Numerical Analysis Homework 6

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Chapter 5

11)

[...]

Solution:

Let $f(x)$ be a smooth enough function in a neighbourhood of a zero $f(x_0) = 0$ and $f'(x_0) \neq 0$. Recall that Newton's method is quadratic in a neighbourhood of x_0 in this case, so we'll solve this question by finding an f with $a^{\frac{1}{n}}$ as a root. Newton's method will then converge quadratically in some neighbourhood of the $a^{\frac{1}{n}}$, as is required. Fixing n, consider $f(x) := x^n - a$. Then by definition, $a^{\frac{1}{n}}$ is a zero of this function. Furthermore, $f'(x) = nx^{n-1}$ and $f'(a^{\frac{1}{n}}) = n(a^{\frac{1}{n}})^{n-1} \neq 0$. Then the iteration scheme defined by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^n - a}{nx_k^{n-1}} = x_k\left(1 - \frac{1}{n}\right) + \frac{a}{nx_k^{n-1}}$$

will suffice.

13)

Let the function $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable for all $x \in U(\xi) := \{x \in \mathbb{R} \mid |x - \xi| \leqslant r\}$ in a neighborhood of a simple zero ξ, $f(\xi) = 0$. Show that the sequence $\{x_n\}$ generated by

$$\begin{aligned} y &= x_n - f'(x_n)^{-1} f(x_n), \\ x_{n+1} &= y - f'(x_n)^{-1} f(y), \end{aligned}$$

converges locally at least cubically to ξ.

Solution:

Lemma 1 (Theorem 8.1 (J.F Traub, "ITERATIVE METHODS FOR THE SOLUTION OF EQUATIONS", 1964, Prentice-Hall.))**.** *Suppose $x_{n+1} = \phi(x_n)$ converges to x_0 with order p (where $f(x_0) = 0$ is multiplicity one) on some neighbourhood of x_0 where f is sufficiently differentiable. Then*

$$x_{n+1} = \psi(x_n) = \phi(x_n) - \frac{f(\phi(x_n))}{f'(x_n)}$$

converges with order $p + 1$.

Proof. We'll prove by induction that $\psi^{(j)}(x_0) = 0$ for $j = 1, 2, \ldots, p$ which is equivalent to being order $p + 1$. First see that

$$\psi'(x) = \phi'(x) - \frac{f'(\phi(x))\phi'(x)f'(x) - f''(x)f(\phi(x))}{f'(x)^2}$$

which evaluates at $x = x_0$ to

$$\psi'(x_0) = \phi'(x_0) - \frac{f'(\phi(x_0))\phi'(x_0)f'(x_0) - f''(x_0)f(\phi(x_0))}{f'(x_0)^2} = 0 - \frac{0-0}{f'(x_0)^2} = 0$$

since x_0 is a simple zero of f, $f'(x_0) \neq 0$.

Next fix an integer $l \leqslant p$ and suppose $\psi^{(k)}(x_0) = 0$ for $k = 1, 2, \ldots, l-1 < p$. We get that

$$\begin{aligned}
\psi(x) &= \phi(x) - \frac{f(\phi(x))}{f'(x)} \\
f'(x)\psi(x) &= f'(x)\phi(x) - f(\phi(x)) \\
\frac{\partial^{(l)}}{\partial x^{(l)}}\bigg|_{x=x_0} f'(x)\psi(x) &= \frac{\partial^{(l)}}{\partial x^{(l)}}\bigg|_{x=x_0} f'(x)\phi(x) - \frac{\partial^{(l)}}{\partial x^{(l)}}\bigg|_{x=x_0} f(\phi(x)) \\
\sum_{k=0}^{l}\binom{l}{k} f^{(l-k+1)}(x_0)\underbrace{\psi^{(k)}(x_0)}_{=0,\ \forall k=1,\ldots,l-1} &= \sum_{k=0}^{l}\binom{l}{k} f^{(l-k+1)}(x_0)\underbrace{\phi^{(k)}(x_0)}_{=0,\ \forall k=1,\ldots,l-1} - \frac{\partial^{(l)}}{\partial x^{(l)}}\bigg|_{x=x_0} f(\phi(x)) \\
f^{(l+1)}(x_0)\psi(x_0) + f^{(1)}(x_0)\psi^{(l)}(x_0) &= f^{(l+1)}(x_0)\phi(x_0) + f^{(1)}(x_0)\phi^{(l)}(x_0) - \frac{\partial^{(l)}}{\partial x^{(l)}}\bigg|_{x=x_0} f(\phi(x)) \\
f^{(l+1)}(x_0)\phi(x_0) + f^{(1)}(x_0)\psi^{(l)}(x_0) &= f^{(l+1)}(x_0)\phi(x_0) + f^{(1)}(x_0)\phi^{(l)}(x_0) - \frac{\partial^{(l)}}{\partial x^{(l)}}\bigg|_{x=x_0} f(\phi(x)) \\
f^{(1)}(x_0)\psi^{(l)}(x_0) &= f^{(1)}(x_0)\phi^{(l)}(x_0) - \frac{\partial^{(l)}}{\partial x^{(l)}}\bigg|_{x=x_0} f(\phi(x)).
\end{aligned}$$

Now Faa di Bruno's formula onto the right most term yields

$$\begin{aligned}
\frac{\partial^{(l)}}{\partial x^{(l)}}\bigg|_{x=x_0} f(\phi(x)) &= \sum_{\sum_{i=1}^{l} i m_i = l} \frac{l!}{\prod_{i=1}^{l} m_i! i!^{m_i}} f^{(m_1+\cdots+m_l)}(\phi(x_0)) \prod_{k=1}^{l} (\phi^{(k)}(x_0))^{m_k}, \text{only non-zero when } m_l = 1 \\
&= \frac{l!}{1!l!^1} f'(x_0)\phi^{(l)}(x_0) \\
&= f'(x_0)\phi^{(l)}(x_0)
\end{aligned}$$

and we now have

$$f'(x_0)\psi^{(l)}(x_0) = f'(x_0)\phi^{(l)}(x_0) - f'(x_0)\phi^{(l)}(x_0) = 0.$$

Since $f'(x_0) \neq 0$ we have that $\psi^{(l)}(x_0) = 0$. Thus by the principle of mathematical induction $\psi^{(k)}(x_0) = 0$ for $k = 1, 2, \ldots, p$ and $x_{n+1} = \psi(x_n)$ converges to x_0 with order $p+1$. □

Then the solution this problem is simple. Since

$$\begin{aligned}
y &= x_n - f'(x_n)^{-1} f(x_n), \\
x_{n+1} &= y - f'(x_n)^{-1} f(y),
\end{aligned}$$

we recognize that y is just Newton's method, which converges quadratically. Replace $\phi(x)$ in the lemma with y to see that $x_{n+1} = y - f'(x_n)^{-1} f(y)$ converges cubically, as required. □

Code

Coding Question:

Let the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(z) = \begin{bmatrix} \exp(x^2 + y^2) - 3 \\ x + y - \sin(3(x+y)) \end{bmatrix}, \quad z = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Calculate a root of $f(z)$ using the modified Newton method. Use an error tolerance of $\epsilon = 10^{-6}$ for the stop condition. Comment on what you observed about choosing the starting point and the order of convergence.