

Supplementary Material for “Diversity begets stability: sublinear growth scaling and competitive coexistence across ecosystems”

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A Conditions for stable coexistence in competitive communities

Here we provide more details on the analytical results discussed in the main text.

A.1 Homogeneous communities: stability criteria

Let us model the competition of S species with abundances x_i (either numerical density N_i or biomass density B_i) in terms of *per capita* growth rates, as follows

$$\frac{1}{x_i} \frac{dx_i}{dt} = \text{inter}_i(x_i) + \text{intra}_i(\mathbf{x}_{-i}) \equiv f_i(\mathbf{x}), \quad 1 \leq i \leq S. \quad (\text{A.1})$$

Here inter_i represents intra-specific effects which only depend on the abundance of population i , and intra_i represents inter-specific effects which only depend on the abundances of its competitors, collectively denoted $\mathbf{x}_{-i} \equiv (x_j)_{j \neq i}$. Because competition can only reduce the growth rate of a population, we assume that both terms are monotonically decreasing functions of their arguments.

Following standard terminology, the system may be called *feasible* if it admits an equilibrium point \mathbf{x}^* whose components x_i are all positive. Further, an equilibrium point \mathbf{x}^* is *(locally) stable* if all eigenvalues of the Jacobian of f evaluated at \mathbf{x}^* (also known as the ‘community matrix’ C^* of the system) have negative real parts. The community matrix has diagonal and off-diagonal elements, respectively

$$C_{ii}^* = x_i^* \frac{\partial \text{intra}_i}{\partial x_i}(x_i^*), \quad C_{ij}^* = x_i^* \frac{\partial \text{inter}_i}{\partial x_j}(\mathbf{x}_{-i}^*). \quad (\text{A.2})$$

Consider the case species are indistinguishable and interactions are uniform ($A_{ij} = A > 0$ for $i \neq j$). In that case all species have equal abundance x^* at equilibrium, and

moreover the eigenvalue λ of the Jacobian J^* with largest real part is given by

$$\lambda = x^* \left(\frac{\partial \text{intra}}{\partial x_i}(x^*) - \frac{\partial \text{inter}}{\partial x_j}(\mathbf{x}^*) \right). \quad (\text{A.3})$$

From this expression derives a general rule of thumb concerning the stability of competitive communities: a multi-species equilibrium can only be stable if *intra-specific regulation is more strongly density-dependent than inter-specific competition*.

In the model introduced in the main text, we have $\text{intra}(x_i) = rg(x_i) - z$ and $\text{inter}(x_{-i}) = -\sum_{j \neq i} A_{ij}x_j$. We can therefore write the local stability condition as

$$rg'(x^*) < -A. \quad (\text{A.4})$$

In the logistic case we have $g'(x^*) = -1/K$, hence stability requires $A < r/K$: inter-specific interactions must be weaker than intra-specific interactions. In the sublinear model, on the other hand, we have $g'(x^*) = (k-1)g(x^*)/x^*$, and using the equilibrium condition $rg(x^*) - z = (S-1)Ax^*$, we arrive at $(1-k)[(S-1)A + z/x^*] > A$. This condition is fulfilled whenever

$$S > 1 + \frac{1}{1-k}. \quad (\text{A.5})$$

(When $z > 0$ this condition is sufficient but not necessary for local stability.) Eq. (A.5) shows that, at least in homogeneous communities, increasing diversity enhances the local stability of multi-species equilibria.

A.2 Homogeneous communities: May vs. anti-May behavior

In this section we show that the GLV model lies on the boundary between two phases: one with May behavior (diversity begets instability), and one with anti-May behavior (diversity begets stability). To this aim we consider a variant of the competition model of the form

$$\frac{1}{x_i} \frac{dx_i}{dt} = x_i^k - \sum_{j=1}^S a_{ij}x_i^a x_j^b = x_i^k - A_{ii}x_i^{a+b} - \sum_{j \neq i} a_{ij}x_i^a x_j^b. \quad (\text{A.6})$$

In this formulation the GLV model corresponds to $k = a = b = 1$ and the sublinear model to $k < 1$ and $a = b = 1$. We compute the Jacobian matrix elements:

$$C_{ii} = kx_i^{k-1} - A_{ii}(a+b)x_i^{a+b-1} - a \sum_{j \neq i} A_{ij}x_i^{a-1}x_j^b, \quad (\text{A.7})$$

$$C_{ij} = -b \sum_{j \neq i} A_{ij}x_i^a x_j^{b-1}. \quad (\text{A.8})$$

If we assume as before a homogenous community with identical interaction strengths (denoted $A_{ii} = A_s$, $A_{ij} = A_c$), the equilibrium abundance reads

$$x^* = [A_s + (S-1)A_c]^{1/(k-a-b)} \quad (\text{A.9})$$

and the corresponding community matrix

$$\begin{aligned} C_{ii} &= k(x^*)^{k-1} - [(a+b)A_s + a(S-1)A_c](x^*)^{a+b-1}, \\ C_{ij} &= -bA(x^*)^{a+b-1} \quad \text{for } i \neq j. \end{aligned}$$

The largest eigenvalue of C^* is given by

$$\begin{aligned} \lambda &= C_{ii}^* - C_{ij}^* \\ &= k(x^*)^{k-1} - [(a+b)A_s + (a(S-1) - b)A_c](x^*)^{a+b-1} \\ &= \left(k - \frac{(a+b)A_s + (a(S-1) - b)A_c}{A_s + (S-1)A_c} \right) (x^*)^{k-1}. \end{aligned}$$

The local stability condition is, therefore,

$$(S-1)(k-a) > (a+b-k)\alpha - b \quad \text{with} \quad \alpha \equiv A_s/A_c. \quad (\text{A.10})$$

We have three possibilities:

$$\begin{aligned} S &> 1 + \frac{(a+b-k)\alpha - b}{k-a} && \text{if } k < a, \\ S &< 1 + \frac{(a+b-k)\alpha - b}{k-a} && \text{if } k > a, \\ \alpha &> 1 && \text{if } k = a. \end{aligned}$$

A series of comments are in order:

- Depending on k and a we have two regimes: one in which increasing S hinders stability ($k > a$, a ‘May’ phase), and one in which increasing S enhances stability ($k < a$, an ‘anti-May’ phase). Notably this transition is independent from b , indicating that only the relative density dependence of self- and cross-regulation in the *per capita* growth of a population is relevant, and not the form of the dependence on other species’ abundance in this *per capita* growth.
- The critical case ($k = a$) recovers the usual stability condition for the homogeneous GLV model ($b = 1$) and generalizes it for general interactions ($b \neq 1$).
- For $a = b = 1$ and $A_s = 0$ we recover the result (A.5) obtained in the previous section.

These results are obtained in the special case where interactions are homogenous (standard deviation $\sigma = 0$). When $\sigma > 0$, the GLV model falls squarely into the May phase, as illustrated e.g. by Fig. 3A of the main text.

A.3 Random communities: DMFT and cavity solution

Dynamical mean field theory (DMFT) and the cavity method have been used to study the GLV model in the large diversity limit ($S \rightarrow \infty$) by several authors [REFS]. Here we show how to derive the analytical results in the sublinear model using these techniques.

A.3.1 Equilibrium distribution

A.3.2 Stability condition

In Ref. [Ahmadian] it is shown that the spectrum of matrices of the form $M + LJR$, where M , L and R are $S \times S$ deterministic matrices and J is a $S \times S$ random matrix with i.i.d. coefficients (with mean zero and variance σ^2) is contained in the region of the complex plane defined by

$$\text{Tr}[(M_z M_z^\dagger)^{-1}] \geq 1/\sigma^2 \quad \text{where } M_z = L^{-1}(zI - M)R^{-1}, \quad (\text{A.11})$$

where Tr denotes the trace, dagger the Hermitian conjugate, and $z \in \mathbb{C}$. In the special case where L , R and M are diagonal, this gives

$$\sum_{i=1}^S \frac{(L_i R_i)^2}{|z - M_i|^2} \geq 1/\sigma^2. \quad (\text{A.12})$$

We can apply these results to generalize the results in Sec. (A.1) to the case of random interactions A . From (A.2) we can write the community matrix as

$$C^* = -D(\mathbf{x}^*)A - D(\mathbf{x}^* \mathbf{r} g'(\mathbf{x}^*)), \quad (\text{A.13})$$

where $D(\mathbf{y}) \equiv \text{diag}(\mathbf{y})$ and the product of vectors is taken element-wise. Now, writing $A = \mu \mathbf{1} - \mu I + J$, and identifying $L = -D(\mathbf{x}^*)$, $R = I$ and $M = \mu I - D(\mathbf{x}^* g'(\mathbf{x}^*))$ we can say that the eigenvalues of the community matrix C^* lie within the domain

$$\sum_i \frac{(x_i^*)^2}{|z - \mu + x_i^* g'(x_i^*)|^2} \geq 1/\sigma^2. \quad (\text{A.14})$$

This domain first touches the right half-plane at $z = 0$ (triggering an instability) when

$$\sum_i \frac{(x_i^*)^2}{(\mu - x_i^* g'(x_i^*))^2} \geq 1/\sigma^2. \quad (\text{A.15})$$

A.4 Random communities: replica solution

B Linking dynamical theory with macroecological laws

This section expands on the way we parametrize the sublinear model using individual-level allometries to recover various macroecological patterns.

- B.1 Production-biomass scaling
- B.2 Species abundance distribution
- B.3 Mean-variance scaling
- B.4 Size-density scaling
- B.5 Size spectra