q,w-AIM for the Second Order Linear Difference Equation

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Introduction of the Hahn Operator

A generalization of the derivative that takes the ordinary form as $q \to 1$ is the *q-derivative*

$$D_q y(x) = \frac{y(qx) - y(x)}{(q-1)x}.$$

It has seen a lot of use in the area of q-calculus. A similar operator is the *shift operator*

$$\Delta_w y(x) = \frac{y(x+w) - y(x)}{w}.$$

Which in the case of $w = 1$ is equivalent to recurrence relations. It was Wolfgang Hahn [1] who introduced an operator that captures the properties of both these operator in the *Hahn operator*

$$\Delta_{q;w} y(x) = \frac{y(qx+w) - y(x)}{(q-1)x + w}.$$

Our use case will be solving second order linear homogeneous equations with functional coefficients in the Hahn operator

$$\Delta^2_{q;w} y(x) = \lambda_0(x)\Delta_{q;w} y(x) + s_0(x) y(x).$$

There is no known general solution for this equation, however we have found that both linearly independent solutions can be found if the *terminating condition*

$$\frac{s_n(x)}{\lambda_n(x)} = \frac{s_{n-1}(x)}{\lambda_{n-1}(x)}$$

is satisified, where

$$\begin{aligned}\lambda_n(x) &= \Delta_{q;w}\lambda_{n-1}(x) + \lambda_{n-1}(qx+w)\lambda_0(x)\\ &\quad + s_{n-1}(qx+w)\\ s_n(x) &= \Delta_{q;w} s_{n-1}(x) + \lambda_{n-1}(qx+w) s_0(x).\end{aligned}$$

The *qw — Exponential*

It was shown by Annaby et al. [2] that, for certain fixed $a \in \mathbb{C}$, the *q,w-exponential* defined as

$$\begin{aligned}e_{q;w}(a;x) &= \frac{1}{\prod_{k=0}^{\infty}(1 - aq^k(x(q-1) - w))}\\ &= \sum_{i=0}^{\infty} \frac{(a(x(1-q) - w)^k}{(q;q)_k}\end{aligned}$$

is the unique solution to

$$\Delta_{q;w} y(x) = ay(x), \quad y\left(\frac{w}{1-q}\right) = 1.$$

The Main Result

Let $x_i = q^i x + w\sum_{k=0}^{i-1} q^k$, the i^{th} composition of $qx + w$. The second order linear homogeneous Hahn difference equation

$$\Delta^2_{q;w} y(x) = \lambda_0(x)\Delta_{q;w} y(x) + s_0(x) y(x)$$

has solution

$$\begin{aligned}y(x) &= \frac{1}{\prod_{i=0}^{\infty}[1 - q^i((q-1)x + w)\frac{s_{n-1}(x_i)}{\lambda_{n-1}(x_i)}]}\\ &\quad + \sum_{i=0}^{\infty} \frac{\frac{q^i((q-1)x+w)}{\lambda_{n-1}(x_i)}}{\prod_{j=0}^{i}\left[1 - q^j((q-1)x + w)\frac{s_{n-1}(x_j)}{\lambda_{n-1}(x_j)}\right]\prod_{j=0}^{\infty}\left[1 + q^{i+j}((q-1)x + w)\frac{\lambda_n(x_{i+j})}{\lambda_{n-1}(x_{i+j})}\right]}\end{aligned}$$

for $0 < q < 1$, $w \geq 0$ provided that

$$\frac{s_n(x)}{\lambda_n(x)} = \frac{s_{n-1}(x)}{\lambda_{n-1}(x)}.$$

The Asymptotic Iteration Method

The Asymptotic Iteration Method (AIM) was first introduced by Dr. Saad et al. [3] in 2003. It is a method to solve second order linear homogeneous differential equations, and we generalized this method to the Hahn operator. If we consider the second order equation, and take the derivative we can see that

$$\begin{aligned}\Delta^3_{q;w} y(x) &= \Delta_{q;w}\left[\Delta_{q;w}\lambda_0(x) y(x) + s_0(x) y(x)\right]\\ &= [\Delta_{q;w}\lambda_0(x) + \lambda_0(qx+w)\lambda_0(x)\\ &\quad + s_0(qx+w)]\Delta_{q;w} y(x)\\ &\quad + [\Delta_{q;w} s_{n-1}(x) + \lambda_{n-1}(qx+w) s_0(x)] y(x)\\ &= \Delta_{q;w}\lambda_1(x) y(x) + s_1(x) y(x)\end{aligned}$$

and if we keep doing this we can arrive at

$$\Delta^{n+2}_{q;w} = \lambda_n(x)\Delta_{q;w} y(x) + s_n(x) y(x).$$

When we consider the ratio

$$\frac{\Delta^{n+2}_{q;w} y(x)}{\Delta^{n+1}_{q;w} y(x)} = \frac{\lambda_n(x)\left[\Delta_{q;w} y(x) + \frac{s_n(x)}{\lambda_n(x)} y(x)\right]}{\lambda_{n-1}(x)\left[\Delta_{q;w} y(x) + \frac{s_{n-1}(x)}{\lambda_{n-1}(x)} y(x)\right]} = \frac{\lambda_n(x)}{\lambda_{n-1}(x)},$$

we can use the terminating condition to actually reduce the second order equation into a first order one, and because this is solvable we may derive the solution.

Constant Coefficients

$$\Delta^2_{q;w} y(x) = \lambda_0\Delta_{q;w} y(x) + s_0 y(x)$$

With λ_0 and s_0 constant, despite not having an explicit n such that $\frac{n(x)}{\lambda_n(x)} = \frac{s_{n-1}(x)}{\lambda_{n-1}(x)}$, we may solve using qwAIM. With both coefficients constant, it follows the qwAIM sequence is

$$\begin{aligned}\lambda_n(x) &= \lambda_{n-1}\lambda_0 + s_{n-1}\\ s_n(x) &= \lambda_{n-1} s_0.\end{aligned}$$

We can then show that if $\frac{s_n(x)}{\lambda_n(x)} = \frac{s_{n-1}(x)}{\lambda_{n-1}(x)}$ then

$$\left(\frac{s_n}{\lambda_n}\right)^2 + \frac{s_n}{\lambda_n}\lambda_0 - s_0 = 0.$$

This is a quadratic in $\frac{s_n}{\lambda_n}$ so the solution is

$$\frac{s_n}{\lambda_n} = \frac{-\lambda_0 \pm \sqrt{\lambda_0^2 + 4s_0}}{2}.$$

We then construct the solution as

$$y(x) = C_1 e_{q,w}\left(\frac{\lambda_0 - \sqrt{\lambda_0^2 + 4s_0}}{2}; x\right) + C_2 e_{q,w}\left(\frac{\lambda_0 + \sqrt{\lambda_0^2 + 4s_0}}{2}; x\right).$$

An example

We will solve

$$\Delta^2_{q;w} y(x) = x\Delta_{q;w} y(x) - y(x).$$

We can take $\lambda_0(x) = x$ and $s_0(x) = -1$. Then this implies $\lambda_1(x) = qx^2 + wx$ and $s_1(x) = -(qx + w)$. It follows that

$$\frac{s_0(x)}{\lambda_0(x)} = \frac{-1}{x} = \frac{-(qx+w)}{qx^2 + wx} = \frac{s_1(x)}{\lambda_1(x)}$$

and so the first independent solution for $0 < q < 1$ is

$$\begin{aligned}y(x) &= \frac{1}{\prod_{i=0}^{\infty}\left[1 - q^i((q-1)x + w)\frac{s_0(x_i)}{\lambda_0(x_i)}\right]}\\ &= \frac{1}{\prod_{i=0}^{\infty}\left[1 - q^i((q-1)x + w)\frac{-1}{x_i}\right]}\\ &= \lim_{k\to\infty}\frac{1}{\prod_{i=0}^{k}\left[1 + \frac{q^i((q-1)x+w)}{(q^i x + w[i]_q)}\right]}\\ &= \lim_{k\to\infty}\frac{x}{q^{1+k} + [k+1]_q w}\\ &= \frac{(1-q)x}{w}.\end{aligned}$$

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