Numerical Analysis Homework 1

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Chapter 2

2)

Interpolate the function $\ln(x)$ by a quadratic polynomial at $x = 10, 11, 12$.

(a) Estimate the error committed for $x = 11.1$ when approximating $\ln(x)$ by the interpolating polynomial

Solution:

In this question we are interesting in finding the polynomial that interpolates $(x_1, f_1) = (10, \ln(10)), (x_2, f_2) = (11, \ln(11)), (x_3, f_3) = (12, \ln(12))$ so we should use the Lagrange interpolating polynomials. Recall that

$$L_i(x) = \frac{\omega(x)}{(x - x_i)\omega'(x_i)}, \quad \omega(x) = \prod_i (x - x_i)$$

then the interpolating polynomial is simply

$$P(x) = \sum_{i=1}^{3} f_i L_i(x).$$

Running this through a computer algebra system ('Symbolic.jl' in Julia, see HW1.ipynb for the source) yields

$$0.89305 + 0.18245x - 0.0041494x^2.$$

To estimate the error we recall theorem (2.1.4.1). For $\bar{x} = 11.1$ we know there exists an $\xi \in I[x_0, x_1, x_2, \bar{x}] = [10, 12]$ such that $\ln(\bar{x}) - P(\bar{x}) = \omega(\bar{x})\frac{f^{(3)}(\xi)}{6}$. Since $\ln'''(x) = \frac{2}{x^3}$ is a monotonic function over this interval, we can bound the error using $\ln'''(10) = \frac{2}{1000}$ as

$$|\ln(\bar{x}) - P(\bar{x})| \leqslant \omega(\bar{x})\frac{\frac{2}{1000}}{6} = \frac{-0.099}{3000} = -3.3\text{e}{-5}$$

$\square$

(b) How does the sign of the error depend on x?

Solution:

Recall

$$P(\bar{x}) - \ln(\bar{x}) = \omega(\bar{x})\frac{f^{(3)}(\xi)}{(3)!}$$

for some ξ in the interval. Since $\frac{f^{(3)}(\xi)}{(3)!}$ is positive for any ξ in the interval, the sign is determined by $\omega(\bar{x})$. Since ω is a third degree monic polynomial with distinct roots, we know that the sign of $\omega(x)$ is $[-, +, -, +]$ if x is in the respective regions $(-\infty, 10), (10, 11), (11, 12), (12, \infty)$. $\square$

3)

Consider a function f which is twice continuously differentiable on the interval $I = [-1, 1]$. Interpolate the function by a linear polynomial through the support points $(x_i, f(x_i)), i = 0, 1, x_0, x_1 \in I$. Verify that

$$\alpha = \frac{1}{2} \max_{\xi \in I} |f''(\xi)| \max_{x \in I} |(x - x_0)(x - x_1)|$$

is an upper bound for the maximal absolute interpolation error on the interval I. Which values x_0, x_1 minimize α? What is the connection between $(x - x_0)(x - x_1)$ and $\cos(2 \arccos(x))$?

Solution:

This is almost the same as the previous question. We will once again rely on the fact that there exists a $\xi \in [-1, 1]$ such that

$$|f(x) - P(x)| = |\omega(x) \frac{f^{(2)}(\xi)}{2!}|$$

then by multiplicity of the absolute value we also get

$$|f(x) - P(x)| = |\omega(x)| \frac{|f^{(2)}(\xi)|}{2!}.$$

Distributing $\max_{\xi, x \in I}$ over the right hand side and expanding ω gives the bounds in the question

$$\begin{aligned} |f(x) - P(x)| &= |\omega(x)| \frac{|f^{(2)}(\xi)|}{2!} \\ \max_{\xi, x \in I} |f(x) - P(x)| &= \max_{\xi, x \in I} |\omega(x)| \frac{|f^{(2)}(\xi)|}{2!} \\ \max_{x \in I} |f(x) - P(x)| &\leqslant \frac{1}{2} \max_{\xi \in I} |f''(\xi)| \max_{x \in I} |(x - x_0)(x - x_1)| \end{aligned}$$

which verifies α is an upperbound as was to be shown. We now look to find what values of $(x_0, x_1) \in C := [-1, 1] \times [-1, 1]$ minimize α. We note that x_0, x_1 has no effect on $f(x)$ so we focus only on the term $\omega(x) = (x - x_0)(x - x_1)$, thus we look to solve the problem

$$\arg\min_{x_0, x_1} \left(\max_{x} \ \omega(x, x_0, x_1) \right).$$

Since $(x - x_0)(x - x_1)$ is a quadratic with both its roots in I, right away we know that it's maximum over x occurs at either end of the interval -1 or 1 for any x_0, x_1. It should be clear that if the vertex $\frac{x_0 + x_1}{2}$ is negative, then the maximum is at $x = 1$, if it is positive the maximum is at $x = -1$, and if it is zero then there are two maxima with the same value. W.L.O.G we can assume $x_0 \leqslant x_1$ and break C into 2 cases: $x_0 + x_1 \leqslant 0$, C_1, and $x_0 + x_1 \geqslant 0$, C_2 (There is no problem that C_1 and C_2 overlap as the maximum of $\omega(1)$ and $\omega(-1)$ coincide in this region).

Case 1: $x_0 + x_1 \leqslant 0$

Since the maximum is at $x = 1$ we solve

$$\arg\min_{x_0 \leqslant x_1, x_0 + x_1 \leqslant 0} (1 - x_0)(1 - x_1).$$

Since ω is smooth and we're solving over a compact set C, the minimum can only occur at the boundary or where the partial derivatives are both zero. Making use of the symmetry of $\omega(x_1, x_2) = \omega(x_2, x_1)$, we know that the partial derivatives can only be zero along the diagonal $x_1 = x_2, \ x_0 \leqslant 0$ thus we can restrict our search to there. We will see that the minima of this region is greater than the boundary, so we don't need to calculate partial derivatives. First we restrict ω to the boundaries and $x_0 = x_1$:

1. $x_0 = -1 : \quad \omega = 2(1 - x_1)$
2. $x_1 = -1 : \quad \omega = 2(1 - x_0)$

3. $x_0 = x_1 : \quad \omega = (1 - x_0)^2$

then minimizing over each region yields the minima

1. $(x_0, x_1) = (-1, 1) : \omega = 0$
2. $(x_0, x_1) = (1, -1) : \omega = 0$
3. $(x_0, x_1) = (0, 0) : \omega = 1$

but since we require $x_0 \leqslant x_1$ we get a single solution and so

$$\underset{x_0 \leqslant x_1, x_0 + x_1 \leqslant 0}{\arg\min} (1 - x_0)(1 - x_1) = \{(-1, 1)\}.$$

Case 2: $x_0 + x_1 \geqslant 0$ The set up is almost the same as the previous problem, except now we solve

$$\underset{x_0 \leqslant x_1, x_0 + x_1 \geqslant 0}{\arg\min} (1 + x_0)(1 + x_1)\,.$$

Again we reason that the minima can only occur along the boundary of C_2 or $x_0 = x_1,\ x_0 \geqslant 0$ so we find

1. $x_0 = 1 : \quad \omega = 2(1 + x_1)$
2. $x_1 = 1 : \quad \omega = 2(1 + x_0)$
3. $x_0 = x_1 : \quad \omega = (1 + x_0)^2$

then minimize in each region to yield the minimum

1. $(x_0, x_1) = (1, -1) : \omega = 0$
2. $(x_0, x_1) = (-1, 1) : \omega = 0$
3. $(x_0, x_1) = (0, 0) : \omega = 1$

then again since $x_0 \leqslant x_1$ we get a unique solution

$$\underset{x_0 \leqslant x_1, x_0 + x_1 \geqslant 0}{\arg\min} (1 + x_0)(1 + x_1) = \{(-1, 1)\}.$$

In both cases we see the choice of x_0, x_1 that minimizes the maximum of ω is $x_0 = -1, x_1 = 1$ corresponding to nodes at either end of the interval.

The connection to $\cos(2\arccos(x))$ has to do with chebyshev polynomials. If we solved our minimization problem in a different way, we would get that the interpolating points that minimize the maximum error are given by the roots of $T_2(\theta) = \cos(2\theta)$ where $x = \cos(\theta)$. Of course, this results in $x = -1$ and $x = 1$ coinciding with our result.

4)

Suppose a function $f(x)$ is interpolated on the interval $[a, b]$ by a polynomial $P_n(x)$ whose degree does not exceed n. Suppose further that f is arbitrarily often differentiable on $[a, b]$ and that there exists M such that $|f^{(i)}(x)| \leqslant M$ for $i = 0, 1, \ldots$ and any $x \in [a, b]$. Can it be shown, without additional hypotheses about the location of the support abscissas $x_i \in [a.b]$ that $P_n(x)$ converges uniformly on $[a, b]$ to $f(x)$ as $n \to \infty$?

Solution:

No. This is a key point of this chapter, that choice of the abscissas have effect on the convergence of $P_n \to f$. The Runge function example in the book does not constitute as a proof as it only states the upperbound does not tend to 0. Here is a proof presented by David E Speyer (https://math.stackexchange.com/q/807784) in my own words:

Let $f(x) = \frac{1}{1+x^2}$ be Runge's function. We'll choose the abscissas and interval such that the error $P_n(x) - f(x)$ can be evaluated analytically. Consider the interpolating polynomial over $[-2, 2]$ and equidistance abscissas

$$x_k = \frac{2k}{n}, \quad -n \leqslant k \leqslant n.$$

Let $P_n(x)$ be the interpolating polynomial and denote $q(x) = \prod_{k=-n}^{n}(x - x_k)$. Since $P_n(x_k) = f(x_k)$ we have that $P_n(x)(x^2+1) - 1$ has zeros at the absiccas. By division properties of polynomials, there exists a polynomial $r(x)$ such that

$$P_n(x)(x^2-1) - 1 = q(x)r(x).$$

Since $\deg(P_n) \leqslant 2n$ and $\deg(q) = 2n+1$ we must have $\deg(r) \leqslant 1$ or $r(x) = ax + b$. Since the absiccas are symettric about zero and $f(x)$ is symettric about zero, we have that $P_n(x)$ is symmetric about zero and thus even. Similarly, $q(x)$ must be odd because it is an odd degree polynomial with roots symmetric about zero. Since x^2+1 is even and -1 won't change the functional parity, we must then have that $r(x)$ is odd and $r(x) = cx$ for some c to be determined. Determining c can be done by substituting $x = i$ into $P_n(x)(x^2-1) - 1 = q(x)r(x)$:

$$\begin{aligned}
P_n(x)(x^2-1) - 1 &= q(x)r(x) \\
P_n(x)(i^2-1) - 1 &= q(i)ci \\
-1 &= q(i)ci \\
c &= \frac{-1}{iq(i)} \\
&= i\frac{1}{\prod_{k=-n}^{n}(i - \frac{2k}{n})} \\
&= \frac{i}{i\prod_{k=1}^{n}(i - \frac{2k}{n})\prod_{k=-n}^{-1}(i - \frac{2k}{n}))} \\
&= \frac{1}{\prod_{k=0}^{n}(i - \frac{2k}{n})\prod_{k=-n}^{0}(i - \frac{2k}{n}))} \\
&= \frac{1}{\prod_{k=1}^{n}(i + \frac{2k}{n})\prod_{k=1}^{n}(i - \frac{2k}{n}))} \\
&= \frac{1}{\prod_{k=1}^{n}(i^2 - \frac{2^2k^2}{n^2})} \\
&= \frac{1}{\prod_{k=1}^{n}(-1 - \frac{2^2k^2}{n^2})} \\
&= \frac{(-1)^n}{\prod_{k=1}^{n}(1 + \frac{2^2k^2}{n^2})}.
\end{aligned}$$

Using a similar rearrangement we can write $q(x) = \prod_{k=-n}^{n}(x - \frac{2k}{n}) = x\prod_{k=1}^{n}(x^2 - \frac{2^2k^2}{n^2})$

$$\begin{aligned}
P_n(x) - f(x) &= \frac{cxq(x)}{1+x^2} && (1) \\
&= \frac{(-1)^n x^2 \prod_{k=1}^{n}(x^2 - \frac{2^2k^2}{n^2})}{(1+x^2)\prod_{k=1}^{n}(1 + \frac{2^2k^2}{n^2})} && (2) \\
&= \frac{(-x^2)^n x^2 \prod_{k=1}^{n}(1 - \frac{2^2k^2}{x^2n^2})}{(1+x^2)\prod_{k=1}^{n}(1 + \frac{2^2k^2}{n^2})} && (3) \\
&= \frac{(-x^2)^n x^2}{(1+x^2)} \prod_{k=1}^{n} \frac{1 - \frac{2^2k^2}{x^2n^2}}{1 + \frac{2^2k^2}{n^2}} && (4)
\end{aligned}$$

there are two ways I see to proceed from here. By the following desmos project (link here) there is evidence that there exists $\varepsilon > 0$ such that $x = 2 - \varepsilon$ causes the above infinite product to diverge as $n \to \infty$. In fact, by extending our interval from $[-2, 2]$ to $[-a, a]$ and adjusting our arguments accordingly, we may also change $2 - \varepsilon$ to $a - \varepsilon$. However, I choose $a = 2$ in hopes that it may be easier to simplify the infinite product but unfortunately I wasn't able to finish this argument, but I thought to mention it incase you (reader) enjoy these kinds of puzzles and may find interest in it yourself.

Another way to proceed is to follow Speyer's solution. We recall that divergence of the infinite product $\prod_{k=1}^{\infty} a_k$ to infinity coincides with that of $\sum_{k=1}^{\infty} \log(a_k)$ (so long as $a_k \neq 0$ i.e $x \neq x_k$) which leads us to study

$$\sum_{k=1}^{\infty} \log(x^2 - \frac{2^2k^2}{n^2}) - \log(1 + \frac{2^2k^2}{n^2}).$$

Speyer makes the observation that these resemble Riemann sums with with x_k as a partition, thus in the limit as $n \to \infty$ we have

$$\begin{aligned}\sum_{k=1}^{n} \log(x^2 - \frac{2^2k^2}{n^2}) - \log(1 + \frac{2^2k^2}{n^2}) &\approx \frac{n}{2}\left(\int_{-2}^{2} \log(x^2 - t^2)dt - \int_{-2}^{2} \log(1 - t^2)dt\right) \\ &= \ldots \text{partial fraction decomposition and standard integrals} \ldots \\ &= \frac{n}{2} \cdot 2\left(x\frac{\ln(|x+2|)}{\ln(|x-2|)} + 2\ln\left(x^2 - 4\right) - 4\right) - C\end{aligned}$$

(see integral calculator for steps) where $C = \int_{-2}^{2} \log(1 - t^2)dt$ is just a constant that doesn't affect convergence. If $x < 2$ then as $n \to \infty$ the infinite sum diverges to $-\infty$ (due to $(\ln(x^2 - 4)$ and the corresponding infinite product is 0 i.e the error converges to zero. Otherwise, taking $n \to \infty$ results in the right hand diverging to positive infinity, as does the product, as does the error. Thus we can see that additional hypotheses are necessary on the location of the support abscissas to insure uniform convergence of $P_n(x)$. □