Chapter 1 Representation Theorem

1.1 Riemann-Stieltjes Integral

Definition 1.1. Riemann-Stieltjes Integral

Let f and g be continuous functions over $[a,b]$ and $P = x_1, \ldots, x_n$ a partition of $[a,b]$. Then for $c_i \in [x_i, x_{i+1}]$

$$\int_a^b f(x)dg(x) = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(c_i)[g(x_{i+1}) - g(x_i)]$$

♣

There are two relevant special cases of the Riemann-Stieltjes(RS) Intgeral

$g(x) = x$ is the normal Riemann integral

and if $g(x)$ is continuously differentiable over $\mathbb{R}$

$$\int_a^b f(x)dg(x) = \int_a^b f(x)g'(x)dx$$

assuming f is Riemann integrable in the first place. What's more, we may expect equality to hold in the case that $g(x)$ only has jump discontinuities. This is not the case.
There is a simple condition for the existence of the RS integral that requires the concept of bounded variation

Definition 1.2. Total Variation

The Total Variation of a function ψ over $[a,b]$ is

$$V_a^b(\psi) = \sup_P \sum_{i=0}^{n_P-1} |\psi(x_{i+1} - x_i)|$$

where P is the set of all partitions of $[a,b]$ and n_P is the size of a parition. ♣

In a sense, the total variation is a measure of the how much $f(x)$ moves over $[a,b]$. This takes a nice form for functions whos derivative are riemann integralable, and makes variation more clear.

Theorem 1.1. I

f is differentiable and f' is riemann integrable then

$$V_a^b(f) = \int_a^b |\frac{df}{dx}|dx.$$

♡

We connect this with our theory of orthogonal polynomials by seeing that if ψ has total variation that is bounded (bounded variation) over every interval then we can construct a moment functional

$$\mathcal{L}(x^n) = \int_{-\infty}^{\infty} x^n d\psi(x)$$

with the condition that such an integral converges. In general, such a function will not have $\Delta_n \neq 0$ however if the set $M = \{x|\psi(x+\delta) - \psi(x-\delta) \quad \forall \delta > 0\}$ is infinite then $\mathcal{L}$ will be positive definite. What's more, M will actually be a supporting set for $\mathcal{L}$.

Definition 1.3. Supporting Set

A set $E \subset \mathbb{R}$ is a supporting set for $\mathcal{L}$ if $\mathcal{L}$ is positive definite on E. ♣

Recall that this is equivelent to $\mathcal{L}(p(x)) > 0$ for all real polynomials $p(x) \geq 0$ on E that do not vanish identically.

We now come to the definition of a *distribution function*

Definition 1.4. Distribution Function

A bounded, non-decreasion function ψ is called a distribution function if all of its moments are finite

$$\mu_n = \int_{-\infty}^{\infty} x^n d\psi(x)$$

♣

Whats more, the previously defined M is called the *spectrum* of ψ, with an element being a *spectral point.* What more can we say about this spectrum?

Example 1.1 What is the spectrum for $f(x) = x^2$?

It is the set of all x such that $(x+\delta)^2 - (x-\delta)^2 > 0$ for all δ. Expanding

$$\begin{aligned}(x+\delta)^2 - (x-\delta)^2 &= x^2 + 2x\delta + \delta^2 - x^2 + 2x\delta - \delta^2 \\ &= 4x\delta.\end{aligned}$$

We see then that the spectrum of x^2 is $x > 0$.

Example 1.2 What is the spectrum for $f(x) = sin(x)$?

We see

$$\sin(x+\delta) - \sin(x-\delta) = 2\cos(x)\sin(\delta)$$

by a trig identity and so for any x that makes $\cos(x)$ positive, we may find a δ that makes $\sin(\delta)$ negative. Thus there is no spectrum for $f(x)$.

Example 1.3 What is the spectrum of x^n?

See that

$$
\begin{aligned}
(x+\delta)^n-(x-\delta)^n &= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}\delta^k-\sum_{k=0}^{n}\binom{n}{k}(-1)^k x^{n-k}\delta^k \\
&= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}\delta^k(1-(-1)^k)
\end{aligned}
$$

so for even k the terms vanish. Putting $k \to 2k+1$ we get

$$\sum_{k=0}^{n}\binom{n}{2k+1}x^{n-2k-1}\delta^{2k+1}$$

If $n=2m+1$ is odd we get

$$\sum_{k=0}^{n}\binom{n}{2k+1}x^{2(m-k)}\delta^{2k+1}=\delta\sum_{k=0}^{n}\binom{n}{2k+1}x^{2(m-k)}\delta^{2k}$$

and so for *any* fixed x, the sum will be positive for all $\delta>0$.

If $n=2m$ is even we get

$$\sum_{k=0}^{n}\binom{n}{2k+1}x^{2(m-k)-1}\delta^{2k+1}=\delta\sum_{k=0}^{n}\binom{n}{2k+1}x^{2(m-k)-1}\delta^{2k}$$

and so it follows that if $x\leq 0$ then the sum will be 0 or negative for any δ. The spectrum then is $x>0$.

By looking at the definition, we can see that we are looking for all the points x on the function, such that if take a ball around x, the right side will always be greater than the left side. Intuitivly then we can expect that all even monic polynomials will have positive support, which atleast agrees with our findings of x^n. In fact, any strictly increasing function will have support over all of $\mathbb{R}$, while a non-decreasing function will have support over all the intervals that don't include their constant parts. The lastly, we notice that if we have two integrators $g_1(x)$ and $g_2(x)=g_1(x)+C$, the RS integral is the same due to $g(x_{i+1})-g(x_i)$ in the definition cancelling out the constants.

Definition 1.5. Substantially equal

Two distribution functiosn are substantially equal if

$$\psi_1(x)=\psi_2(x)+C$$

at all common points of continuity. ♣

1.2 Some Convergence Theorems

This section is composed of some theorems that have to do with convergence necessary for the major theorem of this section. Proofs available in the book.

Theorem 1.2

Let f_n be a sequence of real functions defined on a countable set E. If for each $x \in E$, $\{f_n(x)\}$ is bounded, then $\{f_n\}$ contains a subsequence that converges everywhere in E.

Theorem 1.3

Let ψ_n be a uniformly bounded sequence of non-decreasing functions defined on $(-\infty, \infty)$. Then $\{\psi_n\}$ has a subsequence which converges on $(-\infty, \infty)$to a bounded, non-decreasing function.

Theorem 1.4

Let $\{\psi_n\}$ be a uniformly bounded($\psi_n < C, \forall n$) sequence of non-decreasing functions defined on a compact interval $[a, b]$ and let $\psi_n \to \psi$ on $[a, b]$. Then for every real function f continuous on $[a, b]$

$$\lim_{n\to\infty} \int_a^b f d\psi_n = \int_a^b f d\psi.$$

1.3 The Representation Theorem

The main theorem of this chapter is the Representation Theorem, which I hope answers Nassers question from last week "Can any orthogonal polynomial sequence be represented by an integral from a to b w.r.t some weight function $w(x)$.". This theorem states the following

Theorem 1.5. The Representation Theorem

Every positive-definite moment functional can be represented as a Riemann-Stieltjes integral with a non-decreasing integrator ψ whose spectrum is an infinite set

$$\mathcal{L}[x^k] = \int_{-\infty}^{\infty} x^k d\psi(x), \quad k = 0, 1, 2, \dots$$

and so in the case that $\psi(x)$ is continuously differentiable, Nasser's question is answered. However, I'm not sure if this is the case in the representation theorem. We'll see that the differentiability depends on how a sequence of right-continuous step functions converges. Intuition says yes, but I don't have a clear answer yet.

Proof Let $\mathcal{L}$ be a positive-definite($\mathcal{L}[p(x)] > 0, p(x) > 0$) moment function with moments μ_n. By the Gauss quadrature formula presented by Queen, there exists

$A_{n1}, A_{n2}, \dots, A_{nn}$ such that

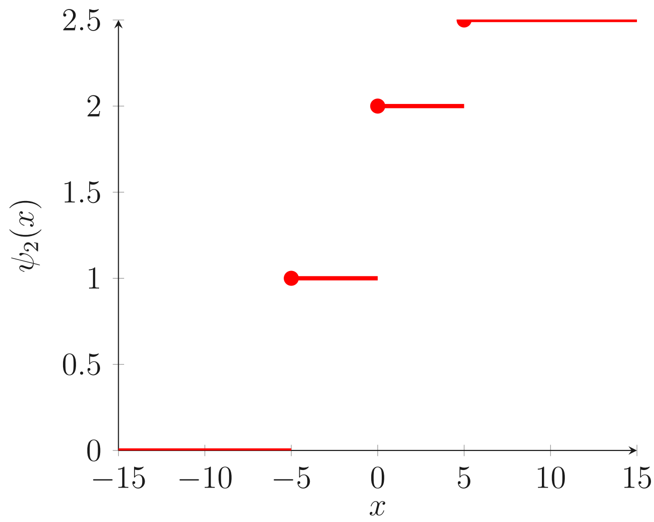
$$\mathcal{L}[x^k] = \mu_k = \sum_{i=1}^{n} A_{ni} x_{ni}^k, \quad k = 0, 1, \dots, 2n-1$$

where $x_{n1} < x_{n2} < \cdots < x_{nn}$ are the zeroes of the nth degree monic OPS corresponding to $\mathcal{L}$. Now define

$$\psi_n(x) = \begin{cases} 0, \quad x < x_{n1} \\ A_{n1} + \cdots + A_{np}, \quad x_{np} \le x < x_{n,p+1} \quad (1 \le p < n) \\ \mu_0, \quad x \ge x_{nn} \end{cases}$$

so $\psi_2(x)$ might look something like Now, $\psi_n(x)$ is a bounded, right continuous, non-

Figure 1.1



decreasing step function whos spectrum is the finite set $\{x_{n1}, \dots, x_{nn}\}$ with jumps $A_{ni} > 0$. Because the spectrum is just $\{x_{ni}\}$ we get

$$\int_{\infty}^{\infty} x^k d\psi_n(x) = \sum_{i=1}^{n} A_{ni} x_{ni}^k = \mu_k.$$

Now, by theorem 1.3 there is a subsequence which converges on $\mathbb{R}$ to a bounded, non-decreasing function ϕ. If the true interval of orthogonality of $\mathcal{L}$ was bounded, then remembering in the RS integral $\sum f(c_i)[\psi(x_{i+1}) - \psi(x_i)]$ we see that outside the true orthogonal interval the integral vanishes. This allows us to extend the bounds to infinity and in addition to theorem 1.4 we conclude

$$\int_{-\infty}^{\infty} x^k d\psi(x) = \mu_k = \mathcal{L}[x^k].$$

If the true interval of orthogonality is *unbounded*, we need to treat this sepretaly.

Lemma 1.1

Let $\mathcal{L}$ be a positive-definite moment functional and let ψ_n be defined as above. Then there is a subsequence that converges on $\mathbb{R}$ to a distribution function ψ which has infinite spectrum and for which the representation theorem is valid. ♡

As we saw before, we have a subsequence $\{\psi_{n_i}\}$ which converges on $\mathbb{R}$ to ψ. Let $\phi_i = \psi_{n_i}$ then we have

$$\int_{\infty}^{\infty} x^k d\phi_i(x) = \mu_k, \quad n_i \geq \frac{k+1}{2}.$$

By theorem 1.4 we have for any compact interval $[\alpha, \beta]$

$$\lim_{i\to\infty} \int_\alpha^\beta x^k d\phi_i(x) = \int_\alpha^\beta x^k d\psi(x).$$

Taking $\alpha < 0 < \beta$ and $n_i > k+1$, we can write

$$\begin{aligned}
|\mu_k - \infty_\alpha^\beta x^k d\psi(x)| &= |\int_{-\infty}^{\infty} x^k d\phi_i(x) - \int_\alpha^\beta x^k d\psi(X)| \\
&\leq |\int_{-\infty}^{\alpha} x^k d\phi_i(x)| + |\int_\beta^\infty x^k d\phi_i(x)| + |\int_\alpha^\beta x^k d\phi_i(x) - \int_\alpha^\beta x^k d\psi(x)|.
\end{aligned}$$

however we also have that

$$\begin{aligned}
|\int_\beta^\infty x^k d\phi_i(x)| &= |\int_\beta^\infty \frac{x^{2(k+1)}}{x^{k+2}} d\phi_i(x)| \\
&\leq \beta^{-(k+2)} |\int_\beta^\infty x^{2(k+1)} d\phi_i(x)| \\
&\leq \beta^{-(k+2)} u_{2(k+1)}.
\end{aligned}$$

and in the same fashion

$$|\int_\beta^\infty x^k d\phi_i(x)| \leq \alpha^{-(k+2)} u_{2(k+1)}.$$

Then

$$|\mu_k - \int_\alpha^\beta x^k d\psi(X)| \leq |\int_\alpha^\beta x^k d\phi_i(x) - \int_\alpha^\beta x^k d\psi(x)| + \left(|\alpha|^{-(k+2)} + \beta^{-(k+2)}\right) \mu_{2k+2}$$

in which we let $i \to \infty$ to elimate the integral term on the right and $\alpha \to -\infty$, $\beta \to \infty$ which gives the desired result. It actually turns out to be quite easy to prove this has infinite spectrum. Suppose it had N spectral points. Then let $p(x)$ be a polynomial with zeroes at the spectral point implying $\mathcal{L}[p(x)^2] = 0$ which contradicts positive-definiteness.