Chapter 1 Support

Last time we talked about the spectrum of a integrator in the Riemann Stieljets integral, which is similar to the riemann integral except changing $x_{i+1} - x_i$ to $g(x_{i+1}) - g(x_i)$

Definition 1.1. Riemann Steiljetz integral

For a function of bounded variation ψ

$$\int_a^b f(x)d\psi(x) = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(c_i)[g(x_{i+1}) - g(x_i)], \quad x_i < c_i < x_{i+1}$$

with a partition of norm $\frac{a-b}{n}$ *of an interval* $[a, b]$. ♣

Definition 1.2. Spectrum

The spectrum of a function ψ *is*

$$\mathfrak{S}(\psi) = \{x|\psi(x+\delta) < \psi(x-\delta) \quad \forall \delta > 0\}$$

♣

Consider the constant function $g(x) = 1$. Then the spectrum is empty because $g(x+\delta) - g(x-\delta) = 1 - 1 = 0$ What happens if we use this function as an integrator?

$$\begin{aligned}
\int_a^b f(x)d\psi(x) &= \lim_{n\to\infty} \sum_{i=0}^{n-1} f(c_i)[g(x_{i+1}) - g(x_i)] \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} f(c_i)[1-1] \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} f(c_i)[0] \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} 0 \\
&= 0.
\end{aligned}$$

What we can see here is that if we integrate over an interval outside the spectrum, then the resulting integral is 0. The reason the definition is defined in terms of δ is due to the nature of partitions. Comparing the definition of support of a measure, we will see similarities.

Definition 1.3. Support of a measure

The support of a function $\psi(x)$ *is*

$$supp\psi(x) = \{x|\psi(x) \neq 0\}$$

♣

So whenever we do something like a lebesgue integral with measure $\psi(x)$ or perhaps the expected value of a random variable χ with PDF $\psi(x)$, the spectrum is the set of all x with probability or measure 0. So, the spectrum is a support for our integrator. If there's an interval where $\psi(x)$ is constant, then it will be outside the spectrum and the integral will be 0. If a function is piece-wise constant with jump continuities, *the spectrum will consist of all the x where the jumps occur.* We don't need to care about the values outside the spectrum because

$$\int_{\mathbb{R}-\mathfrak{S}(\psi)} f(x) d\psi(x) = 0$$

Chapter 2 Representation theorem

Theorem 2.1. Representation Theorem

Suppose $\mathcal{L}\{x^k\} = \mu_k$ where $\mathcal{L}$ is positive definite. Then there exists a non-decreasing $\psi(x)$ of bounded variation such that

$$\int_{-\infty}^{\infty} x^k d\psi(x) = \mathcal{L}\{x^k\}$$

NOTE: $\psi(x)$ is NOT garaunteed to be unique! ♡

An outline of the proof is

Proof Gauss quadrature says $\mu_k = \sum_{i=0}^{n} A_{ni} x_{ni}^k$ for $k = 0 \ldots 2n-1$ where $A_{ni} > 0$ and $x_{n1}, x_{n2}, \ldots x_{nn}$ are the zeroes of $P_n(x)$ in increasing order. Next we define

$$\mathcal{L}[x^k] = \mu_k = \sum_{i=1}^{n} A_{ni} x_{ni}^k, \quad k = 0, 1, \ldots, 2n-1$$

where $x_{n1} < x_{n2} < \cdots < x_{nn}$ are the zeroes of the nth degree monic OPS corresponding to $\mathcal{L}$. Now define

$$\psi_n(x) = \begin{cases} 0, & x < x_{n1} \\ A_{n1} + \cdots + A_{np}, & x_{np} \le x < x_{n,p+1} \quad (1 \le p < n) \\ \mu_0, & x \ge x_{nn} \end{cases}$$

then there exists a subsequence of ψ_n that converges to $\lim_{n\to\infty} \psi_n = \psi$ that gives us the desired result. Also, the spectrum is in the true interval of orthogonality $\mathfrak{S}(\psi) \subset [\xi_1, \eta_1]$

Reminder

Definition 2.1. Representative of $\mathcal{L}$

If

$$\int_{-\infty}^{\infty} x^k d\psi = \mathcal{L}\{x^k\}$$

We call $\psi(x)$ a representative for $\mathcal{L}$ ♣

Definition 2.2. Substantially Equal

Two functions ψ_1 and ψ_2 are called substantially equal if

$$\psi_2 = \psi_1 + C$$

♣

This definition is needed because they will result in the same RS integral

$$\int_{-\infty}^{\infty} f(x) d\psi_1 = \int_{-\infty}^{\infty} f(x) d\psi_2.$$

As we said before, there can be more than one representative for $\mathcal{L}$ and so

Definition 2.3. Determinancy

$\mathcal{L}$ is called determinant if any two representatives are substantially equal. Otherwise, it is indeterminant. ♣

Definition 2.4. Natural Representation

A distribution function is called the natural representation of $\mathcal{L}$ if it is one of the limits of ψ_n. ♣

Chapter 3 Spectral Points and Zeroes

For this section, $\mathcal{L}$ is positive definite. We want to explore how the spectral points interact with the zeros of an OPS. Please try to keep in mind the idea of density for bounded intervals of orthogonality. We start off with a small theorem

Theorem 3.1

Let x_{ni} be the zeros of $P_n(x)$ with moment functional $\mathcal{L}$ represented by ψ. Then

$$\mathfrak{S}(\psi) \bigcap (x_{ni}, x_{n,i+1}) \neq \varnothing.$$

i.e there is atleast 1 spectral point between 2 zeroes. ♡

Proof We do a proof by contradiction. Assume the contrary. Fix two zeroes, $x_{n,i}, x_{n,i+1}$ and let

$$\pi(x) = \frac{P_n(x)}{(x - x_{ni})(x - x_{n,i+1})}$$

such that $\pi(x)P_n(x) > 0$ outside $(x_{n,i}, x_{n,i+1})$. So it follows that $\pi(x)P_n(x) \geq 0$, $x \in \mathfrak{S}(\psi)$. Since $\mathcal{L}$ is positive definite, $\mathcal{L}[\pi(x)P_n(x)] > 0$ over the spectrum (it is a support) but this contradicts $\mathcal{L}[\pi(x)P_n(x)] = 0$ for $deg\pi(x) < n$. This completes the proof.

This is interesting because we can see that for a bounded interval of orthogonality, the zeroes are dense in $\mathbb{R}$. We can also see that the spectrum can never be finite, which we had to prove earlier.

Theorem 3.2

A natural representatative for $\mathcal{L}$ has no spectral points in any open set that does not contain zeroes. ♡

Proof If there are no zeroes in an openset G, then ψ_n is constant on G and so $\psi_n(x_1) - \psi_n(x_2) = 0$ for $x_1, x_2 \in G$. $n \to \infty$ gives $\psi(x_1) - \psi(x_2) = 0$

Actually, the contrapositive of this theorem says something interesting

Theorem 3.3

Every neighbourhood of a spectral point contains a zero for infinitely many n of $P_n(x)$. ♡

What this tells us is something very important. For a natural representative, the spectral points are either the zeroes of $P_n(x)$ or the limit of those zeros.

this part I include for completeness

We next recall the limits

$$\xi_i = \lim_{n\to\infty} x_{ni}, \qquad \eta_j = \lim_{n\to\infty} x_{n,n-j+1}, \qquad i, j = 1, 2, 3, \ldots \tag{4.4}$$

We have (in the extended real number system), $\xi_{i-1} \leqq \xi_i < \eta_j \leqq \eta_{j-1}$, and hence define

$$\sigma = \begin{cases} -\infty & \text{if } \xi_i = -\infty \quad \text{for all } i \\ \lim\limits_{i\to\infty} \xi_i & \text{if } \xi_p > -\infty \quad \text{for some } p \end{cases} \tag{4.5}$$

$$\tau = \begin{cases} +\infty & \text{if } \eta_j = +\infty \quad \text{for all } j \\ \lim\limits_{j\to\infty} \eta_j & \text{if } \eta_q < +\infty \quad \text{for some } q. \end{cases} \tag{4.6}$$

Finally we write

$$\xi_0 = -\infty, \qquad \eta_0 = +\infty$$

so that

$$-\infty = \xi_0 \leqq \xi_1 \leqq \xi_2 \leqq \cdots \leqq \sigma \leqq \tau \leqq \cdots \leqq \eta_2 \leqq \eta_1 \leqq \eta_0 = +\infty.$$

Figure 3.1: Limits

Theorem 3.4

Let ψ be a representative of $\mathcal{L}$

1. *if $\xi_k > \xi_{k+1}$ for some k then $\mathfrak{S}(\psi) \bigcap (\xi_k, \xi_{k+1}] \neq \varnothing$*
2. *If $\xi_k = \xi_{k+1}$ then ξ_k is a limit point of $\mathfrak{S}(\psi)$*
3. *σ is a limit point of $\mathfrak{S}(\psi)$*

♡

Chapter 4 Favards theorem

Theorem 4.1. Favards theorem

Let c_n and λ_n be sequences of complex numbers and let P_n defined by

$$P_n = (x - c_n)P_{n-1} - \lambda_n P_{n-2}, \quad P_{-1} = 1, P_0 = 1$$

then there exists a unique $\mathcal{L}$ such that

$$\mathcal{L}[1] = \lambda_1, \quad \mathcal{L}[P_m P_n] = 0, m \neq n.$$

Farthur more, $\mathcal{L}$ is quasi-definite and P_n is monic if and only if $\lambda_n \neq 0$. We also have $\mathcal{L}$ is positive definite if and only if C_n is real and $\lambda_n > 0, n > 0$ ♡

Proof Inductively, we define $\mathcal{L}[1] = \mu_0 = \lambda_1$ and $\mathcal{L}[P_n(x)] = 0$. We can do so simply by solving

$$\begin{aligned}
\mathcal{L}[P_0] &= \mu_0 = \lambda_1 \\
\mathcal{L}[P_1] &= \mu_1 - c_1\mu_0 = 0 \\
\mathcal{L}[P_2] &= \mu_2 - (c_1 + c_2)\mu_1 + (\lambda_2 - c_1c_2)\mu_0 = 0
\end{aligned}$$

and so on.

Next, we rewrite the recurrence relation as

$$xP_n = P_{n+1} + c_{n+1}P_n + \lambda_{n+1}P_{n-1}$$

see that

$$\begin{aligned}
xP_n &= P_{n+1} + c_{n+1}P_n + \lambda_{n+1}P_{n-1} \\
\mathcal{L}[xP_n] &= \mathcal{L}[P_{n+1} + c_{n+1}P_n + \lambda_{n+1}P_{n-1}] \\
&= \mathcal{L}[P_{n+1}] + c_{n+1}\mathcal{L}[P_n] + \lambda_{n+1}\mathcal{L}[P_{n-1}] \\
&= 0 + 0 + 0 \\
&= 0
\end{aligned}$$

So $\mathcal{L}[xP_n(x)] = 0$ for $n \geq 2$. If we multiply by x^2 instead and use $\mathcal{L}[xP_n(x)] = 0$ we get the same result for $n \geq 3$. Thus

$$\mathcal{L}[x^k P_n(x)] = 0, \quad 0 \leq k < n.$$

For a similar argument, we can get

$$\mathcal{L}[x^n P_n] = \lambda_{n+1}\mathcal{L}[x^{n-1}P_{n-1}] = \lambda_{n+1}\lambda_n\mathcal{L}[x^{n-2}P_{n-2}] = \dots.$$

Finally, using the lemma

Lemma 4.1

$$\mathcal{L}[\pi_n(x)P_n(x)] = a_n\mathcal{L}[x^nP_n(x)]$$

a_n the leading coeffecient of π

We can conclude that

$$\mathcal{L}[P_n^2] = \mathcal{L}[x^nP_n] = \lambda_1\lambda_2\ldots\lambda_{n+1}.$$

This tells us that $\mathcal{L}$ is quasidefinite at the very least, and that P_n is the OPS as long as $\lambda_n \neq 0$. We can also see that the moments are real as long as c_n and λ_n are real, which also shows the positive definiteness.

Chapter 5 Determinacy of $\mathcal{L}$ in the bounded case

Theorem 5.1. Main Theorem

If the true interval of orthogonality of a positive definite moment functional $\mathcal{L}$ is bounded then $\mathcal{L}$ is determinate. ♡

Definition 5.1. Quasi-orthogonal

For $q(x) \neq 0$, q is a quasi-orthogonal polynomial of order $n+1$ if it is of degree at most $n+1$ and

$$\mathcal{L}[x^k q(x)] = 0, \quad k = 0, 1, \ldots n-1$$

♣

Note that orthogonal polynomials are quasi-orthogonal.

Theorem 5.2

$q(x)$ is quasi-orthogonal degree $n+1$ if and only if there exists constants $A, B \neq 0$ such that

$$q(x) = AP_{n+1}(x) + BP_n(x).$$

Also, we can determine A, B such that

$$q(z_0) = 0.$$

♡

Theorem 5.3

The zeroes of a real quasi-orthogonal polynomial are all real and simple. At most one of the zeros lies outside (ξ_1, η_1) ♡

Theorem 5.4. A Quadrature Formula

Let $y_{n0} < y_{n1} < \cdots < y_{nn}$ denote the zeros of $q(x)$ in ascending order. Then there exists B_{ni} constant such that

$$\mathcal{L}[\pi(x)] = \sum_{i=0}^{n} B_{ni}\pi(y_{ni}$$

for every polynomial π of degree at most $2n$ ♡

Note here we can replace π with x^n to get something analgous to the Guass quadrature. Actually, we can even reconstruct the representation theorem from here to construct a ψ representative of $\mathcal{L}$.