Pseudoentropy

PhD Dissertation Talk

University of Warsaw

May 23, 2023

- ✔ Overviews the goals, resources, and deliverables of my PhD project.
- ✔ Demonstrates/sketches interesting techniques used in the dissertation.
- ✔ Defends my position on the dissertation form, in view of reviews received 🛡.
- ❌ Avoids complex definitions and proofs for brevity's sake (see the papers) ⏱.
- ❌ Does not assess my own academic KPIs (see the documentation) 😳.

Outline

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Credits

I am particularly grateful:

- ❤️ for love, to my wife Aneta
- 💰 for funding and know-how, to my advisor Stefan Dziembowski
- 💡 for merit support, to my co-advisor Krzysztof Pietrzak
- 👏 for motivation and recognition, to dozens of people with whom I shared ideas: research collaborators, reviewers, audience of my talks 😊

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$+$ several travel grants from various research institutions

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About Pseudoentropy

- Introduced in [ILL89, HILL99] as a **computational variant of information-theoretic entropy**.
- Recognized as a **useful tool and convenient language** in research around cryptography, computational complexity and information theory. Examples:
  - Pseudorandom generators from one-way functions [HILL99]
  - Computational Dense Model Theorem [RTTV08, Zha11], improving upon the key ingredient in the results of Green-Tao-Ziegler [GT08, TZ08].
- Promising but messy: suffers from **contextual definitions** and **insufficiently developed foundations**.

Goals

My PhD project set these goals:

- **improve understanding of foundational properties** of pseudoentropy notions
- **demonstrate further technical applications**
- optionally, identify **new inspirational application areas**

Contribution

Works presented under the scope of this PhD project:

- ✔ **obtained characterizations and manipulation rules** for pseudoentropy notions, using **convex analysis as a toolbox**, as well as **impossibility results**
- ✔ **simplified some of existing technical proofs**, for instance of Dense Model Theorem and of Computational Simulators
- ✔ **developed machine-learning inspired framework** for proving computational indistinguishability

My self-assesment:

- 🥅 these works contributed to the goals 🧹, ⚙️ and 💎 respectively.
- 🏃 goals were set broadly, leaving still room for improvement

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Background

- Pseudoentropy at least k when the distribution behaves *nearly as well* as with information-theoretic (min)entropy k in *cryptographic games*.
- Program-input games used in definitions
  (a) Distinguish: discriminate between two distributions based on a sample. Pseudoentropy example: for a pseudorandom generator G from d-bit seeds to k-bit outputs, $|\mathbb{E}\mathrm{D}(G(U_d)) - \mathbb{E}\mathrm{D}(U_k)| \leqslant \epsilon$ for small ϵ and comp. bounded D.
  (b) Predict: guess a sampled outcome. Pseudoentropy example: a one way function f satisfies $\mathbf{P}\{f(\mathrm{D}(f(x))) = f(x)\} \leqslant 2^{-k}$ for computationally bounded D and large k.
  (c) Compress: decoding / encoding games, see [Yao82, BSW03].
  (d) ... and even more exotic examples, see [HRVW09].

Outline

1. Acknowledgments 🙏
2. Introduction 🏁
3. Detailed Overview 🔎
   - Preliminaries
   - Geometric Characterizations of Pseudoentropy 🧹⚙️
   - Unpredictability Pseudoentropy 🧹
   - Best Generic Attacks on Pseudoentropy ⚙️
   - Lower Bounds for Pseudoentropy Chain Rules and Transformations ⚙️
   - Simulating Auxiliary Information 💎
4. References 📚
5. Discussion 💬

Outline

- 📖 Indistinguishability quantifies how close are two distributions under a given class of computationally bounded tests.
- ❓ What is the geometrical meaning of indistinguishability?
- ✋ Computational indistinguishability can be **characterized by inseparability by a class of feasible hyperplanes**. The separation margin can be determined analytically too!

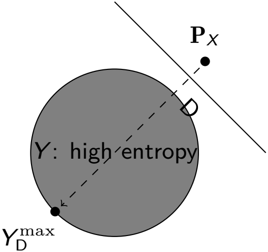
Contribution

The characterizations [Sko15a] has found the following applications:

- Unifying unpredictability-based and indistinguishability-based pseudoentropy notions [SGP15]
- Short proof of the Dense Model Theorem [Sko15c]
- New proofs of Hardcore Lemmas [Sko15c]
- Further applications to key derivation [Sko17b]
- Simplifies other technical arguments [VZ12]

Technique (Sketch)

- In program-input indistinguishability games, it makes sense to **characterize the optimal input player** Y against a given program player D.
- View D as a *separating hyperplane*, *maximize margin* with high-entropy Y.



Symbol/Operator	Crypto	Geometry
X	candidate distribution	
Y	input player	feasible point
D	distinguisher/program player	separating hyperplane
$\mathbf{E}\mathrm{D}(Y)$	expectation	$\mathrm{D}\cdot\mathbf{P}_Y$ (dot-product)
$\epsilon = \mathbf{E}\mathrm{D}(Y) - \mathbf{E}\mathrm{D}(X)$	advantage	separation margin

Figure 1: Geometrical meaning of cryptographic indistinguishability.

- **Closed-form solutions found** in interesting cases by **convex optimization**. For pseudoentropy of at least k bits against attackers $\mathcal{D}$ with advantage ϵ:

$$\forall \mathrm{D}\in\mathcal{D}: \quad \mathbf{E}\mathrm{D}(X) \leqslant 2^{-k}|\mathrm{D}| + \epsilon$$

instead of the standard depth-2 formula $\forall \mathrm{D}\exists Y : \mathbf{H}_\infty(Y) \geqslant k \& \ \mathbf{E}\mathrm{D}(X) \leqslant \mathbf{E}\mathrm{D}(Y) + \epsilon$.

- Characterization depend on feasible distinguishers and the baseline entropy.

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- Applications of pseudoentropy use different notions, most commonly unpredictability-based and indistinguishability-based.
- Are unpredictability and indistinguishability entropies different? Note: usually, distinguishing is easier than predicting[1].
- Surprisingly, **equivalent in high-entropy regimes**!

[1]Think of discriminating between dogs and cats versus predicting the breed.

Contribution

The following result was obtained [SGP15]:

- **equivalence of unpredictability and indistinguishability** pseudoentropy definitions in **high-entropy regimes**, namely $n - O(\log n)$ for n-bit strings,
- **geometric characterizations as a workhorse** of the proof.

Technique (Sketch)

The proof strategy is to *constructively convert a distinguisher into a predictor*:

(a) Indistinguishability fails: $\mathbf{E}\mathsf{D}(X) \geqslant \mathbf{E}\mathsf{D}(Y) + \epsilon$ for all Y of min-entropy k.

(b) $\mathbf{E}\mathsf{D}(X) \geqslant |\mathsf{D}|/2^k + \epsilon$ for boolean D, by geometrical characterizations (!)

(c) Sample A from the image of D, then $\mathbf{P}\{A = X\} > 2^{-k} + \frac{\epsilon}{\#\mathsf{D}}$.

(d) Approximate image sampling by *rejection sampling* ℓ times, then

$$\mathbf{P}\{A = X\} > \left(2^{-k} + \frac{\epsilon}{\#\mathsf{D}}\right) \cdot \left(1 - \frac{\#\mathsf{D}}{2^n}\right)^{\ell}.$$

(e) $\mathbf{P}\{A = X\} > 2^{-k}$ when $\ell \approx 2^{n-k}/\epsilon$ independently of $\#\mathsf{D}$!

⚠ More sophisticated rejection-sampling handles X with auxiliary input Z.

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- Applications of pseudoentropy assume strength parameters that propagate through reduction proofs. Not clear what regimes are non-trivial to start with.
- Can we characterize when quality parameters are non-trivial?
- Yes, by time-advantage tradeoffs - similarly to pseudorandomness!

Contribution

The following result was obtained [Sko17b]:

- generic attacks with time t succeed against psuedoentropy amount k with advantage $\epsilon = O\left(\sqrt{t/2^k}\right)$.
- generalization of the famous time-advantage tradeoffs against pseudorandomness [DTT10].

Technique (Sketch)

The proof leverages **random walk techniques**, see also [Ber97].

(a) Let D : $\{0,1\}^n \rightarrow \{-1,1\}$ be fully random (Rademacher), then $|\mathbf{E}\mathrm{D}(X) - \mathbf{E}\mathrm{D}(Y)| \approx \|\mathbf{P}_X - \mathbf{P}_Y\|_2$ w.h.p. (random walks theory [Haa81]).

(b) By slicing the domain $\{0,1\}^n$ into t random parts and flipping the signs accordingly, we can have $|\mathbf{E}\mathrm{D}'(X) - \mathbf{E}\mathrm{D}'(Y)| \approx t \cdot \frac{\|\mathbf{P}_X - \mathbf{P}_Y\|_2}{\sqrt{t}}$ w.h.p., with $O(t)$ extra memory.

(c) Under mild assumptions on Y (far from having k bits of min-entropy) the attack achieves advantage $\epsilon \approx \sqrt{t/2^k}$. Compare with $\epsilon \approx \sqrt{t/2^n}$ for pseudorandomness!

(d) "Random" can be weakened to $O(1)$-**wise independent**, and the construction complexity is indeed $O(t)$, alternatively complexity t yields $\epsilon = O(\sqrt{t/2^k})$!

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- Applications of pseudoentropy **heavily rely on manipulation rules**, particularly chain rules and transformations [BSW03, FOR12]. However, their use **weakens security guarantees**, due to tradeoffs in quality parameters **caused by reduction proofs**.
- Can we improve known manipulation rules?
- No, not by black-box reductions!

Contribution

The following results were obtained in [PS16]:

- **Impossibility of better proofs** by black-box reductions!
- A **probabilistic construction of an oracle**, of independent interest, inspired by the earlier work on limitations of dense model theorems [Zha11].
- Inspired techniques used in research on black-box limitations of auxiliary input simulators [CCL18].

Techniques (Sketch)

(a) Proofs, in case of indistinguishability-based pseudoentropy, rely on *building distinguishers from distinguishers in other setups*. In particular, we have $D' = \mathbb{I}\{\sum_i w_i D_i > t_0\}$ in the proof of the Dense Model Theorem [Zha11] or transformations [BSW03, Sko15b].

(b) Loosely speaking, black-box reductions *aggregate distinguishers by high-level operations*. To prove the need of many operations (queries) we manipulate distinguishers at low-level by *choosing values probabilistically and sophistically*!

(c) Examples (from the proof, see also [CCL18]):
   (1) $D_i \sim \mathrm{Bern}(1/2+\epsilon)$ on a small set and $D_i \sim \mathrm{Bern}(1/2)$ elsewhere.
   (2) $D_i \sim \mathrm{Bern}(1/2+\epsilon)$ and $D_i \sim \mathrm{Bern}(1/2-\epsilon)$ on complementary random subsets.

⚠ High-level aggregations are essential. For example, without them Dense Model Theorems can fail [IM20].

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- In security proofs, it helps to model leakages as explicit functions of secrets [JP14].
- What leakages can be modelled as functions of secrets?
- Short leakages can be efficiently simulated!

Contribution

The following important results were obtained [Sko16a]:

- Construction of a simulator for m bits of leakage which makes only $2^{O(m)}\epsilon^{-2}$ calls to achieve ϵ-indistinguishability. Significantly improved upon prior works [VZ13, JP14]
- The reasoning, inspired by ML techniques, **builds on the gradient descent algorithm** and was recognized with the *best student paper award at TCC'16*.
- Inspired follow-up works that solved the simulator problem [CCL18], and generalized the learning framework [Sko16b, Sko17a], and studied quantum pseudoentropy (c.f. Chen's dissertation [Che19]).

Technique (Sketch)

(a) The algorithm below demonstrates the procedure

Algorithm 1: Auxiliary Input Simulator

Data:

- Oracle access to distinguishers/test functions $\mathcal{D}$

Result: Simulator h of $Z \in \{0,1\}^m$ given $X \in \{0,1\}^n$

$\mathbf{P}\{h(x) = z\} \leftarrow 2^{-m}$ `// initialize the solution as uniform`

while $\max_{D\in\mathcal{D}} \mathbf{E}D(X,Z) - \mathbf{E}D(X,h(X)) > \epsilon$ `// as long as can distinguish...`

do

  $\mathbf{P}\{h'(x) = z\} \leftarrow \mathbf{P}\{h(x) = z\} - \gamma D(x,z)$ `// improve candidate`

  $\mathbf{P}\{h'(x) = z\} \leftarrow \mathbf{P}\{h'(x) = z\} + \mathsf{Correct}(x,z)$ `// guarantee constraints`

  $h \leftarrow h'$

end

return h

(b) Outputs an efficient simulator p.d.f., with appropriate γ and Correct operation.

(c) Finishes after $2^{O(m)}\epsilon^{-2}$ steps, proved by "energy" arguments.

(d) Resembles boosting: we learn how to (strongly) simulate from (weak) distinguishers

(e) Resembles convex optimization: with D as subgradient, γ as a stepsize, Correct as a projection operation!

Outline

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Outline

Addressing Reviewers Feedback

R: Editorial changes and reference requests.

M: Addressed, thanks for the feedback!

R: A book-style dissertation would be better than a mixture of conference works.

M: I discussed this form with senior researchers, but found *ineffective*:

- 🥕 Gain citations! 🤔 *Time-consuming, better to keep writing papers.*
- 🥕 Get your PhD distinguished. 🤔 *Prestigious conferences not enough?*
- 💦 Take your time to present it better! 🤔 *Why to work harder? We count conference works when granting junior/senior professorships!*

R: Parts of lengthy works might not have been fully reviewed at conferences.

M: Same can happen for junior professorships, but we had extra reviewers 😌.