Solution to National Taiwan University Dept. of Mathematics Graduate Qualifying/Entrance Exam

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Prologue

Because I am bad at exam and failed.

You'll have to download the problems on your own. If you find an error, don't hesitate to pull an issue; if you want to collab, throw a PR. The proofs here are not the cleanest: they just work, and that's all I need.

Note that there are two different type of exams: Qualifying exams (held in November), and Entrance Exam(Held in February). There will be solutions to Qualifying exams, but not now.

Have fun suffering in test :)

Best,

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National Taiwan University Dept. of Mathematics
Graduate Entrance Exam 2013: Advanced Calculus/Analysis
Solutions

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1. The integral can be rewrite as
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-x^2}}{1+(x+y)^2}dxdy$$
by noting that the denominator does not change along the line $x+y=c$ for $c\in\mathbb{R}$, we can take a change of variable $z=x+y$, so that
$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-x^2}}{1+(x+y)^2}dxdy &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-x^2}}{1+z^2}dxdz\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-x^2}}{1+z^2}dzdx\\ &= \int_{-\infty}^{\infty}e^{-x^2}dx\tan^{-1}z|_{z=-\infty}^{\infty}\\ &= \pi\int_{-\infty}^{\infty}e^{-x^2}dx\\ &= \pi\cdot\sqrt{\pi}=\pi^{\frac{3}{2}}\end{aligned}$$
The swap of double integral is guaranteed by Tonelli's theorem. ■

2. We abbreviate $g(x,y)$ as g and $\bar{g}(x)$ as $\bar{g}$ for simplicity.

(a) Notice that $\log\left(\frac{1+g}{1-g}\right)$ and $\tan^{-1}(yg)$ are increasing (and decreasing) simultaneously with g. As x increases, the left hand side of the equation must increase, and g must increase.

(b) Rewrite the equation as
$$\log\left(\frac{1+g}{1-g}\right)=2(y^2+1)x-2y\tan^{-1}(yg)$$
and consider 3 separate cases:

i. If $x>0$, then the right hand side tend to $+\infty$ (as y^2 tends to ∞ faster than y, and $\tan^{-1}$ is bounded), therefore by letting $y\to\infty$ in this equation we have
$$\log\left(\frac{1+\bar{g}}{1-\bar{g}}\right)\to+\infty$$
and solving $\bar{g}$ gives $\bar{g}=1$.

ii. If $x<0$, then the right hand side tend to $-\infty$, therefore by letting $y\to\infty$ in this equation we have
$$\log\left(\frac{1+\bar{g}}{1-\bar{g}}\right)\to-\infty$$
and solving $\bar{g}$ gives $\bar{g}=-1$.

iii. If $x=0$, then we have
$$\begin{aligned}\log\left(\frac{1+g}{1-g}\right)&=-2y\tan^{-1}(yg)\\ \frac{1+g}{1-g}&=e^{-2y\tan^{-1}(yg)}\\ 1+g&=(1-g)e^{-2y\tan^{-1}(yg)}\\ g&=\frac{e^{-2y\tan^{-1}(yg)}-1}{1+e^{-2y\tan^{-1}(yg)}}\end{aligned}$$

and we have $\bar{g}(0) = 0$, which could be verified easily by calculation.
(If $\bar{g}(0) > 0$, then the right hand side tends to -1, which is absurd; if $\bar{g}(0) < 0$, then the right hand side tends to 1 ($\tan^{-1}$ changes sign), which is again, nonsense)

Therefore

$$\bar{g}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

(c) To show that g is differentiable, we consider

$$F(x, y, g(x, y)) = \log\left(\frac{1 + g(x, y)}{1 - g(x, y)}\right) + 2y\tan^{-1}(yg(x, y)) - 2(y^2 + 1)x$$

and

$$\frac{\partial F}{\partial g} = \left(\frac{2}{(1-g)^2}\right)\frac{1-g}{1+g} + 2y^2\frac{1}{1+y^2g^2} = \frac{2}{1-g^2} + \frac{2y^2}{1+y^2g^2}$$

which is nowhere zero within the range. It is easily seen that we can always find $g(x, y)$ such that $F(x, y, g(x, y)) = 0$ for all (x, y) (as $\log(\frac{1+g}{1-g})$ maps $(-1, 1)$ to $\mathbb{R}$), therefore by implicit function theorem, g is differentiable in (x, y).

(d) Take $\frac{\partial}{\partial x}$ on both sides of equation gives ($g' = \frac{\partial}{\partial x}g$)

$$\left(\frac{2g'}{(1-g)^2}\right)\frac{1-g}{1+g} + 2y^2g'\frac{1}{1+y^2g^2} = 2(y^2 + 1)$$

$$g' = \left(1 - g^2\right)\left(1 + y^2g^2\right)$$

By taking limit, we have

$$\lim_{y\to\infty} g' = \lim_{y\to\infty}(1 - \bar{g}^2)(1 + y^2\bar{g}^2)$$

and after some calculation we conclude that

$$\lim_{y\to\infty}\frac{\partial g}{\partial x}(x) = \begin{cases} 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

The limit can be calculated using L'Hôpital's rule. ■

3. (a) If we write

$$f(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx}$$

then the fourier coefficient is

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\,2\int_0^{\pi} e^{-in\pi}dx = \frac{-1}{in\pi}e^{-inx}|_{x=0}^{\pi} = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{2}{inx} & \text{if } n \text{ is odd} \end{cases}$$

therefore

$$f(x) = \frac{2}{i\pi}\sum_{n\in\mathbb{Z}}\frac{1}{(2n+1)}e^{i(2n+1)x}.$$

(b) Let $x = \frac{\pi}{2}$ and we have

$$1 = \frac{2}{i\pi}\left(\cdots\frac{1}{-5}(-i) + \frac{1}{-3}i + \frac{1}{-1}(-i) + \frac{1}{1}i + \frac{1}{3}(-i) + \frac{1}{5}i + \cdots\right)$$

therefore

$$1 = \frac{\mathbf{4}}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

and the desired sum is $\frac{\pi}{4}$. ■

4. (a) Proceed by mathematical induction, the case $n = 2$ is simple, so we assume that $f_k < f_{k-1}$ for some k. This gives

$$f_k(x) = \frac{1}{2}\left(f_{k-1}(x) + \frac{e^x}{f_{k-1}(x)}\right) < f_{k-1}(x)$$

$$f_{k-1}(x)^2 - e^x > 0 \tag{1}$$

Now we want to show that $f_k(x)^2 - e^x > 0$, so $f_{k+1} < f_k$. Indeed,

$$\begin{aligned} f_k(x)^2 - e^x &= \frac{1}{4}\left(f_{k-1}(x) + 2e^x + \frac{e^{2x}}{f_{k-1}(x)}\right) - e^x \\ &= \frac{1}{4}\left(f_{k-1}(x)^2 - 2e^x + \frac{e^{2x}}{f_{k-1}(x)^2}\right) \\ &\geq \frac{1}{4}\left(-e^x + \frac{e^{2x}}{f_{k-1}(x)^2}\right) \qquad \text{by (1)} \\ &\geq \frac{1}{4}e^x\left(1 - \frac{e^x}{f_{k-1}(x)^2}\right) > 0 \end{aligned}$$

as $\dfrac{e^x}{f_{k-1}(x)^2} < 1$ by (1).

(b) Take limit on both sides of the recursive formula leaves

$$f(x) = \frac{1}{2}\left(f(x) + \frac{e^x}{f(x)}\right)$$

and solving this gives $f(x) = e^{\frac{x}{2}}$.

(c) Yes, as $[-1, 1]$ is compact, $f_n(x)$ is continuous for all n, and $\{f_n\}$ is monotone decreasing in n, therefore by Dini's Theorem, the sequence converges to f uniformly. ■

National Taiwan University Dept. of Mathematics
Graduate Entrance Exam 2014: Advanced Calculus/Analysis
Solutions

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1. The function

$$f(x) = \begin{cases} x^2 \cos(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is differentiable and has derivative

$$f'(x) = \begin{cases} 2x \cos(\frac{1}{x}) + \sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

but $\lim_{x \to 0} f'(x)$ does not exist as sin oscillates near 0, therefore not continuous. ∎

2. For every $\epsilon > 0$ we must find $\delta > 0$ such that

$$\left| \frac{x}{\sqrt{x^2+5}} - \frac{2}{3} \right| < \epsilon \text{ whenever } |x-2| < \delta \tag{*}$$

Consider the "close" case $|x-2| < 1$, i.e. $1 < x < 3$. Then

$$\begin{aligned} \left| \frac{x}{\sqrt{x^2+5}} - \frac{2}{3} \right| &= \left| \frac{3x - 2\sqrt{x^2+5}}{3\sqrt{x^2+5}} \right| \\ &= \left| \frac{(3x - 2\sqrt{x^2+5})(3x + 2\sqrt{x^2+5})}{(3\sqrt{x^2+5})(3x + 2\sqrt{x^2+5})} \right| \\ &= \left| \frac{5x^2 - 20}{(3\sqrt{x^2+5})(3x + 2\sqrt{x^2+5})} \right| \end{aligned}$$

Now notice that the numerator is

$$|5x^2 - 20| = 5|(x+2)(x-2)| < 25|x-2|$$

and the denominator is

$$|(3\sqrt{x^2+5})(3x + 2\sqrt{x^2+5})| > 9\sqrt{5} + 30 > 50$$

for all $x \in (1,3)$. Now we can choose

$$\delta = \min(1, \epsilon/2)$$

and it will satisfy (*). ∎

3. False. We can construct a counterexample

$$n_1 = (1,0,0,\cdots), n_2 = (0,1,0,0,\cdots), n_3 = (0,0,1,0,0,\cdots), \cdots$$

The sequence $\{n_i\}_{i=1}^{\infty}$ is in B, but B is not (sequentially) compact as $\{n_i\}_{i=1}\infty$ does not have any convergent subsequence. ∎

4. False. Only if the metric space is *complete*, then any Cauchy sequence will converge in X. A counterexample is given by an non-complete metric space

$$X = (\mathbb{Q}, d) \quad \text{where} \quad d(x,y) = |x-y|$$

then there are seqences in X that converges to $\sqrt{2}$, but $\sqrt{2} \notin \mathbb{Q}$. ∎

5. False. Choose a partition $0 = a_0 < a_1 < \cdots < a_n = 1$. Then we can always find k even such that

$$0 < \frac{1}{(k+\frac{3}{2})\pi} < \frac{1}{(k+\frac{1}{2})\pi} < a_1$$

by Archimedean principle. Then $f\left(\frac{1}{(k+\frac{3}{2})\pi}\right) = -1$, $f\left(\frac{1}{(k+\frac{1}{2})\pi}\right) = 1$. Therefore for any $a_1 > 0$, the upper Riemann sum and the lower Riemann sum will never be equal in $[0, a_1]$ (and there is a partition P such that $U(P, f) = L(P, f)$ on $[a_1, 1]$ by continuity), hence f is not Riemann integrable. ■

6. False. Notice that the outer measure on $B(\epsilon)$ is ϵ, and $[0, 1]$ has outer measure 1. By the fact that if $A \subseteq B$ then $m^*(A) \leq m^*(B)$, we have that for $\epsilon < 1$, B will never cover $[0, 1]$. ■

7. (a)

   (b) We guess that the value is $f(1)$, as the function $K_n(x)$ became concentrate at 1 as $n \to \infty$. We proof our claim below.
   First notice that

$$\int_0^1 K_n(x) = 1 \tag{1}$$

   and

$$\int_0^{1-\delta} K_n(x) \to 0 \text{ as } n \to \infty \tag{2}$$

   for every $\delta < 1$. Now as f is continuous, for any $\epsilon > 0$ we can find $\delta > 0$ such that

$$|f(x) - f(1)| < \epsilon \text{ whenever } |1 - x| < \delta$$

   therefore

$$\begin{aligned}
\left|\int_0^1 K_n(x) f(x) dx - f(1)\right| &= \left|\int_0^1 K_n(x) f(x) - f(1) dx\right| \overset{(1)}{=} \left|\int_0^1 K_n(x)(f(x) - f(1)) dx\right| \\
&\leq \int_{1-\delta}^1 |K_n(x)(f(x) - f(1))| dx + \int_0^{1-\delta} |K_n(x)(f(x) - f(1))| dx \\
&\leq \epsilon + \int_0^{1-\delta} 2M K_n(x) \overset{(2)}{\to} \epsilon \text{ as } n \to \infty
\end{aligned}$$

   where M is the number such that $|f(x)| \leq M$ for all $x \in [0, 1]$. By the arbitrariness of ϵ, we conclude that $\int_0^1 K_n(x) f(x) dx = f(1)$. ■

8. f is integrable means that for every $\epsilon > 0$, there exists a partition $P = \{a = x_0, x_1, \cdots x_n = b\}$ such that

$$\left|\int_a^b f(x) dx - \sum_{i=1}^n m_i (x_i - x_{i-1})\right| < \epsilon$$

where m_i is the minimum of f on $[x_{i-1}, x_i]$. This is the same as

$$0 \leq \int_a^b f(x) - g(x) < \epsilon$$

where $g(x) = \sum_{i=1}^n m_i \chi_{[x_{i-1}, x_i]}$. Then

$$\begin{aligned}
\int_a^b f(x) \sin nx \, dx &\leq \int_a^b (f(x) - g(x)) \sin nx \, dx + \int_a^b g(x) \sin nx \, dx \\
&\leq \int_a^b (f(x) - g(x)) \, dx + \left|\frac{1}{n} \sum_i m_i (\cos nx_i - \cos nx_{i-1})\right|
\end{aligned}$$

the former part is less than ϵ, and the latter can be arbitrary small (we can choose ϵ) as we take n large enough. So the sum is less than 2ϵ, and by the arbitrariness of ϵ, we conclude that

$$\lim_{n\to\infty}\int_a^b f(x)\sin nx\,dx = 0. \quad \blacksquare$$

9. (a) Arezla-Ascoil Theorem states that a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is uniform bounded and equicontinuous if and only if $\{f_n\}_{n\in\mathbb{N}}$ admits a subsequence that converges uniformly.

   (b) This problem is flawed, as we can find a counterexample

$$f_n(x) = -n. \quad \blacksquare$$

10. Consider a C^∞ function F

$$F(x,y,z) = x^3 + y^3 + z^3 - 3xyz - 1.$$

   (a) We calculate the partial derivative

$$\frac{\partial F}{\partial z} = 3z^2 - 3xy$$

   and since $\frac{\partial F}{\partial z}(0,0,1) = 3 \neq 0$, by Implict Function Theorem there exists a neighborhood near $(0,0,1)$ such that $z = z(x,y)$, and it has the same differentiabiliy as F, i.e. C^∞.

   (b) We calculate the partial derivative

$$\frac{\partial F}{\partial x} = 3x^2 - 3yz$$

   and since $\frac{\partial F}{\partial z}(0,0,1) = 0$, x cannot be represent as a function $x = x(y,z)$.

11. (a) The left hand side can be directly calculated as

$$\lim_{h\to 0}\int_c^d \frac{f(x+h,y) - f(x,y)}{h}dy$$

   and if we can pass the limit through the integral, then the assertion is proved. By the continuity of f, we can use mean value theorem to find $z \in [x, x+h]$ such that

$$f_h(z,y) = f_d(x,y) = \frac{f(x+d,y) - f(x,y)}{d}$$

   where f_d is the difference defined at the right. As $\frac{\partial f}{\partial x}$ is continuous and hence bounded in $[a,b]$, we have that for every sequence $\{n_k\} \to 0$, the sequence $\{f_{n_k}(x,y)\}$ is *uniformly* bounded (as it does not depend on n_k by the equation above) and converges pointwise to $\frac{\partial f}{\partial x}$ (by the existence of the partial derivative), therefore by Bounded convergence theorem, the limit can be passed through the integral sign, which proved the assertion.

   All the needed additional requirement in the proof above are the continuity of f and $\dfrac{\partial f}{\partial x}$.

   (b) The equation can be rewritten as

$$\lim_{d\to 0}\lim_{n\to\infty}\int_0^n \frac{f(x+d,y) - f(x,y)}{d}dy = \lim_{n\to\infty}\int_0^n \lim_{d\to 0}\frac{f(x+d,y) - f(x,y)}{d}dy$$

National Taiwan University Dept. of Mathematics
Graduate Entrance Exam 2014: Linear Algebra
Solutions

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1. The first problem is flawed as the solution space of the given two equations is only 1-dimensional.

2. (1) $f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 7\lambda^2 - 15\lambda + 9.$

   (2) The solutions of f are 1 and 3, in which 3 has multiplicity of 2. If $\lambda = 1$ we have

$$A - I = \begin{pmatrix} 0 & 2 & 2 \\ 1 & 1 & -1 \\ -1 & 1 & 3 \end{pmatrix}$$

   and a possible eigenvector is

$$e_1 = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}.$$

   When $\lambda = 3$ we have

$$A - 3I = \begin{pmatrix} -2 & 2 & 2 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$$

   which has two eigenvectors as the rank is 1:

$$e_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

   Therefore we have

$$P = \begin{pmatrix} -2 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}. \quad \blacksquare$$

3. (1) If $A^k = 0$, then $I - A^k = I$, and

$$(I - A)(I + A + A^2 + A^3 + \cdots + A^{k-1}) = I - A^k = I$$

   so $I - A$ is invertible.

   (2) First note that $\operatorname{im}(A^k) \subseteq \operatorname{im}(A^{k-1})$ since $A^k(V) = A^{k-1}(\operatorname{im} A) \subseteq A^{k-1}(V)$, so $\operatorname{rank}(A^k) \leq \operatorname{rank}(A^{k-1})$. But as V is finite dimensional, the sequence of descending subsets (or ranks) must stop, meaning there exists some p such that

$$\operatorname{rank}(A^p) = \operatorname{rank}(A^{p+k}) \quad \text{i.e. } A^p(V) = A^{p+k}(V) \quad \forall k \geq 0 \tag{1}$$

   Now consider the simplest nontrivial case, i.e. $p = 2$. If $\operatorname{rank}(A) = \operatorname{rank}(A^2)$, then

$$\dim(\operatorname{im} A) = \operatorname{rank} A = \operatorname{rank} A^2 = \dim(A(A(V)) = \dim(A(\operatorname{im} A)) = \operatorname{rank}(A|_{\operatorname{im} A})$$

   By viewing $A|_{\operatorname{im} A}$ as a endomorphism on $\operatorname{im} A$ (as $\operatorname{im} A$ is A-invariant), since the rank of A is exactly $\operatorname{im} A$, we conclude that $A|_{\operatorname{im} A}$ is surjective (hence injective since V is finite dimensional), so $\{0\} = \ker A|_{\operatorname{im} A} = \ker A \cap \operatorname{im} A$. Then it is immediate that

$$V = \ker A \oplus \operatorname{im} A.$$

   Then for any $p \geq 2$ that satisfy (1), since $\operatorname{rank}(A^p) = \operatorname{rank}(A^{2p})$, by the simple case we have

$$V = \ker A^p \oplus \operatorname{im} A^p.$$

   as desired. $\blacksquare$

4. Just assume that X is of form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and after calculation we have

$$L\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c-4b & 5b-d+a \\ 4a-5c-4d & 4b+c \end{pmatrix}$$

and if we identify the 2×2 matrix by $\mathbb{R}^4$ then we have

$$L\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 & -4 & -1 & 0 \\ 1 & 5 & 0 & -1 \\ 4 & 0 & -5 & -4 \\ 0 & 4 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

To find a basis of $\operatorname{im} L$ we can plug in the standard $\mathbb{R}^4$ basis and simplify, and we have

$$\left\{\begin{pmatrix} 0 \\ 1 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 5 \\ 1 \end{pmatrix}\right\}$$

as a basis of $\operatorname{im} L$. Then by Dimension Theorem, the dimension of the kernel is 2. ■

5. (1) Let $S = AA^*$ so $S = S^*$. Then $AA^*v = A^*Av = 0 = Sv$. Then

$$\langle Av, Av\rangle = \langle v, Sv\rangle = 0$$

therefore $Av = 0$.

(2) Let k be the minimal positive integer such that $A^k v = 0$. Then

$$S^k v = (A^*A)^k v = A^{*k}A^k v = 0$$

so

$$0 = \langle S^k v, S^{k-2}v\rangle = \langle S^{k-1}v, S^{k-1}v\rangle$$

and $S^{k-1}v = 0$. Then following this pattern we have $Sv = 0$, and by the previous question $Av = 0$.

(3) If the minimal polynomial of A, say f, has repeated roots, i.e. $O = f(A) = (A-kI)^p g(A)$ where $p > 1$, $g(A)$ is some polynomial, then for all $v \in \operatorname{im} g(A)$, we have

$$0 = (A-kI)^p v$$

Then previous problem and the fact that $(A-kI)$ is normal for all k impiles that $(A-kI)v = 0$ for all $v \in \operatorname{im} g(A)$, therefore

$$(A-kI)g(A) = O$$

but this polynomial has degree less than f, contradicting to the hypotheses that f is minimal. Hence the charactristic polynomial of A must not have repeated roots. ■

National Taiwan University Dept. of Mathematics
Graduate Entrance Exam 2015: Linear Algebra
Solutions

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1. If v is a vector that is in $V \cap W$, then this means that there are values $a, b \in \mathbb{R}^4$ such that

$$v = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 6 & 7 & 9 & 10 \\ 0 & 1 & 0 & 2 & 0 & 3 \\ 1 & -2 & 3 & -4 & 5 & -6 \end{pmatrix}^T a = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 \\ -2 & 0 & -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & -2 & -2 \end{pmatrix}^T b$$

and, in a complete matrix,

$$\begin{pmatrix} 1 & 3 & 0 & 1 & 1 & -2 & 1 & 0 \\ 2 & 4 & 1 & -2 & 1 & 0 & 0 & 0 \\ 3 & 6 & 0 & 3 & 1 & -1 & 1 & 1 \\ 4 & 7 & 2 & -4 & 2 & 0 & 0 & 0 \\ 5 & 9 & 0 & 5 & 2 & 1 & 2 & -2 \\ 6 & 10 & 3 & -6 & 3 & 2 & 0 & -2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ -b_1 \\ -b_2 \\ -b_3 \\ -b_4 \end{pmatrix} = 0$$

2. We show that

$$\text{given } A \text{ a integer matrix, } A^{-1} \text{ is an integer matrix if (and only if) } \det(A) = \pm 1$$

Indeed, if $\det(A) = \pm 1$, then $\det(A^{-1}) = \mp 1$. Since all the adjugate matrix has integer entries, A^{-1} is a integer matrix. So we can calculate the determinant of A, which is

$$\det A = 2x^2 - x - 4$$

as $\det A$ can only be -1 (the other one has no integer solution), we have $\det A = -1$, which has a integer solution $x = -1$. Then the inverse matrix is found as

$$\begin{pmatrix} 0 & -4 & 2 & -9 & -5 \\ 1 & -1 & 2 & -6 & -1 \\ 0 & 3 & -2 & 8 & 4 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & -2 & 1 & -4 & -3 \end{pmatrix}. \quad \blacksquare$$

3. A linear operator $T : V \to V$ is diagonalizable if and only if all the vectors can be written as linear combinations of eigenvectors. A property that the transpose operator has is that $T^2 = I$. In other words, T has eigenvalue $+1$ and -1. Now, if M is any matrix, then $M + M^t$ is a eigenvector of $+1$, and $M - M^t$ is a eigenvector of -1. As we have

$$M = \frac{1}{2}(M + M^t) + \frac{1}{2}(M - M^t)$$

we conclude that T is diagonalizable. Notice that if char $F = 2$, then the only eigenvalue is 1, and since the dimesion of all symmetric matrices are only $\frac{n(n+1)}{2}$, the multiplicities did not agree, hence not diagonalizable.
To find a basis, simply list out a basis for symmetric matrices and a basis for antisymmetric matrices and take the union. $\blacksquare$

4. If T is onto, then we want to check if $\ker T^* = \{0\}$ or not. This follows since if $f(T) = 0$, then

$$f(w) = f(T(v)) = 0 \; \forall w \in W$$

by surjectivity of T. So f must be 0.

For the converse, if we have T^* injective, we need to show $\operatorname{im} T = W$. Suppose not, i.e. $\operatorname{im} T \subsetneq W$, then we can construct a linear functional $f \in W^*$ by

$$blablabla$$

5. Consider

$$0 = \det(cA + B) = \det(cI + BA^{-1}) \det(A)$$

This yields $\det(cI + BA^{-1}) = 0$ as A is invertible. This equation then just simply calculate the characterstic polynomial of $-BA^{-1}$. As the degree of the polynomial is at most n, there are at most n number such that the matrix $cA + B$ is not invertible. $\blacksquare$

6. (a)

$$q(TS) = TSp(TS) = Tp(ST)S = 0.$$

(b) Either the minimal polynomial are equal ($ST = TS$), or they differ from a multiple.

7. Existence of the vector follows by consider the vector

$$v = \sum_{1}^{n} c_n v_n.$$

Now if there are two vectors v, v' that satisfy the equations given, then

$$0 = \langle v, v_j \rangle - \langle v', v_j \rangle = \langle v - v', v_j \rangle$$

for all $j = 1, \cdots n$. This simply says that $v - v'$ is orthogonal to all vectors that form a basis, which implies $v - v' = 0$. $\blacksquare$

National Taiwan University Dept. of Mathematics
Graduate Entrance Exam 2016: Linear Algebra
Solutions

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1. Three vectors are linearly dependent if
$$\begin{vmatrix} 1 & 3 & a \\ a & 4 & 3 \\ 0 & a & 1 \end{vmatrix} = 0$$
the resulting equation is $a^3 - 6a + 4 = 0$, which has 3 roots: 2 and $1 \pm \sqrt{3}$. $\blacksquare$

2. If we use the vectors given by the problem, it will be too slow to solve. Instead we use the standard basis $\{1, x, x^2\}$ and do Gram-Schmidt:
$$v_1 = 1$$
$$v_2 = x - \frac{\langle x, 1\rangle}{||1||^2}1 = x - 1$$
$$v_3 = x^2 - \frac{\langle x^2, 1\rangle}{||1||^2}1 - \frac{\langle x^2, x-1\rangle}{||x-1||^2}(x-1) = x^2 - 2x + \frac{2}{3}$$
and after normalizing, we have
$$w_1 = \frac{1}{2}, w_2 = \sqrt{\frac{3}{2}}(x-1), w_3 = \sqrt{\frac{45}{8}}(x^2 - 2x + \frac{2}{3}) \quad \blacksquare$$

3. Note that even if the characterstic polynomial of A is $-(\lambda - 1)(\lambda^2 - 2\lambda + 10)$ which says that A is not diagonalizable over $\mathbb{R}$, but if two matrices are simliar in $\mathbb{R}$, then P must be in $M_3(\mathbb{R})$. So let
$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & -3 & 1 \end{pmatrix}$$
If we can write $A = HDH^{-1}, B = QDQ^{-1}$, then $P^{-1}AP = B = QDQ^{-1} = QH^{-1}DHQ^{-1}$, so $P = HQ^{-1}$. Directly solve for P and Q gives
$$H = \begin{pmatrix} 1 & -i & i \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \ Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & i \\ 0 & 1 & 1 \end{pmatrix}, \ D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+3i & 0 \\ 0 & 0 & 1-3i \end{pmatrix}$$
direct computation then gives
$$HQ^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \blacksquare$$

4. The eigenvalues are $1, -2, -1$, which correspond to three eigenvectors v_1, v_2 and v_3. Also note that A^T has three eigenvectors w_1, w_2 and w_3 corresponding to the same eigenvalue as A. We now claim that
$$x_i y_j^T \text{ is an eigenvalue of } T \text{ for } i, j = 1, 2, 3$$
Indeed
$$T(x_i y_j^T) = Ax_i y_j^T A^{-1} = \lambda x_i \lambda^{-1} y_j^T = x_i y_j^T$$
as
$$(y_j^T A^{-1})^T = (A^{-1})^T y^j = (A^T)^{-1} y_j = \lambda^{-1} y_j \Rightarrow y_j^T A^{-1} = \lambda^{-1} y_j^T$$
since y_j is also an eigenvector of A^T but with $1/\lambda$ as eigenvalue. Therefore we conclude that T is diagonalizable since eigenvectors of A and A^T are linearly independent, so does $\{x_i y_j^T\}_{1 \le i,j \le 3}$. $\blacksquare$

5. Without loss of generality, we can assume that A is a Jordan block of eigenvalue λ, as matrices are simliar if and only if they have the same Jordan canonical form. Our goal is to prove that B has a Jordan form that is exactly A.

   Let $A \in M_{n\times n}(\mathbb{R})$ be a Jordan block of eigenvalue λ. If λ is 0, then since $\operatorname{rank} A = \operatorname{rank} A^2$, the eigenvalue 0 has no new vector in A^2: all eigenvectors are in A, which means that the Jordan block that has 0 as eigenvalue will have size 1. So A must be a zero matrix, and so must be A^3, B^3 and B.

   If $\lambda \neq 0$, a quick calculation of A^3 gives

$$\begin{pmatrix} \lambda^3 & n\lambda^2 & \frac{n(n-1)}{2}\lambda & 0 & \cdots & 0 \\ 0 & \lambda^3 & n\lambda^2 & \frac{n(n-1)}{2}\lambda & 0 & \cdots \\ 0 & 0 & \lambda^3 & n\lambda^2 & \frac{n(n-1)}{2}\lambda & \cdots \\ \vdots & & & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \lambda^3 \end{pmatrix}$$

   This is an upper triangle matrix, so its eigenvalues are exactly the elements in the diagonal. Now to determine its Jordan form, by noting that

$$\operatorname{rank}(A^3 - \lambda^3 I)^k = n - k$$

   we have the Jordan form of A^3, that is

$$J_3 = \begin{pmatrix} \lambda^3 & 1 & 0 & \cdots & 0 \\ 0 & \lambda^3 & 1 & 0 & \cdots \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \lambda^3 \end{pmatrix}$$

   By assumption, J_3 is also the Jordan form of B^3, so we have

$$B^3 = QJ_3Q^{-1}$$

   now if $B = PJ_1P^{-1}$ where J_1 is a Jordan form, then we have

$$B^3 = PJ_1^3P^{-1} = QJ_3Q^{-1}$$

   so J_1^3 is simliar to J_3. In particular, we can let $J_1 = A$, and by the uniqueness of Jordan canonical form, we are done. ■

6. Rewrite the equation as

$$AAB - ABA = ABA - BAA$$

   and we have $A[A, B] = [A, B]A$. We show a lemma first.
   Lemma. If $\operatorname{tr}(A^n) = 0$ for all positive integers n, then A is nilpotent.
   Proof. If not, then A has some non-zero distinct eigenvalues $\lambda_i, i = 1, \cdots, k$. Let n_i be the multiplicity of λ_i. Then we have

$$\begin{cases} n_1\lambda_1 + \cdots + n_r\lambda_r &= 0 \\ \vdots & \vdots \\ n_1\lambda_1^r + \cdots + n_r\lambda_r^r &= 0 \end{cases}$$

   that is,

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_r^2 \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^r & \lambda_2^r & \cdots & \lambda_r^r \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

But

$$\det \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_r^2 \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^r & \lambda_2^r & \cdots & \lambda_r^r \end{pmatrix} = \lambda_1 \cdots \lambda_r \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_r^2 \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{r-1} & \lambda_2^{r-1} & \cdots & \lambda_r^{r-1} \end{pmatrix} \neq 0$$

as this is a Vandermonde matrix. Therefore all n_i's are zero, a contradiction.

Now we need to show that $\mathrm{tr}([A,B]^n) = 0$ for all positive integers n, then by the lemma we are done. For $k \geq 0$, we have

$$[A,B]^{k+1} = (AB - BA)[A,B] = AB[A,B] - BA[A,B] = AB[A,B] - B[A,B]A$$

then the trace is

$$\mathrm{tr}([A,B])^{k+1}) = \mathrm{tr}(AB[A,B]) - \mathrm{tr}(B[A,B]A) = \mathrm{tr}(B[A,B]A) - \mathrm{tr}(B[A,B]A) = 0$$

by the cyclic property of trace. Then $[A,B]$ is nilpotent by lemma, therefore $[A,B]^n = 0$. ■

National Taiwan University Dept. of Mathematics
Graduate Entrance Exam 2017: Linear Algebra
Solutions

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1. The proof is by induction. The case $n = 1$ is trivial, as $\mathbb{R}$ is not a countable union of points ($\mathbb{R}^0$). Assume that the case $n = k - 1$ is true. For the case $n = k$, suppose the contrary, i.e.

$$\mathbb{R}^k = \bigcup_{i \in \mathbb{N}} W_i$$

where $W_i \subsetneq \mathbb{R}^k$ is a $(n-1)$-dimesional subspace (we can assume this without loss of generality). Let G be a $(n-1)$-dimesional subspace that is not in $\{W_i\}_{i\in\mathbb{N}}$. Then we can clearly write

$$G = \mathbb{R}^k \cap G = \bigcup_{i \in \mathbb{N}} (W_i \cap G)$$

As G is not in W_i, every $(W_i \cap G)$ has dimesion less than $n-1$. So we can write a $(n-1)$-dimesional subspace as a countable union of proper subspaces. But this is a contradiction to our induction, so $\mathbb{R}^k$ cannot be represent as a countable union of proper subspace. Therefore there must be some W_i that is the whole space, which proved the assertion. ■

2. First we need to diagonalize the quadratic form. Finding the eigenvectors could be pretty hard, so we try to complete the squares:

$$\begin{aligned}
&ax^2 + by^2 + cz^2 + 2dxy + 2eyz + 2fxz \\
&= a\left(x + \frac{d}{a}y + \frac{f}{a}z\right)^2 + \left(2e - 2\frac{df}{a}\right)yz + \left(b - \frac{d^2}{a}\right)y^2 + \left(c - \frac{f^2}{a}\right)z^2 \\
&= a\left(x + \frac{d}{a}y + \frac{f}{a}z\right)^2 + \left(b - \frac{d^2}{a}\right)\left(y + \frac{e - \frac{df}{a}}{b - \frac{d^2}{a}}z\right)^2 + \left(c - \frac{f^2}{a} - \frac{e - \frac{df}{a}}{b - \frac{d^2}{a}}\right)z^2
\end{aligned}$$

So we perform the change of variable

$$\begin{aligned}
u &= \sqrt{a}\left(x + \frac{d}{a}y + \frac{f}{a}z\right) \\
v &= \sqrt{b - \frac{d^2}{a}}\left(y + \frac{e - \frac{df}{a}}{b - \frac{d^2}{a}}z\right) \\
w &= \sqrt{c - \frac{f^2}{a} - \frac{e - \frac{df}{a}}{b - \frac{d^2}{a}}}z
\end{aligned}$$

and we have the change of variable matrix

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \sqrt{a} & * & * \\ 0 & \sqrt{b - \frac{d^2}{a}} & * \\ 0 & 0 & \sqrt{c - \frac{f^2}{a} - \frac{e - \frac{df}{a}}{b - \frac{d^2}{a}}} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

the determinant of this matrix is

$$\sqrt{abc + 2def - ae^2 - bf^2 - cd^2}$$

Now the quadratic form is converted to $u^2 + v^2 + w^2 = 1$, which is a sphere with radius 1, so it has volume

$$\frac{4\pi}{3} = \det \times Q_{x,y,z}$$

therefore

$$Q = \frac{4\pi}{3\sqrt{abc + 2def - ae^2 - bf^2 - cd^2}} \quad ■$$

3. The determinant
$$\begin{vmatrix} t & 2t^2 & 3t^3 & 4t^4 \\ t^2 & 2t^3 & 3t^4 & 4t \\ t^3 & 2t^4 & 3t & 4t^2 \\ t^4 & 2t & 3t^2 & 4t^3 \end{vmatrix} = 24t^4(1-t^4)^3$$
is only zero for finitely many t. ■

4. The characteristic polynomial is $(\lambda-1)^2(\lambda-2)^2$, and in order to fail the diagonalizability, we must let one eigenvalue's geometric multiplicity less than its algebraic multiplicity:

   - The case $\lambda = 1$ gives the matrix
   $$\begin{pmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
   and we want this matrix to be rank 3. As long as $a \neq 0$, this matrix will be rank 3.
   - The case $\lambda = 2$ gives the matrix
   $$\begin{pmatrix} -1 & a & b & c \\ 0 & -1 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
   and we want this matrix to be rank 3. As long as $f \neq 0$, this matrix will be rank 3.

5. The assumption implys that if $Au = \lambda u$, then
$$(A^7 - I) = (\lambda^7 - 1) = 0$$
and one root of λ is 1. ■

6. (a) True. Since $\ker\psi$ has dimesion $n-1$, $\operatorname{im}\psi$ has dimesion 1, and has a basis $\beta \in \mathbb{R}^n$, all vector in $\operatorname{im}\psi$ can be represent as $\lambda\beta$ for some $\lambda \in \mathbb{R}$. As $\operatorname{im}\psi$ is invariant under ψ, the conclusion is immediate. ■

   (b) True. By the relation
   $$\dim(A+B) = \dim(A) + \dim(B) - \dim(A \cap B)$$
   we know that $\dim(W_1 \cap W_2)$ could be $6, 7$ or 8. And the system is of rank 1 to 4, which means that the system has a solution of dimesion 9 to 6, so it is complete possible to find a system that has exactly $W_1 \cap W_2$ as solution space. ■