Notes on ecosystems stability

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1 Super quadratic self-regulation

Consider the case as in the draft but with production

$$P(n_i) = rn_i - rn_i^l/K^l \qquad \text{(super-regulated)}\,, \tag{1}$$

with $l \geq 2$, recovering the logistic case for $l = 2$. Following the same notation and calculation as in the draft we have, in this case, the following community matrix

$$C = -DA - \frac{(l-1)r}{K^l} D^{l-1}\,. \tag{2}$$

In this case we have that instabilities sets in if

$$\sum_i \frac{1}{|\mu - (l-1)r(n_i^*)^{l-2}/K^l|^2} \geq 1/\sigma^2\,, \tag{3}$$

which correctly recover the logistic case for $l = 2$. The stability condition for $l > 2$ can be approximated by

$$\sigma \leq (l-1)\sqrt{S}\mu r/K^l\,, \tag{4}$$

where we used the fact that

$$(n_i^*)^{l-2} = \sum_j A_{ij} n_j^*/n_i^* - r/n_i^* = \mathcal{O}(\mu S)\,. \tag{5}$$

2 General

Consider the production function

$$P(n_i) = r\left(n_i^k - n_i^l/K^{l-k}\right) \qquad \text{(general)}\,, \tag{6}$$

with $k \leq 1$, $l \geq 2$ and the carrying capacity K either finite or infinite. We recover the logistic case for $k = 1$ and $l = 2$, the sublinear case for $l = 2$ and $K \to \infty$ and the super-regulated case for $k = 1$. The community matrix reads

$$C = -DA - (1-k)rD^{k-1} - \frac{(l-1)r}{K^{l-k}} D^{l-1}\,, \tag{7}$$

therefore the system becomes unstable if

$$\sum_i \frac{1}{|\mu - (1-k)r(n_i^*)^{k-2} - (l-1)r(n_i^*)^{l-2}/K^{l-k}|^2} \geq 1/\sigma^2\,. \tag{8}$$

It seems less straightforward to find an approximate stability condition in terms of S in this case. The fact that we have

$$r(n_i^*)^{k-2} - r(n_i^*)^{l-2}/K^{l-k} = \sum_j A_{ij} n_j^*/n_i^* = \mathcal{O}(\mu S)\,, \tag{9}$$

could be maybe used but the terms appear with weights in the equaition for stability. I will explore further.

3 More general

Consider the system of S species

$$\frac{dn_i}{dt} = P(n_i) - n_i \sum_{j\neq i} A_{ij} n_j\,, \tag{10}$$

where the summation runs over all the species and we leave the production function generic and A is a matrix with zero diagonal and off-diagonal elements gaussianly ditrubuted with mean μ and standard deviation σ. The community matrix reads in this case

$$C = -D(n^*)A - D(P(n^*)/n^* - P'(n^*))\,, \tag{11}$$

where the notation $D(x)$ stands for a diagonal matrix filled with the components of the vector x and $P(n^*) = (P(n_1^*), ..., P(n_S^*))$ and $P'(n^*) = (dP(n_1^*)/dn_1^*, ..., dP(n_S^*)/dn_S^*)$.

We have instability when

$$\sum_i \frac{1}{|\mu - P(n_i^*)/(n_i^*)^2 + P'(n_i^*)/n_i^*|^2} \geq 1/\sigma^2\,. \tag{12}$$

4 Sublinear production across a gradient

Dynamical sublinear production does not generally leads to sublinear production across a biomass gradient. Nonetheless, although not sufficient, it might be necessary for (or at least functional to) sublinear scaling across a biomass gradient.

4.1 Single population

Let's focus on a single population example to calrify the ideas. Consider a production function of the form

$$P(n) = rn^k\,, \tag{13}$$

where n is the biomass density of the population, $k \leq 1$ specify the intensity of sublinear dynamical scaling (linear when $k = 1$) and r is the parameter that allows to move across a biomass gradient. Consider then a self-regulation term of the form

$$Q(n) = sn^l\,. \tag{14}$$

The evolution equaiton for the population is

$$\frac{dn}{dt} = P(n) - Q(n) = rn^k - sn^l\,, \tag{15}$$

which reduces to logistic growth for $k = 1$ and $l = 2$. The equilibrium is given by

$$n^* = \left(\frac{r}{s}\right)^{1/(l-k)} \,, \tag{16}$$

and the stability condition is given by

$$\left.\frac{d\,[P(n) - Q(n)]}{dn}\right|_{n=n^*} < 0\,, \tag{17}$$

which leads, after some calculations, to the condition

$$k < l\,, \tag{18}$$

independent from r and s.

It possible to appreciate from Eq. (16) that $k > l$, apart from making the equilibrium unstable, implies that the stationary population *decreases* for increasing r, which is biologically unreasonable.

The equilibrium production scales, at varying growth rate r, as $P(n^*) = s(n^*)^l$ as can be noted by Eq. (16) for r and then substituting it in the defining Eq. (13) or simply by considering the dynamical equation at stationarity. Notice that, if s is varied instead, the exponent of dynamical and across-biomass-gradient production coincide, consistently with what is proposed in the 2015 paper. I discuss anyway why r is preferable to be changed instead of s elsewhere. Figure 1 shows an example with specific parameters.

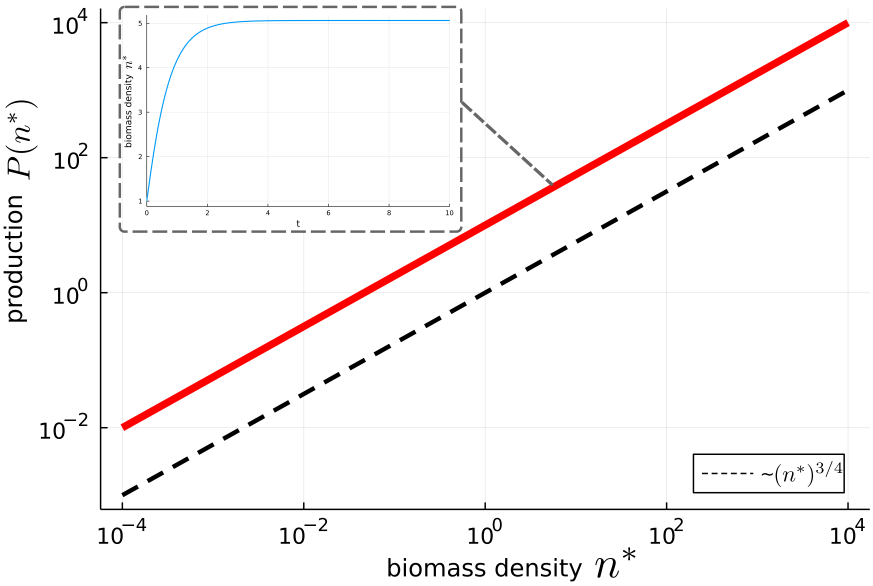

Figure 1: Production versus biomass density for $r \in [1, 100]$, $s = 10$, $k = 1/2$ and $l = 3/4$. In the inset the equilibration process for $r = 10$.

In summary, for an environmental change that amount to a change in r, the production across a biomass gradient scales as $(n^*)^l$. In order for the stationary solution to be stable an exponent $k < l$ for the dynamical produciton is needed. Therefore, in order to have sublinear $P(n^*)$ across a biomass gradient, we need $k < l < 1$.