

Supplementary Material for

“Diversity begets stability: sublinear growth scaling and competitive coexistence across ecosystems”

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A Conditions for stable coexistence in competitive communities

Here we provide more details on the analytical results discussed in the main text.

A.1 Homogeneous communities: stability criteria

Let us model the competition of S species with abundances x_i (either numerical density x_i or biomass density B_i) in terms of *per capita* growth rates, as follows

$$\frac{1}{x_i} \frac{dx_i}{dt} = \text{inter}_i(x_i) + \text{intra}_i(\mathbf{x}_{-i}) \equiv f_i(\mathbf{x}), \quad 1 \leq i \leq S. \quad (\text{A.1})$$

Here inter_i represents intra-specific effects which only depend on the abundance of population i , and intra_i represents inter-specific effects which only depend on the abundances of its competitors, collectively denoted $\mathbf{x}_{-i} \equiv (x_j)_{j \neq i}$. Because competition can only reduce the growth rate of a population, we assume that both terms are monotonically decreasing functions of their arguments.

Following standard terminology, the system may be called *feasible* if it admits an equilibrium point \mathbf{x}^* whose components x_i are all positive. Further, an equilibrium point \mathbf{x}^* is *(locally) stable* if all eigenvalues of the Jacobian of f evaluated at \mathbf{x}^* (also known as the ‘community matrix’ C^* of the system) have negative real parts. The community matrix has diagonal and off-diagonal elements, respectively

$$C_{ii}^* = x_i^* \frac{\partial \text{intra}_i}{\partial x_i}(x_i^*), \quad C_{ij}^* = x_i^* \frac{\partial \text{inter}_i}{\partial x_j}(\mathbf{x}_{-i}^*). \quad (\text{A.2})$$

Consider the case species are indistinguishable and interactions are uniform ($A_{ij} = A > 0$ for $i \neq j$). In that case all species have equal abundance x^* at equilibrium, and moreover the eigenvalue λ of the Jacobian J^* with largest real part is given by

$$\lambda = x^* \left(\frac{\partial \text{intra}}{\partial x_i}(x^*) - \frac{\partial \text{inter}}{\partial x_j}(\mathbf{x}^*) \right). \quad (\text{A.3})$$

From this expression derives a general rule of thumb concerning the stability of competitive communities: a multi-species equilibrium can only be stable if *intra-specific regulation is more strongly density-dependent than inter-specific competition*.

For the model introduced in the main text, we have $\text{intra}(x_i) = rg(x_i) - z$ and $\text{inter}(x_{-i}) = -\sum_{j \neq i} A_{ij}x_j$. We can therefore write the local stability condition as

$$rg'(x^*) < -A. \quad (\text{A.4})$$

In the logistic case we have $g'(x^*) = -1/K$, hence stability requires $A < r/K$: inter-specific interactions must be weaker than intra-specific interactions. In the sublinear model, on the other hand, we have $g'(x^*) = (k-1)g(x^*)/x^*$, and using the equilibrium condition $rg(x^*) - z = (S-1)Ax^*$, we arrive at $(1-k)[(S-1)A + z/x^*] > A$. This condition is fulfilled whenever

$$S > 1 + \frac{1}{1-k}. \quad (\text{A.5})$$

(When $z > 0$ this condition is sufficient but not necessary for local stability.) Eq. (A.5) shows that, at least in homogeneous communities, increasing diversity enhances the local stability of multi-species equilibria.

A.2 Homogeneous communities: May vs. anti-May behavior

In this section we show that the GLV model lies on the boundary between two phases: one with May behavior (diversity begets instability), and one with anti-May behavior (diversity begets stability). To this aim we consider a variant of the competition model of the form

$$\frac{1}{x_i} \frac{dx_i}{dt} = x_i^k - \sum_{j=1}^S a_{ij}x_i^a x_j^b = x_i^k - A_{ii}x_i^{a+b} - \sum_{j \neq i} a_{ij}x_i^a x_j^b. \quad (\text{A.6})$$

In this formulation the GLV model corresponds to $k = a = b = 1$ and the sublinear model to $k < 1$ and $a = b = 1$. We compute the Jacobian matrix elements:

$$C_{ii} = kx_i^{k-1} - A_{ii}(a+b)x_i^{a+b-1} - a \sum_{j \neq i} A_{ij}x_i^{a-1}x_j^b, \quad (\text{A.7})$$

$$C_{ij} = -b \sum_{j \neq i} A_{ij}x_i^a x_j^{b-1}. \quad (\text{A.8})$$

If we assume as before a homogenous community with identical interaction strengths (denoted $A_{ii} = A_s$, $A_{ij} = A_c$), the equilibrium abundance reads

$$x^* = [A_s + (S - 1)A_c]^{1/(k-a-b)} \quad (\text{A.9})$$

and the corresponding community matrix

$$\begin{aligned} C_{ii} &= k(x^*)^{k-1} - [(a+b)A_s + a(S-1)A_c](x^*)^{a+b-1}, \\ C_{ij} &= -bA(x^*)^{a+b-1} \quad \text{for } i \neq j. \end{aligned}$$

The largest eigenvalue of C^* is given by

$$\begin{aligned} \lambda &= C_{ii}^* - C_{ij}^* \\ &= k(x^*)^{k-1} - [(a+b)A_s + (a(S-1) - b)A_c](x^*)^{a+b-1} \\ &= \left(k - \frac{(a+b)A_s + (a(S-1) - b)A_c}{A_s + (S-1)A_c} \right) (x^*)^{k-1}. \end{aligned}$$

The local stability condition is, therefore,

$$(S-1)(k-a) < (a+b-k)\alpha - b \quad \text{with} \quad \alpha \equiv A_s/A_c. \quad (\text{A.10})$$

We have three possibilities:

$$\begin{aligned} S &> 1 + \frac{(a+b-k)\alpha - b}{k-a} && \text{if } k < a, \\ S &< 1 + \frac{(a+b-k)\alpha - b}{k-a} && \text{if } k > a, \\ \alpha &> 1 && \text{if } k = a. \end{aligned}$$

A series of comments are in order:

- Depending on k and a we have two regimes: one in which increasing S hinders stability ($k > a$, a ‘May’ phase), and one in which increasing S enhances stability ($k < a$, an ‘anti-May’ phase). Notably this transition is independent from b , indicating that only the relative density dependence of self- and cross-regulation in the *per capita* growth of a population is relevant, and not the form of the dependence on other species’ abundance in this *per capita* growth.
- The critical case ($k = a$) recovers the usual stability condition for the homogeneous GLV model ($b = 1$) and generalizes it for general interactions ($b \neq 1$).
- For $a = b = 1$ and $A_s = 0$ we recover the result (A.5) obtained in the previous section.

These results are obtained in the special case where interactions are homogenous (standard deviation $\sigma = 0$). When $\sigma > 0$, the GLV model falls squarely into the May phase, as illustrated e.g. by Fig. 3A of the main text.

A.3 Random communities: DMFT and cavity solution

Dynamical mean field theory (DMFT) and the cavity method have been used to study the GLV model in the large diversity limit ($S \rightarrow \infty$) by several authors [REFS]. Here we show how to derive the analytical results in the sublinear model using these techniques.

A.3.1 Equilibrium distribution

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A.3.2 Stability condition

Ahmadian *et al.* [AHMADIAN] consider large matrices of the form $M + LJR$, where M , L and R are deterministic matrices, and J is a random matrix with i.i.d. coefficients with zero mean and variance σ^2 . Generalizing the classical ‘circular law’, they show that the spectrum of such matrices is contained within the region of the complex plane defined by

$$\text{Tr}[(M_\zeta M_\zeta^\dagger)^{-1}] \geq 1/\sigma^2 \quad \text{where } M_\zeta = L^{-1}(\zeta I - M)R^{-1}, \quad (\text{A.11})$$

where Tr denotes the trace, dagger the Hermitian conjugate, and $\zeta \in \mathbb{C}$. In the special case where L , R and M are diagonal, this condition reduces to

$$\sum_{i=1}^S \frac{(L_i R_i)^2}{|\zeta - M_i|^2} \geq 1/\sigma^2 \quad (\text{A.12})$$

for matrices of size S .

Following Stone [STONE], we now apply this result to generalize the stability condition to random interactions A_{ij} . From (A.2) we can write the community matrix as

$$C^* = -\text{diag}(\mathbf{x}^*)[A + \text{diag}(\mathbf{r}g'(\mathbf{x}^*))] \quad (\text{A.13})$$

where the product of vectors is understood component-wise. Denoting $\mathbf{1}$ the rank-one matrix with all entries equal to 1, we can write $A = \mu\mathbf{1} - \mu I + J$ with J as above, hence C^* has the form $M + LJR$ with $L = -\text{diag}(\mathbf{x}^*)$, $R = I$ and $M = \text{diag}(\mathbf{x}^*)[\mu I - \text{diag}(\mathbf{r}g'(\mathbf{x}^*))]$. Thus, we have that the eigenvalues of the community matrix C^* must lie within the domain

$$\sum_{i=1}^S \frac{(x_i^*)^2}{|z - x_i^*[\mu - r_i g'(x_i^*)]|^2} \geq 1/\sigma^2. \quad (\text{A.14})$$

This domain first touches the right half-plane at $z = 0$, hence stability of C^* requires

$$\sum_{i=1}^S [\mu - r_i g'(x_i^*)]^{-2} < \sigma^{-2}. \quad (\text{A.15})$$

In the logistic model, we have $g'(x_i^*) = -1/K_i$, hence (A.15) becomes $\sum_i (\mu - r_i/K_i)^{-2} < \sigma^{-2}$. When growth rates and carrying capacities are all equal, this is just the May condition $\sigma\sqrt{S} + \mu < r/K$, interpreted as stating that interaction strength (μ, σ) and diversity S must be small compared to self-regulation r/K to allow for stable coexistence.

By contrast, in the sublinear model (defined by $g(x) = (x/x_0)^{k-1}$), we have $g'(x) = (k-1)x^{k-2}/x_0^{k-1}$, hence (A.15) reads

$$\sum_{i=1}^S [\mu + (1-k)r_i(x_i^*)^{k-2}/x_0^{k-1}]^{-2} < \sigma^{-2}. \quad (\text{A.16})$$

Making use of the cavity solution $P(x^*)$ obtained in sec. A.3.1, we can write this condition as

$$S \int dP(r, x^*) [\mu + (1-k)r(x^*)^{k-2}/x_0^{k-1}]^{-2} < \sigma^{-2} \quad (\text{A.17})$$

TBC

A.4 Random communities: replica solution

In this section we provide an alternative derivation of the stability condition for random communities based on the replica method, borrowed from spin-glass and recently applied to ecological dynamics [1, 2]. This alternative derivation requires two additional assumptions: the interaction matrix A_{ij} must be symmetric ($A_{ij} = A_{ji}$), an sublinear growth must be stabilized by a finite carrying capacity K , viz. $g(x) = (x/x_0)^k - x/K$. For simplicity we take $r = 1$ and $K = x_0 = 1$ in the following.

Under these assumptions, the evolution of species abundances x_i follows the euqation

$$\frac{dx_i}{dt} = -x_i \left[V_i'(x_i) + \sum_{j \neq i} A_{ij} x_j \right] \quad (\text{A.18})$$

where the one-species potential $V(x_i)$ represents self-regulation, *i.e.*

$$V_i(x_i) = -\frac{x_i^k}{k} + \frac{x_i^2}{2}. \quad (\text{A.19})$$

@ADA: replica calculation below, with notations consistent with main test (for instance μ and σ are *not* rescaled).

B Linking dynamical theory with macroecological laws

This section expands on the way we parametrize the sublinear model using individual-level allometries to recover various macroecological patterns.

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- B.1 Production-biomass scaling**
- B.2 Species abundance distribution**
- B.3 Mean-variance scaling**
- B.4 Size-density scaling**
- B.5 Size spectra**

References

- [1] Giulio Biroli, Guy Bunin, and Chiara Cammarota. “Marginally stable equilibria in critical ecosystems”. *New Journal of Physics* 20.8 (2018), p. 083051.
- [2] Ada Altieri, Felix Roy, Chiara Cammarota, and Giulio Biroli. “Properties of equilibria and glassy phases of the random lotka-volterra model with demographic noise”. *Physical Review Letters* 126.25 (2021), p. 258301.