February 16, 2006

Physics 681-481; CS 483: Discussion of #2

I. Constructing a spooky 2-Qbit state

We can write the state $|\Psi\rangle$ as

$$|\Psi\rangle = \tfrac{1}{\sqrt{3}}\big(|00\rangle + |01\rangle + |10\rangle\big) = \big(\mathbf{1}\otimes\mathsf{H}\big)\big(\sqrt{\tfrac{2}{3}}|0\rangle|0\rangle + \sqrt{\tfrac{1}{3}}|1\rangle\mathsf{H}|0\rangle\big)$$

$$= \big(\mathbf{1}\otimes\mathsf{H}\big)\mathsf{C}_H\Big[\big(\sqrt{\tfrac{2}{3}}|0\rangle + \sqrt{\tfrac{1}{3}}|1\rangle\big)|0\rangle\Big] = (\mathbf{1}\otimes\mathsf{H})\mathsf{C}_H(\mathsf{w}\otimes\mathbf{1})|0\rangle|0\rangle, \tag{1}$$

where w is any one-Qbit unitary transformation that takes $|0\rangle$ into $\sqrt{\tfrac{2}{3}}|0\rangle + \sqrt{\tfrac{1}{3}}|1\rangle$.

To construct a controlled-Hadamard C_H from a controlled-NOT C, note that the NOT operation X is $\mathbf{x}\cdot\sigma$ while the Hadamard transformation is $\mathsf{H} = \tfrac{1}{\sqrt{2}}\big(\mathsf{X}+\mathsf{Z}\big) = \tfrac{1}{\sqrt{2}}\big(\mathbf{x}+\mathbf{z}\big)\cdot\sigma$. It follows from Section A2 of the appendix to chapter 1 that

$$\mathsf{H} = \mathsf{u}\mathsf{X}\mathsf{u}^\dagger, \tag{2}$$

where u is the one-Qbit unitary transformation associated with any rotation that takes $\mathbf{x}$ into $\tfrac{1}{\sqrt{2}}(\mathbf{x}+\mathbf{z})$. Since we also have $\mathbf{1} = \mathsf{u}\mathsf{u}^\dagger$, it follows that

$$\mathsf{C}_H = \big(\mathbf{1}\otimes\mathsf{u}\big)\mathsf{C}\big(\mathbf{1}\otimes\mathsf{u}^\dagger\big). \tag{3}$$

So (1) reduces to the compact form

$$|\Psi\rangle = (\mathbf{1}\otimes\mathsf{H}\mathsf{u})\mathsf{C}(\mathsf{w}\otimes\mathsf{u}^\dagger)|0\rangle|0\rangle, \tag{4}$$

which produces the state $|\Psi\rangle$ by acting on $|0\rangle|0\rangle$ with three one-Qbit unitaries ($\mathsf{H}\mathsf{u}$, w, and $\mathsf{u}^\dagger$) and one (two-Qbit) cNOT gate.

If you want an explicit form for w, its matrix in the computational basis could be

$$\begin{pmatrix} \sqrt{\tfrac{2}{3}} & -\sqrt{\tfrac{1}{3}} \\ \sqrt{\tfrac{1}{3}} & \sqrt{\tfrac{2}{3}} \end{pmatrix}. \tag{5}$$

To get an explicit form for u note that a rotation through $\pi/4$ about the y-axis takes $\mathbf{x}$ into $\tfrac{1}{\sqrt{2}}(\mathbf{x}+\mathbf{z})$. The associated unitary transformation is

$$\mathsf{u} = e^{i(\pi/8)\sigma_y} = \cos(\pi/8) + i\sigma_y\sin(\pi/8). \tag{6}$$

Since the matrix for σ_y in the computational basis is $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, the matrix for u is

$$\begin{pmatrix} \cos(\pi/8) & \sin(\pi/8) \\ -\sin(\pi/8) & \cos(\pi/8) \end{pmatrix}. \tag{7}$$

Since the matrices for X and H are $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ you can easily confirm, if you're not comfortable with Appendix A2 of Chapter 1, that these three matrices do indeed satisfy (2). Verifying this should give you an appreciation for the power of the method described in A2.

II. Measurement gates

(a) Write the computational basis states as $|xyz\rangle$ where x is an r-bit integer, associated with the first r Qbits to be measured, y is an s-bit integer, associated with the next s to be measured, and z is a third $(n-r-s)$-bit integer, associated with the remaining unmeasured Qbits. The initial state $|\Psi\rangle$ is

$$|\Psi\rangle = \sum_{x,y,z} \alpha_{xyz}|xyz\rangle. \tag{8}$$

The Born rule, in its most general form, tells us that if the first r Qbits are measured the result will be x with probability

$$p(x) = \sum_{y,z} |\alpha_{xyz}|^2 \tag{9}$$

and the postmeasurement state will be

$$|x\rangle|\Phi_x\rangle \tag{10}$$

where

$$|\Phi_x\rangle = \frac{1}{\sqrt{p(x)}}\sum_{y,z} \alpha_{xyz}|yz\rangle. \qquad (11).$$

If the result of the first measurement is indeed x, so that the postmeasurement state is (10), then applying the generalized Born rule to (10) and (11) tells us that if the next s Qbits associated with y are measured, the result will be y with probability

$$p(y|x) = \sum_{z} |\alpha_{xyz}/\sqrt{p(x)}|^2, \tag{12}$$

and the postmeasurement state after the second measurement will be

$$|x\rangle|y\rangle|\Phi_{xy}\rangle \tag{13}$$

where

$$|\Phi_{xy}\rangle = \frac{1}{\sqrt{p(y|x)}}\frac{1}{\sqrt{p(x)}}\sum_{z} \alpha_{xyz}|z\rangle. \tag{14}$$

The joint probability of getting x, and then getting y, is related to the conditional probability $p(y|x)$ by

$$p(xy) = p(x)p(y|x), \tag{15}$$

so (12) gives

$$p(xy) = \sum_z |\alpha_{xyz}|^2. \tag{16}$$

(b) If the initial state of the Qbits is (8) then applying the generalized Born rule directly to a measurement of the first $r+s$ Qbits tells us that the result wil be xy with a probability that is indeed given by (16), and that the post-measurement state of the Qbits will be

$$|xy\rangle|\Phi_{xy}\rangle = |xy\rangle\frac{1}{\sqrt{p(xy)}}\sum_z \alpha_{xyz}|z\rangle. \qquad (17).$$

But (15) shows that this is identical to (14).