Studiați convergența seriilor numerice și determinați suma lor, folosind serii de puteri:

(13) $\sum_{n\geq 0} \frac{(-1)^n}{3n+1}$ Conv. și suma

! C. Leibniz $\sum_{n\geq 0} (-1)^n \cdot u_n$, $u_n \searrow 0 \Rightarrow \sum (-1)^n u_n$ conv.

$u_n = \frac{1}{3n+1} \to 0$

$\frac{u_{n+1}}{u_n} = \frac{3n+1}{3n+4} < 1 \Rightarrow u_n \searrow$

$\Rightarrow \sum (-1)^n u_n$ conv.

$\Downarrow$

are sumă.

Cons. seria de puteri

$$\sum_{n\geq 0} (-1)^n \frac{1}{3n+1} \cdot x^{3n+1}.$$

$a_n = (-1)^n \frac{1}{3n+1}$

$\lim \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{3n+1}{3n+4} = 1 \Rightarrow R = 1.$

$x = 1 \Rightarrow \sum (-1)^n \frac{1}{3n+1} \Rightarrow$ conv.

$x = -1 \Rightarrow \sum -\frac{1}{3n+1}. = -\sum \frac{1}{3n+1}.$

! C. comp. la lim. $\lim \frac{u_n}{v_n} = \lim \frac{n}{3n+1} = \frac{1}{3} \neq 0, \infty$

$\Rightarrow$ seriile au ac. nat.

$\sum v_n = \sum \frac{1}{n}$ div.

$\Rightarrow \sum \frac{1}{3n+1}$ div.

Mult. de conv. $(-1; 1]$

$f: (-1,1] \to \mathbb{R}$, $f(x) = \sum_{n\geq 0} (-1)^n \frac{1}{3n+1} x^{3n+1}$

Se derivează termen cu termen

$\Rightarrow f'(x) = \sum (-1)^n \cdot x^{3n} = \frac{1}{1+x^3}$

$\sum (-x^3)^n$ serie geom. cu $S = \frac{1}{1-x}$

$\Rightarrow S = \frac{1}{1+x^3}$

Integrăm

$f(x) = \int \frac{1}{x^3+1} dx = \int \frac{1}{(x+1)(x^2-x+1)} dx = ⊛$

$\frac{1}{(\ )(\ )} = \frac{a}{x+1} + \frac{bx+c}{x^2-x+1}$

$1 = ax^2 - ax + a + bx^2 + bx + cx + c$

$\begin{cases} a+b=0 & \Rightarrow b=-a \\ -a+b+c=0 & \Rightarrow c = a-b = a+a = 2a \\ a+c=1 & 3a=1 \Rightarrow a=\frac{1}{3} ; b=-\frac{1}{3} \end{cases}$

$c = \frac{2}{3}$

⊛ $\frac{1}{3}\int \frac{dx}{x+1} - \frac{1}{3}\int \frac{x-2}{x^2-x+1} dx =$

$\frac{1}{3}\ln(x+1) - \frac{1}{3}\cdot\frac{1}{2}\int \frac{2x-4}{x^2-x+1} dx =$

$\frac{1}{3}\ln(x+1) - \frac{1}{6}\int \frac{2x-1-3}{x^2-x+1} dx =$

$\frac{1}{3}\ln(x+1) - \frac{1}{6}\ln(x^2-x+1) + \frac{1}{2}\int \frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}} =$

$\frac{1}{6}\ln\frac{(x+1)^2}{x^2-x+1} + \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\sqrt{3}} \operatorname{arctg}\frac{2x-1}{\sqrt{3}} + \mathcal{C}$

$$f(0) = \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{-1}{\sqrt{3}} + C = 0$$

$$-\frac{1}{\sqrt{3}} \frac{\pi}{6} + C = 0 \Rightarrow C = \frac{\pi}{6\sqrt{3}}.$$

$$\Rightarrow \text{pt}\cdot x=1 \quad \sum_{n\geq 0} \frac{(-1)^n}{3n+1} = f(1) = \lim_{\substack{x\to 1\\ x<1}} f(x) =$$

$$= \frac{1}{6} \ln \frac{2^2}{1} + \frac{1}{\sqrt{3}} \cdot \operatorname{arctg} \frac{1}{\sqrt{3}} + \frac{\pi}{6\sqrt{3}} =$$

$$= \frac{1}{3} \ln 2 + \frac{\pi}{6\sqrt{3}} + \frac{\pi}{6\sqrt{3}} = \frac{1}{3} \ln 2 + \frac{\pi}{3\sqrt{3}}.$$

(14) $\displaystyle\sum_{n\geq 0} \frac{(n+1)^2}{n!}$

! C. Rap. $\displaystyle \lim \frac{u_{n+1}}{u_n} = l \begin{cases} <1 \Rightarrow \text{conv.} \\ >1 \Rightarrow \text{div.} \\ =1 \Rightarrow \text{nu * poate} \end{cases}$

$$\lim \frac{u_{n+1}}{u_n} = \lim \left(\frac{(n+1)!}{(n+2)^2}\right)^{-1} \cdot \frac{n!}{(n+1)^2} = \frac{(n+2)^2}{(n+1)(n+1)^2} = 0 < 1$$

$\Rightarrow$ conv.

Știm că: $\boxed{e^x = \sum_{n\geq 0} \frac{x^n}{n!}}$ $\Big| \cdot x \Rightarrow x e^x = \sum_{n\geq 0} \frac{x^{n+1}}{n!} \Big| '$

$$\Rightarrow (x+1)e^x = \sum \frac{(n+1)x^n}{n!} \Big| \cdot x \Rightarrow (x^2+x)e^x = \sum \frac{(n+1)x^{n+1}}{n!} \Big| '$$

$$\Rightarrow (x^2+x+2x+1)e^x = \sum \frac{(n+1)^2 x^n}{n!}$$

Pt. $x=1 \Rightarrow 5e = \displaystyle\sum_{n\geq 1.} \frac{(n+1)^2}{n!}$

(15) $\sum_{n\geq 1} \frac{n^2(3^n-2^n)}{6^n} =$

C. Rap. $\lim \frac{u_{n+1}}{u_n} = e$ $\begin{cases} <1 \Rightarrow \text{conv.} \\ >1 \Rightarrow \text{div} \\ =1 \Rightarrow \text{nu se poate} \end{cases}$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^2(3^{n+1}-2^{n+1})\,6^n}{6^{n+1}(n^2)(3^n-2^n)} =$$

$$\frac{(n+1)^2}{n^2}\cdot\frac{1}{6}\cdot\frac{3^{n+1}\left(1-\left(\frac{2}{3}\right)^{n+1}\right)}{3^n\left(1-\left(\frac{2}{3}\right)^n\right)} \to 1\cdot\frac{1}{6}\cdot 3\cdot\frac{1-0}{1-0} = \frac{1}{2}$$

$\frac{1}{2} < 1 \Rightarrow$ conv.

$\Rightarrow \sum_{n\geq 1} n^2\left(\frac{1}{2}\right)^n - \sum_{n\geq 1} n^2\left(\frac{1}{3}\right)^n$

Cons. seria de puteri $\sum n^2 x^n$.

$\lim\left|\frac{a_{n+1}}{a_n}\right| = \lim\frac{(n+1)^2}{n^2} = 1 \Rightarrow R = 1.$

$x = 1 \Rightarrow \sum n^2 = \sum \frac{1}{n^{-2}}$ serie armonică $\alpha = -2 \leq 1 \Rightarrow$ div.

?
$x = -1 \Rightarrow \sum (-1)^n n^2$ $\Rightarrow$ div

$\Rightarrow$ Mult. de conv este $(-1;1)$.

$$\sum x^n = \frac{1}{1-x} \Big| ()' \text{ serie geom.}$$

$$\sum n x^{n-1} = -\frac{-1}{(1-x)^2} = \frac{1}{(1-x)^2} \Big| \cdot x$$

$$\sum n x^n = \frac{x}{(1-x)^2} \Big| ()'$$

$$\sum n^2 x^{n-1} = \frac{(1-x)^{\cancel{2}} + x \cdot 2(1-x)}{(1-x)^{\cancel{4}}} = \frac{x+1}{(1-x)^3} \Big| \cdot x$$

$$\sum n^2 x^n = \frac{x^2+x}{(1-x)^3}$$

$$x = \frac{1}{2} \Rightarrow \sum n^2 \left(\frac{1}{2}\right)^n = \frac{\frac{1}{4}+\frac{1}{2}}{\left(1-\frac{1}{2}\right)^3} = \frac{\frac{3}{4}}{\left(\frac{1}{2}\right)^3} = \frac{3}{4} : \frac{1}{8} = 6.$$

$$x = \frac{1}{3} \Rightarrow \sum n^2 \left(\frac{1}{3}\right)^n = \frac{\frac{1}{9}+\frac{1}{3}}{\left(1-\frac{1}{3}\right)^3} = \frac{\frac{4}{9}}{\left(\frac{2}{3}\right)^3} = \frac{4}{9} : \frac{8}{27} =$$

$$\frac{\cancel{4}}{\cancel{9}} \cdot \frac{\cancel{27}^{3}}{\cancel{8}_{2}} = \frac{3}{2} \Rightarrow S = 6 - \frac{3}{2} = \frac{9}{2}.$$

(16) $\sum_{n\geq 0} \frac{3n^3-n^2+1}{n!}$

! crit. rap. $\lim \frac{u_{n+1}}{u_n} = e \begin{cases} <1 \Rightarrow \text{conv} \\ >1 \Rightarrow \text{div} \\ =1 \Rightarrow \text{nu se poate} \end{cases}$

$\lim_{n\to\infty} \frac{3(n+1)^3-(n+1)^2+1}{(n+1)!} \cdot \frac{n!}{3n^3-n^2+1} = 0 < 1 \Rightarrow \text{conv.}$

$\Rightarrow 3\sum \frac{n^3}{n!} - \sum \frac{n^2}{n!} + \sum \frac{1}{n!} = 3S_1 - S_2 + S_3.$

$\boxed{e^x = \sum \frac{x^n}{n!}} \Big/ ()' \Rightarrow e^x = \sum \frac{nx^{n-1}}{n!} \Big| \cdot x$

$\Rightarrow xe^x = \sum \frac{nx^n}{n!} \Big| ()' \Rightarrow (x+1)e^x = \sum \frac{n^2x^{n-1}}{n!} \Big/ \cdot x$

$\Rightarrow (x^2+x)e^x = \sum \frac{n^2x^n}{n!} \Big| ()' \Rightarrow (x^2+x+2x+1)e^x = \sum \frac{n^3x^{n-1}}{n!}$

$\Rightarrow (x^2+3x+1)e^x = \sum \frac{n^3x^{n-1}}{n!} \Big| \cdot x \Rightarrow (x^3+3x^2+x)e^x = \sum \frac{n^3x^n}{n!}$

pt. $x=1 \Rightarrow S_1 = 5e.$

$(x^2+x)e^x = \sum \frac{n^2x^n}{n!} \Rightarrow$ pt. $x=1 \Rightarrow S_2 = 2e$

$e^x = \sum \frac{x^n}{n!} \overset{x=1}{\Rightarrow} S_3 = e.$

$\Rightarrow S = 3\cdot 5e - 2e + e = \underline{\underline{14e}}$

Dezvoltați în serie de puteri următoarele funcții:

(17) $f(x) = \ln\sqrt{\frac{1+x}{1-x}}$, $x\in(-1;1)$

$$f'(x) = \frac{1}{2}\left(\ln(1+x) - \ln(1-x)\right)'$$

$$= \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right) = \frac{1}{2}\cdot\frac{\cancel{1-x}+\cancel{1+x}}{1-x^2} = \frac{1}{1-x^2}$$

$$f'(x) = \left(1+(-x^2)\right)^{-1}$$

fct. binomială de forma $(1+y)^{\alpha}$, $\alpha\in\mathbb{R}$ de clasă C^{∞}, deci dezvoltabilă în serie de puteri.

$y=-x^2$; $\alpha=-1$. $\Rightarrow f'(x) = \sum x^{2n}$

Integrăm termen cu termen pe $[0,t]$, cu $|t|<1$

$$f(t) = \sum_{n\geq 0}\frac{t^{2n+1}}{2n+1}, \text{ adică } f(x) = \sum_{n\geq 0}\frac{x^{2n+1}}{2n+1}.$$

$$\Rightarrow \ln\sqrt{\frac{1+x}{1-x}} = \sum_{n\geq 0}\frac{x^{2n+1}}{2n+1}$$

(18) $f(x) = \frac{3x}{x^2+5x+6}$, $x \in \mathbb{R} \setminus \{-2,-3\}$.

$f(x)$ este de clasă C^∞ pt. $x \in \mathbb{R} \setminus \{-2,-3\}$.

$\Rightarrow$ dezvoltabilă în serie de puteri.

$$f(x) = \frac{3x}{(x+2)(x+3)} = \frac{a}{x+2} + \frac{b}{x+3}$$

$$3x = a(x+3) + b(x+2)$$

$a+b=3 \quad |\cdot 3 \Rightarrow 3a+3b=9$

$3a+2b=0. \qquad 3a+2b=0$

$/ \quad b = 9 \Rightarrow a = -6$

$$f(x) = -\frac{6}{x+2} + \frac{9}{x+3}$$

$$f(x) = -\frac{6}{2\left(\frac{x}{2}+1\right)} + \frac{9}{3\left(\frac{x}{3}+1\right)}$$

$$f(x) = -3\left(1+\frac{x}{2}\right)^{-1} + 3\left(1+\frac{x}{3}\right)^{-1}.$$

$$f(x) = \underbrace{-3\sum(-1)^n\frac{x^n}{2^n}}_{|x|<2} + \underbrace{3\sum(-1)^n\frac{x^n}{3^n}}_{|x|<3}$$

$$\Rightarrow f(x) = +3\sum_{n\geq 0}(-1)^{n+1}\left(\frac{1}{2^n} - \frac{1}{3^n}\right)x^n, \quad \text{pt. } |x|<2$$

(19) $f(x)=\int_0^x \frac{\operatorname{arctg} t}{t}\,dt \quad , x\in[-1;1]$

$$(\operatorname{arctg} t)' = \frac{1}{1+t^2} = \left(1+(t^2)\right)^{-1}$$

f.ct. bin. $(1+y)^{\alpha}$, $\alpha\in\mathbb{R}$, adică de cls. C^{∞}, deci dezv. în serie de puteri.

$$y=t^2,\ \alpha=-1 \Rightarrow \frac{1}{1+t^2}=\sum_{n\geq 0}(-1)^n t^{2n} \quad \Big|\int$$

$$\Rightarrow \operatorname{arctg} t=\sum(-1)^n \frac{t^{2n+1}}{2n+1}+C \quad \Big|\cdot\frac{1}{t}$$

$$t=0 \Rightarrow C=0.$$

$$\Rightarrow \frac{\operatorname{arctg} t}{t}=\sum(-1)^n \frac{t^{2n}}{2n+1} \quad \Big|\int$$

$$\int_0^x \frac{\operatorname{arctg} t}{t}\,dt=\sum_{n\geq 0}(-1)^n \frac{x^{2n+1}}{(2n+1)^2}, \quad |x|\leq 1.$$

(20) $f(x)=\frac{1}{2x-3}$, $x\in\mathbb{R}\setminus\{\frac{3}{2}\}$.

$$f(x)=\frac{-1}{3-2x}=\frac{-1}{1+2-2x}=-\frac{1}{1+2(1-x)}=$$

$$=-[1+2(1-x)]^{-1}$$ fct. binomială

de forma $(1+y)^{\alpha}$, $\alpha\in\mathbb{R}$, adică de cls. C^{∞}

$\Rightarrow$ deci dezv. în serie de puteri.

serie bin. $y=2(1-x)$, $\alpha=-1$.

$$\Rightarrow f(x)=-\sum(-1)^n[2(1-x)]^n=-\sum 2^n(x-1)^n$$

$$|x-1|<1.$$

$$\Rightarrow -1<x-1<1 \;\Big|+1 \Rightarrow 0<x<2$$

<u>SAU</u>

$$f(x)=\frac{1}{3\left(\frac{2x}{3}-1\right)}=-\frac{1}{3\left(1-\frac{2x}{3}\right)}=-\frac{1}{3}\cdot\left[1+\left(\frac{-2x}{3}\right)\right]^{-1}$$

$y=\frac{-2x}{3}$; $\alpha=-1$. $\Rightarrow f(x)=-\frac{1}{3}\sum_{n\geq 0}\left(\frac{2x}{3}\right)^n$

pt $\left|\frac{2x}{3}\right|<1 \Leftrightarrow -1<\frac{2x}{3}<1 \;\Big|\cdot\frac{3}{2} \Rightarrow -\frac{3}{2}<x<\frac{3}{2}$

<u>Obs!</u> $\Rightarrow$ Putem avea dezv. diferite pe intervale diferite.

$$-a_8 + S_7 < S < a_8 + S_7$$

$$S_7 = a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - a_7 =$$

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \frac{1}{1320} + \frac{1}{9360} - \frac{1}{75600}$$

$$= 0{,}7468228068$$

$$-0{,}0000014589 + 0{,}7468228068 < S < 0{,}7468228068 + 0{,}0000014589$$

$$\underbrace{0{,}7468}213479 < S < \underbrace{0{,}7468}24265 7$$

$$\Rightarrow \int_0^1 e^{-x^2} dx = 0{,}7468 \text{ (cu 4 zecimale exacte)}$$

(22) $\int_0^1 \sin(x^3)\,dx$ cu 3 zec ex.

$$\boxed{\sin y = \sum (-1)^n \frac{y^{2n+1}}{(2n+1)!}} \quad y = x^3 \quad \Big| \int_0^1$$

$$\int_0^1 \sin x^3\,dx = \sum (-1)^n \frac{x^{6n+4}}{(2n+1)!\,(6n+4)} \Big|_0^1 = \sum (-1)^n \frac{1}{(2n+1)!\,(6n+4)}$$

Leibniz

$$a_n = \frac{1}{(2n+1)!\,(6n+4)} \to 0$$

$\Rightarrow$ conv.

$$\frac{a_{n+1}}{a_n} = \left(\frac{(2n+3)!\,(6n+10)}{(2n+1)!\,(6n+4)}\right)^{-1} = \frac{6n+4}{(2n+2)(2n+3)(6n+10)} < 1$$

$\Rightarrow$ are sumă

(21) $\int_0^1 e^{-x^2} dx$ cu 4 zecimale exacte

$\boxed{e^y = \sum_{n\geq 0} \frac{y^n}{n!}}$, cu $y = -x^2 \quad \Big| \int_0^1$

$\int_0^1 e^{-x^2} dx = \int_0^1 \sum \left(\frac{(-x^2)^n}{n!} \right) = \sum \frac{(-1)^n x^{2n+1}}{n!(2n+1)} \Big|_0^1 =$

$= \sum (-1)^n \frac{1}{n!(2n+1)}$ cu $a_n = \frac{1}{n!(2n+1)}$

! C. Leibniz $\Rightarrow u_n = \frac{1}{n!(2n+1)} \to 0$ $\Rightarrow$ conv.

$\frac{u_{n+1}}{u_n} = \frac{n! \, 2n+1}{(n+1)!(2n+3)} < 1 \Rightarrow u_n \searrow$

$\Rightarrow$ are sumă

suma seriei $S = \int_0^1 e^{-x^2} dx$ și șirul sumelor parțiale S_n

Ș.a obține 4 zecimale exacte: $|S - S_n| < a_{n+1}$, $a_{n+1} \leq \frac{1}{10^5}$

$a_0 = 1$; $a_1 = \frac{1}{3}$; $a_2 = \frac{1}{10}$; $a_3 = \frac{1}{42}$

$a_4 = \frac{1}{216}$, $a_5 = \frac{1}{1320}$, $a_6 = \frac{1}{9360}$,

$a_7 = \frac{1}{75600}$, $a_8 = \frac{1}{685440} < \frac{1}{10^5}$

$\Rightarrow a_8 = a_{n+1} \Rightarrow \boxed{n = 7.} \Rightarrow |S - S_7| < a_8$

$S = \int_0^1 \sin(x^3)\,dx$, s_n - șirul sumelor parțiale

Pt. a obține 3 zecimale ex. în aprox. sumei, det. $n \in \mathbb{N}$ pt. care $|S - s_n| < a_{n+1}$ și $a_{n+1} \leq \frac{1}{10^4}$

$$a_0 = \frac{1}{4};\quad a_1 = \frac{1}{3!\,10} = \frac{1}{60};$$

$$a_2 = \frac{1}{5!\,16} = \frac{1}{1920};\quad a_3 = \frac{1}{7!\,22} = \frac{1}{110\,880}$$

$$a_3 = a_{n+1} \Rightarrow \boxed{n=2}$$

$$\Rightarrow |S - S_2| < a_3 \qquad S_2 = \frac{1}{4} - \frac{1}{60} + \frac{1}{1920}$$

$$S_2 = a_0 - a_1 + a_2$$

$$-a_3 + S_2 < S < a_3 + S_2$$

$$-\frac{1}{110880} + 0{,}2338541667 < S < \frac{1}{110880} + 0{,}2338541667$$

$$\underbrace{0{,}233}84\,51479 < S < \underbrace{0{,}233}86\,31855$$

$$\int_0^1 \sin x^3\,dx = 0{,}233. \quad \text{(cu 3 zecimale exacte)}$$

(23) $\int_0^{1/2} \frac{1}{\sqrt{1+x^4}}\,dx$ cu 4 zec ex.

$(1+y)^{-1/2}$

$y = x^4, \alpha = -\frac{1}{2}$

$$\frac{1}{\sqrt{1+x^4}} = 1 + \sum_{n\geq 1} (-1)^n \frac{1\cdot 3\cdot \ldots (2n-1)}{2^n \cdot n!} x^{4n}$$

$x \in (-1,1) \Rightarrow x^4 \in [0,1)$

Intervalul de integrare $[0, \frac{1}{2}] \subset (-1,1) = (-R, R)$

$$\Rightarrow \int \Rightarrow \int_0^{1/2} \frac{1}{\sqrt{1+x^4}}\,dx = \frac{1}{2} + \sum (-1)^n \cdot \frac{1\cdot 3\cdot \ldots (2n-1)}{2^{5n+1}\; n!} \cdot \frac{1}{(4n+1)}$$

C. Leibniz $\quad a_n = \frac{1\cdot 3 \cdot \ldots (2n-1)}{2^{5n+1}\; n!\,(4n+1)} \to 0$

$a_n \downarrow \quad \Rightarrow$ conv $\Rightarrow$ are suma

$\Rightarrow S = \int_0^{1/2} \frac{1}{\sqrt{1+x^4}}\,dx.$, S_n - șirul sumelor parțiale

Pt. a obține 4 zecimale exacte $\Rightarrow$

$|S - S_n| < a_{n+1}$, $a_{n+1} \leq \frac{1}{10^5}$

$a_0 = \frac{1}{2}$ $\quad a_1 = \frac{1}{2^6\cdot 5} = \frac{1}{320}$; $a_2 = \frac{1\cdot 3}{2^{11}\cdot 2\cdot \cancel{9}_3} = \frac{1}{12288}$

$a_3 = \frac{1\cdot \cancel{3}\cdot 5}{2^{16}\cdot \cancel{6}_2 \cdot 13} = \frac{5}{1703936}$ $\qquad a_3 = a_{n+1}$

$\Rightarrow \boxed{n=2}$

$$S_2 = a_0 - a_1 + a_2 = \frac{1}{2} - \frac{1}{320} + \frac{1}{12288} \Rightarrow$$

$$S_2 = 0,4969563802$$

$$|S - S_2| < a_3$$

$$-a_3 + S_2 < S < a_3 + S_2$$

$$-\frac{5}{1903936} + 0,49695663802 < S < \frac{5}{1703936} + 0,49695663802$$

$$\underbrace{0,4969}537036 < S < \underbrace{0,4969}5957724$$

$$\int_0^{1/2} \frac{1}{\sqrt{1+x^4}}\,dx = 0,4969 \quad (\text{cu 4 zec. ex.})$$

(24) $\int_0^{1/2} \frac{\operatorname{arctg} x}{x}\,dx$ cu 3 zec. ex.

$(\operatorname{arctg} x)' = \frac{1}{1+x^2} = (1+(x^2))^{-1}$ serie bin.

$(1+y)^{\alpha}$, $\alpha \in \mathbb{R}$; $y = x^2$, $\alpha = -1$.

$$\boxed{(1+x)^{\alpha} = 1 + \sum_{n\geq 1} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} x^n, \quad |x|<1}$$

$$(1+x^2)^{-1} = 1 + \sum_{n\geq 1} (-1)^n x^{2n} \quad \Big| \int$$

$$\operatorname{arctg} x = \sum_{n\geq 0} \frac{(-1)^n x^{2n+1}}{2n+1} + \mathscr{C}. \quad \Big| \cdot \frac{1}{x}$$

$x=0 \Rightarrow \mathscr{C}=0.$

$$\frac{\operatorname{arctg} x}{x} = \sum \frac{(-1)^n x^{2n}}{2n+1.} \quad \Big| \int_0^{1/2}$$

$$\int_0^{1/2} \frac{\operatorname{arctg} x}{x} = \sum \frac{(-1)^n x^{2n+1}}{(2n+1)^2} = \sum (-1)^n \frac{1}{2^{2n+1}(2n+1)^2}$$

<u>C. Leibniz</u> $a_n = \frac{1}{2^{2n+1}} \cdot \frac{1}{(2n+1)^2} \to 0.$

$a_n \searrow \quad \frac{a_{n+1}}{a_n} = \frac{2^{2n+1}(2n+1)^2}{2^{2n+3}(2n+3)^2} < 1$

$\Rightarrow a_n$ conv $\Rightarrow \sum (-1)^n a_n$ are sumă

$S = \int_0^{1/2} \frac{\operatorname{arctg} x}{x} dx$, S_n - șirul sumelor parțiale

Pt. a obține 3 zec ex.

$|S - S_n| < a_{n+1}$, $a_{n+1} \leq \frac{1}{10^4}$

$a_0 = \frac{1}{2}$; $a_1 = \frac{1}{2^3 \cdot 3^2} = \frac{1}{72}$; $a_2 = \frac{1}{2^5 \cdot 5^2} = \frac{1}{800}$.

$a_3 = \frac{1}{2^7 \cdot 7^2} = \frac{1}{6272}$; $a_4 = \frac{1}{2^9 \cdot 9^2} = \frac{1}{41472}$

$a_5 = \frac{1}{2^{11} \cdot 11^2} = \frac{1}{247808} \Rightarrow a_5 = a_{n+1}$

$\Rightarrow \boxed{n=4}$

$S_4 = a_0 - a_1 + a_2 - a_3 + a_4 = 0,487225785$

$|S - S_4| < a_5$

$0,487225785 - \frac{1}{247808} < S < 0,487225785 + \frac{1}{247808}$

$\underbrace{0,48}7217494 < S < \underbrace{0,48}7229204$

$\Rightarrow \int_0^{1/2} \frac{\operatorname{arctg} x}{x} dx = 0,487$ (cu 3 zec exacte)