DERIVATE PARȚIALE - CURS 6

① $f(x,y) = e^{x-y^2}$

$\frac{\partial f}{\partial x}(x,y) = e^{x-y^2}$

$\frac{\partial f}{\partial y}(x,y) = -2y\, e^{x-y^2}$

② $f(x,y,z) = e^{x^2+y^2} \cdot \sin^2 z$

$\frac{\partial f}{\partial x} = 2x\, e^{x^2+y^2} \cdot \sin^2 z$

$\frac{\partial f}{\partial y} = 2y\, e^{x^2+y^2} \sin^2 z$

$\frac{\partial f}{\partial z} = e^{x^2+y^2} \cdot 2\sin z \cos z$

③ $f(x,y) = xy \cdot \operatorname{arctg}\left(\frac{x+y}{1-xy}\right)$, $xy \neq 1$.

$\frac{\partial f}{\partial x} = y \operatorname{arctg}\left(\frac{x+y}{1-xy}\right) + xy \cdot \frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2} \cdot$

$\cdot \frac{1-\cancel{xy}+y(\cancel{x}+y)}{(1-xy)^2} = y \operatorname{arctg}\left(\frac{x+y}{1-xy}\right) + xy \cdot \frac{1+y^2}{1-2\cancel{xy}+x^2y^2+x^2+2\cancel{xy}+y^2}$

$= y \operatorname{arctg}\left(\frac{x+y}{1-xy}\right) + xy \frac{1+y^2}{x^2+y^2+x^2y^2+1}$.

$$\frac{\partial f}{\partial y} = x \operatorname{arctg} \frac{x+y}{1-xy} + xy \frac{1+x^2}{x^2+y^2+x^2y^2+1}$$

④ $f(x,y,z) = x^{y^z}$

$$\frac{\partial f}{\partial x} = y^z \cdot x^{y^z-1} \qquad \left((x^n)' = n x^{n-1}\right)$$

$$\frac{\partial f}{\partial y} = x^{y^z} \ln x \cdot (y^z)' = x^{y^z} \ln x \cdot z \cdot y^{z-1}$$

$$\left((a^u)' = a^u \ln a \cdot u'\right)$$

$$\frac{\partial f}{\partial z} = x^{y^z} \ln x \, (y^z)' = x^{y^z} \ln x \cdot y^z \ln y$$

$$(a^u)' = a^u \ln a \cdot u'$$

⑤ $f(x,y) = e^x \cos y$

$$\frac{\partial f}{\partial x} = e^x \cos y$$

$$\frac{\partial f}{\partial y} = -e^x \sin y$$

$$\frac{\partial^2 f}{\partial x^2} = e^x \cos y$$

$$\frac{\partial^2 f}{\partial y^2} = -e^x \cos y$$

$$\frac{\partial f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -e^x \sin y$$

⑥ $f(x;y) = x^3 + xy$

$$\frac{\partial f}{\partial x} = 3x^2 + y$$

$$\frac{\partial f}{\partial y} = x$$

$$\frac{\partial^2 f}{\partial x^2} = 6x$$

$$\frac{\partial^2 f}{\partial y^2} = 1$$

$$\frac{\partial^2 f}{\partial y \partial x} = 1 = \frac{\partial^2 f}{\partial x \partial y}$$

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$$f(x,y)=\frac{x-y}{x+y} \qquad (x,y)\neq(0,0)$$

$$\frac{\partial f}{\partial x}=\frac{\cancel{x}+y-\cancel{x}+y}{(x+y)^2}=\frac{2y}{(x+y)^2}$$

$$\frac{\partial f}{\partial y}=\frac{-x-y-(x-y)}{(x+y)^2}=\frac{-2x}{(x+y)^2}$$

$$\frac{\partial^2 f}{\partial x^2}=-\frac{2y\cdot 2(x+y)\cdot 1}{(x+y)^4}=-\frac{4y}{(x-y)^3}$$

$$\frac{\partial^2 f}{\partial y^2}=+\frac{2x\cdot 2(x+y)\cdot 1}{(x+y)^4}=\frac{4x}{(x+y)^3}$$

$$\frac{\partial^2 f}{\partial y\,\partial x}=\frac{\partial^2 f}{\partial x\,\partial y}=\frac{-2(x+y)^2+2x\cdot 2(x+y)\cdot 1}{(x+y)^4}=$$

$$=\frac{-2x-2y+4x}{(x+y)^3}=\frac{2x-2y}{(x+y)^3}=\frac{2(x-y)}{(x+y)^3}$$

⑧ $f(x,y) = \arccos \frac{x}{\sqrt{x^2+y^2}}$, $(x,y) \neq (0,0)$

$$\frac{\partial f}{\partial x} = -\frac{1}{\sqrt{1-\frac{x^2}{x^2+y^2}}} \cdot \left(\frac{x}{\sqrt{\ }}\right)' =$$

$$-\frac{\sqrt{\ }}{|y|} \cdot \frac{\sqrt{\ } - x\cdot \frac{\cancel{2}x}{\cancel{2}\sqrt{\ }}}{x^2+y^2} =$$

$$-\frac{\cancel{\sqrt{x^2+y^2}}}{|y|} \cdot \frac{y^2}{(x^2+y^2)\cancel{\sqrt{x^2+y^2}}} = -\frac{\overset{|y|}{\cancel{y^2}}}{|y|(x^2+y^2)}$$

$$= -\frac{|y|}{x^2+y^2} = \begin{cases} \frac{-y}{x^2+y^2}, & y \geq 0 \\ \frac{y}{x^2+y^2}, & y < 0 \end{cases}$$

$$\frac{\partial f}{\partial y} = -\frac{1}{\sqrt{1-\frac{x^2}{x^2+y^2}}} \left(-\frac{x \cdot \sqrt{\ }'}{x^2+y^2}\right) =$$

$$\frac{x \cdot}{\frac{|y|}{\cancel{\sqrt{\ }}}} \cdot \frac{\cancel{y}}{\cancel{\sqrt{\ }}} \cdot \frac{1}{x^2+y^2} = \frac{x \cdot |y|}{y\ (x^2+y^2)} = \begin{cases} \frac{x}{x^2+y^2}, & y>0 \\ -\frac{x}{x^2+y^2}, & y<0 \end{cases}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -\frac{x^2+y^2-2y\cdot y}{(x^2+y^2)^2} = -\frac{x^2-y^2}{(x^2+y^2)^2} =$$

$$= -\frac{x^2-y^2}{(x^2+y^2)^2} \cdot \frac{|y|}{y}.$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{2x|y|}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial y^2} = -\frac{x \cdot 2y\,|y|}{(x^2+y^2)^2 y} = -\frac{2x|y|}{(x^2+y^2)^2}$$

(9) $f(x,y) = \ln(x^2+y^2) \quad ,(x,y) \neq (0,0)$

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{2(x^2+y^2)-4x^2}{(x^2+y^2)^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{2(x^2+y^2)-4y^2}{(x^2+y^2)^2} = \frac{2(x^2-y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\frac{2y \cdot 2x}{(x^2+y^2)^2} = -\frac{4xy}{(x^2+y^2)^2}$$

(10) $f(x,y)=\frac{1}{\sqrt{x^2+y^2}}$ $(x,y)\neq(0,0)$

$$\frac{\partial f}{\partial y}=-\frac{y}{(x^2+y^2)\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial x}=-\frac{\frac{\cancel{2}x}{\cancel{2}\sqrt{}}}{\sqrt{x^2+y^2}^{\,2}}=-\frac{x}{(x^2+y^2)\sqrt{x^2+y^2}}=-x(x^2+y^2)^{-3/2}$$

$$\frac{\partial^2 f}{\partial x^2}=-\frac{(x^2+y^2)^{\frac{3}{2}}-x\cdot\frac{3}{\cancel{2}}(x^2+y^2)^{\frac{1}{2}}\cdot\cancel{2}x}{(x^2+y^2)^3}$$

$$=-\frac{\sqrt{x^2+y^2}\,(x^2+y^2-3x^2)}{(x^2+y^2)^3}=\frac{2x^2-y^2}{(x^2+y^2)^2\sqrt{x^2+y^2}}$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{2y^2-x^2}{(x^2+y^2)^2\sqrt{x^2+y^2}}$$

$$\frac{\partial^2 f}{\partial y\partial x}=-x\cdot\left(-\frac{3}{\cancel{2}}\right)\cdot(x^2+y^2)^{-\frac{5}{2}}\cdot\cancel{2}y=3xy\,(x^2+y^2)^{-5/2}$$

(11) $f(x,y,z) = xyz$

$\frac{\partial f}{\partial x} = yz$ | $\frac{\partial^2 f}{\partial x^2} = 0$; $\frac{\partial^2 f}{\partial y \partial x} = z$; $\frac{\partial^2 f}{\partial z \partial x} = y$

$\frac{\partial f}{\partial y} = xz$ | $\frac{\partial^2 f}{\partial y^2} = 0$; $\frac{\partial^2 f}{\partial x \partial y} = z$; $\frac{\partial^2 f}{\partial z \partial y} = x$

$\frac{\partial f}{\partial z} = xy$ | $\frac{\partial^2 f}{\partial z^2} = 0$; $\frac{\partial^2 f}{\partial x \partial z} = y$; $\frac{\partial^2 f}{\partial y \partial z} = x$

(14) $f(x,y,z) = y \sin(x+z)$

$\frac{\partial f}{\partial x} = y \cos(x+z)$ | $\frac{\partial^2 f}{\partial x^2} = -y \sin(x+z)$; $\frac{\partial^2 f}{\partial y \partial x} = \cos(x+z)$

$\frac{\partial f}{\partial y} = \sin(x+z)$ | $\frac{\partial^2 f}{\partial y^2} = 0$; $\frac{\partial^2 f}{\partial x \partial y} = \cos(x+z)$

$\frac{\partial f}{\partial z} = y \cos(x+z)$ | $\frac{\partial^2 f}{\partial z^2} = -y \sin(x+z)$; $\frac{\partial^2 f}{\partial x \partial z} = -y \sin(x+z)$

$\frac{\partial^2 f}{\partial z \partial x} = -y \sin(x+z)$

$\frac{\partial^2 f}{\partial z \partial y} = \cos(x+z)$

$\frac{\partial^2 f}{\partial y \partial z} = \cos(x+z)$

(13) $f(x,y) = 2x^2 + xy$ $\quad \frac{\partial f}{\partial x}(1,1);\ \frac{\partial f}{\partial y}(3,2) = ?$

$\frac{\partial f}{\partial x} = 4x + y$ $\quad \Rightarrow \frac{\partial f}{\partial x}(1,1) = 4+1 = 5$ $\quad \frac{\partial f}{\partial y}(3,2) = 3$

$\frac{\partial f}{\partial y} = x$

(14) $f(x,y) = e^{\sin xy}$, $\quad \frac{\partial f}{\partial x}(1,\frac{\pi}{2});\ \frac{\partial f}{\partial y}(1,0) = ?$

$\frac{\partial f}{\partial x} = e^{\sin xy} \cdot \cos xy \cdot y \Rightarrow$

$\frac{\partial f}{\partial x}(1;\frac{\pi}{2}) = e^{\sin 1\cdot\frac{\pi}{2}} \cdot \cos 1\cdot\frac{\pi}{2} \cdot \frac{\pi}{2} =$

$e^{1} \cdot 0 \cdot \frac{\pi}{2} = 0$

$\frac{\partial f}{\partial y} = e^{\sin xy} \cos xy \cdot x$

$\Rightarrow \frac{\partial f}{\partial y}(1,0) = e^{\sin 1\cdot 0} \cdot \cos(1\cdot 0) \cdot 1 =$

$e^{0} \cdot \cos 0 \cdot 1 = 1.$

(15) $f(x,y) = \sqrt{\sin^2 x + \sin^2 y}$ $\quad ; \frac{\partial f}{\partial x}(\frac{\pi}{4};0);\ \frac{\partial f}{\partial y}(\frac{\pi}{4};\frac{\pi}{4})$

$\frac{\partial f}{\partial x} = \frac{\cancel{2}\sin x\cos x}{\cancel{2}\sqrt{\sin^2 x + \sin^2 y}}$ $\Rightarrow \frac{\partial f}{\partial x}(\frac{\pi}{4};0) = \frac{\sin\frac{\pi}{4}\cos\frac{\pi}{4}}{\sqrt{\sin^2\frac{\pi}{4} + \sin^2 0}} = \frac{\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}}{\sqrt{\frac{\cancel{2}}{\cancel{4}_2}}}$

$= \sqrt{2}/2$

$\frac{\partial f}{\partial y} = \frac{\cancel{2}\sin y\cos y}{\cancel{2}\sqrt{\ }}$ $\Rightarrow \frac{\partial f}{\partial y}(\frac{\pi}{4};\frac{\pi}{4}) = \frac{\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}}{\sqrt{\frac{2}{4}+\frac{2}{4}}} = \frac{1}{2}$

(16) $f(x,y) = \ln(1+x^2+y^2)$, $\frac{\partial f}{\partial x}(1,1)$; $\frac{\partial f}{\partial y}(1,1)$

$\frac{\partial f}{\partial x} = \frac{2x}{1+x^2+y^2} \Rightarrow \frac{\partial f}{\partial x}(1,1) = \frac{2}{3}$

$\frac{\partial f}{\partial y} = \frac{2y}{1+x^2+y^2} \Rightarrow \frac{\partial f}{\partial y}(1,1) = \frac{2}{3}$

(17) $f(x,y) = \sqrt[3]{x^2y}$, $\frac{\partial f}{\partial x}(-2,2) = ?$, $\frac{\partial f}{\partial y}(-2,2)$; $\frac{\partial^2 f}{\partial x \partial y}(-2;2)$

$\frac{\partial f}{\partial x} = \frac{2xy}{3\sqrt[3]{(x^2y)^2}} \Rightarrow \frac{\partial f}{\partial x} = \frac{-4 \cdot 2}{3\sqrt[3]{(4 \cdot 2)^2}} = \frac{-\cancel{4} \cdot 2}{3 \cdot \cancel{2}^2} = -\frac{2}{3}$

$\frac{\partial f}{\partial y} = \frac{x^2}{3\sqrt[3]{(x^2y)^2}} \Rightarrow \frac{\partial f}{\partial y}(-2,2) = \frac{4}{3\sqrt[3]{(4 \cdot 2)^2}} = \frac{1}{3}$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{2x \cdot 3\sqrt[3]{x^4 \cdot y^2} - x^2 \cdot \cancel{3} \frac{4x^3y^2}{\cancel{3}\sqrt[3]{x^8y^4}}}{9\sqrt[3]{x^8y^4}}$$

$$= \frac{6x^2\sqrt[3]{xy^2} - \frac{4x^{\cancel{5}\,3}y^{\cancel{2}}}{x^2y\sqrt[3]{x^2y}}}{9x^2y\sqrt[3]{x^2y}} = \frac{6x^2 \cdot xy - 4x^3y}{9x^2y\sqrt[3]{x^4y^2}}$$

$$= \frac{2x^{\cancel{3}}\cancel{y}}{9x^2y \cdot x\sqrt[3]{xy^2}} = \frac{2}{9\sqrt[3]{xy^2}} \Rightarrow \frac{\partial^2 f}{\partial x \partial y} = \frac{2}{-9 \cdot 2} = -\frac{1}{9}$$

(18) $f(x,y) = xy\ln x$, $x \neq 0$, $\frac{\partial^2 f}{\partial x \partial y}(1,1)$, $\frac{\partial^2 f}{\partial y \partial x}(1,1)$

$\frac{\partial f}{\partial x} = y\ln x + xy \cdot \frac{1}{x} = y(\ln x + 1)$

$\frac{\partial^2 f}{\partial y \partial x} = \ln x + 1 \Rightarrow \ln 1 + 1 = 1$

$\frac{\partial f}{\partial y} = x\ln x$

$\frac{\partial f}{\partial x \partial y} = \ln x + x \cdot \frac{1}{x} = \ln x + 1 \Rightarrow 1.$

(19) $f(x,y) = \ln(u^2+v)$, $u(x,y) = e^{x+y^2}$
$v(x,y) = \cancel{x}^2 + y$

$$\begin{cases} \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} \\ \frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} \end{cases}$$

$\Rightarrow \frac{\partial f}{\partial x} = \frac{2u}{u^2+v} \cdot e^{x+y^2} + \frac{1}{u^2+v} \cdot 2x$

$= \frac{2(e^{x+y^2})^2}{(e^{x+y^2})^2 + x^2 + y} + \frac{1}{u^2+v} \cdot 2x$

$= \frac{2(u^2+x)}{u^2+v}$

$\frac{\partial u}{\partial x} = e^{x+y^2}$

$\frac{\partial u}{\partial y} = e^{x+y^2} \cdot 2y$

$\frac{\partial v}{\partial x} = 2x$

$\frac{\partial v}{\partial y} = 1$

$\frac{\partial f}{\partial u} = \frac{2u}{u^2+v}$

$\frac{\partial f}{\partial v} = \frac{1}{u^2+v}$

$$\frac{\partial f}{\partial y} = \frac{2u}{u^2+v} \cdot u \cdot 2y + \frac{1}{u^2+v} \cdot 1 = \frac{4u^2y+1}{u^2+v}.$$

(20) $f(x,y) = \operatorname{arctg} \frac{2u}{v}$, $u = x \sin y$, $v = x\cos y$

$$\frac{\partial f}{\partial u} = \frac{1}{1+\frac{4u^2}{v^2}} \cdot \frac{2}{v} = \frac{2v}{v^2+4u^2}$$

$$\frac{\partial f}{\partial v} = \frac{1}{1+\frac{4u^2}{v^2}} \cdot \left(-\frac{2u}{v^2}\right) = \frac{-2u}{v^2+4u^2}$$

$$\frac{\partial u}{\partial x} = \sin y \, ; \quad \frac{\partial v}{\partial x} = \cos y$$

$$\frac{\partial u}{\partial y} = x\cos y \, ; \quad \frac{\partial v}{\partial y} = -x\sin y$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{2v}{v^2+4u^2} \cdot \sin y + \frac{-2u}{v^2+4u^2} \cdot \cos y$$

$$= \frac{2(v\sin y - u\cos y)}{v^2+4u^2} =$$

$$= \frac{2(x\sin y\cos y - x\sin y\cos y)}{v^2+4u^2} = 0$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial x} \cdot \frac{\partial v}{\partial y} = \frac{2vx\cos y}{v^2+4u^2} + \frac{-2u(-x\sin y)}{v^2+4u^2}$$

$$= \frac{2x^2\cos^2 y + 2x^2\sin^2 y}{v^2+4u^2} = \frac{2\cancel{x^2}}{\cancel{x^2}\cos^2 y + 4\cancel{x^2}\sin^2 y} = \frac{2}{3\sin^2 y+1}$$