Numerical Analysis Homework 3

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Chapter 2

22)

Start with
$$d_j = \frac{6}{h_j + h_{j+1}} \left(\frac{y_{j+1} - y_j}{h_{j+1}} - \frac{y_j - y_{j-1}}{h_j} \right)$$
then replacing $y_i = f(x_i)$ and applying taylors theorem gives
$$\begin{aligned}
d_j &= \frac{6}{h_j + h_{j+1}} \left(\frac{y_{j+1} - y_j}{h_{j+1}} - \frac{y_j - y_{j-1}}{h_j} \right) \\
&= \frac{6}{h_j + h_{j+1}} \left(\frac{f(x_{j+1}) - f(x_j)}{h_{j+1}} - \frac{f(x_j) - f(x_{j-1})}{h_j} \right) \\
&= \frac{6}{h_j + h_{j+1}} \left(\frac{f(x_j) + h_{j+1} f'(x_j) + \frac{h_{j+1}^2}{2} f''(x_j) - f(x_j)}{h_{j+1}} - \frac{f(x_j) - f(x_j) - h_j f'(x_j) - \frac{h_j^2}{2} f''(x_j)}{h_j} + O(h_j^2) + O(h_{j+1}^2) \right) \\
&= \frac{6}{h_j + h_{j+1}} \left(\frac{h_{j+1} f'(x_j) + \frac{h_{j+1}^2}{2} f''(x_j)}{h_{j+1}} - \frac{-h_j f'(x_j) - \frac{h_j^2}{2} f''(x_j)}{h_j} + O(h_j^2) + O(h_{j+1}^2) \right) \\
&= \frac{6}{2(h_j + h_{j+1})} \left(h_{j+1} f''(x_j) + h_j f''(x_j) + O(h_j^2) + O(h_{j+1}^2) \right) \\
&= 3f''(x_j) + \frac{O(h_{j+1}^2) + O(h_j^2)}{h_{j+1} + h_j} \\
&= 3f''(x_j) + O(\|\Delta\|)
\end{aligned}$$
where the last step is justified by considering the larger of the two intervals, h_j and h_{j+1}, and reabsorbing into the big O.

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Fix $j \neq 0, 1, n-1, n$. Then we have $S_j(x_k) = y_k$ so $y_k = 0$ unless $k = j$. We can then write
$$\begin{aligned}
\lambda_k &= \frac{h_{k+1}}{h_k + h_{k+1}} = \frac{h}{2h} = \frac{1}{2} \\
\mu_k &= \frac{1}{2} \\
d_j &= \frac{6}{h_{k+1} + h_k} \left(\frac{y_{k+1} - y_k}{h_{k+1}} - \frac{y_k - y_{k-1}}{h_k} \right) = \frac{6}{2h} \left(\frac{-1}{h} - \frac{1}{h} \right) = \frac{-6}{h^2} \\
d_{j-1} &= \frac{6}{2h} \left(\frac{1}{h} \right) = \frac{6}{2h^2} \\
d_{j+1} &= \frac{6}{2h} \left(\frac{1}{h} \right) = \frac{6}{2h^2} \\
d_k &= 0, \quad j \neq k-1, k, k+1.
\end{aligned}$$

We'll prove $M_k = \frac{-1}{\rho_k} M_{k+1}$ using induction. To get the base case, we use the defining equation of moments $AM = d$ to get

$$\begin{aligned} 2M_1 + \lambda_0 M_2 &= 0 \\ 2M_1 + \frac{1}{2} M_2 &= 0 \\ M_1 &= \frac{-1}{4} M_2 \end{aligned}$$

Then using the defining equations

$$\mu_k M_{k-1} + 2M_k + \lambda_k M_{k+1} = d_k, k = 1, \dots, n-1$$

and taking $k = 1, \dots, j-2$ we can prove the induction step. Suppose $M_{k-1} = \frac{-1}{\rho_{k-1}} M_k$. Then

$$\begin{aligned} \mu_k M_{k-1} + 2M_k + \lambda_k M_{k+1} &= d_k \\ \frac{1}{2} M_{k-1} + 2M_k + \frac{1}{2} M_{k+1} &= 0 \\ \frac{1}{2} \frac{-1}{\rho_{k-1}} M_k + 2M_k + \frac{1}{2} M_{k+1} &= 0 \\ M_k \left(4 - \frac{1}{\rho_{k-1}} \right) &= -M_{k+1} \\ M_k \rho_k &= -M_{k+1} \\ M_k &= \frac{-1}{\rho_k} M_{k+1} \end{aligned}$$

and of course since this only holds when $d_k = 0$ it verifies the first statement up to $j-2$. Now we'll prove

$$M_k = -\frac{1}{\rho_{n-k}} M_{k-1}$$

using induction from n. The base case follows from the last row of $AM = d$

$$\begin{aligned} \mu_n M_{n-1} + 2M_n &= d_n \\ \frac{1}{2} M_{n-1} + 2M_n &= 0 \\ M_n &= \frac{-1}{4} M_{n-1} \end{aligned}$$

then supposing $M_{k+1} = \frac{1}{-\rho_{n-k-1}} M_k$ we get

$$\begin{aligned} \mu_k M_{k-1} + 2M_k + \lambda_k M_{k+1} &= d_k \\ \frac{1}{2} M_{k-1} + 2M_k + \frac{1}{2} M_{k+1} &= 0 \\ \frac{1}{2} M_{k-1} + 2M_k + \frac{1}{-2\rho_{n-k-1}} M_k &= 0 \\ M_{k-1} + 4M_k + \frac{-1}{\rho_{n-k-1}} M_k &= 0 \\ M_k \left(4 - \frac{1}{\rho_{n-k-1}} \right) &= -M_{k-1} \\ M_k &= \frac{-1}{\rho_{n-k}} M_{k-1} \end{aligned}$$

and since this holds until $d_k \neq 0$, we have that this holds for $k = j+2, \dots, n-1$.

Next we'll show that $M_{j-1} = \frac{1}{\rho_{j-1}}(6h^{-2} - M_j)$. Using row $j-1$ from $AM = d$ gives

$$\begin{aligned}
\frac{1}{2}M_{j-2} + 2M_{j-1} + \frac{1}{2}M_j &= d_{j-1} \\
\frac{1}{2}\frac{-1}{\rho_{j-2}}M_{j-1} + 2M_{j-1} + \frac{1}{2}M_j &= \frac{6}{2h^2} \\
\frac{-1}{\rho_{j-2}}M_{j-1} + 4M_{j-1} + M_j &= \frac{6}{h^2} \\
M_{j-1}\left(\frac{-1}{\rho_{j-2}} + 4\right) + M_j &= \frac{6}{h^2} \\
M_{j-1}\rho_{j-1} + M_j &= \frac{6}{h^2} \\
M_{j-1} &= \frac{1}{\rho_{j-1}}(6h^{-2} - M_j)
\end{aligned}$$

as required.

Next we'll show that $M_{j+1} = \frac{1}{\rho_{n-j-1}}(6h^{-2} - M_j)$. As before we start with the $j+1$'th row of $AM = d$

$$\begin{aligned}
\frac{1}{2}M_j + 2M_{j+1} + \frac{1}{2}M_{j+2} &= d_{j+1} \\
\frac{1}{2}M_j + 2M_{j+1} + \frac{1}{2}\frac{-1}{\rho_{n-j-2}}M_{j+1} &= \frac{6}{2h^2} \\
M_j + 4M_{j+1} + \frac{-1}{\rho_{n-j-2}}M_{j+1} &= \frac{6}{h^2} \\
M_j + \left(4 + \frac{-1}{\rho_{n-j-2}}\right)M_{j+1} &= \frac{6}{h^2} \\
M_j + \rho_{n-j-1}M_{j+1} &= \frac{6}{h^2} \\
M_{j+1} &= \frac{1}{\rho_{n-j-1}}\left(\frac{6}{h^2} - M_j\right)
\end{aligned}$$

as required.

Finally we'll show that

$$M_j = \frac{-6}{h^2} \cdot \frac{2 + 1/\rho_{j-1} + 1/\rho_{n-j-1}}{4 - 1/\rho_{j-1} - 1/\rho_{n-j-1}}.$$

Start with the j'th row of $AM = d$ then

$$\begin{aligned}
\frac{1}{2}M_{j-1} + 2M_j + \frac{1}{2}M_{j+1} &= d_j \\
\frac{1}{2}\frac{1}{\rho_{j-1}}(6h^{-2} - M_j) + 2M_j + \frac{1}{2}\frac{1}{\rho_{n-j-1}}(6h^{-2} - M_j) &= -6h^{-2} \\
M_j\left(2 - \frac{1}{2\rho_{j-1}} - \frac{1}{2\rho_{n-j-1}}\right) &= -6h^{-2} - \frac{6h^{-2}}{2\rho_{j-1}} - \frac{6h^{-2}}{2\rho_{n-j-1}} \\
M_j\left(4 - \frac{1}{\rho_{j-1}} - \frac{1}{\rho_{n-j-1}}\right) &= -6h^{-2}2 - \frac{6h^{-2}}{\rho_{j-1}} - \frac{6h^{-2}}{\rho_{n-j-1}} \\
M_j &= \frac{-6}{h^2} \cdot \frac{2 + 1/\rho_{j-1} + 1/\rho_{n-j-1}}{4 - 1/\rho_{j-1} - 1/\rho_{n-j-1}}
\end{aligned}$$

as required. □

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(This question is poorly worded. It should be "Show that $E_{\Delta,f}$, between knots, is of the form..." or anything that seperates the noun in the first preposition phrase from "of" in the second. Otherwise this sentence is meaningless. Also I am assuming there is an error and ε_i should be ϕ_i.)

(a)

(b) **Solution:**

Using the taylor expansions of cosh and sinh gives

$$\psi_i(x) = \frac{2}{\lambda_i^2}(1 + \lambda_i^2 x^2/2 - 1 + O(\lambda_i^4 x^4)) = x^2 + O(\lambda_i^2 x^4)$$

and

$$\phi_i(x) = \frac{6}{\lambda_i^3}(\lambda_i x - \lambda_i x + \lambda_i^3 x^3/6 + O(\lambda_i^5 x^5) = 1 + O(\lambda_i^5 x^5)$$

and taking the limits yields

$$\psi_i(x) = x^2, \quad \phi_i(x) = 1$$

Thus between two knots, the limit $E_{\Delta,f}(x)$ simplifies to

$$E_{\Delta,f} = \alpha_i + \beta_i(x - x_i) + \gamma_i(x - x_i)^2 + \delta_i$$

which is just a quartic spline.

(c) Is this a question?

Computing

Interpolation accuracy tends to increase with number of knots but, as can be seen in figure 1, adding another knot may not decrease the accuracy of the spline.

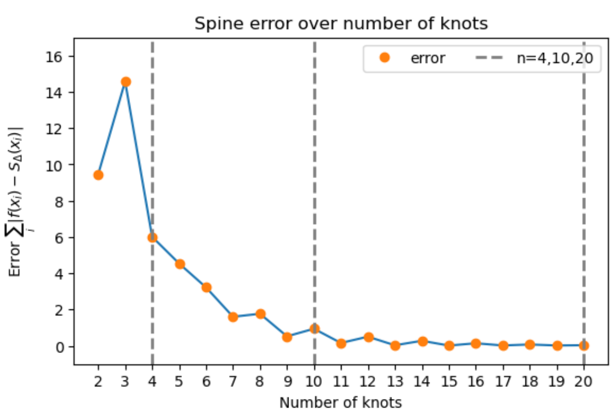

Figure 1: Plot depicting the error of the interpolating spline as the number of knots increases. We can see the error improves as the number of knots increases, but not monotonically. Special values $n = 4, 10, 20$ are indicated with verticle dashed grey lines.

```
import numpy as np
from scipy.interpolate import CubicSpline
import matplotlib.pyplot as plt
```

```
def runge(x: float) -> float:
    return 1 / (1 + 25 * x**2)

errors = []
steps = range(2,21)
for n in steps:
    #First fit the points
    x = np.linspace(-1,1, n)
    y = runge(x)
    cs = CubicSpline(x, y,bc_type=((2, 0.0), (2, 0.0)))
    #Then calculate the error
    xs = np.linspace(-1, 1, 41)
    ybars= cs(xs)
    error = np.sum(np.abs(ybars - runge(xs)))
    errors.append(error)

print(steps)

fig, ax = plt.subplots(figsize=(6.5, 4))
ax.plot(steps, errors)
ax.plot(steps, errors, 'o', label='error')

ax.set_title('Spine error over number of knots')
ax.vlines([4,10,20],-1,17,linestyles='--', color='grey', lw=2, label = 'n=4,10,20')
ax.set_xlabel('Number of knots')
ax.set_ylabel(r'Error $\sum_i|f(x_i) - S_\Delta(x_i)|$')
ax.set_ylim(-1,17)
ax.set_xticks(steps)
ax.legend(loc='best', ncol=2)

plt.show()
```