ex X, Y v.a. indep. cu repartițiile $X \sim \begin{pmatrix} -1 & 1 & 2 \\ \frac{11}{16} & \frac{4}{16} & \frac{1}{16} \end{pmatrix}$, $Y \sim \begin{pmatrix} 1 & 2 & 3 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$

- $E[X] = \sum_x x \cdot P(X=x) = -1 \cdot \frac{11}{16} + 1 \cdot \frac{4}{16} + 2 \cdot \frac{1}{16} = \frac{-5}{16}$

$X^2 \sim \begin{pmatrix} 1 & 4 \\ \frac{15}{16} & \frac{1}{16} \end{pmatrix}$, $E[X^2] = 1 \cdot \frac{15}{16} + 4 \cdot \frac{1}{16} = \frac{19}{16}$

- $Var(X) = E[X^2] - E[X]^2 =$
$= \frac{19}{16} - \left(\frac{-5}{16}\right)^2 = \frac{19}{16} - \frac{25}{256} = \frac{279}{256} = 1{,}09$

- $Var(X+Y) = Var(X) + Var(Y)$, X, Y indep!!!
- $Var(aX+b) = a^2 Var(X)$ → oricând

- Repartiția variabilei aleatoare $X+Y$

$X \backslash Y$	1	2	3
-1	$\frac{11}{64}$	$\frac{11}{32}$	$\frac{11}{64}$
1	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{1}{16}$
2	$\frac{1}{64}$	$\frac{1}{32}$	$\frac{1}{64}$

$P(X=-1, Y=1) = P(X=-1) \cdot P(Y=1) = \frac{11}{16} \cdot \frac{1}{4} = \frac{11}{64}$ ••• pt. toate (adunate $= 1$)

$X+Y \sim \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ \frac{11}{64} & \frac{22}{64} & \frac{15}{64} & \frac{9}{64} & \frac{6}{64} & \frac{1}{64} \end{pmatrix}$

$P(X+Y=0) = P(X=-1, Y=1) = \frac{11}{64}$

La fel, pt. toate posibilitățile, pe rând

$P(X+Y=1) = P(X=-1, Y=2) = \frac{11}{32} = \frac{22}{64}$

$P(X+Y=2) = P(X=-1, Y=3) + P(X=1, Y=1) = \frac{11}{64} + \frac{1}{16} = \frac{15}{64}$

...

Verif: $\frac{11}{64} + \frac{22}{64} + \frac{15}{64} + \frac{9}{64} + \frac{6}{64} + \frac{1}{64} = 1$ ✓

- $XY \sim \begin{pmatrix} -3 & -2 & -1 & 1 & 2 & 3 & 4 & 6 \\ \frac{11}{64} & \frac{22}{64} & \frac{4}{64} & \frac{4}{64} & \frac{9}{64} & \frac{4}{64} & \frac{2}{64} & \frac{1}{64} \end{pmatrix}$; $X^2Y^2 \sim \begin{pmatrix} 1 & 4 & 9 & 16 & 36 \\ \frac{15}{64} & \frac{31}{64} & \frac{15}{64} & \frac{2}{64} & \frac{1}{64} \end{pmatrix}$

$E[XY] = \frac{-3 \cdot 11 + (-2 \cdot 22) + (-1) \cdot 4 + 1 \cdot 4 + 2 \cdot 9 \cdot 3 \cdot 4 + 4 \cdot 2 + 6 \cdot 1}{64} = \frac{-40}{64} = 0{,}625$

$E[X^2Y^2] = \ldots = \frac{342}{64} \approx 5{,}34$

- $Var(XY) = E[(XY)^2] - E[XY] = 5{,}34 - (0{,}625)^2 = 4{,}9493$

- $Var(2X - 4Y) = Var(2X) + Var(-4Y) = 2^2 Var(X) + (-4)^2 Var(Y) = 4 \cdot 1{,}09 + 16 \cdot 0{,}5 = 12{,}36$

---

ex X, Y var. aleatoare $X \sim \begin{pmatrix} -5 & 1 \\ 0{,}24 & 0{,}76 \end{pmatrix}$, $Y \sim \begin{pmatrix} 0 & 6 \\ p_1 & p_2 \end{pmatrix}$ $p_1, p_2 \in (0,1)$

a) $P(X=-5, Y=6) = 0{,}12$ și $E[X|Y=6] = -2$

- aflați p_1, p_2

- $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$X \backslash Y$	0	6	Σ
-5	0.12	0.12	0.24
1	p1-0.12	p2-0.12	0.76
Σ	p1	p2	

$X|Y=6 \sim \begin{pmatrix} -5 & 1 \\ P(X=-5|Y=6) & P(X=1|Y=6) \end{pmatrix} \sim$

$\sim \begin{pmatrix} -5 & 1 \\ \frac{0.12}{p_2} & \frac{p_2 - 0.12}{p_2} \end{pmatrix}$

$E[X|Y=6] = -2 \Leftrightarrow -5 \cdot \frac{0.12}{p_2} + 1 \cdot \frac{p_2 - 0.12}{p_2} = -2 \Leftrightarrow \frac{-0.6 + p_2 - 0.12}{p_2} = -2$

$\Leftrightarrow p_2 - 0.72 = -2p_2 \Leftrightarrow 3p_2 = 0.72 \Leftrightarrow p_2 = 0.24$

$p_1 = 1 - p_2 = 1 - 0.24 = 0.76$

b) $x \sim \begin{pmatrix} -5 & 1 \\ 0.24 & 0.76 \end{pmatrix}$, $y \sim \begin{pmatrix} 0 & 6 \\ 0,76 & 0,24 \end{pmatrix}$

x\y	0	6	Σ
-5	0,12	0,12	0.24
1	0,64	0,12	0.76
Σ	0.76	0.24	

Înlocuim în punctul a) cu valorile obținute pt. p1 și p2, apoi procedăm ca la primul ex, luând valorile din acest tabel.

NU avem vaie $Var(x+y) = Var(x) + Var(x)$ → nu avem var. independ.

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ex | comparații | Cauchy-Schwartz: $E[xy] \le \sqrt{E[x^2]E[y^2]}$

• Jensen: - f convexă $\Rightarrow E[f(x)] \ge f(E[x])$

- f concavă $\Rightarrow E[f(x)] \le f(E[x])$

- f afină $\Rightarrow E[f(x)] = f(E[x])$

• Markov: $P(x \ge a) \le \frac{E[x]}{a} \mid x \ge 0, a > 0$

• Chebyshev: $P(|x-\alpha| \ge \beta) \le \frac{Var(x)}{\beta}$

• Chernoff: $P(x \ge a) \le E[e^{+\infty}] / e^{ta}$, $a, t > 0$

• $E[\log(x)] \le \log(E[x])$ → $\log(x)$ concavă, din ineg. lui Jensen ⇒ ...

• $E[\sqrt{x}] \le \sqrt{E[x]}$ → concavă din Jensen

• $E[\sin^2(x)] + E[\cos^2(x)] = 1$

• $E[x] ? \sqrt{E[x]}$ pt că $E[x] \le \sqrt{E[x]}$, $E[x] \in [0,1]$

$E[x] \ge \sqrt{E[x]}$, $E[x] \in [1, \infty)$

• $P(x \le y) = P(x \ge y)$ → $P(y \ge x) = \sum_{i=1}^{n} P(y \ge x \mid x = i) \cdot P(x=i) = \sum_{i=1}^{n} P(y \ge i) P(x=i)$

$P(x \ge y) = \sum_{i=1}^{n} P(x \ge y \mid y=i) \cdot P(y=i) = \sum_{i=1}^{n} P(x \ge i) P(y=i)$ ⇒ =

• $P(x > c) \le \frac{E[x^3]}{c^3}$ din Markov

• $P(x+y > 10) \le P(x > 5 \text{ sau } y > 5)$ 1. Există cazuri când $x>5$ sau $y>5$ dar $x+y<10$

2. Pt a obține o sumă ≥10, tb. să avem c.p.o val.

≥5 ($1+\underline{10}$, $2+\underline{9}$, $3+\underline{8}$, ..., $\underline{5}+6$, $6+5$ ... $(5+5)$) ⇒ $x+y>0 \subseteq x>5$ sau $y>5$

• $E[x^2(x^2+1)] = E[x^2(y^2+1)]$ → $E[x^2(x^2+1)] = E[x^4 + x^2]$

$E[x^2(y^2+1)] = E[x^2y^2 + x^2]$

x, y au ac. distrib. și i.i.d. ⇒ $x^2 = y^2$ ⇒ =

(liniaritatea valabilă a indep.)

• $E\left[\frac{x}{y}\right] = \frac{E[x]}{E[y]}$ • $E\left[\frac{1}{x}\right] ? \frac{1}{E[x]}$

• $E[\min(x,y)] \ge \min(E[x], E[y])$ (monotonicitate)

• $P(|x+y| > 2) \le \frac{E[(x+4)^4]}{10}$ (din Markov)

- $E[X^Y] \,?\, E[X]^{E[Y]}$
- $\frac{E[X^3]}{E[X^2]} \,?\, \frac{E[X^4]}{E[X^3]}$ pt. că x poate fi ori supraunitar, ori subunitar
- $P(|X-Y|>2) \le \frac{Var(X)}{2}$

continuare comparații

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ex monedă | Aruncăm în mod repetat o monedă cu $P_{succes}=0.93$. Fie X o v.a. ce descrie nr. de succese înainte de al 6-lea eșec într-o secv. de aruncări. Det. repartiția lui X, $E[6X-8]$ și $Var(3X+8)$

sol: Fie x nr de succese înainte de al 6-lea eșec

$$P(X=k) = \binom{k-1}{5} \cdot (1-p)^6 \cdot p^{k-6}, \quad k \ge 6$$

$$E[X] = \frac{p \cdot n}{1-p} = \frac{0,93 \cdot 6}{1-0,93} = \dots$$

$$E[6X-8] = 6 \cdot E[X] - 8 = 6 \cdot \dots - 8 =$$

$$Var(X) = \frac{p \cdot n}{(1-p)^2} = \frac{0,93 \cdot 6}{(0,07)^2} = \dots$$

$$Var(3X+8) = Var(3X) = 9\,Var(X) = 9 \cdot \dots$$

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ex telefoane | Într-un lot de 7 tel., 2 sunt defecte. Tel. sunt testate succesiv până sunt găsite cele 2 defecte. Fie X nr. de teste efectuate pt. identif. primului tel. defect și Y pt. al 2-lea.

a) Rep. comună (X,Y) și rep marginale
b) media + Var lui X și Y, resp. coef. de corel. dintre X și Y
c) media + Var rep. cond. a lui X la $Y=2$

a) $2 \le X+Y \le 6$

$H_i \to$ tel. i este defect

$P(X=1, Y=1) = \frac{2}{7} \cdot \frac{1}{6} = \frac{2}{42} = \frac{1}{21}$

$P(X=1, Y=2) = \frac{2}{7} \cdot \frac{5}{6} \cdot \frac{1}{5} = \frac{2}{42} = \frac{1}{21}$

$P(X=1, Y=3) = \frac{2}{7} \cdot \frac{5}{6} \cdot \frac{4}{5} \cdot \frac{1}{4} = \frac{1}{21}$

$P(X=1, Y=4) = \frac{2}{7} \cdot \frac{5}{6} \cdot \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{21}$

$P(X=1, Y=5) = \frac{2}{7} \cdot \frac{5}{6} \cdot \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{21}$

X\Y	0	1	2	3	4	5	Σ
1	0	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{2}{21}$	$\frac{6}{21}$
2	0	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{2}{21}$	0	$\frac{5}{21}$
3	0	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{2}{21}$	0	0	$\frac{4}{21}$
4	0	$\frac{1}{21}$	$\frac{2}{21}$	0	0	0	$\frac{3}{21}$
5	$\frac{1}{21}$	$\frac{2}{21}$	0	0	0	0	$\frac{3}{21}$
Σ	$\frac{1}{21}$	$\frac{6}{21}$	$\frac{6}{21}$	$\frac{4}{21}$	$\frac{3}{21}$	$\frac{2}{21}$	

$$X \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \frac{6}{21} & \frac{5}{21} & \frac{4}{21} & \frac{3}{21} & \frac{3}{21} \end{pmatrix}$$

$$Y \sim \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ \frac{1}{21} & \frac{6}{21} & \frac{5}{21} & \frac{4}{21} & \frac{3}{21} & \frac{2}{21} \end{pmatrix}$$

b) $E[X] = \frac{6+10+12+12+15}{21} = \frac{55}{21} = 2{,}61$ | $E[Y] = \frac{6+10+12+12+10}{21} = \frac{50}{21} = 2{,}38$

$X^2 \sim \begin{pmatrix} 1 & 4 & 9 & 16 & 25 \\ \frac{6}{21} & \frac{5}{21} & \frac{4}{21} & \frac{3}{21} & \frac{3}{21} \end{pmatrix}$ $\quad Y^2 \sim \begin{pmatrix} 0 & 1 & 4 & 9 & 16 & 25 \\ \frac{1}{21} & \frac{6}{21} & \frac{5}{21} & \frac{4}{21} & \frac{3}{21} & \frac{2}{21} \end{pmatrix}$

$E[X^2] = \frac{6+20+36+48+75}{21} = 8{,}80$ $\quad E[Y^2] = \frac{1+6+20+36+48+50}{21} = 7{,}61$

$Var(X) = E[X^2] - E[X]^2 = 1{,}99$ $\quad Var(Y) = 1{,}95$

$$\boxed{\begin{aligned} &\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} \qquad \text{formule} \quad \cdot E[h(X,Y)] = \sum_{x,y} h(x,y)\cdot P(X=x, Y=y) \\ &Cov(X,Y) = E[(X-E[X])(Y-E[Y])] = E[XY] - E[X]\cdot E[Y] \end{aligned}}$$

$E[XY] = \sum xy\,P(X=x, Y=y) = 1\cdot 0\cdot 0 + 1\cdot 1\cdot \frac{1}{21} + 1\cdot 2\cdot \frac{1}{21} + 1\cdot 3\cdot\frac{1}{21} + \dots = 5$

$Cov(X,Y) = 5 - 1{,}99\cdot 1{,}95 = 1{,}12$

$\rho(X,Y) = \frac{1{,}12}{\sqrt{1{,}99\cdot 1{,}95}} = 0{,}57$

c) $E[X \mid Y=2]$

$X \mid Y=2 \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \frac{1/21}{5/21} & \frac{1/21}{5/21} & \frac{1/21}{5/21} & \frac{2/21}{5/21} & \frac{0}{5/21} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} & 0 \end{pmatrix}$

$E[X \mid Y=2] = \frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{8}{5} + 0 = \frac{14}{5} = 2{,}8$

$Var(X \mid Y=2) = E[(X \mid Y=2)^2] - E[X \mid Y=2]^2$

$(X \mid Y=2)^2 \sim \begin{pmatrix} 1 & 4 & 9 & 16 & 25 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} & 0 \end{pmatrix}$

$E[(X \mid Y)^2] = \frac{1+4+9+32}{5} = 9{,}2$, $Var(X \mid Y=2) = 9{,}2 - 7{,}84 = 1{,}36$

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formule | $P(X \cup Y) = P(X) + P(Y)$; $P(X \cap Y) = P(X)\cdot P(Y)$
sau

- Prob. cond: $P(A|B) = \frac{P(A\cap B)}{P(B)} \overset{*}{=} \frac{P(B|A)\cdot P(A)}{P(B|A)\cdot P(A) + P(B|A^c)\cdot P(A^c)}$
- $P(A) = \sum_{i=1}^{n} P(A|B_i)\cdot P(B_i)$
- A, B indep $\Leftrightarrow P(B|A) = P(B) \Leftrightarrow \frac{P(A\cap B)}{P(B)} = P(A) \Leftrightarrow P(A\cap B) = P(A)\cdot P(B)$
- f. de masă: $f(x) = P(X=x)$