Stochastic Process

Poisson Processes (1)

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Chapter 2. Poission processes

§ 2.1 **Introduction**

§ 2.2 Definition and properties of a Poission process

Introduction

Example

甲，乙，丙三人玩游戏。试讨论如下几种规则的公平性：

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甲，乙，丙三人玩游戏。试讨论如下几种规则的公平性：

情形1：丙是仲裁，掷一均匀硬币多次。

1. 若第一次出现正面，则甲胜；否则，乙胜。
2. 若前三次出现的是“正正反”，则甲胜；若前三次出现的是“正正正”，则乙胜；其他情形，和。
3. 依次记录掷硬币结果，若“正正反”先出现，则甲胜；若“正正正”先出现则乙胜；
4. 依次记录掷硬币结果，若“正正反”先出现，则甲胜；若“正反反”先出现，则乙胜。

Introduction

Example

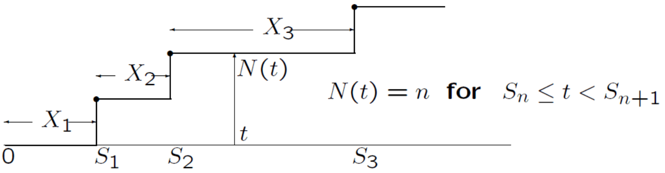
情形2：甲、乙各持一个硬币，各自掷，并依据各自的硬币观察序列

1. 若甲第一次出现正面，则甲胜；若乙第一次出现正面，乙胜。其他情形（包括同时出现正面），和。
2. 若甲前三次出现的是“正正反”，则甲胜；若乙前三次出现的是“正正正”，则乙胜；其他情形（包括同时观察到），和。
3. 若甲先观察到“正正反”，则甲胜；若乙先观察到“正正正”，则乙胜；同时观察到，和。
4. 若甲先观察到“正正反”，则甲胜；若乙先观察到“正反反”，则乙胜。同时观察到，和。
5. 若甲先观察到“正反正”，则甲胜；乙若先观察到“正反正”，则乙胜；同时观察到，和。

Arrival Processes

Definition

An arrival process *is a sequence of increasing r.v.s* $0 < S_1 < S_2 < \dots$ *where* $S_{i-1} < S_i$ *means that* $S_i - S_{i-1} = X_i$ *is a positive r.v.,* $F_{X_i}(0) = 0$*. The differences* X_i *are call* interarrival times *and the* S_i *are called* arrival epochs*.*

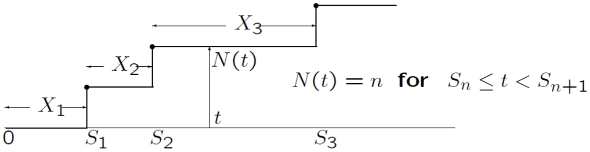


For each $t > 0$, $N(t)$ is the number of arrivals in $(0, t]$. We call $\{N(t) : t > 0\}$ an *arrival counting process*.

Arrival Processes

Remark

The process starts at time 0 and that multiple arrivals cannot occur simultaneously, or we will permit simultaneous arrivals or arrivals at time 0 as events of zero probability, but these can be ignored.



The figure shows how the arrival epochs, interarrival times, and counting variables are interrelated for a generic stair case function.

Remark

- $X_i = S_i - S_{i-1}$ *for* $i \geqslant 2$ *and* $X_1 = S_1$; $S_n = \sum_{i=1}^{n} X_i$.
- $\{S_n \leqslant t\} = \{N(t) \geqslant n\}$ *for all* $n \geqslant 1,\ t > 0$.

If $S_n = \tau$ *for some* $\tau \leqslant t$, *then* $N(\tau) = n$ *and* $N(t) \geqslant n$.

An arrival process can be specified by the joint distributions of the arrival epochs, or of the interarrival times, or of the counting random variables.

Chapter 2. Poission processes

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§ 2.2 Definition and properties of a Poission process

The general class of arrival processes is too complicated to make much further progress. We simplify as follows:

Definition

A *renewal process* is an arrival process for which the sequence of interarrival times $X_1, X_2, \ldots$ are i.i.d..

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Definition

A *Poisson process* is a renewal process for which each X_i has an exponential distribution,

$$\Pr\{X > x\} = F_X^c(x) = \exp(-\lambda x) \quad \text{for } x \geqslant 0,$$

where λ is a fixed parameter called the rate.

or each X_i has the PDF: $f_X(x) = \lambda \exp(-\lambda x) \quad \text{for } x \geqslant 0.$

Memoryless Property

The remarkable simplicity of Poisson processes is closely related to the 'memoryless' property:

Definition

A r.v. X is *memoryless* if X is positive and, for all real $t > 0$ and $x > 0$,

$$\Pr\{X > t + x\} = \Pr\{X > t\}\Pr\{X > x\}.$$

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Since the interarrival interval for a Poisson process is exponential, i.e., $\Pr\{X > x\} = \exp(-\lambda x)$ for $x \geqslant 0$,

$$\exp(-\lambda(x + t)) = \exp(-\lambda t)\exp(-\lambda x).$$

An arbitrary r.v. X is memoryless *iff.* it is exponential.

The reason for the word 'memoryless' is more apparent when using conditional probabilities,

$$\Pr\{X > t + x | X > t\} = \Pr\{X > x\}.$$

If people in a checkout line have exponential service, and you have waited 15 minute for the person in front, what is his or her remaining service time?

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If people in a checkout line have exponential service, and you have waited 15 minute for the person in front, what is his or her remaining service time?

Same as when service started. The remaining waiting time has no 'memory' of previous waiting.

Has your time waiting been wasted?

Why do you move to another line if someone takes a long time?

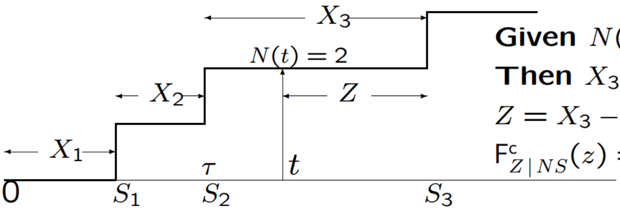
Theorem 1

For a Poisson process of rate λ, and any given $t > 0$, the length of the interval from t until the first arrival after t is a positive r.v. Z with the CDF

$$F_Z(z) = 1 - \exp(-\lambda z) \quad \text{for } z \geqslant 0.$$

This r.v. is independent of both $N(t)$ and the $N(t)$ arrival epochs before time t. It is also independent of the set of r.v.s $\{N(\tau) : \tau \leqslant t\}$.

Idea of proof: Conditional on $N(t) = n$ and $S_n = \tau$, i.e., the number n of arrivals in $(0, t]$ and the time, τ of the most recent arrival in $(0, t]$.



Given $N(t)=2, S_2 = \tau$.
Then $X_3 > t - \tau$ **and**
$Z = X_3 - (t - \tau)$
$F^c_{Z\,|\,NS}(z) = \exp(-\lambda z)$

Remark

- *This theorem essentially extends the idea of 'memorylessness' to the entire Poisson process.*

  That is, starting at any $t > 0$, the interval Z to the next arrival is also an exponential r.v. of rate λ. Z is independent of everything before t.
- *Let Z_m be the time from the $(m-1)$th arrival after t to the mth arrival epoch after t. We can see that $Z_1, Z_2, \ldots$ are unconditionally i.i.d and also independent of $\{N(\tau) : \tau \leqslant t\}$ and $S_1, S_2, \ldots, S_{N(t)}$.*
- *Thus the interarrival process starting at t with first interarrival Z, and continuing with subsequent interarrivals is also a Poisson process.*

Stationary and Independent Increments

Definition

A counting process $\{N(t) : t > 0\}$ has the *stationary increment property* if $N(t') - N(t)$ has the same CDF as $N(t' - t)$ for every $t' > t > 0$.

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Let

$$\widetilde{N}(t, t') = N(t') - N(t).$$

Stationary increments means that $\widetilde{N}(t, t')$ has the same distribution as $N(t' - t)$.

Thus, the distribution of the number of arrivals in an interval depends on the size of the interval but NOT the starting point.

Stationary and Independent Increments

Definition

A counting process $\{N(t) : t > 0\}$ has the *independent increment property* if, for every integer $k > 0$ and every k-tuple of times $0 < t_1 < \cdots < t_k$, the k-tuples of r.v.s $N(t_1), \widetilde{N}(t_1, t_2), \ldots, \widetilde{N}(t_{k-1}, t_k)$ are statistically independent.

Stationary and Independent Increments

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A counting process $\{N(t) : t > 0\}$ has the *independent increment property* if, for every integer $k > 0$ and every k-tuple of times $0 < t_1 < \cdots < t_k$, the k-tuples of r.v.s $N(t_1), \widetilde{N}(t_1, t_2), \ldots, \widetilde{N}(t_{k-1}, t_k)$ are statistically independent.

This implies that the number of arrivals in each of a set of non-overlapping intervals are independent r.v.s.

Theorem 2

Poisson processes have stationary and independent increments.

Erlang Distribution

Definition

For a Poisson process, $S_n = \sum_{i=1}^{n} X_i$, where X_i are i.i.d. r.v.s with the PDF

$$f_X(x) = \lambda \exp(-\lambda x),$$

then the S_n has the Erlang density *($S_n \sim \Gamma(n, \lambda)$):*

$$f_{S_n}(t) = \frac{\lambda^n t^{n-1} \exp(-\lambda t)}{(n-1)!}.$$

Idea of proof: $S_1 = X_1$ and $S_n = S_{n-1} + X_n$, where S_{n-1} and X_n are independent. Use induction,

$$f_{S_n}(t) = \int_0^t f_{S_{n-1}}(x) \cdot f_{X_n}(t-x)\mathrm{d}x$$

The joint density of $X_1, X_2, \ldots, X_n$ is

$$
\begin{aligned}
f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) &= \lambda^n \exp(-\lambda x_1 - \lambda x_2 - \cdots - \lambda x_n) \\
&= \lambda^n \exp(-\lambda s_n) \quad \text{where } s_n = \sum_{i=1}^{n} x_i.
\end{aligned}
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Theorem 3

Let $S_1, S_2, \ldots, S_n, \ldots$ be the arrival epoches of a Poisson process with the rate λ. Then the joint density of $S_1, S_2, \ldots, S_n$ is

$$f_{S_1,\ldots,S_n}(s_1, \ldots, s_n) = \lambda^n \exp(-\lambda s_n) \quad \text{for } 0 \leqslant s_1 \leqslant s_2 \leqslant \cdots \leqslant s_n.$$

Given that the nth arrival is at s_n, the other $n-1$ arrivals are uniformly distributed in $(0, s_n)$, subject to the ordering. Integrating, we get the Erlang marginal density.

Theorem 4

For a Poisson process of rate λ, and for any $t > 0$, the PMF for $N(t)$, i.e., the number of arrivals in $(0, t]$, is given by the Poisson PMF,

$$P_{N(t)}(n) = \frac{(\lambda t)^n \exp(-\lambda t)}{n!}.$$

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Idea of proof: use $\{N(t) \geqslant n\} = \{S_n \leqslant t\}$, then

$$\sum_{i=n}^{\infty} P_{N(t)}(i) = \int_0^t f_{S_n}(\tau)\mathrm{d}\tau.$$

$$\begin{aligned}
P_{N(t)}(n) &= \int_0^t f_{S_n}(\tau) - f_{S_{n+1}}(\tau)\mathrm{d}\tau \\
&= \int_0^t \left[\frac{\lambda^n \tau^{n-1}\exp(-\lambda\tau)}{(n-1)!} - \frac{\lambda^{n+1}\tau^n \exp(-\lambda\tau)}{n!}\right]\mathrm{d}\tau \\
&= \frac{\lambda^n}{n!}\int_0^t \exp(-\lambda\tau)\mathrm{d}(\tau^n) + \tau^n \mathrm{d}(\exp(-\lambda\tau)) = \frac{(\lambda t)^n \exp(-\lambda t)}{n!}.
\end{aligned}$$

Theorem 5

If an arrival process has the stationary and independent increment properties and if $N(t)$ has the Poisson PMF for given λ and all $t > 0$, then the process is Poisson.

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Theorem 6

If an arrival process has the stationary and independent increment properties and satisfies

$$\Pr\{\widetilde{N}(t, t+\delta) = n\} = \begin{cases} 1 - \lambda\delta + o(\delta) & \text{for} \quad n = 0, \\ \lambda\delta + o(\delta) & \text{for} \quad n = 1, \\ o(\delta) & \text{for} \quad n \geqslant 2. \end{cases}$$

then it is Poisson.

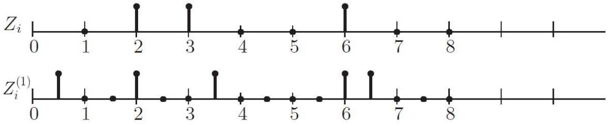
Shrinking Bernoulli Process

We can view a Bernoulli process as an arrival process, an arrival occurs at discrete time n if and only if $Z_n = 1$. Thus $S_n = Z_1 + \cdots + Z_n$ is the number of arrivals up to and including time n.

$$P_{N(t)}(k) = \binom{\lfloor t \rfloor}{k} p^k (1-p)^{\lfloor t \rfloor - k}, \text{ for } k < \lfloor t \rfloor$$

.

Now we 'shrink' the time scale of the process so that for some integer $j > 0$, $Z_i^{(j)}$ is an arrival or no arrival at time $i2^{-j}$.



In order to keep the arrival rate constant, we let $p = \lambda 2^{-j}$ for the jth process.

The expected number of arrivals per unit time is then λ.

Theorem 7 (Poisson's theorem)

Consider the sequence of shrinking Bernoulli processes with arrival probability $\lambda 2^{-j}$ and time-slot size 2^{-j}. Then for every fixed time $t > 0$ and fixed number of arrivals n, the counting PMF $P_{N_j(t)}(n)$ approaches the Poisson PMF (of the parameter λt) with increasing j,

$$\lim_{j\to\infty} P_{N_j(t)}(n) = \frac{(\lambda t)^n \mathrm{e}^{-\lambda t}}{n!}.$$

Proof.

For each j, the jth Bernoulli process has an associated Bernoulli counting process

$$N_j(t) = \sum_{i=1}^{\lfloor t2^j \rfloor} Z_i^{(j)}$$

$$P_{N_j(t)}(n) = \binom{\lfloor t2^j \rfloor}{n} p^n (1-p)^{\lfloor t2^j \rfloor - n}, \text{ for } n < \lfloor t \rfloor$$

where $p = \lambda 2^{-j}$.

Thus,

$$\begin{aligned}
\lim_{j\to\infty} P_{N_j(t)}(n) &= \lim_{j\to\infty} \binom{\lfloor t2^j \rfloor}{n} (\lambda 2^{-j})^n (1-\lambda 2^{-j})^{\lfloor t2^j \rfloor - n} \\
&= \lim_{j\to\infty} \binom{\lfloor t2^j \rfloor}{n} \left(\frac{\lambda 2^{-j}}{1-\lambda 2^{-j}}\right)^n \exp[\lfloor t2^j \rfloor \ln(1-\lambda 2^{-j})]
\end{aligned}$$

$$\lim_{j\to\infty} P_{N_j(t)}(n) = \lim_{j\to\infty} \binom{\lfloor t2^j \rfloor}{n} \left(\frac{\lambda 2^{-j}}{1-\lambda 2^{-j}}\right)^n \exp[\lfloor t2^j \rfloor \ln(1-\lambda 2^{-j})]$$

$$(\text{use } \ln(1-\lambda 2^{-j}) = -\lambda 2^{-j} + o(2^{-j}).)$$

$$= \lim_{j\to\infty} \binom{\lfloor t2^j \rfloor}{n} \left(\frac{\lambda 2^{-j}}{1-\lambda 2^{-j}}\right)^n \exp(-\lambda t)$$

$$= \lim_{j\to\infty} \frac{\lfloor t2^j \rfloor \cdot \lfloor t2^j - 1 \rfloor \cdots \lfloor t2^j - n + 1 \rfloor}{n!} \left(\frac{\lambda 2^{-j}}{1-\lambda 2^{-j}}\right)^n \exp(-\lambda t)$$

$$(\text{for } 0 \leqslant i \leqslant n-1,\ \lim_{j\to\infty} \lfloor t2^j - i \rfloor \left(\frac{\lambda 2^{-j}}{1-\lambda 2^{-j}}\right) = \lambda t.)$$

$$= \frac{(\lambda t)^n}{n!} \exp(-\lambda t).$$

Thank you for your attention!