Notes on qw-Aim

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1 Introduction

We focus on the shifted and scaled operator $\Delta_{q;w}$ defined as

$$\Delta_{q;w} f(x) = \frac{f(qx+w) - f(x)}{(q-1)x + w}$$

where $0 < q < 1$ and $w \geq 0$. We also define $w_0 = \frac{w}{1-q}$

1.1 Useful identities

1.1.1 q pochhammer

$$(a;q)_n = \prod_{i=0}^{n-1} (1 - aq^k)$$

1.1.2 $y_1(qx+w)$ and $y_1(x)$

Denote $y_1(x) = \prod_{k=0}^{m-1} \frac{1}{1+q^k((1-q)x-w)\frac{s_{n-1}(x_k)}{\lambda_{n-1}(x_k)}}$ then

$$\begin{aligned}
y_1(qx+w) &= \prod_{k=0}^{m-1} \frac{1}{1 + q^{k+1}((1-q)x - w)\frac{s_{n-1}(x_{k+1})}{\lambda_{n-1}(x_{k+1})}} \\
&= \frac{1 + ((1-q)x - w)\frac{s_{n-1}(x)}{\lambda_{n-1}(x)}}{1 + q^m((1-q)x - w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}} \prod_{k=0}^{m-1} \frac{1}{1 + q^k((1-q)x - w)\frac{s_{n-1}(x_k)}{\lambda_{n-1}(x_k)}} \\
&= \frac{1 + ((1-q)x - w)\frac{s_{n-1}(x)}{\lambda_{n-1}(x)}}{1 + q^m((1-q)x - w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}} y_1(x)
\end{aligned}$$

1.1.3 $\Delta_{q;w} y_1(x) and y_1(x)$

$$\begin{aligned}
\Delta_{q;w} y_1(x) &= \frac{y_1(qx+w) - y_1(x)}{(q-1)x+w} \\
&= \frac{\frac{1+((1-q)x-w)\frac{s_{n-1}(x)}{\lambda_{n-1}(x)}}{1+q^m((1-q)x-w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}} y_1(x) - y_1(x)}{(q-1)x+w} \\
&= y_1(x) \frac{\frac{1+((1-q)x-w)\frac{s_{n-1}(x)}{\lambda_{n-1}(x)}}{1+q^m((1-q)x-w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}} - 1}{(q-1)x+w} \\
&= y_1(x) \frac{\frac{1+((1-q)x-w)\frac{s_{n-1}(x)}{\lambda_{n-1}(x)}}{1+q^m((1-q)x-w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}} - \frac{1+q^m((1-q)x-w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}}{1+q^m((1-q)x-w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}}}{(q-1)x+w} \\
&= y_1(x) \frac{\frac{((1-q)x-w)\frac{s_{n-1}(x)}{\lambda_{n-1}(x)} - q^m((1-q)x-w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}}{1+q^m((1-q)x-w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}}}{(q-1)x+w} \\
&= y_1(x) \frac{\frac{s_{n-1}(x)}{\lambda_{n-1}(x)} - q^m \frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}}{1+q^m((1-q)x-w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}}
\end{aligned}$$

1.2 Identities on $\Delta_{q;w}$

1.2.1 Equivalence to other operators

$$\begin{aligned}
\Delta_{q,w} f(x) &= \frac{f(qx+w) - f(x)}{(q-1)x+w} \\
\Delta_{1,1} f(x) &= \frac{f(1x+1) - f(x)}{(1-1)x+1} = f(x+1) - f(x) = \Delta f(x) \\
\Delta_{1,w} f(x) &= \frac{f(1x+w) - f(x)}{(1-1)x+w} = \frac{f(x+w) - f(x)}{w} = \Delta_w f(x) \\
\Delta_{q,0} f(x) &= \frac{f(qx) - f(x)}{(q-1)x} = D_q f(x), \quad q \neq 1 \\
\lim_{w\to 0} \Delta_{1,w} f(x) &= \lim_{w\to 0} \frac{f(x+w) - f(x)}{w} = \frac{d}{dx} f(x)
\end{aligned}$$

1.2.2 Linearity of $\Delta_{q;w}$

$$\begin{aligned}
\Delta_{q;w}[af(x) + bg(x)] &= \frac{af(qx+w) + bg(qx+w) - (af(x) + bg(x))}{(q-1)x+w} \\
&= \frac{af(qx+w) - af(x)}{(q-1)x+w} + \frac{bg(qx+w) - bg(x)}{(q-1)x+w} \\
&= a\frac{f(qx+w) - f(x)}{(q-1)x+w} + b\frac{g(qx+w) - g(x)}{(q-1)x+w} \\
&= a\Delta_{q;w} f(x) + b\Delta_{q;w} g(x)
\end{aligned}$$

1.2.3 Product rule $\Delta_{q;w} f(x)g(x)$

$$\begin{aligned}
\Delta_{q;w} f(x)g(x) &= \frac{f(qx+w)g(qx+w) - f(x)g(x)}{(q-1)x+w} \\
&= f(qx+w)\Delta_{q;w} g(x) + \frac{f(qx+w)g(x)}{(q-1)x+w} - \frac{f(x)g(x)}{(q-1)x+w} \\
&= f(qx+w)\Delta_{q;w} g(x) + g(x)\Delta_{q;w} f(x) \\
&= g(qx+w)\Delta_{q;w} f(x) + f(x)\Delta_{q;w} g(x)
\end{aligned}$$

We note that this agrees with the 4 special cases and their respective product rules

$$\begin{aligned}\Delta_{1,1} f(x)g(x) &= g(1x+1)\Delta f(x) + f(x)\Delta g(x)\\ &= g(x+1)\Delta f(x) + f(x)\Delta g(x)\\ &= f(x+1)\Delta g(x) + g(x)\Delta f(x)\end{aligned}$$

$$\begin{aligned}\Delta_{1,w} f(x)g(x) &= g(1x+w)\Delta_w f(x) + f(x)\Delta_w g(x)\\ &= g(x+w)\Delta_w f(x) + f(x)\Delta_w g(x)\\ &= f(x+w)\Delta_w g(x) + g(x)\Delta_w f(x)\end{aligned}$$

$$\begin{aligned}\Delta_{q,0} f(x)g(x) &= g(qx+0)D_q f(x) + f(x)D_q g(x)\\ &= g(qx)D_q f(x) + f(x)D_q g(x)\\ &= f(qx)D_q g(x) + g(x)D_q f(x)\end{aligned}$$

$$\begin{aligned}\lim_{w\to 0}\Delta_{1,w} f(x)g(x) &= \lim_{w\to 0} g(1x+w)\Delta_{1,w} f(x) + f(x)\Delta_{1,w} g(x)\\ &= g(x+0)\lim_{w\to 0}\Delta_{1,w} f(x) + f(x)\lim_{w\to 0}\Delta_{1,w} g(x)\\ &= g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x)\end{aligned}$$

1.2.4 Reciprocal Rule $\Delta_{q;w}\frac{1}{f(x)}$

$$\begin{aligned}\Delta_{q;w}\frac{1}{f(x)} &= \frac{\frac{1}{f(qx+w)} - \frac{1}{f(x)}}{(q-1)x+w}\\ &= \frac{1}{(q-1)x+w}\cdot\frac{f(x)-f(qx+w)}{f(qx+w)f(x)}\\ &= \frac{-\Delta_{q;w} f(x)}{f(qx+w)f(x)}\end{aligned}$$

This agrees with reciprical rules from specific cases

$$\begin{aligned}\Delta_{1;1}\frac{1}{f(x)} &= \frac{-\Delta f(x)}{f((1-1)x+1)f(x)}\\ &= \frac{-1\Delta f(x)}{f(x+1)f(x)}\end{aligned}$$

$$\begin{aligned}\Delta_{1;w}\frac{1}{f(x)} &= \frac{-\Delta_w f(x)}{f((1-1)x+w)f(x)}\\ &= \frac{-\Delta_w f(x)}{f(x+w)f(x)}\end{aligned}$$

$$\begin{aligned}\Delta_{q;0}\frac{1}{f(x)} &= \frac{-D_q f(x)}{f(qx+0)f(x)}\\ &= \frac{-D_q f(x)}{f(qx)f(x)}\end{aligned}$$

$$\begin{aligned}\lim_{w\to 0}\Delta_{1;w}\frac{1}{f(x)} &= \lim_{w\to 0}\frac{-\Delta_{1;w} f(x)}{f(1x+w)f(x)}\\ &= \frac{-\frac{d}{dx}f(x)}{f(x)f(x)}\\ &= \frac{-\frac{df}{dx}}{f(x)^2}\end{aligned}$$

1.2.5 Quotient Rule $\Delta_{q;w}\frac{f(x)}{g(x)}$

$$\begin{aligned}
\Delta_{q;w}\frac{f(x)}{g(x)} &= \Delta_{q;w}f(x)\frac{1}{g(x)} \\
&= \frac{1}{g(qx+w)}\Delta_{q;w}f(x) + f(x)\Delta_{q;w}\frac{1}{g(x)} \\
&= \frac{1}{g(qx+w)}\Delta_{q;w}f(x) + f(x)\frac{-\Delta_{q;w}g(x)}{g(qx+w)g(x)} \\
&= \frac{g(x)\Delta_{q;w}f(x) - f(x)\Delta_{q;w}g(x)}{g(qx+w)g(x)}
\end{aligned}$$

agreeing with the 4 specific cases

$$\begin{aligned}
\Delta_{1;1}\frac{f(x)}{g(x)} &= \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x+1)g(x)} \\
&= \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x+1)g(x)}
\end{aligned}$$

$$\begin{aligned}
\Delta_{1;w}\frac{f(x)}{g(x)} &= \frac{g(x)\Delta_w f(x) - f(x)\Delta_w g(x)}{g(x+w)g(x)} \\
&= \frac{g(x)\Delta_w f(x) - f(x)\Delta_w g(x)}{g(x+w)g(x)}
\end{aligned}$$

$$\begin{aligned}
\Delta_{q;0}\frac{f(x)}{g(x)} &= \frac{g(x)D_q f(x) - f(x)D_q g(x)}{g(qx+0)g(x)} \\
&= \frac{g(x)D_q f(x) - f(x)D_q g(x)}{g(qx)g(x)}
\end{aligned}$$

$$\begin{aligned}
\lim_{w\to 0}\Delta_{1;w}\frac{f(x)}{g(x)} &= \lim_{w\to 0}\frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{g(1x+w)g(x)} \\
&= \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{g(x)g(x)} \\
&= \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{g(x)^2}
\end{aligned}$$

1.2.6 Relationship between $f(qx+w)$ and $\Delta_{q;w}f(x)$

Theorem 1. *The following are equivalent*

$$\begin{aligned}
y(q^2x+qw+w) &= f(x)y(qx+w) + g(x)y(x) \\
\Delta^2_{q;w}y(x) &= \frac{f(x)-(1+q)}{q((q-1)x+w)}\Delta_{q;w}y(x) + \frac{g(x)+f(x)-1}{q((q-1)x+w)^2}y(x)
\end{aligned}$$

and

$$\begin{aligned}
\Delta^2_{q;w}y(x) &= \lambda(x)\Delta_{q;w}y(x) + s(x)y(x) \\
y(q^2x+qw+w) &= y(qx+w)\left[(1+q) + q((q-1)x+w)\lambda(x)\right] \\
&\quad + y(x)\left[q((q-1)x+w)^2 s(x) - q((q-1)x+w) - q\right]
\end{aligned}$$

$$\begin{aligned}
y(q^2x+qw+w) &= f(x)y(qx+w) + g(x)y(x) \\
\frac{y(q^2x+qw+w)-(1+q)y(qx+w)+qy(x)}{q((q-1)x+w)^2} &= \frac{-(1+q)y(qx+w)+qy(x)+f(x)y(qx+w)+g(x)y(x)}{q((q-1)x+w)^2} \\
\Delta^2_{q;w}y(x) &= \frac{[f(x)-(1+q)]y(qx+w)+[q+g(x)]y(x)-[f(x)-(1+q)]y(x)+[f(x)-(1+q)]y(x)}{q((q-1)x+w)^2}
\end{aligned}$$

$$\Delta^2_{q;w}y(x) = \frac{f(x)-(1+q)}{q((q-1)x+w)}\Delta_{q;w}y(x) + \frac{[q+g(x)+f(x)-(1+q)]}{q((q-1)x+w)^2}y(x)$$
$$\Delta^2_{q;w}y(x) = \frac{f(x)-(1+q)}{q((q-1)x+w)}\Delta_{q;w}y(x) + \frac{g(x)+f(x)-1}{q((q-1)x+w)^2}y(x)$$

$$\Delta^2_{q;w}y(x) = \lambda(x)\Delta_{q;w}y(x) + s(x)y(x)$$
$$\frac{y(q^2x+qw+w)-(1+q)y(qx+w)+qy(x)}{q((q-1)x+w)^2} = \lambda(x)\frac{y(qx+w)-y(x)}{(q-1)x+w} + s(x)y(x)$$

$$((q-1)x+w)y(q^2x+qw+w) - ((q-1)x+w)(1+q)y(qx+w) + ((q-1)x+w)qy(x)$$
$$= q((q-1)x+w)^2\lambda(x)(y(qx+w)-y(x)) + q((q-1)x+w)((q-1)x+w)^2s(x)y(x)$$

$$((q-1)x+w)y(q^2x+qw+w) = y(qx+w)\left[(1+q)((q-1)x+w) + q((q-1)x+w)^2\lambda(x)\right]$$
$$+y(x)\left[q((q-1)x+w)^3s(x) - q((q-1)x+w)^2 - ((q-1)x+w)q\right]$$

$$y(q^2x+qw+w) = y(qx+w)\left[(1+q) + q((q-1)x+w)\lambda(x)\right]$$
$$+y(x)\left[q((q-1)x+w)^2s(x) - q((q-1)x+w) - q\right]$$

1.2.7 $\Delta_{q;w}f(qx+w) + \Delta_{q;w}f(x)$

Theorem 2.
$$\Delta_{q;w}f(qx+w) + \Delta_{q;w}f(x) = q((q-1)x+w)\Delta^2_{q;w}f(x) + (q+1)\Delta_{q;w}f(x)$$

Proof.

$$f(qx+w) = ((q-1)x+w)\Delta_{q;w}f(x) + f(x)$$

$$\begin{aligned}
\Delta_{q;w}f(qx+w) &= \Delta_{q;w}[((q-1)x+w)\Delta_{q;w}f(x) + f(x)] \\
&= q((q-1)x+w)\Delta^2_{q;w}f(x) + (q-1)\Delta_{q;w}f(x) + \Delta_{q;w}f(x) \\
&= q((q-1)x+w)\Delta^2_{q;w}f(x) + q\Delta_{q;w}f(x)
\end{aligned}$$

$$\Delta_{q;w}f(qx+w) + \Delta_{q;w}f(x) = q((q-1)x+w)\Delta^2_{q;w}f(x) + (q+1)\Delta_{q;w}f(x)$$

□

2 First Order Difference Equation Homogeneous

Theorem 3. *The solution to the first order homogeneous general difference equation*

$$\Delta_{q;w}y(x) = \lambda(x)y(x)$$

is given by

$$y(x) = y(x_0)\frac{\prod_{i=0}^{\infty} 1 + (q^i[x_0(q-1)+w])\lambda(q^ix_0 + w[i]_q)}{\prod_{i=0}^{\infty} 1 + (q^i[x(q-1)+w])\lambda(q^ix + w[i]_q)}$$

where $y(x_0)$ is a given initial condition.

We look to solve in general the equation

$$\Delta_{q;w}y(x) = \lambda(x)y(x)$$

which is equivalent to

$$\begin{aligned}\frac{y(qx+w)-y(x)}{(q-1)x+w} &= \lambda(x)y(x)\\ y(qx+w)-y(x) &= ((q-1)x+w)\lambda(x)y(x)\\ y(qx+w) &= [1+((q-1)x+w)\lambda(x)]y(x)\end{aligned}$$

Suppose we are given an initial condition $y(x_0)$, then consider

$$y(qx_0+w) = [1+((q-1)x_0+w)\lambda(x_0)]y(x_0)$$

$$\begin{aligned}y(q(qx_0+w)+w) &= y(q^2x_0+qw+w)\\ &= [1+((q-1)(qx_0+w)+w)\lambda(qx_0+w)]y(qx_0+w)\\ &= [1+((q-1)(qx_0+w)+w)\lambda(qx_0+w)][1+((q-1)x_0+w)\lambda(x_0)]y(x_0)\end{aligned}$$

$$\begin{aligned}y(q(q(qx_0+w)+w)+w) &= y(q^3x_0+q^2w+qw+w)\\ &= [1+((q-1)(qx_0+w)+w)\lambda(qx_0+w)]y(q^2x_0+qw+w)\\ &= [1+((q-1)(q^2x_0+wq+w)+w)\lambda(q^2x_0+qw+w)][1+((q-1)(qx_0+w)+w)\lambda(qx_0+w)]\\ &\times[1+((q-1)x_0+w)\lambda(x_0)]y(x_0)\end{aligned}$$

and in general

$$\begin{aligned}y(q^kx_0+w\sum_{i=0}^{k-1}q^i) &= y(x_0)\prod_{i=0}^{k-1}\left[1+\left(w+(q-1)\left(q^ix_0+w\sum_{j=0}^{i-1}q^j\right)\right)\lambda(q^ix_0+w\sum_{j=0}^{i-1}q^j)\right]\\ &= y(x_0)\prod_{i=0}^{k-1}\left[1+\left(w+q^ix_0(q-1)+w[q+q^2+q^3\ldots q^i-1-q\cdots-q^{i-1}]\right)\lambda(q^ix_0+w\sum_{j=0}^{i-1}q^j)\right]\\ &= y(x_0)\prod_{i=0}^{k-1}\left[1+\left(w+q^ix_0(q-1)+w[q^i-1]\right)\lambda(q^ix_0+w\sum_{j=0}^{i-1}q^j)\right]\\ &= y(x_0)\prod_{i=0}^{k-1}\left[1+\left[q^i(w+x_0(q-1))\right]\lambda\left(x_i\right)\right].\end{aligned}$$

Letting $k\to\infty$ we get

$$y(w_0) = y(x_0)\prod_{i=0}^{\infty}\left[1+\left[q^i(w+x_0(q-1))\right]\lambda\left(x_i\right)\right]$$

Taking the original derivation using $x\to xq$ we arrive at

$$y(q^kx+w\sum_{i=0}^{k-1}q^i) = y(x)\prod_{i=0}^{k-1}\left[1+\left[q^i(w+x(q-1))\right]\lambda\left(x_i\right)\right].$$

Taking $k\to\infty$ again we get

$$\begin{aligned}y(w_0) &= y(x)\prod_{i=0}^{\infty}\left[1+\left[q^i(w+x(q-1))\right]\lambda\left(q^ix+w[i]_q\right)\right]\\ y(x) &= \frac{y(w_0)}{\prod_{i=0}^{\infty}\left[1+\left[q^i(w+x(q-1))\right]\lambda\left(q^ix+w[i]_q\right)\right]}\\ y(x) &= y(x_0)\frac{\prod_{i=0}^{\infty}\left[1+\left[q^i(w+x_0(q-1))\right]\lambda\left(q^ix_0+w[i]_q\right)\right]}{\prod_{i=0}^{\infty}\left[1+\left[q^i(w+x(q-1))\right]\lambda\left(q^ix+w[i]_q\right)\right]}\end{aligned}$$

3 First Order Difference Equation Non-Homogeneous

Theorem 4. *The solution to the Hahn difference equation*

$$\Delta_{q;w}y(x) = \lambda(x)y(x) + g(x)$$

is given by

$$y(x) = \frac{y(w_0)}{\prod_{i=0}^{\infty}[1+q^i((q-1)x+w)\lambda(x_i')]} - \sum_{i=0}^{\infty}\frac{q^i((q-1)x+w)g(x_i')}{\prod_{j=0}^{i}[1+q^j((q-1)x+w)\lambda(x_j')]}$$

Proof. It is sufficient to show that $y(qx+w) = [1+((q-1)x+w)\lambda(x)]y(x) + [(q-1)x+w]g(x)$.

$$\begin{aligned}
y(qx+w) &= \frac{y(w_0)}{\prod_{i=0}^{\infty}[1+q^i((q-1)(qx+w)+w)\lambda(x_{i+1}')]} - \sum_{i=0}^{\infty}\frac{q^i((q-1)(qx+w)+w)g(x_{i+1}')}{\prod_{j=0}^{i}[1+q^j((q-1)(qx+w)+w)\lambda(x_{j+1}')]}\\
&= \frac{y(w_0)}{\prod_{i=0}^{\infty}[1+q^{i+1}((q-1)x+w)\lambda(x_{i+1}')]} - \sum_{i=0}^{\infty}\frac{q^{i+1}((q-1)x+w)g(x_{i+1}')}{\prod_{j=0}^{i}[1+q^{j+1}((q-1)x+w)\lambda(x_{j+1}')]}\\
&= \frac{y(w_0)}{\prod_{i=1}^{\infty}[1+q^i((q-1)x+w)\lambda(x_i')]} - \sum_{i=1}^{\infty}\frac{q^i((q-1)x+w)g(x_i')}{\prod_{j=1}^{i-1}[1+q^j((q-1)x+w)\lambda(x_j')]}\\
&= [1+[(q-1)x+w]\lambda(x)]\left\{\frac{y(w_0)}{\prod_{i=0}^{\infty}[1+q^i((q-1)x+w)\lambda(x_i')]} - \sum_{i=0}^{\infty}\frac{q^i((q-1)x+w)g(x_i')}{\prod_{j=0}^{i}[1+q^j((q-1)x+w)\lambda(x_j')]}\right\}\\
&+ ((q-1)x+w)g(x)\\
&= (1+[(q-1)x+w]\lambda(x))y(x) + ((q-1)x+w)g(x)
\end{aligned}$$

□

Derivation

We rewrite the above equation as

$$y(qx+w) = [1+[(q-1)x+w]\lambda(x)]y(x) + [(q-1)x+w]g(x)$$

now given an initial condition $y(x_0)$ we may see that

$$\begin{aligned}
y(q(qx_0+w)+w) &= [1+[(q-1)(qx_0+w)+w]\lambda(qx_0+w)]y(qx_0+w) + [(q-1)(qx_0+w)+w]g(qx_0+w)\\
&= [1+[(q-1)(qx_0+w)+w]\lambda(qx_0+w)]([1+[(q-1)x_0+w]\lambda(x_0)]y(x_0)\\
&+ [(q-1)x_0+w]g(x_0)) + [(q-1)(qx_0+w)+w]g(qx_0+w)\\
&= [1+[(q-1)(qx_0+w)+w]\lambda(qx_0+w)][1+[(q-1)x_0+w]\lambda(x_0)]y(x_0)\\
&+ [1+[(q-1)(qx_0+w)+w]\lambda(qx_0+w)][(q-1)x_0+w]g(x_0)\\
&+ [(q-1)(qx_0+w)+w]g(qx_0+w)
\end{aligned}$$

applying the same substitution again yields

$$\begin{aligned}
y(q(q(qx_0+w)+w)+w) &= [1+[(q-1)(q(qx_0+w)+w)+w]\lambda(q(qx_0+w)+w)]y(q(qx_0+w)+w)\\
&+ [(q-1)(q(qx_0+w)+w)+w]g(q(qx_0+w)+w)\\
\\
&= [1+[(q-1)(q(qx_0+w)+w)+w]\lambda(q(qx_0+w)+w)][[1+[(q-1)(qx_0+w)+w]\lambda(qx_0+w)]\\
&\times[1+[(q-1)x_0+w]\lambda(x_0)]y(x_0)\\
&+[1+[(q-1)(qx_0+w)+w]\lambda(qx_0+w)][(q-1)x_0+w]g(x_0)\\
&+[(q-1)(qx_0+w)+w]g(qx_0+w)]\\
&+ [(q-1)(q(qx_0+w)+w)+w]g(q(qx_0+w)+w)\\
\\
&= [1+[(q-1)(q(qx_0+w)+w)+w]\lambda(q(qx_0+w)+w)][[1+[(q-1)(qx_0+w)+w]\lambda(qx_0+w)]\\
&\times[1+[(q-1)x_0+w]\lambda(x_0)]y(x_0)\\
&+ [1+[(q-1)(q(qx_0+w)+w)+w]\lambda(q(qx_0+w)+w)][1+[(q-1)(qx_0+w)+w]\lambda(qx_0+w)]\\
&\times[(q-1)x_0+w]g(x_0)
\end{aligned}$$

$$+ [1 + [(q-1)(q(qx_0+w)+w)+w]\lambda(q(qx_0+w)+w)][(q-1)(qx_0+w)+w]g(qx_0+w)]$$
$$+ [(q-1)(q(qx_0+w)+w)+w]g(q(qx_0+w)+w)$$

and in general we may write

$$\begin{aligned} y\left(q^k x_0 + w\sum_{i=0}^{k-1}\right) &= y(x_0)\prod_{i=0}^{k-1}\left[1+\left[(q-1)\left(q^i x_0 + w\sum_{j=0}^{i-1} q^j\right)+w\right]\lambda\left(q^i x_0 + w\sum_{j=0}^{i-1} q^j\right)\right] \\ &+ \sum_{i=0}^{k-1}\left\{\left[(q-1)\left(q^i x_0 + w\sum_{j=0}^{i-1} q^j\right)+w\right] g\left(q^i x_0 + w\sum_{j=0}^{i-1} q^j\right)\right. \\ &\times \left.\prod_{l=i+1}^{k-1}\left[1+\left[(q-1)\left(q^l x_0 + w\sum_{j=0}^{l-1} q^j\right)+w\right]\lambda\left(q^l x_0 + w\sum_{j=0}^{l-1} q^j\right)\right]\right\} \end{aligned}$$

then using $(q-1)x_k + w = q^k(x_0(q-1)+w)$

$$\begin{aligned} y(x_k) &= y(x_0)\prod_{i=0}^{k-1}\left[1+q^i(x_0(q-1)+w)\lambda(x_i)\right] \\ &+ \sum_{i=0}^{k-1}\left\{q^i(x_0(q-1)+w)g(x_i)\prod_{l=i+1}^{k-1}\left[1+q^l(x_0(q-1)+w)\lambda(x_l)\right]\right\}. \end{aligned}$$

We take $k \to \infty$ to get

$$y(w_0) = y(x_0)\prod_{i=0}^{\infty}\left[1+q^i(x_0(q-1)+w)\lambda(x_i)\right] + \sum_{i=0}^{\infty}\left\{q^i(x_0(q-1)+w)g(x_i)\prod_{l=i+1}^{\infty}\left[1+q^l(x_0(q-1)+w)\lambda(x_l)\right]\right\}.$$

Return to the original equation, we substitute $x \to qx+w$ k times and let $x'_k = q^k x + w\sum_{i=0}^{k-1} q^i$ to yield

$$\begin{aligned} y\left(q^k x + w\sum_{i=0}^{k-1}\right) &= y(x)\prod_{i=0}^{k-1}\left[1+\left[(q-1)\left(q^i x + w\sum_{j=0}^{i-1} q^j\right)+w\right]\lambda\left(q^i x + w\sum_{j=0}^{i-1} q^j\right)\right] \\ &+ \sum_{i=0}^{k-1}\left\{\left[(q-1)\left(q^i x + w\sum_{j=0}^{i-1} q^j\right)+w\right] g\left(q^i x + w\sum_{j=0}^{i-1} q^j\right)\right. \\ &\times \left.\prod_{l=i+1}^{k-1}\left[1+\left[(q-1)\left(q^l x + w\sum_{j=0}^{l-1} q^j\right)+w\right]\lambda\left(q^l x + w\sum_{j=0}^{l-1} q^j\right)\right]\right\} \end{aligned}$$

and $k \to \infty$

$$\begin{aligned} y(w_0) &= y(x)\prod_{i=0}^{\infty}\left[1+q^i(x(q-1)+w)\lambda(x'_i)\right] + \sum_{i=0}^{\infty}\left\{q^i(x(q-1)+w)g(x'_i)\prod_{l=i+1}^{\infty}\left[1+q^l(x(q-1)+w)\lambda(x'_l)\right]\right\} \\ y(x)\prod_{i=0}^{\infty}\left[1+q^i(x(q-1)+w)\lambda(x'_i)\right] &= y(w_0) - \sum_{i=0}^{\infty}\left\{q^i(x(q-1)+w)g(x'_i)\prod_{l=i+1}^{\infty}\left[1+q^l(x(q-1)+w)\lambda(x'_l)\right]\right\} \\ y(x) &= \frac{y(w_0) - \sum_{i=0}^{\infty}\left\{q^i(x(q-1)+w)g(x'_i)\prod_{l=i+1}^{\infty}\left[1+q^l(x(q-1)+w)\lambda(x'_l)\right]\right\}}{\prod_{i=0}^{\infty}\left[1+q^i(x(q-1)+w)\lambda(x'_i)\right]} \\ y(x) &= \frac{y(w_0)}{\prod_{i=0}^{\infty}[1+q^i((q-1)x+w)\lambda(x'_i)]} - \sum_{i=0}^{\infty}\frac{q^i((q-1)x+w)g(x'_i)}{\prod_{j=0}^{i}[1+q^j((q-1)x+w)\lambda(x'_j)]} \end{aligned}$$

4 DAIM

Theorem 5. *Let $x'_k = q^k x + w\sum_{i=0}^{k-1} q^i$. The solution to the difference equation*

$$\Delta^2_{q;w} y(x) = \lambda_0(x)\Delta_{q;w} y(x) + s_0(x)y(x)$$

is given by

$$y(x) = \frac{y(w_0)}{\prod_{i=0}^{\infty}[1 - q^i((q-1)x + w)\frac{s_{n-1}(x'_i)}{\lambda_{n-1}(x'_i)}]} - \sum_{i=0}^{\infty} \frac{\frac{q^i((q-1)x+w)\Delta^{n+1}_{q;w} y(w_0)}{\lambda_{n-1}(x'_i)\prod_{i=0}^{\infty}\left[1+q^i((q-1)x'_i+w)\frac{\lambda_n(x'_i)}{\lambda_{n-1}(x'_i)}\right]}}{\prod_{j=0}^{i}\left[1 - q^j((q-1)x + w)\frac{s_{n-1}(x'_j)}{\lambda_{n-1}(x'_j)}\right]}$$

provided that

$$\frac{s_n}{\lambda_n} = \frac{s_{n-1}}{\lambda_{n-1}}.$$

Note that we can write the first solution as

$$y(x) = \frac{y(w_0)}{\left(\frac{((q-1)x+w)s_{n-1}(x'_i)}{\lambda_{n-1}(x'_i)}; q\right)_\infty}$$

Proof. **4.0.1 λ_n, s_n sequences**

We consider the derivatives of the difference equation

$$\Delta^2_{q:w} y(x) = \lambda_0(x)\Delta_{q:w} y(x) + s_0(x)y(x)$$

$$\begin{aligned}
\Delta^3_{q:w} y(x) &= \Delta_{q:w}[\lambda_0(x)\Delta_{q:w} y(x)] + \Delta_{q:w}[s_0(x)y(x)] \\
&= \lambda_0(qx+w)\Delta^2_{q:w} y(x) + \Delta_{q:w} y(x)\Delta_{q:w}\lambda_0(x) + s_0(qx+w)\Delta_{q:w} y(x) + y(x)\Delta_{q:w} s_0(x) \\
&= \lambda_0(qx+w)[\lambda_0(x)\Delta_{q;w} y(x) + s_0(x)y(x)] + \Delta_{q:w} y(x)\Delta_{q:w}\lambda_0(x) + s_0(qx+w)\Delta_{q:w} y(x) + y(x)\Delta_{q:w} s_0(x) \\
&= \Delta_{q:w} y(x)[\Delta_{q:w}\lambda_0(x) + \lambda_0(qx+w)\lambda_0(x) + s_0(qx+w)] + y(x)[\lambda_0(qx+w)s_0(x) + \Delta_{q:w} s_0(x)] \\
&= \lambda_1(x)\Delta_{q:w} y(x) + s_1(x)y(x)
\end{aligned}$$

$$\begin{aligned}
\Delta^4_{q:w} y(x) &= \Delta_{q:w}[\lambda_1(x)\Delta_{q:w} y(x)] + \Delta_{q:w}[s_1(x)y(x)] \\
&= \lambda_1(qx+w)\Delta^2_{q:w} y(x) + \Delta_{q:w} y(x)\Delta_{q:w}\lambda_1(x) + s_1(qx+w)\Delta_{q:w} y(x) + y(x)\Delta_{q:w} s_1(x) \\
&= \lambda_1(qx+w)[\lambda_0(x)\Delta_{q;w} y(x) + s_0(x)y(x)] + \Delta_{q:w} y(x)\Delta_{q:w}\lambda_1(x) + s_1(qx+w)\Delta_{q:w} y(x) + y(x)\Delta_{q:w} s_1(x) \\
&= \Delta_{q:w} y(x)[\Delta_{q:w}\lambda_1(x) + \lambda_1(qx+w)\lambda_0(x) + s_1(qx+w)] + y(x)[\lambda_1(qx+w)s_0(x) + \Delta_{q:w} s_1(x)] \\
&= \lambda_2(x)\Delta_{q:w} y(x) + s_2(x)y(x) \\
&\dots \\
\Delta^{n+2}_{q:w} y(x) &= \lambda_n(x)\Delta_{q:w} y(x) + s_n(x)y(x)
\end{aligned}$$

where

$$\begin{aligned}
\lambda_n(x) &= \Delta_{q:w}\lambda_{n-1}(x) + \lambda_{n-1}(qx+w)\lambda_0(x) + s_{n-1}(qx+w) \\
s_n(x) &= \Delta_{q:w} s_{n-1}(x) + \lambda_{n-1}(qx+w)s_0(x)
\end{aligned}$$

4.0.2 The case $\frac{s_n}{\lambda_n} = \frac{s_{n-1}}{\lambda_{n-1}}$

We consider the ratio

$$\begin{aligned}
\frac{\Delta^{n+2}_{q;w} y(x)}{\Delta^{n+1}_{q;w} y(x)} &= \frac{\lambda_n(x)\Delta_{q:w} y(x) + s_n(x)y(x)}{\lambda_{n-1}(x)\Delta_{q:w} y(x) + s_{n-1}(x)y(x)} \\
&= \frac{\lambda_n(x)\left[\Delta_{q:w} y(x) + \frac{s_n(x)}{\lambda_n(x)}(x)y(x)\right]}{\lambda_{n-1}(x)\left[\Delta_{q:w} y(x) + \frac{s_{n-1}(x)}{\lambda_{n-1}(x)}(x)y(x)\right]}.
\end{aligned}$$

If $\frac{s_n}{\lambda_n} = \frac{s_{n-1}}{\lambda_{n-1}}$ for some n, the ratio simplifies to

$$\frac{\Delta_{q;w}^{n+2} y(x)}{\Delta_{q;w}^{n+1} y(x)} = \frac{\lambda_n(x)}{\lambda_{n-1}(x)}$$

which is a first order homogeneous difference equation in $\Delta_{q;w}^{n+1} y(x)$:

$$\Delta_{q;w} \Delta_{q;w}^{n+1} y(x) = \frac{\lambda_n(x)}{\lambda_{n-1}(x)} \Delta_{q;w}^{n+1} y(x)$$

4.0.3 Solving the homogeneous equation

The solution to a first order homogeneous equation is known

$$\Delta_{q;w}^{n+1} y(x_k) = \frac{\Delta_{q;w}^{n+1} y(w_0)}{\prod_{i=0}^{\infty} [1 + q^i((q-1)x + w) \frac{\lambda_n(x_i')}{\lambda_{n-1}(x_i')}]}.$$

However, we can rewrite this equation as

$$\begin{aligned}
\Delta_{q;w}^{n+1} y(x) &= \frac{\Delta_{q;w}^{n+1} y(w_0)}{\prod_{i=0}^{\infty} [1 + q^i((q-1)x + w) \frac{\lambda_n(x_i')}{\lambda_{n-1}(x_i')}]} \\
\lambda_{n-1}(x) \Delta_{q:w} y(x) + s_{n-1}(x) y(x) &= \frac{\Delta_{q;w}^{n+1} y(w_0)}{\prod_{i=0}^{\infty} [1 + q^i((q-1)x + w) \frac{\lambda_n(x_i')}{\lambda_{n-1}(x_i')}]} \\
\Delta_{q:w} y(x) &= -\frac{s_{n-1}(x)}{\lambda_{n-1}(x)} y(x) + \frac{\Delta_{q;w}^{n+1} y(w_0)}{\lambda_{n-1}(x) \prod_{i=0}^{\infty} [1 + q^i((q-1)x + w) \frac{\lambda_n(x_i')}{\lambda_{n-1}(x_i')}]}
\end{aligned}$$

which is a first order non-homogeneous equation.

4.0.4 Solving the non-homogeneous equation

We know the solution to the non-homogeneous case, thus

$$\Delta_{q:w} y(x) = -\frac{s_{n-1}(x)}{\lambda_{n-1}(x)} y(x) + \frac{\Delta_{q;w}^{n+1} y(w_0)}{\lambda_{n-1}(x) \prod_{i=0}^{\infty} [1 + q^i((q-1)x + w) \frac{\lambda_n(x_i')}{\lambda_{n-1}(x_i')}]}$$

has solution

$$y(x) = \frac{y(w_0)}{\prod_{i=0}^{\infty} [1 - q^i((q-1)x + w) \frac{s_{n-1}(x_i')}{\lambda_{n-1}(x_i')}]} - \sum_{i=0}^{\infty} \frac{\frac{q^i((q-1)x+w) \Delta_{q;w}^{n+1} y(w_0)}{\lambda_{n-1}(x_i') \prod_{i=0}^{\infty} \left[1 + q^i((q-1)x_i' + w) \frac{\lambda_n(x_i')}{\lambda_{n-1}(x_i')}\right]}}{\prod_{j=0}^{i} \left[1 - q^j((q-1)x + w) \frac{s_{n-1}(x_j')}{\lambda_{n-1}(x_j')}\right]}$$

which is the final solution. □

5 Non-continuous DAIM solution

Theorem 6. *Let $x_k = q^k x_0 + w \sum_{i=0}^{k-1} q^i$. The solution to the difference equation*

$$\Delta_{q;w}^2 y(x) = \lambda_0(x) \Delta_{q;w} y(x) + s_0(x) y(x)$$

is given by

$$\begin{aligned}
y(x_k) = y(x_0) &\prod_{i=0}^{k-1} \left[1 - q^i((q-1)x_0 + w) \frac{s_{n-1}(x_i)}{\lambda_{n-1}(x_i)}\right] \\
&+ \left[\Delta_{q;w}^{n+1} y(x_0)\right] \sum_{i=0}^{k-1} \left\{ q^i((q-1)x_0 + w) \frac{\prod_{j=0}^{i-1} \left[1 + \left[q^j((x_0(q-1) + w)\right] \frac{\lambda_n(x_j)}{\lambda_{n-1}(x_j)}\right]}{\lambda_{n-1}(x_i)} \right. \\
&\left. \times \prod_{l=i+1}^{k-1} \left[1 - q^l((q-1)x_0 + w) \frac{s_{n-1}(x_l)}{\lambda_{n-1}(x_l)}\right] \right\}
\end{aligned}$$

provided that

$$\frac{s_n}{\lambda_n} = \frac{s_{n-1}}{\lambda_{n-1}}.$$

Proof. **5.0.1 λ_n, s_n sequences**

We consider the derivatives of the difference equation

$$\Delta^2_{q:w} y(x) = \lambda_0(x)\Delta_{q:w} y(x) + s_0(x)y(x)$$

$$\begin{aligned}
\Delta^3_{q:w} y(x) &= \Delta_{q:w}[\lambda_0(x)\Delta_{q:w} y(x)] + \Delta_{q:w}[s_0(x)y(x)] \\
&= \lambda_0(qx+w)\Delta^2_{q:w} y(x) + \Delta_{q:w} y(x)\Delta_{q:w}\lambda_0(x) + s_0(qx+w)\Delta_{q:w} y(x) + y(x)\Delta_{q:w} s_0(x) \\
&= \lambda_0(qx+w)[\lambda_0(x)\Delta_{q;w} y(x) + s_0(x)y(x)] + \Delta_{q:w} y(x)\Delta_{q:w}\lambda_0(x) + s_0(qx+w)\Delta_{q:w} y(x) + y(x)\Delta_{q:w} s_0(x) \\
&= \Delta_{q:w} y(x)[\Delta_{q:w}\lambda_0(x) + \lambda_0(qx+w)\lambda_0(x) + s_0(qx+w)] + y(x)[\lambda_0(qx+w)s_0(x) + \Delta_{q:w} s_0(x)] \\
&= \lambda_1(x)\Delta_{q:w} y(x) + s_1(x)y(x)
\end{aligned}$$

$$\begin{aligned}
\Delta^4_{q:w} y(x) &= \Delta_{q:w}[\lambda_1(x)\Delta_{q:w} y(x)] + \Delta_{q:w}[s_1(x)y(x)] \\
&= \lambda_1(qx+w)\Delta^2_{q:w} y(x) + \Delta_{q:w} y(x)\Delta_{q:w}\lambda_1(x) + s_1(qx+w)\Delta_{q:w} y(x) + y(x)\Delta_{q:w} s_1(x) \\
&= \lambda_1(qx+w)[\lambda_0(x)\Delta_{q;w} y(x) + s_0(x)y(x)] + \Delta_{q:w} y(x)\Delta_{q:w}\lambda_1(x) + s_1(qx+w)\Delta_{q:w} y(x) + y(x)\Delta_{q:w} s_1(x) \\
&= \Delta_{q:w} y(x)[\Delta_{q:w}\lambda_1(x) + \lambda_1(qx+w)\lambda_0(x) + s_1(qx+w)] + y(x)[\lambda_1(qx+w)s_0(x) + \Delta_{q:w} s_1(x)] \\
&= \lambda_2(x)\Delta_{q:w} y(x) + s_2(x)y(x) \\
&\dots \\
\Delta^{n+2}_{q:w} y(x) &= \lambda_n(x)\Delta_{q:w} y(x) + s_n(x)y(x)
\end{aligned}$$

where

$$\begin{aligned}
\lambda_n(x) &= \Delta_{q:w}\lambda_{n-1}(x) + \lambda_{n-1}(qx+w)\lambda_0(x) + s_{n-1}(qx+w) \\
s_n(x) &= \Delta_{q:w} s_{n-1}(x) + \lambda_{n-1}(qx+w)s_0(x)
\end{aligned}$$

5.0.2 The case $\frac{s_n}{\lambda_n} = \frac{s_{n-1}}{\lambda_{n-1}}$

We consider the ratio

$$\begin{aligned}
\frac{\Delta^{n+2}_{q;w} y(x)}{\Delta^{n+1}_{q;w} y(x)} &= \frac{\lambda_n(x)\Delta_{q:w} y(x) + s_n(x)y(x)}{\lambda_{n-1}(x)\Delta_{q:w} y(x) + s_{n-1}(x)y(x)} \\
&= \frac{\lambda_n(x)\left[\Delta_{q:w} y(x) + \frac{s_n(x)}{\lambda_n(x)}(x)y(x)\right]}{\lambda_{n-1}(x)\left[\Delta_{q:w} y(x) + \frac{s_{n-1}(x)}{\lambda_{n-1}(x)}(x)y(x)\right]}.
\end{aligned}$$

If $\frac{s_n}{\lambda_n} = \frac{s_{n-1}}{\lambda_{n-1}}$ for some n, the ratio simplifies to

$$\frac{\Delta^{n+2}_{q;w} y(x)}{\Delta^{n+1}_{q;w} y(x)} = \frac{\lambda_n(x)}{\lambda_{n-1}(x)}$$

which is a first order homogeneous difference equation in $\Delta^{n+1}_{q;w} y(x)$:

$$\Delta_{q;w}\Delta^{n+1}_{q;w} y(x) = \frac{\lambda_n(x)}{\lambda_{n-1}(x)}\Delta^{n+1}_{q;w} y(x)$$

□

6 Linear independence of solutions

Theorem 7. *Let $x_i = q^i x + w[i]_q$. The two solutions to*

$$\Delta^2_{q;w} y(x) = \lambda_0(x)\Delta_{q;w} y(x) + s_0(x)y(x)$$

given by

$$y_1(x) = \frac{1}{\prod_{k=0}^{m-1}\left[1 + q^k((1-q)x - w)\frac{s_{n-1}(x_k)}{\lambda_{n-1}(x_k)}\right]}$$

and

$$y_2(x) = \sum_{i=0}^{m-1} \frac{\frac{q^i((q-1)x+w}{\lambda_{n-1}(x_i)}}{\prod_{j=0}^{i}\left[1 + q^j((1-q)x - w)\frac{s_{n-1}(x_j)}{\lambda_{n-1}(x_j)}\right]\prod_{j=0}^{m-1}\left[1 - q^{i+j}((1-q)x - w)\frac{\lambda_n(x_{i+j})}{\lambda_{n-1}(x_{i+j})}\right]}$$

are linearly independent.

Proof. To prove linear independence of our two solutions, we must show that the *qw-casorati determinant* does not vanish

$$\begin{vmatrix} y_1(x) & y_1(qx+w) \\ y_2(x) & y_2(qx+w) \end{vmatrix}.$$

$$\begin{aligned}
y_1(x)y_2(qx+w) - y_1(qx+w)y_2(x) &= \frac{1}{\prod_{k=0}^{m-1}\left[1 + q^k((1-q)x - w)\frac{s_{n-1}(x_k)}{\lambda_{n-1}(x_k)}\right]} \\
&\times \sum_{i=0}^{m-1} \frac{\frac{q^{i+1}((q-1)x+w}{\lambda_{n-1}(x_{i+1})}}{\prod_{j=0}^{i}\left[1 + q^{j+1}((1-q)x - w)\frac{s_{n-1}(x_{j+1})}{\lambda_{n-1}(x_{j+1})}\right]\prod_{j=0}^{m-1}\left[1 - q^{i+j+1}((1-q)x - w)\frac{\lambda_n(x_{i+j+1})}{\lambda_{n-1}(x_{i+j+1})}\right]} \\
&- \frac{1}{\prod_{k=0}^{m-1}\left[1 + q^{k+1}((1-q)x - w)\frac{s_{n-1}(x_{k+1})}{\lambda_{n-1}(x_{k+1})}\right]} \\
&\times \sum_{i=0}^{m-1} \frac{\frac{q^i((q-1)x+w}{\lambda_{n-1}(x_i)}}{\prod_{j=0}^{i}\left[1 + q^j((1-q)x - w)\frac{s_{n-1}(x_j)}{\lambda_{n-1}(x_j)}\right]\prod_{j=0}^{m-1}\left[1 - q^{i+j}((1-q)x - w)\frac{\lambda_n(x_{i+j})}{\lambda_{n-1}(x_{i+j})}\right]} \\
&= \frac{1}{\prod_{k=0}^{m-1}\left[1 + q^k((1-q)x - w)\frac{s_{n-1}(x_k)}{\lambda_{n-1}(x_k)}\right]} \\
&\times \sum_{i=1}^{m} \frac{\frac{q^i((q-1)x+w}{\lambda_{n-1}(x_i)}}{\prod_{j=0}^{i-1}\left[1 + q^{j+1}((1-q)x - w)\frac{s_{n-1}(x_{j+1})}{\lambda_{n-1}(x_{j+1})}\right]\prod_{j=0}^{m-1}\left[1 - q^{i+j}((1-q)x - w)\frac{\lambda_n(x_{i+j})}{\lambda_{n-1}(x_{i+j})}\right]} \\
&- \frac{1}{\prod_{k=1}^{m}\left[1 + q^k((1-q)x - w)\frac{s_{n-1}(x_k)}{\lambda_{n-1}(x_k)}\right]} \\
&\times \sum_{i=0}^{m-1} \frac{\frac{q^i((q-1)x+w}{\lambda_{n-1}(x_i)}}{\prod_{j=0}^{i}\left[1 + q^j((1-q)x - w)\frac{s_{n-1}(x_j)}{\lambda_{n-1}(x_j)}\right]\prod_{j=0}^{m-1}\left[1 - q^{i+j}((1-q)x - w)\frac{\lambda_n(x_{i+j})}{\lambda_{n-1}(x_{i+j})}\right]} \\
&= \frac{1}{\prod_{k=0}^{m-1}\left[1 + q^k((1-q)x - w)\frac{s_{n-1}(x_k)}{\lambda_{n-1}(x_k)}\right]} \\
&\times \sum_{i=1}^{m} \frac{\frac{q^i((q-1)x+w}{\lambda_{n-1}(x_i)}}{\prod_{j=1}^{i}\left[1 + q^j((1-q)x - w)\frac{s_{n-1}(x_j)}{\lambda_{n-1}(x_j)}\right]\prod_{j=0}^{m-1}\left[1 - q^{i+j}((1-q)x - w)\frac{\lambda_n(x_{i+j})}{\lambda_{n-1}(x_{i+j})}\right]} \\
&- \frac{1}{\prod_{k=1}^{m}\left[1 + q^k((1-q)x - w)\frac{s_{n-1}(x_k)}{\lambda_{n-1}(x_k)}\right]} \\
&\times \sum_{i=0}^{m-1} \frac{\frac{q^i((q-1)x+w}{\lambda_{n-1}(x_i)}}{\prod_{j=0}^{i}\left[1 + q^j((1-q)x - w)\frac{s_{n-1}(x_j)}{\lambda_{n-1}(x_j)}\right]\prod_{j=0}^{m-1}\left[1 - q^{i+j}((1-q)x - w)\frac{\lambda_n(x_{i+j})}{\lambda_{n-1}(x_{i+j})}\right]} \\
&= \frac{1}{\prod_{k=1}^{m-1}\left[1 + q^k((1-q)x - w)\frac{s_{n-1}(x_k)}{\lambda_{n-1}(x_k)}\right]}
\end{aligned}$$

$$
\begin{aligned}
&\times \left\{ \sum_{i=1}^{m} \frac{\frac{q^i((q-1)x+w}{\lambda_{n-1}(x_i)}}{\prod_{j=0}^{i}\left[1+q^j((1-q)x-w)\frac{s_{n-1}(x_j)}{\lambda_{n-1}(x_j)}\right]\prod_{j=0}^{m-1}\left[1-q^{i+j}((1-q)x-w)\frac{\lambda_n(x_{i+j})}{\lambda_{n-1}(x_{i+j})}\right]} \right.\\
&- \frac{1}{\left[1+q^m((1-q)x-w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}\right]}\\
&\left. \times \sum_{i=0}^{m-1} \frac{\frac{q^i((q-1)x+w}{\lambda_{n-1}(x_i)}}{\prod_{j=0}^{i}\left[1+q^j((1-q)x-w)\frac{s_{n-1}(x_j)}{\lambda_{n-1}(x_j)}\right]\prod_{j=0}^{m-1}\left[1-q^{i+j}((1-q)x-w)\frac{\lambda_n(x_{i+j})}{\lambda_{n-1}(x_{i+j})}\right]} \right\}\\
&= \frac{1}{\prod_{k=1}^{m-1}\left[1+q^k((1-q)x-w)\frac{s_{n-1}(x_k)}{\lambda_{n-1}(x_k)}\right]}\\
&\times \sum_{i=1}^{m-1} \frac{\frac{q^i((q-1)x+w}{\lambda_{n-1}(x_i)}}{\prod_{j=0}^{i}\left[1+q^j((1-q)x-w)\frac{s_{n-1}(x_j)}{\lambda_{n-1}(x_j)}\right]\prod_{j=0}^{m-1}\left[1-q^{i+j}((1-q)x-w)\frac{\lambda_n(x_{i+j})}{\lambda_{n-1}(x_{i+j})}\right]}\\
&\times \left\{ \frac{\frac{q^m((q-1)x+w}{\lambda_{n-1}(x_m)}}{\prod_{j=0}^{m}\left[1+q^j((1-q)x-w)\frac{s_{n-1}(x_j)}{\lambda_{n-1}(x_j)}\right]\prod_{j=0}^{m-1}\left[1-q^{m+j}((1-q)x-w)\frac{\lambda_n(x_{m+j})}{\lambda_{n-1}(x_{m+j})}\right]} \right.\\
&- \frac{1}{\left[1+q^m((1-q)x-w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}\right]}\\
&\left. \times \frac{\frac{q^0((q-1)x+w}{\lambda_{n-1}(x)}}{\prod_{j=0}^{0}\left[1+q^j((1-q)x-w)\frac{s_{n-1}(x_j)}{\lambda_{n-1}(x_j)}\right]\prod_{j=0}^{m-1}\left[1-q^{0+j}((1-q)x-w)\frac{\lambda_n(x_{0+j})}{\lambda_{n-1}(x_{0+j})}\right]} \right\}\\
&= \frac{1}{\prod_{k=1}^{m-1}\left[1+q^k((1-q)x-w)\frac{s_{n-1}(x_k)}{\lambda_{n-1}(x_k)}\right]}\\
&\times \sum_{i=1}^{m-1} \frac{\frac{q^i((q-1)x+w}{\lambda_{n-1}(x_i)}}{\prod_{j=0}^{i}\left[1+q^j((1-q)x-w)\frac{s_{n-1}(x_j)}{\lambda_{n-1}(x_j)}\right]\prod_{j=0}^{m-1}\left[1-q^{i+j}((1-q)x-w)\frac{\lambda_n(x_{i+j})}{\lambda_{n-1}(x_{i+j})}\right]}\\
&\times \left\{ \frac{\frac{q^m((q-1)x+w}{\lambda_{n-1}(x_m)}}{\prod_{j=0}^{m}\left[1+q^j((1-q)x-w)\frac{s_{n-1}(x_j)}{\lambda_{n-1}(x_j)}\right]\prod_{j=0}^{m-1}\left[1-q^{m+j}((1-q)x-w)\frac{\lambda_n(x_{m+j})}{\lambda_{n-1}(x_{m+j})}\right]} \right.\\
&- \frac{1}{\left[1+q^m((1-q)x-w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}\right]}\\
&\left. \times \frac{\frac{((q-1)x+w}{\lambda_{n-1}(x)}}{\prod_{j=0}^{0}\left[1+q^j((1-q)x-w)\frac{s_{n-1}(x_j)}{\lambda_{n-1}(x_j)}\right]\prod_{j=0}^{m-1}\left[1-q^{0+j}((1-q)x-w)\frac{\lambda_n(x_{0+j})}{\lambda_{n-1}(x_{0+j})}\right]} \right\}.
\end{aligned}
$$

The second term is 0 only if $x = w_0 = \frac{w}{1-q}$ this expression is 0 only if

$$q^m((q-1)x+w) - ((q-1)x+w) = (q^m-1)((q-1)x+w) = 0$$

which only occurs at $x = w_0$. Thus, the solution are linearly independent for all $x = q^n x + w[n]_q$ $\square$

7 Variation of Parameters for the second solution

We assume we know one of the solutions $y_1(x)$ and farther assume that the second solution is of the form $y_2(x) = y_1(x)v(x)$ for some $v(x)$ to be determined with $\delta_m = 0$. It follows that

$$
\begin{aligned}
\Delta_{q;w}^2 v(x)y_1(x) &= \lambda_0(x)\Delta_{q;w}v(x)y_1(x) + s_0(x)v(x\\
\Delta_{q;w}[y_1(qx+w)\Delta_{q;w}v(x) + v(x)\Delta_{q;w}y_1(x)] &= \lambda_0(x)\Delta_{q;w}v(x)y_1(x) + s_0(x)v(x\\
y_1(q^2x+qw+w)\Delta_{q;w}^2 v(x) + \Delta_{q;w}v(x)\Delta_{q;w}y_1(qx+w) + \Delta_{q;w}y_1(qx+w)\Delta_{q;w}v(x) + v(x)\Delta_{q;w}^2 y_1(x) &= \lambda_0(x)\Delta_{q;w}v(x)y_1(x) + s_0(x)v(x\\
y_1(q^2x+qw+w)\Delta_{q;w}^2 v(x) + 2\Delta_{q;w}v(x)\Delta_{q;w}y_1(qx+w) + v(x)\Delta_{q;w}^2 y_1(x) &= \lambda_0(x)y_1(qx+w)\Delta_{q;w}v(x)
\end{aligned}
$$

$$
\begin{aligned}
&+ \lambda_0(x)v(x)\Delta_{q;w}y_1(x)\\
&+ s_0(x)y_1(x)v(x)\\
y_1(q^2x+qw+w)\Delta^2_{q;w}v(x) + 2\Delta_{q;w}v(x)\Delta_{q;w}y_1(qx+w) &= \lambda_0(x)y_1(qx+w)\Delta_{q;w}v(x)\\
y_1(q^2x+qw+w)\Delta^2_{q;w}v(x) &= \Delta_{q;w}v(x)[\lambda_0(x)y_1(qx+w)\\
&- 2\Delta_{q;w}y_1(qx+w)]
\end{aligned}
$$

so

$$
\Delta^2_{q;w}v(x) = \Delta_{q;w}v(x)\frac{[\lambda_0(x)y_1(qx+w) - 2\Delta_{q;w}y_1(qx+w)]}{y_1(q^2x+qw+w)}
$$

which is a first order homogeneous difference equation in $\Delta_{q;w}v(x)$. Thus

$$
\Delta_{q;w}v(x) = \frac{\Delta_{q;w}v(x_n)}{\prod_{k=0}^{n-1} 1 + q^k((q-1)x+w)\left[\lambda_0(x_k) - \frac{\Delta_{q;w}y_1(x_k)+\Delta_{q;w}y_1(x_{k+1})}{y_1(x_{k+1})}\right]}
$$

and thus

$$
v(x) = v(x_n) - \sum_{i=0}^{n-1} \frac{q^i((q-1)x+w)\Delta_{q;w}v(x_n)}{\prod_{k=0}^{n-1} 1 + q^{k+i}((q-1)x+w)\left[\lambda_0(x_{k+i}) - \frac{\Delta_{q;w}y_1(x_{k+i})+\Delta_{q;w}y_1(x_{k+i+1})}{y_1(x_{k+i+1})}\right]}.
$$

Now, we can construct our second solution as

$$
\begin{aligned}
y_2(x) = y_1(x)v(x) &= \frac{1}{\prod_{k=0}^{n-1}\left[1 + q^k((1-q)x-w)\frac{s_{m-1}(x_k)}{\lambda_{m-1}(x_k)}\right]}v(x)\\
&= \frac{1}{\prod_{k=0}^{n-1}\left[1 + q^k((1-q)x-w)\frac{s_{m-1}(x_k)}{\lambda_{m-1}(x_k)}\right]}\\
&\times v(x_n) - \sum_{i=0}^{n-1} \frac{q^i((q-1)x+w)\Delta_{q;w}v(x_n)}{\prod_{k=0}^{n-1} 1 + q^{k+i}((q-1)x+w)\left[\lambda_0(x_{k+i}) - \frac{\Delta_{q;w}y_1(x_{k+i})+\Delta_{q;w}y_1(x_{k+i+1})}{y_1(x_{k+i+1})}\right]}
\end{aligned}
$$

The first term is our original solution, so we only focus on simplification of the second and remove its constant

$$
\begin{aligned}
y_1(x)v(x) = &\frac{1}{\prod_{k=0}^{n-1}\left[1 + q^k((1-q)x-w)\frac{s_{m-1}(x_k)}{\lambda_{m-1}(x_k)}\right]}\\
&\times \sum_{i=0}^{n-1} \frac{q^i((q-1)x+w)}{\prod_{k=0}^{n-1} 1 + q^{k+i}((q-1)x+w)\left[\lambda_0(x_{k+i}) - \frac{\Delta_{q;w}y_1(x_{k+i})+\Delta_{q;w}y_1(x_{k+i+1})}{y_1(x_{k+i+1})}\right]}
\end{aligned}
$$

we bring the first product inside the denominator and simplify using

$$
\Delta_{q;w}y_1(x) = y_1(x)\frac{\frac{s_{m-1}(x)}{\lambda_{m-1}(x)} - q^n\frac{s_{m-1}(x_n)}{\lambda_{m-1}(x_n)}}{\left[1 + q^n((1-q)x-w)\frac{s_{m-1}(x_n)}{\lambda_{m-1}(x_n)}\right]}
$$

so

$$
\sum_{i=0}^{n-1} \frac{q^i((q-1)x+w)}{\prod_{k=0}^{n-1}\left[1 - q^k((q-1)x+w)\frac{s_{m-1}(x_k)}{\lambda_{m-1}(x_k)}\right][1 + q^{k+i}((q-1)x+w)\left[\lambda_0(x_{k+i}) - \frac{\Delta_{q;w}y_1(x_{k+i})+\Delta_{q;w}y_1(x_{k+i+1})}{y_1(x_{k+i+1})}\right]]}.
$$

Then using

$$
\begin{aligned}
\Delta_{q;w}y_1(x) &= y_1(x)\frac{\frac{s_{m-1}(x)}{\lambda_{m-1}(x)} - q^n\frac{s_{m-1}(x_n)}{\lambda_{m-1}(x_n)}}{\left[1 + q^n((1-q)x-w)\frac{s_{m-1}(x_n)}{\lambda_{m-1}(x_n)}\right]}\\
&= y_1(qx+w)\frac{1 + q^n((1-q)x-w)\frac{s_{m-1}(x_n)}{\lambda_{m-1}(x_n)}}{1 + ((1-q)x-w)\frac{s_{m-1}(x)}{\lambda_{m-1}(x)}}\frac{\frac{s_{m-1}(x)}{\lambda_{m-1}(x)} - q^n\frac{s_{m-1}(x_n)}{\lambda_{m-1}(x_n)}}{\left[1 + q^n((1-q)x-w)\frac{s_{m-1}(x_n)}{\lambda_{m-1}(x_n)}\right]}\\
&= y_1(qx+w)\frac{\frac{s_{m-1}(x)}{\lambda_{m-1}(x)} - q^n\frac{s_{m-1}(x_n)}{\lambda_{m-1}(x_n)}}{1 + ((1-q)x-w)\frac{s_{m-1}(x)}{\lambda_{m-1}(x)}}
\end{aligned}
$$

$$\frac{\Delta_{q;w}y_1(x)}{y_1(qx+w)} = \frac{\frac{s_{m-1}(x)}{\lambda_{m-1}(x)} - q^n\frac{s_{m-1}(x_n)}{\lambda_{m-1}(x_n)}}{1+((1-q)x-w)\frac{s_{m-1}(x)}{\lambda_{m-1}(x)}}$$

yielding

$$\frac{\Delta_{q;w}y_1(x_{k+i})}{y_1(x_{k+i+1})} = \frac{\frac{s_{m-1}(x_{k+i})}{\lambda_{m-1}(x_{k+i})} - q^n\frac{s_{m-1}(x_{k+i+n})}{\lambda_{m-1}(x_{k+i+n})}}{1+q^{k+i}((1-q)x-w)\frac{s_{m-1}(x_{k+i})}{\lambda_{m-1}(x_{k+i})}}$$

and so

$$\begin{aligned}
\frac{\Delta_{q;w}y_1(x_{k+i})+\Delta_{q;w}y_1(x_{k+i+1})}{y_1(x_{k+i+1})} &= \frac{\Delta_{q;w}y_1(x_{k+i})}{y_1(x_{k+i+1})} + \frac{\frac{s_{m-1}(x_{k+i+1})}{\lambda_{m-1}(x_{k+i+1})} - q^n\frac{s_{m-1}(x_{k+i+n+1})}{\lambda_{m-1}(x_{k+i+n+1})}}{\left[1+q^{n+i+k+1}((1-q)x-w)\frac{s_{m-1}(x_{k+i+n+1})}{\lambda_{m-1}(x_{k+i+n+1})}\right]} \\
&= \frac{\frac{s_{m-1}(x_{k+i+1})}{\lambda_{m-1}(x_{k+i+1})} - q^n\frac{s_{m-1}(x_{k+i+n+1})}{\lambda_{m-1}(x_{k+i+n+1})}}{\left[1+q^{n+i+k+1}((1-q)x-w)\frac{s_{m-1}(x_{k+i+n+1})}{\lambda_{m-1}(x_{k+i+n+1})}\right]} \\
&+ \frac{\frac{s_{m-1}(x_{k+i})}{\lambda_{m-1}(x_{k+i})} - q^n\frac{s_{m-1}(x_{k+i+n})}{\lambda_{m-1}(x_{k+i+n})}}{1+q^{k+i}((1-q)x-w)\frac{s_{m-1}(x_{k+i})}{\lambda_{m-1}(x_{k+i})}}
\end{aligned}$$

substituting back in to $y_2(x)$ we get

$$y_2(x) = \sum_{i=0}^{n-1} \frac{q^i((q-1)x+w)}{\prod_{k=0}^{n-1}\left[1-q^k((q-1)x+w)\frac{s_{m-1}(x_k)}{\lambda_{m-1}(x_k)}\right]\left[1+q^{k+i}((q-1)x+w)\left[\lambda_0(x_{k+i}) - \dots\right.\right.}$$

$$\overline{\left.\left.\frac{\frac{s_{m-1}(x_{k+i+1})}{\lambda_{m-1}(x_{k+i+1})}-q^n\frac{s_{m-1}(x_{k+i+n+1})}{\lambda_{m-1}(x_{k+i+n+1})}}{\left[1+q^{n+i+k+1}((1-q)x-w)\frac{s_{m-1}(x_{k+i+n+1})}{\lambda_{m-1}(x_{k+i+n+1})}\right]} - \frac{\frac{s_{m-1}(x_{k+i})}{\lambda_{m-1}(x_{k+i})}-q^n\frac{s_{m-1}(x_{k+i+n})}{\lambda_{m-1}(x_{k+i+n})}}{1+q^{k+i}((1-q)x-w)\frac{s_{m-1}(x_{k+i})}{\lambda_{m-1}(x_{k+i})}}\right]\right]}$$

8 $\delta_n = 0 \implies \delta_m = 0 \quad m > n$

Theorem 8. *Suppose for some n that $\delta_n = \lambda_n(x)s_{n-1}(x) - s_n(x)\lambda_{n-1}(x) = 0$. Then for all m greater than n, $\delta_m = 0$.*

Proof. Supposed $\delta_n = \lambda_n(x)s_{n-1}(x) - s_n(x)\lambda_{n-1}(x) = 0$. We show that $\delta_{n+1} = 0$. We see that

$$\begin{aligned}
\delta_{n+1} &= \lambda_{n+1}(x)s_n(x) - s_{n+1}(x)\lambda_n(x) \\
&= \Delta_{q;w}\lambda_n(x)s_n(x) + \lambda_n(qx+w)s_n(x)\lambda_0(x) + s_n(qx+w)s_n(x) \\
&- \lambda_n(x)\Delta_{q;w}s_n(x) - \lambda_n(x)\lambda_n(qx+w)s_0(x) \\
&= \Delta_{q;w}\lambda_n(x)s_n(x) - \lambda_n(x)\Delta_{q;w}s_n(x) \\
&+ \lambda_n(qx+w)s_n(x)\lambda_0(x) + s_n(qx+w)s_n(x) - \lambda_n(x)\lambda_n(qx+w)s_0(x) \\
&= \Delta_{q;w}\left(\frac{\lambda_n(x)}{s_n(x)}\right)s_n(x)s_n(qx+w) \\
&+ \lambda_n(qx+w)s_n(x)\lambda_0(x) + s_n(qx+w)s_n(x) - \lambda_n(x)\lambda_n(qx+w)s_0(x) \\
&= s_n(x)s_n(qx+w)\left[\Delta_{q;w}\left(\frac{\lambda_n(x)}{s_n(x)}\right)\right. \\
&+ \left.\frac{\lambda_n(qx+w)}{s_n(qx+w)}\lambda_0(x) + 1 - \frac{\lambda_n(x)\lambda_n(qx+w)}{s_n(x)s_n(qx+w)}s_0(x)\right].
\end{aligned}$$

Using $\delta_n = 0$ we then have

$$\delta_{n+1} = s_n(x)s_n(qx+w)\left(\Delta_{q;w}\left(\frac{\lambda_{n-1}(x)}{s_{n-1}(x)}\right) + 1 + \frac{\lambda_{n-1}(qx+w)}{s_{n-1}(qx+w)}\left(\lambda_0(x) - \frac{\lambda_{n-1}(x)}{s_{n-1}(x)}s_0(x)\right)\right)$$

$$
\begin{aligned}
&= s_n(x)s_n(qx+w)\Bigg(\left(\frac{\Delta_{q;w}\lambda_{n-1}(x)s_{n-1}(x) - \lambda_{n-1}(x)\Delta_{q;w}s_{n-1}(x)}{s_{n-1}(x)s_{n-1}(qx+w)}\right) + 1 \\
&+ \frac{\lambda_{n-1}(qx+w)}{s_{n-1}(qx+w)}\left(\lambda_0(x) - \frac{\lambda_{n-1}(x)}{s_{n-1}(x)}s_0(x)\right)\Bigg) \\
&= s_n(x)s_n(qx+w)\Bigg(\frac{\Delta_{q;w}\lambda_{n-1}(x) - \lambda_{n-1}(qx+w)\lambda_0(x) + s_{n-1}(qx+w)}{s_{n-1}(qx+w)} \\
&- \frac{\lambda_{n-1}(x)(\Delta_{q;w}s_{n-1}(x) + \lambda_{n-1}(qx+w)s_0(x))}{s_{n-1}(x)s_{n-1}(qx+w)}\Bigg) \\
&= s_n(x)s_n(qx+w)\left(\frac{\lambda_n(x)}{s_{n-1}(qx+w)} - \frac{\lambda_{n-1}(x)s_n(x)}{s_{n-1}(x)s_{n-1}(qx+w)}\right) \\
&= s_n(x)s_n(qx+w)\left(\frac{\lambda_n(x)s_{n-1}(x) - \lambda_{n-1}(x)s_n(x)}{s_{n-1}(x)s_{n-1}(qx+w)}\right) \\
&= s_n(x)s_n(qx+w)\left(\frac{\delta_n}{s_{n-1}(x)s_{n-1}(qx+w)}\right) \\
&= 0
\end{aligned}
$$

as required. □

9 Examples

9.1 The qw-exponential function

We will define the following function

$$
\begin{aligned}
e_{q;w}(x) &= \left(\prod_{i=0}^{\infty}(1 - q^i((q-1)x+w)\right)^{-1} \\
&= \frac{1}{(((q-1)x+w)^i;q)_\infty} \\
&= \sum_{i=0}^{\infty}\frac{((q-1)x+w)^i}{(q;q)_i}
\end{aligned}
$$

the series is convergent for $w < 1$. It is the unique solution to

$$\Delta_{q;w}y(x) = y(x), \quad y(w_0) = 1$$

9.2 An equation of Euler type

We set out to solve

$$y(q^2x + qw + w) = \frac{2x(x+a)}{x(x+1)}y(qx+w) - \frac{(x+a-1)(x+a)}{x(x+1)}y(x)$$

which we can rewrite as

$$\Delta^2_{q;w}y(x) = \frac{\frac{2x(x+a)}{x(x+1)} - (1+q)}{q((q-1)x+w)}\Delta_{q;w}y(x) + \frac{\frac{(x+a-1)(x+a)}{x(x+1)} + \frac{2x(x+a)}{x(x+1)} - 1}{q((q-1)x+w)^2}y(x)$$

9.3 Hypergeometric Difference Equation

We look to solve the difference equation

$$(a_2x^2 + a_1x + a_0)\Delta^2_{q;w}y(x) + (b_1x + b_0)\Delta_{q;w}y(x) - c_0y(x) = 0$$

which we can rewrite as

$$\Delta^2_{q;w}y(x) = -\frac{(b_1x+b_0)}{(a_2x^2+a_1x+a_0)}\Delta_{q;w}y(x) + \frac{c_0}{(a_2x^2+a_1x+a_0)}y(x)$$

giving

$$
\begin{aligned}
\lambda_0(x) &= -\frac{(b_1 x + b_0)}{(a_2 x^2 + a_1 x + a_0)} \\
s_0(x) &= \frac{c_0}{(a_2 x^2 + a_1 x + a_0)}
\end{aligned}
$$

9.3.1 $\delta_n = \lambda_n s_{n-1} - s_n \lambda_{n-1}$

Using mathematica we can calculate the first few δ_n in the sequence

$$
\begin{aligned}
&\delta_1 = 0, \quad c_0 = b_1 \\
&\delta_2 = 0, \quad c_0 = (b_1 + a_2)(1+q) \\
&\delta_3 = 0, \quad c_0 = (b_1 + a_2 + a_2 q)(1+q+q^2) \\
&\delta_4 = 0, \quad c_0 = (b_1 + a_2 + a_2 q + a_2 q^2)(1+q+q^2+q^3) \\
&\delta_5 = 0, \quad c_0 = (b_1 + a_2 + a_2 q + a_2 q^2 + a_2 q^3)(1+q+q^2+q^3+q^4)
\end{aligned}
$$

it seems that (no proof yet)

$$\delta_n = 0, \quad c_0 = (b_1 + a_2[n-1]_q)[n]_q$$

9.3.2 Solutions

The first linearly independent solution is for $n = 1$ is

$$
\begin{aligned}
y(x) &= \frac{1}{\prod_{i=0}^{\infty}[1 - q^i((q-1)x + w)\frac{s_0(x_i)}{\lambda_0(x_i)}]} \\
&= \frac{1}{\prod_{i=0}^{\infty}\left[1 + q^i((q-1)x + w)\frac{c_0}{b_1(x_i)+b_0}\right]} \\
&= \lim_{k\to\infty}\prod_{i=0}^{k}\left[1 + q^i((q-1)x + w)\frac{b_1}{b_1(x_i)+b_0}\right]^{-1} \\
&= \lim_{k\to\infty}\frac{(q-1)(b_0 + b_1(qx+w))}{b_0(q-1) + wb_1(q^{k+1}-1) + b_1 q^{k+1}x(q-1)} \\
&= \frac{(q-1)(b_0 + b_1(qx+w))}{b_0(q-1) - wb_1} \\
&= \frac{b_0(q-1) - b_1 w + qb_1(qx+w) - b_1 qx}{b_0(q-1) - b_1 w} \\
&= \frac{b_1(q-1)q}{b_0(q-1) - b_1 w}x + \frac{(q-1)(b_0 + b_1 w)}{b_0(q-1) - b_1 w} \\
&= x + \frac{b_0 + b_1 w}{b_1 q}
\end{aligned}
$$

if $w = 0$ we get the solution to the q-hypergeometric solution

$$y(x) = x + \frac{b_0}{b_1 q}$$

taking the limit $q \to 1$ yields the normal hypergeometric

$$y(x) = x + \frac{b_0}{b_1}$$

$n = 2$ has solution

$$
\begin{aligned}
y(x) &= \frac{1}{\prod_{i=0}^{\infty}[1 - q^i((q-1)x + w)\frac{s_1(x_i)}{\lambda_1(x_i)}]} \\
&= \frac{1}{\prod_{i=0}^{\infty}[1 + q^i((q-1)x + w)\frac{(a_2+b_1)(1+q)(a_1+b_0+a_2+w+b_1 w+a_2 x_i'+a_2 q x_i'+b_1 q x_i')}{(b_0(a_1+b_0)+a_0(a_2+(a_2+b_1)q)+(a_2+b_1)(b_0(w+x+qx)+x_i'(a_1+a_1 q+b_1 w+(a_2+(a_2+b_1)q)x_i')))}}
\end{aligned}
$$

and for each n we have solution

$$y_n(x) = \frac{1}{\prod_{i=0}^{\infty}[1 - q^i((q-1)x + w)\frac{s_{n-1}(x_i)}{\lambda_{n-1}(x_i)}]}$$

9.3.3 When is this a polynomial?

9.4 Hermite

We consider the Hermite difference equation as

$$\Delta_{q;w} y^2(x) = ax\Delta_{q;w} y(x) + by(x)$$

where we take

$$\begin{aligned}\lambda_0(x) &= ax\\ s_0(x) &= b.\end{aligned}$$

Using mathematica, we see that the following δ_n can be constructed

$$\begin{aligned}\delta_1 &= b(b+a)\\ \delta_2 &= -b(b+a)(b+a+aq)\\ \delta_3 &= b(b+a)(b+a+aq)(b+a+aq+aq^2)\\ \delta_4 &= -b(b+a)(b+a+aq)(b+a+aq+aq^2)(b+a+aq+aq^2+aq^3)\end{aligned}$$

in general I will state without proof that

$$\delta_n = (-1)^{n+1}\prod_{i=0}^{n}(b+a[i]_q)$$

and we see that $\delta_n = 0$ if $b = -a[n]_q$.
We take for the first solution $b = -a$ and we may write

$$\begin{aligned}y_1(x) &= \frac{1}{\prod_{i=0}^{\infty}\left[1 - q^i((q-1)x+w)\frac{s_0(q^ix+w[i]_q)}{\lambda_0(q^ix+w[i]_q)}\right]}\\ &= \frac{1}{\prod_{i=0}^{\infty}\left[1 - q^i((q-1)x+w)\frac{-a}{a(q^ix+w[i]_q)}\right]}\\ &= \frac{1}{\prod_{i=0}^{\infty}\left[1 + \frac{q^i((q-1)x+w)}{(q^ix+w[i]_q)}\right]}\\ &= \lim_{k\to\infty}\frac{1}{\prod_{i=0}^{k}\left[1 + \frac{q^i((q-1)x+w)}{(q^ix+w[i]_q)}\right]}\\ &= \lim_{k\to\infty}\frac{1}{\frac{q^{1+k}+[k+1]_q w}{x}}\\ &= \lim_{k\to\infty}\frac{x}{q^{1+k}+[k+1]_q w}\\ &= \frac{(1-q)x}{w}.\end{aligned}$$

We may check

$$\begin{aligned}\Delta_{q;w}^2\frac{(1-q)x}{w} - ax\Delta_{q;w}\frac{(1-q)x}{w} + a\frac{(1-q)x}{w} &= 0 - ax\frac{1-q}{w} + a\frac{(1-q)x}{w}\\ &= 0.\end{aligned}$$

Take $b = -a[2]_q$ then the solution is

$$\begin{aligned}y_2(x) &= \frac{1}{\prod_{i=0}^{\infty}\left[1 - q^i((q-1)x+w)\frac{s_1(x_i')}{\lambda_1(x_i')}\right]}\\ &= \frac{1}{\prod_{i=0}^{\infty}\left[1 - q^i((q-1)x+w)\frac{ab(w+qx_i')}{a+b+a^2x_i'(qx_i'+w)}\right]}\end{aligned}$$

$$= \frac{1}{\prod_{i=0}^{\infty}\left[1 + q^i((q-1)x+w)\frac{a[2]_q(x'_{i+1})}{1-[2]_q+ax'_ix'_{i+1}}\right]}$$

9.4.1 Non-continuous solution attempt

We see the solution as

$$\begin{aligned}
y(x_k) &= y(x_0)\prod_{i=0}^{k-1}\left[1 - q^i((q-1)x_0+w)\frac{s_{n-1}(x_i)}{\lambda_{n-1}(x_i)}\right] \\
&= y(x_0)\prod_{i=0}^{k-1}\left[1 - q^i((q-1)x_0+w)\frac{-a}{a(q^ix_0+w[i]_q)}\right] \\
&= y(x_0)\prod_{i=0}^{k-1}\left[1 + \frac{q^i((q-1)x_0+w)}{q^ix_0+w[i]_q}\right] \\
&= y(x_0)\left[q^k + \frac{w[k]_q}{x_0}\right]
\end{aligned}$$

9.5 q-Laguerre

The q-laguerre polynomials satisfy the difference equation

$$(1+q^a+q^{a+b})y(x) = q^a(1+x)y(qx) + y(q^{-1}x).$$

I will rewrite into a similar form with $x \to qx+w$.

$$\begin{aligned}
-q^a(1+qx+w)y(q^2x+qw+w) &= -(1+q^a+q^{a+b})y(qx+w) + y(x) \\
y(q^2x+qw+w) &= \frac{(1+q^a+q^{a+b})}{q^a(1+qx+w)}y(qx+w) - \frac{1}{q^a(1+qx+w)}y(x)
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\Delta^2_{q;w}y(x) &= \frac{\frac{(1+q^a+q^{a+b})}{q^a(1+qx+w)} - (1+q)}{q((q-1)x+w)}\Delta_{q;w}y(x) + \frac{\frac{(1+q^a+q^{a+b})}{q^a(1+qx+w)} - \frac{1}{q^a(1+qx+w)} - 1}{((q-1)x+w)^2}y(x) \\
&= \frac{q^{-(a+1)} + q^{-(b+1)} - (2x+1+w+wq^{-1})}{(1+qx+w)((q-1)x+w)}\Delta_{q;w}y(x) + \frac{q^b - (qx+w)}{(1+qx+w)((q-1)x+w)^2}y(x)
\end{aligned}$$

Using mathematica, we can generate the coefficient sequence with

$$\begin{aligned}
\lambda_0(x) &= \frac{q^{-(a+1)} + q^{-(b+1)} - (2x+1+w+wq^{-1})}{(1+qx+w)((q-1)x+w)} \\
s_0(x) &= \frac{q^b - (qx+w)}{(1+qx+w)((q-1)x+w)^2}
\end{aligned}$$

which reports

$$\delta_1(x) = \frac{q^{-a-b-3}\left(q^a\left(q^{3b+1} + q^{2b}\left(w - q\left(\left(3q^2+q-2\right)x + 4qw + q + 2w - 1\right)\right)\right) + q^b\left((3q-2)q^3x^2 + 3q^2w^2 + (2q+1)(3q-2)qwx - q\right.\right.}{((q-1)x+w)^4(qx+w+1)\left(q^2x+qw+w+1\right)}$$

Clearly this is non-zero for all values of b. Attempts to calculate δ_2, δ_3 proved difficult and unpromising.

9.6 q-Laguerre Accidental

I made a mistake in the derivation here, but i found the terminating condition to my quite interesting so I will save this for later.
The q-laguerre polynomials satisfy the difference equation

$$(1+q^a+q^{a+b})y(x) = q^a(1+x)y(qx) + y(q^{-1}x).$$

I will rewrite into a similar form with $x \to qx + w$.

$$
\begin{aligned}
-q^a(1+qx+w)y(q^2x+qw+w) &= -(1+q^a+q^{a+b})y(qx+w) + y(x) \\
y(q^2x+qw+w) &= \frac{(1+q^a+q^{a+b})}{q^a(1+qx+w)}y(qx+w) - \frac{1}{q^a(1+qx+w)}y(x)
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
\Delta^2_{q;w}y(x) &= \frac{\frac{1+q^a+q^{a+b}}{q^a(1+qx+w)} - 1}{[(q-1)x+w]}\Delta_{q;w}y(x) + \frac{\frac{(1+q^a+q^{a+b})}{q^a(1+qx+w)} - \frac{1}{q^a(1+qx+w)}}{((q-1)x+w)^2}y(x) \\
\Delta^2_{q;w}y(x) &= \frac{\frac{1+q^a+q^{a+b}}{q^a(1+qx+w)} - 1}{[(q-1)x+w]}\Delta_{q;w}y(x) + \frac{\frac{(q^a+q^{a+b})}{q^a(1+qx+w)}}{((q-1)x+w)^2}y(x)
\end{aligned}
$$

$$
\Delta^2_{q;w}y(x) = \frac{q^b+q^{-a}-(qx+w)}{[(q-1)x+w][1+qx+w]}\Delta_{q;w}y(x) + \frac{1+q^b}{[(q-1)x+w]^2[1+qx+w]}y(x)
$$

Using mathematica, we can generate the coeffecient sequence with

$$
\begin{aligned}
\lambda_0(x) &= \frac{q^b+q^{-a}-(qx+w)}{[(q-1)x+w][1+qx+w]} \\
s_0(x) &= \frac{1+q^b}{[(q-1)x+w]^2[1+qx+w]}
\end{aligned}
$$

which reports $\delta_1(x) = 0$ if $b = \frac{i\pi}{ln(q)}$. This is trivial as $b = \frac{i\pi}{ln(q)}$ then $s_0(x) = 0$ Attempts to calculate δ_2, δ_3 proved difficult and unpromising.

9.7 Constant Coefficients case

We consider the equation

$$
\Delta^2_{q;w}y(x) = \lambda_0\Delta_{q;w}y(x) + s_0y(x).
$$

With λ_0 and s_0 constant then λ_n and s_n is constant so, we can see the DAIM sequence is

$$
\begin{aligned}
\lambda_n(x) &= \Delta_{q:w}\lambda_{n-1}(x) + \lambda_{n-1}(qx+w)\lambda_0(x) + s_{n-1}(qx+w) \\
&= \lambda_{n-1}\lambda_0 + s_{n-1}
\end{aligned}
$$

$$
\begin{aligned}
s_n(x) &= \Delta_{q:w}s_{n-1}(x) + \lambda_{n-1}(qx+w)s_0(x) \\
&= \lambda_{n-1}s_0.
\end{aligned}
$$

which implies

$$
\begin{aligned}
\frac{\lambda_n}{s_n} &= \frac{\lambda_{n-1}\lambda_0 + s_{n-1}}{\lambda_{n-1}s_0} \\
\frac{\lambda_n}{s_n} &= \frac{\lambda_0}{s_0} + \frac{s_{n-1}}{\lambda_{n-1}s_0} \\
s_0\frac{\lambda_n}{s_n} &= \lambda_0 + \frac{s_{n-1}}{\lambda_{n-1}} \\
\left(\frac{s_{n-1}}{\lambda_{n-1}}\right)^2 + \frac{s_{n-1}}{\lambda_{n-1}}\lambda_0 - s_0\frac{s_{n-1}}{\lambda_{n-1}}\frac{s_n}{\lambda_n} &= 0
\end{aligned}
$$

and if $\delta_n = 0$ then

$$
\left(\frac{s_{n-1}}{\lambda_{n-1}}\right)^2 + \frac{s_{n-1}}{\lambda_{n-1}}\lambda_0 - s_0 = 0
$$

which is a quadratic in $\frac{s_n}{\lambda_n}$ so the solution is

$$
\frac{s_n}{\lambda_n} = \frac{-\lambda_0 \pm \sqrt{\lambda_0^2 + 4s_0}}{2}
$$

We then construct the first linearly independent solution

$$
\begin{aligned}
y_1(x) &= \frac{1}{\prod_{i=0}^{\infty}\left[1 - q^i((q-1)x + w)\frac{s_0(q^i x + w[i]_q)}{\lambda_0(q^i x + w[i]_q)}\right]} \\
&= \frac{1}{\prod_{i=0}^{\infty}\left[1 - q^i((q-1)x + w)\frac{-\lambda_0 + \sqrt{\lambda_0^2 + 4s_0}}{2}\right]} \\
&= e_{q,w}\left(\frac{\lambda_0 - \sqrt{\lambda_0^2 + 4s_0}}{2}; x\right)
\end{aligned}
$$

and the second as

$$
y_2(x) = e_{q,w}\left(\frac{\lambda_0 + \sqrt{\lambda_0^2 + 4s_0}}{2}; x\right)
$$

We verify the solution

$$
\begin{aligned}
&\Delta_{q;w}^2 e_{q,w}\left(\tfrac{\lambda_0 + \sqrt{\lambda_0^2 + 4s_0}}{2}; x\right) - \lambda_0 \Delta_{q;w} e_{q,w}\left(\tfrac{\lambda_0 + \sqrt{\lambda_0^2 + 4s_0}}{2}; x\right) - s_0 e_{q,w}\left(\tfrac{\lambda_0 + \sqrt{\lambda_0^2 + 4s_0}}{2}; x\right) \\
&= e_{q,w}\left(\tfrac{\lambda_0 + \sqrt{\lambda_0^2 + 4s_0}}{2}; x\right)\left[\left(\tfrac{\lambda_0 + \sqrt{\lambda_0^2 + 4s_0}}{2}\right)^2 - \lambda_0\left(\tfrac{\lambda_0 + \sqrt{\lambda_0^2 + 4s_0}}{2}\right) - s_0\right] \\
&= e_{q,w}\left(\tfrac{\lambda_0 + \sqrt{\lambda_0^2 + 4s_0}}{2}; x\right) 0 \\
&= 0
\end{aligned}
$$

9.7.1 Equivalence to other solutions

In the case of $w = 0$ we see that the solution is equivalent to the constant coeffecients of the D_q equation.

$$
\begin{aligned}
y(x) &= \frac{1}{\prod_{i=0}^{\infty}\left[1 - q^i((q-1)x + w)\frac{-\lambda_0 + \sqrt{\lambda_0^2 + 4s_0}}{2}\right]} \\
&= \frac{1}{\prod_{i=0}^{\infty}\left[1 - q^i((q-1)x)\frac{-\lambda_0 + \sqrt{\lambda_0^2 + 4s_0}}{2}\right]} \\
&= e_q\left(\frac{\lambda_0 - \sqrt{\lambda_0^2 + 4s_0}}{2}; x\right)
\end{aligned}
$$

taking the limit as $q \to 1$ yields

$$
y(x) = e^{\left(\frac{\lambda_0 - \sqrt{\lambda_0^2 + 4s_0}}{2}\right)x}
$$

which is the solution to the classical derivative equation. If we instead take $q = 1$, we consider the non-continuous solution and get

$$
\begin{aligned}
y(x_0 + kw) &= y(x_0)\prod_{i=0}^{k-1}\left[1 - q^i((q-1)x_0 + w)\frac{s_{n-1}(x_i)}{\lambda_{n-1}(x_i)}\right] \\
&= y(x_0)\prod_{i=0}^{k-1}\left[1 + w\frac{\lambda_0 - \sqrt{\lambda_0^2 + 4s_0}}{2}\right] \\
&= y(x_0)\left[1 + w\frac{\lambda_0 - \sqrt{\lambda_0^2 + 4s_0}}{2}\right]^k
\end{aligned}
$$

as expected. Finally we can take $w = 1$ to get the solution of the classical difference operator

$$
\begin{aligned}
y(x_0 + k) &= y(x_0)\left[1 + \frac{\lambda_0 - \sqrt{\lambda_0^2 + 4s_0}}{2}\right]^k \\
y(x) &= y(x_0)\left[1 + \frac{\lambda_0 - \sqrt{\lambda_0^2 + 4s_0}}{2}\right]^{x - x_0}
\end{aligned}
$$

9.7.2 Discussion

I think it is important to look at this in terms of the beta derivative for later. We have 2 solutions and they only difference when $\beta(x)^n$ converges in the limit. We should also notice that we can rewrite the limits in the product of the Δ solution in terms of the initial condition

$$y(x) = \left[1 + \frac{\lambda_0 - \sqrt{\lambda_0^2 + 4s_0}}{2}\right]^{x-x_0}.$$

How do we generalize this to the other solutions? Can we use it in the limit case?

9.8 Simple Eq 1

We consider

$$\Delta_{q;w}^2 y(x) = x\Delta_{q;w} y(x) + ax(x-w)y(x)$$

so

$$\begin{aligned}\lambda_0(x) &= x\\ s_0(x) &= ax(x-w).\end{aligned}$$

Using mathematica, we see that if

$$a = \frac{\frac{q^2x^2}{qx+w-x} + \frac{w^2}{qx+w-x} - \frac{qx^2}{qx+w-x} + \frac{2qwx}{qx+w-x} - \frac{wx}{qx+w-x}}{-q^2wx^2 + q^2x^3 - qw^2x + qwx^2}$$

then $\delta_1 = 0$. This results in the solution

$$\begin{aligned}y(x) &= \frac{1}{\prod_{i=0}^{\infty}\left[1 - q^i((q-1)x+w)\frac{s_0(q^ix+w[i]_q)}{\lambda_0(q^ix+w[i]_q)}\right]}\\ &= \frac{1}{\prod_{i=0}^{\infty}\left[1 - q^i((q-1)x+w)\frac{1}{q(q^ix+w[i]_q)}\right]}\end{aligned}$$

9.8.1 Simple Eq 2

We consider

$$\Delta_{q;w}^2 y(x) = x\Delta_{q;w} y(x) + ax(qx-w)y(x)$$

so

$$\begin{aligned}\lambda_0(x) &= x\\ s_0(x) &= ax(qx-w).\end{aligned}$$

Using mathematica, we see that if

$$a = \frac{\frac{q^3x^2}{qx+w-x} - \frac{q^2x^2}{qx+w-x} + \frac{2q^2wx}{qx+w-x} + \frac{qw^2}{qx+w-x} - \frac{qwx}{qx+w-x}}{q^4x^3 + q^3wx^2 - q^2w^2x - q^2wx^2 - qw^3 + w^3}$$

then $\delta_1 = 0$. This results in the solution

$$\begin{aligned}y(x) &= \frac{1}{\prod_{i=0}^{\infty}\left[1 - q^i((q-1)x+w)\frac{s_0(q^ix+w[i]_q)}{\lambda_0(q^ix+w[i]_q)}\right]}\\ &= \frac{1}{\prod_{i=0}^{\infty}\left[1 - q^i((q-1)x+w)\frac{q}{(1-q)w+q^2(q^ix+w[i]_q)}\right]}\\ &= \lim_{k\to\infty}\frac{1}{\frac{(1-q)(q-qw+q^2x)}{w(-2q^2+2q-1)+q^{2+k}((q-1)x+w)}}\\ &= \frac{w(-2q^2+2q-1)}{(1-q)(q-qw+q^2x)}\end{aligned}$$

9.8.2 Simple Eq 3

We consider

$$\Delta^2_{q;w} y(x) = x\Delta_{q;w} y(x) + ax((q-1)x+w)y(x)$$

so

$$\begin{aligned}\lambda_0(x) &= x\\ s_0(x) &= ax((q-1)x+w).\end{aligned}$$

Using mathematica, we see that if

$$a = \frac{\frac{q^3x^2}{qx+w-x} - \frac{2q^2x^2}{qx+w-x} + \frac{2q^2wx}{qx+w-x} + \frac{qw^2}{qx+w-x} - \frac{w^2}{qx+w-x} + \frac{qx^2}{qx+w-x} - \frac{3qwx}{qx+w-x} + \frac{wx}{qx+w-x}}{q^4x^3 + 3q^3wx^2 - 2q^3x^3 + 3q^2w^2x - 4q^2wx^2 + q^2x^3 + qw^3 - 2qw^2x + qwx^2}$$

then $\delta_1 = 0$. This results in the solution

$$\begin{aligned}y(x) &= \frac{1}{\prod_{i=0}^{\infty}\left[1 - q^i((q-1)x+w)\frac{s_0(q^ix+w[i]_q)}{\lambda_0(q^ix+w[i]_q)}\right]}\\ &= \frac{1}{\prod_{i=0}^{\infty}\left[1 - q^i((q-1)x+w)\frac{q-1}{q((q-1)(q^ix+[i]_qw)+w)}\right]}\\ &= \lim_{k\to\infty} q^k\\ &= 0\end{aligned}$$

9.8.3 Simple Eq 4

We consider

$$\Delta^2_{q;w} y(x) = x\Delta_{q;w} y(x) + ax((q-1)x+w)y(x)$$

so

$$\begin{aligned}\lambda_0(x) &= -x\\ s_0(x) &= a(qx+w).\end{aligned}$$

Using mathematica, we see that if

$$a = \frac{w}{q^3x^2 + 2q^2wx + qw^2 + qwx + w^2}$$

then $\delta_1 = 0$. This results in the solution

$$\begin{aligned}y(x) &= \frac{1}{\prod_{i=0}^{\infty}\left[1 - q^i((q-1)x+w)\frac{s_0(q^ix+w[i]_q)}{\lambda_0(q^ix+w[i]_q)}\right]}\\ &= \frac{1}{\prod_{i=0}^{\infty}\left[1 - q^i((q-1)x+w)\frac{-w}{(q^ix+w[i]_q)(w+qw+q^2(q^ix+w[i]_q)}\right]}\\ &= \lim_{k\to\infty}\frac{1}{\frac{(w+qw+q^2x)(q^{1+k}((q-1)x+w)-w)}{(qx+w)(q^{2+k}((q-1)x+w)-w)}}\\ &= \frac{(qx+w)(-w)}{(w+qw+q^2x)(-w)}\\ &= \frac{qx+w}{q^2x+qw+w}\end{aligned}$$

9.9 An example

We consider the difference equation

$$\Delta^2_{q;w} y(x) = \frac{a-qx}{qx((q-1)x+w)}\Delta_{q;w} y(x) + \frac{a+b}{qx((q-1)x+w)} y(x)$$

so we take

$$\lambda_0(x) = \frac{a-qx}{qx((q-1)x+w)}$$

$$s_0(x) = \frac{a+b}{qx((q-1)x+w)}.$$

Using mathematica, we can construct the following terminating conditions

$$\begin{aligned}
\delta_1 &= \frac{(a+b)(a+b-q)}{q^3x(qx+w)((q-1)x+w)^2} \\
\delta_2 &= -\frac{(a+b)(a+b-q)(a+b-q^2-q^3)}{q^6x(qx+w)((q-1)x+w)^3(q^2x+qw+w)} \\
\delta_3 &= \frac{(a+b)(a+b-q)(a+b-q^2-q^3)(a+b-q^3-q^4-q^5)}{q^{10}x(qx+w)((q-1)x+w)^4(q^2x+qw+w)(q^3x+q^2w+qw+w)} \\
\delta_4 &= -\frac{(a+b)(a+b-q)(a+b-q^2-q^3)(a+b-q^3-q^4-q^5)(a+b-q^4-q^5-q^6-q^7)}{q^{15}x(qx+w)((q-1)x+w)^5(q^2x+qw+w)(q^3x+q^2w+qw+w)(q^4x+q^3w+q^2w+qw+w)}
\end{aligned}$$

we can see that the terminating conditions for these are

$$\begin{aligned}
\delta_1 = 0 &\implies b = -a, \quad \text{or} \quad b = -a+q \\
\delta_2 = 0 &\implies b = -a+q^2+q^3 \\
\delta_3 = 0 &\implies b = -a+q^3+q^4+q^5 \\
\delta_4 = 0 &\implies b = -a+q^4+q^5+q^6+q^7
\end{aligned}$$

In the case $b = -a+q$, we can write the solution as

$$\begin{aligned}
y_1(x) &= \frac{1}{\prod_{i=0}^{\infty}\left[1-q^i((q-1)x+w)\frac{s_0(q^ix+w[i]_q)}{\lambda_0(q^ix+w[i]_q)}\right]} \\
&= \frac{1}{\prod_{i=0}^{\infty}\left[1-q^i((q-1)x+w)\frac{q}{a-q(q^ix+w[i]_q)}\right]} \\
&= \frac{(q-1)(qx-a)}{a(q-1)+qw}
\end{aligned}$$

which in monic form is

$$y_1(x) = x + \frac{a}{q}.$$

For the case $b = -a+q^2+q^3$ we can write

$$\begin{aligned}
y_2(x) &= \frac{1}{\prod_{i=0}^{\infty}\left[1-q^i((q-1)x+w)\frac{s_1(q^ix+w[i]_q)}{\lambda_1(q^ix+w[i]_q)}\right]} \\
&= \frac{(q-1)^2(a^2-aq(1+q)(qx+w)+q^3x((q+2)w+(q^2+q-1)x))}{a^2(q-1)^2+aq(q^2-1)w+q^3w^2}
\end{aligned}$$

which has monic form

$$x^2 + \frac{q^3(q+2)-aq^2(q+1)}{q^3(q^2+q-1)}x + \frac{a^2-aq(q+1)w}{q^3(q^2+q-1)}$$

9.10 A sequence of polynomials

Theorem 9. *Let* $P_k(x) = \prod_{j=0}^{k-1}\left(x - w\sum_{i=0}^{j-1} q^i\right), \quad k \geq 0.$ *Then*

$$P_0(x) = 1, \quad \Delta_{q;w}P_k(x) = \left(\sum_{i=0}^{k-1} q^j\right) P_{k-1}(x), \quad k \geq 1$$

9.10.1 Motivation

Why would we be interested in such a sequence of polynomials? In the case of the normal derivative we have

$$\frac{d}{dx}x^n = nx^{n-1}.$$

For the q-derivative we have that

$$D_q x^n = [n]_q x^{n-1}.$$

When we consider the forward difference operator Δ we find that

$$\Delta x^n = (x+1)^n - x^n = \sum_{i=0}^{n} \binom{n}{i} x^i - x^n = \sum_{i=0}^{n-1} \binom{n}{i} x^i$$

which is not a nice formula as before. When we consider a vectorspace of polynomials, we often choose the basis $\{1, x, x^2, \ldots, x^n\}$ because it allows for easy calculation of the derivative on any linear combination. The same is true for the q-derivative. However, it is clear how this choice of basis is not natural for the forward difference operator. We ask, what basis can we choose that makes taking our derivative the easiest? For the forward difference, this is the falling factorial $(x)_n = \prod_{i=0}^{n-1}(x-k)$ as it satisfies $\Delta(x)_n = n(x)_{n-1}$. Thus we may easily calculate the matrix representation of this operator on our new basis. For the hahn operator, we have thus found our most natural choice for the basis.

10 Criterion for polynomial Solutions

10.1 The derivative of a polynomial of degree n is a polynomial of degree $n-1$

Theorem 10. *Let $B(x)$ be a polynomial of degree n. Then $\Delta_{q;w}B(x)$ is a polynomial of degree $n-1$.*

It follows that $\Delta_{q;w}^{n+1}B(x) = 0$.

Proof. Let $B(x)$ be written as a linear combination of the polynomials $P_k(x) = \prod_{j=0}^{k-1}\left(x - w\sum_{i=0}^{j-1} q^i\right)$. Then

$$\begin{aligned}
\Delta_{q;w}B(x) &= \Delta_{q;w}\sum_{k=0}^{n} a_n P_k(x) \\
&= \sum_{k=0}^{n} a_n \Delta_{q;w} P_k(x) \\
&= \sum_{k=0}^{n} a_n \left(\sum_{i=0}^{k-1} q^i\right) P_{k-1}(x)
\end{aligned}$$

which is a polynomial of degree $n-1$. ☐

10.2 If a solution is a polynomial of degree n then $\delta_n = 0$

Theorem 11. *If the difference equation*

$$\Delta_{q;w}^2 y(x) = \lambda_0(x)\Delta_{q;w}y(x) + s_0(x)y(x)$$

has a polynomial solution of degree $n > 0$ then

$$\delta_n = \lambda_n(x)s_{n-1}(x) - \lambda_{n-1}(x)s_n(x) = 0$$

Proof. We begin with

$$\begin{aligned}
\Delta_{q;w}^{n+1}y(x) &= \lambda_{n-1}(x)\Delta_{q;w}y(x) + s_{n-1}(x)y(x) \\
\Delta_{q;w}^{n+2}y(x) &= \lambda_n(x)\Delta_{q;w}y(x) + s_n(x)y(x)
\end{aligned}$$

then multiply the first and second equation by $s_n(x)$ and $s_{n-1}(x)$ respectively

$$\begin{aligned}
s_n(x)\Delta_{q;w}^{n+1}y(x) &= s_n(x)\lambda_{n-1}(x)\Delta_{q;w}y(x) + s_n(x)s_{n-1}(x)y(x) \\
s_{n-1}(x)\Delta_{q;w}^{n+2}y(x) &= s_{n-1}(x)\lambda_n(x)\Delta_{q;w}y(x) + s_{n-1}(x)s_n(x)y(x).
\end{aligned}$$

Taking the difference we see that

$$\begin{aligned}s_n(x)\Delta_{q;w}^{n+1}y(x) - s_{n-1}(x)\Delta_{q;w}^{n+2}y(x) &= s_n(x)\lambda_{n-1}(x)\Delta_{q;w}y(x) + s_n(x)s_{n-1}(x)y(x) - s_{n-1}(x)\lambda_n(x)\Delta_{q;w}y(x) - s_{n-1}(x)s_n(x)y(x)\\ &= s_n(x)\lambda_{n-1}(x)\Delta_{q;w}y(x) - s_{n-1}(x)\lambda_n(x)\Delta_{q;w}y(x)\\ &= (s_n(x)\lambda_{n-1}(x) - s_{n-1}(x)\lambda_n(x))\Delta_{q;w}y(x).\end{aligned}$$

Since $y(x)$ is a polynomial of degree n, it follows that $\Delta_{q;w}^{n+1}y(x) = 0$ and $\Delta_{q;w}^{n+2}y(x) = 0$. Because $\Delta_{q;w}y(x) \neq 0$ we see that

$$\delta_n(x) = s_n(x)\lambda_{n-1}(x) - s_{n-1}(x)\lambda_n(x) = 0$$

□

10.3 If $s_n(x)\lambda_{n-1}(x) \neq 0$ and $\delta_n = 0$ then there is a polynomial solution

Theorem 12. *If $s_n(x)\lambda_{n-1}(x) \neq 0$ and $\delta_n = 0$ then the difference equation*

$$\Delta_{q;w}^2 y(x) = \lambda_0(x)\Delta_{q;w}y(x) + s_0(x)y(x)$$

has a polynomial solutiong whose degree is at most n

10.3.1 Attempt 1

Proof. We start with $\delta_n = 0$ giving

$$s_n(x)\Delta_{q;w}^{n+1}y(x) - s_{n-1}(x)\Delta_{q;w}^{n+2}y(x) = (s_n(x)\lambda_{n-1}(x) - s_{n-1}(x)\lambda_n(x))\Delta_{q;w}y(x) = 0$$

Then

$$\begin{aligned}s_n(x)\Delta_{q;w}^{n+1}y(x) &= s_{n-1}(x)\lambda_n(x)\Delta_{q;w}y(x) + s_{n-1}(x)s_n(x)y(x)\\ &= s_{n-1}(x)y(x)\left(\lambda_n(x)\frac{\Delta_{q;w}y(x)}{y(x)} + s_n(x)\right)\\ \lambda_n(x)\Delta_{q;w}^{n+1}y(x) &= \lambda_n(x)\lambda_{n-1}(x)\Delta_{q;w}y(x) + \lambda_n(x)s_{n-1}(x)y(x)\\ &= \lambda_{n-1}\end{aligned}$$

□

10.3.2 Attempt 2

Proof. We have that

$$\begin{aligned}s_n(x)\Delta_{q;w}^{n+1}y(x) &= s_n(x)\lambda_{n-1}(x)\Delta_{q;w}y(x) + s_n(x)s_{n-1}(x)y(x)\\ &= s_{n-1}(x)y(x)\left(\lambda_n(x)\frac{\Delta_{q;w}y(x)}{y(x)} + s_n(x)\right)\end{aligned}$$

and using the fact that

$$\begin{aligned}\frac{\Delta_{q;w}y_1(x)}{y_1(x)} &= \frac{\frac{s_{n-1}(x)}{\lambda_{n-1}(x)} - q^m\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}}{\left[1 + q^m((1-q)x - w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}\right]}\\ &= \frac{s_{n-1}(x)}{\lambda_{n-1}(x)}\frac{1 - q^m\frac{\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}}{\frac{s_{n-1}(x)}{\lambda_{n-1}(x)}}}{\left[1 + q^m((1-q)x - w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}\right]}\end{aligned}$$

and $s(x) = s(x_m), \lambda(x) = \lambda(x_m)$, we can see that

$$s_n(x)\Delta_{q;w}^{n+1}y(x) = s_{n-1}(x)y(x)\left(\lambda_n(x)\frac{\Delta_{q;w}y(x)}{y(x)} + s_n(x)\right)$$

$$
\begin{aligned}
&= s_{n-1}(x)y(x)\left(\lambda_n(x)\frac{s_{n-1}(x)}{\lambda_{n-1}(x)}\frac{1-q^m}{\left[1+q^m((1-q)x-w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}\right]}+s_n(x)\right)\\
&= \frac{s_{n-1}(x)}{\lambda_{n-1}(x)}y(x)\left(\lambda_n(x)s_{n-1}(x)\frac{1-q^m}{\left[1+q^m((1-q)x-w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}\right]}+s_n(x)\lambda_{n-1}(x)\right)\\
&= \frac{s_{n-1}(x)}{\lambda_{n-1}(x)\left[1+q^m((1-q)x-w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}\right]}y(x)\Bigg(\lambda_n(x)s_{n-1}(x)\left(1-q^m\right)\\
&+\left[1+q^m((1-q)x-w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}\right]s_n(x)\lambda_{n-1}(x)\Bigg)\\
&= \frac{s_{n-1}(x)}{\lambda_{n-1}(x)\left[1+q^m((1-q)x-w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}\right]}y(x)\Bigg(\lambda_n(x)s_{n-1}(x)-q^m\lambda_n(x)s_{n-1}(x)\\
&+s_n(x)\lambda_{n-1}(x)+\left[q^m((1-q)x-w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}\right]s_n(x)\lambda_{n-1}(x)\Bigg)\\
&= \frac{s_{n-1}(x)}{\lambda_{n-1}(x)\left[1+q^m((1-q)x-w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}\right]}y(x)\Bigg(\delta_n(x)(1+q^m((q-1)x+w)\\
&-\lambda_n(x)s_{n-1}(x)q^m((q-1)x+w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x)})+s_n(x)\lambda_{n-1}(x)+\delta_n(x)(1-q^m)-\lambda_n(x)s_{n-1}(x)-s_n\lambda_{n-1}(x)q^m\Bigg)\\
&= \frac{s_{n-1}(x)}{\lambda_{n-1}(x)\left[1+q^m((1-q)x-w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}\right]}y(x)\Bigg(\delta_n(x)(1+q^m((q-1)x+w)\\
&-\lambda_n(x)s_{n-1}(x)q^m((q-1)x+w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x)})+\delta_n(x)+\delta_n(x)(1-q^m)-s_n\lambda_{n-1}(x)q^m\Bigg)\\
&= \frac{s_{n-1}(x)}{\lambda_{n-1}(x)\left[1+q^m((1-q)x-w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}\right]}y(x)\Bigg(\delta_n(x)(1+q^m((q-1)x+w)\\
&+\delta_n(x)+\delta_n(x)(1-q^m)+\delta_n(x)q^m\left(1+((1-q)x-w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}\right)\\
&+s_n(x)\lambda_{n-1}(x)q^m((1-q)x-w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}-\lambda_n(x)s_{n-1}(x)q^m\Bigg)\\
&= \frac{s_{n-1}(x)}{\lambda_{n-1}(x)\left[1+q^m((1-q)x-w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}\right]}y(x)\Bigg(\delta_n(x)(1+q^m((q-1)x+w)\\
&+\delta_n(x)+\delta_n(x)(1-q^m)+\delta_n(x)q^m\left(1+((1-q)x-w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}\right)\\
&+q^m\left(\delta_n(x)\left(-1-((1-q)x-w)\frac{s_{n-1}(x_m)}{\lambda_{n-1}(x_m)}\right)+\lambda_n(x)s_{n-1}(x)((1-q)x-w)\frac{s_{n-1}(x)}{\lambda_{n-1}(x)}-s_n(x)\lambda_{n-1}(x)\right)\Bigg)
\end{aligned}
$$

which implies that $y(x)$ is a polynomial of degree at most n. □

11 Umbral Calculus notes

11.1 Sheffer Sequence

Supposed we have a polynomial sequence $P_k(x)$ and the associated linear operator D such that

$$DP_k(x) = kP_{k-1}(x).$$

We denote a *shift operator* T_a as

$$T_a f(x) = f(x+a)$$

and say that the linear operator D is *shift-equivariant* if

$$T_a D = D T_a,$$

that is, D commutes with every shift operator.

Definition.
A sequence of polynomials $P_k(x)$ is called a *Sheffer sequence* if the linear operator characterized by

$$DP_k(x) = kP_{k-1}(x)$$

is *shift equivariant.*

Definition.
A sequence of polynomials $P_k(x)$ is called a *Appell sequence* if

$$\frac{d}{dx}P_k(x) = kP_{k-1}(x).$$

It's clear that all Appell sequences are Sheffer sequences, but not all Sheffer sequences are Appell sequences. We see that our defined sequence of polynomials is not an Appell sequence, as taking the limit as $q \to 1$ of the q-derivative "removes" the q from our polynomial.

12 Theorems regarding telescoping of the solutions product

We start with the solution

$$y(x) = \prod_{i=0}^{\infty} \frac{1}{\left[1 - q^i((q-1)x+w)\frac{s_{n-1}(x_i)}{\lambda_{n-1}(x_i)}\right]}.$$

We can rewrite this in the following form

$$\begin{aligned}
y(x) &= \prod_{i=0}^{\infty} \frac{1}{\left[1 - q^i((q-1)x+w)\frac{s_{n-1}(x_i)}{\lambda_{n-1}(x_i)}\right]} \\
&= \lim_{k\to\infty} \prod_{i=0}^{k} \frac{\lambda_{n-1}(x_i)}{[\lambda_{n-1}(x_i) - q^i((q-1)x+w)s_{n-1}(x_i)]} \\
&= \lim_{k\to\infty} \prod_{i=0}^{k} \frac{\lambda_{n-1}(x_i)}{[\lambda_{n-1}(x_i) - q^i((q-1)x+w)s_{n-1}(x_i)]} \\
&= \lim_{k\to\infty} \frac{\lambda_{n-1}(x)}{[\lambda_{n-1}(x) - ((q-1)x+w)s_{n-1}(x)]} \cdot \frac{\lambda_{n-1}(qx+w)}{[\lambda_{n-1}(qx+w) - q((q-1)x+w)s_{n-1}(qx+w)]} \\
&\times \frac{\lambda_{n-1}(q^2+qw+w)}{[\lambda_{n-1}(q^2x+qw+w) - q^2((q-1)x+w)s_{n-1}(q^2x+qw+w)]} \cdots \\
&\times \frac{\lambda_{n-1}(x_k)}{[\lambda_{n-1}(x_k) - q^k((q-1)x+w)s_{n-1}(x_k)]}
\end{aligned}$$

if we want a telescoping product, we need that

$$\lambda_{n-1}(qx+w) = \lambda_{n-1}(x) - ((q-1)x+w)s_{n-1}(x)$$

this also implies

$$\lambda_{n-1}(x_k) = \lambda_{n-1}(x_{k-1}) - q^{k-1}((q-1)x+w)s_{n-1}(x_{k-1})$$

which is equivalent to

$$\frac{\lambda_{n-1}(qx+w) - \lambda_{n-1}(x)}{((q-1)x+w)} = -s_{n-1}(x).$$

I believe this gives us a way to generate "function pairs" which will telescope in the final product. To test this, we consider $\lambda_0(x) = \frac{1}{x}$ then

$$\frac{\lambda_0(qx+w) - \lambda_0(x)}{((q-1)x+w)} = \frac{\frac{1}{qx+w} - \frac{1}{x}}{((q-1)x+w)}$$

$$
\begin{aligned}
&= \frac{1}{((q-1)x+w)} \frac{x-(qx+w)}{x(qx+w)} \\
&= \frac{1}{((q-1)x+w)} \left[\frac{(1-q)x-w}{x(qx+w)}\right] \\
&= \frac{-1}{x(qx+w)}
\end{aligned}
$$

inspiring us to take $s_0(x) = \frac{-1}{x(qx+w)}$. We will try to solve the difference equation

$$
\Delta_{q;w}^2 y(x) = \frac{1}{x}\Delta_{q;w} y(x) + \frac{1}{x(qx+w)} y(x)
$$

which has the terminating conditions

$$
\begin{aligned}
\delta_1 &= -\frac{q-1}{x(qx+w)^2\left(q^2x+qw+w\right)} \\
\delta_2 &= -\frac{q-1}{x(qx+w)^2\left(q^2x+qw+w\right)^2\left(q^3x+q^2w+qw+w\right)} \\
\delta_3 &= -\frac{(q-1)\left(q^4+q^3+q^2+1\right)}{x(qx+w)^2\left(q^2x+qw+w\right)^2\left(q^3x+q^2w+qw+w\right)^2\left(q^4x+q^3w+q^2w+qw+w\right)}
\end{aligned}
$$

up until $n=6$, we have no terminating condition. Actually, the form above says that the product telescopes if

$$
\Delta_{q;w}\lambda_{n-1}(x) = s_{n-1}(x).
$$

Theorem 13. *Suppose* $\Delta_{q;w}\lambda_{n-1}(x) = -s_{n-1}(x)$. *Then*

$$
\delta_n = 0 \iff s_0(x) = \frac{\lambda_n(x)\Delta_{q;w}\lambda_{n-1}(x) - \lambda_{n-1}(x)\Delta_{q;w}^2\lambda_{n-1}(x)}{\lambda_{n-1}(x)\lambda_{n-1}(qx+w)} \tag{1}
$$

Proof. Supposing the terminating condition $\delta_n = 0$, we get

$$
\begin{aligned}
s_n(x)\lambda_{n-1}(x) &= \lambda_n(x)s_{n-1}(x) \\
s_n(x)\lambda_{n-1}(x) &= -\lambda_n(x)\Delta_{q;w}\lambda_{n-1}(x) \\
s_n(x) &= -\frac{\lambda_n(x)\Delta_{q;w}\lambda_{n-1}(x)}{\lambda_{n-1}(x)} \\
\Delta_{q;w}s_{n-1}(x) + \lambda_{n-1}(qx+w)s_0(x) &= -\frac{\lambda_n(x)\Delta_{q;w}\lambda_{n-1}(x)}{\lambda_{n-1}(x)} \\
-\Delta_{q;w}^2\lambda_{n-1}(x) + \lambda_{n-1}(qx+w)s_0(x) &= -\frac{\lambda_n(x)\Delta_{q;w}\lambda_{n-1}(x)}{\lambda_{n-1}(x)} \\
s_0(x) &= -\frac{\lambda_n(x)\Delta_{q;w}\lambda_{n-1}(x)}{\lambda_{n-1}(x)\lambda_{n-1}(qx+w)} + \frac{\Delta_{q;w}^2\lambda_{n-1}(x)}{\lambda_{n-1}(qx+w)} \\
s_0(x) &= \frac{\lambda_{n-1}(x)\Delta_{q;w}^2\lambda_{n-1}(x) - \lambda_n(x)\Delta_{q;w}\lambda_{n-1}(x)}{\lambda_{n-1}(x)\lambda_{n-1}(qx+w)}
\end{aligned}
$$

□

12.0.1 Observation of previous theorem

Using (2) of theorem 12, we can see that the sequence $\lambda_n(x)$ is completely independent of s_n

$$
\begin{aligned}
\lambda_n(x) &= \Delta_{q;w}\lambda_{n-1}(x) + \lambda_{n-1}(qx+w)\lambda_0(x) + s_{n-1}(qx+w) \\
&= \Delta_{q;w}\lambda_{n-1}(x) + \lambda_{n-1}(qx+w)\lambda_0(x) + \Delta_{q;w}\lambda_{n-1}(qx+w) \\
&= q((q-1)x+w)\Delta_{q;w}^2\lambda_{n-1}(x) + (q+1)\Delta_{q;w}\lambda_{n-1}(x) + \lambda_{n-1}(qx+w)\lambda_0(x)
\end{aligned}
$$

then we can rewrite $s_0(x)$ as

$$
s_0(x) = \frac{\lambda_n(x)\Delta_{q;w}\lambda_{n-1}(x) - \lambda_{n-1}(x)\Delta_{q;w}^2\lambda_{n-1}(x)}{\lambda_{n-1}(x)\lambda_{n-1}(qx+w)}
$$

$$= \frac{\left[q((q-1)x+w)\Delta^2_{q;w}\lambda_{n-1}(x) + (q+1)\Delta_{q;w}\lambda_{n-1}(x) + \lambda_{n-1}(qx+w)\right]\Delta_{q;w}\lambda_{n-1}(x) - \lambda_{n-1}(x)\Delta^2_{q;w}\lambda_{n-1}(x)}{\lambda_{n-1}(x)\lambda_{n-1}(qx+w)}.$$

We also see that

$$s_m(x) = \Delta_{q;w}s_{m-1}(x) + \lambda_{m-1}(qx+w)s_0(x)$$

so if $s_0(x)$ is in terms of $\lambda_{n-1}(x)$ it follows that $s_m(x)$ is also in terms of $\lambda_{n-1}(x)$

12.0.2 Example

Let $\lambda_0(x) = x(x-w)$

Theorem 14. *Suppose $\delta_n = 0$ for some n and*

$$\Delta_{q;w}\lambda_{n-1}(x) = s_{n-1}(x)$$

then a solution to the difference equation

$$\Delta^2_{q;w}y(x) = \lambda_0(x)\Delta_{q;w}y(x) + s_0(x)y(x)$$

is

$$y_1(x) = \frac{\lambda_{n-1}(x)}{\lambda_{n-1}(w_0)}$$

Proof. By DAIM, $\delta_n = 0$ implies the solution is written

$$\begin{aligned}
y(x) &= \prod_{i=0}^{\infty} \frac{1}{\left[1 + q^i((q-1)x+w)\frac{s_{n-1}(x_i)}{\lambda_{n-1}(x_i)}\right]} \\
&= \lim_{k\to\infty}\prod_{i=0}^{k} \frac{\lambda_{n-1}(x_i)}{[\lambda_{n-1}(x_i) + q^i((q-1)x+w)s_{n-1}(x_i)]} \\
&= \lim_{k\to\infty}\prod_{i=0}^{k} \frac{\lambda_{n-1}(x_i)}{[\lambda_{n-1}(x_i) + q^i((q-1)x+w)s_{n-1}(x_i)]} \\
&= \lim_{k\to\infty} \frac{\lambda_{n-1}(x)}{[\lambda_{n-1}(x) + ((q-1)x+w)s_{n-1}(x)]} \cdot \frac{\lambda_{n-1}(qx+w)}{[\lambda_{n-1}(qx+w) + q((q-1)x+w)s_{n-1}(qx+w)]} \\
&\times \frac{\lambda_{n-1}(q^2+qw+w)}{[\lambda_{n-1}(q^2x+qw+w) + q^2((q-1)x+w)s_{n-1}(q^2x+qw+w)]}\cdots \\
&\times \frac{\lambda_{n-1}(x_k)}{[\lambda_{n-1}(x_k) + q^k((q-1)x+w)s_{n-1}(x_k)]}.
\end{aligned}$$

Now by the condition $\Delta_{q;w}\lambda_{n-1}(x) = s_{n-1}(x)$ we have

$$\begin{aligned}
\Delta_{q;w}\lambda_{n-1}(x) &= s_{n-1}(x) \\
\frac{\lambda_{n-1}(qx+w) - \lambda_{n-1}(x)}{((q-1)x+w)} &= s_{n-1}(x) \\
\lambda_{n-1}(qx+w) - \lambda_{n-1}(x) &= ((q-1)x+w)s_{n-1}(x) \\
\lambda_{n-1}(qx+w) &= \lambda_{n-1}(x) + ((q-1)x+w)s_{n-1}(x).
\end{aligned}$$

Note that taking $x \to q^{k-1}x + w[k-1]_q$ implies

$$\lambda_{n-1}(x_k) = \lambda_{n-1}(x_{k-1}) + q^k((q-1)x+w)s_{n-1}(x_{k-1})$$

for arbitrary k. $\square$

We this condition, we see that the solution telescopes to

$$y(x) = \lim_{k\to\infty} \frac{\lambda_{n-1}(x)}{[\lambda_{n-1}(x) + ((q-1)x+w)s_{n-1}(x)]} \cdot \frac{\lambda_{n-1}(qx+w)}{[\lambda_{n-1}(qx+w) + q((q-1)x+w)s_{n-1}(qx+w)]}$$

$$
\begin{aligned}
&\times \frac{\lambda_{n-1}(q^2+qw+w)}{[\lambda_{n-1}(q^2x+qw+w)+q^2((q-1)x+w)s_{n-1}(q^2x+qw+w)]}\cdots \\
&\times \frac{\lambda_{n-1}(x_k)}{[\lambda_{n-1}(x_k)+q^k((q-1)x+w)s_{n-1}(x_k)]} \\
&= \lim_{k\to\infty} \frac{\lambda_{n-1}(x)}{\lambda_{n-1}(x_k)+q^k((q-1)x+w)s_{n-1}(x_k)} \\
&= \frac{\lambda_{n-1}(x)}{\lambda_{n-1}(w_0)}
\end{aligned}
$$

13 An investigation of the hyper geometric equation

We consider a series solution using the basis constructed before. Were gunna solve

$$D_{q;w}^2 P_n(x) + \frac{2\epsilon x+\gamma}{ex^2+2fx+g} D_{q;w}P_n(x) - \frac{[n](2\epsilon+e[n-1])}{ex^2+2fx+g}P_n(x).$$

First we rewrite the nth degree polynomial $P_n(x)$ as a linear combination of the basis functions previously defined written as $B_n(x)$ so

$$P_n(x) = \sum_{i=0}^{n} a_i B_i(x).$$

The equation then becomes

$$\sum_{i=0}^{n-2}[i+2][i+1]a_{i+2}B_i(x) + \frac{2\epsilon x+\gamma}{ex^2+2fx+g}\sum_{i=0}^{n-1}[i+1]a_{i+1}B_i(x) - \frac{[n](2\epsilon+e[n-1])}{ex^2+2fx+g}\sum_{i=0}^{n}a_iB_i(x) = 0 \tag{2}$$

13.0.1 Switch indices for n

We change the indices to get

$$
\begin{aligned}
&\sum_{i=0}^{n-2}[i+2][i+1]a_{i+2}B_i(x) + \frac{2\epsilon x+\gamma}{ex^2+2fx+g}\sum_{i=0}^{n-2}[i+1]a_{i+1}B_i(x) - \frac{[n](2\epsilon+e[n-1])}{ex^2+2fx+g}\sum_{i=0}^{n-2}a_iB_i(x) \\
&\qquad + \frac{-[n](2\epsilon+e[n-1])}{ex^2+2fx+g}\left(a_nB_n(x)+a_{n-1}B_{n-1}(x)\right) + \frac{2\epsilon x+\gamma}{ex^2+2fx+g}[n]a_nB_{n-1}(x) = 0
\end{aligned}
$$

and collect all under one sum as

$$
\begin{aligned}
&\sum_{i=0}^{n-2}\left[[i+2][i+1]a_{i+2} + \frac{2\epsilon x+\gamma}{ex^2+2fx+g}[i+1]a_{i+1} - \frac{[n](2\epsilon+e[n-1])}{ex^2+2fx+g}a_i\right]B_i(x) \\
&\qquad + \frac{-[n](2\epsilon+e[n-1])}{ex^2+2fx+g}\left(a_nB_n(x)+a_{n-1}B_{n-1}(x)\right) + \frac{2\epsilon x+\gamma}{ex^2+2fx+g}[n]a_nB_{n-1}(x) = 0.
\end{aligned}
$$

Now we multiply throughout by $ex^2+2fx+g$ to get

$$
\begin{aligned}
&\sum_{i=0}^{n-2}\left[(ex^2+2fx+g)[i+2][i+1]a_{i+2} + (2\epsilon x+\gamma)[i+1]a_{i+1} - [n](2\epsilon+e[n-1])a_i\right]B_i(x) \\
&\qquad -[n](2\epsilon+e[n-1])\left(a_nB_n(x)+a_{n-1}B_{n-1}(x)\right) + (2\epsilon x+\gamma)[n]a_nB_{n-1}(x) = 0.
\end{aligned}
$$

If we substitute by the definition of $B_i(x) = \prod_{k=0}^{i-1}(x-w[k])$ we see that

$$
\begin{aligned}
&\sum_{i=0}^{n-2}\left[(ex^2+2fx+g)[i+2][i+1]a_{i+2} + (2\epsilon x+\gamma)[i+1]a_{i+1} - [n](2\epsilon+e[n-1])a_i\right]\prod_{k=0}^{i-1}(x-w[k]) \\
&-[n](2\epsilon+e[n-1])\left(a_n\prod_{k=0}^{n-1}(x-w[k]) + a_{n-1}\prod_{k=0}^{n-2}(x-w[k])\right) + (2\epsilon x+\gamma)[n]a_n\prod_{k=0}^{n-2}(x-w[k]) = 0.
\end{aligned}
$$

If we divide by $\prod_{k=0}^{n-2}(x-w[k])$ we get

$$\sum_{i=0}^{n-2}\left[(ex^2+2fx+g)[i+2][i+1]a_{i+2} + (2\epsilon x+\gamma)[i+1]a_{i+1} - [n](2\epsilon+e[n-1])a_i\right]\frac{1}{\prod_{k=i}^{n-2}(x-w[k])}$$

$$-[n](2\epsilon+e[n-1])\left(a_n(x-w[n-1])+a_{n-1}\right)+(2\epsilon x+\gamma)[n]a_n=0.$$

If we take out the $n-2$ term we get

$$\sum_{i=0}^{n-3}\left[(ex^2+2fx+g)[i+2][i+1]a_{i+2}+(2\epsilon x+\gamma)[i+1]a_{i+1}-[n](2\epsilon+e[n-1])a_i\right]\frac{1}{\prod_{k=i}^{n-2}(x-w[k])}$$
$$+(ex^2+2fx+g)[n][n-1]a_n+(2\epsilon x+\gamma)[n-1]a_{n-1}-[n](2\epsilon+e[n-1])a_{n-2}$$
$$-[n](2\epsilon+e[n-1])\left(a_n(x-w[n-1])+a_{n-1}\right)+(2\epsilon x+\gamma)[n]a_n=0.$$

collecting terms yields

$$\sum_{i=0}^{n-3}\left[(ex^2+2fx+g)[i+2][i+1]a_{i+2}+(2\epsilon x+\gamma)[i+1]a_{i+1}-[n](2\epsilon+e[n-1])a_i\right]\frac{1}{\prod_{k=i}^{n-2}(x-w[k])}$$
$$+a_n((ex^2+2fx+g)[n][n-1]-[n](2\epsilon+e[n-1])(x-w[n-1])+(2\epsilon x+\gamma)[n])$$
$$+a_{n-1}((2\epsilon x+\gamma)[n-1]-[n](2\epsilon+e[n-1]))$$
$$+a_{n-2}(-[n](2\epsilon+e[n-1])))=0$$