Appendix A

Partial Order

Good order is the foundation of all good things. — Reflections on the Revolution in France, Edmund Burke.

A.1 Introduction

It is essential to understand the theory of partially ordered sets to study distributed systems. In this section, we give a concise introduction to this theory.

A partial order is simply a relation with certain properties. A *relation* R over any set X is a subset of $X \times X$. For example, let

$$X = \{a, b, c\}.$$

Then, one possible relation is

$$R = \{(a, c), (a, a), (b, c), (c, a)\}.$$

It is sometimes useful to visualize a relation as a graph on the vertex set X such that there is a directed edge from x to y iff $(x, y) \in R$. The graph corresponding to the relation R in the previous example is shown in Figure A.1

A relation is *reflexive* if for each $x \in X$, $(x, x) \in R$. In terms of a graph, this means that there is a self-loop on each node. If X is the set of natural numbers, then "x divides y" is a reflexive relation. R is *irreflexive* if for each $x \in X$, $(x, x) \notin R$. In terms of a graph, this means that there are no self-loops. An example on the set of natural

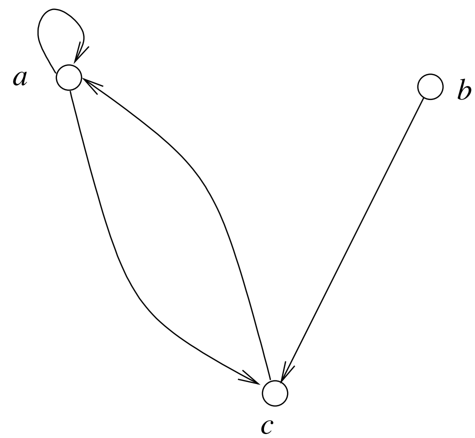


Figure A.1: The graph of a relation

numbers is the relation "x less than y." Note that a relation may be neither reflexive nor irreflexive.

A relation R is *symmetric* if $(x, y) \in R$ implies $(y, x) \in R$ for all $x, y \in X$. An example of a symmetric relation on the set of natural number is

$$R = \{(x, y) \mid x \ mod\ 5 = y \ mod\ 5\}.$$

A symmetric relation can be represented using an undirected graph. R is *antisymmetric* if for all x and y, $(x, y) \in R$ and $(y, x) \in R$ implies $x = y$. For example, the relation *less than or equal to* defined on the set of natural numbers is anti-symmetric. A relation R is *asymmetric* if for any x, y, $(x, y) \in R$ implies $(y, x) \notin R$. The relation *less than* is asymmetric. Note that an asymmetric relation is always irreflexive. A relation R is *transitive* if $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$ for all x, y and z. The relations *less than* and *equal to* on natural numbers are transitive.

A relation R is an *equivalence* relation if it is reflexive, symmetric, and transitive. When R is an equivalence relation, we use $x \cong y(R)$ (or simply $x \cong y$ when R is clear from the context) to denote that $(x, y) \in R$. Furthermore, for each $x \in X$, we use $[x](R)$, called the *equivalence class of* x, to denote the set of all $y \in X$ such that $y \cong x(R)$. It can be seen that the set of all such equivalence classes forms a *partition* of X. We use $|R|$, called the *index* of equivalence relation R, to denote the cardinality of equivalence classes under R. The relation on $\mathcal{N}$ defined as

$$\forall x, y \in \mathcal{N} : (x, y) \in R \Leftrightarrow [x \bmod 5 = y \bmod 5]$$

is an example of an equivalence relation. It partitions the set of natural numbers into five equivalence classes.

Given any binary relation R on a set X, we define its irreflexive transitive closure, denoted by R^+, as follows. For all $x, y \in X : (x, y) \in R^+$ iff there exists a sequence $x_0, x_1, ..., x_j, j \geq 1$ such that

$$\forall i : 0 \leq i < j : (x_i, x_{i+1}) \in R.$$

Thus $(x, y) \in R^+$ iff there is a nonempty path from x to y in the graph of the relation R. We define the reflexive transitive closure, denoted by R^*, as

$$R^* = R^+ \cup \{(x, x) \mid x \in X\}$$

. Thus $(x, y) \in R^*$ iff y is reachable from x by taking zero or more edges in the graph of the relation R.

A.2 Definition of Partial Orders

A relation R is a *reflexive partial order* if it is reflexive, antisymmetric, and transitive. The *divides* relation on the set of natural numbers is a reflexive partial order. A relation R is an *irreflexive partial order* if it is irreflexive and transitive. The *less than* relation on the set of natural numbers is an irreflexive partial order. When R is a reflexive partial order we use $x \leq y(R)$ (or simply $x \leq y$ when R is clear from the context) to denote that $(x, y) \in R$. A reflexive partially ordered set, *poset* for short, is denoted by $(X, \leq)$. When R is an irreflexive partial order we use $x < y(R)$ (or simply $x < y$ when R is clear from the context) to denote that $(x, y) \in R$. The set X together with the partial order is denoted by $(X, <)$. In this book, we use a partial order (poset) to mean an irreflexive partial order (poset) unless otherwise stated.

A relation is a *total order* if R is a partial order and for all distinct $x, y \in X$, either $(x, y) \in R$ or $(y, x) \in R$. The natural order on the set of integers is a total order, but the "divides" is only a partial order.

Finite posets are often depicted graphically using a *Hasse diagram*. To define Hasse diagrams, we first define a relation *covers* as follows. For any two elements x, y, y covers x if $x < y$ and $\forall z \in X : x \leq z < y$ implies $z = x$. In other words, there should not be any element z with

$x < z < y$. A Hasse diagram of a poset is a graph with the property that there is an edge from x to y iff y covers x. Furthermore, when drawing the figure in an Euclidean plane, x is drawn lower than y when y covers x. For example, consider the following poset $(X, \leq)$.

$$X \stackrel{\text{def}}{=} \{p, q, r, s\}; \quad \leq \stackrel{\text{def}}{=} \{(p, q), (q, r), (p, r), (p, s)\}.$$

Its Hasse diagram is shown in Figure A.2.

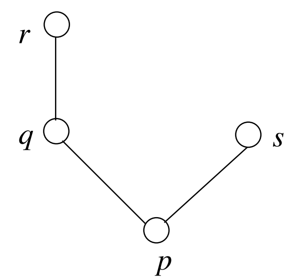


Figure A.2: Hasse diagram

Let $x, y \in X$ with $x \neq y$. If either $x < y$ or $y < x$, we say x and y are *comparable*. On the other hand, if neither $x < y$ nor $x > y$, then we say x and y are incomparable, and write $x||y$. A poset $(X, <)$ is called a *chain* if every distinct pair of points from X is comparable. Similarly, we call a poset an *antichain* if every distinct pair of points from X is incomparable.

A chain C of a poset $(X, <)$ is a *maximum chain* if no other chain contains more points than C. We use a similar definition for *maximum antichain*. The *height* of the poset is the number of points in the maximum chain, and the *width* of the poset is the number of points in a maximum antichain.

A.3 Lattices

We now define two operators on subsets of the set X—*infimum* (or *inf*) and *supremum* (or *sup*). Let $Y \subseteq X$, where $(X, \leq)$ is a poset. For any $m \in X$, we say that $m = inf\ Y$ iff

1. $\forall y \in Y : m \leq y$.
2. $\forall m' \in X : (\forall y \in Y : m' \leq y) \Rightarrow m' \leq m$.

The condition (1) says that m is a lower bound of the set Y. The condition (2) says that if m' is another lower bound of Y, then it is less than m. For this reason, m is also called the *greatest lower bound* (*glb*) of the set Y. It is easy to check that the infimum of Y is unique whenever it exists. Observe that m is not required to be an element of Y.

The definition of *sup* is similar. We say that $s = sup\ Y$ iff

1. $\forall y \in Y : y \leq s$
2. $\forall s' \in X : (\forall y \in Y : y \leq s') \Rightarrow s \leq s'$

Again, s is also called the *least upper bound* (*lub*) of the set Y. We denote the *glb* of $\{a, b\}$ by $a \sqcap b$, and *lub* of $\{a, b\}$ by $a \sqcup b$. In the set of natural numbers ordered by the *divides* relation, the *glb* corresponds to finding the greatest common divisor (gcd) and the *lub* corresponds to finding the least common multiple of two natural numbers. The greatest lower bound or the least upper bound may not always exist. In Figure A.3, the set $\{e, f\}$ does not have any upper bound. In the third poset in Figure A.4, the set $\{b, c\}$ does not have any least upper bound (although both d and e are upper bounds).

$$X = \{a, b, c, d, e, f\}$$

$$R = \left\{ \begin{array}{l} (a,b), (a,c), (b,d), \\ (c,f), (c,e), (d,e) \\ (a,d), (a,e), (a,f), \\ (b,e) \end{array} \right\}$$

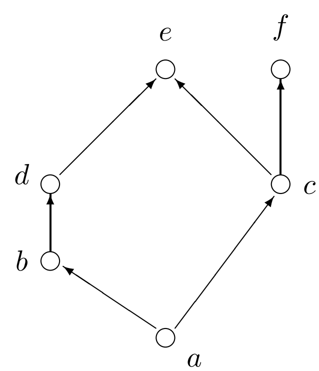


Figure A.3: A poset that is not a lattice.

We say that a poset $(X, \leq)$ is a *lattice* iff $\forall x, y \in X : x \sqcup y$ and $x \sqcap y$ exist. The first two posets in Figure A.4 are lattices, whereas the third one is not.

If $\forall x, y \in X : x \sqcup y$ exists, then we call it a *sup semilattice*. If $\forall x, y \in X : x \sqcap y$ exists then we call it an *inf semilattice*. A lattice is

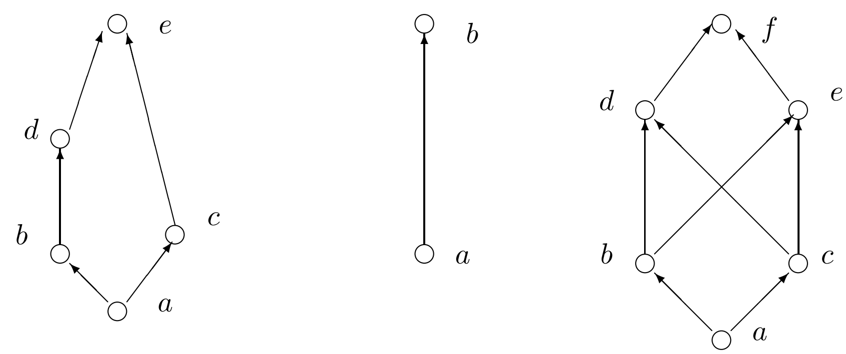


Figure A.4: Only the first two posets are lattices.

distributive if it satisfies the distributive law,

$$\forall x, y, z \in X : x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z).$$

It is easy to verify that the above condition is equivalent to

$$\forall x, y, z \in X : x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z).$$

Thus in a distributive lattice, $\sqcup$ and $\sqcap$ operators distribute over each other.

Any power-set lattice is distributive. The lattice of natural numbers with $\leq$ defined as the relation *divides* is also distributive. Some examples of nondistributive lattices (see Figure A.5) are:

1. *Diamond*

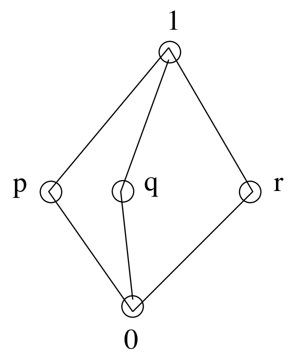
$$X = \{0, p, q, r, 1\}$$

$$\leq = \{(0, p), (0, q), (0, r), (p, 1), (q, 1), (r, 1), (0, 1)\}.$$

2. *Pentagon*

$$X = \{0, p, q, r, 1\},$$

$$\leq = \{(0, p), (0, q), (0, r), (p, 1), (q, 1), (r, 1), (0, 1), (p, q)\}.$$



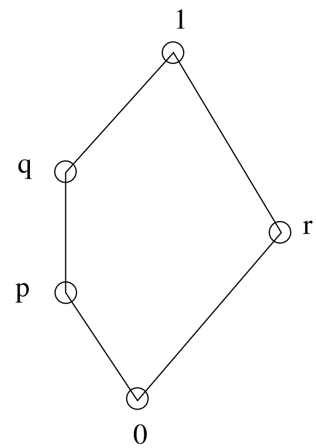


Figure A.5: Examples of nondistributive lattices

A.4 Properties of Functions on Posets

We now discuss properties of functions on posets. Let $(X, \prec_X)$ and $(Y, \prec_Y)$ be two posets.

Definition A.1 *A function $f : X \to Y$ is called monotone iff*

$$\forall x_1, x_2 \in X : x_1 \prec_X x_2 \Rightarrow f(x_1) \prec_Y f(x_2)$$

In other words, monotone functions preserve the ordering. An example of a monotone function on the set of integers is addition by any constant. This is because

$$x_1 \leq x_2 \Rightarrow x_1 + c \leq x_2 + c$$

for any integers x_1, x_2 and c.

A.5 Down-Sets and Up-Sets

Let $(X, <)$ be any poset. We call a subset $Y \subseteq X$ a down-set if

$$z \in Y \wedge y < z \Rightarrow y \in Y.$$

Similarly, we call $Y \subseteq X$ an up-set if

$$y \in Y \wedge y < z \Rightarrow z \in Y.$$

In the discussion of distributed systems, down-sets play an important role. We use $\mathcal{O}(X)$ to denote the set of all down-sets of X.

We now give a simple but important lemma.

Lemma A.2 *Let* $(X, <)$ *be any poset. Then,* $(\mathcal{O}(X), \subseteq)$ *is a distributive lattice.*

Proof: We need to show that if Y and Z are down-sets, then $Y \cup Z$ and $Y \cap Z$ are also down-sets. To prove that $Y \cup Z$ is a down-set, let $z \in Y \cup Z$ and $y < z$. There are two cases : $z \in Y$ or $z \in Z$. If $z \in Y$, then because Y is a down-set, $y \in Y$. Therefore, $y \in Y \cup Z$. The other case also leads to the same conclusion. Therefore, $Y \cup Z$ is a down-set.

We leave it for the reader to show that $Y \cap Z$ is also a down set. Distributivity of $(\mathcal{O}(X), \subseteq)$ follows from distributivity of $\cap$ over $\cup$.

■

A.6 Problems

A.1. Show that if P and Q are posets defined on set X, then so is $P \cap Q$.

A.2. Show that for all posets P on set X, there exists a total order Q on X such that $P \subseteq Q$.

A.3. Show that if C_1 and C_2 are down-sets for any poset $(E, <)$, then so is $C_1 \cap C_2$.

A.4. Consider the poset defined by the *divides* relations on the set of positive integers. Show that this poset is a lattice.

A.5. The *transitive closure* of a relation R on a finite set can also be defined as the smallest transitive relation on S that contains R. Show that the transitive closure is uniquely defined. We use "smaller" in the sense that R_1 is smaller than R_2 if $|R_1| < |R_2|$.

A.7 Bibliographic Remarks

The reader should consult Davey and Priestley [DP90] for a more comprehensive introduction to theory of posets and lattices.