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Physics 681-481; CS 483: Discussion of #1

I. Tensor products and positional notation.

We represent the digits 5, 3, and 2 by 10-component column vectors $\mathbf{c}$, $\mathbf{b}$, and $\mathbf{a}$, given by

$$\mathbf{c} = \begin{pmatrix} 0\\0\\0\\0\\0\\1\\0\\0\\0\\0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0\\0\\0\\1\\0\\0\\0\\0\\0\\0 \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} 0\\0\\1\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}, \tag{1}$$

Now the rule for forming the tensor product of two 10-dimensional column vectors

$$\mathbf{u} = \begin{pmatrix} u_0\\u_1\\u_2\\\vdots \end{pmatrix} \text{ and } \mathbf{v} = \begin{pmatrix} v_0\\v_1\\v_2\\\vdots \end{pmatrix} \tag{2}$$

is that $\mathbf{u}\otimes\mathbf{v}$ is the $10\times 10 = 100$-dimensional column vector given by

$$\mathbf{u}\otimes\mathbf{v} = \begin{pmatrix} u_0\mathbf{v}\\u_1\mathbf{v}\\u_2\mathbf{v}\\u_3\mathbf{v}\\\vdots\\u_9\mathbf{v} \end{pmatrix} = \begin{pmatrix} u_0v_0\\u_0v_1\\u_0v_2\\\vdots\\u_1v_0\\u_1v_1\\u_1v_2\\\vdots\\u_9v_9 \end{pmatrix}, \tag{3}$$

Since only b_3 and a_2 are non-zero, and both of these are 1, the tensor product $\mathbf{b}\otimes\mathbf{a}$ is a 100-dimensional column vector with all components 0 except the one 32 down from the top. Since only c_5 is non-zero, repeating this for $\mathbf{c}\otimes\big(\mathbf{b}\otimes\mathbf{a}\big)$ establishes that $\mathbf{c}\otimes\mathbf{b}\otimes\mathbf{a}$ is indeed a 1000-dimensional column vector with all components 0 except the one 532 down from the top.

More generally, this establishes that for 10-vectors

$$c_ib_ja_k = \big(\mathbf{c}\otimes\mathbf{b}\otimes\mathbf{a}\big)_{100i+10j+k}. \tag{4}$$

So if only c_5, b_3, and a_2 are nonzero, then only component 532 of $\mathbf{c}\otimes\mathbf{b}\otimes\mathbf{a}$ will be nonzero.

II. Manipulating elementary operators.

If

$$\mathsf{C}_{ij} = \overline{\mathsf{n}}_i + \mathsf{X}_j \mathsf{n}_i, \tag{5}$$

then

$$\mathsf{C}_{ij}\mathsf{C}_{j\,i}\mathsf{C}_{ij} = \big(\overline{\mathsf{n}}_i + \mathsf{X}_j\mathsf{n}_i\big)\big(\overline{\mathsf{n}}_j + \mathsf{X}_i\mathsf{n}_j\big)\big(\overline{\mathsf{n}}_i + \mathsf{X}_j\mathsf{n}_i\big) \tag{6}$$

Consider first the 4 terms in (6) containing either one X or three:

$$\big(\mathsf{X}_j\mathsf{n}_i\big)\overline{\mathsf{n}}_j\overline{\mathsf{n}}_i,\quad \overline{\mathsf{n}}_i\big(\mathsf{X}_i\mathsf{n}_j\big)\overline{\mathsf{n}}_i,\quad \overline{\mathsf{n}}_i\overline{\mathsf{n}}_j\big(\mathsf{X}_j\mathsf{n}_i\big),\quad \big(\mathsf{X}_j\mathsf{n}_i\big)\big(\mathsf{X}_i\mathsf{n}_j\big)\big(\mathsf{X}_j\mathsf{n}_i\big). \tag{7}$$

Every one of these terms vanishes as a consequence of these facts: $\mathsf{n}_i\overline{\mathsf{n}}_i = 0$, n_i and $\overline{\mathsf{n}}_i$ commute with all the operators appearing in (6) other than X_i, and

$$\mathsf{X}_i\mathsf{n}_i = \overline{\mathsf{n}}_i\mathsf{X}_i, \quad \mathsf{X}_i\overline{\mathsf{n}}_i = \mathsf{n}_i\mathsf{X}_i. \tag{8}$$

The remaining 4 terms in (6) can be simplified using these facts together with

$$\mathsf{n}_i^2 = \mathsf{n}_i, \quad \overline{\mathsf{n}}_i^2 = \overline{\mathsf{n}}_i, \quad \mathsf{X}_j^2 = 1. \tag{9}$$

One has:

$$\overline{\mathsf{n}}_i\overline{\mathsf{n}}_j\overline{\mathsf{n}}_i = \overline{\mathsf{n}}_i\overline{\mathsf{n}}_j, \tag{10}$$

$$\overline{\mathsf{n}}_i\big(\mathsf{X}_i\mathsf{n}_j\big)\big(\mathsf{X}_j\mathsf{n}_i\big)) = \mathsf{X}_i\mathsf{X}_j\mathsf{n}_i\overline{\mathsf{n}}_j, \tag{11}$$

$$\big(\mathsf{X}_j\mathsf{n}_i\big)\overline{\mathsf{n}}_j\big(\mathsf{X}_j\mathsf{n}_i\big) = \mathsf{n}_i\mathsf{n}_j, \tag{12}$$

$$\big(\mathsf{X}_j\mathsf{n}_i\big)\big(\mathsf{X}_i\mathsf{n}_j\big)\overline{\mathsf{n}}_i = \mathsf{X}_i\mathsf{X}_j\overline{\mathsf{n}}_i\mathsf{n}_j. \tag{13}$$

Adding together the four terms in (10)-(13) gives the SWAP operator S_{ij} in the form

$$\mathsf{S}_{ij} = \mathsf{n}_i\mathsf{n}_j + \overline{\mathsf{n}}_i\overline{\mathsf{n}}_j + (\mathsf{X}_i\mathsf{X}_j)(\mathsf{n}_i\overline{\mathsf{n}}_j + \overline{\mathsf{n}}_i\mathsf{n}_j), \tag{14}$$

III. The Toffoli gate.

T_{210} acts as the identity on all eight 3-Cbit states $|0\rangle_3, \ldots, |7\rangle_3$ except that it exchanges $|110\rangle = 6_3$ and $|111\rangle = |7\rangle_3$ Therefore its matrix is

$$\mathsf{T}_{210} \to \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0 \end{pmatrix}. \tag{15}$$

Similarly, the Toffoli gate T_{201} interchanges $|101\rangle = |5\rangle_3$ and $|111\rangle = |7\rangle_3$, so its matrix is

$$\mathsf{T}_{201} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \tag{16}$$

and T_{102} interchanges $|011\rangle = |3\rangle_3$ and $|111\rangle = |7\rangle_3$, so its matrix is

$$\mathsf{T}_{102} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{17}$$