Asymptotic Iteration Method for Hahn Difference Equations

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What's the Asymptotic Iteration Method?

A way to solve a special class of differential equations

$$y''(x) = \lambda_0(x)y'(x) + s_0(x)y(x)$$

What's the Asymptotic Iteration Method?

AIM Sequence

Define $\lambda_n(x)$ to be

$$\lambda_n(x) = \lambda'_{n-1}(x) + s_{n-1}(x) + \lambda_0(x)\lambda_{n-1}(x)$$

and $s_n(x)$ to be

$$s_n(x) = s'_{n-1}(x) + s_0(x)\lambda_{n-1}(x).$$

What's the Asymptotic Iteration Method?

Terminating Condition and main conclusion

If $\frac{s_n(x)}{\lambda_n(x)} = \frac{s_{n-1}(x)}{\lambda_{n-1}(x)}$ then the solution is

$$y(x) = \exp\left(-\int^x \frac{s_n(t)}{\lambda_n(t)} dt\right) \times \left[C_2 + C_1 \int^x \exp\left(\int^t \left(\lambda_0(\tau) + 2\frac{s_n(\tau)}{\lambda_n(\tau)}\right) d\tau\right) dt\right]$$

What's a Hahn Difference Equation?

The Hahn Difference Operator

The Hahn operator $\Delta_{q;w}$ was introduced by Wolfgang Hahn in 1948 and is defined as

$$\Delta_{q;w}y(x) = \frac{y(qx+w)-y(x)}{qx+w-x}, \quad 0<q<1, w>0$$

It's a generalization of the q-derivative

$$D_q y(x) = \frac{y(qx)-y(x)}{qx-x}$$

and the finite difference operator

$$\Delta_w y(x) = \frac{y(x+w)-y(x)}{w}$$

What’s a Hahn Difference Equation?

The Hahn Difference Operator

We can even get the ordinary derivative by taking the limit of both these

$$\begin{tikzcd}
& D_q y(x) \arrow[dr, "q \to 1"] & \\
\Delta_{q;w} y(x) \arrow[ur, "w \to 0"] \arrow[dr, "q \to 1"'] \arrow[rr, dashed, "q \to 1 \ \ w \to 0", "w \to 0 \ \ q \to 1"'] & & \frac{d}{dx} y(x) \\
& \Delta_w y(x) \arrow[ur, "w \to 0"'] &
\end{tikzcd}$$

What's a Hahn Difference Equation?

Why was it created?

So what did Hahn do with this operator?

What's a Hahn Difference Equation?

Why was it created?

Über Orthogonalpolynome, die q-Differenzengleichungen genügen.

Von Wolfgang Hahn in Berlin.

(Eingegangen am 9. 8. 1948.)

§ 1. Problemstellung.

Es sei $M_0, M_1, \ldots, M_n, \ldots$ eine Zahlenfolge. Die Hankelschen Determinanten

$$D_{0,n} = |M_{i+k}| \qquad (i, k = 0, 1, \ldots, n-1)$$

mögen für alle n von Null verschieden sein. Die Polynome

$$p_n(x) = \frac{1}{D_{0,n}} \begin{vmatrix} M_0 & M_1 & \ldots & M_n \\ M_1 & M_2 & \ldots & M_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ M_{n-1} & M_n & \ldots & M_{2n-1} \\ 1 & x & \ldots & x^n \end{vmatrix} \tag{1.1}$$

genügen bekanntlich einer dreigliedrigen linearen Rekursionsformel

$$p_n(x) = (x + \alpha_n)\, p_{n-1}(x) + \beta_n\, p_{n-2}(x). \tag{1.2}$$

Sind alle $D_{0,n}$ positiv (und damit alle β_n negativ), so existieren mindestens eine „Belegungsfunktion" $\psi(x)$ und ein reelles Intervall (a, b) derart, daß

$$\int_a^b x^n \, d\psi(x) = M_n, \qquad \int_a^b p_n(x)\, p_m(x)\, d\psi(x) = 0 \qquad (m \neq n).$$

I'm not sure!

What's a Hahn Difference Equation?

The Difference Equation

The second order linear homogeneous Hahn difference equation is

$$\Delta^2_{q;w} y(x) = \lambda(x)\Delta_{q;w} y(x) + s(x)y(x)$$

In general there is no known solution

qw-AIM

The extension of the Asymptotic Iteration Method(AIM) to Hahn operator is called qw-AIM.

qw-AIM

Main Result

Let $x_i = q^i x + w \sum_{k=0}^{i-1} q^k$, the i^{th} composition of $qx + w$. The second order linear homogeneous Hahn difference equation

$$y(x) = \frac{1}{\prod_{i=0}^{\infty}\left[1 - q^i((q-1)x + w)\frac{s_{n-1}(x_i)}{\lambda_{n-1}(x_i)}\right]} + \sum_{i=0}^{\infty} \frac{\frac{q^i((q-1)x+w)}{\lambda_{n-1}(x_i)}}{\prod_{j=0}^{i}\left[1 - q^j((q-1)x + w)\frac{s_{n-1}(x_j)}{\lambda_{n-1}(x_j)}\right]\prod_{j=0}^{\infty}\left[1 + q^{i+j}((q-1)x + w)\frac{\lambda_n(x_{i+j})}{\lambda_{n-1}(x_{i+j})}\right]}$$

for $0 < q < 1$, $w > 0$ provided that

$$\frac{s_n(x)}{\lambda_n(x)} = \frac{s_{n-1}(x)}{\lambda_{n-1}(x)} \text{ \emph{equivalently} } \delta_n := s_n(x)\lambda_{n-1}(x) - \lambda_n(x)s_{n-1}(x) = 0.$$

First $x_i = q^i x + w \sum_{k=0}^{i-1} q^k$ is just the $i'th$ composition of $qx + w$

$$
\begin{aligned}
x_0 &= x \\
x_1 &= qx + w \\
x_2 &= q(qx + w) + w = q^2 x + qw + w \\
x_3 &= q(q(qx + w) + w) + w = q^3 x + q^2 w + qw + w \\
\ldots \\
x_i &= q^i x + w \sum_{k=0}^{i-1} q^k.
\end{aligned}
$$

Note that if $i \to \infty$ and x_i converges then $x_i \to \frac{w}{1-q} := w_0$

qw-AIM

How do we get there?

Start by differentiating the original equation

$$\Delta^2_{q:w} y(x) = \lambda_0(x)\Delta_{q:w} y(x) + s_0(x)y(x)$$

$$\begin{aligned}
\Delta^3_{q:w} y(x) &= \Delta_{q:w}[\lambda_0(x)\Delta_{q:w} y(x)] + \Delta_{q:w}[s_0(x)y(x)] \\
&= \lambda_0(qx+w)\Delta^2_{q:w} y(x) + \Delta_{q:w} y(x)\Delta_{q:w}\lambda_0(x) \\
&+ s_0(qx+w)\Delta_{q:w} y(x) + y(x)\Delta_{q:w} s_0(x) \\
&= \lambda_0(qx+w)[\lambda_0(x)\Delta_{q;w} y(x) + s_0(x)y(x)] \\
&+ \Delta_{q:w} y(x)\Delta_{q:w}\lambda_0(x) + s_0(qx+w)\Delta_{q:w} y(x) \\
&+ y(x)\Delta_{q:w} s_0(x) \\
&= \Delta_{q:w} y(x)[\Delta_{q:w}\lambda_0(x) + \lambda_0(qx+w)\lambda_0(x) + s_0(qx+w)] \\
&+ y(x)[\lambda_0(qx+w)s_0(x) + \Delta_{q:w} s_0(x)] \\
&= \lambda_1(x)\Delta_{q:w} y(x) + s_1(x)y(x)
\end{aligned}$$

This is where we get our qw-AIM sequence

$$\Delta_{q:w}^{n+2} y(x) = \lambda_n(x)\Delta_{q:w} y(x) + s_n(x)y(x)$$

so

$$\begin{aligned}
\lambda_n(x) &= \Delta_{q:w}\lambda_{n-1}(x) + \lambda_{n-1}(qx+w)\lambda_0(x) + s_{n-1}(qx+w) \\
s_n(x) &= \Delta_{q:w} s_{n-1}(x) + \lambda_{n-1}(qx+w)s_0(x)
\end{aligned}$$

Consider the ratio

$$\frac{\Delta_{q;w}^{n+2}y(x)}{\Delta_{q;w}^{n+1}y(x)} = \frac{\lambda_n(x)\Delta_{q:w}y(x) + s_n(x)y(x)}{\lambda_{n-1}(x)\Delta_{q:w}y(x) + s_{n-1}(x)y(x)}$$

$$= \frac{\lambda_n(x)\left[\Delta_{q:w}y(x) + \frac{s_n(x)}{\lambda_n(x)}(x)y(x)\right]}{\lambda_{n-1}(x)\left[\Delta_{q:w}y(x) + \frac{s_{n-1}(x)}{\lambda_{n-1}(x)}(x)y(x)\right]}.$$

If the *terminating condition* $\frac{s_n}{\lambda_n} = \frac{s_{n-1}}{\lambda_{n-1}}$ is satisfied we have

$$\frac{\Delta_{q;w}^{n+2}y(x)}{\Delta_{q;w}^{n+1}y(x)} = \frac{\lambda_n(x)}{\lambda_{n-1}(x)} \text{ or } \Delta_{q;w}\Delta_{q;w}^{n+1}y(x) = \frac{\lambda_n(x)}{\lambda_{n-1}(x)}\Delta_{q;w}^{n+1}y(x).$$

This is actually a first order difference equation in $\Delta_{q;w}^{n+1}y(x)$!

qw-AIM

The first order homogeneous difference equation

We can actually solve the first order difference equation

$$\Delta_{q;w}^{n+1} y(x) = \frac{\Delta_{q;w}^{n+1} y(w_0)}{\prod_{i=0}^{\infty}[1 + q^i((q-1)x + w)\frac{\lambda_n(x_i')}{\lambda_{n-1}(x_i')}]}.$$

With some more manipulation...

qw-AIM

The first order homogeneous difference equation

$$\Delta_{q;w}^{n+1} y(x) = \frac{\Delta_{q;w}^{n+1} y(w_0)}{\prod_{i=0}^{\infty}[1 + q^i((q-1)x + w)\frac{\lambda_n(x_i')}{\lambda_{n-1}(x_i')}]}$$

$$\lambda_{n-1}(x)\Delta_{q:w} y(x) + s_{n-1}(x) y(x) = \frac{\Delta_{q;w}^{n+1} y(w_0)}{\prod_{i=0}^{\infty}[1 + q^i((q-1)x + w)\frac{\lambda_n(x_i')}{\lambda_{n-1}(x_i')}]}$$

$$\Delta_{q:w} y(x) = -\frac{s_{n-1}(x)}{\lambda_{n-1}(x)} y(x) + \frac{\Delta_{q;w}^{n+1} y(w_0)}{\lambda_{n-1}(x)\prod_{i=0}^{\infty}[1 + q^i((q-1)x + w)\frac{\lambda_n(x_i')}{\lambda_{n-1}(x_i')}]}$$

This is a *non-homogeneous first order equation*.
We can solve this too!

qw-AIM

The first order non-homogeneous difference equation

We know the solution to the non-homogeneous case, thus

$$\Delta_{q:w} y(x) = -\frac{s_{n-1}(x)}{\lambda_{n-1}(x)} y(x) + \frac{\Delta_{q;w}^{n+1} y(w_0)}{\lambda_{n-1}(x) \prod_{i=0}^{\infty} [1 + q^i((q-1)x + w) \frac{\lambda_n(x_i')}{\lambda_{n-1}(x_i')}]}$$

has solution

$$y(x) = \frac{y(w_0)}{\prod_{i=0}^{\infty} [1 - q^i((q-1)x + w) \frac{s_{n-1}(x_i')}{\lambda_{n-1}(x_i')}]} - \sum_{i=0}^{\infty} \frac{\frac{q^i((q-1)x+w)\Delta_{q;w}^{n+1} y(w_0)}{\lambda_{n-1}(x_i') \prod_{i=0}^{\infty} \left[1+q^i((q-1)x_i'+w) \frac{\lambda_n(x_i')}{\lambda_{n-1}(x_i')}\right]}}{\prod_{j=0}^{i} \left[1 - q^j((q-1)x + w) \frac{s_{n-1}(x_j')}{\lambda_{n-1}(x_j')}\right]}$$

which is the final solution.

qw-AIM

What does this mean?

If we have a difference equation, we can check if it's qw-AIM sequence satisfies the terminating condition $\delta_n = 0$. If it does then we have the solution!

qw-AIM

Example 1

For $q < 1$

$$\Delta_{q;w}^2 y(x) = x\Delta_{q;w} y(x) - y(x)$$

$$\lambda_0(x) = x$$

and

$$s_0(x) = -1.$$

So $\lambda_1(x) = qx^2 + wx$ and $s_1(x) = -(qx + w)$. It follows that

$$\frac{s_0(x)}{\lambda_0(x)} = \frac{-1}{x} = \frac{-(qx + w)}{qx^2 + wx} = \frac{s_1(x)}{\lambda_1(x)}$$

qw-AIM

Example 1

Our solution is

$$
\begin{aligned}
y(x) &= \frac{1}{\prod_{i=0}^{\infty}\left[1 - q^i((q-1)x + w)\frac{s_0(x_i)}{\lambda_0(x_i)}\right]} \\
&= \frac{1}{\prod_{i=0}^{\infty}\left[1 - q^i((q-1)x + w)\frac{-1}{x_i}\right]} \\
&= \lim_{k\to\infty} \frac{1}{\prod_{i=0}^{k}\left[1 + \frac{q^i((q-1)x+w)}{(q^i x + w[i]_q)}\right]} \\
&= \lim_{k\to\infty} \frac{x}{q^{1+k} + [k+1]_q w} \\
&= \frac{(1-q)x}{w}.
\end{aligned}
$$

qw-AIM

Hyper-geometric Equation

$$(ex^2 + 2fx + g)\Delta^2_{q;w}y(x) + (2\epsilon x + \gamma)\Delta_{q;w}y(x) + \tau y(x) = 0$$

qw-AIM

Hyper-geometric Equation

$$\Delta_{q;w}^2 y(x) = \underbrace{-\frac{2\epsilon x + \gamma}{ex^2 + 2fx + g}}_{\lambda_0(x)} \Delta_{q;w} y(x) \underbrace{-\frac{\tau}{ex^2 + 2fx + g}}_{s_0(x)} y(x)$$

qw-AIM

Hyper-geometric Equation

$\delta_1 = s_1(x)\lambda_0(x) - \lambda_1(x)s_0(x) = \tau(2\epsilon + \tau)$ so $\delta_1 = 0$ if $\tau = 0$ or $\tau = -2\epsilon$

After substitution and simplification, the solution is

$$y(x) = x + \frac{\gamma}{2\epsilon}$$

Going a step farthur $\delta_2 = \tau(2\epsilon + \tau)((1+q)(e+2\epsilon)+\tau)$
so if $\tau = -(1+q)(e+2\epsilon)$ we can get the solution

qw-AIM

Hyper-geometric equation

$$y(x) = x^2 + \frac{(2f+\gamma)(1+q)+2\epsilon w}{(1+q)e+2\epsilon q} x + \frac{2\epsilon g q+\gamma(2f_\gamma+2\epsilon w)+e((1+q)g+\gamma w)}{(e+2\epsilon)((1+q)e+2\epsilon q)}$$

And in general we can get a solution for each τ as

$$\tau_n = -\left(\sum_{j=0}^{n-1} q^j\right)\left(2\epsilon + e\sum_{j=0}^{n-2} q^j\right)$$

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