Non-Modellable Risk Factor (NMRF) measurement using Gaussian Process Regression (GPR)

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Abstract

One innovation defined in the new market risk rules by the Fundamental Review of the Trading Book (FRTB) is the Non-Modellable Risk Factor (NMRF) framework. This new concept introduces a methodology to differentiate between modellable and non-modellable risk factors in the Internal Models Approach (IMA). The modellability assessment is based on two dimensions: the time observability and the price availability. This study is focused on the modelling approach for NMRFs respecting the regulation requirements. First, we present the FRTB requirements and the European Bank Authority (EBA) model suggested for NMRFs. We then introduce the concept and theory of Gaussian Process Regression (GPR). Thereafter, we show the motivation and we also explain the adequate model for this issue respecting the regulator requirements. Finally, we present results by comparing our model with the EBA one and relate extensions to other issues.

JEL classification: C10; C15; G10; G18; G28.

Keywords: Market Risk; Fundamental Review of the Trading Book (FRTB); Risk measurement; Stress Testing; Expected Shortfall; Machine Learning; Gaussian Process Regression.

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I. *Introduction*

The Fundamental Review of the Trading Book (FRTB January 2016) has rebuilt the Internal Models Approach (IMA), and this new regulation covers all internal models, developed by banks, for the market risk. One of the innovations brought in by the FRTB is that banks must distinguish between the modellable risk factors and non-modellable risk factors (NMRFs). However, the regulator defines NMRFs indirectly by giving requirement criteria of modellable risk factors (MRFs). Hence, the modellable one has to respect two properties, called the risk factor eligibility test (RFET) by the Basel committee in the last Consultative Document (CD) for revisions to the minimum capital requirements for market risk (March 2018):

- An MRF must have at least 24 observable real prices per year (12 months) over the calibration period.
- The time interval between two consecutive observations must be less then month.

However, the Basel committee makes the following changes on the RFET roles in The last version of the FRTB (January 2019):

- An MRF must have at least 24 observable real prices per year (12 months) over the calibration period with one price by day, and 90-day must contain more than four real price observations over the previous 12 months.
- Or the bank must count 100 real price observations with one price by day.

Therefore, the RFET makes sure that the factor is liquid and observable to be adequate for modeling. Otherwise the factors that do not respect the RFET are deemed NMRFs. The regulator also allows using an adequate proxy by replacing NMRFs by the basis between MRFs and NMRFs. Hence, this basis will be considered as NMRFs. Furthermore, the regulator clarifies the meaning of the “real price” in this last CD since they had not done so in the first version of the FRTB regulation. They require that the observation must be “representative” to be considered as a real price. In fact, the CD declines two cases when the factor is not directly observed:

- The first one is when the observation factor is deduced from derivatives underlying, for example, the implied volatility of an equity. In this case, the banks can use the quoted price of options to deduce this factor. However, they must build a process mapping between these factors and their corresponding products.
- The second one is when the market quotes a similar risk factor. For example, credit default swap (CDS) spreads with one- or five-year maturity are not directly quoted, and as solution banks use linear interpolation or extrapolation to rebuild data. The regulator allows also to use the closest quoted maturities and make the observation a real price. Moreover, they define two alternatives to make the similarity eligible for these risk factors as well as to conform to the

RFET. Banks are allowed to build a bucket for each risk factor, which must be approved by the regulator as the first alternative. The regulator will propose buckets that the bank has to refer to as the second alternative.

The data eligibility defined by the CD targets the modellable and non-modellable risk factors, and only observations meeting these conditions must be used for calibration and risk measurement.

The NMRFs must be quantified on a standalone basis using a stress scenario without correlation and are excluded from Expected Shortfall (ES) capital charge calculation. The calibration of the stress must be consistent, otherwise the maximum loss possible will be required for the bank. The diversification is not permitted for all NMRFs, but the regulator makes exception for NMRFs not associated to the idiosyncratic risk (credit spread and equity) and accepts the diversification for those. The horizon liquidity for an NMRF has to be the largest period between two observations. The aggregate capital requirement for a set of NMRFs with I elements of idiosyncratic credit spread risks, J elements of idiosyncratic equity risks and K risk factors remaining in model-eligible desks that are NMRFs is equal to:

$$SES = \sqrt{\sum_{i=1}^{I} ISES_{NM,i}^2} + \sqrt{\sum_{j=1}^{J} ISES_{NM,j}^2} + \sqrt{\left(\rho \times \sum_{k=1}^{K} SES_{NM,k}\right)^2 + (1-\rho^2)\sum_{k=1}^{K} SES_{NM,k}^2}$$

where $SES_{NM,k}$ represents the stress test scenario capital charge for k from K, $ISES_{NM,i}$ is the stress test scenario capital charge for i from I, $ISES_{NM,j}$ is the stress test scenario capital charge for j from J, and $\rho = 0.6$.

As we see, the Basel committee requires a stress scenario (SS) for NMRF capitalization and does not give an approach to calibrate it. The committee suggests only that the SS should be consistent, else a maximum loss scenario will have to be applied. However, the European Bank Authority (EBA) published a proposed SS shock calibration for NMRFs in the discussion paper “Implementation in the European Union of the revised market risk and counterparty credit risk framework” (December 2017) and names it the Risk Factor Based Approach (RFBA). Indeed, the EBA deems to scale the return observations (absolute or log-return) with the square root of the liquidity horizon LH for each NMRF. So, if we denote $r_i(D_t)$ the return of the NMRF i at date D_t, then the LH_i is defined as:

$$LH_i = \max_{t \in \{1 \ldots N+1\}} (D_t - D_{t-1})$$

The last version of the FRTB (January 2019) redefined the LH as the maximum between the liquidity horizon assigned to the risk factor in the [MAR33.12] and 20, capped to the maturity of the instrument.

This data transformation leads to:

$$\begin{cases} r_i(D_t) = \left(obs(D_t) - obs(D_{t-1})\right) \times \sqrt{\dfrac{LH_i}{(D_t - D_{t-1})}}; Absolute\ return \\ r_i(D_t) = \log\left(\dfrac{obs(D_t)}{obs(D_{t-1})}\right) \times \sqrt{\dfrac{LH_i}{(D_t - D_{t-1})}}; log\ return \end{cases}$$

where $obs(D_t)$ gives the observation value at D_t.

The standard deviation of these returns is estimated by:

$$\hat{\sigma} = \sqrt{\frac{1}{N - 1.5} \times \sum_{t=1}^{N} (r_i(D_t) - \bar{r})^2}$$

where $\bar{r}$ is the mean of returns.

Given these points, the EBA defines the downgrade and the upgrade shocks to determine the shock interval. This interval for the absolute shock is represented as:

$$CSSRFR_i = [obs_i(D_{N+1}) - CS_i, obs_i(D_{N+1}) + CS_i]$$

where $CS_i = C_{ESequiv} \times \hat{\sigma} \times \left(1 + \frac{\Phi^{-1}(CL_{sigma})}{\sqrt{2(N-1.5)}}\right)$ is the maximum shock, $C_{ESequiv}$ represents the approximate ratio of the ES to the standard deviation and is floored to 3, and CL_{sigma} is the confidence level of not underestimating the true standard deviation of the risk factor returns and must be conservative (for example 90% or more).

Based on all these elements, the future shock scenario at D_{N+2} is the one that maximises the loss and is given by:

$$SS_i(D_{N+2}) = \kappa_i \times \max_{obs_i \in CSSRFR_i} \left(P\&L(obs_i)\right)$$

where $P\&L(obs_i) = PV\left(obs_i, obs_{j \neq i} fixed\right) - PV\left(obs_i(D_{N+1}), obs_{j \neq i} fixed\right)$ is the profit and loss generated for all portfolio positions attached to this NMRF when the shock is applied, and $\kappa_i = \max\left(1, \frac{ES[P\&L(obs_i)]}{\max_{obs_i \in CSSRFR_i}\left(P\&L(obs_i)\right)}\right)$ represents the non-linearity adjustment factor.

In addition, the EBA permits to use a fallback approach in the case of the RFBA not providing satisfy results. Hence, the alternative solution must be more conservative than RFBA and respect the FRTB standards. Besides, the bank must give an approved methodology. In fact, the EBA proposes two options for a Risk Factor Fallback Approach (RFFA):

- The first one defines an adverse stress scenario for an NMRF that maximises loss. This approach is straightforward to implement, but it may not produce a good result for instruments like interest rate swaps, because of the non-boundary of theoretical loss.
- The second option allows to implement the stress scenario shock interval (like RFBA) to compute the maximum loss.

We keep the second option to suggest an alternative approach since the EBA proposes to take more consideration for this option. They also require that the RFFA must keep the same model principle in the IMA for the ES. It means indirectly that the NMRF has a posterior Gaussian distribution, and calibration of the mean and the standard deviation for this distribution is needed. Thereafter, we resort to the Gaussian Process Regression to implement our RFFA since this approach deals with the incomplete data issues to give a smooth calibration for the posterior distribution.

II. *Gaussian Process Regression theory*

1. Context and motivation

The most important issue in the market risk modelling area is the prediction of risk factors to allow quantifying the future Profit and Loss (P&L). Indeed, there are three approaches to deal with this issue. The first one is regression, including linear and non-linear methods. The second one uses times series theory to define the model's predictions. The third one is based on Brownian motion diffusion using stochastic calculustheory. However, the calibration is a fundamental step to make the model accurate for the three theories. We know that this step needs historical data, but sometimes we have to deal with missing data and it can decrease the accuracy of our model as well as the prediction accuracy. The usual approach is regression to deal with this issue, and we then extrapolate to get the prediction of the new value. The simplest way is to implement a linear regression, but this method is not efficient in the case of non-linear historical observations. Thereby, we can resort to polynomial regression to handle more complex historical data shapes. The issue of these regressions is the need of many calibration parameters to target a good prediction, and they are not smooth enough to learn more on the historical data. Hence, Gaussian Process Regression (GPR) is developed to fill these weaknesses since it uses fewer parameters for calibration. Besides, GPR employs supervised learning in machine learning theory on the training data.

2. Gaussian Process definition

A Gaussian Process (GP) is simply defined as a set of finite random variables following a joint Gaussian distribution. The mean and covariance function are the structural aim of the GP. Hence, if we deem that some observable random variables $f(x)$ at point x (for example time series) follow a Gaussian Process $f(x) \sim \mathcal{GP}\big(m(x), K(x, x')\big)$, then the mean and the covariance are defined as:

$$\begin{cases} m(x) = \mathbb{E}[f(x)] \\ K(x,x') = \mathbb{E}\left[\left(f(x) - m(x)\right) \times \left(f(x') - m(x')\right)\right] \end{cases}$$

Most uses cases consider $m(x) = 0$ for simplification, and there are many covariance function forms implemented in the literature that will be discussed later. The GP verifies the marginality property and supposes that the study of the larger set does not change the distribution of the smaller one. The linear regression illustrates a sample example of the Gaussian Process and is given by:

$$f(x) = x^\top * w;\ y = f(x) + \varepsilon$$

where x represents the vector of observation points, $w \sim \mathcal{N}(0, \Sigma)$ gives the weight vector, y is the observed values and $\varepsilon \sim \mathcal{N}(0, \sigma)$.

We then say $f(x)$ and $f(x')$ are jointly Gaussian with the following mean and covariance function:

$$\begin{cases} m(x) = \mathbb{E}[f(x)] = x^\top * \mathbb{E}[w] = 0 \\ K(x,x') = \mathbb{E}[f(x) \times f(x')] = x^\top * \mathbb{E}[w * w^\top] * x = x^\top * \Sigma * x \end{cases}$$

3. Gaussian Process Regression

Gaussian Process Regression uses the GP as the kernel of regression and is based on the multivariate Gaussian distribution to define the model's predictions. Indeed, we can approximate a new point $\boldsymbol{f}(\boldsymbol{x}^*)$ on $\boldsymbol{x}^*$, and we have:

$$\begin{bmatrix} f(x) \\ f(x^*) \end{bmatrix} \sim \mathcal{N}\left(0, \begin{bmatrix} K(x,x) & K(x^*,x) \\ K(x,x^*) & K(x^*,x^*) \end{bmatrix}\right)$$

where $x = (x_1, \dots, x_n)$ is a vector of n elements and K represents the covariance matrix,

$$K(x,x) = \begin{bmatrix} K(x_1,x_1) & \cdots & K(x_1,x_n) \\ \vdots & \ddots & \vdots \\ K(x_n,x_1) & \cdots & K(x_n,x_n) \end{bmatrix}$$

$$K(x,x^*) = \left(K(x_1,x^*), \dots, K(x_n,x^*)\right);\ K(x^*,x) = \left(K(x^*,x_1), \dots, K(x^*,x_n)\right)$$

As result, the conditional probability of $f(x^*)|x^*, x, f(x)$ also follows a Gaussian distribution with the following mean and variance:

$$f(x^*)|x^*, x, f(x) \sim \mathcal{N}(\mu, \sigma)$$

where

$$\begin{cases} \mu = K(x,x^*) * K^{-1}(x,x) * f(x) \\ \sigma^2 = K(x^*,x^*) - K(x,x^*) * K^{-1}(x,x) * K(x^*,x) \end{cases}$$

However, the covariance function $K: (x, \theta) \to K(x, \theta)$ depends on the vector of parameters θ that we should calibrate to include the observed data specification of the GPR model. Thus, we use the maximum likelihood method to estimate these parameters:

$$log\big(p(f(x)|x,\theta)\big) = \sum_{i=1}^{n} log\big(p(f(x_i)|x_i,\theta)\big)$$
$$= -\frac{1}{2}\Big(\big(f(x)\big)^{\top} * K^{-1}(x,x,\theta) * f(x) + log(|K(x,x,\theta)|) + n \times log(2\pi)\Big)$$

The covariance matrix $K(x,x,\theta)$ must be invertible to implement these calculations. We should run an optimization program to compute the model parameters. Additionally, the relative derivative is given by:

$$\frac{\partial log\big(p(f(x)|x,\theta)\big)}{\partial\theta} = \frac{1}{2}\left(\alpha^{\top} * \frac{\partial K(x,x,\theta)}{\partial\theta} * \alpha - tr\left(K^{-1}(x,x,\theta) * \frac{\partial K(x,x,\theta)}{\partial\theta}\right)\right)$$
$$= \frac{1}{2} tr\left(\big(\alpha^{\top} * \alpha - K^{-1}(x,x,\theta)\big) * \frac{\partial K(x,x,\theta)}{\partial\theta}\right)$$

where $\alpha = K^{-1}(x,x,\theta) * f(x)$.

The complexity of the calculation time is $O(n^3)$ for matrix inversion and $O(n^2)$ for matrix derivatives. An estimation of the predicted value of x^* is equal to $y^* = K(x,x^*) * K^{-1}(x,x) * f(x)$ with α confidence level interval error $y^* \pm \Phi^{-1}\left(1-\frac{\alpha}{2}\right) \times \sigma$.

4. Covariance function choice

One of the important components of the GPR is the covariance function, also known by the name kernel. This function must respect the symmetric property since it will define the covariance matrix. The choice of the covariance function allows good fitting of the model with the observation. We find many functions in the literature, and we summarize the most common ones in the following table:

Function names	Expression
Constant	σ^2
Linear	$x^{\top} * x$
Gaussian noise	$\sigma^2 \times \delta_{x,x'}$
Squared exponential	$\sigma^2 \times \exp\left(-{d^2}/{2l^2}\right)$
Ornstein–Uhlenbeck	$\exp\left(-{\lvert d\rvert}/{l}\right)$
Matérn	$\frac{2^{1-\vartheta}}{\Gamma(\vartheta)}\left(\frac{\sqrt{2\vartheta}\lvert d\rvert}{l}\right)^{\vartheta} K_{\vartheta}\left(\frac{\sqrt{2\vartheta}\lvert d\rvert}{l}\right)$
Periodic	$\exp\left(-\frac{2sin^2\left({d}/{2}\right)}{l^2}\right)$
Rational quadratic	$(1+\lvert d\rvert)^{-\alpha}, \alpha \geq 0$

Table 1: Covariance functions.

In Table 1, $d = x - x'$, $\delta_{x,x'} = \mathbb{1}_{x=x'}$, $\Gamma(\vartheta)$ represents the gamma function and K_ϑ is the modified Bessel function.

We have to calibrate parameters $\theta = \{\sigma, l, \vartheta, \alpha\}$ for each case of functions using the maximum likelihood. Albeit, we can build a new covariance function based on an old one by applying some mathematical operation like addition or multiplication. For example, when we have a population representing a long term and also periodicity with Gaussian noise, we can use the following kernel function built using the addition of three functions:

$$K(x, x') = \sigma^2 \times \exp\left(-\frac{d^2}{2l_1^2}\right) + \exp\left(-\frac{2sin^2\left({}^{d}/_{2}\right)}{l_2^2}\right) + \sigma_2^2 \times \delta_{x,x'}$$

where σ, l_1, l_2 are model parameters to be calibrated based on training data.

The kernel function could be more complex in order to build a good prediction model. Nevertheless, it must respect the symmetric property and the invertibility of the covariance matrix.

III. *Modelling NMRFs using GPR*

1. Framework implementation and model calibration

In this section, we present the steps of the framework implementation. The first step is data validation and risk factor classification based on the Real Price Criteria (RPC) and RFET rules. We propose the following process to build this step:

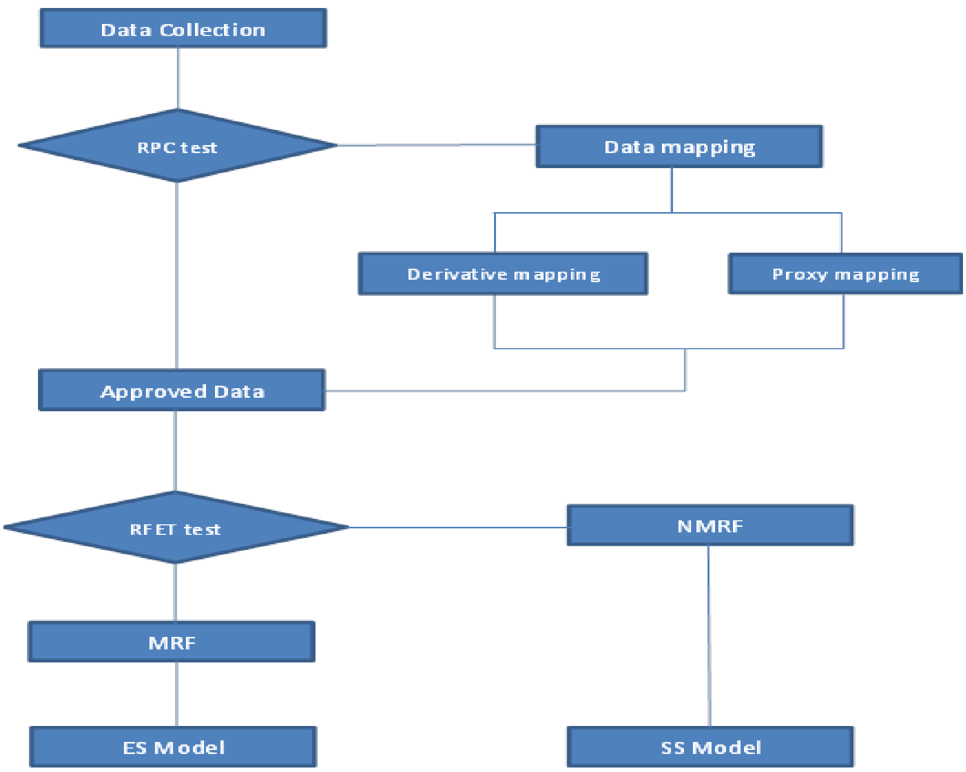

Figure 1: NMRF process selection and building.

In our case, the NMRF is the CDS spread with five-year tenor. We chose the ten-year historical horizon data including the same period crisis of the Expected Shortfall to conform with the ES IMA requirement and ten days as the observation frequency. First, the liquidity horizon should be calculated since we must rescale the return observations to LH to calibrate the shocks for the RFBA. In our case, the liquidity horizon is equal to:

$$LH = 60$$

We opt for the absolute return, and we plot both of the rescaled time series returns to show the evolution on time and the density:

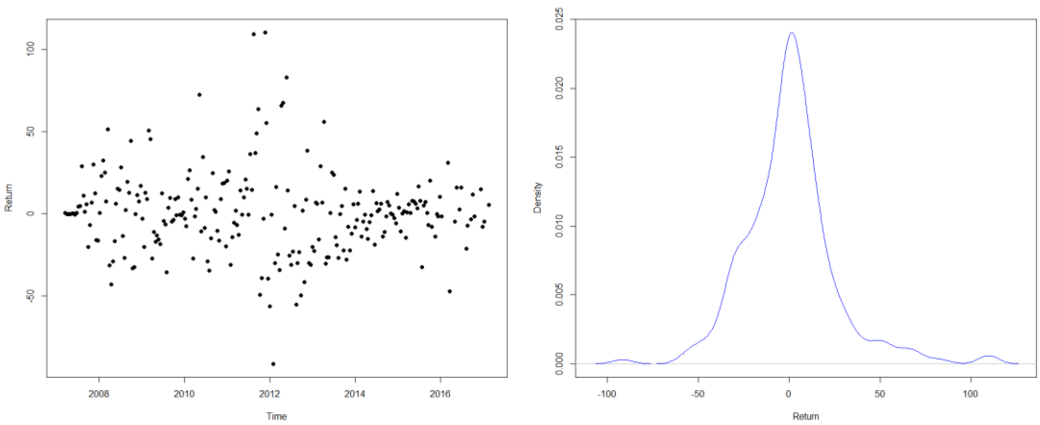

Figure 2: Time series return evolution and density.

The rescaled data allow us to compute the standard deviation estimation $\hat{\sigma} = 25.23$. We then consider the following values respectively for the confidence level and the ES of the return standard deviation, $CL_{sigma} = 90\%$ and $C_{ESequiv} = 3$, to implement the EBA model. Given all these elements, we then compute the value of the maximum shock $CS_i = 79.78$ bps (basis points).

Thereafter, we have to calibrate the GPR model to define the posterior parameters $\theta_{GPR} = (\mu, \sigma)$. We then need to choose the covariance function, and we opt for the following one since we observe a long term and periodicity in our historical time series:

$$K(x, x') = \sigma_1^2 \times \exp\left(-\frac{d^2}{2l_1^2} - 2sin^2(l_2 \times \pi \times d)\right)$$

where $x = (D_1, \dots, D_{N+1})$ is the observation date vector, $\theta = (l_1, l_2, \sigma_1)$ represents the set of parameters that must be calibrated, and the covariance matrix is equal to:

$$K(x, x) = \begin{bmatrix} K(D_1, D_1) & \cdots & K(D_1, D_{N+1}) \\ \vdots & \ddots & \vdots \\ K(D_{N+1}, D_1) & \cdots & K(D_{N+1}, D_{N+1}) \end{bmatrix}$$

The return observations of the CDS spread scaled to LH represents $f(x) = r_i$ in our case, and we use these to build the calibration. The calculations give the following parameters, $\theta = (1, 1, 628.78)$. Our predicted point is $x^* = D_{N+2} = D_{N+1} + LH$, and the covariance vectors are:

$$K(x, x^*) = \left(K(D_1, D_{N+2}), \dots, K(D_{N+1}, D_{N+2})\right);\ K(x^*, x) = \left(K(D_{N+2}, D_1), \dots, K(D_{N+2}, D_{N+1})\right)$$

Hence, we deduce the mean and variance values of the predicted distribution:

$$\begin{cases} \mu = \ 0.1438 \\ \sigma^2 = \ 629.531 \end{cases}$$

The following figure shows the evolution of the estimated values in each point of D_i with 95% confidence interval error:

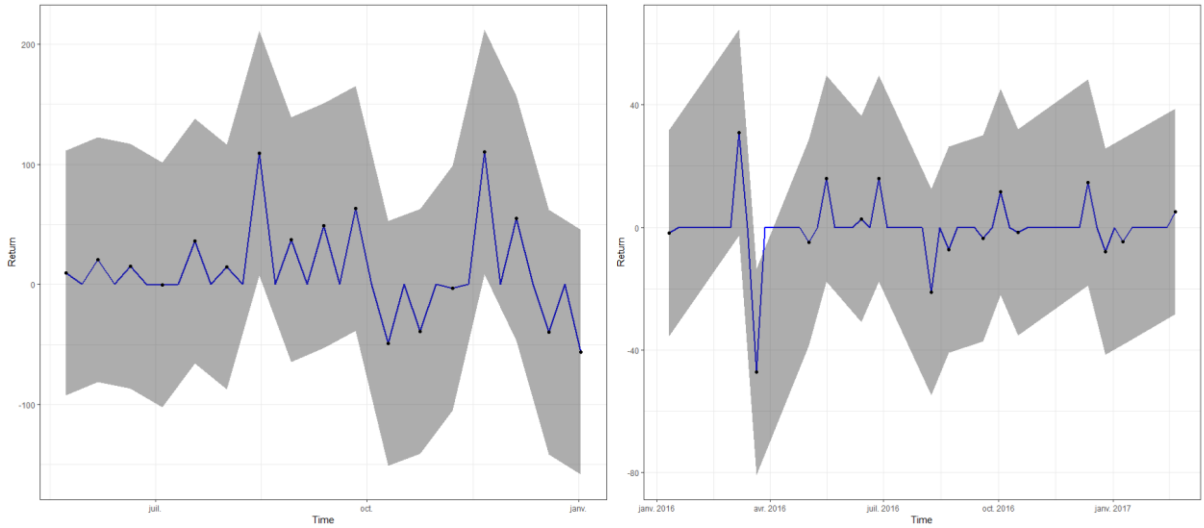

Figure 3: Time series GPR reproduction on daily observation.

We plot in the left the sovereign crisis that represents our stressed period, and the right graph draws the last year of the observation. The solid line indicates an estimation of daily points based on the Gaussian Process.

2. Results analysis

These calibrations permit to compute the stress scenario loss for the RFBA and RFFA models. We also deem the CDS as an instrument to compute the P&L, and the set of its risk factors is composed of the spread and the interest rate. However, we keep the spread as the NMRF, and we have the following relation:

$$P\&L_{CDS} = \Delta PV = \Delta s \times DV01 + \Delta s^2 \times \frac{\partial^2 PV}{\partial s^2}$$

where $DV01 = \frac{\partial PV}{\partial s}$ gives the sensitivity to the spread variation and $\frac{\partial^2 PV}{\partial s^2}$ represents the convexity term.

First, we need to compute κ_i in the case of a non-linear instrument ($\kappa_i = 1$ for pure delta position). For this, we deem $\max_{obs_i \in CSSRFR_i}\left(P\&L_{CDS}(obs_i)\right) = P\&L_{CDS}(ES_{97.5\%}[obs_i]) =$

2.08%, and we compute $ES_{97.5\%}[P\&L_{CDS}(obs_i)] = 0.85\%$ using the historical distribution. Therefore, we obtain the following value of kappa, $\kappa_i = 2.45$. Therefore, we deduce the loss shock scenario of the EBA model, $SES_{EBA,i}(D_{N+2}) = \kappa_i \times \max_{obs_i \in CSSRFR_i}\left(P\&L_{CDS}(obs_i)\right) = 1.958\%$.

We opt for the Monte Carlo method to generate return shocks and then compute the P&L for the GPR model since we suppose that the posterior distribution of shocks follows a normal distribution with (μ, σ) as parameters. We fix the number of simulations to $M = 1\,000\,000$, and the $P\&L_{CDS}$ density is plotted in the following graphic with $DV01 = 1\%$; $\frac{\partial^2 PV}{\partial s^2} = 10\%$:

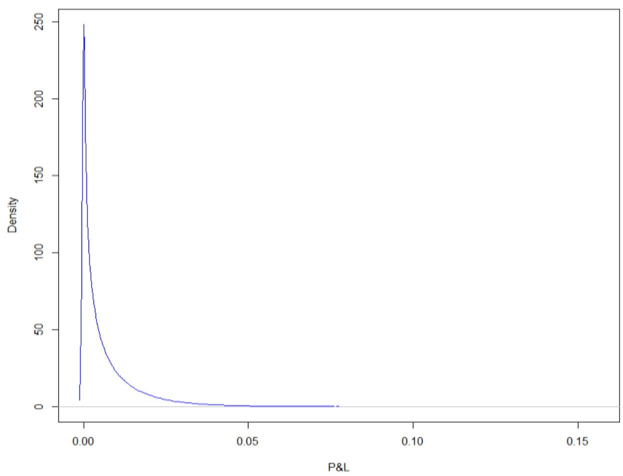

Figure 4: CDS P&L density.

The loss shock scenario is given by the Expected Shortfall with 97.5% as quantile:

$$SES_{GPR,i}(D_{N+2}) = ES_{97.5\%}[P\&L_{CDS}] = 4.42\%$$

The convexity has an important impact on the value of the Stressed Expected Shortfall (SES). We draw the following figure to show the evolution of the SES according to the convexity, and we conclude that the SES increases with the convexity in our case:

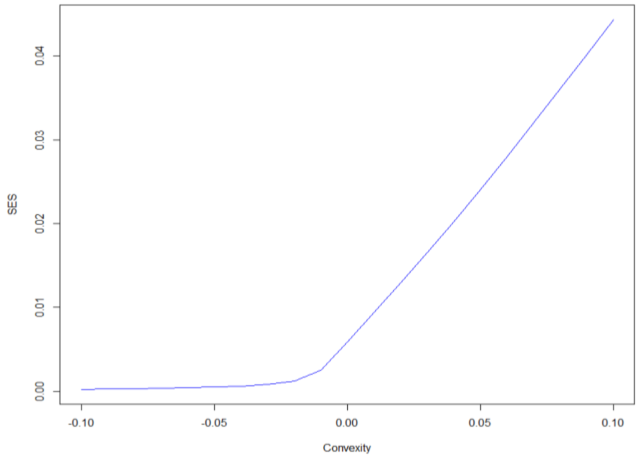

Figure 5: SES evolution according to the convexity.

We conclude that the GPR model gives more important value of the SES since we have $SES_{GPR,i}(D_{N+2}) > SES_{EBA,i}(D_{N+2})$. That means the RFFA is more conservative than RFBA in this case of study. Hence, we should keep the GPR model to compute the capital risk of this NMRF as the EBA recommended.

IV. *Conclusion*

We introduce an application of machine learning theory to risk market measurement. The need was raised automatically from the FRTB and the EBA regulation for the NMRF modelling issue. We introduce the FRTB text guidelines for NMRFs in the IMA. Then, we present both the Risk Factor Based Approach (RFBA) and the Risk Factor Fallback Approach (RFFA). We then choose the Gaussian Process Regression to answer this requirement as it is defined on the last regulation of market risk and we saw that the choice of the covariance function is very important to have an accurate result. We then applied this to the CDS instrument, and we conclude that the GPR model could work well as the FRBA model. We also deduce that the GPR model is more conservative than RFFA since we obtain more important Stressed Expected Shortfall. However, this study can be a gateway to apply other theories of machine learning on the many other risk measurement issues to make computational optimization of some measures like Credit Value Adjustment (CVA).

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