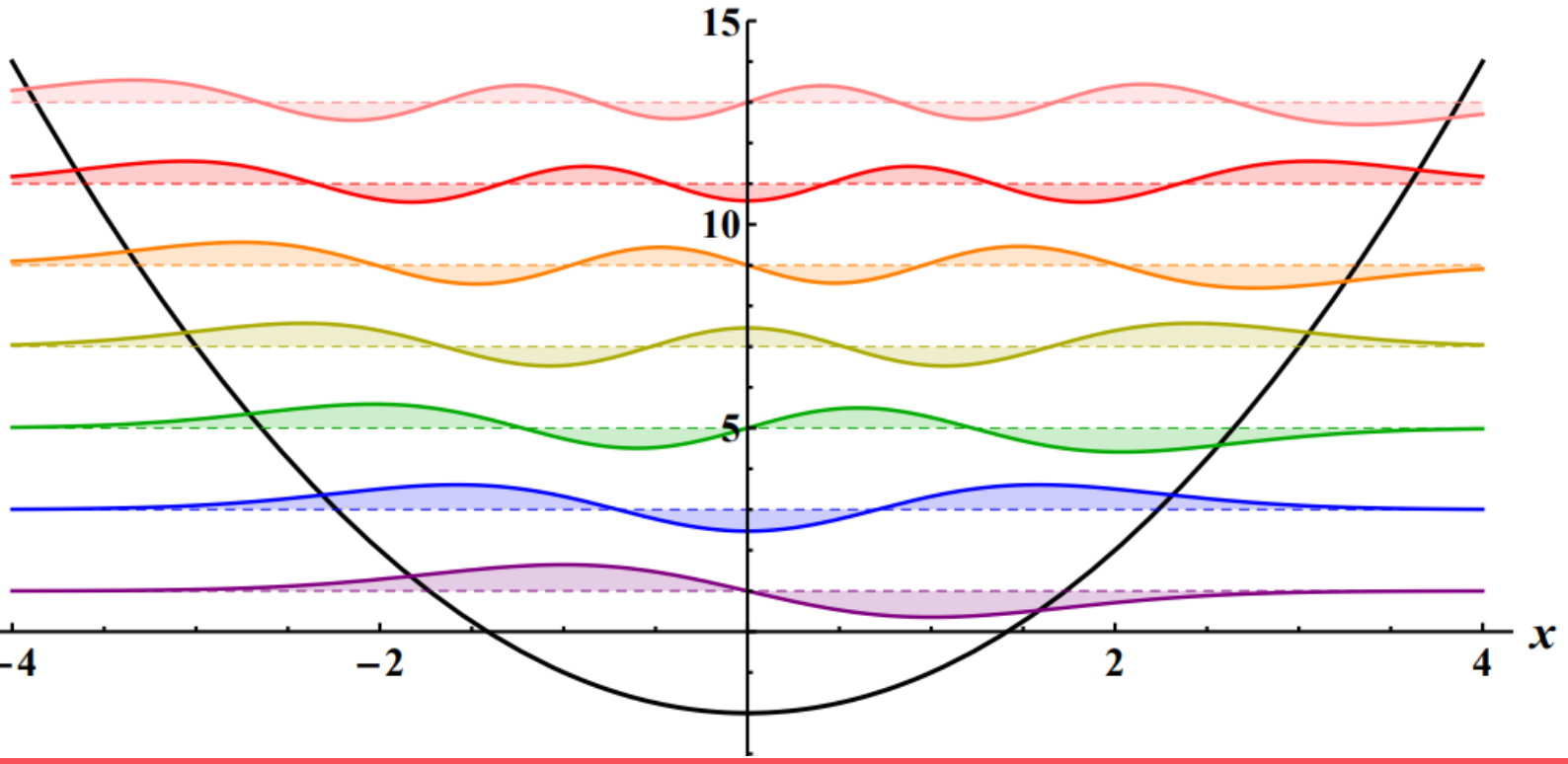

Quantum Mechanics

Course Notes for Phys-3220

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Date: Nov 30, 2020

Version: 0.0

Bio: These are course notes for UPEI's quantum mechanics course with Dr. Sheldon Opps.

Cover page courtesy of Kyle Bryenton.

Contents

Chapter 1 Math Prerequisites

We assume the reader is familiar with vector spaces. There will be many definitions, however they will be necessary to understand their use in quantum mechanics. We will first set the groundwork necessary for the *Hilbert Space*, the space in which quantum vectors live. We start with a theorem of complex numbers

Theorem 1.1

Let $z = a + ib$ be a complex number and $\overline{z} = a - ib$ be its conjugate. Further more, the absolute value of modulus of z is $|z| = \sqrt{a^2 + b^2}$ Then

$$z * \overline{z} = |z|^2$$

♡

Proof

$$\begin{aligned} z * \overline{z} &= (a + ib)(a - ib) \\ &= a^2 - iab + iab - i^2b^2 \\ &= a^2 - i^2b^2 \\ &= a^2 + b^2 \\ &= \sqrt{a^2 + b^2}^2 \\ &= |z|^2 \end{aligned}$$

Definition 1.1

An inner product space is a pair $(V, \langle\cdot|\cdot\rangle)$ where V is a vector space over a field F with a function $\langle\cdot|\cdot\rangle : V \times V \to F$ called the inner product. Let $x, y, z \in V$ and $a \in F$. Then the inner product must satisfy the following conditions

- *(1) Linearity in the first argument*

$$\langle ax|y\rangle = a\langle x|y\rangle$$

$$\langle x + y|z\rangle = \langle x|z\rangle + \langle y|z\rangle$$

- *(2) Conjugate symmetry/Hermitian Symmetry*

$$\langle x|y\rangle = \overline{\langle y|x\rangle}$$

- *(3) Positive Definite*

$$\langle x|x\rangle > 0, \quad x \neq 0.$$

The inner product space is called real or complex if $F = \mathbb{R}$ or $F = \mathbb{C}$ respectively. ♣

Example 1.1 Consider the vector space $\mathbb{R}^2$. The dot product is actually an inner product

on $\mathbb{R}^2$ and so $(\mathbb{R}^2, \_ \cdot \_)$ is an inner product space. We give an example of the inner product of two vectors

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 * 1 + 1 * 1 = 3.$$

Another inner product space is $\mathbb{R}$ with the inner product defined as $\langle x|y\rangle = xy$. It is a good excersise to check that this satisfies the 3 conditions of an inner product.

Definition 1.2

The norm induced by an inner product is

$$||x|| = \sqrt{\langle x|x\rangle}$$

♣

Example 1.2 The absolute value of a 2D vector in $\mathbb{R}^2$ is the norm induced by the dot product

$$\left|\begin{pmatrix} x \\ y \end{pmatrix}\right| = \sqrt{x^2 + y^2}$$

$$= \sqrt{\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}}.$$

Definition 1.3

A metric space is a pair $(M, d(x, y))$ where M is a set and $d(x, y)$ is a function mapping $x, y \in M$ to $\mathbb{R}$ called the distance function or the metric. Equivalently, $d : M \times M \to \mathbb{R}$. The metric must satisfy the following conditions

- *(1) Identity of indiscernibles*

$$d(x, y) = 0 \iff x = y$$

- *(2) Symmetry*

$$d(x, y) = d(y, x)$$

- *(3) Triangle inequality*

$$d(x, z) \leq d(x, y) + d(y, z)$$

♣

Example 1.3 The real numbers $\mathbb{R}$ is a metric space with the *Euclidean distance* defined by $d(x, y) = |x - y|$.

Example 1.4 The *discrete metric* is defined as

$$\rho(x, y) = \begin{cases} \rho(x, y) = 0, & x = y \\ \rho(x, y) = 1, & x \neq y \end{cases}$$

and is an interesting example as it is a metric on *any set.*

Definition 1.4

A metric space $(M, d(\cdot,\cdot))$ is complete if every Cauchy sequence in M has a limit in M.

♣

Definition 1.5

The metric induced by a norm is

$$d(x, y) = ||x - y||.$$

Likewise, the metric induced by the inner product is the metric induced by the norm induced by the inner product

$$d(x, y) = \sqrt{\langle x - y, x - y\rangle}$$

♣

We now have enough to define the *Hilbert Space*. While a solid understanding of a metric space or complete spaces is not required, it may help the reader in understanding the definition of the space we will be working in.

Definition 1.6

A Hilbert Space is real or complex inner product space which is also a complete metric space with respect to the metric induced by the inner product.

♣

Remark In quantum mechanics, it is customary to denote the vectors in a Hilbert space as the *ket vector* $|\psi\rangle \in \mathbb{E}$.

Definition 1.7

A linear functional of a vector space V over F is a linear function $f : V \to F$

♣

Definition 1.8

The Dual Space $V^\star$ of a vector space V is the set of all linear functionals on V. The dual space is a vectorspace with the operations

$$(f + g)(x) = f(x) + g(x)$$
$$(f \cdot g)(x) = f(x) \cdot g(x).$$

It is important to note that the dual space of any vector space will always exist and that V and its dual space are isomorphic

$$V \cong V^\star$$

♣

Remark The dual of a ket $|\psi\rangle$ is the *bra* $\langle\psi| \in \mathbb{E}^*$.

Finally, we may explain the use of the definitions within quantum mechanics. A quantum state $|\psi\rangle$ is an element of the Hilbert space $\mathbb{E}$ which will change depending on the situation being modelled.

Example 1.5 An electron is a *spin-*$\frac{1}{2}$ particle which can be represented by the vector $|\psi\rangle$.

When measuring electrons with a Stern-Gerlach device, we may find it to be *spin up* or *spin down* represented by $|+z\rangle$ and $|-z\rangle$ respectively. Our Hilbert space is *spanned by* $|+z\rangle$ *and* $|-z\rangle$ and so

$$|\psi\rangle = c_1 |+z\rangle + c_2 |-z\rangle$$

for some constants c_1, c_2. We say that $|\psi\rangle$ is a *linear combination* or *superposition* of the basis vectors $|+z\rangle$ and $|-z\rangle$.

Definition 1.9

The superposition of 2 kets $|v_1\rangle$ and $|v_2\rangle$ is

$$c_1 |v_1\rangle + c_2 |v_2\rangle .$$

♣

We also require that $\mathbb{E} \cong L^2$ the space of square integrable functions. The associated bra vectors are elements of the dual space of $\mathbb{E}$, $\mathbb{E}^\star$. When we bring a bra $\langle\phi|$ and a ket $|\psi\rangle$ together in a braket $\langle\phi|\psi\rangle$, we are actually talking about the inner product $\langle\cdot|\cdot\rangle$, with ϕ in the first slot, $\langle\phi|\cdot\rangle$, acting on $|\psi\rangle$.

We return to the general quantum state $|\phi\rangle$of an electron

$$|\phi\rangle = c_1 |-z\rangle + c_2 |+z\rangle .$$

It is a good exercise to show that for the *co-vector of* $|\psi\rangle$, $\langle\psi|$, can then be written

$$\langle\psi| = \overline{c_+} \langle+z| + \overline{c_-} \langle-z| .$$

The inner product of two state vectors would be the complex number known as the *probability amplitude*. When we talk about scenarios where the ket vectors are functions or *wave functions*, the inner product is often defined as

$$\langle\psi(x), \phi(x)\rangle = \int_{-\infty}^{\infty} \overline{\psi(x)}\phi(x)dx$$

where $\overline{\psi(x)}$ is the complex conjugate of $\psi(x)$. Note that in the case of $\langle\psi(x)|\psi(x)\rangle$ we have

$$\langle\psi(x)|\psi(x)\rangle = \int_{-\infty}^{\infty} \overline{\psi(x)}\psi(x)dx = \int_{-\infty}^{\infty} |\psi(x)|^2 dx.$$

Recall that any n dimensional vectorspace is isomorphic to $\mathbb{R}^n$. We will see how we can represent kets in an n dimensional Hilbert space as an n^{th} order vector v. Then the inner product of two vectors v and w is

$$w^T v = (w_1, w_2, \ldots, w_n) \begin{pmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{pmatrix} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

In the context of quantum mechanics, which is heavily dependent on probability, we require that

$$|\psi|^2 = \langle\psi|\psi\rangle = 1.$$

Intuitively, we are saying that $|\langle\psi|\phi\rangle|^2$ is the probability that the quantum state vector $|\phi\rangle$ is in the state $|psi\rangle$. So, the probability that $|\phi\rangle$ is in its own state $|\phi\rangle$ *must* be 1. If you are interested, you can read more online about *the L^2 space of functions* and *probability measures.*

Chapter 2 Stern-Gerlach Experiments

2.1 Magnetic moment and intrinsic spin

You should be familiar with the Stern-Gerlach experiments and the associated Stern-Gerlach device.

We first introduce the idea that particles have an *intrinsic property of spin*. It is not the same spin as we would use to describe a spinning top or rotation of the earth. It is *analogous* to spin. The *magnetic moment* μ of a charged particle is related to the orbital angular momentum L by the following: Let q be the charge, m mass, of a particle moving in a circular orbit of radius r with speed v. Then let the period $T = \frac{2\pi r}{v}$, $A = \pi r^2$, I the current of the charge q, and $L = mvr$. We have that

$$\mu = \frac{IA}{c} = \frac{q\pi r^2}{Tc} = \frac{qvr}{2c} = \frac{q}{2mc}L.$$

Since the magnetic moment and the orbital angular momentum are parallel or antiparallel depending on the sign of the charge q, we may say the following about their relation

Remark Let $\boldsymbol{\mu}$ be the magnetic moment of a particle and $\mathbf{L}$ the orbital angular momentum, both vectors. Then

$$\boldsymbol{\mu} = \frac{q}{2mc}\mathbf{L}.$$

We can derive a similar formula for the relationship between intrinsic spin angular momentum and the magnetic moment.

Remark Let $\mathbf{S}$ be the intrinsic spin angular momentum of a microscopic particle. Then

$$\boldsymbol{\mu} = \frac{gq}{2mc}\mathbf{S}.$$

Note the new parameter g. g is experimentally determined and is found to be 2.00 for an electron, 5.58 for a proton, and -3.82 for a neutron.

In their experiments, Stern and Gerlach measured the component of the intrinsic spin angular moment of an electron along the z-axis of their device. Common sense would expect for a continuum of silver atoms on their plate as z component of the magnetic moment can vary between $-|\boldsymbol{\mu}|$ and $|\boldsymbol{\mu}|$. What they actually found was that the intrinsic spin angular moment only took two discrete values $\frac{\hbar}{2}$ and $\frac{-\hbar}{2}$, where $\hbar$ is the reduced Planck constant with value $\hbar = 1.054571817 \times 10^{-34}$ [**planckConst**].

2.2 Bra and Ket

2.2.1 The probability of finding $|\phi\rangle$ in state $|\psi\rangle$

We start with the *ket*. We represent the state of a particle in quantum mechanics with the ket vector $|\phi\rangle$. If we want to say that $|\phi\rangle$ has positive intrinsic spin in the z direction, we write $S_z |\phi\rangle = \frac{\hbar}{2}$. S_z is an operator that acts on ket vectors. We say that a measurement of S_z is made on $|\phi\rangle$. For every ket vector $|\phi\rangle$ there is a *bra* vector $\langle\phi|$. A bra vector $\langle\psi|$ will meet with a ket vector $|\phi\rangle$ to form an inner product $\langle\psi|\phi\rangle$ which is a complex number. The value of $\langle\psi|\phi\rangle$ is the *probability amplitude*. The probability for a particle in the state $|\phi\rangle$ to be in the state $|\psi\rangle$ is $|\langle\psi|\phi\rangle|^2$.

Example 2.1 An important question we want to know the answer to is given a quantum state vector that represents an electron $|\psi\rangle$, whats the probability that it's spin up? Spin down? To answer this, we remember the concept of super position for an electron

$$|\psi\rangle = c_+ |+z\rangle + c_- |-z\rangle .$$

We know that the probability that $|\psi\rangle$ is in the state $|+z\rangle$ is $\langle+z|\psi\rangle^2$. We must remember that the inner product $\langle\phi|\psi\rangle$ is the *linear function* $\langle\phi|\cdot\rangle : \mathbb{E} \to \mathbb{R}$ acting on the quantum space vector $|\phi\rangle$. Thus we apply $\langle+z|\cdot\rangle$ to both sides

$$\begin{aligned}
\langle+z|\psi\rangle &= \langle+z|c_+ |+z\rangle + c_- |-z\rangle\rangle , \quad \text{apply linearity} \\
&= c_+ \langle+z| + z\rangle + c_1 \langle+z| - z\rangle \\
&= c_+ \cdot 1 + c_- \cdot 0 \\
&= c_+,
\end{aligned}$$

so the probability $|\psi\rangle$ has spin up is $|\langle+z|\psi\rangle|^2 = |c_+|^2 = c_+ \cdot \overline{c_+}$. We can do the same with $\langle-z|$ to get $|c_-|^2$.

Example 2.2 Let $|\phi\rangle = \frac{2i}{5} |+z\rangle + \frac{5}{3} |-z\rangle$. *What is the probability of finding $|\phi\rangle$ in the spin up state? What about spin down?*

We start with spin up. We remember that the probability is $|\langle+z|\phi\rangle|^2$ so

$$\begin{aligned}
|\langle+z|\phi\rangle|^2 &= |c_+|^2 \\
&= \frac{2i}{5} \cdot \frac{-2i}{5} \\
&= \frac{4}{25}.
\end{aligned}$$

We leave it as an exercise to verify that the probability that it is spin down is $\frac{21}{25}$.

Recall that

$$|\psi\rangle = c_+ |+z\rangle + c_- |-z\rangle$$

has the covector

$$\langle\psi| = \overline{c_+} \langle+z| + \overline{c_-} \langle-z| .$$

If we use this to calculate $\langle\psi|\psi\rangle$, we can see that

$$\begin{aligned}\langle\psi|\psi\rangle &= \overline{c_+} \langle+z|\psi\rangle + \overline{c_-} \langle-z|\psi\rangle , \quad \langle+z|\psi\rangle = c_+, \quad \langle-z|\psi\rangle = c_-, \\ &= \overline{c_+}c_+ + \overline{c_-}c_- \\ &= |c_+|^2 + |c_-|^2 = 1\end{aligned}$$

where the final step is due to our condition that $\langle\psi|\psi\rangle = 1$. By the *Hermitian symmetry* property $\langle x|y\rangle = \overline{\langle y|x\rangle}$ of the inner product we also have that $\langle+z|\psi\rangle = c_+$ implies $\langle\psi|+z\rangle = \overline{\langle+z|\psi\rangle} = \overline{c_+}$. This gives the following important result.

Theorem 2.1

$$\langle\psi|\psi\rangle = |\langle+z|\psi\rangle|^2 + |\langle-z|\psi\rangle|^2 = 1$$

♡

It's interesting to note that if we remember the identity

$$\cos(\theta)^2 + \sin(\theta)^2 = 1$$

and the fact that the absolute value of the inner product is always positive, we can see that the pair $(|\langle+z|\psi\rangle|, |\langle-z|\psi\rangle|)$ is always on the positive quadrant of the unit circle.

If we remember the Stern-Gerlach experiments, we found that an electron in the state $|+x\rangle$ has a 50% chance to be spin up in z and 50% chance to be in spin down in z. We can write this as

$$|\langle+z| + x\rangle|^2 = \frac{1}{2}$$

but since $|x\rangle = c_+ |+z\rangle + c_- |-z\rangle$ we have

$$|\langle+z| + x\rangle|^2 = \overline{\langle+z| + x\rangle} * \langle+z| + x\rangle = \overline{c_+} * c_+.$$

Doing the same for c_- we find that

$$\overline{c_+} * c_+ = \frac{1}{2} \tag{2.1}$$

$$\overline{c_-} * c_- = \frac{1}{2} \tag{2.2}$$

and so we can actually choose whateverc_+ and c_- we want as long as it satisfies (2.1) and (2.2). The general solution to this is that $c_+ = \frac{e^{i\theta_+}}{\sqrt{2}}$ and $c_- = \frac{e^{i\theta_-}}{\sqrt{2}}$ for arbitrary θ_- and $\theta_+ \in \mathbb{R}$. The book uses the variable $\delta_\pm$ and calls it the *phase*. Verify this result by

calcualting $\overline{c_+}c_+$ and think geometrically about why this makes sense. So

$$|+x\rangle = \frac{e^{i\theta_+}}{\sqrt{2}}|+z\rangle + \frac{e^{i\theta_-}}{\sqrt{2}}|-z\rangle . \tag{2.3}$$

When considering the SG experiments, we have only so far been working with the x and z axis. The labeling of axis is quite arbitrary,and so we can just as easily replace x with z or y with x, etc. So, we expect to have

$$|\langle +y| + x\rangle|^2 = \frac{1}{2}.$$

Convince yourself using the same idea as (2.3) that we can write the $\langle +y|$ as

$$\begin{aligned}\langle +y| &= \frac{e^{-i\gamma_+}}{\sqrt{2}}\langle +z| + \frac{e^{-i\gamma_-}}{\sqrt{2}}\langle -z| \\ &= \frac{e^{-i\gamma_+}}{\sqrt{2}}\left[\langle +z| + e^{-i(\gamma_- - \gamma_+}\langle -z|\right].\end{aligned}$$

We then have the following theorem

Theorem 2.2

*Let $\delta = \delta_- - \delta_+$ and $\gamma = \gamma_- + \gamma_+$ which we reffer to as the **relative phases** between $|+z\rangle$ and $|-z\rangle$. Then*

$$|\langle +y| + x\rangle|^2 = \frac{1}{2}[1 + \cos(\delta - \gamma)]$$

♡

Proof

$$\begin{aligned}|\langle +y| + x\rangle|^2 &= \left[\frac{e^{i(\delta_+ - \gamma_+)}}{2}\left(1 + e^{i(\delta-\gamma)}\right)\right]\left[\frac{e^{-i(\delta_+ - \gamma_+)}}{2}\left(1 + e^{-i(\delta-\gamma)}\right)\right] \\ &= \frac{1}{4}\left[1 + e^{i(\delta-\gamma)}\right]\left[1 + e^{-i(\delta-\gamma)}\right] \\ &= \frac{1}{2}\left[1 + \cos(\delta - \gamma)\right]\end{aligned}$$

The consequences of this theorem is that we required $|\langle +y| + x\rangle|^2 = \frac{1}{2}$ and thus $1 + \cos(\delta - \gamma) = 1$ which is the same as $\cos(\delta - \gamma) = 0$. We remember that $\cos(x)$ is only 0 when $x = \pm\frac{(2n+1)\pi}{2}$ and so we take $\delta - \gamma = \frac{\pi}{2}$. Now keep in mind that δ and γ are arbitrary parameters chosen by us, and the most common convention is to take $\delta = 0$. Finally, we can rewrite our x ket vector and y ket vectors as

$$\begin{aligned}|+x\rangle &= \frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle \\ |+y\rangle &= \frac{1}{\sqrt{2}}|+z\rangle + \frac{i}{\sqrt{2}}|-z\rangle\end{aligned}$$

2.2.2 Calculating Probability

What do we expect the value of $S_z|+x\rangle$ to be? $\frac{\hbar}{2}$ or $\frac{-\hbar}{2}$? We can answer this question with the *expectation value*

Definition 2.1

Let X be a random variable with a finite number of outcomes $x_1, x_2, \ldots, x_n$ and $P(x_i)$ be the probability of x_i occurring. Then the expected value or expectation value is

$$\langle X \rangle = \sum_{i=0}^{n} x_i P(x_i).$$

If X is a random variable with a probability density function $f(x)$, then the expectation value is

$$\langle X \rangle = \int_{-\infty}^{\infty} x f(x) dx$$

♣

Example 2.3 What is the expectation value $\langle S_z \rangle$ for $|+x\rangle$? We had said that $S_z |+x\rangle$ is a random variable with two outcomes $\frac{-\hbar}{2}$ and $\frac{\hbar}{2}$, each with probability $\frac{1}{2}$. So

$$\begin{aligned}
\langle S_z |+x\rangle \rangle &= \sum_{i=0}^{n} x_i P(x_i) \\
&= \frac{1}{2} \cdot \frac{-\hbar}{2} + \frac{1}{2} \cdot \frac{\hbar}{2} \\
&= \frac{-\hbar}{4} + \frac{\hbar}{4} \\
&= 0.
\end{aligned}$$

Likewise, we can talk about the *standard deviation* of a random variable. In quantum mechanics, we call the standard deviation *uncertainty*.

Definition 2.2

The uncertainty $(\Delta X)^2$ of a random variable X is

$$(\Delta X)^2 = \langle X^2 \rangle - \langle X \rangle^2$$

where

$$\langle X^2 \rangle = \sum_{i=0}^{n} x_i P(x_i)^2$$

for discrete variables and

$$\langle X^2 \rangle = \int_{-\infty}^{\infty} x^2 f(x) dx$$

for continuous variables. ΔX is the variance.

♣

Example 2.4 Let $\Phi(x,t) = \sqrt{\lambda} e^{-\lambda|x|} e^{-iwt}$ with $\lambda, w > 0$ be a wave function. Find the uncertainty of $\Phi(x,t)$.

Calculating the required values and substituting we see that

$$\begin{aligned}(\Delta X)^2 &= \langle x^2 \rangle - \langle x \rangle^2 \\ &= \frac{1}{2\lambda^2} - 0^2 \\ &= \frac{1}{2\lambda^2}.\end{aligned}$$

Thus the uncertainty is $\Delta X = \sqrt{\frac{1}{2\lambda^2}} = \frac{1}{\sqrt{2}\lambda}$.

Example 2.5 What is the uncertainty of S_z for $|+x\rangle$

Chapter 3 Matrix Mechanics

Quantum Mechanics is done in a Hilbert Space, which has all the properties of a Vector space. Thus, we may use a very powerful theorem to represent our ket vectors as elements of $\mathbb{C}^{\aleph}$

Theorem 3.1

If V is an n-dimensional vectorspace over the field $\mathbb{F}$ then

$$V \cong \mathbb{F}^n$$

♡

Proof Put the proof here.

So, because our Hilbert space is a vectorspace over $\mathbb{C}$, it is isomorphic to $\mathbb{C}^n$, n the dimension of the Hilbert space. Another important theorem relates our linear operators $S_z, S_x, \ldots$

Theorem 3.2

Let $T : V \to V$ be a linear operator and V an n-dimension vectorspace. Then there exists a nxn matrix A such that

$$T(v) = Av.$$

♡

Proof proof here

Theorem 3.3

Let V be a vector space with basis $\{|b_1\rangle, |b_2\rangle, \ldots, |b_n\rangle\}$ and T a linear operator on V. Then the matrix representation of T is

$$T(v) = Av = \begin{bmatrix} \langle b_1| T |b_1\rangle & \langle b_1| T |b_2\rangle & \ldots & \langle b_1| T |b_n\rangle \\ \langle b_2| T |b_1\rangle & \langle b_2| T |b_2\rangle & \ldots & \langle b_2| T |b_n\rangle \\ \ldots & \ldots & \ldots & \ldots \\ \langle b_n| T |b_1\rangle & \langle b_n| T |b_2\rangle & \ldots & \langle b_n| T |b_n\rangle \end{bmatrix} v$$

where $T_{n\times n}$ is the nxn matrix containing T as entries.

♡

Proof Proof

The great part about this theorem is that *we may find the matrix representation of a linear operator just by knowing its action on the basis vectors.*

3.1 Basis

The basis of any Hilbert space in quantum mechanics will be orthonormal, typically *pure states.*

Example 3.1 For the electron, some typical basis states are $|+z\rangle$ and $|-z\rangle$, $|+y\rangle$ and $|-y\rangle$, or $|+x\rangle$ and $|-x\rangle$ which correspond to S_z, S_y, and S_x respectively.

Example 3.2 The polarization of light can be modelled with the basis states $|x\rangle$ and $|y\rangle$, where $|x\rangle$ passes through a linear polarizer oriented in an x direction (we choose x).

By choosing our basis kets we have determined the vector representations and matrix representations along with it. A typical choice of basis for a spin-$\frac{1}{2}$ particle is

$$|\phi\rangle = c_1 |+z\rangle + c_2 |-z\rangle$$

which we may represent as

$$|\phi\rangle = \begin{pmatrix} \langle +z|\phi\rangle \\ \langle -z|\phi\rangle \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

The basis vectors would then be represented as

$$|+z\rangle = \begin{pmatrix} \langle +z|+z\rangle \\ \langle -z|+z\rangle \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|-z\rangle = \begin{pmatrix} \langle +z|-z\rangle \\ \langle -z|-z\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We call this the S_z basis. Likewise, there exists the basises S_y and S_x. We give some useful changes of basis

Definition 3.1

$$\begin{pmatrix} |+x\rangle \\ |-x\rangle \end{pmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{pmatrix} |+z\rangle \\ |-z\rangle \end{pmatrix}$$

$$\begin{pmatrix} |+z\rangle \\ |-z\rangle \end{pmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{pmatrix} |+x\rangle \\ |-x\rangle \end{pmatrix}$$

$$\begin{pmatrix} |+y\rangle \\ |-y\rangle \end{pmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{bmatrix} \begin{pmatrix} |+z\rangle \\ |-z\rangle \end{pmatrix}$$

$$\begin{pmatrix} |+z\rangle \\ |-z\rangle \end{pmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix} \begin{pmatrix} |+y\rangle \\ |-y\rangle \end{pmatrix}$$

$$\begin{pmatrix} |+x\rangle \\ |-x\rangle \end{pmatrix} = \begin{bmatrix} \frac{1}{2} - i\frac{1}{2} & \frac{1}{2} + i\frac{1}{2} \\ \frac{1}{2} + i\frac{1}{2} & \frac{1}{2} - i\frac{1}{2} \end{bmatrix} \begin{pmatrix} |+y\rangle \\ |-y\rangle \end{pmatrix}$$

$$\begin{pmatrix} |+y\rangle \\ |-y\rangle \end{pmatrix} = \begin{bmatrix} \frac{1}{2} + i\frac{1}{2} & \frac{1}{2} - i\frac{1}{2} \\ \frac{1}{2} - i\frac{1}{2} & \frac{1}{2} + i\frac{1}{2} \end{bmatrix} \begin{pmatrix} |+x\rangle \\ |-x\rangle \end{pmatrix}$$

♣

Note that this also determines the change of basis for the bra vectors aswell. If $|\phi\rangle = A|\psi\rangle$ then

$$\begin{aligned} \langle\phi| &= |\phi\rangle^\dagger \\ &= (A|\psi\rangle)^\dagger \\ &= |\psi\rangle^\dagger A^\dagger \\ &= \langle\psi| A^\dagger. \end{aligned}$$

3.2 Rotation Operators

We introduce the linear operator $\hat{R}(\theta\hat{k})$ which rotates a quantum state θ radians counter clockwise around the z-axis. Likewise, $\hat{R}(\theta\hat{i})$ and $\hat{R}(\theta\hat{j})$ rotate around the x and y axes. Consider a soace spanned by $|+x\rangle, |+y\rangle$, and $|+z\rangle$. Rotating this space $\frac{\pi}{2}$ degrees around the y axis would then transform the z axis into the x axis. We may write this as

$$\hat{R}\left(\frac{\pi}{2}j\right)|+z\rangle = |+x\rangle.$$

We use the same reasoning to show

$$\hat{R}\left(\frac{\pi}{2}j\right)|-z\rangle = |-x\rangle.$$

For a general ket $|\phi\rangle$, we can use the $|\pm z\rangle$ basis and the previous results to show that

$$\begin{aligned} \hat{R}\left(\frac{\pi}{2}j\right)|\phi\rangle &= \hat{R}\left(\frac{\pi}{2}j\right)(c_1|+z\rangle + c_2|-z\rangle), \quad \textit{by linearity of } \hat{R}\left(\frac{\pi}{2}j\right) \\ &= c_1\hat{R}\left(\frac{\pi}{2}j\right)|+z\rangle + c_2\hat{R}\left(\frac{\pi}{2}j\right)|-z\rangle \\ &= c_1|+x\rangle + c_2|-x\rangle. \end{aligned}$$

Therefore to determine the state of a rotated ket, we need only determine the rotation on its basis.

Example 3.3 Determine $\hat{R}\left(\frac{\pi}{2}j\right)|\phi\rangle$ where $|\phi\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{-1}{\sqrt{2}}|-z\rangle$.

Solution *Solution*

Example 3.4 Determine $\hat{R}(\pi k)|+y\rangle$ and $\hat{R}(\pi k)|-y\rangle$ using the axis reasoning from before. Use this result to rotate $|\phi\rangle = \frac{1}{2}|+y\rangle + \frac{\sqrt{3}}{2}$ around the z-axis.

Solution *Solution*

When rotating a quantum state, we still expect it to satisfy the normalization condition $\langle\phi|\phi\rangle = 1$. Remember that a vector that is acted on by a linear operator Av has the co-vector $v^T A^\dagger$ where $A^\dagger$ is the *conjugate transpose* of A.

Definition 3.2

The conjugate transpose of a complex matrix $A = a_{ij}$ is $A^\dagger = \bar{A}^T = \bar{a_{ji}}$. ♣

Example 3.5 Find the conjugate transpose of $A = \begin{pmatrix} 1 & 2i \\ 2 & i \end{pmatrix}$

Solution *We know that the conjugate transpose of A is $\bar{A}^T$ so*

$$\begin{aligned} \bar{A}^T &= \overline{\begin{pmatrix} 1 & 2i \\ 2 & i \end{pmatrix}}^T \\ &= \overline{\begin{pmatrix} 1 & 2 \\ 2i & i \end{pmatrix}} \\ &= \begin{pmatrix} 1 & 2 \\ -2i & -i \end{pmatrix} \end{aligned}$$

Definition 3.3

An operator A is Hermitian if $A = A^\dagger$. ♣

Actually, Hermitian operators have some useful properties

Theorem 3.4

- *The eigenvalues of A are real*
- *The eigenvectors are orthogonal*
- *$\langle\phi|A\phi\rangle$ is real for all $|\phi\rangle$*
- *$(A_{n\times m}B_{m\times p})^\dagger = B^\dagger A^\dagger$ for* ♡

Note that the last holds for vectors as well, as vectors are $n \times 1$ matrices. In the language of quantum mechanics, we require that if $\langle\phi|\phi\rangle = 1$ then

$$\langle\phi|\,\hat{R}^\dagger(\pi k)\,\hat{R}(\pi k)\,|\phi\rangle = 1$$

as well. In order for this condition to be satisfied, we require that

$$\hat{R}^\dagger(\pi k)\,\hat{R}(\pi k) = 1$$

which means that

$$\hat{R}^\dagger(\pi k) = \hat{R^{-1}}(\pi k).$$

When an operator satisfies this property, we say that it is *Unitary*.

Definition 3.4

An operator A is Unitary if $A^\dagger = A^{-1}$

♣

The condition that an operator is unitary in quantum mechanics is due to the requirement that probability must be conserved. In this case, probability is conserved under rotation. We also are interested in writing $\hat{R}^\dagger(\theta k)$ in a form which does not require the conjugate transpose. The following theorem is left as an exercise.

Theorem 3.5

$$\hat{R}^\dagger(\theta k) = \hat{R}(-\theta k)$$

♡

We end this section by giving the general form of the rotation operator which frees ourselves of rotation around a particular axis and satisfies the previous properties of unitary and conservation of probability.

Definition 3.5

We denote the rotation operator rotating a vector clockwise around $\hat{n}$ *by* θ *radians*

$$\hat{R}(\theta n).$$

♣

3.3 The Generator of Rotation

We move from rotation vectors by an angle θ to *infitesimal rotations* $d\theta$.

Definition 3.6

An infinitesimal rotation around the z*-axis is given by*

$$\hat{R}(d\theta k) = 1 - \frac{i}{\hbar}\hat{J}_z d\theta,$$

where $\hat{J}_z$ *is the angular momentum.*

Furthermore, an infinitesimal rotation about a vector $\vec{n}$ *is given by*

$$\hat{R}(d\theta \vec{n}) = 1 - \frac{i}{\hbar} n \cdot J d\theta$$

♣

Theorem 3.6

$\hat{J}_z$ *is Hermitian.*

♡

Proof For conservation of probability we require that

$$\hat{R}^\dagger(d\theta k)\hat{R}(d\theta k) = 1$$

then

$$\begin{aligned}1 &= \hat{R}^{\dagger}(d\theta k)\hat{R}(d\theta k)\\ &= (1+\frac{i}{\hbar}\hat{J}_z^{\dagger}d\theta)(1-\frac{i}{\hbar}\hat{J}_z d\theta)\\ &= 1-\frac{i}{\hbar}\hat{J}_z d\theta+\frac{i}{\hbar}\hat{J}_z^{\dagger}d\theta+\frac{1}{\hbar^2}\hat{J}_z\hat{J}_z^{\dagger}d\theta^2.\end{aligned}$$

We then disregard the $d\theta^2$ to get

$$1 = 1+\frac{i}{\hbar}d\theta\left(\hat{J}_z^{\dagger}-\hat{J}_z\right)$$

giving

$$\frac{i}{\hbar}d\theta\left(\hat{J}_z^{\dagger}-\hat{J}_z\right)=0.$$

Because $\frac{i}{\hbar}d\theta$ is non-zero, it follows that $\hat{J}_z^{\dagger}-\hat{J}_z=0$ for the required result

$$\hat{J}_z^{\dagger}=\hat{J}_z$$

Now, rather than considering $d\theta$, suppose we wish to rotate a vector by the angle θ. We must then rotate it by $\frac{\theta}{N}$, N times. We can write this as

$$\hat{R}(\theta k)=\underbrace{\hat{R}\left(\frac{\theta}{N}k\right)\hat{R}\left(\frac{\theta}{N}k\right)\ldots\hat{R}\left(\frac{\theta}{N}k\right)}_{N times}$$

if we write this in terms of the generator of rotation we find

$$\begin{aligned}\hat{R}(\theta k) &= \underbrace{\hat{R}\left(\frac{\theta}{N}k\right)\hat{R}\left(\frac{\theta}{N}k\right)\ldots\hat{R}\left(\frac{\theta}{N}k\right)}_{N times}\\ &= \underbrace{\left(1-\frac{i}{\hbar}\frac{\theta}{N}\hat{J}_z\right)\left(1-\frac{i}{\hbar}\frac{\theta}{N}\hat{J}_z\right)\ldots\left(1-\frac{i}{\hbar}\frac{\theta}{N}\hat{J}_z\right)}_{N times}\\ &= \left(1-\frac{i}{\hbar}\frac{\theta}{N}\hat{J}_z\right)^N.\end{aligned}$$

Note that the exponential function e^x may be defined as $\lim_{n\to\infty}\left[1-\frac{x}{n}\right]^n$ so by taking the limit as $N\to\infty$ of $\left(1-\frac{i}{\hbar}\frac{\theta}{N}\hat{J}_z\right)^N$ we can see that

$$\hat{R}(\theta k)=\lim_{N\to\infty}\left(1-\frac{i}{\hbar}\frac{\theta}{N}\hat{J}_z\right)^N=e^{\frac{-i}{\hbar}\theta\hat{J}_z}.$$

Thus we have the following theorem

Theorem 3.7

$$\hat{R}(\theta k)=e^{\frac{-i}{\hbar}\theta\hat{J}_z}.$$

♡

Raising a number to a operator may seen ambiguous, however it is customary to

define this in terms of the Taylor series of $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$e^{\frac{-i}{\hbar}\theta \hat{J}_z} = \sum_{n=0}^{\infty} \frac{\left(\frac{-i}{\hbar}\theta \hat{J}_z\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{((-i)^n \theta^n)}{\hbar^n n!} \hat{J}_z{}^n$$

Based on the Stern-Gerlach experiments, we define the eigenvalue of the spin angular momentum operator to be $\frac{\hbar}{2}$

Definition 3.7

$$\hat{J}_z \left|\pm z\right\rangle = \pm \frac{\hbar}{2} \left|\pm z\right\rangle .$$

♣

With this definition, it is easy to find the eigenkets and eigenvalues of $\hat{R}(\theta k)$

Theorem 3.8

The eigenkets and associated eigenvalues of the rotation operator are

$$\hat{R}(\theta k) \left|+z\right\rangle = e^{\frac{-i\theta}{2}} \left|+z\right\rangle$$
$$\hat{R}(\theta k) \left|-z\right\rangle = e^{\frac{i\theta}{2}} \left|-z\right\rangle$$

♡

$$\begin{aligned}
\hat{R}(\theta k) \left|+z\right\rangle &= e^{\frac{-i}{\hbar}\theta \hat{J}_z} \left|+z\right\rangle \\
&= \sum_{n=0}^{\infty} \frac{((-i)^n \theta^n)}{\hbar^n n!} \hat{J}_z{}^n \left|+z\right\rangle \\
&= \sum_{n=0}^{\infty} \frac{((-i)^n \theta^n)}{\hbar^n n!} \left(\frac{\hbar}{2}\right)^n \left|+z\right\rangle \\
&= \sum_{n=0}^{\infty} \frac{((-i)^n \theta^n)}{\hbar^n n!} \frac{\hbar^n}{2^n} \left|+z\right\rangle \\
&= \sum_{n=0}^{\infty} \frac{((-i)^n \theta^n)}{n!} \frac{1}{2^n} \left|+z\right\rangle \\
&= \sum_{n=0}^{\infty} \frac{\left(\frac{-i\theta}{2}\right)^n}{n!} \left|+z\right\rangle \\
&= e^{\frac{-i\theta}{2}} \left|+z\right\rangle .
\end{aligned}$$

The other eigenvalue is found in a similar way.

Note how a rotation of $\theta = 4\pi$ is necessary to return the ket to it's original state.

We end this section by giving the matrix representation of the generators of rotation

Theorem 3.9

$$J_z \xrightarrow[S_z]{} \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$J_z \xrightarrow[S_y]{} \frac{\hbar}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$J_z \xrightarrow[S_x]{} \frac{\hbar}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

♡

Proof

$$\begin{aligned} J_z \xrightarrow[S_z]{} & \begin{bmatrix} \langle +z| J_z |+z\rangle & \langle +z| J_z |-z\rangle \\ \langle -z| J_z |+z\rangle & \langle -z| J_z |-z\rangle \end{bmatrix} \\ &= \begin{bmatrix} \frac{\hbar}{2}\langle +z| + z\rangle & \frac{-\hbar}{2}\langle +z| - z\rangle \\ \frac{\hbar}{2}\langle -z| + z\rangle & \frac{-\hbar}{2}\langle -z| - z\rangle \end{bmatrix} \\ &= \begin{bmatrix} \frac{\hbar}{2} & 0 \\ 0 & \frac{-\hbar}{2} \end{bmatrix} \\ &= \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} J_z \xrightarrow[S_y]{} & \begin{bmatrix} \langle +y| J_z |+y\rangle & \langle +y| J_z |-y\rangle \\ \langle -y| J_z |+y\rangle & \langle -y| J_z |-y\rangle \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix} J_z \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix} \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{bmatrix} \\ &= \frac{\hbar}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \end{aligned}$$

J_z's x representation is left as an exercise.

3.4 Some Special Operators

In this section we will discuss the Projection operator, the Identity operator, and Ladder operators. These operators and their matrix representations have important uses in Quantum Mechanics. We start with the Identity operator, which maps a vector to itself $Iv = v$ and a matrix to itself $I \cdot M = I \cdot M = M$. The identity operator has matrix

representation

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

and you can check that it satisfies the previous properties. It's interesting that all n-matrices over $\mathbb{C}$ (Or any field) is an associative algebra under matrix multiplication. In this algebra, I is the multiplicative identity. We also have the 0 matrix defined by $0_{ij} = 0$ being the additive identity. The reason it is not a *commutative algebra* is because *matrix multiplication is not commutative.* So for two matrices A and B, it is not guaranteed that $AB = BA$. If two matrices *are* commutative, see that $AB = BA$ implies $AB - BA = 0$. Here is where we define the *commutator*

Definition 3.8

The commutator of 2 elements of an associative algebra, A and B, is

$$[A, B] = AB - BA.$$

♣

See that A and B commuting is equivalent to their commutator being 0. We will sometimes see $AB = [A, B] + BA$ or $BA = AB - [A, B]$ used in calculations where we know the commutator.

Example 3.6 Show that identity matrix commutes with all matrices

Solution *To show commutativity, we can show that $[I, A] = 0$ for some matrix A*

$$\begin{aligned} [I, A] &= I \cdot A - A \cdot I \\ &= A - A \\ &= 0. \end{aligned}$$

Thus I commutes with A. I is actually defined to be commutative with all matrices.

It is useful to have these identities for the commutator

Theorem 3.10

1. $[A + B, C] = [A, C] + [B, C]$
2. $[A, A] = 0$
3. $[A, B] = -[B, A]$
4. $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$
5. $[A, BC] = [A, B]C + B[A, C]$
6. $[A, BCD] = [A, B]CD + B[A, C]D + BC[A, D]$
7. $[A, B_1 B_2 \dots B_n] = \sum_{i=1}^{n} B_1 B_2 \dots B_{i-1} [A, B_i] B_{i+1} \dots B_{n-1} B_n$

♡

Proof

1.
$$\begin{aligned}[A+B,C] &= (A+B)C - C(A+B)\\ &= AC + BC - CA - CB\\ &= AC - CA + BC - CB\\ &= [A,C] + [B,C]\end{aligned}$$
2. $[A,A] = AA - AA = 0$
3. $-[B,A] = -(BA - AB) = -BA + AB = AB - BA = [A,B]$
4. This is left as an exercise. This is called the Jacobi identity.
5.
$$\begin{aligned}[A,B]C + B[A,C] &= (AB - BA)C + B(AC - CA)\\ &= ABC - BAC + BAC - BCA\\ &= ABC - BCA\\ &= [A,BC]\end{aligned}$$
6. Left as an exercise.
7. Left as an exercise.

Theorem 3.11

$$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z$$
$$[\hat{J}_y, \hat{J}_z] = i\hbar\hat{J}_x$$
$$[\hat{J}_z, \hat{J}_x] = i\hbar\hat{J}_y$$

♡

3.4.1 Projection Operators

We now look at the projection operator. One may ask themselves, given a quantum state $|\psi\rangle = 2|+z\rangle + (1+2i)|-z\rangle$ how can we recover either $2|+z\rangle$ or $(1+2i)|-z\rangle$? The former can be done with a *Projection operator* $P_+ = |+z\rangle\langle+z|$. See that

$$\begin{aligned}|+z\rangle\langle+z|\,|\psi\rangle &= |+z\rangle\langle+z|\,[2|+z\rangle + (1+2i)|-z\rangle]\\ &= |+z\rangle\,[\langle+z|\,2|+z\rangle + \langle+z|\,(1+2i)|-z\rangle]\\ &= |+z\rangle\,[2\langle+z|+z\rangle + (1+2i)\langle+z|-z\rangle]\\ &= |+z\rangle\,[2\cdot 1 + (1+2i)\cdot 0]\\ &= |+z\rangle\,2 = 2|+z\rangle\end{aligned}$$

as desired. We can get a similar similar result for the projection operator associated with $|-z\rangle$, $P_+ = |-z\rangle\langle -z|$. We can generalize this projection operation to any quantum state by replacing $|\pm z\rangle$ with *any quantum state* $|\phi\rangle$

Definition 3.9

The Projection operator mapping $|\psi\rangle = a_1|b_1\rangle + a_2|b_2\rangle + \cdots + a_n|b_n\rangle$ *to* $a_i|b_i\rangle$ *is*

$$P_{b_i} = |b_i\rangle\langle b_i|.$$

In general, the projection of a quantum state $|\psi\rangle$ *onto some vector* $|\phi\rangle$ *is*

$$P_\phi|\psi\rangle = |\phi\rangle\langle\phi|\psi\rangle$$

♣

✍ **Exercise 3.1** Find the matrix representation of P_+ in the S_z basis.

Solution

$$\begin{aligned}
P_+ \underset{S_z}{\longrightarrow} &= \begin{pmatrix} \langle +z|P_+|-z\rangle & \langle +z|P_+|+z\rangle \\ \langle -z|P_+|+z\rangle & \langle -z|P_+|-z\rangle \end{pmatrix} \\
&= \begin{pmatrix} \langle +z|1|+z\rangle & \langle +z|0|-z\rangle \\ \langle -z|1|+z\rangle & \langle -z|0|-z\rangle \end{pmatrix} \\
&= \begin{pmatrix} 0\langle +z|+z\rangle & 1\langle +z|+z\rangle \\ 1\langle -z|+z\rangle & 0\langle -z|-z\rangle \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1\cdot 1 \\ 1\cdot 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

✍ **Exercise 3.2** Find the matrix representation of P_- in the S_z basis.

✍ **Exercise 3.3** Find the matrix representation of P_+ in the S_y basis.

✍ **Exercise 3.4** Find the matrix representation of the projection operator P_0 for a spin-1 particle $|\psi\rangle = c_1|-1\rangle + c_2|0\rangle + c_3|+1\rangle$

Notice the similarity in the way in which we project a vector $\hat{v}$ onto a vector $\hat{n}$

$$\hat{n}(\hat{n}\cdot\hat{v})$$

with the way we project $|\psi\rangle$ onto $|\phi\rangle$

$$|\phi\rangle\langle\phi|\psi\rangle.$$

This is due to the generalized notion of projection of a vector in an inner product space, where the differences are only due to notation. We remember that a vectorspace has associated with it a dual-vector space that contains elements called co-vectors which are linear functionals (mapping vectors to scalars). There is a bijection between these two

spaces, so each vector v has associated with it a co-vector v'. We have shown before that $w'(v)$ is an inner product $\langle w|v\rangle$. In $\mathbb{C}^n$, the dual space is the functionals $w \cdot \_$ and so we can see that an inner product is $w \cdot v$. So projecting v onto w can be generalized as $w \langle w|v\rangle$.

3.4.2 Ladder Operators

Finally, we begin our discussion on *Ladder operators*. We will see that it's useful to have an operator that increases the eigenvalues of a vector by a discrete value. We'll start by introduction a new notation. We write $|n\rangle$ to be the vector with eigenvalue n for an operator N

$$N|n\rangle = n|n\rangle .$$

So for J_z, we can write $|+z\rangle$ as $|\frac{\hbar}{2}\rangle$ because $J_z|+z\rangle = \frac{\hbar}{2}|+z\rangle$.

Now suppose that two operators N and X have the commutation relation

$$[N, X] = cX$$

for some scalar c. Then

$$\begin{aligned} NX|n\rangle &= (XN + [N, X])|n\rangle \\ &= XN|n\rangle + [N, X]|n\rangle \\ &= Xn|n\rangle + cX|n\rangle \\ &= (n + c)X|n\rangle . \end{aligned}$$

This statement says that *by applying X to $|n\rangle$, we get a new vector with an eigenvalue $n + c$.* We can even write this as

$$X|n\rangle = |n + c\rangle .$$

This leads to the definition

Definition 3.10

X is a ladder operator for N if

$$X|n\rangle = |n + c\rangle .$$

If c is positive, we say X is a raising operator and if c is negative X is a lowering operator.

♣

✍ **Exercise 3.5** Show that if X is a raising operator for N then N is a lowering operator for X.

✍ **Exercise 3.6** Show that $J_+ = J_x + iJ_y$ is a raising operator for J_z

3.5 Expectation value

We return to a moment to the expectation value of an operator, $\langle A \rangle$, which was $\sum_{i=0}^{n} a_i P(a_i)$ where a_i are the n possible outcomes of A. We remember that in quantum mechanics, the outcomes of an operator are defined to be the *eigenvalues* of that operator and so we can actually write

$$\langle A \rangle = \sum_{i=0}^{n} \lambda_i P(\lambda_i).$$

Next, the probability of obtaining an eigenvalue λ_i for a quantum state $|\psi\rangle$ was $P(\lambda_i) = |\langle \lambda_i | \psi \rangle|^2$ which gives us the general form for the expectation values

$$\sum_{i=0}^{n} \lambda_i |\langle \lambda_i | \psi \rangle|^2.$$

In this form, we can get a very useful theorem

Theorem 3.12

Let A be an observable on a quantum state $|\psi\rangle$. Then

$$\langle A \rangle = \langle \psi | A | \psi \rangle .$$

♡

Proof http://physics.mq.edu.au/ jcresser/Phys301/Chapters/Chapter14.pdf

$$\begin{aligned}
\langle A \rangle &= \sum_{i=0}^{n} \lambda_i |\langle \lambda_i | \psi \rangle|^2 \\
&= \sum_{i=0}^{n} \lambda_i \langle \psi | \lambda_i \rangle \langle \lambda_i | \psi \rangle \\
&= \sum_{i=0}^{n} \langle \psi | A | \lambda_i \rangle \langle \lambda_i | \psi \rangle \\
&= \langle \psi | A \sum_{i=0}^{n} | \lambda_i \rangle \langle \lambda_i | \psi \rangle \\
&= \langle \psi | A | \psi \rangle .
\end{aligned}$$

We can use this to simplify the calculations of the expectation values.

Example 3.7 Determine $\langle S_z \rangle$for the following states

1. $|\phi\rangle = |+z\rangle$
2. $|\phi\rangle = \frac{1}{2}|+z\rangle + \frac{i\sqrt{3}}{2}|-z\rangle$
3. $|\phi\rangle = |+x\rangle$
4. $|\phi\rangle = \frac{1}{2}|+x\rangle + \frac{i\sqrt{3}}{2}|-x\rangle$

3.6 Outputs of multiple Stern-Gerlach devices

We know that the outputs of a SG device depends on the orientation of it's magnetic components. We will denote the SG device oriented in the $|\hat{n}\rangle$ direction *SGn*, such that $\hat{J}_n |\hat{n}\rangle = \frac{\hbar}{2} |\hat{n}\rangle$. Then the probability of $\hat{J}\cdot n$ attaining $\frac{\hbar}{2}$ for some quantum state $|\hat{n_0}\rangle$ is

$$|\langle \hat{n}|\psi\rangle|^2.$$

Another way of saying this is that $|\psi\rangle$ has a $|\langle \hat{n}|\psi\rangle|^2$ chance of passing through SGn. Therefore if we were to put N_0 particles through SGn, we would expect to have $N_0 \times |\langle \hat{n}|\psi\rangle|^2$ particles pass through. What's more, all of these particles would be in the state $|\hat{n}\rangle$. We can generalize this result to determine the fraction of particles in some state $|\psi\rangle$ that will pass through a line of m SG devices each oriented in the directions $\hat{n_1}$, $\hat{n_2}$, ..., $\hat{n_m}$. Like before, the number of particles that pass through the first device is $N_0 \times |\langle \hat{n_1}|\psi\rangle|^2$ and are all in the state $|\hat{n_1}\rangle$. When these particles pass through the next device, they do so with a probability of $|\langle \hat{n_2}|\hat{n_1}\rangle|^2$ which gives the number of particles passing through to be $N_0 \times |\langle \hat{n_1}|\psi\rangle|^2 \times |\langle \hat{n_2}|\hat{n_1}\rangle|^2$. Continue this pattern, replace $|\psi\rangle$ with $|\hat{n_0}\rangle$ for notational convenience, and arrive at the following result

Theorem 3.13

Suppose N_0 particles in the state $|\hat{n_0}\rangle$ are sent through m Stern-Gerlach devices SGn_1, SGn_2, ... SGn_m. Then the number of particles exiting all the devices is

$$N_0 \prod_{i=0}^{m-1} |\langle n_{i+1}|n_i\rangle|^2.$$

♡

We can simplify this result to another form which requires us only to calculate the inner product rather than the norm

Corollary 3.1

The number of particles exiting the SG devices is

$$N_0 \left| \prod_{i=0}^{m-1} \langle n_{i+1}|n_i\rangle \right|^2 .$$

♡

Proof

$$\begin{aligned}
\prod_{i=0}^{m-1} |\langle n_{i+1}|n_i\rangle|^2 &= \prod_{i=0}^{m-1} \langle n_{i+1}|n_i\rangle \langle n_i|n_{i+1}\rangle \\
&= \prod_{i=0}^{m-1} \langle n_{i+1}|n_i\rangle \overline{\langle n_{i+1}|n_i\rangle} \\
&= \prod_{i=0}^{m-1} \langle n_{i+1}|n_i\rangle \overline{\prod_{i=0}^{m-1} \langle n_{i+1}|n_i\rangle} \\
&= \left| \prod_{i=0}^{m-1} \langle n_{i+1}|n_i\rangle \right|^2 .
\end{aligned}$$

3.7 Polarization of Light

We can use this quantum mechanical method to look at the polarization of light, but note that *polarization states of light are not spin-$\frac{1}{2}$ particles*. In fact, they are spin-1 particles. Consider a beam for photos travelling in the z direction. We can define light to be in the $|x\rangle$ state when it passes through a polarized aligned in the x direction and define $|y\rangle$ to be the state after passing a polarizer aligned in the y direction. We then know that $|\langle x|y\rangle|^2 = 0$ and farther more define $\langle x|x\rangle = 1$ and $\langle y|y\rangle = 1$. One of the important questions we can ask regarding the polarization of light is what happens when we rotate the beam $|\psi\rangle = c_1 |x\rangle + c_2 |y\rangle$ along the z-axis into a new state $|\psi'\rangle = a_1 |x'\rangle + a_2 |y'\rangle$. This can be done just by using the rotation matrix in 2D space

$$\begin{pmatrix} |x'\rangle \\ |y'\rangle \end{pmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{pmatrix} |x\rangle \\ |y\rangle \end{pmatrix}$$

which rotates a 2D vector θ radians in the counter clockwise direction. Note, we cannot use this matrix for spin-$\frac{1}{2}$ particles.

> Although one might expect $\hat{R}(\hat{n}, 360°) = 1$ (a rotation of 360° is the identity operator), this is not assumed in quantum mechanics, and it turns out it is often not true: When the total angular momentum quantum number is a half-integer (1/2, 3/2, etc.), $\hat{R}(\hat{n}, 360°) = -1$, and when it is an integer, $R(\hat{n}, 360°) = +1$. Mathematically, the structure of rotations in the universe is not SO(3), the group of three-dimensional rotations in classical mechanics. Instead, it is SU(2), which is identical to SO(3) for small rotations, but where a 360° rotation is mathematically distinguished from a rotation of 0°. (A

rotation of 720° is, however, the same as a rotation of 0°.) -Wikipedia

Another cool polarization of light is *circular polarization states.* These are not simple rotations about the z-axis, but offsetting *the period of the amplitudes of the 2 electromagnetic components of the photon.* Consider the photon described by

$$|\psi\rangle = E_0 e^{\theta i z} + i E_0 e^{\phi i z}.$$

We take the real part of this equation to describe the true physical state of the photon. We see that θ and ϕ determine the phase of both waves. Remember, multiplying by i rotates a complex number $\frac{\pi}{2}$ radians counter clockwise and so if $\theta = \phi$ then the 2 functions are offset by $90°$.

✍ **Exercise 3.7** Prove it.

Consider the function $e^{i\theta}$ from the reals to the complex plane and how it progresses along θ. An important property of this function is that *rotation along the θ axis is the same as translation along the θ axis.* A picture of this is quite helpful. You can also see this in Archimedes's screw, which uses rotation of a spiral to produce translation to move water. We can rotate each of the components of this photon using complex numbers to then offset the phase. This rotation is done by the explained as

$$\begin{pmatrix} |R\rangle \\ |L\rangle \end{pmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{bmatrix} \begin{pmatrix} |x\rangle \\ |y\rangle \end{pmatrix}$$

where $|R\rangle$ and $|L\rangle$ are *right and left circularly polarized* light respectively. If we rotate circularly polarized light into a new state $\begin{pmatrix} |R'\rangle \\ |L'\rangle \end{pmatrix}$ then we can see

$$\begin{aligned}
\begin{pmatrix} |R'\rangle \\ |L'\rangle \end{pmatrix} &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{bmatrix} \begin{pmatrix} |x'\rangle \\ |y'\rangle \end{pmatrix} \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{pmatrix} |x\rangle \\ |y\rangle \end{pmatrix} \\
&= \begin{bmatrix} \frac{\cos(\theta) - i\sin(\theta)}{\sqrt{2}} & \frac{\sin(\theta) + i\cos(\theta)}{\sqrt{2}} \\ \frac{\cos(\theta) + i\sin(\theta)}{\sqrt{2}} & \frac{\sin(\theta) - i\cos(\theta)}{\sqrt{2}} \end{bmatrix} \begin{pmatrix} |x\rangle \\ |y\rangle \end{pmatrix} \\
&= \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{bmatrix} \begin{pmatrix} |x\rangle \\ |y\rangle \end{pmatrix} \\
&= \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \begin{pmatrix} |R\rangle \\ |L\rangle \end{pmatrix}.
\end{aligned}$$

Note that if we write this as the rotation operator we get

$$|R'\rangle = \hat{R}(\theta k)\,|R\rangle = e^{-i\hat{J}_z\hbar}\,|R\rangle$$
$$|L'\rangle = \hat{L}(\theta k)\,|L\rangle = e^{i\hat{J}_z\hbar}\,|L\rangle$$

so we need $e^{-i\hat{J}_z\hbar}\,|R\rangle = e^{i\theta}\,|R\rangle$ or that

$$J_z\,|R\rangle = \hbar\,|R\rangle\,.$$

We use the same reasoning to get

$$J_z\,|L\rangle = -\hbar\,|L\rangle\,.$$

✍ **Exercise 3.8** Find the matrix representation of J_z for $|R\rangle\,, |L\rangle$ and $|x\rangle\,, |y\rangle\,.$

Chapter 4 Angular Momentum

4.1 The 3D Rotation Group SO(3)

Townsend dedicates the first topic of Angular Momentum to discussing the commutation of rotations and generators. In this chapter, he discusses many aspects of *Lie algebras*, *Lie groups* and properties of a special *group* called SO(3) without referring to them by name. For my own interest, I will be discussing in terms of these mathematical objects while still covering the same conclusions. I will assume familiarity with basic groups.

4.1.1 SO(3)

Definition 4.1

The 3D Rotation Group, SO(3), or special orthogonal group of order 3 is the group of all transformations about the origin of $\mathbb{R}^3$ *that*

- *Preserves the origin*
- *Euclidean distance*
- *Orientation*

under the operation of composition. ♣

One very important property of SO(3) is that it is *not abelian*; that is, any two elements of SO(3) are not guarantee to commute. One can see this by performing a 90° rotation counter-clockwise around the positive x-axis and then another 90° rotation about the positive z-axis. Doing so in the opposite order will give a different result. You can see this by drawing a 3D coordinate axes and applying each transformation one at a time.

I hope you will agree that this group is composed of linear transformations, and therefore has matrix representations for some choice of basis in $\mathbb{R}^3$. If we choose an *orthonormal basis* then each rotation can be described by an *orthogonal matrix* with determinant 1. Remember this can be characterized by

$$MM^T = \mathbb{I}.$$

This leads to an equivalent definition of SO(3)

Definition 4.2

SO(3) is the group of all orthogonal 3×3 matrices with determinant 1 under matrix multiplication.

♣

Actually, you will see similar notation this is a subgroup of a some other important families of groups.

Definition 4.3

The Orthogonal group in dimension n, denoted O(n), is the group of $n \times n$ orthogonal matrices under matrix multiplication.

- $MM^T = \mathbb{I}$

♣

and the generalized group of SO(3)

Definition 4.4

The special orthogonal group of order n, denoted SO(n), is the group of all $n \times n$ orthogonal matrices with determinant 1 under matrix multiplication.

- $MM^T = \mathbb{I}$
- $\det(M) = 1$

♣

You can check that the second condition preserves the euclidean distance. These groups have a lot of topological properties and if you look at the wikipedia page a lot of research has been done on this kind of stuff, mostly due to their use in physics. Going forward, we introduce a theorem due to Euler

Theorem 4.1 (Euler's Rotation Theorem)

Every 3 dimensional rotation can be described by an axis of rotation $\hat{n}$ together with an angle of rotation ϕ counter-clockwise about $\hat{n}$. We write this transformation

$$\hat{R}(\phi, \hat{n}).$$

♡

This is actually a very useful theorem. Right away, we know

- $\hat{R}(0, \hat{n})$ is the identity
- $\hat{R}(\phi, \hat{n}) = \hat{R}(-\phi, -\hat{n})$
- $\hat{R}(\phi + \pi, \hat{n}) = \hat{R}(\pi - \phi, -\hat{n})$

which can be used to show

- $\hat{n}$ is arbituary if $\phi = 0$
- $\hat{n}$ is unique if $0 < \phi < \pi$
- $\hat{R}(\pi, \hat{n}) = \hat{R}(\pi, -\hat{n})$.

✍ **Exercise 4.1** Prove it.

As mentioned before, SO(3) is a *Lie group* which is a group with some idea of continuity.

Definition 4.5 (Lie Group)

A Lie group is a group that is locally similar to $\mathbb{R}^n$ where the operation and it's inverse are smooth. ♣

In essence, Lie groups are "smooth" versions of groups where we are able to do calculus. We also talk about Lie algebra's, that have a neat connection

Definition 4.6 (Lie Algebra)

A Lie Algebra is a vector space $\mathfrak{g}$ over a field $\mathbb{F}$ with a binary operator $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket that satisfies the following

1. $[ax + by, z] = a[x, z] + b[y, z]$
   $[z, ax + by] = a[z, x] + b[z, y]$
2. $[x, x] = 0$
3. $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0.$

♣

If you remember, the *commutator* in the previous chapter satisfies all these properties and we will see constitutes a Lie algebra for SO(3) which we denote $\mathfrak{so}(3)$. Actually, it is a very important theorem that every Lie group is associated with a Lie algebra and vice-verse. This is *Lie's third theorem* but we won't prove it. The importance of this theorem is how we think of Lie algebra's in an informal sense. First we give a incomplete definition of the tangent space. For a sphere, we can imagine a plane that touches the sphere at a single point p and so we say the plane is *tangent* to the sphere. This plane is a 2D vectorspace and contains all the possible directions in which we can tangently pass through the point p. If we are comfortable with this, then we can get a nice theorem

Theorem 4.2 (Lie group- Lie algebra correspondance)

To every Lie group we can associate a Lie algebra whos vectorspace is the tangent space of the Lie group at the identity element. In the case of invertible matrices, the Lie bracket is given by the commutator. ♡

One way to think of this is that the elements of this Lie algebra are elements of the group that infinitesimally close to the identity and the Lie bracket, which working with invertible matrices, is just how those infinitesimal elements commute. When we look back, we remembered that we found

$$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z$$
$$[\hat{J}_y, \hat{J}_z] = i\hbar\hat{J}_x$$
$$[\hat{J}_z, \hat{J}_x] = i\hbar\hat{J}_y.$$

I wanted to show how this shows that the span of $\hat{J}_x$, $\hat{J}_y$, $\hat{J}_z$, forms a Lie algebra, but I

think all the proofs are inaccessible to me. Either way, this Lie algebra is isomorphic to SO(3) and it can be shown that any operators satisfying those commutation relations is isomorphic to SO(3). If we consider the matrix representation of $\mathfrak{so}(3)$, the largest eigenvalue of $\hat{J}_z$ is m where m can be $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. It then follows that

$$J^2 = \hat{J}_z^2 + \hat{J}_y^2 + \hat{J}_x^2 = m(m+1)\hbar^2 I.$$

For some value m, the matrix representation of $\hat{J}_z$ is $(2m+1)$-dimensional.

4.2 Commuting Operators

We start with a theorem that will extend our notation of $|n\rangle$ having the eigenvalue n

Theorem 4.3

Suppose $[A, B] = 0$ where $|a\rangle$ is the only eigenstate of A. Then $|a\rangle$ is also an eigenstate of B with it's own eigenvalue b. It follows that this state can be written

$$A|a,b\rangle = a|a,b\rangle$$
$$B|a,b\rangle = b|a,b\rangle.$$

♡

Proof We start by showing that if A and B commute, then $B|a\rangle$ is an eigenvector of A

$$A|a\rangle = a|a\rangle$$
$$BA|a\rangle = Ba|a\rangle$$
$$AB|a\rangle = aB|a\rangle$$
$$A(B|a\rangle) = a(B|a\rangle).$$

Now using the fact that $|a\rangle$ is the *unique* eigenstate of A, we must have that $B|a\rangle$ is a scalar multiple of $|a\rangle$. In other words,

$$B|a\rangle = b|a\rangle$$

for some b. This b is the eigenvalue of B and so

$$A|a,b\rangle = a|a,b\rangle$$
$$B|a,b\rangle = b|a,b\rangle$$

as required. Actually, the fact that two commuting Hermitian operators have a complete set of eigenstates in common is called the Spectral theorem. This is quite an important theorem in linear algebra, but what we have proven is enough for our purposes.

4.3 The Eigenvalues and Eigenstates of Angular Momentum

Definition 4.7

The squared total angular momentum is

$$\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$$

♣

Theorem 4.4

$\hat{J}^2$ commutes with $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$.

♡

Proof We will show the proof with $\hat{J}_z$, but the same method is applied for all three for the desired result

$$\begin{aligned}
[\hat{J}_z, \hat{J}^2] &= [\hat{J}_z, \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2] \\
&= [\hat{J}_z, \hat{J}_x^2] + [\hat{J}_z, \hat{J}_y^2] + [\hat{J}_z, \hat{J}_z^2] \\
&= 0 + [\hat{J}_z, \hat{J}_x]\hat{J}_x + \hat{J}_x[\hat{J}_z, \hat{J}_x] + [\hat{J}_z, \hat{J}_y]\hat{J}_y + \hat{J}_y[\hat{J}_z, \hat{J}_y] \\
&= i\hbar\hat{J}_y\hat{J}_x + i\hbar\hat{J}_x\hat{J}_y - i\hbar\hat{J}_x\hat{J}_y - i\hbar\hat{J}_y\hat{J}_x \\
&= i\hbar[\hat{J}_y\hat{J}_x + \hat{J}_x\hat{J}_y - \hat{J}_x\hat{J}_y - \hat{J}_y\hat{J}_x] \\
&= i\hbar \cdot 0 = 0.
\end{aligned}$$

as required.

Note that this also implies $[\hat{J}^2, \hat{J}_z] = 0$. So, we have that the two operators commute. By our previous theorem, they must share a set of eigenvalues. We let the eigenvalue of $\hat{J}$ be λ and the eigenvalue of $\hat{J}_z$ be m. So we have $\hat{J}\,|\lambda, m\rangle = \lambda\,|\lambda, m\rangle$, so

$$\hat{J}^2\,|\lambda, m\rangle = \lambda\hbar^2\,|\lambda, m\rangle$$
$$\hat{J}_z\,|\lambda, m\rangle = m\hbar\,|\lambda, m\rangle$$

where $\hbar$ was added to give the eigenvalues the dimension of $\hbar$.

Theorem 4.5

$$\hat{J}_x \xrightarrow[S_z]{} \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \hat{J}_y \xrightarrow[S_z]{} \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \hat{J}_z \xrightarrow[S_z]{} \hbar\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

♡

Corollary 4.1

$$\hat{J}^2 = 2\hbar^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

♡

Check that $\hat{J}_z$ has the eigenstates and values

$$\hat{J}_z \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \hbar \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\hat{J}_z \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\hat{J}_z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -\hbar \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

By our theorem of shared eigenvalues, $\hat{J}_z$ and $\hat{J}^2$ both share eigenvectors which we leave to you to verify. What's more, if we define

Definition 4.8

$$\hat{J}_+ = \hat{J}_x - i\hat{J}_y = \sqrt{2}\hbar \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\hat{J}_- = \hat{J}_x + i\hat{J}_y = \sqrt{2}\hbar \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

♣

we can find that they constitute *ladder operators* for $\hat{J}_z$. See that the eigenvalue of e_3 is $-\hbar$, but then

$$J_+ e_3 = \sqrt{2}\hbar e_2$$

$$J_+ e_2 = \sqrt{2}\hbar e_1$$

$$J_+ e_1 = \vec{0}$$

which is what we would expect. Notice that this operator is *not* Hermitian

$$\hat{J}_+^\dagger = \sqrt{2}\hbar \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^\dagger = \sqrt{2}\hbar \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \hat{J}_-.$$

Another useful thing about these ladder operators is their commutation relations with $\hat{J}_z$

Theorem 4.6

$$[\hat{J}_z, \hat{J}_\pm] = \pm\hbar\hat{J}_\pm$$

♡

Proof

$$[\hat{J}_z, \hat{J}_\pm] = [\hat{J}_z, \hat{J}_x \pm i\hat{J}_y] = i\hbar\hat{J}_y \pm i(-i\hbar\hat{J}_x) = \pm\hbar\hat{J}_\pm$$

Using the commutation relation and applying the same method from chapter 2, we can see

$$\begin{aligned} \hat{J}_z\hat{J}_+ |\lambda, m\rangle &= \left(\hat{J}_+\hat{J}_z + \hbar\hat{J}_+\right)|\lambda, m\rangle \\ &= \left(\hat{J}_+ m\hbar + \hbar\hat{J}_+\right)|\lambda, m\rangle \\ &= (m+1)\hbar\hat{J}_+ |\lambda, m\rangle \end{aligned}$$

Theorem 4.7

$$[\hat{J}_\pm, \hat{J}^2] = 0$$

♡

Proof proof

Since $\hat{J}_\pm$ and $\hat{J}^2$ commute, it follows that $\hat{J}_\pm$ is still an eigenstate of $\hat{J}^2$. An important property of this raising operator is that it "hits a limit" once the ladder operator is applied to the eigenket with the largest eigenvalue, returning zero. We can actually find this limit for m interms of the eigenvalue λ, by considering the following:

$$\langle\lambda, m| \hat{J}_x^2 + \hat{J}_y^2 |\lambda, m\rangle \geq 0$$

which is true because $|\lambda, m\rangle \neq |0\rangle$. Then, using $\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$, we get

$$\begin{aligned} \langle\lambda, m| \hat{J}_x^2 + \hat{J}_y^2 |\lambda, m\rangle &= \langle\lambda, m| \hat{J}^2 - \hat{J}_z^2 |\lambda, m\rangle \\ &= \langle\lambda, m| \left(\hat{J}^2 |\lambda, m\rangle - \hat{J}_z^2 |\lambda, m\rangle\right) \\ &= \langle\lambda, m| \left(\lambda\hbar |\lambda, m\rangle - m^2\hbar^2 |\lambda, m\rangle\right) \\ &= \langle\lambda, m| (\lambda - m^2)\hbar^2 |\lambda, m\rangle \\ &= (\lambda - m^2) \underbrace{\hbar^2}_{\geq 0} \underbrace{\langle\lambda, m|\lambda, m\rangle}_{\geq 0} \geq 0 \end{aligned}$$

and so $\lambda - m^2 \geq 0$ or

$$m^2 \leq \lambda.$$

Now to find this maximum value of m, we suppose that j is the maximum value

$$\hat{J}_+ |\lambda, j\rangle = 0.$$

We can use this identity

$$\hat{J}_- \hat{J}_+ = \hat{J}^2 - \hat{J}_z^2 - \hbar \hat{J}_z,$$

which follows from the definition of $\hat{J}_\pm$, to see that

$$\begin{aligned} \hat{J}_i \hat{J}_+ |\lambda, j\rangle &= \left(\hat{J}^2 - \hat{J}_z^2 - \hbar \hat{J}_z\right) |\lambda, j\rangle \\ &= (\lambda - j^2 - j) \underbrace{\hbar^2}_{\neq 0} \underbrace{|\lambda.j\rangle}_{\neq 0} = 0 \end{aligned}$$

and so it follows that $\lambda - j^2 - j = 0$ or $\lambda = j(j+1)$. We can use similar reasoning and proof to get the *minimal value of* m, j', to be

$$\lambda = j'(j' - 1).$$

If we equate these two results, $\lambda = j'(j'-1)$ and $\lambda = j(j+1)$, we see

$$j'(j'-1) = j(j+1)$$

implying that $j' = -j$ or $j' = j + 1$. We said that j' is the minimum value of m, and so we disregard the second solution and take the first $j' = j$. What we have found is that the upper bound of this raising operator is j, and it's lower bound is $-j$ and therefore we may apply $\hat{J}_z$ to the minimum eigenket $j - (-j) = 2j$ times. The improtant observation here is that $\hat{J}_+$ is applied an integer number of times, so j must then be either an integer or a half integer

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots.$$

Finally, this tells us that the m values for each j run from j to $-j$ in integral steps

$$m = \underbrace{-j, -j+1, -j+2, \ldots, j-2, j-1, j}_{2j+1 \text{ states}}.$$

We can see that given a value of j, the possible values of m are

$$\begin{aligned} j &= 0, \quad m = 0 \\ j &= \frac{1}{2}, \quad m = \frac{-1}{2}, \frac{1}{2} \\ j &= 1, \quad m = -1, 0, 1 \\ j &= 2, \quad m = \frac{-3}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{3}{2}. \end{aligned}$$

Since we have this nice relationship between j and m, we are going to *switch notation*

to $|j,m\rangle$. Note that now we have

$$\hat{J}^2 |j,m\rangle = j(j+1)\hbar^2 |j,m\rangle$$
$$\hat{J}_z |j,m\rangle = m\hbar |j,m\rangle .$$

Our first example using this new notation is the spin-$\frac{1}{2}$ particle. The eigenstates are then written $|\frac{1}{2},\frac{1}{2}\rangle$ and $|\frac{1}{2},\frac{-1}{2}\rangle$. It then follows that the magnitude squared of the spin of the particle is

$$\hat{J}^2 |\frac{1}{2},\frac{1}{2}\rangle = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 |\frac{1}{2},\frac{1}{2}\rangle = \frac{3\hbar^2}{4} |\frac{1}{2},\frac{1}{2}\rangle$$

so the magnitude is $\frac{\sqrt{3}\hbar}{2}$. Also, noting that

$$\hat{J}_z |\frac{1}{2},\frac{1}{2}\rangle = \frac{\hbar}{2} |\frac{1}{2},\frac{1}{2}\rangle$$
$$\hat{J}_z |\frac{1}{2},\frac{-1}{2}\rangle = \frac{-\hbar}{2} |\frac{1}{2},\frac{-1}{2}\rangle$$

and so these states are actually the spin up and down states $|\pm z\rangle$ that we identified in previous chapters.

Now, we are also interested in the coeffecients in

$$\hat{J}_+ |j,m\rangle = c_+ |j,m+1\rangle$$
$$\hat{J}_- |j,m\rangle = c_- |j,m-1\rangle .$$

Theorem 4.8

$$\hat{J}_+ |j,m\rangle = \sqrt{j(j+1)-m(m+1)}\hbar |j,m+1\rangle$$
$$\hat{J}_- |j,m\rangle = \sqrt{j(j+1)-m(m-1)}\hbar |j,m-1\rangle .$$

♡

Proof We wish to solve for c_+ and c_- in

$$\hat{J}_+ |j,m\rangle = c_+\hbar |j,m+1\rangle$$
$$\hat{J}_- |j,m\rangle = c_-\hbar |j,m-1\rangle .$$

We first solve for c_+ by considered $\langle j,m| \hat{J}_-\hat{J}_+ |j,m\rangle$

$$\begin{aligned}\langle j,m| \hat{J}_-\hat{J}_+ |j,m\rangle &= \langle j,m| \hat{J}_+^\dagger\hat{J}_+ |j,m\rangle \\ &= \langle j,m| \hat{J}_+^\dagger c_+\hbar |j,m+1\rangle \\ &= \langle j,m+1| c_+^* c_+\hbar^2 |j,m+1\rangle \\ &= c_+^* c_+ \langle j,m+1|j,m+1\rangle .\end{aligned}$$

Next, using the identity $\hat{J}_-\hat{J}_+ = \hat{J}^2 - \hat{J}_z^2 - \hbar\hat{J}_z$ we get

$$\begin{aligned}\langle j,m|\,\hat{J}_-\hat{J}_+\,|j,m\rangle &= \langle j,m|\,\hat{J}^2 - \hat{J}_z^2 - \hbar\hat{J}_z\,|j,m\rangle \\ &= \langle j,m|\,[j(j+1) - m^2 - m]\hbar^2\,|j,m\rangle \\ &= [j(j+1) - m^2 - m]\hbar^2\,\langle j,m|j,m\rangle \\ &= [j(j+1) - m(m+1)]\hbar^2\,\langle j,m|j,m\rangle\end{aligned}$$

and so $\langle j,m|\,\hat{J}_-\hat{J}_+\,|j,m\rangle = c_+^*c_+\langle j,m+1|j,m+1\rangle = [j(j+1)-m^2-m]\hbar^2\langle j,m|j,m\rangle$. Since $\langle j,m|j,m\rangle = \langle j,m+1|j,m+1\rangle$, we must have that

$$c_+^*c_+ = j(j+1) - m(m+1)$$

which implies

$$c_+ = \sqrt{j(j+1) - m(m+1)}$$

as required. The proof is the same for c_-.

Note that if $m = j$, then the coeffecient c_+ vanishes for $\hat{J}_+$, and the same happens for $m = -j$ and $\hat{J}_-$. Using this result, we can actually know that these eigenstates $|j,m\rangle$ form a basis (not necessarily orthogonal)

Theorem 4.9

The eigenstates

$$|j,-j\rangle\,, |j,-j+1\rangle\,, \ldots, |j,j\rangle$$

form a basis ♡

Proof Consider

$$\langle j,m'|\,\hat{J}_+\,|j,m\rangle$$

then

$$\begin{aligned}\langle j,m'|\,\hat{J}_+\,|j,m\rangle &= \langle j,m'|\,\sqrt{j(j+1) - m(m+1)}\hbar\,|j,m+1\rangle \\ &= \sqrt{j(j+1) - m(m+1)}\hbar\,\langle j,m'|j,m+1\rangle\,.\end{aligned}$$

Now if $m' \neq m+1$, then $\langle j,m'|j,m+1\rangle = 0$. If $m' = m+1$ then $\langle j,m'|j,m+1\rangle = 1$, thus the eigenstates for a basis.

Using this basis, we can find the matrix representation of $\hat{J}_+$ and $\hat{J}_-$. Here we will start replace the notation S_z basis with J_z. Remember that S_z was the basis $|+z\rangle$ and $|-z\rangle$, which we have shown was the same as $|\frac{1}{2},-\frac{1}{2}\rangle$ and $|\frac{1}{2},\frac{1}{2}\rangle$ and so they are the same basis.

Theorem 4.10

The matrix representation of $\hat{J}_+$ and $\hat{J}_-$ in the spin-1 basis is

$$\hat{J}_+ \underset{J_z}{\longrightarrow} \sqrt{2}\hbar \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\hat{J}_- \underset{J_z}{\longrightarrow} \sqrt{2}\hbar \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

♡

Proof By direct calculation

$$\begin{aligned}
\hat{J}_+ \underset{J_z}{\longrightarrow} &\begin{bmatrix} \langle 1,1|\hat{J}_+|1,1\rangle & \langle 1,1|\hat{J}_+|1,0\rangle & \langle 1,1|\hat{J}_+|1,-1\rangle \\ \langle 1,0|\hat{J}_+|1,1\rangle & \langle 1,0|\hat{J}_+|1,0\rangle & \langle 1,0|\hat{J}_+|1,-1\rangle \\ \langle 1,-1|\hat{J}_+|1,1\rangle & \langle 1,-1|\hat{J}_+|1,0\rangle & \langle 1,-1|\hat{J}_+|1,-1\rangle \end{bmatrix} \\
= &\begin{bmatrix} \langle 1,1|0|0\rangle & \langle 1,1|\sqrt{2-(0)(0+1)}\hbar|1,1\rangle & \langle 1,1|\sqrt{2-(-1)(-1+1)}\hbar|1,0\rangle \\ \langle 1,0|0|0\rangle & \langle 1,0|\sqrt{2-(0)(0+1)}\hbar|1,1\rangle & \langle 1,0|\sqrt{2-(-1)(-1+1)}\hbar|1,0\rangle \\ \langle 1,-1|0|0\rangle & \langle 1,-1|\sqrt{2-(0)(0+1)}\hbar|1,1\rangle & \langle 1,-1|\sqrt{2-(-1)(-1+1)}\hbar|1,0\rangle \end{bmatrix} \\
= &\begin{bmatrix} 0 & \sqrt{2}\hbar\langle 1,1|1,1\rangle & \sqrt{2}\hbar\langle 1,1|1,0\rangle \\ 0 & \sqrt{2}\hbar\langle 1,0|1,1\rangle & \sqrt{2}\hbar\langle 1,0|1,0\rangle \\ 0 & \sqrt{2}\hbar\langle 1,-1|1,1\rangle & \sqrt{2}\hbar\langle 1,-1|1,0\rangle \end{bmatrix} \\
= &\begin{bmatrix} 0 & \sqrt{2}\hbar\cdot 1 & \sqrt{2}\hbar\cdot 0 \\ 0 & \sqrt{2}\hbar\cdot 0 & \sqrt{2}\hbar\cdot 1 \\ 0 & \sqrt{2}\hbar\cdot 0 & \sqrt{2}\hbar\cdot 0 \end{bmatrix} \\
= &\sqrt{2}\hbar \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned}$$

Then using $\hat{J}_- = \hat{J}_+^\dagger$ we see that

$$\hat{J}_- \underset{J_z}{\longrightarrow} \sqrt{2}\hbar \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^\dagger = \sqrt{2}\hbar \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

which completes the proof.

4.4 Uncertainty Principles

We begin by recalling a statement we made before about the angular momentum: *the angular momentum of a particle does not point in a definite direction.* What this means is that we cannot know the values of the 3 components of angular momentum; we are limited to knowing their their standard deviation and expected values. It turns out that this limit is

$$\Delta \hat{J}_x \Delta \hat{J}_y = \frac{\hbar}{2} |\langle \hat{J}_z \rangle|$$

and it follows directly from the commutator relation

$$[\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z.$$

To prove this, we will show that this actually is a special case of a more general theorem called the *Rovertson undertainty relation*

Theorem 4.11 (Robertson Uncertainty Relation)

Let A, B, and C be Hermitian operators satisfying

$$[A, B] = iC$$

then

$$\Delta A \Delta B \geq \frac{|\langle C \rangle|}{2}.$$

♡

Proof Let

$$|a\rangle = (A - \langle A \rangle) |\psi\rangle$$
$$|b\rangle = (B - \langle B \rangle) |\psi\rangle$$

where

$$\langle A \rangle = \langle \psi | A | \psi \rangle$$
$$\langle B \rangle = \langle \psi | B | \psi \rangle .$$

It follows that

$$\langle a|a\rangle = \langle \psi | (A - \langle A \rangle)^2 | \psi \rangle = (\Delta A)^2$$
$$\langle b|b\rangle = \langle \psi | (B - \langle B \rangle)^2 | \psi \rangle = (\Delta B)^2$$
$$\langle a|b\rangle = \langle \psi | (A - \langle A \rangle)(B - \langle B \rangle) | \psi \rangle .$$

We can simplify $\langle a|b\rangle$ into

$$\begin{aligned}
\langle a|b\rangle &= \langle\psi|\left(A-\langle A\rangle\right)\left(B-\langle B\rangle\right)|\psi\rangle \\
&= \langle\psi|\, AB - A\langle B\rangle - \langle A\rangle B + \langle A\rangle\langle B\rangle\,|\psi\rangle \\
&= \langle\psi|\, AB\,|\psi\rangle - \langle\psi|\, A\langle B\rangle\,|\psi\rangle - \langle\psi|\,\langle A\rangle B\,|\psi\rangle + \langle\psi|\,\langle A\rangle\langle B\rangle\,|\psi\rangle \\
&= \langle\psi|\, AB\,|\psi\rangle - \langle B\rangle\,\langle\psi|\, A\,|\psi\rangle - \langle A\rangle\,\langle\psi|\, B\,|\psi\rangle + \langle A\rangle\langle B\rangle\,\langle\psi|\psi\rangle \\
&= \langle AB\rangle - \langle B\rangle\langle A\rangle - \langle A\rangle\langle B\rangle + \langle A\rangle\langle B\rangle \\
&= \langle AB\rangle - \langle B\rangle\langle A\rangle \\
&= \langle BA + iC\rangle - \langle B\rangle\langle A\rangle \\
&= \langle BA\rangle + i\langle C\rangle - \langle B\rangle\langle A\rangle \\
&= \langle BA\rangle - \langle B\rangle\langle A\rangle + i\langle C\rangle
\end{aligned}$$

and so

$$|\langle a|b\rangle|^2 = \left(\langle BA\rangle - \langle B\rangle\langle A\rangle\right)^2 + \langle C\rangle^2 \geq \frac{\langle C\rangle^2}{4}.$$

Using then Cauchy-Schwarz

$$\langle a|a\rangle\,\langle b|b\rangle \geq |\langle a|b\rangle|^2$$

we get

$$(\Delta A)^2(\Delta B)^2 \geq |\langle a|b\rangle|^2 \geq \frac{\langle C\rangle^2}{4}$$

or just

$$\Delta A \Delta B \geq \frac{|\langle C\rangle|}{2}$$

as required.

Apply this to $[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z$ we get

Theorem 4.12

$$\Delta\hat{J}_x\Delta\hat{J}_y \geq \frac{\hbar}{2}|\langle\hat{J}_z\rangle|.$$

♡

Another important consequence of this theorem is the Heisenberg uncertainty principle, which follows from that fact that $[\hat{x}, \hat{p}_x] = i\hbar$ where $\hat{x}$ is the position operator and $\hat{p}_x$ is the momentum operator. Actually, the Robertson uncertainty relation is a special case of an even stronger inequality, the *Schrödinger uncertainty relation*

Theorem 4.13 (The Schrödinger Uncertainty Relation)

Let A and B be Hermitian operators with standard deviations ΔA and ΔB respectively. Then

$$\Delta A \Delta B \geq |\frac{1}{2}\langle\{A, B\}\rangle - \langle A\rangle\langle B\rangle|^2 + |\frac{1}{2i}\langle[A, B]\rangle|^2$$

where $\{A, B\}$ *is the anticommutator* $\{A, B\} = AB + BA$ ♡

Proof As before, let

$$|a\rangle = (A - \langle A\rangle)\,|\psi\rangle$$
$$|b\rangle = (B - \langle B\rangle)\,|\psi\rangle$$

then

$$\langle a|a\rangle = (\Delta A)^2$$
$$\langle b|b\rangle = (\Delta B)^2.$$

We will once again use the Cauchy-Schwarz inequality to get the desired result, requiring us to look at $\langle a|b\rangle$. First, we note that

$$|z|^2 = \Re(z)^2 + \Im(z)^2 = \left(\frac{z + z^*}{2}\right)^2 + \left(\frac{z - z^*}{2i}\right)^2$$

and so

$$|\langle a|b\rangle|^2 = \left(\frac{\langle a|b\rangle + \langle b|a\rangle}{2}\right)^2 + \left(\frac{\langle a|b\rangle - \langle b|a\rangle}{2i}\right)^2.$$

We showed before that

$$\langle a|b\rangle = \langle AB\rangle - \langle A\rangle\langle B\rangle$$

and similarly we can also show that

$$\langle b|a\rangle = \langle BA\rangle - \langle A\rangle\langle B\rangle.$$

Substituing this into $|\langle a|b\rangle|^2$

$$\begin{aligned}
|\langle a|b\rangle|^2 &= \left(\frac{\langle a|b\rangle + \langle b|a\rangle}{2}\right)^2 + \left(\frac{\langle a|b\rangle - \langle b|a\rangle}{2i}\right)^2 \\
&= \left(\frac{\langle AB\rangle - \langle A\rangle\langle B\rangle + \langle BA\rangle - \langle A\rangle\langle B\rangle}{2}\right)^2 + \left(\frac{\langle AB\rangle - \langle A\rangle\langle B\rangle - (\langle BA\rangle - \langle A\rangle\langle B\rangle)}{2i}\right)^2 \\
&= \left(\frac{\langle AB\rangle + \langle BA\rangle - 2\langle A\rangle\langle B\rangle}{2}\right)^2 + \left(\frac{\langle AB\rangle - \langle A\rangle\langle B\rangle - \langle BA\rangle + \langle A\rangle\langle B\rangle}{2i}\right)^2 \\
&= \left(\frac{\langle AB\rangle + \langle BA\rangle - 2\langle A\rangle\langle B\rangle}{2}\right)^2 + \left(\frac{\langle AB\rangle - \langle BA\rangle}{2i}\right)^2 \\
&= \left(\frac{\{A, B\} - 2\langle A\rangle\langle B\rangle}{2}\right)^2 + \left(\frac{\langle [A, B]\rangle}{2i}\right)^2
\end{aligned}$$

which, by Cauchy-Schwarz, implies

$$(\Delta A)^2(\Delta B)^2 \geq \left(\frac{\{A, B\} - 2\langle A\rangle\langle B\rangle}{2}\right)^2 + \left(\frac{\langle [A, B]\rangle}{2i}\right)^2$$

or

$$\Delta A \Delta B \geq \sqrt{\left(\frac{\{A,B\} - 2\langle A\rangle\langle B\rangle}{2}\right)^2 + \left(\frac{\langle [A,B]\rangle}{2i}\right)^2}.$$

An important distinction to make about this relation is that $B\,|\psi\rangle$ must be in the *domain of the unbounded operator A and vice versa.*

Definition 4.9

An operator A is bounded if there exists some C such that

$$\langle A\psi|A\psi\rangle^2 \leq C\,\langle\psi|\psi\rangle^2$$

for all $|\psi\rangle$.

♣

If this condition does not hold, we are actually able to produce some counter examples to the Schrödinger uncertainty relation.

Example 4.1 Counter example Imagine a particle in the plane that travels along a unit circle, repeating every 2π radians. This particle has the wave function $\psi(\theta)$ where $\theta \in [0, 2\pi]$. Let us define the two operators A and B as

$$A\psi(\theta) = \theta\psi(\theta)$$

and

$$B\psi(\theta) = -i\hbar\frac{d}{d\theta}\psi(\theta).$$

where *the domain of B is all functions satisfying* $\psi(0) = \psi(2\pi)$ and $\psi'(0) = \psi'(2\pi)$. If we find the commutator of these two operators, we can see that

$$\begin{aligned}
[A,B] &= AB - BA \\
&= \theta(-i\hbar)\frac{d}{d\theta} - (-i\hbar)\frac{d}{d\theta}\theta \\
&= 0 + i\hbar\frac{d}{d\theta}\theta \\
&= i\hbar \cdot 1 \\
&= i\hbar
\end{aligned}$$

and so we expect $\Delta A \Delta B \geq \frac{\hbar}{2}$. Now, the (normalized) eigenvalues of B can be found as $\psi_n(\theta) = \frac{1}{\sqrt{2}}e^{2\pi i n\theta}$ but the operator A is bounded above by 2π. Therefore, the *uncertainty of A is finite while the uncertainty of B is 0!* This means that we have $\Delta A \Delta B = \Delta A \cdot 0 = 0 \leq \frac{\hbar}{2}$ which contradicts the Robertson uncertainty relation. Of course, this does not mean quantum mechanics is incorrect, just that we need to be careful of the domains of our operators we are interested in. B was defined for functions $\psi(0) = \psi(2\pi)$ and $\psi'(0) = \psi'(2\pi)$ but for $A\psi_n(\theta) = \theta\frac{1}{\sqrt{2}}e^{2\pi i n\theta}$ we see that

$$0 \cdot \psi_n(0) = 0 \neq 2\pi\frac{1}{\sqrt{2}}e^{2\pi i n 2\pi}$$

because $e^{2\pi i n 2\pi}$ is non-zero. So $A\psi(\theta)$ is not in the domain of B and therefore are not garauntee to follow the Robertson uncertain relation.

4.5 Determining S_x and S_y from the eigenstates

We introduce another change in notation. We used the operators $\hat{J}_x$, $\hat{J}_y$, $\hat{J}_z$ to refer to angular momentum *in general* but, as noted before, there is *orbital angular momentum* and *intrinsic spin angular momentum.* When talking strictly about one or the other, we use $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ for orbital angular momentum and we use $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$ for instrisic spin angular momentum. We also change $|j, m\rangle$ to $|s, m\rangle$. In this section, we will show how we can derive the matrix representations of S_x and S_y by using the definition of S_+ and S_-

$$\hat{S}_+ = \hat{S}_x + i\hat{S}_y$$
$$\hat{S}_- = \hat{S}_x - i\hat{S}_y.$$

First, see that this implies

$$\hat{S}_x = \frac{\hat{S}_+ + \hat{S}_-}{2}$$
$$\hat{S}_y = \frac{\hat{S}_+ - \hat{S}_-}{2i}$$

and so in calculating the matrix representation, we can replace $\hat{S}_x$ and $\hat{S}_y$ with their forms of S_+ and S_-, since we know how they act on the basis states!

Example 4.2 Find the matrix representation of S_x and S_y for the eigenstates of a spin-$\frac{1}{2}$ particle.

Solution

$$\begin{aligned}
S_x \xrightarrow[S_z]{} &\begin{bmatrix} \langle \frac{1}{2}, \frac{1}{2}| S_x |\frac{1}{2}, \frac{1}{2}\rangle & \langle \frac{1}{2}, \frac{1}{2}| S_x |\frac{1}{2}, \frac{-1}{2}\rangle \\ \langle \frac{1}{2}, \frac{-1}{2}| S_x |\frac{1}{2}, \frac{1}{2}\rangle & \langle \frac{1}{2}, \frac{-1}{2}| S_x |\frac{1}{2}, \frac{-1}{2}\rangle \end{bmatrix} \\
= &\begin{bmatrix} \langle \frac{1}{2}, \frac{1}{2}| \frac{\hat{S}_+ + \hat{S}_-}{2} |\frac{1}{2}, \frac{1}{2}\rangle & \langle \frac{1}{2}, \frac{1}{2}| \frac{\hat{S}_+ + \hat{S}_-}{2} |\frac{1}{2}, \frac{-1}{2}\rangle \\ \langle \frac{1}{2}, \frac{-1}{2}| \frac{\hat{S}_+ + \hat{S}_-}{2} |\frac{1}{2}, \frac{1}{2}\rangle & \langle \frac{1}{2}, \frac{-1}{2}| \frac{\hat{S}_+ + \hat{S}_-}{2} |\frac{1}{2}, \frac{-1}{2}\rangle \end{bmatrix} \\
= &\begin{bmatrix} \frac{\hbar}{2}\langle \frac{1}{2}, \frac{1}{2}| \left[0 + \sqrt{(\frac{1}{2})(\frac{1}{2}+1) - \frac{1}{2}(\frac{1}{2}-1)}\right] |\frac{1}{2}, \frac{-1}{2}\rangle & \frac{\hbar}{2}\langle \frac{1}{2}, \frac{1}{2}| \left[\sqrt{(\frac{1}{2})(\frac{1}{2}+1) - \frac{-1}{2}(\frac{-1}{2}+1)} + 0\right] |\frac{1}{2}, \frac{1}{2}\rangle \\ \frac{\hbar}{2}\langle \frac{1}{2}, \frac{-1}{2}| \left[0 + \sqrt{(\frac{1}{2})(\frac{1}{2}+1) - \frac{1}{2}(\frac{1}{2}-1)}\right] |\frac{1}{2}, \frac{-1}{2}\rangle & \frac{\hbar}{2}\langle \frac{1}{2}, \frac{-1}{2}| \left[\sqrt{(\frac{1}{2})(\frac{1}{2}+1) - \frac{-1}{2}(\frac{-1}{2}+1)} + 0\right] |\frac{1}{2}, \end{bmatrix} \\
= &\begin{bmatrix} 0 & \frac{\hbar}{2}\sqrt{(\frac{1}{2})(\frac{1}{2}+1) - \frac{-1}{2}(\frac{-1}{2}+1)} \\ \frac{\hbar}{2}\sqrt{(\frac{1}{2})(\frac{1}{2}+1) - \frac{1}{2}(\frac{1}{2}-1)} & 0 \end{bmatrix} \\
= &\frac{\hbar}{2}\begin{bmatrix} 0 & \frac{\hbar}{2}\sqrt{\frac{3}{4}+\frac{1}{4}} \\ \frac{\hbar}{2}\sqrt{\frac{3}{4}+\frac{1}{4}} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\end{aligned}$$

For S_y

$$S_y \xrightarrow[S_z]{} \frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}.$$

If we put all 3 together,

Theorem 4.14

The matrix represenentations of S_x, S_y, and S_z are

$$S_x \xrightarrow[S_z]{} \frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

$$S_y \xrightarrow[S_z]{} \frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$$

$$S_z \xrightarrow[S_z]{} \frac{\hbar}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

♡

actually, these 3 matrices without the constant are the *pauli matrices*

Definition 4.10 (Pauli Matrices)

The Pauli matrices are

$$\sigma_x = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

$$\sigma_y = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$$

$$\sigma_z = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

.

♣

They have the following properties

- $I, \sigma_x, \sigma_y, \sigma_z$ form a basis for all 2×2 Hermitian matrices, where all coeffecients are real
- $i\sigma_x, i\sigma_y, i\sigma_z$ form a basis for the real Lie algebra $\mathfrak{su}(2)$, (they are the infiniteseimal generators of SU(2)).
- $\sigma_a = \begin{bmatrix}\delta_{az} & \delta_{ax} - i\delta_{ay}\\ \delta_{ax} + i\delta_{ay} & \delta_{az}\end{bmatrix}$, $a = x, y, z$
- $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = -i\sigma_x\sigma_y\sigma_z = I$
- $[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma_c$
- $\{\sigma_a, \sigma_b\} = 2\delta_{ab}I$
- $\sigma_a\sigma_b = \delta_{ab}I + i\epsilon_{abc}\sigma_c$, to prove this, add the commutator and anticommutator

We end this chapter by wondering what proportian of spin-1 particles in the state $|1,1\rangle_z$

will divide into when passed through a SGy device. Essentially, we want to calculate

$$|\langle 1,1|_y |1,1\rangle_z|^2$$
$$|\langle 1,0|_y |1,1\rangle_z|^2$$
$$|\langle 1,-1|_y |1,1\rangle_z|^2$$

and to do this, we require switching from the S_z basis to the S_y basis. We can actually calculate this quite easily, because we know S_+ and S_- matrix representations. Remember that

$$S_y = \frac{S_+ - S_y}{2i}$$

so

$$\begin{aligned} S_y \xrightarrow[S_z]{} \frac{1}{2i}\left(\sqrt{2}\hbar \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} - \sqrt{2}\hbar \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right) \\ = \frac{1}{2i}\sqrt{2}\hbar \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \\ = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}. \end{aligned}$$

To figure out the S_z representation of $|1,1\rangle_y$, we can just solve the eigenvalue equation using

$$S_y |1,1\rangle_y = \hbar |1,1\rangle_y$$

where $|1,1\rangle_y$ is represented by $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and we can just solve for the eigenvector.

Chapter 5 Time Evolution

Definition 5.1

The time evolution operator, $\hat{U}(t)$, translates a ket vector $|\psi\rangle$ forward in time t units

$$\hat{U}(t)\,|\psi(0)\rangle = |\psi(t)\rangle\,.$$

♣

Of course, to preserve probability we require that $\hat{U}(t)$ is unitary

$$\hat{U}(t)^\dagger \hat{U}(t) = 1.$$

Like with the generators of rotation, we introduce the generator of time translation as

$$\hat{U}(dt) = 1 - \frac{i}{\hbar}\hat{H}dt$$

which leads us to consider translating the ket at a time t forward in time dt as

$$\hat{U}(t+dt) = \hat{U}(dt)\hat{U}(t) = \left(1 - \frac{i}{\hbar}\hat{H}dt\right)\hat{U}(t)$$

leading to a differential equation as follows

$$\begin{aligned}
\hat{U}(t+dt) &= \left(1 - \frac{i}{\hbar}\hat{H}dt\right)\hat{U}(t) \\
\hat{U}(t+dt) &= \hat{U}(t) - \frac{i}{\hbar}\hat{H}dt\hat{U}(t) \\
\hat{U}(t+dt) - \hat{U}(t) &= -\frac{i}{\hbar}\hat{H}dt\hat{U}(t) \\
\frac{\hat{U}(t+dt) - \hat{U}(t)}{dt} &= -\frac{i}{\hbar}\hat{H}\hat{U}(t) \\
\frac{d\hat{U}(t)}{dt} &= \frac{-i}{\hbar}\hat{H}\hat{U}(t).
\end{aligned}$$

If we input kets we get the *Schrödinger equation*, which we call the *Time Dependent Schrödinger equation* because $|\psi\rangle$ is a function of t.

Definition 5.2 (Time Dependent Schrödinger equation)

The Schrödinger equation is the

$$i\hbar\frac{d}{dt}\,|\psi(t)\rangle = \hat{H}\,|\psi(t)\rangle$$

where $\hat{H}$ is the Hamiltonian operator corresponding to the energy of the system.

♣

The choice of H will depend on the system we are modelling. If it turns out to be *time independent* then we can solve explicitly. This is just a first order differential equation with constant coeffecients so

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}\,|\psi(0)\rangle\,.$$

To solve this then explicitly only requires for us to know the action of $\hat{H}$ on $|\psi(0)\rangle$ and $|\psi(0)\rangle\, itself$. Actually, this Hamiltonian has an expectation value that is also independent of time

$$\langle\psi(t)|\,\hat{H}\,|\psi(t)\rangle = \langle\psi(0)|\,\hat{U}(t)^{\dagger}\hat{H}\hat{U}(t)\,|\psi(0)\rangle = \langle\psi(0)|\,\hat{H}\,|\psi(0)\rangle$$

due to the fact this it commutes with $\hat{U}(t)$

$$\hat{U}(t)\hat{H} = \hat{H}\hat{U}(t).$$

This leads to us associating the Hamiltonian with the *energy of a system.*

Definition 5.3 (Energy)

The energy of a system is

$$\langle E\rangle = \langle\psi|\,\hat{H}\,|\psi\rangle$$

♣

Definition 5.4 (Energy Eigenstates)

The eigenstates of the Hamiltonian are the energy eigenstates satisfying

$$\hat{H}\,|E\rangle = E\,|E\rangle\,.$$

♣

If an energy eigenstate is the intial state of our system, then finding an explicit form for $|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}\,|\psi(0)\rangle$ is easy

$$\begin{aligned}
|\psi(t)\rangle &= e^{-i\hat{H}t/\hbar}\,|\psi(0)\rangle \\
&= e^{-i\hat{H}t/\hbar}\,|E\rangle \\
&= \sum_{n=0}^{\infty}\frac{(-i\hat{H}t/\hbar)^n}{n!}\,|E\rangle \\
&= \sum_{n=0}^{\infty}\frac{(-it/\hbar)^n}{n!}\hat{H}^n\,|E\rangle \\
&= \sum_{n=0}^{\infty}\frac{(-it/\hbar)^n}{n!}E^n\,|E\rangle \\
&= e^{-iEt/\hbar}\,|E\rangle
\end{aligned}$$

and we see that the solution only changes my a constant as time progesses. We call such a state a *stationary state*.

5.1 Time Dependence of Expectation Values

Using the Schrödinger equation, we can determine which observables are dependent on time

Theorem 5.1 (Time Dependence of Observables)

Let A be some observable and $|\psi(t)\rangle$ some quantum state satisfying the Schrödinger equation. Then

$$\frac{d}{dt}\langle A\rangle = \frac{i}{\hbar}\langle\psi(t)|\,[\hat{H},\hat{A}]\,|\psi(t)\rangle + \langle\psi(t)|\,\frac{\partial A}{\partial t}\,|\psi(t)\rangle$$

♡

Proof

$$\begin{aligned}
\frac{d}{dt}\langle A\rangle &= \frac{d}{dt}\langle\psi(t)|\,A\,|\psi(t)\rangle \\
&= \left(\frac{d}{dt}\langle\psi(t)|\right)A\,|\psi(t)\rangle + \langle\psi(t)|\,\frac{\partial A}{\partial t}\,|\psi(t)\rangle + \langle\psi(t)|\,A\left(\frac{d}{dt}\,|\psi(t)\rangle\right) \\
&= \left(\frac{1}{-i\hbar}\langle\psi(t)|\,\hat{H}\right)A\,|\psi(t)\rangle + \langle\psi(t)|\,\frac{\partial A}{\partial t}\,|\psi(t)\rangle + \langle\psi(t)|\,A\left(\frac{1}{i\hbar}\,|\psi(t)\rangle\,\hat{H}\right) \\
&= \left(\frac{-1}{i\hbar}\langle\psi(t)|\,\hat{H}\right)A\,|\psi(t)\rangle + \langle\psi(t)|\,A\left(\frac{1}{i\hbar}\,|\psi(t)\rangle\,\hat{H}\right) + \langle\psi(t)|\,\frac{\partial A}{\partial t}\,|\psi(t)\rangle \\
&= \frac{i}{\hbar}\langle\psi(t)|\,[\hat{H},\hat{A}]\,|\psi(t)\rangle + \langle\psi(t)|\,\frac{\partial A}{\partial t}\,|\psi(t)\rangle\,.
\end{aligned}$$

This allows us to see that the expectation value of an observable changes with time if and only if it commutes with the hamiltonian and the observable itself does not change with time.

5.2 The Hamiltonian

We dedicate this section to some more background information on the Hamiltonian before proceeding to an example. The Hamiltonian of a system is an operator that corresponds to the total energy of the system, so it is the *sum of the potential energy $\hat{V}$ and kinetic energy $\hat{T}$*

Definition 5.5

$$\hat{H} = \hat{T} + \hat{V}.$$

♣

The kinetic energy operator $\hat{T}$ is actually definited in terms of the *momentum operator* $\hat{p} = -i\hbar\nabla$ as

$$\hat{T} = \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m}\nabla^2.$$

Note that the Laplacian will have different forms when using different coordinate systems, so in cartesian coordianates

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

but in 2D polar coordinates

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

The *potential energy* is written in a general form $\hat{V}(\vec{r}, t)$ due to the fact that this operator will change depending on which system we're attempting to model. The vector $\vec{r}$ contains spatial information about the particle. If we are modeling a particle in a magnetic field for example, our potential energy will be related to how that particle reacts to the field. This lets us write

Definition 5.6

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r}, t).$$

♣

Fortunately, the Hamiltonian extends to N-particles in the following way

Theorem 5.2

The Hamiltonian for N-particles whos kinetic energy is not dependent on interaction between particles is

$$\hat{H} = -\frac{\hbar^2}{2}\sum_{n=1}^{N}\frac{1}{m_n}\nabla_n^2 + \hat{V}(\vec{r}_1, \vec{r}_2, \ldots \vec{r}_N, t)$$

♡

Proof The potential operator simply becomes a function of all position variables. Now for kinetic energy, we have that total kinetic energy of the whole system is stored in the kinetic energy of each particle. Farthermore, we assumed that the particles do not interact and therefore and so we may write $\hat{T} = \sum_{n=0}^{N}\hat{T}_n$. As before, each of these particles has kinetic energy $\hat{T}_n = \frac{\hat{p}_n \cdot \hat{p}_n}{2m_n}$ where $p\hat{p}$ has a new subscript due to the coordinates of each particle. If we write

$$\hat{p_n} = -\frac{\hbar^2}{2m_n}\nabla_n^2 = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial y_n^2} + \frac{\partial^2}{\partial z_n^2}\right)$$

we see that we need to specfiy the coordinates and weight of each particle. Putting everything together yields the desired result.

Farthur more, if the particles do not interact at all, then the potential energy simply

becomes the potential energy of each of the particles, so we simply get

$$\begin{aligned}\hat{H} &= \hat{T} + \hat{V} \\ &= \frac{\hbar^2}{2}\sum_{n=0}^{N}\frac{1}{m_n}\nabla_n^2 + \sum_{n=0}^{N}\hat{V}_n \\ &= \sum_{n=0}^{N}\left(-\frac{\hbar^2}{2m_n}\nabla_n^2 + V_n\right) \\ &= \sum_{n=0}^{N}\left(\hat{T}_n + \hat{V}_n\right) \\ &= \sum_{n=0}^{N}\hat{H}_n\end{aligned}$$

5.3 Precession of a spin-$\frac{1}{2}$ particle in a magnetic field

In this section, we'll assume some knowledge of magnetic fields as a simple example of how Schrodinger's equation dictates the precession of spin in a magnetic field. Suppose a particle that is in the state $|+z\rangle$ is placed in a magnetic field $B = B_0\vec{k}$ and take the charge of the particle to be $q = -e$. Then, we know that the Hamiltonian is the sum of the kinetic and potential energys. Becaues the particle is stationary, the kinetic energy must be 0. This leaves us with the potential energy, which must be $-\hat{\mu}\cdot\vec{B}$ or

$$\hat{H} = -\hat{mu}\cdot\vec{B} = -\frac{gq}{2mc}\hat{S}\cdot\hat{B} = \frac{ge}{2mc}B_0\hat{S}_z$$

where we let on the constants on the left side be equal to w_0. Now that we have the Hamiltonian for our system, we can see how $|+z\rangle$ progresses through time based on the Schrödinger equation. We have to solve the initial value problem

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \frac{ge}{2mc}B_0\hat{S}_z|\psi(t)\rangle\,, \quad |\psi(0)\rangle = |+z\rangle\,.$$

Recall that, since this Hamiltonian is time independent, this has solution

$$|\psi(t)\rangle = e^{-iw_0\hat{S}_z/\hbar}|+z\rangle$$

and since $|+z\rangle$ is an eigenstate of $\hat{S}_z$

$$S_z|+z\rangle = \frac{\hbar}{2}|+z\rangle$$

we get the solution

$$|\psi(t)\rangle = e^{\frac{-tiw_0\hbar}{2\hbar}}|+z\rangle = e^{\frac{-tiw_0}{2}}|+z\rangle\,.$$

The closed form solution actually tells us some interesting things about the behaviour of particle. We see that only the phase changes with time, repeating every $\frac{4\pi}{w_0}$. What's

more, we can easily determine the probability that $|\psi(t)\rangle$ is in the state $|+z\rangle$ to be 1

$$\begin{aligned}
|\langle +z|\psi(t)\rangle|^2 &= \langle +z|\psi(t)\rangle \langle \psi(t)| + z\rangle \\
&= \langle +z| e^{\frac{-tiw_0}{2}} |+z\rangle \langle +z| e^{\frac{tiw_0}{2}} |+z\rangle \\
&= e^{\frac{-tiw_0}{2}} e^{\frac{tiw_0}{2}} \langle +z| + z\rangle \langle +z| + z\rangle \\
&= 1 \cdot 1 \cdot 1 \\
&= 1.
\end{aligned}$$

Actually, this follows directly from that fact that $\hat{U}(t)$ is unitary. Another thing this solution tells us is the *energy* of the system. Remember that we defined the energy of the system to be $\langle E\rangle = \langle \psi(t)| \hat{H} |\psi(t)\rangle$ and so we can substitute our solution for $|\psi(t)\rangle$ to see

$$\begin{aligned}
\langle E\rangle &= \langle \psi(t)| \hat{H} |\psi(t)\rangle \\
&= \langle +z| e^{\frac{tiw_0}{2}} w_0 \hat{S}_z e^{\frac{-tiw_0}{2}} |+z\rangle \\
&= w_0 e^{\frac{tiw_0}{2}} e^{\frac{-tiw_0}{2}} \langle +z| \hat{S}_z |+z\rangle \\
&= w_0 \langle +z| \hat{S}_z |+z\rangle \\
&= w_0 \langle +z| \frac{\hbar}{2} |+z\rangle \\
&= w_0 \frac{\hbar}{2} \langle +z| + z\rangle \\
&= \frac{w_0 \hbar}{2}.
\end{aligned}$$

What about the expected value of $\hat{S}_z$?

$$\begin{aligned}
\langle S_z\rangle &= \langle \psi(t)| \hat{S}_z |\psi(t)\rangle \\
&= \langle +z| e^{\frac{tiw_0}{2}} \hat{S}_z e^{\frac{-tiw_0}{2}} |+z\rangle \\
&= \langle +z| \hat{S}_z |+z\rangle \\
&= \frac{\hbar}{2} \langle +z| |+z\rangle \\
&= \frac{\hbar}{2}.
\end{aligned}$$

What would our solutions be if we started with the initial state $|\psi(0)\rangle = |+x\rangle$?. Our Hamiltonian would remain the same, however our solution for the IVP would be different due to different initial conditions. We see that

$$|\psi(t)\rangle = e^{-itw_0 S_z/\hbar} |+x\rangle .$$

However, the advantage of linear algebra allows us to *choose a basis representaiton of* $|+x\rangle$ *that are the eigenvalues of* S_z! Remember from the earlier chapters

$$|+x\rangle = \frac{1}{\sqrt{2}} |+z\rangle + \frac{1}{\sqrt{2}} |-z\rangle$$

and so substituting into the equation

$$\begin{aligned}
|\psi(t)\rangle &= e^{-itw_0 S_z/\hbar} |+x\rangle \\
&= e^{-itw_0 S_z/\hbar} \left(\frac{1}{\sqrt{2}} |+z\rangle + \frac{1}{\sqrt{2}} |-z\rangle \right) \\
&= e^{-itw_0 S_z/\hbar} \frac{1}{\sqrt{2}} |+z\rangle + e^{-itw_0 S_z/\hbar} \frac{1}{\sqrt{2}} |-z\rangle \\
&= e^{-itw_0 \frac{\hbar}{2\hbar}} \frac{1}{\sqrt{2}} |+z\rangle + e^{-itw_0 \frac{-\hbar}{2\hbar}} \frac{1}{\sqrt{2}} |-z\rangle \\
&= \frac{e^{\frac{-itw_0}{2}}}{\sqrt{2}} |+z\rangle + \frac{e^{\frac{itw_0}{2}}}{\sqrt{2}} |-z\rangle .
\end{aligned}$$

What would the energy and expect values of S_x, S_y, and S_z of these be? We can calculate them!

$$\begin{aligned}
\langle E\rangle &= \langle\psi(t)| \hat{H} |\psi(t)\rangle \\
&= \left(\frac{e^{\frac{itw_0}{2}}}{\sqrt{2}} \langle +z| + \frac{e^{\frac{-itw_0}{2}}}{\sqrt{2}} \langle -z| \right) w_0 \hat{S}_z \left(\frac{e^{\frac{-itw_0}{2}}}{\sqrt{2}} |+z\rangle + \frac{e^{\frac{itw_0}{2}}}{\sqrt{2}} |-z\rangle \right) \\
&= \left(\frac{e^{\frac{itw_0}{2}}}{\sqrt{2}} \langle +z| + \frac{e^{\frac{-itw_0}{2}}}{\sqrt{2}} \langle -z| \right) w_0 \left(\frac{e^{\frac{-itw_0}{2}}}{\sqrt{2}} \frac{\hbar}{2} |+z\rangle + \frac{e^{\frac{itw_0}{2}}}{\sqrt{2}} \frac{-\hbar}{2} |-z\rangle \right) \\
&= w_0 \frac{e^{\frac{itw_0}{2}}}{\sqrt{2}} \frac{e^{\frac{-itw_0}{2}}}{\sqrt{2}} \frac{\hbar}{2} \langle +z| + z\rangle + w_0 \frac{e^{\frac{itw_0}{2}}}{\sqrt{2}} \frac{-\hbar}{2} \frac{e^{\frac{-itw_0}{2}}}{\sqrt{2}} \langle -z| - z\rangle \\
&= \frac{w_0\hbar}{4} + \frac{-w_0\hbar}{4} \\
&= 0
\end{aligned}$$

Right away we can see that $|\langle +z|\psi(t)\rangle|^2 = \frac{1}{2}$ and $|\langle -z|\psi(t)\rangle|^2 = \frac{1}{2}$. Notice how these are time independent. But when we calculate for the x eigenstates

$$\begin{aligned}
\langle +x|\psi(t)\rangle &= \left(\frac{1}{\sqrt{2}} \langle +z| + \frac{1}{\sqrt{2}} \langle -z| \right) \left(\frac{e^{\frac{-itw_0}{2}}}{\sqrt{2}} |+z\rangle + \frac{e^{\frac{itw_0}{2}}}{\sqrt{2}} |-z\rangle \right) \\
&= \frac{1}{\sqrt{2}} \frac{e^{\frac{-itw_0}{2}}}{\sqrt{2}} + \frac{1}{\sqrt{2}} \frac{e^{\frac{itw_0}{2}}}{\sqrt{2}} \\
&= \frac{1}{2} \left(e^{\frac{-itw_0}{2}} + e^{\frac{itw_0}{2}} \right) \\
&= \frac{1}{2} \left(\cos(\frac{-tw_0}{2}) + i\sin(\frac{-tw_0}{2}) + \cos(\frac{tw_0}{2}) + i\sin(\frac{tw_0}{2}) \right) \\
&= \frac{1}{2} \left(\cos(\frac{tw_0}{2}) - i\sin(\frac{tw_0}{2}) + \cos(\frac{tw_0}{2}) + i\sin(\frac{tw_0}{2}) \right) \\
&= \frac{1}{2} 2\cos(\frac{tw_0}{2}) \\
&= \cos(\frac{tw_0}{2})
\end{aligned}$$

so

$$|\langle +x|\psi(t)\rangle|^2 = \cos^2\left(\frac{tw_0}{2}\right).$$

This is quite cool! The first example of a time dependent spin state we have come across. Doing a similar calculation we can find

$$|\langle -x|\psi(t)\rangle|^2 = \sin^2\left(\frac{tw_0}{2}\right).$$

Finally then its easy to find that

$$\langle S_x\rangle = \cos^2\left(\frac{tw_0}{2}\right)\left(\frac{\hbar}{2}\right) + \sin^2\left(\frac{tw_0}{2}\right)\left(\frac{-\hbar}{2}\right) = \frac{\hbar}{2}\cos(w_0 t).$$

Again, we skip the calculations for $\langle \hat{S}_y\rangle$ and get all three

$$\langle \hat{S}_x\rangle = \frac{\hbar}{2}\cos(w_0 t)$$
$$\langle \hat{S}_y\rangle = \frac{\hbar}{2}\sin(w_0 t)$$
$$\langle \hat{S}_z\rangle = 0.$$

What does this really tell us? It's that the spin of the particle is precessing around the $z - axis$ with a period $\frac{2\pi}{t}$.