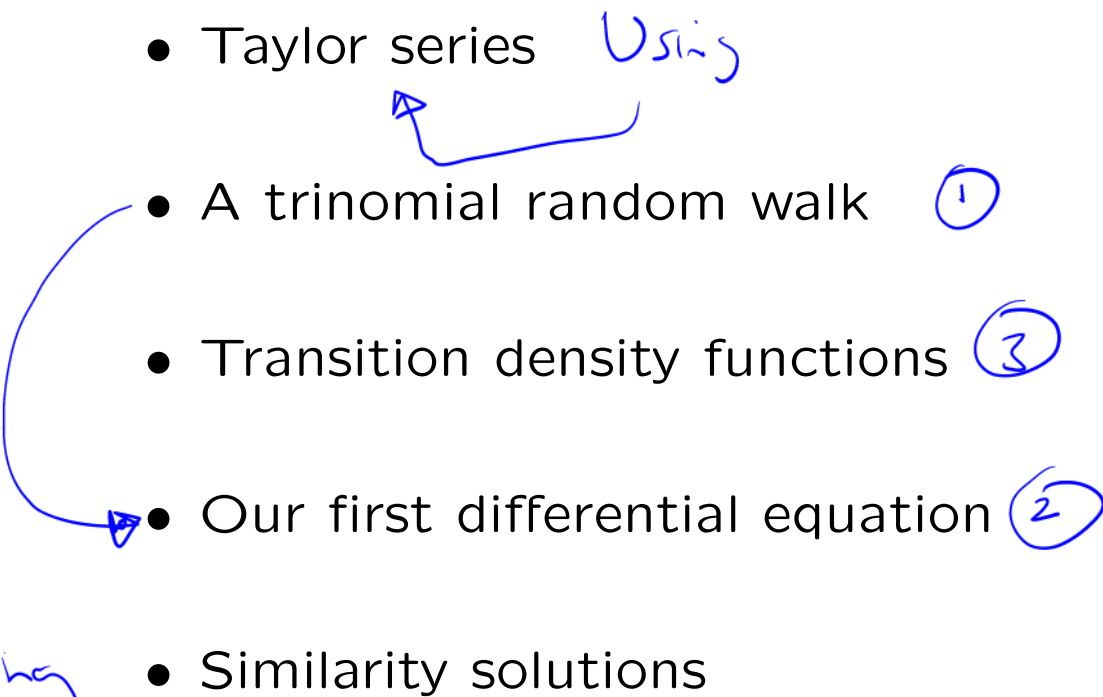CQF Module 1 Lecture 3

Random Walks and Transition Density Functions

# In this lecture. . .

- Taylor series  *Using*

- A trinomial random walk  ①

- Transition density functions  ③

- Our first differential equation  ②

*Using*

- Similarity solutions

By the end of this lecture you will be able to

- use Taylor series for expanding functions in a power series

  *algebraically*

- interpret partial differential equations ( self-contained )

- transform simple partial differential equations into even simpler ordinary differential equations

$$pde \longrightarrow Series \; of \quad ODEs$$

## Introduction

This lecture is an introduction to some simple ideas and some simple tools that we shall be seeing again and again in our journey through the subject of quantitative finance.

Quantitative finance can be relatively painless if approached correctly.

This lecture shows how an applied mathematician approaches problems with the aim of getting to a useful model or result with the minimum of fuss.

## Importance in quant finance

In quantitative finance the function that we are interested in, instead of being $f$, is $V$, the value of an option.

The independent variable is no longer $x$ but is $S$, the price of the underlying asset.

From day to day the asset price changes by a small amount. This asset price change is just $\delta S$ (instead of $\delta x$).

$$\longrightarrow 2 \text{ variables}$$

So we can use $\boxed{\text{Taylor series}}$ for option prices, we just have different names for the variables!

$f^{n}s \text{ of } \quad \text{several} \quad \text{variables} \mid \text{Multivariate Calc.}$

# Randomness in finance

Modern finance theory, especially derivatives theory, is based on the random movement of financial quantities.

We are now going to explore the simple idea of the random walk and see its relationship to differential equations.

This is achieved via the concept of a **transition** **probability density function**.

prob.

$P(t, x)$

D.E.

$P(x)$

x

① ● The <u>trinomial</u> random walk

  3 states

③ ● The transition probability density function

  p.d.

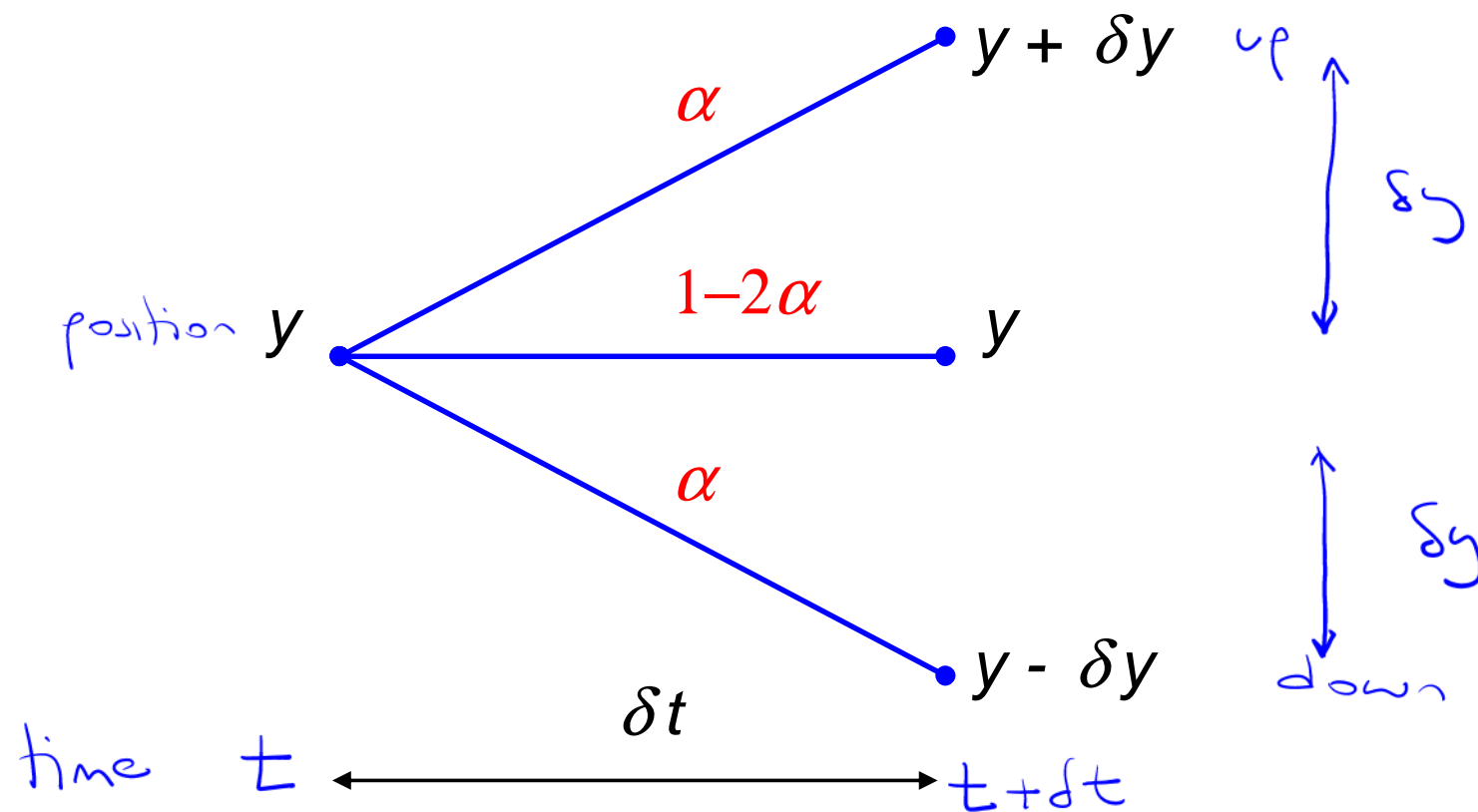② ● An equation for the transition probability density function

<u>State:</u>   Is   a   two tuple

a.   position   $y$
                       $\Big\}$  $(y, t)$
b.   time   $t$

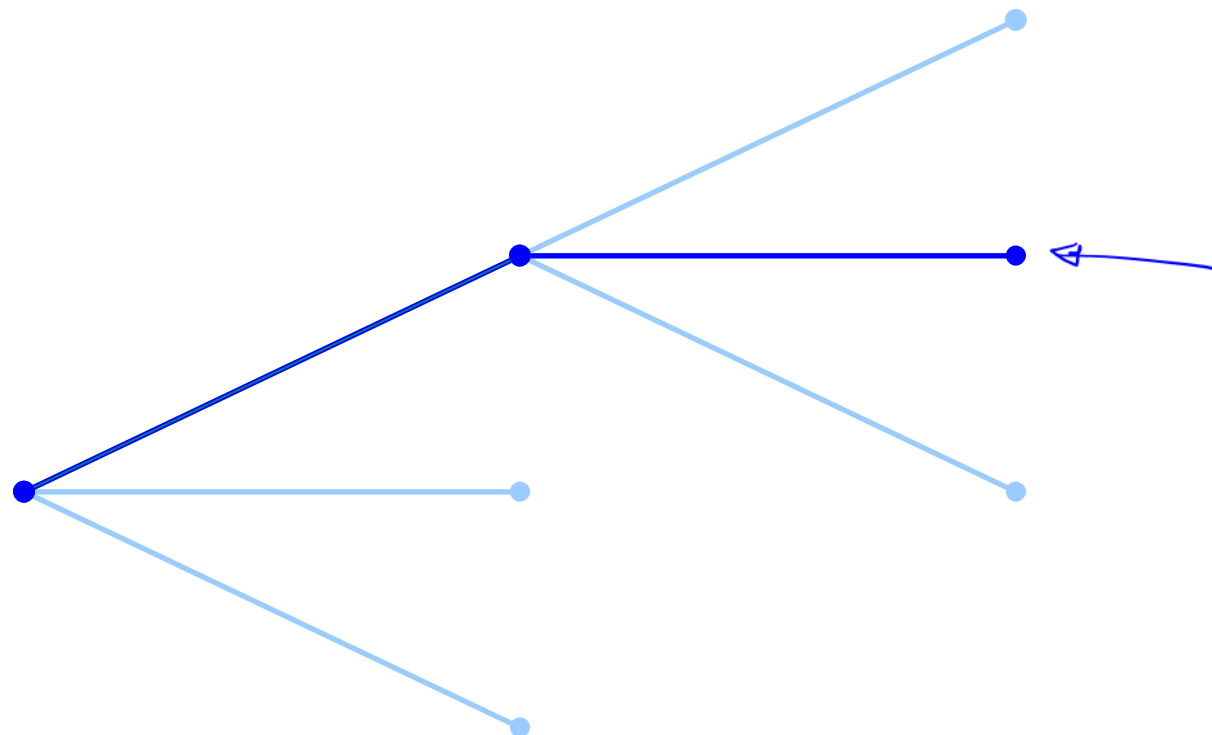<u>State 1</u> :   current / past   $(y, t)$

<u>State 2</u> :   future   $(y', t')$

# The trinomial random walk



$y$ is the value of our random variable. $\delta t$ is a time step. $\alpha$ is a probability. $\delta y$ is the size of the move in $y$.
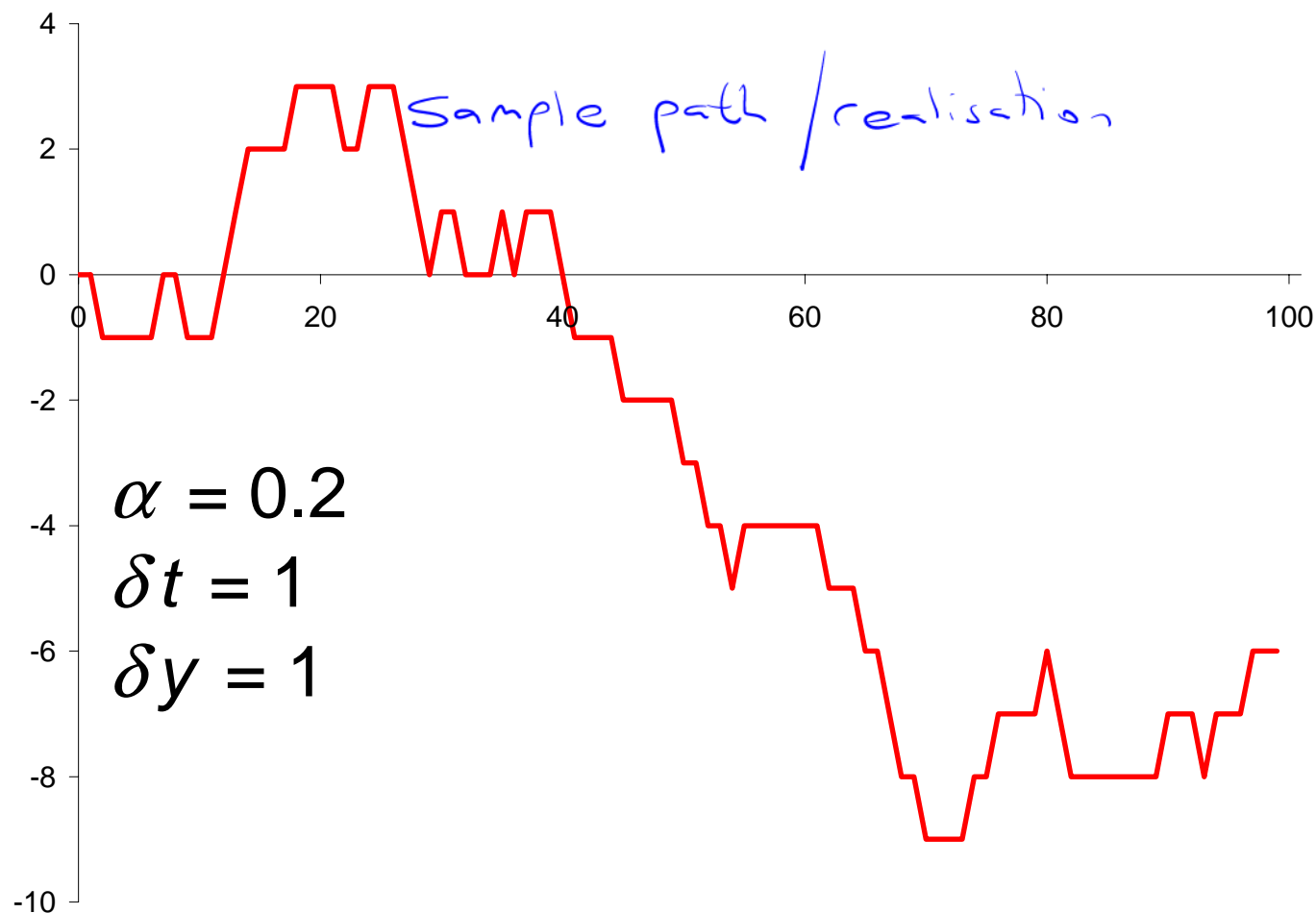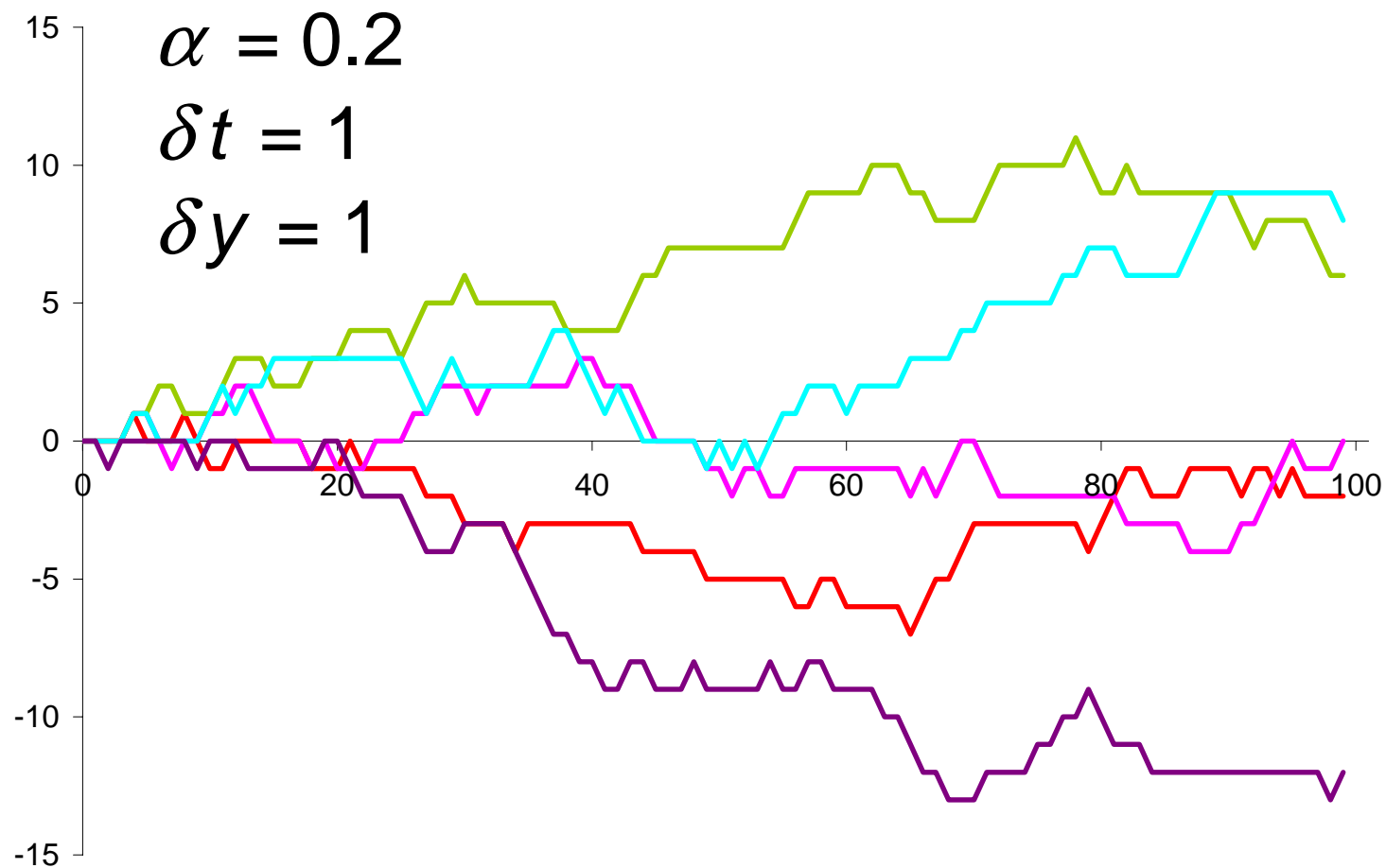
$t$ $\quad$ $t+\delta t$ $\quad$ $t+2\delta t$

Suppose the top branch is chosen after one time step. After the second there are three places that $y$ could be.

Sample path / realisation

$\alpha = 0.2$

$\delta t = 1$

$\delta y = 1$

After lots of time steps we might end up with a picture like this.
This is a random walk.

$\alpha = 0.2$
$\delta t = 1$
$\delta y = 1$

Often we are interested in the probabilistic properties of the random walk rather than the outcome of a single realization.

# The transition probability density function

To analyze the probabilistic properties of the random walk, we introduce the **transition probability density function** $p(y, t; y', t')$ defined by

$$y, t \quad ; \quad y', t'$$
$$\underbrace{\qquad}_{past} \qquad \underbrace{\qquad}_{future}$$

$$\text{Prob}(a < y' < b \text{ at time } t' | y \text{ at time } t) = \int_a^b p(y, t; y', t')\, dy'.$$

In words this is "the probability that the random variable $y'$ lies between $a$ and $b$ at time $t'$ in the future, given that it started out with value $y$ at time $t$."

If $y', t'$ vars. $\implies$ $y, t$ is fixed

If $y, t$ vars $\implies$ $y', t'$ fixed

**Warning:** The trinomial random walk presented here is 'discrete' in the sense that time and random variable only take discrete values.

We will be moving over to a continuous-time and continuous-variable model shortly.

discrete

random

Think of $y$ and $t$ as being current values with $y'$ and $t'$ being future values. The transition probability density function can be used to answer the question,

"What is the probability of the variable $y'$ being in a specified range at time $t'$ in the future given that it started out with value $y$ at time $t$?"

$t \longrightarrow$

$t + \delta t$

The transition probability density function $p(y,t;y',t')$ satisfies two equations, one involving derivatives with respect to the future state and time ($y'$ and $t'$) and called the **forward equation**, and the other involving derivatives with respect to the current state and time ($y$ and $t$) and called the **backward equation**.

fwd eq$^\frown$     $y',t' \longleftarrow$     $(y,t)$ fixed

bwd eq$^\frown$     $y,t \longleftarrow$     $(y',t')$ fixed

## From the trinomial model
## to the transition probability density function

The variable $y$ can either rise, fall or take the same value after a time step $\delta t$. These movements have certain probabilities associated with them.

We are going to assume that the probability of a rise and a fall are both the same, $\alpha < \frac{1}{2}$. (But, of course, this can be generalized. Why would we want to generalize this?)

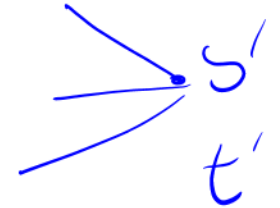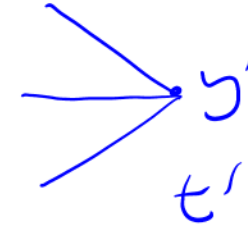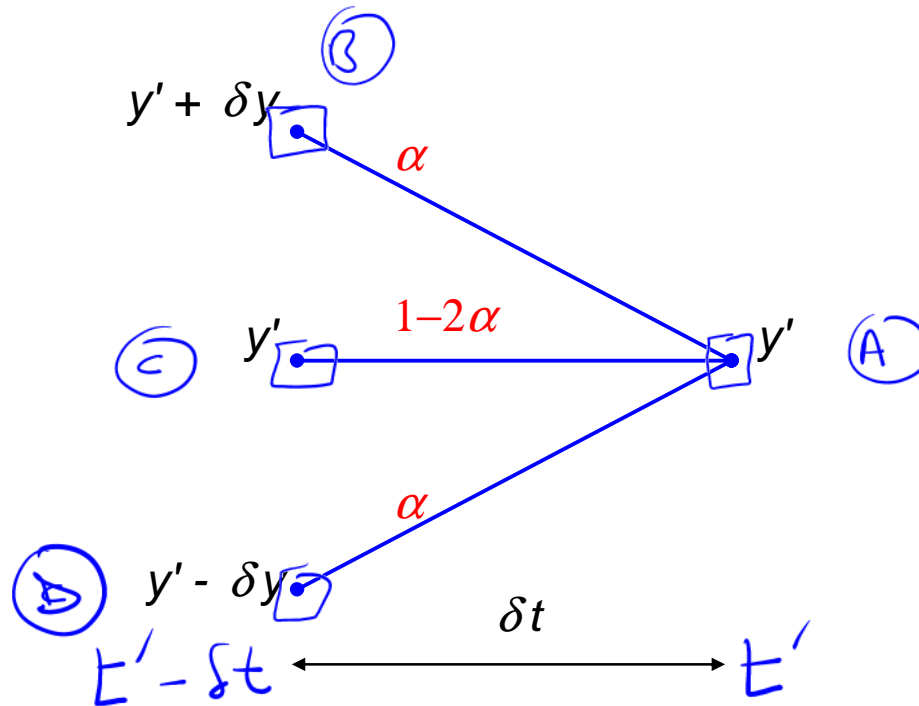$\therefore$ finance variables are more complex.

## The forward equation

The variable $y$ takes the value $y'$ at time $t'$, but how did it get there?

current

$y$

•$y$

$t$

fixed

I.C.



In our trinomial walk we can only get to the point $y'$ from the three values $y' + \delta y$, $y'$ and $y' - \delta y$.

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The probability of being at $y'$ at time $t'$ is related to the probabilities of being at the previous three values and *moving in the right direction*:

$$\text{(A)} \quad p(y,t;y',t') = \overset{\text{(B)}}{\underset{\downarrow \text{down}}{\alpha}} \, p(y,t;y'+\delta y, t'-\delta t)$$

$$+(1-2\alpha)p(y,t;\overline{y'},t'-\delta t) + \alpha \, p(y,t;y'-\delta y,t'-\delta t). \quad (1)$$

staying $\overset{\text{(C)}}{\longrightarrow}$ up $\uparrow$ (D)

Expand about each point in the brackets

about $t' - \delta t$

T.S.E : $1^{st}$ order in time $\boxed{\delta t}$

$2^{nd}$ order in the position $\boxed{\delta y^2}$

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We can easily expand each of the terms in Taylor series about the point $y'$, $t'$. For example,

$$y' \quad \text{and} \quad t'$$

Ⓑ $p(y, t; y' + \delta y, t' - \delta t) \approx p(y, t; y', t') - \delta t \frac{\partial p}{\partial t'} \pm \delta y \frac{\partial p}{\partial y'} + \frac{1}{2} \delta y^2 \frac{\partial^2 p}{\partial y'^2} + \cdots,$$

Ⓓ $p(y, t; y' - \delta y, t' - \delta t) \approx p(y, t; y', t') - \delta t \frac{\partial p}{\partial t'} - \delta y \frac{\partial p}{\partial y'} + \frac{1}{2} \delta y^2 \frac{\partial^2 p}{\partial y'^2} + \cdots$

and

Ⓒ $p(y, t; y', t' - \delta t) \approx p(y, t; y', t') - \delta t \frac{\partial p}{\partial t'} + \cdots.$

Subst Ⓑ, Ⓒ, Ⓓ in (1)

ignore $(s, t)$ ∴ they don't change

Equation (1) becomes (and we are using shorthand notation $p(y, t; y', t') = p$)

$$p = \alpha(p - \delta t \frac{\partial p}{\partial t'} + \delta y \frac{\partial p}{\partial y'} + \tfrac{1}{2}\delta y^2 \frac{\partial^2 p}{\partial y'^2} + \cdots)$$

$$+ (1 - 2\alpha)(p - \delta t \frac{\partial p}{\partial t'} + \cdots)$$

$$+ \alpha(p - \delta t \frac{\partial p}{\partial t'} - \delta y \frac{\partial p}{\partial y'} + \tfrac{1}{2}\delta y^2 \frac{\partial^2 p}{\partial y'^2} + \cdots).$$

Lots of these terms cancel out!  $\therefore$ of the symmetry in our random walk

We are left with

$$\frac{\partial p}{\partial t'} = \alpha\, \delta y^2 \frac{\partial^2 p}{\partial y'^2} + \cdots$$

$$\underbrace{\quad}_{\text{higher order terms}}$$

$$\frac{\delta y^2}{\delta t}$$

Now we explicitly mention that we are really interested in the continuous limit, as increments in time and $y$ get smaller and smaller. ① $\delta y \to 0$ quicker than $\delta t \to 0$     random walk collapses to zero

The above equation only makes sense if

② $\delta t \to 0$ quicker than numerator $\to$ random walk grows indef.

No random walk

③ $\frac{\delta y^2}{\delta t}$ is a constant     $\boxed{\dfrac{\alpha\, \delta y^2}{\delta t}}$

tends to some finite limit as the time step and the $y$ increment $\delta y$ go to zero.     $\dfrac{\delta y^2}{\delta t} \sim O(1) \implies \delta y^2 \sim O(\delta t)$

or     $\delta y \sim O(\sqrt{\delta t})$

Let's define

$$\frac{\alpha \, \delta y^2}{\delta t} = c^2,$$

$$\therefore \quad \alpha \, \frac{\delta y^2}{\delta t} > 0$$

for some finite, non-zero $c$.

The final equation is now

$$\frac{\partial p}{\partial t'} = c^2 \frac{\partial^2 p}{\partial y'^2}.$$

heat / diffusion eq$^n$

Fwd Kolmogorov Eq$^n$    FkE

Fwd Eq$^n$

This is the **Fokker–Planck** or **forward Kolmogorov equation**. It is a forward parabolic partial differential equation, requiring initial conditions at time $t$ and to be solved for $t' > t$.

This equation is to be used if there is some special state now and you want to know what could happen later. For example, you know the current value of $y$ and want to know the distribution of values at some later date.

**Observations:**

- This is a partial differential equation for $p$ as a function of two independent variables $y'$ and $t'$.

  $p(s', t')$

- It is an example of a diffusion $|$ equation.

  *heat*

  $\uparrow$ hidden in here

  are $(s, t)$

- $y$ and $t$ are rather like parameters in this problem, think of them as starting quantities for the random walk. or initial condition

- This is a diffusion equation. You need that special relationship between $\alpha$, $\delta t$ and $\delta y$ to get this equation.

- This is also an example of Brownian motion.

  $\delta t^{y_2}$

- When we get on to financial applications the quantity $c$ will be related to volatility.
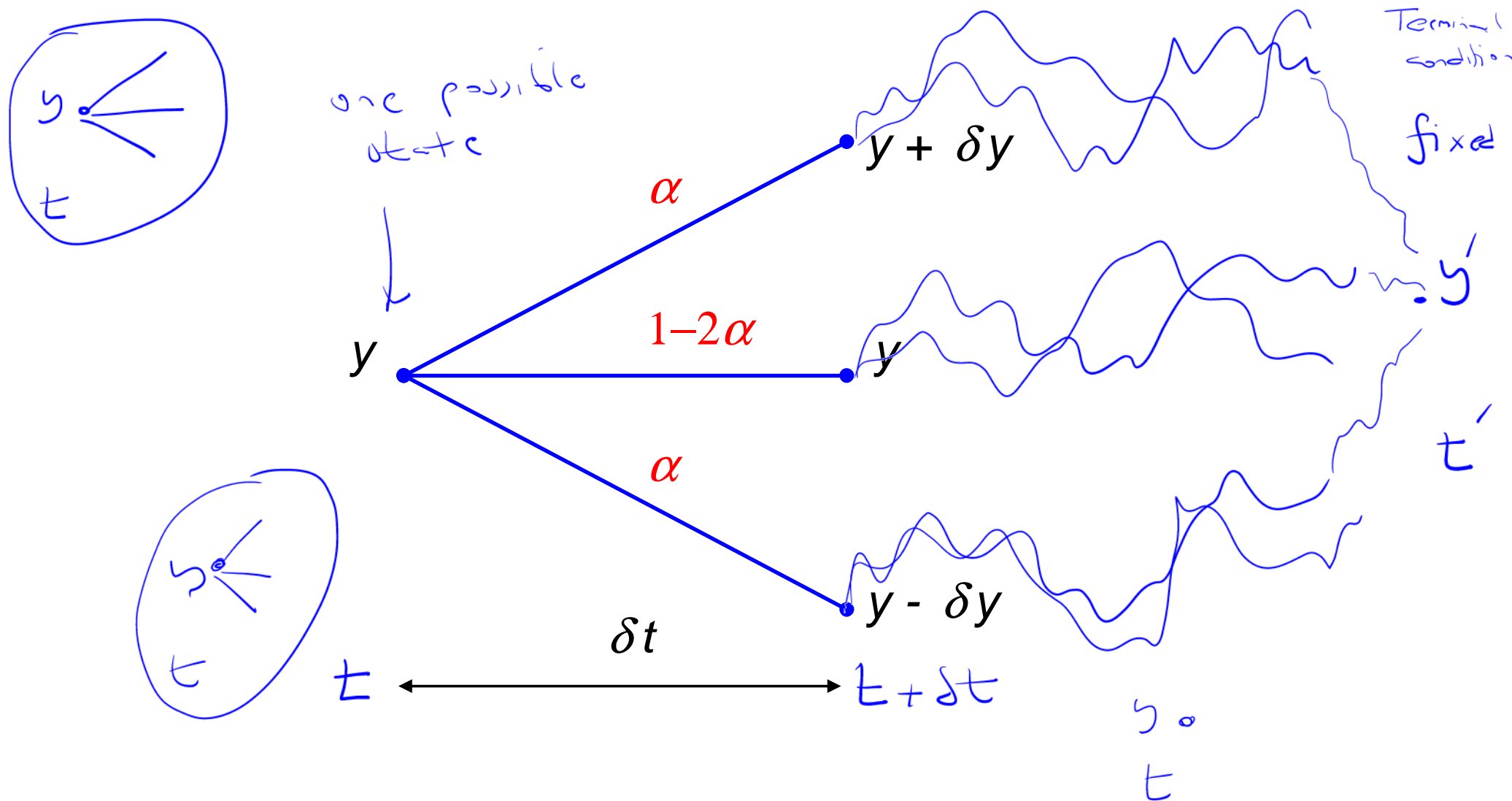
  In mod 3.1

  $$\frac{\partial p}{\partial t} = \frac{1}{2} \sigma^2 \frac{\partial^2 p}{\partial s^2}$$

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## The backward equation

Now we come to find the backward equation. This will be useful if we want to calculate probabilities of reaching a specified final state from various initial states. It will be a backward parabolic partial differential equation requiring conditions imposed in the future, and solved backwards in time.

Whereas the forward equation had independent variable $t'$ and $y'$ the backward equation has variables $t$ and $y$.

The derivation uses the trinomial random walk directly as drawn here.



$y + \delta y$

$\alpha$

$1{-}2\alpha$

$y$

$\alpha$

$y - \delta y$

$\delta t$

$y$

one possible state

Terminal condition

fixed

$y'$

$t'$

$t$        $t + \delta t$

$y_0$

$t$

The justification for the relationship between the probabilities of being at the four 'nodes' is more subtle than in the derivation of the forward equation.
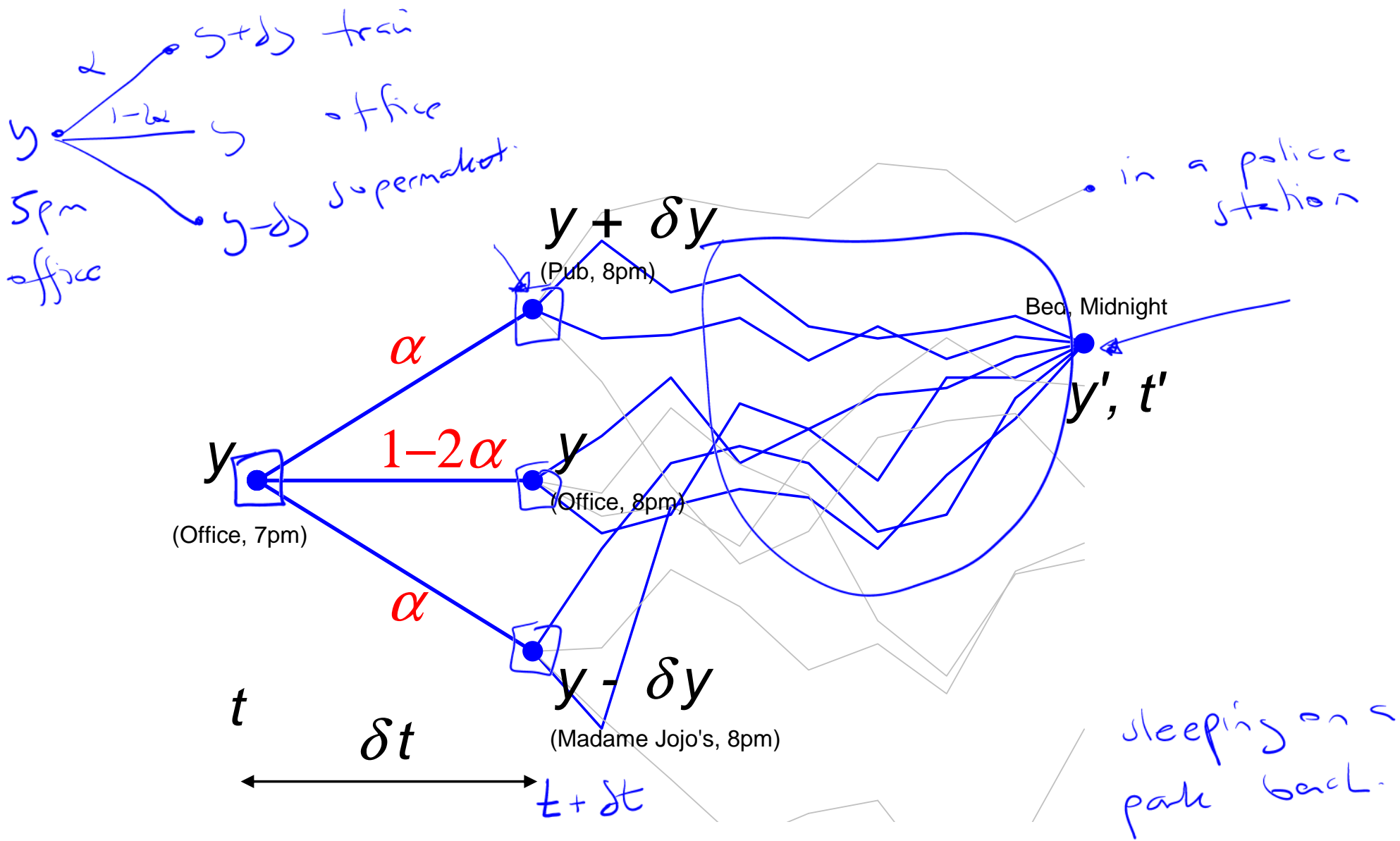
So, let's look at a concrete example.

At 7pm you are in the office. (This is the point $(y, t)$.)

At 8pm you will be at one of three places: The Pub; Still at the office; Madame Jojo's. (These are the points $(y + \delta y, t + \delta t)$, $(y, t + \delta t)$ and $(y - \delta y, t + \delta t)$.)

We are going to look at the probability that at midnight you are tucked up in bed. (This is the point $(y', t')$.)

fixed

$y + \delta y$

(Pub, 8pm)

$y$

(Office, 7pm)

$1-2\alpha$

$y$

(Office, 8pm)

$\alpha$

$\alpha$

$y - \delta y$

(Madame Jojo's, 8pm)

Bed, Midnight

$y', t'$

in a police station

sleeping on a park bench.

$t$

$\delta t$

$t + \delta t$

$y$

$\alpha$ → $y + \delta y$ train

$1-2\alpha$ → $y$ office

$y - \delta y$ supermarket

5pm office

Remember that $p(y, t; y', t')$ represents the probability of being at the future point $(y', t')$, bed at midnight, given that you started at $(y, t)$, the office at 7pm.

You can only get to the bed at midnight via either the pub, the office or Madame Jojo's at 8pm.

What happens after 8pm doesn't matter (you may not even remember!), we are only concerned with the probability that you are in bed at midnight, not how you got there.

In the previous figure there are lots of different paths, only the ones ending up in bed are of interest to us.

In words:

The probability of going from the office at 7pm to bed at midnight is the probability of going to the pub from the office and then to bed at midnight plus the probability of staying in the office and then getting to bed at midnight plus the probability of going to Madame Jojo's from the office and then to bed at midnight.

In symbols we can write this as

bed @ 12 am

$$p(y, t; y', t') = \alpha \, p(y + \delta y, t + \delta t; y', t')$$

$$+(1 - 2\alpha)p(y, t + \delta t; y', t') + \alpha \, p(y - \delta y, t + \delta t; y', t').$$

pob.

office

MJ

The Taylor series expansion leads to the **backward Kolmogorov equation**.

① $\delta y^2 \to 0$ quicker than $\delta t$

② $\delta y \to 0$ slower than $\delta t$

③ $\dfrac{\delta y^2}{\delta t} \sim O(1)$    scenario

$\delta y^2 \sim O(\delta t)$

$\delta y \sim O(\sqrt{\delta t})$

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The end result is

$$-\frac{\partial p}{\partial t} = c^2 \frac{\partial^2 p}{\partial s^2}$$

$$\frac{\partial p}{\partial t} + c^2 \frac{\partial^2 p}{\partial y^2} = 0.$$

B.k.e

Exactly the same as the forward equation, but with a sign change.

(The sign change makes all the difference between a forward and a backward diffusion equation.)

**Warning:** More general random walks lead to slightly more complicated forward and backward equations, and their relationship is no longer as simple as a change of sign.

$p$rob's are fns of the position

$$\phi^+(s,t) \qquad \phi^-(s,t)$$

## Generalization

We will look at more general random walks in later lectures. But why would we want to consider them?

- Financial quantities are more interesting than the trinomial model here.

- At the very least, equity prices cannot go negative, unlike the variable $y$ here.  $IRs$ can be $-ve$.

- We might need different models for different financial quantities, equities, interest rates,...

# Similarity solutions

That was our first partial differential equation.

As a general rule, we are not going to spend much time finding explicit solutions to equations—our emphasis will be on number crunching—but it is well worth looking at the forward diffusion equation in some detail. In particular, it is helpful to solve the equation in a simple case because *Create a new variable by combining existing variables in a special way, so that that the dimension of the problem reduces by one.*

- it illustrates a very useful technique, similarity solutions

- it highlights the important role that the normal distribution plays

*Solution in terms of Gaussian curves.*

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The equation to be solved is

*We use an invariant scaling.*

$$p = p(y', t')$$

*Model PDE to
solve*

$$\frac{\partial p}{\partial t'} = c^2 \frac{\partial^2 p}{\partial y'^2}. \quad \langle 1 \rangle$$

This equation has an infinite number of solutions. It has different solutions for different **initial conditions** and different **boundary conditions** .

The initial condition tells you how the solution starts off. We must specify $p$ as a function of $y'$ at some point in time, $t'$.

Boundary conditions tell you how the function behaves on specified $y'$ boundaries. Diffusion equations typically need two boundary conditions.

**Warning:** We are now going to do lots of (very fast) manipulations of functions, including solving ordinary differential equations. Do not panic!

We will look for a solution of a special form.

And because it works.

We take the unknown fn. $p(y', t')$ and decompose to

1. time dep./ scaling
2. 'regular' pdf $f(\cdot)$

$\left. \begin{array}{l} \text{some fn. in time} \\ \text{times a pdf for} \\ \text{some new R.V.} \end{array} \right\}$

$a = -\frac{1}{2}$

$b = \frac{1}{2}$

We are now going to find a very simple solution. It is very simple and very special because unlike most solutions of the diffusion it does not depend on *two* independent variable $y'$ and $t'$ but on a combination of them.

The new variable is

Let us seek a solution of the form

$$\xi = \frac{y'}{t'^b}$$

time
s.t
$$p = t'^a f\left(\frac{y'}{t'^b}\right).$$

new
R.v

$P = t'^a f(\xi)$

Here $a$ and $b$ are constants, T.B.D

time
Scaling

Note that $f$ is a function of only the one variable.

Subst $p$ into <1> ie F.K.E

From $\xi = y' t'^{-b}$

$$\frac{\partial \xi}{\partial y'} = t'^{-b} \quad ; \quad \frac{\partial \xi}{\partial t'} = -b y' t'^{-b-1}$$

$$p = t'^a f\left(\frac{y'}{t'^b}\right) = t'^a f(\xi)$$

we have

$$\frac{\partial p}{\partial y'} = t'^a \frac{\partial}{\partial y'} f = t'^a \frac{df}{d\xi} \frac{\partial \xi}{\partial y'} = t'^{a-b} \frac{df}{d\xi}$$

$$\frac{\partial p}{\partial y'} = t'^{a-b} \frac{df}{d\xi} \leftarrow$$

where

$$\xi = \frac{y'}{t'^b}.$$

**Note:** Derivatives of $f$ are ordinary, not partial, since $f$ only has one argument.

Then

$$\frac{\partial}{\partial y'}\left(t'^{a-b}\frac{df}{d\xi}\right) = t'^{a-b}\frac{\partial \xi}{\partial y'}\frac{d}{d\xi}\frac{df}{d\xi} = t'^{a-b}\,t'^{-b}\frac{d^2f}{d\xi^2}$$

$$= t'^{-b}$$

$$\frac{\partial^2 p}{\partial y'^2} = t'^{a-2b}\frac{d^2 f}{d\xi^2}.$$

(†)

Also

$$\frac{\partial p}{\partial t'} = at'^{a-1}f(\xi) - by't'^{a-b-1}\frac{df}{d\xi}.$$

(††)

$$\frac{\partial}{\partial t'}\left[t'^{a}f(\xi)\right] = at'^{a-1}f(\xi) + t'^{a}\frac{df}{d\xi}\frac{\partial \xi}{\partial t'}$$

$$=$$

Subst. (†), (††) in ⟨1⟩

Let's substitute these into our partial differential equation and see what happens.

$$at'^{a-1} f(\xi) - by't'^{a-b-1} \frac{df}{d\xi} = c^2 t'^{a-2b} \frac{d^2 f}{d\xi^2}.$$

We know

$$y't'^{-b} = \xi$$

This is not really any type of Diff Eqⁿ

$$at'^{a-1} f - b\xi t'^{a-1} \frac{df}{d\xi} = c^2 t'^{a-2b} \frac{d^2 f}{d\xi^2}$$

We can cancel out all the $t'$ terms

if we set $a - 1 = a - 2b$

$$\Rightarrow \boxed{b = \frac{1}{2}}$$

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Let's cancel some $t'$s and write $y' = t'^b \xi$:

$$a f(\xi) - b\xi \frac{df}{d\xi} = c^2 t'^{-2b+1} \frac{d^2 f}{d\xi^2}.$$

So far we have only been fooling around 'changing variables.' The next step is important.

The left-hand side of this equation is only a function of $\xi$, whereas the right-hand side depends on both $\xi$ and $t'$. This is only possible if the right-hand side is also independent of $t'$.

And this is only possible if $b = \frac{1}{2}$.

If we can solve

$$af(\xi) - \frac{1}{2}\xi\frac{df}{d\xi} = c^2\frac{d^2 f}{d\xi^2}$$

Our ODE becomes :

then we have found a solution of our original equation in the form

$$p = t'^a f\left(\frac{y'}{t'^b}\right) = t'^a f\left(\frac{y'}{\sqrt{t'}}\right)$$

$$p = t'^a f\left(\frac{y'}{\sqrt{t'}}\right).$$

$p$ is a transition pdf

$\xi$ R.V

$\therefore$ $f(\xi)$ pdf

And this isn't just a single solution, it is a whole family of solutions because we can choose the constant $a$.

$$a \; ?$$

However, for our present problem, only one value of $a$ is relevant.

Remember that $p$ represents a probability. That means that its integral must be one:

$$\text{for all } t' \quad \boxed{\int_{-\infty}^{\infty} p(y', t')\, dy' = 1} = \int_{-\infty}^{\infty} t'^a f\left(\frac{y'}{\sqrt{t'}}\right)\, dy'.$$

Let $u = \dfrac{y'}{\sqrt{t'}}$

$\sqrt{t'}\, du = dy'$

Change variables by writing $y' = t'^{1/2} u$ to get

$t'^{1/2} \int_{-\infty}^{\infty} t'^a f(u)\, du$

$t'^{a+\frac{1}{2}} = 1 = t'^0$

$\therefore \quad a + \frac{1}{2} = 0$

$$\boxed{t'^{a+1/2} \int_{-\infty}^{\infty} f(u)\, du = 1.} \quad \text{for all } t'$$

Conclusion? This is only possible if $a = -\frac{1}{2}$.

Our ordinary differential equation is now

$$-\frac{1}{2}f(\xi) - \frac{1}{2}\xi\frac{df}{d\xi} = c^2\frac{d^2f}{d\xi^2}.$$

This can be written as

$$\text{LH·S} \quad -\frac{1}{2}\left(f + \xi\frac{df}{d\xi}\right)$$

$$\underbrace{\phantom{-\frac{1}{2}\left(f + \xi\frac{df}{d\xi}\right)}}$$

$$\frac{d}{d\xi}\left(\xi f(\xi)\right)$$

$$\longrightarrow \quad -\frac{1}{2}\frac{d\left(\xi\,f(\xi)\right)}{d\xi} = c^2\frac{d^2f}{d\xi^2}.$$

(That was lucky!) $\therefore$ we can now integrate both sides

$$-\frac{1}{2}\xi f(\xi) = c^2\frac{df}{d\xi} + \text{const.}$$

This can be integrated once to give

$$-\frac{1}{2}\xi\, f(\xi) = c^2 \frac{df}{d\xi} + const.$$

$\xi$ R·V

$f(\xi)$ p.d.f.

$\xi \to \infty \begin{cases} f \to 0 \\ \frac{df}{d\xi} \to 0 \end{cases}$

$\overset{=0}{\overbrace{\hspace{1cm}}}$ over the $c^2\frac{df}{d\xi}$ term

$\therefore \; const = 0$

(There's an arbitrary constant of integration that could go in here but for the answer we want this is zero.)

This can be rewritten as

$$c^2 \frac{d(\ln f)}{d\xi} = -\frac{1}{2}\xi$$

$e^{-\frac{\xi^2}{4c^2}+k} = e^{-\frac{\xi^2}{4c^2}} \times \left( e^k \right)$

$A$

$c^2\frac{df}{d\xi} = -\frac{1}{2}\xi f(\xi)$

$\frac{df}{f} = -\frac{\xi}{2c^2}\,d\xi$

Now $\int$

$$\int \frac{df}{f} = -\frac{1}{2c^2}\int \xi\, d\xi$$

$$\log f(\xi) = -\frac{\xi^2}{4c^2}+k$$

Take exp of both sides

$$f(\xi) = A\exp\left(-\xi^2/4c^2\right)$$

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$$\int_{\mathbb{R}} = \int_{-\infty}^{\infty}$$

...and integrated again to give

$$\ln f(\xi) = -\frac{\xi^2}{4c^2} + \text{ an arbitrary constant of integration}$$

Normalising const.

or

$$\int_{\mathbb{R}} f(\xi)\, d\xi = 1$$

$$A \int_{\mathbb{R}} e^{-\xi^2/4c^2}\, d\xi = 1$$

$$f(\xi) = A \exp\left(-\frac{\xi^2}{4c^2}\right).$$

Use integration by parts

The constant $A$ is chosen so that the integral of $f$ is one.

Let $\quad x = \dfrac{\xi}{2c} \implies \quad 2c\, dx = d\xi$

$$2c\, A \underbrace{\int_{-\infty}^{\infty} e^{-x^2}\, dx}_{\sqrt{\pi}} = 1$$

$$2c\sqrt{\pi}\, A = 1 \qquad \therefore \quad A = \frac{1}{2c\sqrt{\pi}}$$

$$f(\xi) = \frac{1}{2c\sqrt{\pi}} e^{-\frac{\xi^2}{4c^2}}$$

Cutting to the chase, and going back to the original $t'$ and $y'$, we have

$$p = \frac{1}{2c\sqrt{\pi\,t'}}\exp\left(-\frac{y'^2}{4c^2t'}\right).$$

$$\xi = \frac{y'}{\sqrt{t'}}$$

Do you recognize this expression?

$$p(y',t') = \frac{1}{\sqrt{t'}}\frac{1}{2\sqrt{\pi}}\exp\left(-\frac{\xi^2}{4c^2}\right)$$

$$= \frac{1}{2c\sqrt{\pi t'}}\exp\left[-\frac{1}{4c^2}\left(\frac{y'}{\sqrt{t'}}\right)^2\right]$$

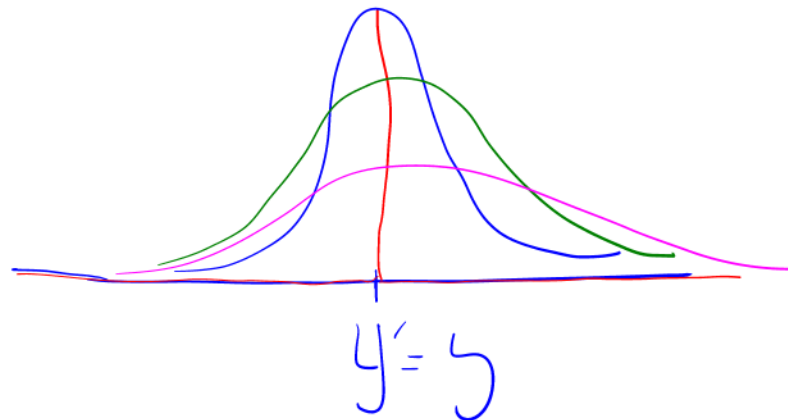$$\frac{1}{2c\sqrt{\pi t'}} \exp\left[-\frac{1}{4c^2}\left(\frac{y'}{\sqrt{t'}}\right)^2\right]$$

I.C :     at $t'=t$   we know $y'=y$

$$p(y,t\,;\,y',t') = \frac{1}{2c\sqrt{\pi|t'-t|}} \exp\left[-\frac{1}{4c^2}\frac{(y'-y)^2}{|t'-t|}\right]$$

We have incorporated the I.C.

At $y'=y$     $p = \dfrac{1}{2c\sqrt{\pi|t'-t|}}$    As $t'\to t$   p blows up.

<span style="color:red">Dirac delta<br>fn.</span>



$y'=y$

$$p(y,t \; ; y',t') = \frac{1}{2c\sqrt{\pi|t'-t|}} \exp\left[-\frac{1}{4c^2} \frac{(y'-y)^2}{|t'-t|}\right]$$

c.f. Normal dist$^n$ $N(\mu, \sigma^2)$

$$-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2} \qquad : \qquad -\frac{1}{2}\frac{(y'-0)^2}{2c^2 t'}$$

i.e mean = 0

Variance = $2c^2 t'$

$$y' \sim N(0, 2c^2 t')$$

It is very like

$$\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\phi^2}{2}\right).$$

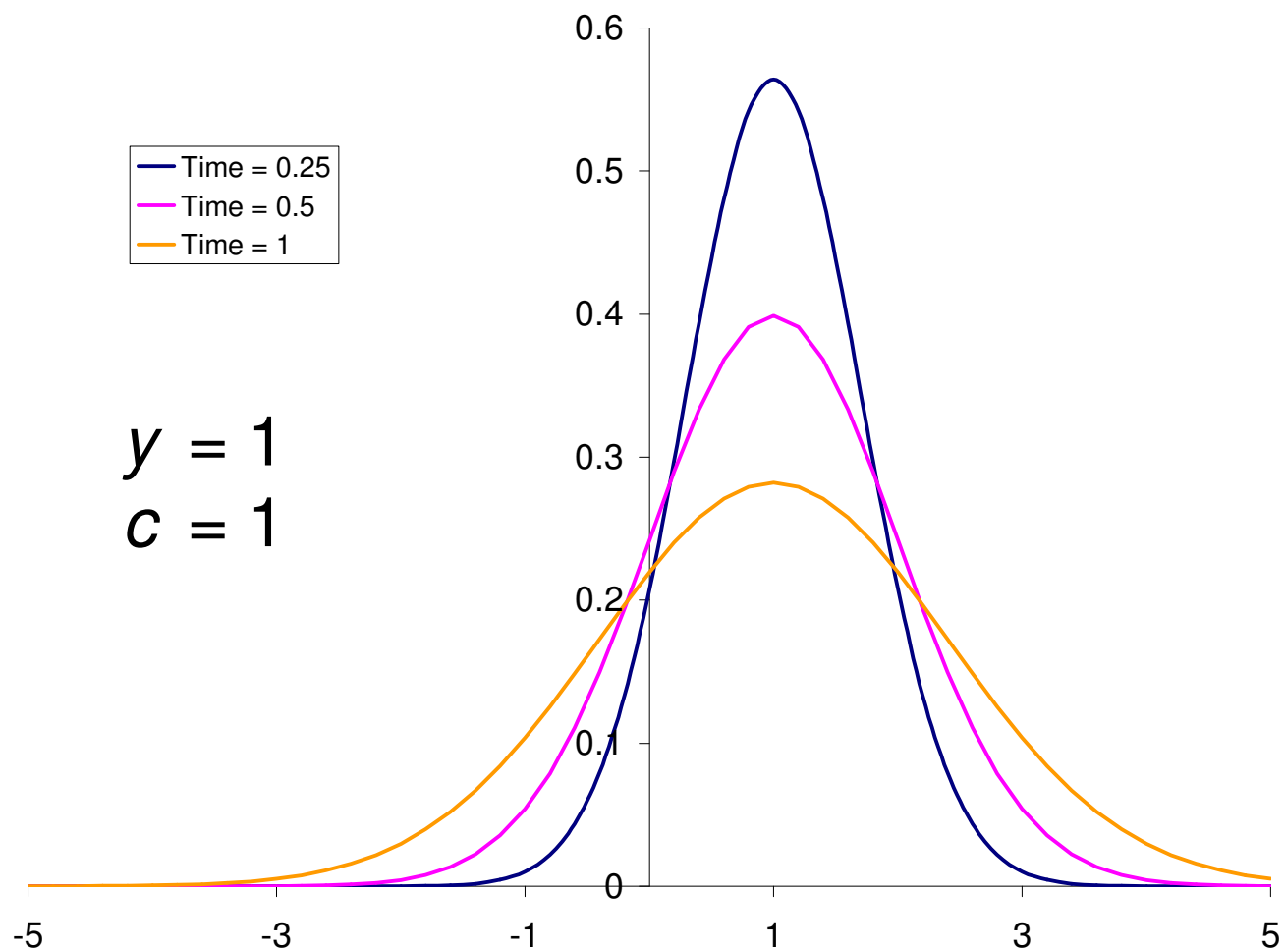It is the probability density function for the normal distribution!

So

- $y'$ is normally distributed

- with mean of zero

- and standard deviation of $c\sqrt{2t'}$

Minor generalization. . . suppose that $y'$ has value $y$ at time $t$ then we have

$$p(y, t; y', t') = \frac{1}{2c\sqrt{\pi(t' - t)}} \exp\left(-\frac{(y' - y)^2}{4c^2(t' - t)}\right).$$

And this is our transition probability density functionfor our random walk!

$$y = 1$$
$$c = 1$$

## Summary

Please take away the following important ideas

- Taylor series as a way of finding out values of a function knowing only local information (such as gradients, and higher derivatives). It is an approximation only.

- Random walks have associated differential equations for their probability density functions, and are naturally related to the normal distribution.

- Generally partial differential equations are hard to solve explicitly, but sometimes they can be simplified to ordinary differential equations.