Neural networks and isogeometric analysis

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K. Doległo, A. Paszyńska, M. Paszyński, L. Demkowicz, Deep neural networks for smooth approximation of physics with higher order and continuity B-spline base functions, **https://arxiv.org/abs/2201.00904** (2022)

Physics Informed Neural Networks for strong form of PDE

Strong form: Find $u \in C^2(0,1)$:

$$\underbrace{-\epsilon \frac{d^2 u(x)}{dx^2}}_{\text{diffusion}=\epsilon} \quad \underbrace{+1 \frac{du(x)}{dx}}_{\text{advection"wind"}=1} \quad = 0, x \in (0,1), -\epsilon \frac{du}{dx}(0) + u(0) = 1.0, \ u(1) = 0$$

Now, neural network IS the solution

$u(x) = NN(x) = A_n \sigma\left(A_{n-1}\sigma(...\sigma(A_1 x + B_1)... + B_{n-1}\right) + B_n$

$$LOSS_{PDE}(x) = (-\epsilon \frac{d^2 NN(x)(x)}{dx^2} + \frac{d NN(x)(x)}{dx})^2,$$
$$LOSS_{BC0} = (-\epsilon \frac{d NN(0)}{dx} + NN(0) - 1.0)^2,$$
$$LOSS_{BC1} = (NN(1)s)^2,$$
$$LOSS(x) = (LOSS_{PDE}(x)^2 + LOSS_{BC0}^2 + LOSS_{BC1}^2).$$

1. $\epsilon = 1.0$, $\#N$ = number of points = 10 (selected randomly at beginning of each epoch)
   - One epoch is: random selection of 10 points, evaluation of LOSS functions at the 10 points. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).
2. $\epsilon = 1.0$, $\#N$ = number of points = 100 (selected randomly at beginning of each epoch)
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1. $\epsilon = 0.1$, $\#N$ = number of points = 10 (selected randomly at beginning of each epoch)
    - One epoch is: random selection of 10 points, evaluation of LOSS functions at the 10 points. Please run 100 epochs.
    - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
    - Please measure the total time of training
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1. $\epsilon = 0.01$, $\#N$ = number of points = 10 (selected randomly at beginning of each epoch)
   - One epoch is: random selection of 10 points, evaluation of LOSS functions at the 10 points. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
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1. $\epsilon = 1.0$, $\#N$ = number of points = 10 (now, 10 points are distributed as x_i, where $x_0 = 0$, $x_1 = \frac{1}{2}$,$x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (0, 0.5, 0.75, 0,875, 0,9375, 0,96875, 0,984375, 0,9921875, 0,99609375, 1.0)
    - One epoch is: 10 adaptive points, evaluation of LOSS functions at the 10 points. Please run 100 epochs.
    - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
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1. $\epsilon = 0.1$, $\#N$ = number of points = 10 (now, 10 points are distributed as x_i, where $x_0 = 0, x_1 = \frac{1}{2}, x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (0, 0.5, 0.75, 0,875, 0,9375, 0,96875, 0,984375, 0,9921875, 0,99609375, 1.0)
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    - One epoch is: 10 adaptive points, evaluation of LOSS functions at the 10 points. Please run 100 epochs.
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Variational Physics Informed Neural Networks for weak form of PDE

Weak form: Find $u \in H^1(0,1)$:

$$\int_0^1 \epsilon \frac{du(x)}{dx}\frac{dv(x)}{dx}dx + \int_0^1 \frac{du(x)}{dx}v(x)dx + u(0)v(0) = v(0)$$
$$\forall v \in V$$

Now, neural network IS the solution

$u(x) = NN(x) = A_n\sigma\left(A_{n-1}\sigma(...\sigma(A_1x + B_1)... + B_{n-1}\right) + B_n$

$$b_{weak}(v) = \int \left(\epsilon \frac{dNN(x)}{dx}\frac{dv}{dx} + \frac{dNN(x)}{dx}v\right)dx + NN(0)v(0),$$

$$l_{weak}(v) = v(0), LOSS_{weak}(v) = \left(b_{weak}(v) - l_{weak}(v)\right)^2,$$

$$LOSS_{BC0} = (-\epsilon\frac{dNN(0)}{dx} + NN(0) - 1.0)^2, \quad LOSS_{BC1} = (NN(1))^2,$$

$$LOSS(v) = LOSS_{strong}(x) + LOSS_{weak} + LOSS_{BC0} + LOSS_{BC1}.$$

1. $\epsilon = 1.0$, $\#N$ = number of intervals = 10 [0 0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0 1.0] (linear B-splines based on 11 knot points)
    - One epoch is: test with 11 test functions build from 11 knot points, evaluation of LOSS functions with 11 test functions. Please run 100 epochs.
    - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
    - Please measure the total time of training
    - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).
2. $\epsilon = 1.0$, $\#N$ = number of intervals = 100 [0 0 0.01 0.02 0.03 ... 0.98 0.99 1.0 1.0] (linear B-splines based on 101 knot points)
    - One epoch is: test with 101 test functions build from 101 knot points, evaluation of LOSS functions with 101 test functions. Please run 100 epochs.
    - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
    - Please measure the total time of training
    - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon = 1.0$, $\#N$ = number of intervals = 1000 [0 0 0.001 0.002 0.003 ... 0.998 0.999 1.0 1.0] (linear B-splines based on 1001 knot points)
    - One epoch is: test with 1001 test functions build from 1001 knot points, evaluation of LOSS functions with 1001 test functions. Please run 100 epochs.
    - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
    - Please measure the total time of training
    - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon = 1.0$, $\#N$ = number of intervals = 10 [0 0 0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0 1.0 1.0] (quadratic B-splines based on 11 knot points)
   - One epoch is: test with 12 test functions build from 11 knot points, evaluation of LOSS functions with 12 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).
2. $\epsilon = 1.0$, $\#N$ = number of intervals = 100 [0 0 0 0.01 0.02 0.03 ... 0.98 0.99 1.0 1.0 1.0] (quadratic B-splines based on 101 knot points)
   - One epoch is: test with 102 test functions build from 101 knot points, evaluation of LOSS functions with 102 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon = 1.0$, $\#N$ = number of intervals = 1000 [0 0 0 0.001 0.002 0.003 ... 0.998 0.999 1.0 1.0 1.0] (quadratic B-splines based on 1002 knot points)
    - One epoch is: test with 1002 test functions build from 1002 knot points, evaluation of LOSS functions with 1002 test functions. Please run 100 epochs.
    - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
    - Please measure the total time of training
    - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon = 1.0$, $\#N$ = number of intervals = 10 [0 0 0 0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0 1.0 1.0 1.0] (cubic B-splines based on 11 knot points)
   - One epoch is: test with 11 test functions build from 13 knot points, evaluation of LOSS functions with 13 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).
2. $\epsilon = 1.0$, $\#N$ = number of intervals = 100 [0 0 0 0 0.01 0.02 0.03 ... 0.98 0.99 1.0 1.0 1.0 1.0] (cubic B-splines based on 101 knot points)
   - One epoch is: test with 103 test functions build from 101 knot points, evaluation of LOSS functions with 103 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best

1. $\epsilon = 1.0$, $\#N$ = number of intervals = 1000 [0 0 0 0 0.001 0.002 0.003 ... 0.998 0.999 1.0 1.0 1.0 1.0] (cubic B-splines based on 1003 knot points)
    - One epoch is: test with 1003 test functions build from 1003 knot points, evaluation of LOSS functions with 1003 test functions. Please run 100 epochs.
    - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
    - Please measure the total time of training
    - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon = 0.1$, $\#N$ = number of points $= 10$ (now, 11 knot points are distributed as x_i, where $x_0 = 0, x_1 = \frac{1}{2}$,$x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. [0 0 0.5 0.75 0.875 0.9375 0.96875 0.984375 0.9921875 0,99609375 0,998046875 1.0 1.0] We use linear B-splines
    - One epoch is: 11 linear B-splines, evaluation of LOSS functions with 11 test functions. Please run 100 epochs.
    - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
    - Please measure the total time of training
    - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solution (horizontal axis $= x \in (0, 1)$, vertical axis $= NN(x)$).

1. $\epsilon = 0.1$, $\#N$ = number of points = 100 (now, 101 knot points are distributed as x_i, where $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (99 intervals) [0 0 0.5 0.75 0.875 0.9375 0.96875 0.984375 0.9921875 0.99609375 0.998046875 ... 1.0 1.0] We use linear B-splines
   - One epoch is: 101 linear B-splines, evaluation of LOSS functions with 101 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).
2. $\epsilon = 0.1$, $\#N$ = number of points = 1000 (now, 1001 test functions are distributed as x_i, where $x_0 = 0$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (999 intervals) [0 0 0.5 0.75 0.875 0.9375 ... 1.0 1.0] We use linear B-splines
   - One epoch is: 1001 linear B-splines, evaluation of LOSS functions at the 1001 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimial). It

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   - One epoch is: 12 quadratic B-splines , evaluation of LOSS functions with 12 test functions. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
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   - One epoch is: 102 quadratic B-splines, evaluation of LOSS functions with 102 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
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2. $\epsilon = 0.1$, $\#N$ = number of points = 1001 (now, 1001 knot points are distributed as x_i, where $x_0 = 0$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (999 intervals) [0 0 0 0.5 0.75 0.875 0.9375 ... 1.0 1.0 1.0] We use quadratic B-splines
   - One epoch is: 1002 quadratic B-splines, evaluation of LOSS functions with 1002 test functions. Please run 100 epochs.
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    - One epoch is: 13 cubic B-splines , evaluation of LOSS functions with 13 test functions. Please run 100 epochs.
    - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
    - Please measure the total time of training
    - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon = 0.1$, $\#N$ = number of points = 101 (now, 101 knot points are distributed as x_i, where $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (100 intervals) [0 0 0 0 0.5 0.75 0.875 0.9375 ... 1.0 1.0 1.0 1.0] We use cubic B-splines
   - One epoch is: 103 cubic B-splines, evaluation of LOSS functions with 103 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).
2. $\epsilon = 0.1$, $\#N$ = number of points = 1001 (now, 1001 knot points are distributed as x_i, where $x_0 = 0$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (999 intervals) [0 0 0 0 0.5 0.75 0.875 0.9375 ... 1.0 1.0 1.0 1.0] We use cubic B-splines
   - One epoch is: 1003 cubic B-splines, evaluation of LOSS functions with 1003 test functions. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimial). It

1. $\epsilon = 0.01$, $\#N =$ number of points $= 10$ (now, 11 knot points are distributed as x_i, where $x_0 = 0, x_1 = \frac{1}{2}$,$x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. [0 0 0.5 0.75 0.875 0.9375 0.96875 0.984375 0.9921875 0,99609375 0,998046875 1.0 1.0] We use linear B-splines
    - One epoch is: 11 linear B-splines, evaluation of LOSS functions with 11 test functions. Please run 100 epochs.
    - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
    - Please measure the total time of training
    - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solution (horizontal axis $= x \in (0, 1)$, vertical axis $= NN(x)$).

1. $\epsilon = 0.01$, $\#N$ = number of points = 100 (now, 101 knot points are distributed as x_i, where $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (99 intervals) [0 0 0.5 0.75 0.875 0.9375 0.96875 0.984375 0.9921875 0.99609375 0.998046875 ... 1.0 1.0] We use linear B-splines
   - One epoch is: 101 linear B-splines, evaluation of LOSS functions with 101 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).
2. $\epsilon = 0.01$, $\#N$ = number of points = 1000 (now, 1001 test functions are distributed as x_i, where $x_0 = 0$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (999 intervals) [0 0 0.5 0.75 0.875 0.9375 ... 1.0 1.0] We use linear B-splines
   - One epoch is: 1001 linear B-splines, evaluation of LOSS functions at the 1001 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimial). It

1. $\epsilon = 0.01$, $\#N$ = number of points = 10 (now, 11 knot points are distributed as x_i, where $x_0 = 0, x_1 = \frac{1}{2}$,$x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. [0 0 0 0.5 0.75 0.875 0.9375 0.96875 0.984375 0.9921875 0.99609375 0.998046875 1.0 1.0 1.0] We use quadratic B-splines
   - One epoch is: 12 quadratic B-splines , evaluation of LOSS functions with 12 test functions. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon = 0.01$, $\#N$ = number of points = 101 (now, 101 knot points are distributed as x_i, where $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (100 intervals) [0 0 0 0.5 0.75 0.875 0.9375 ... 1.0 1.0 1.0] We use quadratic B-splines
   - One epoch is: 102 quadratic B-splines, evaluation of LOSS functions with 102 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).
2. $\epsilon = 0.01$, $\#N$ = number of points = 1001 (now, 1001 knot points are distributed as x_i, where $x_0 = 0$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (999 intervals) [0 0 0 0.5 0.75 0.875 0.9375 ... 1.0 1.0 1.0] We use quadratic B-splines
   - One epoch is: 1002 quadratic B-splines, evaluation of LOSS functions with 1002 test functions. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimial). It

1. $\epsilon = 0.01$, $\#N$ = number of points = 10 (now, 11 knot points are distributed as x_i, where $x_0 = 0, x_1 = \frac{1}{2}$,$x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. [0 0 0 0 0.5 0.75 0.875 0.9375 0.96875 0.984375 0.9921875 0.99609375 0.998046875 1.0 1.0 1.0 1.0] We use cubic B-splines
    - One epoch is: 13 cubic B-splines , evaluation of LOSS functions with 13 test functions. Please run 100 epochs.
    - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
    - Please measure the total time of training
    - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon = 0.01$, $\#N$ = number of points = 101 (now, 101 knot points are distributed as x_i, where $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (100 intervals) [0 0 0 0 0.5 0.75 0.875 0.9375 ... 1.0 1.0 1.0 1.0] We use cubic B-splines
   - One epoch is: 103 cubic B-splines, evaluation of LOSS functions with 103 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solution (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).
2. $\epsilon = 0.01$, $\#N$ = number of points = 1001 (now, 1001 knot points are distributed as x_i, where $x_0 = 0$, $x_i = x_{i-1} + (x_i - x_{i-1})/2$, e.g. (999 intervals) [0 0 0 0 0.5 0.75 0.875 0.9375 ... 1.0 1.0 1.0 1.0] We use cubic B-splines
   - One epoch is: 1003 cubic B-splines, evaluation of LOSS functions with 1003 test functions. Please run 100 epochs.
   - Please plot the convergence of PINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimial). It

IGA-DNN-VPINN

Weak form: Find $u \in H^1(0,1)$:

$$\int_0^1 \epsilon \frac{du(x)}{dx}\frac{dv(x)}{dx}dx + \int_0^1 \frac{du(x)}{dx}v(x)dx + u(0)v(0) = v(0)$$

$$\forall v \in V$$

Now, we plug a combination of B-splines in place of the solution $u(x) = \sum_{i=1,\dots,N} u_i B_{i,p}(x)$

$$b_{weak}(v) = \int \left(\epsilon \sum_i u_i \frac{dB_{i,p}(x)}{dx}\frac{dv}{dx} + \sum_i u_i \frac{dB_{i,p}(x)}{dx} v \right) dx + u_1 v(0),$$

$$l_{weak}(v) = v(0), LOSS_{weak}(v) = \left(b_{weak}(v) - l_{weak}(v)\right)^2,$$

$$LOSS_{BC0} = (-\epsilon u_1 \frac{dB_{i,p}(0)}{dx} + u_1 - 1.0)^2, LOSS_{BC1} = (u_N)^2,$$

$$LOSS(v) = LOSS_{strong}(x) + LOSS_{weak} + LOSS_{BC0} + LOSS_{BC1}.$$

All the experiments like in VPINN

Parametric-IGA-VPINN

Weak form: Find $u \in H^1(0,1)$:

$$\int_0^1 \epsilon \frac{du(x)}{dx}\frac{dv(x)}{dx}dx + \int_0^1 \frac{du(x)}{dx}v(x)dx + u(0)v(0) = v(0)$$

$$\forall v \in V$$

Now, NN input is ϵ, NN output are N coefficients of B-splines
$\epsilon \rightarrow NN(\epsilon)_i \quad u(x) = \sum_{i=1,\ldots,N} NN(\epsilon)_i B_{i,p}(x)$

$$b_{weak}(v) = \int \left(\epsilon \sum_i NN(\epsilon)_i \frac{dB_{i,p}(x)}{dx}\frac{dv}{dx} + \sum_i NN(\epsilon)_i \frac{dB_{i,p}(x)}{dx} v \right) dx + NN(\epsilon)_1 v(0), \quad l_{weak}(v) = v(0)$$

$$LOSS_{weak}(v) = (b_{weak}(v) - l_{weak}(v))^2,$$

$$LOSS_{BC0} = (-\epsilon NN(\epsilon)_1 \frac{dB_{1,p}(0)}{dx} + NN(\epsilon)_1 - 1.0)^2, LOSS_{BC1} = (NN(\epsilon)_N)^2,$$

$$LOSS(v) = LOSS_{strong}(x) + LOSS_{weak} + LOSS_{BC0} + LOSS_{BC1}.$$

1. $\epsilon \in (0.01, 1.0)$, $\#N$ = number of intervals = 10 [0 0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0 1.0] (linear B-splines based on 11 knot points)
    - One epoch is: test with 11 test functions build from 11 knot points, evaluation of LOSS functions with 11 test functions. Please run 100 epochs.
    - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
    - Please measure the total time of training
    - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solutions for $\epsilon = 0.01$, for $\epsilon = 0.1$, and for $\epsilon = 1.0$ (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon \in (0.01, 1.0)$, $\#N$ = number of intervals = 100 [0 0 0.01 0.02 0.03 ... 0.98 0.99 1.0 1.0] (linear B-splines based on 101 knot points)
    - One epoch is: test with 101 test functions build from 101 knot points, evaluation of LOSS functions with 101 test functions. Please run 100 epochs.
    - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
    - Please measure the total time of training
    - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solution for $\epsilon = 0.01$, for $\epsilon = 0.1$, and for $\epsilon = 1.0$ (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon \in (0.01, 1.0)$, $\#N$ = number of intervals = 1000 [0 0 0.001 0.002 0.003 ... 0.998 0.999 1.0 1.0] (linear B-splines based on 1001 knot points)
    - One epoch is: test with 1001 test functions build from 1001 knot points, evaluation of LOSS functions with 1001 test functions. Please run 100 epochs.
    - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
    - Please measure the total time of training
    - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solution for $\epsilon = 0.01$, for $\epsilon = 0.1$, and for $\epsilon = 1.0$ (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon \in (0.01, 1.0)$, $\#N$ = number of intervals = 10 [0 0 0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0 1.0 1.0] (quadratic B-splines based on 11 knot points)
    - One epoch is: test with 12 test functions build from 11 knot points, evaluation of LOSS functions with 12 test functions. Please run 100 epochs.
    - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
    - Please measure the total time of training
    - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solutions for $\epsilon = 0.01$, for $\epsilon = 0.1$, and for $\epsilon = 1.0$ (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon \in (0.01, 1.0)$, $\#N$ = number of intervals = 100 [0 0 0 0.01 0.02 0.03 ... 0.98 0.99 1.0 1.0 1.0] (quadratic B-splines based on 101 knot points)
   - One epoch is: test with 102 test functions build from 101 knot points, evaluation of LOSS functions with 102 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solution for $\epsilon = 0.01$, for $\epsilon = 0.1$, and for $\epsilon = 1.0$ (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon \in (0.01, 1.0)$, $\#N$ = number of intervals = 1000 [0 0 0 0.001 0.002 0.003 ... 0.998 0.999 1.0 1.0 1.0] (quadratic B-splines based on 1001 knot points)
    - One epoch is: test with 1002 test functions build from 1001 knot points, evaluation of LOSS functions with 1002 test functions. Please run 100 epochs.
    - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
    - Please measure the total time of training
    - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solution for $\epsilon = 0.01$, for $\epsilon = 0.1$, and for $\epsilon = 1.0$ (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon \in (0.01, 1.0)$, $\#N$ = number of intervals = 10 [0 0 0 0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0 1.0 1.0 1.0] (cubic B-splines based on 11 knot points)
   - One epoch is: test with 13 test functions build from 11 knot points, evaluation of LOSS functions with 13 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solutions for $\epsilon = 0.01$, for $\epsilon = 0.1$, and for $\epsilon = 1.0$ (horizontal axis = $x \in (0,1)$, vertical axis = $NN(x)$).

1. $\epsilon \in (0.01, 1.0)$, $\#N$ = number of intervals = 100 [0 0 0 0 0.01 0.02 0.03 ... 0.98 0.99 1.0 1.0 1.0 1.0] (cubic B-splines based on 101 knot points)
   - One epoch is: test with 103 test functions build from 101 knot points, evaluation of LOSS functions with 103 test functions. Please run 100 epochs.
   - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
   - Please measure the total time of training
   - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solution for $\epsilon = 0.01$, for $\epsilon = 0.1$, and for $\epsilon = 1.0$ (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

1. $\epsilon \in (0.01, 1.0)$, $\#N$ = number of intervals = 1000 [0 0 0 0 0.001 0.002 0.003 ... 0.998 0.999 1.0 1.0 1.0 1.0] (cubic B-splines based on 1001 knot points)
    - One epoch is: test with 1003 test functions build from 1001 knot points, evaluation of LOSS functions with 1003 test functions. Please run 100 epochs.
    - Please plot the convergence of VPINN training (horizontal axis = number of epochs, vertical axis = LOSS function value)
    - Please measure the total time of training
    - Select a single best solution (where the loss functions was minimial). It does not have to be the last point of training! Please plot the best solution for $\epsilon = 0.01$, for $\epsilon = 0.1$, and for $\epsilon = 1.0$ (horizontal axis = $x \in (0, 1)$, vertical axis = $NN(x)$).

Eriksson-Johnson model problem

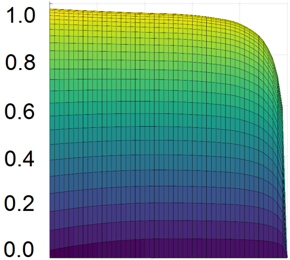
Given $\Omega = (0,1)^2$, $\beta = (1,0)^T$, we seek the solution of the advection-diffusion problem

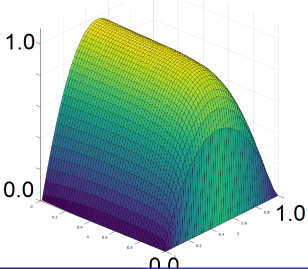
$$\frac{\partial u}{\partial x} - \epsilon \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0$$

with Dirichlet boundary conditions

$$u = 0 \text{ for } x \in (0,1), y \in \{0,1\} \qquad u = sin(\Pi y) \text{ for } x = 0$$

The problem is driven by the inflow Dirichlet boundary condition. It develops a boundary layer of width ϵ at the outflow $x = 1$.





Experiments

All the methods and experiments as for 1D problem, but for VPINN experiments we use knot vector [0 0 0.25 0.5 0.75 1.0 1.0] in y direction for linear B-splines, [0 0 0 0.25 0.5 0.75 1.0 1.0 1.0] in y direction for quadratic B-splines, and [0 0 0 0 0.25 0.5 0.75 1.0 1.0 1.0 1.0] in y direction for cubic B-splines. The knot vectors in x directions identical as for 1D problem.

Physics Informed Neural Networks for strong form of PDE

Strong form: Given $\Omega = (0,1)^2$, $\beta = (1,0)^T$, we seek $\Omega \ni (x,y) \rightarrow u(x,y)$ such that $\frac{\partial u}{\partial x} - \epsilon \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0$ with Dirichlet boundary conditions

$$u = 0 \text{ for } x \in (0,1), y \in \{0,1\} \qquad u = sin(\Pi y) \text{ for } x = 0$$

Now, neural network IS the solution

$$u(x,y) = NN(x,y) = A_n \sigma \left(A_{n-1} \sigma (\ldots \sigma (A_1 \begin{bmatrix} x \\ y \end{bmatrix} + B_1) \ldots + B_{n-1} \right) + B_n$$

$$LOSS_{PDE}(x,y) = (\frac{\partial NN(x,y)}{\partial x} - \epsilon \frac{\partial^2 NN(x,y)(x)}{\partial x^2} + -\epsilon \frac{\partial^2 NN(x,y)(x)}{\partial y^2})^2,$$

$$LOSS_{BC0y}(0,y) = (NN(0,y) - \sin(\Pi y))^2, LOSS_{BC1y}(1,y) = (NN(1,y))^2,$$

$$LOSS_{BCx0}(x,0) = (NN(x,0))^2, LOSS_{BCx1}(x,1) = (NN(x,1))^2,$$

$$LOSS(x,y) = LOSS_{PDE}(x,y) + LOSS_{BC0y}(0,y) + LOSS_{BC1y}(1,y) + \\ + LOSS_{BCx0}(x,0) + LOSS_{BCx1}(x,1)$$

Variational Physics Informed Neural Networks for weak form of PDE

Weak form: Find $u \in H^1([0,1]^2)$: $b(u,v) = l(v)$

$$b(u,v) = \int_{[0,1]^2} \epsilon \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} dxdy + \int_{[0,1]^2} \epsilon \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} dxdy + \int_{[0,1]^2} \frac{\partial u}{\partial x} v dxdy$$

$$l(v) = \int_{[0,1]^2} [\epsilon \left(\sin(\Pi y) \frac{\partial v}{\partial x} dxdy - \Pi \cos(\Pi y) \frac{\partial v}{\partial y} \right) + \sin(\Pi y) v] dxdy, \forall v \in V$$

Now, neural network IS the solution

$$u(x,y) = NN(x,y) = A_n \sigma \left(A_{n-1} \sigma(\ldots \sigma(A_1 \begin{bmatrix} x \\ y \end{bmatrix} + B_1) \ldots + B_{n-1} \right) + B_n$$

$$LOSS_{strong} = (\int_{[0,1]^2} [\frac{\partial NN(x,y)}{\partial x} - \epsilon \left(\frac{\partial^2 NN(x,y)}{\partial x^2} + \frac{\partial^2 NN(x,y)}{\partial y^2} \right)] v dxdy)^2$$

$$LOSS_{weak}(v) = \left(b(NN(x,y),v) - l(NN(x,y),v)\right)^2,$$

$$LOSS_{BC}(x,y) = (NN(0,y) - \sin(\Pi y))^2 + (NN(1,y))^2 + (NN(x,0))^2 + (NN(x,1))^2$$

IGA-DNN-VPINN

Weak form: Find $u \in H^1([0,1]^2)$: $b(u,v) = l(v)$

$$b(u,v) = \int_{[0,1]^2} \epsilon \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} dxdy + \int_{[0,1]^2} \epsilon \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} dxdy + \int_{[0,1]^2} \frac{\partial u}{\partial x} v\, dxdy$$

$$l(v) = \int_{[0,1]^2} [\epsilon \left(\sin(\Pi y)\frac{\partial v}{\partial x} dxdy - \Pi \cos(\Pi y)\frac{\partial v}{\partial y}\right) + \sin(\Pi y) v] dxdy, \forall v \in V$$

Now, we plug a combination of B-splines in place of the solution $u(x,y) = \sum_{i=1,\ldots,N_x; j=1,\ldots,N_y} u_{ij} B_i(x) B_j(y)$

$$LOSS_{strong} = (\int_{[0,1]^2} [\frac{\partial \sum u_{ij} B_i B_j}{\partial x} - \epsilon \left(\frac{\partial^2 \sum u_{ij} B_i B_j}{\partial x^2} + \frac{\partial^2 \sum u_{ij} B_i B_j}{\partial y^2} \right)] v)^2$$

$$LOSS_{weak}(v) = \left(b(\sum u_{ij} B_i B_j, v) - l(\sum u_{ij} B_i B_j, v) \right)^2,$$

$$LOSS_{BC}(x,y) = (\sum u_{ij} B_i(0) B_j(y) - \sin(\Pi y))^2 + (\sum u_{ij} B_i(1) B_j(y))^2 + (\sum u_{ij} B_i(x) B_j(0))^2 + (\sum u_{ij} B_i(x) B_j(1))^2$$

Parametric-IGA-VPINN

Weak form: Find $u \in H^1([0,1]^2)$: $b(u,v) = l(v)$

$$b(u,v) = \int_{[0,1]^2} \epsilon \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} dxdy + \int_{[0,1]^2} \epsilon \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} dxdy + \int_{[0,1]^2} \frac{\partial u}{\partial x} v\, dxdy$$

$$l(v) = \int_{[0,1]^2} [\epsilon \left(\sin(\Pi y) \frac{\partial v}{\partial x} dxdy - \Pi \cos(\Pi y) \frac{\partial v}{\partial y} \right) + \sin(\Pi y) v] dxdy, \forall v \in V$$

Now, NN input is ϵ, NN output are N coefficients of B-splines

$\epsilon \rightarrow NN(\epsilon)_{ij} \quad u(x,y) = \sum_{i=1,\ldots,N_x; j=1,\ldots,N_y} NN(\epsilon)_{ij} B_{i,p}(x) B_{j,p}(y)$

$$LOSS_{strong} = \left(\int_{[0,1]^2} \sum N(\epsilon)_{ij} \left(\frac{\partial B_i B_j}{\partial x} - \epsilon \left(\frac{\partial^2 B_i B_j}{\partial x^2} + \frac{\partial^2 B_i B_j}{\partial y^2} \right) \right) v \right)^2$$

$$LOSS_{weak}(v) = \left(b(\sum u_{ij} B_i B_j, v) - l(\sum u_{ij} B_i B_j, v) \right)^2, LOSS_{BC}(x,y) =$$

$$(\sum NN(\epsilon)_{ij} B_i(0) B_j(y) - \sin(\Pi y))^2 + (\sum NN(\epsilon)_{ij} B_i(1) B_j(y))^2 +$$

$$(\sum NN(\epsilon)_{ij} B_i(x) B_j(0))^2 + (\sum NN(\epsilon)_{ij} B_i(x) B_j(1))^2$$