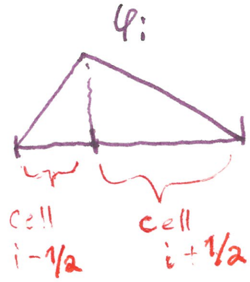
Assembling source vector

$$b_i = \int_a^b g(\varphi_i + \mu \tau \nabla \varphi_i)\, dx$$

Case 1 | for $i \in \{1 : N-1\}$, φ_i looks like this:



- g, τ are cell-wise constant

left part:

$$\int_{x_{i-1}}^{x_i} g(\varphi_i + \mu \tau \nabla \varphi_i)\, dx = g_{i-1/2} \left[\frac{h_{i-1/2}}{2} + \frac{\mu \tau_{i-1/2}}{\cancel{h_{i-1/2}}} \cdot \cancel{h_{i-1/2}} \right]$$

- slope of $\varphi_i = \frac{1}{h_{i-1/2}}$
- $\int_{x_{i-1}}^{x_i} 1\, dx = h_{i-1/2}$

right part:

$$\int_{x_i}^{x_{i+1}} g(\varphi_i + \mu \tau \nabla \varphi_i)\, dx = g_{i+1/2} \left[\frac{h_{i+1/2}}{2} - \frac{\mu \tau_{i+1/2}}{\cancel{h_{i+1/2}}} \cancel{h_{i+1/2}} \right]$$

by a similar argument.

- slope is now negative

Case 2

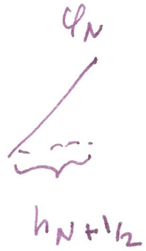
For $i=0$:

$$b_0 = \xi_{-1/2}\left[\frac{h_{-1/2}}{2} - \mu\,\tau_{-1/2}\right]$$

similar to case 1 : right part

Case 3

For $i=N$:



$$b_N = \xi_{N+1/2}\left[\frac{h_N}{2} + \mu\,\tau_{N+1/2}\right]$$

Remark: Don't forget boundary conditions!

Remark: In the code, we don't use half-integer indexing; we instead count by cells, so

$$\xi_{i-1/2} = \xi[i-1]$$

$$\xi_{i+1/2} = \xi[i] \qquad \text{etc...}$$

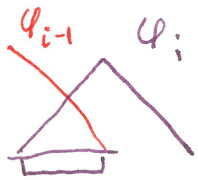
Assembling Matrix

$$T_{ij} = \int_a^b (\mu \cdot \nabla \varphi_j + \sigma^t \varphi_j)(\varphi_i + \mu \tau \nabla \varphi_i)\, dx$$

Case 1: $i \in \{1 : N-1\}$

The "stencil" of i contains $\{i, i-1, i+1\}$

Subcase $j = i-1$



cell $i-1/2$ $= \operatorname{supp} \varphi_i \cap \operatorname{supp} \varphi_{i-1}$

$$\therefore\ T_{i,i-1} = \int_{x_{i-1}}^{x_i} (\mu \cdot \nabla \varphi_{i-1} + \sigma^t \varphi_{i-1})(\varphi_i + \mu \ldots)\, d$$

$$\int_{x_{i-1}}^{x_i} \mu \nabla \varphi_{i-1} \varphi_i \, dx = \mu \frac{-1}{h_{i-1/2}} \int_{x_{i-1}}^{x_i} \varphi_i \, dx = -\mu \frac{1}{\cancel{h_{i-1/2}}} \frac{\cancel{h_{i-1/2}}}{2} = \frac{-\mu}{2}$$

$$\int_{x_{i-1}}^{x_i} \sigma^t \varphi_{i-1} \varphi_i \, dx = \sigma^t_{i-1/2} \int_{x_{i-1}}^{x_i} \varphi_{i-1} \varphi_i \, dx$$

$$= \sigma^t_{i-1/2} \int_0^1 (1-\hat{x})\hat{x}\, d\hat{x} \cdot \frac{dx}{d\hat{x}}$$

$$(x = h_{i-1/2}\hat{x} + \text{shift})$$

$$= \sigma^t_{i-1/2} \frac{1}{6} h_{i-1/2}$$

Case 1 ctd

Sub case $j = i-1$ ctd

$$\int_{x_{i-1}}^{x_i} \sigma^t \varphi_{i-1} \, \mu \tau \nabla \varphi_i \, dx = \cancel{\sigma^t_{i-1/2} \, \mu \, (\nabla \varphi_i)_{cell}} \int$$

$$= \sigma^t_{i-1/2} \, \mu \, \frac{\tau_{i-1/2}}{h_{i-1/2}} \int_{x_{i-1/2}}^{x_i} \varphi_{i-1/2} \, dx$$

$$= \sigma^t_{i-1/2} \, \mu \, \frac{1}{2} \cdot \tau_{i-1/2}$$

$$\int_{x_{i-1}}^{x_i} (\mu \cdot \nabla \varphi_{i-1})(\mu \tau \nabla \varphi_i) dx = \mu^2 \tau_{i-1/2} \frac{-1}{(h_{i-1/2})^2} \int_{x_{i-1}}^{x_i} 1 \, dx$$

$$= -\mu^2 \tau_{i-1/2} / h_{i-1/2}$$

Thus:

$$\boxed{\begin{aligned} T_{i,i-1} &= \frac{-\mu}{2} + \frac{h_{i-1/2}}{6} \sigma^t_{i-1/2} + \sigma^t_{i-1/2} \frac{\mu \, \tau_{i-1/2}}{2} - \mu^2 \tau_{i-1/2} / h_{i-1/2} \\ &= -\frac{\mu}{2} + \frac{h_{i-1/2}}{6} \sigma^t_{i-1/2} + \mu \, \tau_{i-1/2} \left(\sigma^t_{i-1/2}/2 - \mu / h_{i-1/2} \right) \end{aligned}}$$

Case 1:

<u>Subcase $j = i$</u>

$\varphi_i = \varphi_j$

$$T_{ii} = \int_{x_{i-1}}^{x_{i+1}} (\mu \nabla \varphi_i + \sigma^t \varphi_i)(\varphi_i + \mu\tau \nabla \varphi_i)\, dx$$

On the left piece:

$$\int_{x_{i-1}}^{x_i} \mu \nabla \varphi_i \, \varphi_i \, dx = \mu \frac{1}{h_{i-1/2}} \int_{x_{i-1}}^{x_i} \varphi_i \, dx = \frac{\mu}{2}$$

$$\int_{x_{i-1}}^{x_i} \sigma^t \varphi_i^2 \, dx = \sigma^t_{i-1/2} \int_0^1 \hat{x}^2 \, d\hat{x} \, \frac{dx}{d\hat{x}}$$

$$= \sigma^t_{i-1/2} \frac{1}{3} h_{i-1/2}$$

$$\int_{x_{i-1}}^{x_i} \sigma^t \varphi_i \, \mu\tau \nabla \varphi_i \, dx = \sigma^t_{i-1/2} \mu \tau_{i-1/2} \frac{1}{2}$$

$$\int_{x_{i-1}}^{x_i} \mu^2 \tau (\nabla \varphi_i)^2 \, dx = \mu^2 \tau_{i-1/2} \left(\frac{1}{h_{i-1/2}}\right)^2 \int_{x_{i-1}}^{x_i} 1 \, dx$$

$$= \mu^2 \tau_{i-1/2} / h_{i-1/2}$$

$\therefore$ left piece $\rightsquigarrow$ $\underline{\dfrac{\mu}{2} + \dfrac{1}{3}\sigma^t_{i-1/2} h_{i-1/2} + \mu \tau_{i-1/2}\left(\dfrac{\sigma^t_{i-1/2}}{2} + \dfrac{\mu}{h_{i-1/2}}\right)}$

Case 1 ctd

<u>Subcase $j>i$ ctd</u>

The right piece looks similar but here $\nabla\varphi_i < 0$:

$$\int_{x_i}^{x_{i+1}} \mu \nabla\varphi_i \, \varphi_i \, dx = -\frac{\mu}{2} \qquad \int_{x_i}^{x_{i+1}} \sigma^t \varphi_i \, \mu\tau\nabla\varphi_i \, dx = -\sigma^t_{i+1/2}\,\mu\,\tau_{i+1/2}\,\frac{1}{2}$$

$$\int_{x_i}^{x_{i+1}} \sigma^t \varphi_i^2 \, dx = \sigma^t_{i+1/2}\,\frac{1}{3}\,h_{i+1/2} \qquad \int_{x_{i-1}}^{x_i} \mu^2\tau(\nabla\varphi_i)^2 \, dx = \frac{\mu\,\tau_{i+1/2}}{h_{i+1/2}}$$

$$\underline{\text{right piece} \rightsquigarrow \; -\frac{\mu}{2} + \frac{1}{3}\sigma^t_{i+1/2}h_{i+1/2} + \mu\,\tau_{i+1/2}\left(-\frac{\sigma^t_{i+1/2}}{2} + \frac{\mu}{h_{i+1/2}}\right)}$$

In total:

$$\boxed{\begin{aligned} T_{ii} = \; & \frac{1}{3}\left(\sigma^t_{i+1/2}h_{i+1/2} + \sigma^t_{i-1/2}h_{i-1/2}\right) \\ & + \mu\,\tau_{i-1/2}\left(\frac{\sigma^t_{i-1/2}}{2} + \frac{\mu}{h_{i-1/2}}\right) \\ & + \mu\,\tau_{i+1/2}\left(-\frac{\sigma^t_{i+1/2}}{2} + \frac{\mu}{h_{i+1/2}}\right) \end{aligned}}$$

Case 1 ctd

Subcase $j = i+1$

This is similar to the first subcase, but "flipped"

$$T_{i,i+1} = \int_{x_i}^{x_{i+1}} (\mu \cdot \nabla \varphi_{i+1} + \sigma^t \varphi_{i+1})(\varphi_i + \mu \ldots) dx$$

$$\int_{x_i}^{x_{i+1}} \mu \nabla \varphi_{i+1} \varphi_{i} \, dx = \mu \left(\frac{1}{\cancel{h_{i+1/2}}} \right) \frac{\cancel{h_{i+1/2}}}{2} = \frac{\mu}{2}$$

$$\int_{x_i}^{x_{i+1}} \sigma^t \varphi_{i+1} \varphi_i = \sigma^t_{i+1/2} \frac{h_{i+1/2}}{6}$$

$$\int_{x_i}^{x_{i+1}} \sigma^t \varphi_{i+1} \mu \tau \nabla \varphi_i \, dx = \sigma^t_{i+1/2} \mu \tau_{i+1/2} \left(-\frac{1}{2} \right)$$

$$\int_{x_i}^{x_{i+1}} \mu \cdot \nabla \varphi_{i+1} \mu \tau \nabla \varphi_i \, dx = -\mu^2 \tau_{i+1/2} / h_{i+1/2}$$

$$T_{i,i+1} = \frac{\mu}{2} + \frac{h_{i+1/2}}{2} \sigma^t_{i+1/2} + \mu \tau_{i+1/2} \sigma^t / -2 - \mu^2 \tau_{i+1/2} / h_{i+1/2}$$

$$= \frac{\mu}{2} + \frac{h_{i+1/2}}{6} \sigma^t_{i+1/2} + \mu \tau_{i+1/2} \left(\frac{-\sigma^t}{2} - \frac{\mu}{h_{i+1/2}} \right)$$

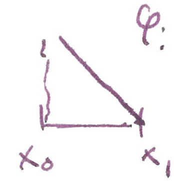
Case 2

$i = 0$

The "stencil" of i here is $\{i, i+1\}$ so there are 2 subcases:

subcase 1: $j = i\ (=0)$

$$T_{00} = \int_{x_0}^{x_1} (\mu \nabla \varphi_i + \sigma^t \varphi_i)(\varphi_i + \mu \tau \nabla \varphi_i)\, dx$$



$$\int_{x_0}^{x_1} \mu \nabla \varphi_0 \, \varphi_0 \, dx = -\frac{\mu}{2}$$

$$\int_{x_0}^{x_1} \sigma^t \varphi_0 \varphi_i \, dx = \sigma^t_{-1/2} \int_0^1 \hat{x}^2 \, d\hat{x} \frac{dx}{d\hat{x}} = \sigma^t_{-1/2} \left(\frac{1}{3}\right) h_{-1/2}$$

$$\int_{x_0}^{x_1} \sigma^t \varphi_i \, \mu \tau \nabla \varphi_i \, dx = \sigma^t_{-1/2} \, \mu \, \tau_{-1/2} \int_{x_0}^{x_1} \varphi_i \nabla \varphi_i \, dx$$

$$= -\sigma^t_{-1/2} \, \mu \, \tau_{-1/2} / 2$$

$$\int_{x_0}^{x_1} \mu^2 \tau (\nabla \varphi_i)^2 \, dx = h_{-1/2} \, \mu^2 \tau \left(-\frac{1}{h_{-1/2}}\right)^2$$

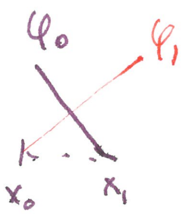
$$= \mu^2 \tau / h_{-1/2}$$

$$\therefore \boxed{T_{00} = -\frac{\mu}{2} + \frac{\sigma^t_{-1/2} h_{-1/2}}{3} + \mu \tau \left(-\frac{\sigma^t_{-1/2}}{2} + \frac{\mu}{h_{-1/2}} \right)}$$

(compare right piece from earlier case)

Case 2 $\quad c \neq 0$

Subcase 2: $j = i+1 = 1$



$$T_{01} = \int_{x_0}^{x_1} (\mu \nabla \varphi_1 + \sigma^t \varphi_1)(\varphi_0 + \mu \tau \nabla \varphi_0)\, dx$$

$$\int_{x_0}^{x_1} \mu \nabla \varphi_1 \varphi_0 \, dx = \mu/2$$

$$\int_{x_0}^{x_1} \sigma^t \varphi_1 \varphi_0 \, dx = \sigma^t_{-1/2} \int_0^1 x(1-x)\, d\hat{x} \frac{dx}{d\hat{x}} = \frac{\sigma^t_{-1/2}}{6} \cdot h_{-1/2}$$

$$\cancel{\int_{x_0}^{x_1} \mu \nabla \varphi_1 \varphi_0 \, dx}$$

$$\int_{x_0}^{x_1} \mu \nabla \varphi_1 \, \mu \tau \nabla \varphi_0 \, dx = -\mu^2 \tau_{-1/2} / h_{-1/2}$$

$$\int_{x_0}^{x_1} \sigma^t \varphi_1 \, \mu \tau \nabla \varphi_0 \, dx = -\sigma^t_{-1/2} \mu \tau_{-1/2} / 2$$

$$\boxed{T_{0,1} = \frac{\mu}{2} + \frac{\sigma^t_{-1/2}}{6} h_{-1/2} + \mu \tau_{-1/2} \left(-\mu / h_{-1/2} - \sigma^t_{-1/2} / 2 \right)}$$

Case 3) $i = N$

Stencil: $\{N, N-1\}$

Subcase $j = N$

φ_N

$$T_{NN} = \int_{x_{N-1}}^{x_N} (\mu \nabla \varphi_N + \sigma^t \varphi_N)(\varphi_N + \mu \tau \nabla \varphi_N)\, dx$$

similar to T_{00} case, but $\nabla \varphi_N > 0$:

$$\boxed{T_{NN} = \frac{\mu}{2} + \frac{\sigma^t_{N-1/2}}{6} h_{N-1/2} + \mu \tau_{N-1/2}\left(\mu / h_{N-1/2} + \sigma^t_{N-1/2} / 2 \right)}$$

Subcase $j = N-1$

φ_{N-1} φ_N

just like $j = i-1$ left piece

$$\boxed{\begin{aligned} T_{N,N-1} &= \int_{x_{N-1}}^{x_N} (\mu \nabla \varphi_{N-1} + \sigma^t \varphi_{N-1})(\varphi_N + \mu \tau \nabla \varphi_N)\, dx \\ &= -\frac{\mu}{2} + \frac{\sigma^t_{N-1/2}}{6} h_{N-1/2} + \mu \tau_{N-1/2}\left(\sigma^t_{N-1/2} / 2 - \mu / h_{N-1/2} \right) \end{aligned}}$$

Summary

$$T_{00} = -\frac{\mu}{2} + \frac{\sigma^t_{-1/2}\, h_{-1/2}}{3} + \mu \tau_{-1/2}\left(\frac{-\sigma^t_{-1/2}}{2} + \frac{\mu}{h_{-1/2}} \right)$$

$$T_{0,1} = \frac{\mu}{2} + \frac{\sigma^t_{-1/2}\, h_{-1/2}}{6} + \mu \tau_{-1/2}\left(-\frac{\sigma^t_{-1/2}}{2} - \frac{\mu}{h_{-1/2}} \right)$$

$$T_{i,i-1} = -\frac{\mu}{2} + \frac{h_{i-1/2}}{6}\sigma^t_{i-1/2} + \mu \tau_{i-1/2}\left(\frac{\sigma^t_{i-1/2}}{2} - \frac{\mu}{h_{i-1/2}} \right)$$

$$T_{i,i} = \frac{1}{3}\left(\sigma^t_{i+1/2} h_{i+1/2} + \sigma^t_{i-1/2} h_{i-1/2} \right) + \mu \tau_{i-1/2}\left(\frac{\sigma^t_{i-1/2}}{2} + \frac{\mu}{h_{i-1/2}} \right) + \mu \tau_{i-1/2}\left(\frac{-\sigma^t_{i+1/2}}{2} + \frac{\mu}{h_{i+1/2}} \right)$$

$$T_{i,i+1} = \frac{\mu}{2} + \frac{h_{i+1/2}\,\sigma^t_{i+1/2}}{6} + \mu \tau_{i+1/2}\left(-\frac{\sigma^t}{2} - \frac{\mu}{h_{i+1/2}} \right)$$

$$T_{N,N-1} = -\frac{\mu}{2} + \frac{h_{i-1/2}\,\sigma^t_{i-1/2}}{6} + \mu \tau_{N-1/2}\left(\sigma^t_{N-1/2}/2 - \frac{\mu}{h_{N-1/2}} \right)$$

$$T_{N,N} = \frac{\mu}{2} + \frac{\sigma^t_{N-1/2}\, h_{N-1/2}}{6} + \mu \tau_{N-1/2}\left(\frac{\sigma^t_{N-1/2}}{2} + \frac{\mu}{h_{N-1/2}} \right)$$

don't forget BCs!