Euler spiral

The Euler spiral is a special curve whose curvature increases linearly with the length of the curve away from the origin, and is widely used in railway engineering. It is defined by the Fresnel integrals as parametric equations

$$C(t) = \int_0^t \cos u^2 \, \mathrm{d}u, \quad S(t) = \int_0^t \sin u^2 \, \mathrm{d}u.$$

The parameter t represents the length of the curve from the origin. The integrals can be expanded into power series form by integrating the Taylor series for $\sin u^2$ and $\cos u^2$ to get

$$\begin{aligned} C(t) &= \sum_{n=0}^{\infty} \frac{(-1)^n \, t^{4n+1}}{(2n)!(4n+1)} &&= t - \frac{t^5}{10} + \frac{t^9}{216} - \cdots, \\ S(t) &= \sum_{n=0}^{\infty} \frac{(-1)^n \, t^{4n+3}}{(2n+1)!(4n+3)} &&= \frac{t^3}{3} - \frac{t^7}{42} + \frac{t^{11}}{1320} - \cdots. \end{aligned}$$

The Euler spiral can be normalised to scale it up, like for example to model a railway easement curve. If a section of the spiral starting with zero curvature has length L_0 and radius of curvature R_0, the factor is defined as

$$\frac{1}{a^2} = 2L_0R_0 \quad \implies \quad a = \frac{1}{\sqrt{2L_0R_0}}.$$

For the parametric equations, the length t is multiplied by the factor a and the overall result is divided by a:

$$\begin{aligned} \frac{1}{a}\int_0^{at} \cos u^2 \, \mathrm{d}u &= \frac{1}{a}\left(at - \frac{(at)^5}{10} + \frac{(at)^9}{216} - \cdots\right), \\ \frac{1}{a}\int_0^{at} \sin u^2 \, \mathrm{d}u &= \frac{1}{a}\left(\frac{(at)^3}{3} - \frac{(at)^7}{42} + \frac{(at)^{11}}{1320} - \cdots\right). \end{aligned}$$

Easement curve definition

The easement curve in TS2016 uses truncated forms of these power series. An easement curve with speed tolerance 200 km/h and ending with radius of curvature 800 m has a length of 298.5 m. Since the length of curve increases cubically with the speed tolerance, we include the speed tolerance V in terms of km/h with the length L_0 and find the factor

$$\frac{1}{a^2} = 2 \times 298.5 \times \left(\frac{V}{200}\right)^3 \times 800, \quad \therefore \; a = \frac{100}{\sqrt{597}} \, V^{-3/2}. \tag{a}$$

If the easement curve starts from the origin along the z axis and curves clockwise, it is defined as pair of parametric equations on the x and z axes (the horizontal axes in TS2016)

$$x(t) = \frac{1}{3}a^2t^3, \quad z(t) = t - \frac{1}{10}a^4t^5; \tag{b}$$

where t is the length of the curve. The anticlockwise version can be acquired by reversing the sign of the $x(t)$ function.

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Tangential angle

The tangential angle ψ, which is the angle between the tangent at a point t on the curve and the z axis, is defined as

$$\psi: \ \frac{(x', z')}{\sqrt{x'^2 + z'^2}} \equiv (\sin\psi, \cos\psi).$$

The functions $x'(t)$ and $z'(t)$ are the first order derivatives of $x(t)$ and $z(t)$ respectively. With $x'(t) = a^2t^2$ and $z'(t) = 1 - \frac{1}{2}a^4t^4$, the tangential angle is

$$\psi(t) = \arccos\left(\frac{2 - (at)^4}{\sqrt{4 + (at)^8}}\right). \tag{c}$$

Curvature

The curvature κ, which is the inverse of the radius of curvature, is defined on the x and z axes as

$$\kappa \equiv \frac{|x'z'' - z'x''|}{(x'^2 + z'^2)^{3/2}}.$$

However, using this definition with the parametric equations (b) provides results that do not match up with what TS2016 shows.

Instead TS2016 uses a simpler definition, one that assumes the curve to be a normalised Euler spiral, ie the power series are not truncated. Given that the curvature is the inverse of the radius of curvature and is proportional to the length of the easement curve, it follows that the length t multiplied by the radius of curvature R should be constant for any point on the curve:

$$tR = 298.5 \times \left(\frac{V}{200}\right)^3 \times 800 = 0.02985 \times V^3,$$

where the initial values from equation (a) are used. With $\kappa = 1/R$, we can define the curvature in terms of t:

$$\kappa(t) = \frac{t}{0.02985\ V^3} \quad \text{and} \quad t = 0.02985\ V^3\kappa. \tag{d}$$

References

[1] Wikipedia, “Track transition curve,” 2016.
`https://en.wikipedia.org/wiki/Track_transition_curve`.

[2] Wikipedia, “Euler spiral,” 2016.
`https://en.wikipedia.org/wiki/Euler_spiral`.

[3] Wikipedia, “Curvature,” 2016.
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[4] Wikipedia, “Tangential angle,” 2013.
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