Recall that in this case the input sequence σ is partitioned as $\sigma = \mathbf{P}_{\text{first}} \mathbf{P}_1 \dots \mathbf{P}_m \mathbf{P}_{\text{last}}$, where $\mathbf{P}_{\text{first}}$ is the subsequence until the first flip to 1, \mathbf{P}_{last} is the subsequence between the last flip to 1 and the end of the sequence, and each \mathbf{P}_i is the subsequence between two consecutive flips to 1. From Lemma 8.11, we have $\text{ALG}(\mathbf{P}_i) \leq (4 + 2\alpha/\lambda) \text{OPT}(\mathbf{P}_i)$. In the remainder, we upper bound the ratio separately for $\mathbf{P}_{\text{first}}$ and \mathbf{P}_{last} , as well as the case where ALG never flips.

We begin with the first part of the input, $\mathbf{P}_{\text{first}} = \sigma(0, t_{\text{first}}]$. Recall that ALG starts serving σ by outputting 0 until it flips to 1 at t_{first} for the first time. We distinguish two cases for $\mathbf{P}_{\text{first}}$.

$\mathbf{P}_{\text{first}}$ is short. In this case, $\text{ALG}(\mathbf{P}_{\text{first}}) \leq 2\lambda + \alpha$. OPT begins in state 0 and either pays $\text{mis}(\mathbf{W}_{\text{first}}) = 2\lambda$ mismatches to 1-requests in $\mathbf{W}_{t_{\text{first}}}$ or it performs a flip to 1. In any case of OPT's actions in $\mathbf{P}_{\text{first}}$, by distinguishing the two cases $\alpha < 2\lambda$ and $\alpha \geq 2\lambda$, we obtain

$$\frac{\text{ALG}(\mathbf{P}_{\text{first}})}{\text{OPT}(\mathbf{P}_{\text{first}})} \leq \frac{\text{mis}(\mathbf{W}_{t_{\text{first}}}) + \alpha}{\min\{2\lambda, \alpha\}} \leq \frac{2\lambda + \alpha}{\min\{2\lambda, \alpha\}} \leq \max\left\{4, \frac{\alpha}{\lambda}\right\}. \quad (9)$$

$\mathbf{P}_{\text{first}}$ is long. Consider the partitioning $\mathbf{P}_{\text{first}} = \mathbf{U}\mathbf{V}\mathbf{W}$, where $|\mathbf{U}|$ is a multiple of 3λ , $|\mathbf{V}| < 3\lambda$ and $\mathbf{W} = \mathbf{W}_{t_{\text{first}}}$. If OPT serves the entire $\mathbf{V}\mathbf{W}$ in one state (either 0 or 1) then $\text{OPT}(\mathbf{V}\mathbf{W}) \geq \text{OPT}(\mathbf{W}) \geq \lambda$ (from Lemma 8.7). Otherwise, OPT flips in $\mathbf{V}\mathbf{W}$ and $\text{OPT}(\mathbf{V}\mathbf{W}) \geq \alpha \geq \lambda$. After applying Lemma 8.8 to \mathbf{U} and using $\text{ALG}(\mathbf{P}_{\text{first}}) = \text{mis}(\mathbf{U}) + \text{mis}(\mathbf{V}\mathbf{W}) + \alpha \leq \text{mis}(\mathbf{U}) + 4\lambda + \alpha$, we obtain

$$\frac{\text{ALG}(\mathbf{P}_{\text{first}})}{\text{OPT}(\mathbf{P}_{\text{first}})} \leq \frac{\text{mis}(\mathbf{U}) + \text{mis}(\mathbf{V}) + 4\lambda + \alpha}{\text{mis}(\mathbf{U})/2 + \lambda} \leq \frac{4\lambda + \alpha}{\lambda} = 4 + \frac{\alpha}{\lambda}. \quad (10)$$

Lastly, we bound costs for $\mathbf{P}_{\text{last}} = \sigma(r_m, n]$ as follows. If $|\mathbf{P}_{\text{last}}| < 3\lambda$ then ALG pays up to 3λ mismatches and possibly performs a last flip (to 0) at $t_m \leq |\sigma|$. Using $\lambda \leq \alpha$, we obtain $\text{ALG}(\mathbf{P}_{\text{last}}) \leq |\mathbf{P}_{\text{last}}| + \alpha \leq 3\lambda + \alpha \leq 4\alpha$.

Else, $|\mathbf{P}_{\text{last}}| \geq 3\lambda$. Consider the partitioning $\mathbf{P}_{\text{last}} = \mathbf{U}'\mathbf{V}'$ where $|\mathbf{U}'|$ is a multiple of 3λ and $|\mathbf{V}'| < 3\lambda$. ALG possibly flips to 0 one last time at t_{last} , in a segment \mathbf{S} that is either the segment \mathbf{V}' or a block of \mathbf{U}' . In either case, $\text{mis}(\mathbf{S}) \leq |\mathbf{S}| \leq 3\lambda$ and $\text{ALG}(\mathbf{S}) = \text{mis}(\mathbf{S}) + \alpha \leq 3\lambda + \alpha \leq 4\alpha$. Using $\text{mis}(\mathbf{V}') < 2\lambda \leq 2\alpha$ and Corollary 8.8, we obtain

$$\text{ALG}(\mathbf{P}_{\text{last}}) = \text{ALG}(\mathbf{U}' \setminus \mathbf{S}) + \text{ALG}(\mathbf{S}) + \text{mis}(\mathbf{V}') \leq \text{ALG}(\mathbf{U}' \setminus \mathbf{S}) + 6\alpha. \quad (11)$$

By an argument similar to that of \mathbf{P}_{last} , the bound (11) holds also for the case ALG never flips in σ , as ALG does not incur any flipping cost.

Combining our bounds. From the upper bounds (9), (10), (11), and by applying Lemma 8.11 to each phase \mathbf{P}_i , we conclude $\text{ALG}(\sigma) \leq (4 + \frac{2\alpha}{\lambda}) \text{OPT}(\sigma) + 6\alpha$, where the additive is a consequence of (11). ◀

Since $\lambda = \min\{T/6, \alpha\}$, we obtain the following ratios for small and large values of T separately.

► **Corollary 8.13.** *The time-local algorithm ALG is c -competitive, where $c = 6$ for $T \geq 6\alpha$, $c = 4 + \frac{12\alpha}{T}$ for $6 \leq T \leq 6\alpha$, and $c = 4 + 2\alpha$ for $1 \leq T \leq 6$.*

8.4 Discussion on the Answer Set Choice

Alternatively to using request-answer games, some online problems are often more convenient to formulate as *task systems* [14]. A task system consists of a set of configurations, transition costs between configurations and the processing cost of each type of request in every possible configuration. In this case, the output is a sequence of configurations chosen by the algorithm, and the cost of a solution is the sum of all transition and processing costs incurred. An online problem given by a task system does not in general have a unique encoding as a request-answer game, and vice versa. In this work we operate in the request-answer game framework, and unless otherwise specified, we typically assume that algorithms output the current configuration, that is, the set Y of output values coincides with (some encoding of) the configurations.

The lower bound discussed in Section 8.1 assumes a particular encoding of the online file migration as a request-answer game. Namely, the answer set is coincident with the set of nodes in the network, and is equivalent to the algorithm's configuration (placement of the file).

Although this encoding is natural, it causes certain types of problems — when faced with a visible horizon with requests scattered over various nodes, without a clear majority, it still must uniquely determine the file location. Dealing with such situations involves e.g. distinguishing a default configuration in case there's no clear majority in the visible horizon, to avoid excessive file migration.

If we consider a different set of answers, the competitive ratio of the problem may be improved. This is in contrast to the classic algorithms setting, where the competitive ratio is indifferent to the problem encoding.

Consider an additional answer “do not move the file”, denoted SKIP, that instructs the file to stay in its current location. Note that this answer is configuration-dependent, and we may need to track an arbitrarily long sequence of answers to determine the actual file location. For this reason, it is impossible to encode the online problem as a local problem (cf. Section 3.1).

With this answer set, the lower bound from Section 8.1 no longer holds. In the remainder of this section, we adapt a classic randomized algorithm by Westbrook [44] to the time-local setting, and we show that it is 3-competitive against the adaptive online adversary even with $T = 1$, i.e., the access to the last request is sufficient.

Let the Behavioral Coin Flip algorithm be defined as follows. Upon receiving a request from any node, we move the file to this node with probability $1/(2\alpha)$, and with probability $1 - 1/(2\alpha)$ we keep the file in its previous location.

► **Theorem 8.14.** *The Behavioral Coin Flip algorithm with $T = 1$ is 3-competitive against the adaptive online adversary for online file migration encoded with the answer set including SKIP. Furthermore, no algorithm can obtain a competitive ratio below 3 against the adaptive online adversary.*

An elegant proof of the first part of the theorem was given by Westbrook [44]. The second part follows by adapting the lower bound of 3 for deterministic algorithms [6] to the adaptive online setting, and uses an important technique of averaging the cost of 3 offline algorithms.

8.5 Transferring Results from Classic Online Algorithms

The lower bounds for classic online algorithms imply lower bounds for time-local algorithms. We first review classic results and algorithms for online file migration, and then provide insights into the case $\alpha < 1$ in the classic online setting (it is usually assumed that $\alpha \geq 1$).

We study a variant of online file migration in networks consisting of 2 nodes [9]. For this problem, a 3-competitive deterministic algorithm can be obtained using a work function algorithm for metrical task system [14]. The result is tight: a lower bound of 3 holds even for 2 node networks [11], and it uses the technique of averaging costs of multiple offline algorithms, introduced in [32]. In the randomized setting, the threshold work function algorithm obtains the competitive ratio that approaches $\frac{2e-1}{e-1} \approx 2.581$ as the length of the input sequence grows [30].

More generally, it is known that online file migration in arbitrary networks admits a 4-competitive deterministic algorithm [10]. The best known lower bound for deterministic algorithms is $3 + \Omega(1)$ and requires 4 nodes [34]. Randomized $(1 + \phi)$ -competitive and 3-competitive algorithms exist against the oblivious offline adversary and the adaptive online adversary, respectively [44], where $\phi \approx 1.6$ is the golden ratio. The result against the adaptive online adversary is tight: there is a lower bound of 3 [6]. Against the oblivious offline adversary, a lower bound of $2 + \frac{1}{2\alpha}$ exists.

Next, we present a lower bound that holds for the classic variant, and hence for the time-local setting as well.

► **Theorem 8.15.** *Consider any deterministic online algorithm A for online file migration with file size α . If A is c -competitive, then $c \geq 1 + 1/\alpha$ for $\alpha \in (0, 1/2]$, and $c \geq \min\{2 + 2\alpha, 1 + 3/(2\alpha)\}$ for $\alpha \in (1/2, 1)$.*

Proof. Consider an input sequence σ_L for any $L \in \mathbb{N}$, constructed in the following way. We start by issuing 1-requests until A migrates the file to node 1. Then, we proceed by issuing 0-requests until A migrates the file to the node 0. We repeat these steps L times. Note that A must eventually perform a migration, otherwise it is not competitive (an optimal offline algorithm pays at most $2\alpha \cdot L$ for σ_L). In the remainder of the proof, we assume that A eventually performs a migration, and consequently σ_L is finite.

We partition σ_L into phases P_1, \dots, P_L in the following way. The first phase begins with the first request and each phase ends when ALG migrates the file to 0. We analyze the ratio of A to OPT on each phase separately. For any $i \leq L$, consider the i th phase $P := P_i$. Let x be the number of 1-requests and y be the number of 0-requests in P . Then $x, y \geq 1$ and $x + y \geq 2$. Recall that A first serves a request and then decides whether to migrate the file or not. Consequently, it incurs the cost 2 in each phase for serving requests remotely, and performs two migrations, and its total cost is $x + y + 2\alpha \geq 2 + 2\alpha$.

Let OPT be any optimal offline solution. Note that OPT never pays more than 2α in any phase: it can always migrate the file to 1 prior to serving all 1-requests for free, and then to 0 prior to serving all 0-requests for free. Thus, for any $\alpha > 0$, we have

$$\frac{\text{ALG}(\sigma_L)}{\text{OPT}(\sigma_L)} \geq \min_i \frac{\text{ALG}(P_i)}{\text{OPT}(P_i)} \geq (2 + 2\alpha)/2\alpha = 1/\alpha + 1.$$

Next, we provide a stronger bound when $1/2 < \alpha < 1$, by distinguishing two cases.

Case 1. OPT has the file at the node 0 when it enters the phase. If $x = 1$ then OPT does not benefit by migrating the file, as otherwise, it would incur for 0-requests in addition to α . Therefore, it pays 1 for serving the (single) 1-request remotely, and the ratio is $(2 + 2\alpha)/1$. Else, $x \geq 2 > 2\alpha$, and ALG pays $x + y + 2\alpha \geq 3 + 2\alpha$. Since OPT never pays more than 2α for any phase, we have $\text{ALG}(P)/\text{OPT}(P) \geq (3 + 2\alpha)/2\alpha = 3/2\alpha + 1$.

Case 2. OPT has the file at the node 1 when it enters the phase. OPT serves all 1-requests in the phase for free. If $y = 1$, then either OPT serves the 0-request remotely without migrating the file, paying 1, or it migrates the file and pays $\alpha < 1$. In either case, it pays

at most 1 and $\text{ALG}(P)/\text{OPT}(P) \geq (2+2\alpha)/1$. Else, $y \geq 2$, and OPT migrates the file to the node 0 before serving the 0-requests; as otherwise it would incur $y \geq 2 > 2\alpha$, more than migrating the file twice. Since $x+y \geq 3$, we have $\text{ALG}(P)/\text{OPT}(P) \geq (3+2\alpha)/\alpha = 3/\alpha + 2$. Hence, in all cases for $\alpha \in (1/2, 1)$, we have $\text{ALG}(P)/\text{OPT}(P) \geq \min\{2+2\alpha, 1+3/(2\alpha)\}$, and consequently for all inputs σ_L for any $L \in \mathbb{N}$ we have

$$\frac{\text{ALG}(\sigma_L)}{\text{OPT}(\sigma_L)} \geq \min_i \frac{\text{ALG}(P_i)}{\text{OPT}(P_i)} \geq \min\{2+2\alpha, 1+3/(2\alpha)\}.$$

By combining the results for $\alpha \in (0, 1/2]$ and $\alpha \in (1/2, 1)$, we conclude the lemma. \blacktriangleleft

A lower bound of 3 is presented in [11] for $\alpha \geq 1$, and we note that it holds also for $\alpha < 1$. We summarize all known lower bounds in the following corollary.

► **Corollary 8.16.** *No deterministic classic online algorithm for online file migration can achieve a competitive ratio less than $\max\{3, 1+1/\alpha\}$, for $\alpha > 0$, or less than $\min\{2+2\alpha, 1+3/(2\alpha)\}$ for $\alpha \in (0.5, 1)$.*

9 Conclusions

In this work, we initiated the study of time-local online algorithms, and its connections with distributed graph algorithms. We used a special case of the online file migration problem as a running example. We saw that even this simple problem already exhibits a wide range of behaviors in the time-local setting, and identified new questions that need further study, also for classic online algorithms; among them is the analysis of the problem for the range $1/2 < \alpha < 1$ and how different types of randomized time-local algorithms relate to each other and to the existence of competitive deterministic time-local algorithms.

Beyond these, there is a wide variety of online problems that would be a good fit with the framework of time-local algorithms, and where the approach of synthesizing optimal algorithms may similarly lead to new discoveries. Moreover, our work suggests also new directions for the study of local distributed graph algorithms. Clocked time-local algorithms are a natural application for distributed algorithms in the supported **LOCAL** model and its slightly weaker variant, numbered **LOCAL** model, introduced here, and the capabilities of these models in distributed optimization are not fully understood yet.

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