Neural networks and isogeometric analysis

Maciej Paszyński
home.agh.edu.pl/paszynsk
e-mail: paszynsk@agh.edu.pl

a2s.agh.edu.pl
Institute of Computer Science
AGH University of Science and Technology, Kraków, Poland

Physics Informed Neural Networks
Variational Physics Informed Neural Networks

M. Raissi, P. Perdikaris, G.E.Karniadakis, Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear PDEs, **Journal of Computational Physics** 378(1) (2019)

E. Kharazmi, Z. Zhang, G. E. Karniadakis, Variational Physics-Informed Neural Networks For Solving PDEs **arxiv.org/abs/1912.00873** (2019)

K. Doległo, A. Paszyńska, M. Paszyński, L. Demkowicz, Deep neural networks for smooth approximation of physics with higher order and continuity B-spline base functions, **https://arxiv.org/abs/2201.00904** (2022)

Physics Informed Neural Networks for strong form of PDE

Strong form: Find $u \in C^2(0,1)$:

$$\underbrace{-\epsilon \frac{d^2 u(x)}{dx^2}}_{\text{diffusion}=\epsilon} \underbrace{+1 \frac{du(x)}{dx}}_{\text{advection"wind"}=1} = 1, x \in (0,1), -\epsilon \frac{du}{dx}(0) + u(0) = 1.0, \ u(1) = 0$$

Now, neural network IS the solution
$$u(x) = NN(x) = A_n \sigma \left(A_{n-1}\sigma(...\sigma(A_1 x + B_1)... + B_{n-1}) + B_n \right.$$

$$LOSS_{PDE}(x) = (-\epsilon \frac{d^2 NN(x)(x)}{dx^2} + \frac{d NN(x)(x)}{dx} - 1.0)^2,$$

$$LOSS_{BC0} = (-\epsilon \frac{d NN(0)}{dx} + NN(0) - 1.0)^2,$$

$$LOSS_{BC1} = (NN(1)s)^2,$$

$$LOSS(x) = (LOSS_{PDE}(x)^2 + LOSS_{BC0}^2 + LOSS_{BC1}^2).$$

1. $\epsilon = 1.0$, $\#N$ = number of points = 10
   (selected randomly at beginning of each epoch)
   - One epoch is: random selection of 10 points, evaluation of LOSS functions at the 10 points. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

2. $\epsilon = 1.0$, $\#N$ = number of points = 100
   (selected randomly at beginning of each epoch)
   - One epoch is: random selection of 100 points, evaluation of LOSS functions at the 100 points. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon = 1.0$, $\#N =$ number of points $= 1000$
   (selected randomly at beginning of each epoch)
   - One epoch is: random selection of 1000 points, evaluation of LOSS functions at the 1000 points. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0,1)$, vertical axis = $NN(x)$).

1. $\epsilon = 0.1$, $\#N$ = number of points = 10

   (selected randomly at beginning of each epoch)

   - One epoch is: random selection of 10 points, evaluation of LOSS functions at the 10 points. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

2. $\epsilon = 0.1$, $\#N$ = number of points = 100

   (selected randomly at beginning of each epoch)

   - One epoch is: random selection of 100 points, evaluation of LOSS functions at the 100 points. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon = 0.1$, $\#N$ = number of points = 1000
   (selected randomly at beginning of each epoch)
   - One epoch is: random selection of 1000 points, evaluation of LOSS functions at the 1000 points. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon = 0.01$, $\#N =$ number of points $= 10$
   (selected randomly at beginning of each epoch)

   - One epoch is: random selection of 10 points, evaluation of LOSS functions at the 10 points. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

2. $\epsilon = 0.01$, $\#N =$ number of points $= 100$
   (selected randomly at beginning of each epoch)

   - One epoch is: random selection of 100 points, evaluation of LOSS functions at the 100 points. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon = 0.01$, $\#N =$ number of points $= 1000$
   (selected randomly at beginning of each epoch)
   - One epoch is: random selection of 1000 points, evaluation of LOSS functions at the 1000 points. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon = 1.0$, $\#N =$ number of points $= 10$ (now, 10 points are distributed as x_i, where $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (0, 0.5, 0.75, 0,875, 0,9375, 0,96875, 0,984375, 0,9921875, 0,99609375, 1.0)

   - One epoch is: 10 adaptive points, evaluation of LOSS functions at the 10 points. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon = 1.0$, $\#N$ = number of points = 100 (now, 100 points are distributed as x_i, where $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (99 intervals) 0, 0.5, 0.75, 0,875, 0,9375, 0,96875, 0,984375, 0,9921875, 0,99609375, 0,998046875, ..., 1.0)

   - One epoch is: 100 adaptive points, evaluation of LOSS functions at the 100 points. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0,1)$, vertical axis = $NN(x)$).

1. $\epsilon = 1.0$, $\#N =$ number of points $= 1000$ (now, 1000 points are distributed as x_i, where $x_0 = 0$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (999 intervals) (0, 0.5, 0.75, 0,875, 0,9375, 0,96875, 0,984375, 0,9921875, 0,99609375, 0,998046875, 1.0)

   - One epoch is: 1000 adaptive points, evaluation of LOSS functions at the 1000 points. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon = 0.1$, $\#N$ = number of points = 10 (now, 10 points are distributed as x_i, where $x_0 = 0, x_1 = \frac{1}{2}, x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (0, 0.5, 0.75, 0,875, 0,9375, 0,96875, 0,984375, 0,9921875, 0,99609375, 1.0)
   - One epoch is: 10 adaptive points, evaluation of LOSS functions at the 10 points. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon = 0.1$, $\#N =$ number of points $= 100$ (now, 100 points are distributed as x_i, where $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (99 intervals) 0, 0.5, 0.75, 0,875, 0,9375, 0,96875, 0,984375, 0,9921875, 0,99609375, 0,998046875, ..., 1.0)

   - One epoch is: 100 adaptive points, evaluation of LOSS functions at the 100 points. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon = 0.1$, $\#N =$ number of points $= 1000$ (now, 1000 points are distributed as x_i, where $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (999 intervals) (0, 0.5, 0.75, 0,875, 0,9375, 0,96875, 0,984375, 0,9921875, 0,99609375, 0,998046875, 1.0)

   - One epoch is: 1000 adaptive points, evaluation of LOSS functions at the 1000 points. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon = 0.01$, $\#N$ = number of points = 10 (now, 10 points are distributed as x_i, where $x_0 = 0, x_1 = \frac{1}{2}, x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (0, 0.5, 0.75, 0,875, 0,9375, 0,96875, 0,984375, 0,9921875, 0,99609375, 1.0)
   - One epoch is: 10 adaptive points, evaluation of LOSS functions at the 10 points. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon = 0.01$, $\#N$ = number of points = 100 (now, 100 points are distributed as x_i, where $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (99 intervals) 0, 0.5, 0.75, 0,875, 0,9375, 0,96875, 0,984375, 0,9921875, 0,99609375, 0,998046875, ..., 1.0)
   - One epoch is: 100 adaptive points, evaluation of LOSS functions at the 100 points. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0,1)$, vertical axis = $NN(x)$).

1. $\epsilon = 0.01$, $\#N =$ number of points $= 1000$ (now, 1000 points are distributed as x_i, where $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (999 intervals) (0, 0.5, 0.75, 0,875, 0,9375, 0,96875, 0,984375, 0,9921875, 0,99609375, 0,998046875, 1.0)

   - One epoch is: 1000 adaptive points, evaluation of LOSS functions at the 1000 points. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

Variational Physics Informed Neural Networks for weak form of PDE

Weak form: Find $u \in H^1(0,1)$:

$$\int_0^1 \epsilon \frac{du(x)}{dx}\frac{dv(x)}{dx}dx + \int_0^1 \frac{du(x)}{dx}v(x)dx + u(0)v(0) = \int_0^1 v(x)dx + v(0)$$

$$\forall v \in V$$

Now, neural network IS the solution

$$u(x) = NN(x) = A_n\sigma\left(A_{n-1}\sigma(...\sigma(A_1x + B_1)... + B_{n-1}) + B_n\right.$$

$$b_{weak}(v) = \int \left(\epsilon\frac{dNN(x)}{dx}\frac{dv}{dx} + \frac{dNN(x)}{dx}v\right)dx + NN(0)v(0),$$

$$l_{weak}(v) = \int_0^1 NN(x)dx + v(0), \quad LOSS_{weak}(v) = (b_{weak}(v) - l_{weak}(v))^2,$$

$$LOSS_{BC0} = \left(-\epsilon\frac{dNN(0)}{dx} + NN(0) - 1.0\right)^2, \quad LOSS_{BC1} = (NN(1))^2,$$

$$LOSS(v) = LOSS_{strong}(x) + LOSS_{weak} + LOSS_{BC0} + LOSS_{BC1}.$$

1. $\epsilon = 1.0$, $\#N$ = number of intervals = 10 [0 0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0 1.0] (linear B-splines based on 11 knot points)
   - One epoch is: test with 11 test functions build from 11 knot points, evaluation of LOSS functions with 11 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

2. $\epsilon = 1.0$, $\#N$ = number of intervals = 100 [0 0 0.01 0.02 0.03 ... 0.98 0.99 1.0 1.0] (linear B-splines based on 101 knot points)
   - One epoch is: test with 101 test functions build from 101 knot points, evaluation of LOSS functions with 101 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon = 1.0$, $\#N$ = number of intervals = 1000 [0 0 0.001 0.002 0.003 ... 0.998 0.999 1.0 1.0] (linear B-splines based on 1001 knot points)
   - One epoch is: test with 1001 test functions build from 1001 knot points, evaluation of LOSS functions with 1001 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon = 1.0$, $\#N$ = number of intervals = 10 [0 0 0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0 1.0 1.0] (quadratic B-splines based on 11 knot points)
   - One epoch is: test with 12 test functions build from 11 knot points, evaluation of LOSS functions with 12 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

2. $\epsilon = 1.0$, $\#N$ = number of intervals = 100 [0 0 0 0.01 0.02 0.03 ... 0.98 0.99 1.0 1.0 1.0] (quadratic B-splines based on 101 knot points)
   - One epoch is: test with 102 test functions build from 101 knot points, evaluation of LOSS functions with 102 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon = 1.0$, $\#N$ = number of intervals = 1000 [0 0 0 0.001 0.002 0.003 ... 0.998 0.999 1.0 1.0 1.0] (quadratic B-splines based on 1002 knot points)

   - One epoch is: test with 1002 test functions build from 1002 knot points, evaluation of LOSS functions with 1002 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon = 1.0$, $\#N =$ number of intervals $= 10$ [0 0 0 0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0 1.0 1.0 1.0] (cubic B-splines based on 11 knot points)
   - One epoch is: test with 11 test functions build from 13 knot points, evaluation of LOSS functions with 13 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

2. $\epsilon = 1.0$, $\#N =$ number of intervals $= 100$ [0 0 0 0 0.01 0.02 0.03 ... 0.98 0.99 1.0 1.0 1.0 1.0] (cubic B-splines based on 101 knot points)
   - One epoch is: test with 103 test functions build from 101 knot points, evaluation of LOSS functions with 103 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best

1. $\epsilon = 1.0$, $\#N =$ number of intervals $= 1000$ [0 0 0 0 0.001 0.002 0.003 ... 0.998 0.999 1.0 1.0 1.0 1.0] (cubic B-splines based on 1003 knot points)

   - One epoch is: test with 1003 test functions build from 1003 knot points, evaluation of LOSS functions with 1003 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon = 0.1$, $\#N$ = number of points = 10 (now, 11 knot points are distributed as x_i, where $x_0 = 0, x_1 = \frac{1}{2}, x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. [0 0 0.5 0.75 0.875 0.9375 0.96875 0.984375 0.9921875 0,99609375 0,998046875 1.0 1.0] We use linear B-splines

   - One epoch is: 11 linear B-splines, evaluation of LOSS functions with 11 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0,1)$, vertical axis = $NN(x)$).

1. $\epsilon = 0.1$, $\#N$ = number of points = 100 (now, 101 knot points are distributed as x_i, where $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (99 intervals) [0 0 0.5 0.75 0.875 0.9375 0.96875 0.984375 0.9921875 0.99609375 0.998046875 ... 1.0 1.0] We use linear B-splines
   - One epoch is: 101 linear B-splines, evaluation of LOSS functions with 101 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

2. $\epsilon = 0.1$, $\#N$ = number of points = 1000 (now, 1001 test functions are distributed as x_i, where $x_0 = 0$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (999 intervals) [0 0 0.5 0.75 0.875 0.9375 ... 1.0 1.0] We use linear B-splines
   - One epoch is: 1001 linear B-splines, evaluation of LOSS functions at the 1001 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It

1. $\epsilon = 0.1$, $\#N =$ number of points $= 10$ (now, 11 knot points are distributed as x_i, where $x_0 = 0, x_1 = \frac{1}{2}, x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. [0 0 0 0.5 0.75 0.875 0.9375 0.96875 0.984375 0.9921875 0.99609375 0.998046875 1.0 1.0 1.0] We use quadratic B-splines

   - One epoch is: 12 quadratic B-splines , evaluation of LOSS functions with 12 test functions. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon = 0.1$, $\#N$ = number of points = 101 (now, 101 knot points are distributed as x_i, where $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (100 intervals) [0 0 0 0.5 0.75 0.875 0.9375 ... 1.0 1.0 1.0] We use quadratic B-splines
   - One epoch is: 102 quadratic B-splines, evaluation of LOSS functions with 102 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0,1)$, vertical axis = $NN(x)$).

2. $\epsilon = 0.1$, $\#N$ = number of points = 1001 (now, 1001 knot points are distributed as x_i, where $x_0 = 0$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (999 intervals) [0 0 0 0.5 0.75 0.875 0.9375 ... 1.0 1.0 1.0] We use quadratic B-splines
   - One epoch is: 1002 quadratic B-splines, evaluation of LOSS functions with 1002 test functions. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It

1. $\epsilon = 0.1$, $\#N$ = number of points = 10 (now, 11 knot points are distributed as x_i, where $x_0 = 0, x_1 = \frac{1}{2}, x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. [0 0 0 0 0.5 0.75 0.875 0.9375 0.96875 0.984375 0.9921875 0.99609375 0.998046875 1.0 1.0 1.0 1.0] We use cubic B-splines

   - One epoch is: 13 cubic B-splines , evaluation of LOSS functions with 13 test functions. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon = 0.1$, $\#N$ = number of points = 101 (now, 101 knot points are distributed as x_i, where $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (100 intervals) [0 0 0 0 0.5 0.75 0.875 0.9375 ... 1.0 1.0 1.0 1.0] We use cubic B-splines
   - One epoch is: 103 cubic B-splines, evaluation of LOSS functions with 103 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0,1)$, vertical axis = $NN(x)$).

2. $\epsilon = 0.1$, $\#N$ = number of points = 1001 (now, 1001 knot points are distributed as x_i, where $x_0 = 0$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (999 intervals) [0 0 0 0 0.5 0.75 0.875 0.9375 ... 1.0 1.0 1.0 1.0] We use cubic B-splines
   - One epoch is: 1003 cubic B-splines, evaluation of LOSS functions with 1003 test functions. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It

1. $\epsilon = 0.01$, $\#N$ = number of points = 10 (now, 11 knot points are distributed as x_i, where $x_0 = 0, x_1 = \frac{1}{2}, x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. [0 0 0.5 0.75 0.875 0.9375 0.96875 0.984375 0.9921875 0,99609375 0,998046875 1.0 1.0] We use linear B-splines
   - One epoch is: 11 linear B-splines, evaluation of LOSS functions with 11 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon = 0.01$, $\#N$ = number of points = 100 (now, 101 knot points are distributed as x_i, where $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (99 intervals) [0 0 0.5 0.75 0.875 0.9375 0.96875 0.984375 0.9921875 0.99609375 0.998046875 ... 1.0 1.0] We use linear B-splines

   - One epoch is: 101 linear B-splines, evaluation of LOSS functions with 101 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

2. $\epsilon = 0.01$, $\#N$ = number of points = 1000 (now, 1001 test functions are distributed as x_i, where $x_0 = 0$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (999 intervals) [0 0 0.5 0.75 0.875 0.9375 ... 1.0 1.0] We use linear B-splines

   - One epoch is: 1001 linear B-splines, evaluation of LOSS functions at the 1001 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It

1. $\epsilon = 0.01$, $\#N$ = number of points = 10 (now, 11 knot points are distributed as x_i, where $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. [0 0 0 0.5 0.75 0.875 0.9375 0.96875 0.984375 0.9921875 0.99609375 0.998046875 1.0 1.0 1.0] We use quadratic B-splines
   - One epoch is: 12 quadratic B-splines , evaluation of LOSS functions with 12 test functions. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0,1)$, vertical axis = $NN(x)$).

1. $\epsilon = 0.01$, $\#N$ = number of points = 101 (now, 101 knot points are distributed as x_i, where $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (100 intervals) [0 0 0 0.5 0.75 0.875 0.9375 ... 1.0 1.0 1.0] We use quadratic B-splines
   - One epoch is: 102 quadratic B-splines, evaluation of LOSS functions with 102 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

2. $\epsilon = 0.01$, $\#N$ = number of points = 1001 (now, 1001 knot points are distributed as x_i, where $x_0 = 0$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (999 intervals) [0 0 0 0.5 0.75 0.875 0.9375 ... 1.0 1.0 1.0] We use quadratic B-splines
   - One epoch is: 1002 quadratic B-splines, evaluation of LOSS functions with 1002 test functions. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It

1. $\epsilon = 0.01$, $\#N =$ number of points $= 10$ (now, 11 knot points are distributed as x_i, where $x_0 = 0, x_1 = \frac{1}{2}, x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. [0 0 0 0 0.5 0.75 0.875 0.9375 0.96875 0.984375 0.9921875 0.99609375 0.998046875 1.0 1.0 1.0 1.0] We use cubic B-splines

    - One epoch is: 13 cubic B-splines , evaluation of LOSS functions with 13 test functions. Please run 100 epochs.
    - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
    - Please measure the total time of training
    - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon = 0.01$, $\#N$ = number of points = 101 (now, 101 knot points are distributed as x_i, where $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (100 intervals) [0 0 0 0 0.5 0.75 0.875 0.9375 ... 1.0 1.0 1.0 1.0] We use cubic B-splines
   - One epoch is: 103 cubic B-splines, evaluation of LOSS functions with 103 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

2. $\epsilon = 0.01$, $\#N$ = number of points = 1001 (now, 1001 knot points are distributed as x_i, where $x_0 = 0$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (999 intervals) [0 0 0 0 0.5 0.75 0.875 0.9375 ... 1.0 1.0 1.0 1.0] We use cubic B-splines
   - One epoch is: 1003 cubic B-splines, evaluation of LOSS functions with 1003 test functions. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It

Weak form: Find $u \in H^1(0, 1)$:

$$\int_0^1 \epsilon \frac{du(x)}{dx} \frac{dv(x)}{dx} dx + \int_0^1 \frac{du(x)}{dx} v(x) dx + u(0)v(0) = \int_0^1 v(x) dx + v(0)$$

$$\forall v \in V$$

Now, we plug a combination of B-splines in place of the solution
$u(x) = \sum_{i=1,\dots,N} u_i B_{i,p}(x)$

$$b_{weak}(v) = \int \left(\epsilon \sum_i u_i \frac{dB_{i,p}(x)}{dx} \frac{dv}{dx} + \sum_i u_i \frac{dB_{i,p}(x)}{dx} v \right) dx + u_1 v(0),$$

$$l_{weak}(v) = \int_0^1 \sum_i u_i B_{i,p} dx + v(0), LOSS_{weak}(v) = (b_{weak}(v) - l_{weak}(v))^2,$$

$$LOSS_{BC0} = (u_N - 1.0)^2, [\text{since } du_1/dx == 0] \quad LOSS_{BC1} = (u_1)^2,$$

$$LOSS(v) = LOSS_{strong}(x) + LOSS_{weak} + LOSS_{BC0} + LOSS_{BC1}.$$

All the experiments like in VPINN

Parametric-IGA-VPINN

Weak form: Find $u \in H^1(0,1)$:

$$\int_0^1 \epsilon \frac{d\,u(x)}{dx}\frac{d\,v(x)}{dx}dx + \int_0^1 \frac{d\,u(x)}{dx}v(x)dx + u(0)v(0) = \int_0^1 v(x)dx + v(0)$$

$$\forall v \in V$$

Now, NN input is ϵ, NN output are N coefficients of B-splines

$$\epsilon \to NN(\epsilon)_i \quad u(x) = \sum_{i=1,\ldots,N} NN(\epsilon)_i B_{i,p}(x)$$

$$b_{weak}(v) = \int \left(\epsilon \sum_i NN(\epsilon)_i \frac{d B_{i,p}(x)}{dx}\frac{dv}{dx} + \sum_i NN(\epsilon)_i \frac{d B_{i,p}(x)}{dx}v \right) dx +$$

$$NN(\epsilon)_1 v(0), \quad l_{weak}(v) = \int_0^1 \sum_i NN(\epsilon)_i B_{i,p}dx + v(0)$$

$$LOSS_{weak}(v) = (b_{weak}(v) - l_{weak}(v))^2,$$

$$LOSS_{BC0} = (NN(\epsilon)_N - 1.0)^2, \quad LOSS_{BC1} = (NN(\epsilon)_1)^2,$$

$$LOSS(v) = LOSS_{strong}(x) + LOSS_{weak} + LOSS_{BC0} + LOSS_{BC1}.$$

1. $\epsilon \in (0.01, 1.0)$, $\#N =$ number of intervals $= 10$ [0 0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0 1.0] (linear B-splines based on 11 knot points)
   - One epoch is: test with 11 test functions build from 11 knot points, evaluation of LOSS functions with 11 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solutions for $\epsilon = 0.01$, for $\epsilon = 0.1$, and for $\epsilon = 1.0$ (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon \in (0.01, 1.0)$, $\#N =$ number of intervals $= 100$ [0 0 0.01 0.02 0.03 ... 0.98 0.99 1.0 1.0] (linear B-splines based on 101 knot points)
   - One epoch is: test with 101 test functions build from 101 knot points, evaluation of LOSS functions with 101 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution for $\epsilon = 0.01$, for $\epsilon = 0.1$, and for $\epsilon = 1.0$ (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon \in (0.01, 1.0)$, $\#N$ = number of intervals = 1000 [0 0 0.001 0.002 0.003 ... 0.998 0.999 1.0 1.0] (linear B-splines based on 1001 knot points)
   - One epoch is: test with 1001 test functions build from 1001 knot points, evaluation of LOSS functions with 1001 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution for $\epsilon = 0.01$, for $\epsilon = 0.1$, and for $\epsilon = 1.0$ (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon \in (0.01, 1.0)$, $\#N$ = number of intervals = 10 [0 0 0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0 1.0 1.0] (quadratic B-splines based on 11 knot points)

   - One epoch is: test with 12 test functions build from 11 knot points, evaluation of LOSS functions with 12 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solutions for $\epsilon = 0.01$, for $\epsilon = 0.1$, and for $\epsilon = 1.0$ (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon \in (0.01, 1.0)$, $\#N$ = number of intervals = 100 [0 0 0 0.01 0.02 0.03 ... 0.98 0.99 1.0 1.0 1.0] (quadratic B-splines based on 101 knot points)

    - One epoch is: test with 102 test functions build from 101 knot points, evaluation of LOSS functions with 102 test functions. Please run 100 epochs.
    - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
    - Please measure the total time of training
    - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution for $\epsilon = 0.01$, for $\epsilon = 0.1$, and for $\epsilon = 1.0$ (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon \in (0.01, 1.0)$, $\#N$ = number of intervals = 1000 [0 0 0 0.001 0.002 0.003 ... 0.998 0.999 1.0 1.0 1.0] (quadratic B-splines based on 1001 knot points)

   - One epoch is: test with 1002 test functions build from 1001 knot points, evaluation of LOSS functions with 1002 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution for $\epsilon = 0.01$, for $\epsilon = 0.1$, and for $\epsilon = 1.0$ (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon \in (0.01, 1.0)$, $\#N$ = number of intervals = 10 [0 0 0 0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0 1.0 1.0 1.0] (cubic B-splines based on 11 knot points)
   - One epoch is: test with 13 test functions build from 11 knot points, evaluation of LOSS functions with 13 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solutions for $\epsilon = 0.01$, for $\epsilon = 0.1$, and for $\epsilon = 1.0$ (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon \in (0.01, 1.0)$, $\#N$ = number of intervals = 100 [0 0 0 0 0.01 0.02 0.03 ... 0.98 0.99 1.0 1.0 1.0 1.0] (cubic B-splines based on 101 knot points)
   - One epoch is: test with 103 test functions build from 101 knot points, evaluation of LOSS functions with 103 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution for $\epsilon = 0.01$, for $\epsilon = 0.1$, and for $\epsilon = 1.0$ (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon \in (0.01, 1.0)$, $\#N$ = number of intervals = 1000 [0 0 0 0 0.001 0.002 0.003 ... 0.998 0.999 1.0 1.0 1.0 1.0] (cubic B-splines based on 1001 knot points)
   - One epoch is: test with 1003 test functions build from 1001 knot points, evaluation of LOSS functions with 1003 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimal). It does not have to be the last point of training! Please plot the best solution for $\epsilon = 0.01$, for $\epsilon = 0.1$, and for $\epsilon = 1.0$ (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).
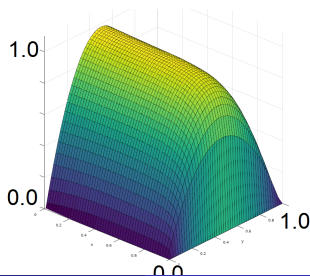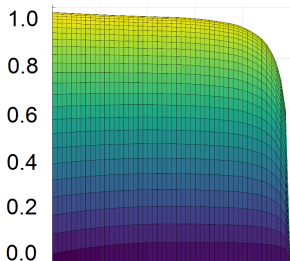
Eriksson-Johnson model problem

Given $\Omega = (0, 1)^2$, $\beta = (1, 0)^T$, we seek the solution of the advection-diffusion problem

$$\frac{\partial u}{\partial x} - \epsilon \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0$$

with Dirichlet boundary conditions

$$u = 0 \text{ for } x \in (0, 1), y \in \{0, 1\} \qquad u = sin(\Pi y) \text{ for } x = 0$$

The problem is driven by the inflow Dirichlet boundary condition. It develops a boundary layer of width ϵ at the outflow $x = 1$.

Experiments

All the methods and experiments as for 1D problem, but for VPINN experiments we use knot vector [0 0 0.25 0.5 0.75 1.0 1.0] in y direction for linear B-splines, [0 0 0 0.25 0.5 0.75 1.0 1.0 1.0] in y direction for quadratic B-splines, and [0 0 0 0 0.25 0.5 0.75 1.0 1.0 1.0 1.0] in y direction for linear B-splines. The knot vectors in x directions identical as for 1D problem.