

# Supplementary Material for “Diversity begets stability: sublinear growth scaling and competitive coexistence across ecosystems”

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## A Conditions for stable coexistence in competitive communities

Here we provide more details on the analytical results discussed in the main text.

### A.1 Homogeneous communities: stability criteria

Let us model the competition of  $S$  species with abundances  $x_i$  (either numerical density  $N_i$  or biomass density  $B_i$ ) in terms of *per capita* growth rates

$$\frac{1}{x_i} \frac{dx_i}{dt} = \text{inter}_i(x_i) + \text{intra}_i(\mathbf{x}_{-i}), \quad 1 \leq i \leq S. \quad (1)$$

Here  $\text{inter}_i$  represents intra-specific effects which only depend on the abundance of population  $i$ , and  $\text{intra}_i$  represents inter-specific effects which only depend on the abundances of its competitors, collectively denoted  $\mathbf{x}_{-i} \equiv (x_j)_{j \neq i}$ . Because competition can only reduce the growth rate of a population, we assume that both terms are monotonically decreasing functions of their arguments.

Following standard terminology, we call the system feasible if it admits an equilibrium point  $\mathbf{x}^*$  whose components  $x_i$  are all positive. Further, an equilibrium point  $\mathbf{x}^*$  is locally stable if all eigenvalues of the Jacobian of  $f$  evaluated at  $\mathbf{x}^*$  (aka the community matrix  $C^*$ ) have negative real parts. The community matrix has diagonal and off-diagonal elements

$$C_{ii}^* = x_i^* \frac{\partial \text{intra}_i}{\partial x_i}(x_i^*), \quad C_{ij}^* = x_i^* \frac{\partial \text{inter}_i}{\partial x_j}(x_{-i}^*), \quad (2)$$

respectively.

Consider the case species are indistinguishable and interactions are uniform ( $A_{ij} = A > 0$ ). In that case all species have equal abundance  $x^*$  at equilibrium, and moreover the eigenvalue  $\lambda$  of the Jacobian  $J^*$  with largest real part is given by

$$\lambda = x^* \left( \frac{\partial \text{intra}}{\partial x_i}(x^*) - \frac{\partial \text{inter}}{\partial x_i}(x^*) \right). \quad (3)$$

From this expression derives a general rule of thumb concerning the stability of competitive communities: a multi-species equilibrium can only be stable if intra-specific regulation effect are more strongly density dependent than inter-specific competition effects. This rule is stated as Eq. (3) in the main text.

For the model introduced in the main text we have  $\text{intra}(x_i) = rg(x_i)$  (assuming a vanishing death rate  $z = 0$ ) and  $\text{inter}(x_{-i}) = -\sum_{j \neq i} A_{ij}x_j$ . We can therefore write the local stability condition as

$$rg'(x^*) < -A \quad (\text{logistic}). \quad (4)$$

In the logistic case we have  $g'(x^*) = -1/K$ , hence stability requires  $A < r/K$ : inter-specific interactions must be weaker than intra-specific interactions. In the sublinear model, on the other hand, we have  $g'(x^*) = (k-1)g(x^*)/x^*$ , and using the equilibrium condition  $rg(x^*) = (S-1)Ax^*$ , we arrive at  $(k-1)(S-1)A < -A$ , i.e.

$$S > 1 + \frac{1}{1-k} \quad (\text{sublinear}). \quad (5)$$

This inequality shows that, at least in homogeneous communities, increasing diversity enhances the local stability of multi-species equilibria.

## A.2 Homogeneous communities: May vs. anti-May behavior

In this section we show that the GLV model lies on the boundary between two phases: one with May behavior (diversity begets instability), and one with anti-May behavior (diversity begets stability). To this aim we consider a variant of the competition model reading

$$f_i(x) = x_i^k - \sum_{j=1}^S a_{ij}x_i^a x_j^b = x_i^k - A_{ii}x_i^{a+b} - \sum_{j \neq i} a_{ij}x_i^a x_j^b. \quad (6)$$

In this formulation the GLV model corresponds to  $k = a = b = 1$  and the sublinear model in the main text to  $k < 1$  and  $a = b = 1$ . We compute the Jacobian matrix elements:

$$C_{ii} = kx_i^{k-1} - A_{ii}(a+b)x_i^{a+b-1} - a \sum_{j \neq i} A_{ij}x_i^{a-1}x_j^b, \quad (7)$$

$$C_{ij} = -b \sum_{j \neq i} A_{ij}x_i^a x_j^{b-1}. \quad (8)$$

If we assume as before a homogenous community with identical interaction strengths (denoted  $A_{ii} = A_s$ ,  $A_{ij} = A_c$ ), the equilibrium abundance reads

$$x^* = [A_s + (S - 1)A_c]^{1/(k-a-b)} \quad (9)$$

and the corresponding community matrix reads

$$C_{ii} = k(x^*)^{k-1} - [(a+b)A_s + a(S-1)A_c](x^*)^{a+b-1}, \quad (10)$$

$$C_{ij} = -bA(x^*)^{a+b-1}. \quad (11)$$

The largest eigenvalue of  $C^*$  is given by

$$\lambda = C_{ii}^* - C_{ij}^* \quad (12)$$

$$= k(x^*)^{k-1} - [(a+b)A_s + (a(S-1) - b)A_c](x^*)^{a+b-1} \quad (13)$$

$$= \left( k - \frac{(a+b)A_s + (a(S-1) - b)A_c}{A_s + (S-1)A_c} \right) (x^*)^{k-1}. \quad (14)$$

The local stability condition therefore reads

$$(S-1)(k-a) > (a+b-k)\alpha - b \quad \text{with} \quad \alpha \equiv A_s/A_c. \quad (15)$$

We have three possibilities:

$$S > 1 + \frac{(a+b-k)\alpha - b}{k-a} \quad \text{if} \quad k < a \quad (16)$$

$$S < 1 + \frac{(a+b-k)\alpha - b}{k-a} \quad \text{if} \quad k > a \quad (17)$$

$$\alpha > 1 \quad \text{if} \quad k = a \quad (18)$$

A series of comments are in order:

- Depending on  $k$  and  $a$  we have two regimes: one in which increasing  $S$  hinders stability ( $k > a$ , a ‘May’ phase), and one in which increasing  $S$  enhances stability ( $k < a$ , an ‘anti-May’ phase). Notably this transition is independent from  $b$ , indicating that only the relative density dependence of self- and cross-regulation in the *per capita* growth of a population is relevant, and not the form of the dependence on other species’ abundance in this *per capita* growth.
- The critical case ( $k = a$ ) recovers the usual stability condition for the homogeneous GLV model ( $b = 1$ ) and generalizes it for general interactions ( $b \neq 1$ ).
- For  $A_s$  and  $a = b = 1$  we recover the results obtained in the previous section,  $S > 1 + 1/(1-k)$ .

These results are obtained in the special case where interactions are homogenous (standard deviation  $\sigma = 0$ ). When  $\sigma > 0$ , the GLV model falls squarely into the May phase, as illustrated e.g. by Fig. 3A of the main text.

### **A.3 Random communities: DMFT and cavity solution**

Dynamical mean field theory (DMFT) and the cavity method have been used to study the GLV model in the large diversity limit ( $S \rightarrow \infty$ ) by several authors [REFS]. Here we show how to derive the analytical results in the sublinear model using these techniques.

Both the cavity method and DMFT starts from the observation that, in a complex community with a large number of species, the competition pressure felt by each population is dominated by a ‘mean field’ representing the aggregate effect of all other species.

### **A.4 Random communities: replica solution**

## **B Linking dynamical theory with macroecological laws**

### **B.1 Production-biomass scaling**

### **B.2 Species abundance distribution**

### **B.3 Mean-variance scaling**

### **B.4 Size-density scaling**

### **B.5 Size spectra**