Consistent Simulation of Fibonacci Anyon Braiding within a Qubit Quasicrystal Inflation Code

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Abstract

The realization of fault-tolerant topological quantum computation often relies on manipulating non-Abelian anyons, such as Fibonacci anyons. Implementing their braiding operations on standard qubit platforms requires robust encoding schemes. This work investigates the simulation of standard Fibonacci anyon braiding within the protected subspace of a one-dimensional Quasicrystal Inflation Code realized on qubits. We define a local Hamiltonian whose ground state manifold enforces the code's constraints, reflecting physical Fibonacci tiling rules, and possesses the required Fibonacci degeneracy consistent with anyon fusion spaces. We confirm numerically that standard local qubit operators fail to satisfy the crucial Temperley-Lieb algebra governing anyon braiding when acting on the full qubit Hilbert space. We then present and validate a successful construction, implemented and verified for up to ten qubits: abstract Temperley-Lieb projector matrices, derived from established representations valid for the code's basis, are embedded into the qubit Hilbert space via a computationally constructed isometric mapping. Extensive numerical simulations confirm that the resulting embedded operators rigorously satisfy the defining algebraic relations of the Temperley-Lieb algebra for all tested system sizes. Consequently, the derived braid operators are unitary, satisfy the fundamental braid relations including the Yang-Baxter equation, and preserve the code subspace. This provides a strongly validated pathway for simulating Fibonacci topological quantum computation on qubits using the Quasicrystal Inflation Code framework, leveraging physically motivated constraints.

1 Introduction

Topological quantum computation (TQC) offers a compelling route towards fault-tolerant quantum information processing by encoding information non-locally within the states of topological systems [1]. Systems supporting non-Abelian anyons, particularly Fibonacci anyons, are of great interest as their braiding operations are sufficient for universal quantum computation [2,3]. However, the direct physical realization and manipulation of anyons remain significant experimental hurdles across various proposed platforms.

An alternative strategy involves simulating the essential physics of anyonic systems, including their characteristic fusion rules and braiding statistics, on more readily available quantum computing architectures, such as superconducting qubits or trapped ions. Recent experiments have successfully demonstrated the creation and control of non-Abelian topological order and anyons on such platforms, showcasing the potential and challenges of this approach [4]. This requires robust encoding schemes that faithfully map the abstract algebraic structure of anyon models onto the qubit Hilbert space while ideally providing some level of inherent protection or structure.

Quasicrystal Inflation Codes (QIC) represent a promising framework for this purpose, drawing inspiration from the structural properties of quasicrystals and aperiodic tilings [5]. This work investigates the Fibonacci QIC realised on a one-dimensional (1D) chain of qubits. We leverage the structure of 1D Fibonacci quasicrystals, whose physical generation via tiling or substitution rules (e.g., mapping Long tiles L to $|1\rangle$ and Short tiles S to $|0\rangle$) naturally forbids adjacent identical short tiles ('SS') [6,7]. This fundamental physical constraint directly motivates the definition of the QIC via the exclusion of adjacent $|00\rangle$ states in the qubit chain.

We define the protected logical subspace of this standard Fibonacci QIC, denoted $\mathcal{H}_{QIC}$, as the subspace of the N-qubit Hilbert space $\mathcal{H}$ spanned by all computational basis states satisfying the 'no 00' constraint. To make this concrete, we introduce a local Hamiltonian $\mathcal{H}_{QIC}$ (detailed in Sec. 2, Eq. (1)) which energetically penalizes $|00\rangle$ configurations, ensuring $\mathcal{H}_{QIC}$ is its zero-energy ground state manifold. Crucially, the dimension of this physically motivated subspace is $\dim(\mathcal{H}_{QIC}) = F_{N+2}$ (where F_n is the n-th Fibonacci number, $F_1 = 1, F_2 = 1$), exactly matching the dimension of the fusion space of N abstract Fibonacci anyons [3]. This establishes $\mathcal{H}_{QIC}$ as a valid encoding space for the standard Fibonacci anyon model.

Simulating TQC within $\mathcal{H}_{QIC}$, however, requires implementing operators that correctly represent anyon braiding. These operations are mathematically described by the Temperley-Lieb (TL) algebra $TL_N(t)$ with parameter $t = \phi^{-2}$ (where ϕ is the golden ratio) [8, 9]. A known challenge arises when attempting to realize this algebra on qubits within constrained subspaces. While models like the Golden Chain [3] define local projectors satisfying the TL algebra in the anyon fusion basis, directly adapting such local constructions to act on qubits within $\mathcal{H}_{QIC}$ does not automatically ensure the crucial multi-site TL relations (Eqs. (4)-(5)) are preserved. Indeed, as confirmed numerically in this work (Sec. 5), simple local qubit operators fail to satisfy these defining algebraic relations when restricted to $\mathcal{H}_{QIC}$. This work addresses this crucial implementation gap. We present and rigorously validate a method to consistently represent and simulate the required $TL_N(\phi^{-2})$ algebra and the associated Braid Group operations within the Hamiltonian-defined QIC subspace $\mathcal{H}_{QIC}$. Our approach involves: (i) constructing the abstract F_{N+2}-dimensional Temperley-Lieb projector matrices (P_n^{anyon}) that act correctly on the QIC basis states according to established diagrammatic rules [9], and (ii) embedding these abstract operators into the full 2^N-dimensional qubit Hilbert space using a computationally derived isometry V via the transformation $P'_n = V P_n^{anyon} V^\dagger$ (Sec. 4).

Through extensive numerical simulations performed for systems up to N=10 qubits (Sec. 7), we demonstrate that these embedded operators P'_n rigorously satisfy the defining relations of the $TL_N(\phi^{-2})$ algebra (Eqs. (2)-(5)). Consequently, the braid operators B'_n constructed from them (Eq. (6)) are unitary, satisfy the Yang-Baxter equation and required commutation relations, and demonstrably preserve the protected QIC subspace $\mathcal{H}_{QIC}$ during operation. This provides a concrete, numerically validated, and scalable framework for simulating Fibonacci anyon TQC on standard qubit platforms, leveraging the physically motivated structure of quasicrystal-inspired constraints. Practical aspects are further explored via Qiskit simulations and resource estimation for N=3 (Secs. 8-9), and the relation to alternative QIC models is discussed (Sec. 10).

2 The QIC Hamiltonian and Protected Subspace

We consider a system of N qubits with the total Hilbert space $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$. We propose a QIC Hamiltonian based on the fundamental structural rule of 1D Fibonacci quasicrystals (tilings or sequences). These physical systems, often generated by substitution rules like $L \to LS, S \to L$, naturally forbid adjacent identical "short" tiles ('SS') [6, 7]. Mapping the short tile to the qubit state $|0\rangle$ ($S \leftrightarrow |0\rangle$) and the long tile to $|1\rangle$ ($L \leftrightarrow |1\rangle$), this physical constraint translates directly to forbidding adjacent $|00\rangle$ states in the qubit chain.

Our Hamiltonian, $\mathcal{H}_{QIC}$, is designed to energetically enforce exactly this physically motivated constraint. Let Z_i be the Pauli-Z operator on qubit i. The local projector onto the forbidden $|00\rangle$ state at adjacent sites $i, i+1$ is given by $\Pi^{(00)}_{i+1,i} = \frac{1}{4}(I + Z_{i+1})(I + Z_i)$. The QIC Hamiltonian is then defined as a sum of these local penalty terms:

$$\mathcal{H}_{QIC} = \lambda \sum_{i=0}^{N-2} \Pi^{(00)}_{i+1,i} \quad (\lambda > 0) \tag{1}$$

The ground state manifold of $\mathcal{H}_{QIC}$ (with energy 0) defines the QIC subspace, denoted $\mathcal{H}_{QIC} \subset \mathcal{H}$. By construction, $\mathcal{H}_{QIC}$ is spanned by the set of all computational basis states $|s_{N-1} \dots s_0\rangle$ of length N that contain no adjacent '00' substring.

The dimension of this subspace follows the Fibonacci recurrence and is given by $\dim(\mathcal{H}_{QIC}) = F_{N+2}$, where F_n are the standard Fibonacci numbers with $F_0 = 0, F_1 = 1$ (sequence 1, 1, 2, 3, 5, 8, ...). This dimensionality is the well-known result for binary sequences avoiding '00' and, critically, precisely matches the dimension of the fusion space of N Fibonacci anyons [3]. An orthonormal basis $\{|q_a\rangle\}_{a=1}^{F_{N+2}}$ for $\mathcal{H}_{QIC}$, corresponding for instance to the lexicographically sorted valid 'no 00' binary strings, can be algorithmically constructed.

Thus, this explicit Hamiltonian $\mathcal{H}_{QIC}$ provides a concrete qubit realization whose protected code space $\mathcal{H}_{QIC}$ is directly motivated by physical Fibonacci quasicrystal constraints and possesses the correct dimensionality required for simulating the standard N-anyon Fibonacci model.

3 Required Anyon Operators and Algebra

Logical operations simulating the braiding of Fibonacci anyons within the protected subspace $\mathcal{H}_{QIC}$ must adhere to the algebraic structure defined by the Temperley-Lieb (TL) algebra $TL_N(t)$ [8, 9]. This algebra provides the mathematical framework for describing the fusion and braiding of anyons. For the Fibonacci model relevant here, the TL parameter is $t = \phi^{-2}$, where $\phi = (1 + \sqrt{5})/2$ is the golden ratio. The algebras for different N form a nested structure $\mathcal{A}_N = \mathrm{Alg}(P_0, \dots, P_{N-2}) \subset \mathcal{A}_{N+1}$, analogous to the Jones tower construction in operator algebras.

3.1 Temperley-Lieb Projectors (P_n)

The generators of the algebra $\mathcal{A}_N$ are the Temperley-Lieb projectors P_n ($n = 0, \dots, N-2$). Abstractly, these operators act on the F_{N+2}-dimensional QIC basis $\mathcal{H}_{QIC}$ (equivalent to the N-anyon fusion basis) and satisfy the following defining relations:

$$P_n^2 = P_n \qquad \text{(Idempotent)} \tag{2}$$

$$P_n^\dagger = P_n \qquad \text{(Hermitian)} \tag{3}$$

$$P_n P_{n\pm1} P_n = \phi^{-2} P_n \qquad \text{(TL Relation)} \tag{4}$$

$$P_n P_m = P_m P_n \qquad \text{for } |n-m| \geq 2 \tag{5}$$

Physically, within the anyon model, the projector P_n acts locally on anyons at positions n and $n+1$, projecting onto the identity (vacuum) fusion channel I, while $(I - P_n)$ projects onto the Fibonacci anyon (τ) channel. (Note: These operators are analogous to E_n in [5], Eq. (1-3)).

3.2 Braid Operators (B_n)

Unitary operators B_n representing the physical process of braiding adjacent anyons (at positions n and $n+1$) are constructed from the TL projectors P_n. A standard representation consistent with the $SU(2)_3$ Chern-Simons theory underlying Fibonacci anyons assigns distinct phases to the two possible fusion channels [1, 9]. This yields the braid operator:

$$B_n = R_I P_n + R_\tau (I - P_n) = e^{-4\pi i/5} P_n + e^{3\pi i/5} (I - P_n) \tag{6}$$

where I is the identity operator on the F_{N+2}-dimensional space, and $R_I = e^{-4\pi i/5}$ and $R_\tau = e^{3\pi i/5}$ are the topological phases (R-matrix eigenvalues) associated with the identity and τ fusion channels, respectively. These operators B_n satisfy the Braid Group relations required for valid TQC operations (including the Yang-Baxter equation $B_n B_{n+1} B_n = B_{n+1} B_n B_{n+1}$ and the commutation relation $B_n B_m = B_m B_n$ for $|n-m| \geq 2$) if and only if the underlying projectors P_n satisfy the TL algebra relations (2)-(5).

From the definition (6), one can directly verify the consistency relation:

$$P_n B_n = B_n P_n = e^{-4\pi i/5} P_n. \tag{7}$$

This confirms that P_n projects onto the subspace (the identity fusion channel) where the braid operator B_n acts simply as multiplication by the phase $R_I = e^{-4\pi i/5}$.

4 Embedding into Qubit Space

To simulate these operations on qubits, we map the abstract operators into the 2^N-dimensional Hilbert space $\mathcal{H}$.

1. **Isometry (V):** We construct the isometry $V : \mathbb{C}^{F_{N+2}} \to \mathcal{H}_{QIC} \subset \mathcal{H}$ mapping the abstract F_{N+2}-dimensional basis (ordered by QIC strings) to the corresponding orthonormal state vectors $\{|q_a\rangle\}$ within $\mathcal{H}$. V satisfies $V^\dagger V = I_{F_{N+2}}$.

2. **Embedded Operators (P'_n, B'_n):** The abstract operators P_n^{anyon} and B_n^{anyon} (acting on $\mathbb{C}^{F_{N+2}}$) are embedded into $\mathcal{H}$ as $2^N \times 2^N$ matrices:

$$P'_n = V P_n^{anyon} V^\dagger \tag{8}$$

$$B'_n = V B_n^{anyon} V^\dagger \tag{9}$$

These operators act non-trivially only within the QIC subspace $\mathcal{H}_{QIC}$ (i.e., $P'_n = P'_n \Pi_{QIC} = \Pi_{QIC} P'_n$ where $\Pi_{QIC} = VV^\dagger$). If P_n^{anyon} satisfy the TL algebra, P'_n will satisfy it when restricted to $\mathcal{H}_{QIC}$. Similarly, B'_n will satisfy the braid relations when acting within $\mathcal{H}_{QIC}$. Note that B'_n derived this way is equivalent to $B'_n = e^{-4\pi i/5} P'_n + e^{3\pi i/5}(\Pi_{QIC} - P'_n)$.

5 Failure of Standard Local Qubit Operators

Initial attempts to directly implement the TL projectors using standard local qubit operators within $\mathcal{H}_{QIC}$ proved unsuccessful. For instance, using the standard local two-qubit spin-singlet projector P_n^{qubit} as a candidate for P'_n fails the TL relation (4): $||P_n^{qubit} P_{n+1}^{qubit} P_n^{qubit} - \phi^{-2} P_n^{qubit}|| \not\approx 0$. This failure prevents the construction of valid braid operators using Eq. (6) with P_n^{qubit}. This exemplifies the general difficulty: faithfully representing the non-local constraints imposed by the TL algebra's multi-site relations within the constrained qubit subspace $\mathcal{H}_{QIC}$ cannot typically be achieved by simple, local qubit operators, highlighting the need for a more sophisticated approach than direct adaptation of local actions inspired by models such as [3].

6 N=3 QIC Implementation and Verification

The correct approach, successfully implemented and verified for $N = 3$, involves using the appropriate abstract operators P_n^{anyon} before embedding.

1. The explicit 5×5 matrix representations P_0^{anyon}, P_1^{anyon} were constructed based on the TL generator action rules from Kauffman & Lomonaco [9] (Theorem 2) applied to the N=3 ordered QIC basis ('['010', '011', '101', '110', '111']'). These matrices satisfy (2)-(3) by construction.

2. The 8×8 embedded operators $P'_n = V P_n^{anyon} V^\dagger$ and $B'_n = V B_n^{anyon} V^\dagger$ were computed.

3. **Numerical Verification:** Python/SciPy simulations confirmed that these embedded operators satisfy all required algebraic properties numerically to high precision ($< 10^{-15}$):

   - $(P'_n)^\dagger = P'_n$ and $(P'_n)^2 = P'_n$.

- $P'_n P'_{n\pm1} P'_n = \phi^{-2} P'_n$ (TL relation holds).
- $B'_n (B'_n)^\dagger = I$ (Unitarity holds).
- $B'_0 B'_1 B'_0 = B'_1 B'_0 B'_1$ (Yang-Baxter holds).

4. **Subspace Preservation:** Further simulations using Qiskit confirmed that applying the braid operator B'_n to an equal superposition state $|\psi\rangle \in \mathcal{H}_{QIC}$ yielded a state $|\psi'\rangle = B'_n |\psi\rangle$ that remained in the QIC subspace ($\langle\psi'| \mathcal{H}_{QIC} |\psi'\rangle \approx 0$).

7 Scaling and Verification for $N > 3$

To rigorously test the scalability and consistency of the framework, the simulation was extended beyond N=3. The generalized procedure for constructing the abstract P_n^{anyon} matrices based on [9] rules (Theorem 2, using the middle-strand rules applied with flanking 'P's) was implemented. The simulation pipeline – generating the QIC basis (dimension F_{N+2}), constructing the isometry V ($2^N \times F_{N+2}$), computing P_n^{anyon}, and building the embedded operators $P'_n = V P_n^{anyon} V^\dagger$ and $B'_n = V B_n^{anyon} V^\dagger$ – was executed successfully for N=4, N=5, and N=6.

For each system size, comprehensive numerical checks were performed:

- The generated abstract matrices P_n^{anyon} were verified to satisfy the Temperley-Lieb algebra relations (2)-(4) (including $P_n P_m = P_m P_n$ for $|n - m| \geq 2$) to high numerical precision ($< 10^{-15}$).
- The embedded operators P'_n correctly behaved as projectors within the QIC subspace.
- The embedded braid operators B'_n were confirmed to be unitary ($B'_n (B'_n)^\dagger = I$) and satisfy the required braid group relations: the Yang-Baxter equation ($B'_n B'_{n+1} B'_n = B'_{n+1} B'_n B'_{n+1}$) and, critically, the **commutation relation for non-adjacent indices** ($B'_n B'_m = B'_m B'_n$ for $|n - m| \geq 2$). These checks also passed with high precision.

Furthermore, the simulation was successfully scaled up to **N=10**. This involved:

- Generating $F_{12} = 144$ QIC basis states.
- Constructing the 1024×144 isometry V.
- Building and verifying all 9 abstract Temperley-Lieb projectors $P_0^{anyon}, \ldots, P_8^{anyon}$ (size 144×144).
- Building and verifying all 9 embedded braid operators $B'_0, \ldots, B'_8$ (size 1024×1024).

All algebraic checks, including the full set of non-adjacent commutation relations for the B'_n operators, passed successfully for N=10, again with numerical errors below 10^{-14}. The computation was performed efficiently using sparse matrix representations in Python/SciPy.

Finally, the Qiskit-based energy verification was performed for N=10 using the QIC Hamiltonian (1). Both the initial equal superposition state within $\mathcal{H}_{QIC}$ and the state resulting from applying a braid operator (B'_4) were confirmed to have an energy expectation value numerically equal to zero. This confirms that the derived braid operators correctly act within the protected QIC ground state subspace even for larger system sizes.

These results provide strong evidence that the implemented framework, deriving operators from the Kauffman & Lomonaco representation and embedding them via the QIC isometry, correctly captures the required algebraic structure for Fibonacci anyon braiding and scales consistently to larger numbers of qubits.

8 Qiskit Simulation of $N = 3$ Braiding and Projection

Building on the verified algebraic structure of the embedded operators for $N = 3$, we performed a Qiskit simulation [10] demonstrating the full cycle of state preparation, braiding, and projection within the QIC framework.

8.1 Methodology

The simulation proceeded as follows:

1. **State Preparation:** A basis state $|101\rangle$ from the QIC subspace $\mathcal{H}_{QIC}$ was selected. Its representation in the $F_4 = 5$-dimensional QIC basis, denoted $|101\rangle_{\text{QIC}}$, was mapped to the full $2^3 = 8$-dimensional Hilbert space $\mathcal{H}$ using the verified isometry V:

$$|\psi_{\text{init}}\rangle = V\,|101\rangle_{\text{QIC}} \in \mathcal{H}.$$

2. **Circuit Initialization:** A 3-qubit 'QuantumCircuit' was created and initialized to $|\psi_{\text{init}}\rangle$ using 'circuit.initialize()'.

3. **Braiding:** The verified 8×8 braid operator B'_0 was applied to all qubits using 'circuit.unitary()'. The resulting state is

$$|\psi_{\text{braided}}\rangle = B'_0\,|\psi_{\text{init}}\rangle\,.$$

   The norm was verified to be 1, confirming unitarity.

4. **Projection:** The embedded projector P'_1 (corresponding to P_{N-1} for $N = 3$) was applied:

$$\left|\psi_{\text{projected}}\right\rangle = P'_1\,|\psi_{\text{braided}}\rangle\,.$$

5. **Probability and Normalization:** The projection probability was computed as

$$p = \|\left|\psi_{\text{projected}}\right\rangle\|^2.$$

   The final state was normalized:

$$|\psi_{\text{final}}\rangle = \frac{\left|\psi_{\text{projected}}\right\rangle}{\sqrt{p}} \quad \text{(assuming } p > 0\text{)}.$$

6. **Verification and Analysis:**

   - The energy expectation $\langle\psi_{\text{final}}|\,\mathcal{H}_{QIC}\,|\psi_{\text{final}}\rangle$ was computed via Qiskit's 'AerEstimatorV2', confirming that the state remains within the QIC subspace (energy ≈ 0).
   - The final state was projected back into the QIC basis via $V^\dagger$:

$$|\psi_{\text{final}}\rangle_{\text{QIC}} = V^\dagger\,|\psi_{\text{final}}\rangle\,.$$

8.2 Results for Initial State $|101\rangle$, Braid B'_0, Projector P'_1

Key outcomes of the simulation:

- The norm of the state remained 1 after applying B'_0.
- The projection probability was $p = \|P'_1 B'_0\,|\psi_{\text{init}}\rangle\|^2 \approx 0.381966$, matching the expected value ϕ^{-2}.
- The energy of $|\psi_{\text{final}}\rangle$ under $\mathcal{H}_{QIC}$ was verified to be near zero.
- The final state in the QIC basis was:

$$|\psi_{\text{final}}\rangle_{\text{QIC}} \approx (-0.191 + 0.588j)\,|101\rangle + (-0.243 + 0.748j)\,|111\rangle\,,$$

  with coefficients rounded to three decimal places.

8.3 Interpretation

This simulation demonstrates a complete operational cycle for $N = 3$ within the QIC framework. Specifically:

1. Valid QIC states can be initialized and manipulated using standard quantum circuits.
2. The embedded braid operators B'_k act unitarily and preserve the QIC subspace.
3. The projectors P'_k yield probabilities consistent with fusion rules (e.g., $\approx \phi^{-2}$ for the identity channel).
4. The final state remains in $\mathcal{H}_{QIC}$ and can be analyzed in the abstract QIC basis.

These results provide concrete simulation-based evidence supporting the viability of Fibonacci anyon braiding logic within the proposed QIC qubit encoding.

9 N=3 Simulation Benchmarks: Fusion and Resource Estimation

To further validate the QIC framework and assess its potential for implementation, we performed two key simulations using the verified $N = 3$ operators (P'_0, P'_1, B'_0, B'_1) within the Qiskit ecosystem.

9.1 Fusion Probability Simulation

We investigated the probabilities associated with fusion outcomes after braiding, specifically by calculating the probability

$$p = \left\| P'_1 B'_k \left| s \right\rangle \right\|^2$$

for projecting the state onto the subspace associated with the identity fusion channel (the image of P'_1) after applying a braid B'_k ($k = 0, 1$) to an initial QIC basis state $|s\rangle$. The simulation iterated through all 5 basis states:

$$|s\rangle \in \{|010\rangle, |011\rangle, |101\rangle, |110\rangle, |111\rangle\}.$$

The results showed variation in the projection probability depending on the specific initial state and the applied braid, yielding values approximately

$$p \approx 0, \quad p \approx \phi^{-2} \approx 0.382, \quad p \approx \phi^{-1} \approx 0.618, \quad p \approx \phi^{-3} \approx 0.236, \quad p \approx 1,$$

where ϕ denotes the golden ratio. This state-dependence is expected, as the probability reflects the squared norm of the component of the *braided* state $B'_k |s\rangle$ lying within the image of P'_1. The simulation correctly quantified these distinct probabilities for all 10 combinations, confirming the expected behavior based on the action of the verified operators and providing numerical confirmation consistent with Fibonacci anyon fusion rules applied dynamically after braiding.

9.2 Hardware Resource Estimation (IBM Eagle)

To estimate the resources required for near-term hardware implementation, we used Qiskit's transpiler to decompose the $N = 3$ embedded braid operators B'_0 and B'_1 (represented as 8×8 unitary matrices) into the native gate set of a specific IBM Quantum backend. We targeted `ibm_sherbrooke`, a 127-qubit Eagle processor featuring heavy-hexagon connectivity and a basis gate set {`ecr`, `id`, `rz`, `sx`, `x`}. We used `optimization_level=3`.

The transpilation results were:

- **For B'_0:**
  - Transpiled Depth: ≈ 157

  - Operations Count: Rz: 125, Sx: 76, ECR: 33, X: 1
  - 2-Qubit Gate Count: 33 (ECR gates)

- **For B'_1:**
  - Transpiled Depth: ≈ 114
  - Operations Count: Rz: 103, Sx: 69, ECR: 26
  - 2-Qubit Gate Count: 26 (ECR gates)

These results provide a concrete estimate of the gate complexity. The number of 2-qubit gates (26–33 ECRs) is non-trivial but potentially feasible for near-term experiments, although the circuit depth indicates that coherence times and gate fidelity will be significant factors. The transpiler automatically accounts for the heavy-hexagon connectivity, likely inserting SWAP operations implemented via the basis gates, contributing to the overall counts and depth. This estimation provides a valuable baseline for planning potential hardware implementations and assessing noise impact.

10 Relation to Physical Fibonacci Sequences and Alternative QICs

The QIC framework validated in this paper is based on the Hilbert subspace $\mathcal{H}_{QIC}$, defined by enforcing a single constraint that excludes adjacent $|00\rangle$ configurations. As discussed in Sec. 2, this "no 00" rule is directly motivated by the fundamental "no SS" property (with $S \leftrightarrow |0\rangle$) observed in 1D Fibonacci tilings generated by standard inflation or substitution rules, such as $S \to L$, $L \to LS$ [6,7]. The resulting Hilbert space has dimension $\dim(\mathcal{H}_{QIC}) = F_{N+2}$, where F_n denotes the n-th Fibonacci number. This dimension is essential, as it matches the fusion space of N Fibonacci anyons and supports consistent simulation of the Temperley–Lieb algebra $TL_N(\phi^{-2})$ [3].

However, one may consider an alternative encoding more tightly aligned with the structure of sequences generated by substitution rules. While these rules enforce the "no 00" constraint, they also inherently avoid the formation of "111" substrings (corresponding to "LLL"). This observation motivates the definition of a more restrictive code subspace that simultaneously enforces both "no 00" and "no 111" constraints. We refer to this as the *Plastic QIC*, as it is associated with sequences governed by the Plastic number growth law [11].

The dual-constraint subspace $\mathcal{H}_{\mathrm{QIC}}^{\mathrm{Plastic}} \subset \mathcal{H}$ has dimension a_N, which follows the recurrence associated with the Plastic number: $a_N \sim \rho_{\mathrm{PL}}^N$, where $\rho_{\mathrm{PL}} \approx 1.3247$ [11]. This is asymptotically smaller than the Fibonacci dimension $F_{N+2} \sim \phi^N/\sqrt{5}$, with $\phi \approx 1.618$, for $N \geq 5$.

While the Plastic QIC offers an appealing connection to the deeper combinatorics of Fibonacci tilings, its dimensional mismatch with the standard fusion space of N Fibonacci anyons means it is not suitable for simulating the $TL_N(\phi^{-2})$ algebra. For this reason, the present work focuses exclusively on the F_{N+2}-dimensional subspace $\mathcal{H}_{QIC}$, as it provides the correct structure for faithfully encoding the standard Fibonacci anyon model.

In summary, this paper rigorously validates the simulation of the Fibonacci anyon model within the F_{N+2}-dimensional QIC defined by the physically motivated "no 00" constraint. We acknowledge the alternative, more constrained Plastic QIC (dimension a_N), which reflects additional structural properties of Fibonacci sequences and merits further study in future work [11].

11 Conclusion and Outlook

This work successfully validates a method for the consistent representation and simulation of the standard Fibonacci Temperley-Lieb algebra ($TL_N(\phi^{-2})$) and Braid Group operations within a protected Quasicrystal Inflation Code (QIC) subspace realized on qubits. We defined the QIC Hilbert space $\mathcal{H}_{QIC}$ using a concrete Hamiltonian ($\mathcal{H}_{QIC}$, Eq. (1)) that enforces the 'no 00' constraint, a rule directly motivated by the physical structure ('no SS' tiling space rule, $S \leftrightarrow |0\rangle$) of 1D Fibonacci quasicrystals. This subspace correctly possesses the required F_{N+2} dimension matching the N-anyon Fibonacci fusion space [5]. Our key contribution lies in demonstrating that the embedding of abstract $F_{N+2} \times F_{N+2}$ projector matrices P_n^{anyon} (derived from Kauffman & Lomonaco [9]) into the 2^N-dimensional qubit space via a computationally constructed isometry V ($P'_n = V P_n^{anyon} V^\dagger$) successfully enforces the complete TL algebra, overcoming the inconsistencies observed when attempting to use simpler local qubit operators or direct adaptations of local projector definitions from models like [3] within the constrained subspace. Through extensive numerical simulations scaling up to **N=10** qubits, we rigorously verified that the embedded operators P'_n satisfy the TL algebra relations and the derived braid operators B'_n are unitary, satisfy the Yang-Baxter equation and far-commutation relations, and preserve the zero-energy QIC subspace defined by $\mathcal{H}_{QIC}$.

This provides a concrete, validated pathway for implementing standard Fibonacci TQC on a qubit substrate. The QIC approach, grounded in the physically motivated 'no 00' constraint, offers potential advantages by leveraging the natural structural rules found in quasicrystalline systems. Furthermore, the inherent aperiodic order underlying the QIC basis states may offer distinct pathways towards robustness against certain classes of errors compared to other encoding schemes [5].

The validated simulations form a strong basis for pursuing the QIC approach towards fault-tolerant TQC. Immediate next steps include:

1. Developing efficient quantum circuit decompositions for the embedded braid operators B'_n. The explicit sparse unitary matrices ($2^N \times 2^N$, validated up to N=10) serve as targets for quantum circuit synthesis algorithms, aiming to minimize gate complexity (particularly two-qubit gates like CNOT or ECR) for near-term hardware.

2. Simulating these decomposed braiding circuits on NISQ hardware or advanced classical simulators (statevector/tensor-network) to benchmark the performance and fidelity of logical operations within the QIC code space under realistic noise conditions.

3. Investigating resource requirements for fault-tolerant implementations, exploring how the specific structure of the QIC code and the non-local nature of the braid operators might interact with quantum error correction codes.

4. Further theoretical analysis of the physical realizability of the $\mathcal{H}_{QIC}$ Hamiltonian using realistic qubit interactions and the robustness of the QIC subspace against plausible physical error models.

5. Exploring the distinct algebraic structures and simulation possibilities for alternative QIC models, such as the dual-constraint Plastic QIC discussed in Sec. 10 [11].

In summary, this research demonstrates a consistent and numerically verified method for simulating Fibonacci anyon braiding on qubits using a QIC defined by physically motivated constraints from Fibonacci quasicrystals. It establishes the operator embedding technique as a viable tool and provides a concrete, validated simulation framework, strengthening the potential of using quasicrystal-inspired codes for achieving topological quantum computation on programmable quantum hardware.

The code used for the numerical simulations reported in this study is available from the corresponding author upon reasonable request.

A Explicit Matrices and Basis for N=3 Simulation

This appendix provides the explicit basis states and operator matrices for the $N = 3$ qubit QIC simulation discussed in the main text (e.g., Sec. 8, 9). These forms facilitate verification and visualization of the calculations. The matrices provided were generated using the verified simulation code implementing the methods described, particularly the Kauffman & Lomonaco rules [9] for the anyonic projectors.

A.1 N=3 QIC Basis States

The $N = 3$ QIC Hilbert subspace $\mathcal{H}_{QIC}$ is spanned by computational basis states $|s_2 s_1 s_0\rangle$ that do not contain adjacent $|00\rangle$. The dimension is $F_{3+2} = F_5 = 5$. Using lexicographical ordering, the basis states $\{|q_a\rangle\}_{a=1}^5$ are:

$$\{|q_1\rangle = |010\rangle, |q_2\rangle = |011\rangle, |q_3\rangle = |101\rangle, |q_4\rangle = |110\rangle, |q_5\rangle = |111\rangle\}$$

A.2 N=3 Isometry Matrix V

The 8×5 isometry V maps the 5-dimensional QIC basis to the full 8-dimensional Hilbert space $(\mathbb{C}^2)^{\otimes 3}$. Its columns are the basis vectors $|q_a\rangle$ embedded in the standard computational basis $\{|000\rangle, \ldots, |111\rangle\}$. We use the convention where the vector index i for state $|s_2 s_1 s_0\rangle$ is determined by interpreting the reversed string $s_0 s_1 s_2$ as a binary number (e.g., $|q_2\rangle = |011\rangle$ corresponds to index $110_2 = 6$).

$$V = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

By construction, $V^\dagger V = I_5$, where I_5 is the 5×5 identity matrix.

A.3 N=3 Anyonic Projectors P_k^{anyon}

The 5×5 anyonic Temperley-Lieb projectors P_k^{anyon} acting on the ordered QIC basis $\{|q_1\rangle, \ldots, |q_5\rangle\}$ were generated by the simulation code based on [9]. They numerically satisfy the defining relations (Eqs. (2)-(5) in the main text). Let $\phi = (1 + \sqrt{5})/2$ be the golden ratio. Define the constants $t = \phi^{-2}$, $x = \phi^{-3/2}$, and $y = \phi^{-1}$. The projectors are:

$$P_0^{anyon} = \begin{pmatrix} t & 0 & 0 & x & 0 \\ 0 & t & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 \\ x & 0 & 0 & y & 0 \\ 0 & x & 0 & 0 & y \end{pmatrix}$$

$$P_1^{anyon} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & x \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & y \end{pmatrix}$$

A.4 N=3 QIC Hamiltonian H_{QIC}

The $N = 3$ QIC Hamiltonian $H_{QIC} = \sum_{i=0}^{1} \Pi_{i+1,i}^{(00)}$ (Eq. (1) with penalty strength $\lambda = 1$), whose zero-energy ground state manifold defines $\mathcal{H}_{QIC}$, can be expressed using Pauli operators (with Z_k acting on qubit k, indexed $k = 0, 1, 2$, corresponding to tensor product order $Z_2 \otimes Z_1 \otimes Z_0$) as:

$$H_{QIC} = \frac{1}{2}III + \frac{1}{4}IIZ + \frac{1}{2}IZI + \frac{1}{4}ZII + \frac{1}{4}IZZ + \frac{1}{4}ZZI$$

where III represents the 8×8 identity matrix I_8, IIZ represents $I \otimes I \otimes Z$, IZI represents $I \otimes Z \otimes I$, IZZ represents $I \otimes Z \otimes Z$, etc. The QIC basis states $\{|q_a\rangle\}$ are, by construction, eigenstates of H_{QIC} with eigenvalue 0.

A.5 Embedded Operators P'_k, B'_k

The 8×8 embedded projector and braid operators, P'_k and B'_k, acting on the full Hilbert space $(\mathbb{C}^2)^{\otimes 3}$, are constructed from the components above using the relations $P'_k = VP_k^{anyon}V^\dagger$ and the formula for B'_k derived from Eq. (6) ($B'_k = R_I P'_k + R_\tau(\Pi_{QIC} - P'_k)$ where $\Pi_{QIC} = VV^\dagger$ is the projector onto the QIC subspace). Their explicit 8×8 forms are omitted for brevity. As discussed in the main text (e.g., Sec. 8, 9), their algebraic properties (TL algebra, Unitarity, Yang-Baxter) and subspace preservation under H_{QIC} were numerically verified.

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