Numerical Analysis Homework 4

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Chapter 3

3)

Solution:

We consider theorem 2.1.4.1 on two knots $\{a, b\}$. Denote $P(x) = \frac{f(a)(x-b)}{(a-b)} + \frac{f(b)(x-a)}{(b-a)}$ the interpolating polynomial. We have that for every x there exists $\eta(x)$ such that

$$f(x) - P(x) = \frac{w(x)f''(\eta(x))}{2}.$$

We'll integrate both sides from a to b then solve in parts

$$\int_a^b f(x) - P(x)dx = \int_a^b \frac{(x-a)(x-b)f''(\eta(x))}{2}dx.$$

$\int_a^b f(x)dx$ is the true integral so we'll just denote it I. $\int_a^b P(x)dx$ is just the trapezoidal rule so we can immediately see that

$$\int_a^b P(x)dx = \frac{(b-a)(f(b)+f(a))}{2}.$$

Now on the right hand side we recall the following mean value theorem for integrals

Theorem 1. *Let f be continuous and g a function which does not change sign on $[a, b]$. Then there exists c such that*

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx.$$

By taking f to be $f''(\eta(x))$ and $g(x) = (x-a)(x-b)$ we get that there exists an $\tilde{x}$ such that

$$I - \frac{(b-a)(f(b)+f(a))}{2} = \frac{f''(\bar{x})}{2}\int_a^b (x-a)(x-b)dx.$$

The integral on the right is

$$\begin{aligned}
\int_a^b (x-a)(x-b)dx &= \int_a^b x^2 - (a+b)x + ab dx \\
&= \left[\frac{x^3}{3} - (a+b)\frac{x^2}{2} + abx\right]_b^a \\
&= \left[\frac{b^3}{3} - (a+b)\frac{b^2}{2} + ab^2\right] - \left[\frac{a^3}{3} - (a+b)\frac{a^2}{2} + a^2b\right] \\
&= -\frac{b^3 - 3ab^2}{6} - \frac{3a^2b - a^3}{6} \\
&= -\frac{(b-a)^3}{6}
\end{aligned}$$

giving the desired result

$$I - \frac{(b-a)(f(b)+f(a))}{2} = -\frac{(b-a)^3 f''(\bar{x})}{12}.$$

Now, it remains to show that $f''(\eta(x))$ is itself continuous, justifying our use of the mean value theorem. Since

$$f(x) - P(x) = \frac{w(x) f''(\eta(x))}{2}.$$

we can just rearrange to get

$$\frac{2(f(x)-P(x))}{w(x)} = f''(\eta(x)).$$

and we have that $f''(\eta(x))$ is continuous atleast on the open interval (a, b). We know that $\eta(x) \in [a, b]$, so $f''(\eta(x))$ is certainly defined on $[a, b]$, but it remains to show that it is continuous, thus we need to verify that the limit exists at the end points $x \to a$ (or b) from within the interval. Since taking the limit of top and bottom is indeterminate $\frac{0}{0}$ we can use lhopitals rule to see the limit is

$$2\frac{f''(a) - P'(a)}{2a-(a+b)} = 2\frac{f''(a) - \left(\frac{f(a)-f(b)}{a-b}\right)}{a-b}$$

for $x \to a$ and

$$2\frac{f''(b) - P'(b)}{2b-(a+b)} = 2\frac{f''(b) - \left(\frac{f(a)-f(b)}{a-b}\right)}{b-a}$$

for $x \to b$. Thus $f''(\eta(x))$ is continuous over $[a, b]$ and we have what was to be shown. □

14)

Consider a weight function $\omega(x) \geqslant 0$ which satisfies (3.6.1a) and (3.6.1b). Show that (3.6.1c) is equivalent to

$$\int_b^a \omega(x)\,dx > 0.$$

Hint: The mean-value theorem of integral calculus applied to suitable subintervals of $[a, b]$.

Solution:

3.6.1a) $\omega(x)$ is measurable on $[a, b]$ finite or infinite.

3.6.1b) All moments are finite and exist.

We want to show

$$\int_b^a \omega(x)\,dx > 0.$$

if and only if for all nonnegative polynomials on $[a, b]$ $s(x)$

$$\int_b^a s(x)\omega(x)dx = 0 \iff s(x) = 0.$$

Suppose $\int_b^a \omega(x)\,dx > 0$ andd $\int_b^a s(x)\omega(x)dx = 0$. Then using the integral mean value theorem there exists a $c \in [a, b]$ such that

$$\int_b^a s(x)w(x)dx = s(c)\int_b^a w(x)dx = 0$$

so it must be that $s(c) = 0$. Now since $\omega(x)$ and $s(x)$ are both positive on the whole interval, we can repeat this argument for any suitable subinterval of $[a, b]$. Since it holds on arbitrary intervals, it must be that $s(x) = 0$ everywhere on the interval and thus $s(x) = 0$.

Now suppose for all nonnegative polynomials on $[a,b]$ $s(x)$

$$\int_b^a s(x)\omega(x)dx = 0 \iff s(x) = 0.$$

Since $\omega(x) \geqslant 0$ we have that $\int_b^a \omega(x)dx \geqslant 0$ with $\int_b^a \omega(x)dx = 0$ if and only if $\omega(x) = 0$. Thus to rule out this case and conclude $\int_b^a \omega(x) > 0$ suppose that $\omega(x) = 0$. Then for any nonnegative polynomial $s(x) \neq 0$ we have $\int_b^a s(x)w(x)dx = 0$ but $s(x) \neq 0$. Thus $\int_b^a \omega(x)dx > 0$.

18)

Consider Gaussian integration in the interval $[-1,+1]$ with the weight function

$$\omega(x) \equiv \frac{1}{\sqrt{1-x^2}}.$$

In this case, the orthogonal polynomials $p_j(x)$ are the classical Chebyshev polynomials $T_0(x) \equiv 1$, $T_1(x) \equiv x$, $T_2(x) \equiv 2x^2-1$, $T_3(x) \equiv 4x^3-3x$, $\ldots$, $T_{j+1}(x) \equiv 2xT_j(x) - T_{j-1}(x)$, up to scalar factors.

(a) Prove that $p_j(x) \equiv (1/2^{j-1})T_j(x)$ for $j \geqslant 1$. What is the form of the tridiagonal matrix (3.6.19) in this case?

Solution:

By linearity we know that $\langle p_n, p_m\rangle = a_n a_m \langle T_n, T_m\rangle = \alpha_{nm}\delta_{nm}$ where $\alpha_{nn} = \begin{cases} a_0^2\pi, & n = 0 \\ \frac{a_n^2\pi}{2}, & n \neq 0 \end{cases}$ so p_j is orthogonal. We want to show that p_j is the sequence of monic orthogonal polynomials for this weight function, thus it will suffice to show that the coeffecient of the term of degree j in T_j is 2^{j-1}. We'll do this by induction: For a base case, see that $T_1(x) = x = 2^{1-1}x$. Then suppose the highest coeffecient of $T_j(x)$ is 2^{j-1}. Then looking at the two term recurrence relation, T_{j+1} must have highest coeffecient $2 \cdot 2^{j-1}$

$$T_{j+1}(x) = \underbrace{2xT_j(x)}_{deg(2xT_j(x))=j+1} - \underbrace{T_{j-1}(x)}_{deg(T_{j-1}(x))=j-1}$$

since T_{j-1} has degree $j-1$ and thus cannot affect the coeffecients of degree j or $j+1$ in $2xT_j(x)$. It follows that the coeffecient of degree $j+1$ in T_{j+1} is $2 \cdot 2^{j-1} = 2^j$ thus by the principle of mathematical induction we have that the highest coeffecient of $T_j(x)$ is 2^{j-1} for all $j \geqslant 1$. Thus $p_j(x) = \frac{1}{2^{j-1}}T_j(x)$ form a sequence of monic orthogonal polynomials as was to be shown.

To show the tridiagonal matrix, we just need to write the recurrence relation for p_j

$$\begin{aligned} p_{j+1} &= 2^{-j}T_{j+1} \\ &= 2^{-j}2xT_j(x) - 2^{-j}T_{j-1}(x) \\ &= 2^{-j}2x2^{j-1}p_j - 2^{-j}2^{j-2}p_{j-1}(x) \\ &= xp_j - \frac{1}{4}p_{j-1} \end{aligned}$$

thus $\delta_i = 0$ and $\gamma_i = \frac{1}{2}$ for all $i \geqslant 0$ in 3.6.5b). The tridiagonal matrix is thus

$$J_n = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \ldots & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & 0 & \frac{1}{2} \\ 0 & \ldots & \ldots & \frac{1}{2} & 0 \end{bmatrix}$$

(b) For $n = 3$, determine the equation system (3.6.13). Verify that $w_1 = w_2 = w_3 = \pi/3$. (In the Chebyshev case, the weights w_i are equal for general n.)

Solution:

Recall that (3.6.13) is

$$\sum_{i=1}^{n} p_k(x_i) w_i = \begin{cases} (p_0, p_0), & k = 0 \\ 0, & k = 1, 2, \dots, n-1 \end{cases}$$

where x_i are the zeros of $p_n(x)$. We should calculate (p_0, p_0):

$$\begin{aligned}
(p_0, p_0) &= (1, 1) \\
&= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx \\
&= [\arcsin(x)]_{-1}^{1} \\
&= \frac{\pi}{2} - -\frac{\pi}{2} \\
&= \pi
\end{aligned}$$

We want to find the w_i so we need to calculate both the zeros of p_n and the values of $p_k(x_i)$. I'll find the zeros first. Since p_n is a scaling of T_n, it suffices to find when $T_n(x) = 0$ $(n = 3)$. Now $T_3(x) = 4x^3 - 3x$ (`https://en.wikipedia.org/wiki/Chebyshev_polynomials#Examples` Refer to any table of chebyshev polynomials). We see that $T_3(x)$ can be factored as

$$T_3(x) = 4x(x - \frac{\sqrt{3}}{2})(x + \frac{\sqrt{3}}{2})$$

giving us the three roots

$$x_1 = -\frac{\sqrt{3}}{2}, \quad x_2 = 0, \quad x_3 = \frac{\sqrt{3}}{2}.$$

then using the fact that

$$\begin{aligned}
p_0(x) &= 1 \\
p_1(x) &= x \\
p_2(x) &= x^2 - \frac{1}{2}
\end{aligned}$$

we get

$$\begin{bmatrix} p_0(x_1) & p_0(x_2) & p_0(x_3) \\ p_1(x_1) & p_1(x_2) & p_1(x_3) \\ p_2(x_1) & p_2(x_2) & p_2(x_3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

and the inverse of this matrix is calculated as

$$\frac{1}{3} \begin{bmatrix} 1 & -\sqrt{3} & 2 \\ 1 & 0 & -4 \\ 1 & \sqrt(3) & 2 \end{bmatrix}$$

and thus the w_i are found as

$$\begin{aligned}
\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} &= \frac{1}{3} \begin{bmatrix} 1 & -\sqrt{3} & 2 \\ 1 & 0 & -4 \\ 1 & \sqrt(3) & 2 \end{bmatrix} \begin{bmatrix} \pi \\ 0 \\ 0 \end{bmatrix} \\
&= \frac{1}{3} \begin{bmatrix} \pi \\ \pi \\ \pi \end{bmatrix} \\
&= \begin{bmatrix} \frac{\pi}{3} \\ \frac{\pi}{3} \\ \frac{\pi}{3} \end{bmatrix}
\end{aligned}$$

as required.

Solution:

Code

Coding Question:

Evaluate following integrals using Gaussian Quadrature. Compare results by using the 4 point Gaussian Quadrature and the composite 2 point Gaussian Quadrature with 2 subintervals.

$$I_1 = \int_{-1}^{1} e^{x^2} \ln(2-x)\,dx$$
$$I_2 = \int_{1}^{3} \frac{1}{\sqrt{1+x^4}}\,dx$$

Solution:

```
    import numpy as np
from scipy.integrate import quad
```

We need to integrate

$$I_1 = \int_{-1}^{1} e^{x^2} \ln(2-x)\,dx$$
$$I_2 = \int_{1}^{3} \frac{1}{\sqrt{1+x^4}}\,dx$$

over 4 points Gaussian Quadrature and the composite 2 point Gaussian quadrature with 2 subintervals.

I_1:

To approximate I_1 I'll use the Legendre polynomials $L_n(x)$ as they already match the intervals. Since we'll use 4 point quadrature, we'll need to evaluate $f_1(x) = e^{x^2} \ln(2-x)$ at the zeros of $L_4(x)$. Recall that they are given in the lecture notes as

$$x_4 = -x_1 = 0.861\,136\,3116, \quad x_3 = -x_2 = 0.339\,981\,0436$$

```
    zeros = [-0.8611363116,
         -0.3399810435,
         0.3399810435,
         0.8611363116,]
```

and the weights are given by

$$w_1 = w_4 = 0.347\,854\,8451, w_2 = w_3 = 0.652\,145\,1549$$

```
    weights = [0.3478548451,
          0.6521451549,
          0.6521451549,
          0.3478548451,]
```

Finally we need to evaluate $f_1(x)$ at the zeros

```
    def f1(x):
    return np.exp(x**2)*np.log(2-x)
points = [f1(xi) for xi in zeros]
```

Finally approximate the integral as

$$\sum_{i=1}^{n} w_i \cdot f(x_i)$$

```
    I1 = np.dot(weights, points)
print("Quadrature      =", I1)
wolfram = 1.85572
print("Wolfram's answer=", wolfram)

:Quadrature      = 1.8559447713840982
:Wolfram's answer= 1.85572
```

I_2 :

This part is more complicated. Since the bounds of I_2 is $[1, 3]$, we should first split the integral into two parts (since we are doing 2 subintervals) then perform a change of variables such that the integrals are over $[-1, 1]$. Denote $f_2(x) = \frac{1}{\sqrt{1+x^4}}$. Then

$$\begin{aligned}\int_1^2 f(x)dx &= \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x + \frac{3}{2})dx \\ &= \frac{1}{2}\int_{-1}^{1} f(\frac{1}{2}x + \frac{3}{2})dx\end{aligned}$$

and

$$\begin{aligned}\int_2^3 f(x)dx &= \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x + \frac{5}{2})dx \\ &= \frac{1}{2}\int_{-1}^{1} f(\frac{1}{2}x + \frac{5}{2})dx\end{aligned}$$

.

The weights and zeros of the $L_2(x)$ are

$$w_1 = w_2 = 1$$

and

$$x_1 = x_2 = 0.577\,350\,2692$$

so calculating the quadrature is simply

```
    zeros = [-0.5773502692,
         0.5773502692,]
weights = [1.0, 1.0]

def f2(x):
    return 1/np.sqrt(1+x**4)
```

```
int1_points = [f2(xi*0.5 +1.5) for xi in zeros]
int2_points = [f2(xi*0.5 +2.5) for xi in zeros]

I2 = 0.5 * np.dot(weights, int1_points) + 0.5 * np.dot(weights, int2_points)

print("Quadrature      =", I2)
wolfram = 0.594113
print("Wolfram's answer=", wolfram)

:Quadrature      = 0.5946956791823543
:Wolfram's answer= 0.594113
```