$\boxed{ex}$ $X \sim \begin{pmatrix} -1 & 0 & 1 \\ 0.3 & 0.2 & 0.5 \end{pmatrix}$, Se cer repartițiile $5X-2$, X^3, $X+X^2$, media și varianța
și $P\left(X < \frac{1}{8} \mid X \geq -\frac{1}{8}\right) = ?$ E Var

$y = g(x)$

- $g_1(x) = 5X-2$, $y_1 = 5X-2 \in \{-7, -2, 3\}$, $5X-2 \sim \begin{pmatrix} -7 & -2 & 3 \\ 0.3 & 0.2 & 0.5 \end{pmatrix}$

$E[5X-2] = 5E[X] - 2 = 5\cdot(-1\cdot 0.3 + 0\cdot 0.2 + 1\cdot 0.5) - 2 = 5\cdot 0.2 - 2 = -1$

* $Var(5X-2) = 5^2 Var(x) = 25\left(E[X^2] - E[X]^2\right) = 25(0.8 - (0.2)^2) = 25(0.8 - 0.04) = 25\cdot 0.76$

$E[X] = -1\cdot 0.3 + 0\cdot 0.2 + 1\cdot 0.5 = 0.2$
$E[X^2] = 0\cdot 0.2 + 1\cdot 0.8 = 0.8$ $X^2 \sim \begin{pmatrix} 0 & 1 \\ 0.2 & 0.8 \end{pmatrix}$

* sau $E[h(x)] = \sum_x h(x) P(X=x) = (-1)^2\cdot 0.3 + 0^2 \cdot 0.2 + 1^2 \cdot 0.5 = 0.8$

$\boxed{E[X] = \sum_x x \cdot P(X=x)}$
$\boxed{Var(X) = E[X^2] - E[X]^2}$
$\boxed{Var(aX+b) = a^2 Var(X)}$

- $X^3 \sim \begin{pmatrix} -1 & 0 & 1 \\ 0.3 & 0.2 & 0.5 \end{pmatrix}$ este repartizat la fel ca și X

$\Rightarrow E[X^3] = E[X] = 0.2$ și $Var(X^3) = Var(X) = 0.76$

- $X^2 \sim \begin{pmatrix} 0 & 1 \\ 0.2 & 0.8 \end{pmatrix}$ $X \sim \begin{pmatrix} -1 & 0 & 1 \\ 0.3 & 0.2 & 0.5 \end{pmatrix}$

$X + X^2 \sim \begin{pmatrix} 0 & 2 \\ 0.5 & 0.5 \end{pmatrix}$

$X + X^2 \in \{\overset{-1}{0}, \overset{0}{0}, \overset{1}{2}\}$

Var$[X+Y] = Var(X) + Var(Y)$ X, Y indep.

$P(X+X^2 = 2) = P(X=1) = 0.5$

$E[X+X^2] = 0\cdot 0.5 + 2\cdot 0.5 = 1$ sau $E[X+X^2] = E[X] + E[X^2] = 0.2 + 0.8 = 1$

$Var(\underbrace{X+X^2}_{Y}) = \underset{\substack{\parallel \\ 2}}{E[Y^2]} - \underset{\substack{\parallel \\ 1}}{E[Y]}^2 = 2 - 1 = 1$

$E[Y^2] = 0^2\cdot 0.5 + 2^2 \cdot 0.5 = 2$

- $P\left(\overbrace{X < \frac{1}{8}}^{B} \mid \overbrace{X \geq -\frac{1}{8}}^{A}\right) = P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P\left(X<\frac{1}{8} \text{ și } X \geq -\frac{1}{8}\right)}{P\left(X \geq -\frac{1}{8}\right)}$

$X \in \{-1, 0, 1\} \Rightarrow \left\{X < \frac{1}{8} \text{ și } X \geq -\frac{1}{8}\right\} = \{X=0\}$; $\left\{X \geq -\frac{1}{8}\right\} = \{X=0\} \cup \{X=1\}$

$P\left(X < \frac{1}{8} \mid X \geq -\frac{1}{8}\right) = \frac{P(X=0)}{P(X=0) + P(X=1)} = \frac{0.2}{0.2 \cdot 0.5} = \frac{2}{7}$

★ $X \sim \begin{pmatrix} x_1 & x_2 & \dots & x_m \end{pmatrix}$ $Y \sim \begin{pmatrix} y_1 & y_2 & \dots & y_n \end{pmatrix}$

$X + Y = \{x_i + y_j,\ i \in \{1 \dots m\}, j \in \{1 \dots n\}\}$

$P(X+Y = a) = \sum_{x+y=a} P(X=x, Y=y)$

$\left\{ \begin{array}{l} X = (-1\ 0\ 1),\ Y = (-2\ 0\ 2),\ X+Y \in \{-3, -1, 1, -2, 0, 2, -1, 1, 3\} \\ X+Y \in \{-3, -2, -1, 0, 1, 2, 3\} \\ P(X+Y = -1) = P(X=-1, Y=0) + P(X=1, Y=-2) \end{array} \right.$

$P(\underset{\text{sau}}{X \cup Y}) = P(X) + P(Y)$

$P(\underset{\text{și}}{X \cap Y}) = P(X) \cdot P(Y)$

ex: (X,Y)

$X \backslash Y$	2	4	6	Σ
0	0.1	0.2	0.1	0.4
1	0.1	0.1	0.1	0.3
2	0.1	0.1	0	0.2
3	0.05	0	0.05	0.1
Σ	0.35	0.4	0.25	

a) Repartițiile marginale pt X și Y, $E[X]$, $Var(X)$, $E[Y]$, $Var(Y)$

* Tabelul da valori repr. într-adevăr a rep. comună?

b) Rep. condiționată a lui Y la $X=1$
X la $Y=4$

c) Calc. media cond. $E[Y|X]$ și $Var(Y|X)$

d) Calc. coeficientul de corelație.

e) Verif dacă are loc egalitatea: $Var(Y) = E[Var(Y|X)] + Var(E[Y|X])$ → adv

* $f(x,y) = P(X=x, Y=y)$

Tb să verif i) $f(x,y) \geq 0 \ (\forall) x,y$; ii) $\sum_{x,y} f(x,y) = 1$

a) Rep. marg. a lui X (facem suma pe linii)

$$X \sim \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0.4 & 0.3 & 0.2 & 0.1 \end{pmatrix}$$

Rep. marg. a lui Y (suma pe coloane)

$$Y \sim \begin{pmatrix} 2 & 4 & 6 \\ 0.35 & 0.4 & 0.25 \end{pmatrix}$$

d) Coef. de corelație: $\rho(X,Y) = \dfrac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$ ★

★ $Cov(X,Y) = E[(X-E[X])(Y-E[Y])] = E[XY] - E[X] \cdot E[Y]$

$$E[XY] = \sum_{x,y} xy\, P(X=x, Y=y) = 0 \cdot 2 \cdot 0.1 + 0 \cdot 4 \cdot 0.2 + 0 \cdot 6 \cdot 0.1 \ldots$$

$$E[h(X,Y)] = \sum_{x,y} h(x,y) \cdot P(X=x, Y=y)$$

b) Rep. cond. a lui Y la $X=1$

$X \backslash Y$	2	4	6	
1	$\frac{0.1}{0.3}$	$\frac{0.1}{0.3}$	$\frac{0.1}{0.3}$	0.3

$$Y|X=1 \sim \begin{pmatrix} 2 & 4 & 6 \\ \frac{0.1}{0.3} & \frac{0.1}{0.3} & \frac{0.1}{0.3} \end{pmatrix} = \begin{pmatrix} 2 & 4 & 6 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$Y|X=3 \sim \begin{pmatrix} 2 & 4 & 6 \\ \frac{0.05}{0.1} & \frac{0}{0.1} & \frac{0.05}{0.1} \end{pmatrix} = \begin{pmatrix} 2 & 4 & 6 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$X|Y=4$

$X \backslash Y$	2	4	6
0		0.2	0.4
1		0.1	0.4
2		0.1	⋮
3		0	
		0.4	

$$X|Y=4 \sim \begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{0.2}{0.4} & \frac{0.1}{0.4} & \frac{0.1}{0.4} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix}$$

c) $E[Y|X]$ este o variab. aleatoare care ia valorile $E[Y|X=x]$

$E[Y|X=0] = 2P(Y=2|X=0) + 4P(Y=4|X=0) + 6P(Y=6|X=0) = 4$

$E[Y|X=1] = 2 \cdot \frac{1}{3} + 4 \cdot \frac{1}{3} + 6 \cdot \frac{1}{3} = 4$

$E[Y|X=2] = 3$

$E[Y|X=3] = 4$

$$E[Y|X] \sim \begin{pmatrix} 3 & 4 \\ P(X=2) & P(X \neq 2) \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 0.2 & 0.8 \end{pmatrix}$$

$Var(Y|x) = ?$ este tot o var. aleatoare care ia valorile $Var(Y|X=x)$

$Var(Y|x=0) = E[y^2|x=0] - E[y|x=0]^2 = 2^2 \cdot P(y=2|x=0) + 4^2 \cdot P(y=4|x=0) + 6^2 \cdot P(y=6|x=0) - E[y|x=0]^2 =$

$= 2^2 \cdot \frac{0,1}{0,4} + 4^2 \cdot \frac{0,2}{0,4} + 6^2 \cdot \frac{0,1}{0,4} - 4^2 = 1 + 8 + 9 - 16 = 2$

$Var(Y|x=1) = 2,66$; $Var(Y|x=2) = 1$; $Var(Y|x=3) = 4$

$$Var(Y|x) \sim \begin{pmatrix} 1 & 2 & 2.66 & 4 \\ P(x=2) & P(x=0) & P(x=1) & P(x=3) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2.66 & 4 \\ 0.2 & 0.4 & 0.3 & 0.1 \end{pmatrix}$$

<u>Prob. cond.</u> $\cdot P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A) \cdot P(A)}{P(B|A) \cdot P(A) + P(B|A^c) \cdot P(A^c)}$

$\cdot P(A) = \sum_{i=1}^{n} P(A|B_i) \cdot P(B_i)$ $\quad \cdot A, B$ indep $\Leftrightarrow P(B|A) = P(B) \Leftrightarrow \frac{P(A \cap B)}{P(B)} = P(A)$

$\Leftrightarrow P(A \cap B) = P(A) \cdot P(B)$

<u>Var. aleat. discrete</u> $\cdot$ F. de masă: $f(x) = P(X=x)$

$\cdot$ F. de repartiție a lui x: $F: \mathbb{R} \to [0,1]$, $F(x) = P(X \leq x)\ (\forall) x \in \mathbb{R}$

$\cdot$ v.a. indep x și $y \Leftrightarrow P(x \in A, Y \in B) = P(x \in A) \cdot P(Y \in B) \Leftrightarrow P(X \leq x, Y \leq y) = P(X \leq x) \cdot P(Y \leq y)$

$\cdot$ medie $E[x] = \sum_x x \cdot P(X=x)$ | $E[aX + bY] = aE[x] + bE[Y]$ | $E[g(x)] = \sum_x g(x) \cdot P(X=x)$

$\cdot$ x, y indep $\Rightarrow E[xy] = E[x] \cdot E[y]$

<u>Repartiții comune, marginale și condiționate - discret</u>

$\cdot$ F. de masă a vect. (x,y): rep. comună: $P_{x,y}(x,y) = P(X=x, Y=y) = f_{x,y}(x,y)$

$\cdot$ rep. marginală: $X: P(X=x) = P_x(x) = \sum_y P(X=x, Y=y)$

$\cdot$ rep. cond. a lui X la $Y=y$: $P_{X|Y} = P(X=x|Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)} = \frac{P_{x,y}(x,y)}{P_y(y)}$

$\cdot$ media: $E[g(x,y)] = \sum_x \sum_y g(x,y) P_{x,y}(x,y)$

$\cdot$ media cond.: $E[X|Y=y] = \sum_x x \cdot P_{X|Y}(x|y)$ | $E[X] = \sum_y E[X|Y=y] \cdot P(Y=y)$

<u>Covarianța și corelația</u> $\cdot$ x și y necorelate $\Rightarrow Cov(x,y) = 0$

$\cdot Cov = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X] \cdot E[Y]$

$\cdot Var(x,y) = Var(x) + Var(y) + 2Cov(x,y)$ | coef. de corelație: $\rho(x,y) = \frac{Cov(x,y)}{\sqrt{Var(x)}\sqrt{Var\,y}}$

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