① aruncăm în mod repetat o monedă $P_{succes} = 0.93$. Fie X v.a. ce descrie nr. de succese înainte de al 6-lea eșec

?→ rep. lui X, $E[6X-8]$, $Var(3X+8)$

sol: X – nr. de succese înainte de...

$$P(X=k) = \binom{k-1}{5}(1-p)^6 \cdot p^{k-6}$$

$$P(X=k) = \binom{k-1}{5} \cdot (0.07)^6 \cdot (0,93)^{k-6}$$

...

$$E[X] = \frac{p \cdot m}{1-p} = \frac{0,93 \cdot 6}{1-0,93} = \dots$$

$$E[6X-8] = 6 \cdot E[X] - 8 = 6 \cdot 79,\dots$$

$$Var(X) = \frac{p \cdot m}{(1-p^2)} = \frac{0,93 \cdot 6}{(0,07)^2} = \dots$$

② Lot 7 tlf, 2 defecte, fie X nr. de teste efectuate și Y nr de teste efectuate pentru identificarea celui de-al doilea tlf.

?→ rep. comună a (X,Y) și marginale

→ $E[X]$, $Var(X)$, $\rho(X,Y)$

→ $E[X|Y=2]$, $Var(X|Y=2)$

a) $2 \le X+Y \le 6$

H_i – telefonul i este defect

$$P(X=1, Y=1) = \frac{2}{7} \cdot \frac{1}{6} = \frac{2}{42} = \frac{1}{21}$$

$$P(X=1, Y=2) = \frac{2}{7} \cdot \frac{5}{6} \cdot \frac{1}{5} = \frac{1}{21}$$

$$P(X=1, Y=3) = \frac{2}{7} \cdot \frac{5}{6} \cdot \frac{4}{5} \cdot \frac{1}{4} = \frac{1}{21}$$

$$P(X=1, Y=4) = \frac{2}{7} \cdot \frac{5}{6} \cdot \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{21}$$

$$P(X=1, Y=6) = \frac{2}{7} \cdot \frac{5}{6} \cdot \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{21}$$

X|Y	0	1	2	3	4	5	Σ
1	0	1/21	1/21	1/21	1/21	2/21	6/21
2	0	1/21	1/21	1/21	~~2/21~~	0	5/21
3	0	1/21	1/21	2/21	0	0	4/21
4	0	1/21	2/21	0	0	0	3/21
5	1/21	2/21	0	0	0	0	3/21
Σ	1/21	6/21	5/21	4/21	3/21	2/21	

$$X \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \frac{6}{21} & \frac{5}{21} & \frac{4}{21} & \frac{3}{21} & \frac{3}{21} \end{pmatrix}$$

$$Y \sim \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ \frac{1}{21} & \frac{6}{21} & \frac{5}{21} & \frac{4}{21} & \frac{3}{21} & \frac{2}{21} \end{pmatrix}$$

b) $E[X] = \frac{6+10+12+12+15}{21} = \dots$

$E[Y] = \dots$

$X^2 \sim \dots$, $Y^2 \sim \dots$, $E[X^2] = \dots$, $E[Y^2] = \dots$, $Var(X) = E[X^2] - E[X]^2$

$Var(Y) = \dots$

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X) Var(Y)}}$$

$$E[XY] = \sum xy \cdot P(X=x, Y=y) = 1 \cdot 0 \cdot 0 + 1 \cdot 1 \cdot \frac{1}{21} + 1 \cdot 2 \cdot \frac{1}{21} + \dots = 5$$

$$Cov(X,Y) = 5 - 1,99 \cdot 1,95 = 1,12$$

$$\rho(X,Y) = \frac{1,12}{\sqrt{1,99 \cdot 1,99}} = 0,57$$

c) $E[X|Y=2] = ?$

$$X|Y=2 \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \frac{1/21}{5/21} & \frac{1/21}{5/21} & \frac{1/21}{5/21} & \frac{2/21}{5/21} & \frac{0}{5/21} \end{pmatrix} = \dots$$

$$E[X|Y=2] = \frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{8}{5} + 0 = \frac{14}{5} = 2,8$$

$$Var(X|Y=2) = E[(X|Y=2)^2] - E[X|Y=2]^2$$

$$(X|Y=2)^2 \sim \begin{pmatrix} 1 & 4 & 9 & 16 & 25 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} & 0 \end{pmatrix}$$

$E[(X|Y=2)^2] = \dots$, $Var(X|Y=2) = \dots$

③ X – v.a. care are densitatea de repartiție

$$f(x) = \frac{x}{64} e^{-\frac{x^2}{128}} \mathbb{1}_{\{x \ge 0\}}$$

$$? \to \frac{F^{-1}(0,75) - F^{-1}(0,25)}{\sqrt{Var(X)}} = ?$$

sol.: $\left.\begin{array}{l} F(x) = \int_{-\infty}^{\infty} f(t)dt \\ x \ge 0 \end{array}\right\} \Rightarrow F(x) = \int_0^x f(t)dt =$

$$= \int_0^x \frac{t}{64} e^{-\frac{t^2}{128}} dt = -e^{-\frac{t^2}{128}}\Big|_0^x = -e^{-\frac{x^2}{128}} + 1 =$$

$$= 1 - e^{-\frac{x^2}{128}}$$

$$F^{-1}(x) = y \,|\cdot F \Rightarrow x = F(y) = 1 - e^{-\frac{y^2}{128}}$$

$$e^{-\frac{y^2}{128}} = 1 - x \Rightarrow -\frac{y^2}{128} = \ln(1-x)$$

$$y = \sqrt{128 \ln\left(\frac{1}{1-x}\right)} = 8\sqrt{2\ln\left(\frac{1}{1-x}\right)} \Rightarrow$$

$$\Rightarrow F^{-1}(x) = 8\sqrt{2\ln\left(\frac{1}{1-x}\right)}$$

$$F^{-1}\left(\frac{3}{4}\right) = 8\sqrt{2\ln 4}, \quad F^{-1}\left(\frac{1}{4}\right) = 8\sqrt{2\ln\frac{4}{3}}$$

$$Var(X) = E[X^2] - E[X]^2$$

$$E[X] = \int_0^{\infty} x f(x) dx = \int_0^{\infty} x \frac{x}{64} e^{-\frac{x^2}{128}} dx =$$

$$= \int_0^{\infty} (-x)\left(e^{-\frac{x^2}{128}}\right)' dx =$$

$$= \underbrace{-x e^{-\frac{x^2}{128}}\Big|_0^{\infty}}_{0} + \int_0^{\infty} e^{-\frac{x^2}{128}} =$$

$$= \int_0^{\infty} (-64)\left(e^{-\frac{x^2}{128}}\right)' dx = -64 \cdot e^{-\frac{x^2}{128}}\Big|_0^{\infty} +$$

$$\int_0^{\infty} e^{-\frac{x^2}{128}} dx = -64 + 64 \cdot 8 \underbrace{\int_0^{\infty} e^{-\frac{y^2}{2}} dx}_{\frac{\sqrt{2\pi}}{2}}$$

$$= -64 + \frac{512\sqrt{2\pi}}{2}$$

$$\cancel{E[X^2] = \int_0^{\infty} x^2 f(x) dx = \int_0^{\infty} (-x^2)\left(e^{-\frac{x^2}{128}}\right)' dx = \underbrace{-x^2 \cdot e^{-\frac{x^2}{128}}\Big|_0^{\infty}}_{0} + 64^2 \underbrace{\int_0^{\infty} (-1)\left(e^{-\frac{x^2}{128}}\right) dx}_{8 \cdot \frac{\sqrt{2\pi}}{2}} = -64^2 + 64^2 \cdot 8 \cdot \frac{\sqrt{2\pi}}{2}}$$

$$Var(X) = -64^2 + 64^2 \cdot 8 \cdot \frac{\sqrt{2\pi}}{2} + 64^2 - 64 \cdot 8 \cdot \frac{\sqrt{2\pi}}{2} + \frac{2 \cdot 64^2 \cdot 8\sqrt{2\pi}}{2}$$

$$Var(X) = -2 \cdot 64^2 + \frac{\sqrt{2\pi}}{2}(64^2 \cdot 8 - 512 + 2 \cdot 64^2 \cdot 8)$$, înlocuim și calculăm raportul

$$E[X^2] = \int_0^{\infty} x^2 f(x) dx = \int_0^{\infty} (-x^2)\left(e^{-\frac{x^2}{128}}\right)' dx = \underbrace{-x^2 \cdot e^{-\frac{x^2}{128}}\Big|_0^{\infty}}_{0} + 64^2 \int_0^{\infty} (-1)\left(e^{-\frac{x^2}{128}}\right)' dx = 64^2 \cdot (-1) \cdot e^{-\frac{x^2}{128}}\Big|_0^{\infty} + 64^2$$

$$\underbrace{\int_0^{\infty} e^{-\frac{x^2}{128}} dx}_{8\frac{\sqrt{2\pi}}{2}} = -64^2 + 64^2 \cdot 8 \cdot \frac{\sqrt{2\pi}}{2}$$

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④ alegeri pain Pois(906), 0.64 șanse pt x, 0,36 pt y, indep.
Fie V dif de voturi
? -> repartiția cuplului dat de var. ale. care det. nr. voturi
-> arăt. că v.a. ce determină nr. de vot. sunt indep.
-> $E[V]$ și $Var[V]$

$\lambda = 906$, x - olban, y - câțu

a) $P(x=i, y=j) = P(x=i \mid y=j \mid V=i+j) P(V=i+j)$

$P(x=i=y=j \mid V=i+j) = P(x=i \mid V=i+j) = P(y=j \mid V=i+j)$

$P(x=i, y=j) = \binom{j+i}{i} p^i (1-p)^j e^{-\lambda} \frac{\lambda^{i+j}}{(i+j)!} =$

$= \frac{\cancel{(i+j)!}}{i!\,j!} p^i (1-p)^j e^{-\lambda} \cdot \frac{\lambda^{i+j}}{\cancel{(i+j)!}} =$

$= e^{-\lambda p} \frac{(\lambda p)^i}{i!} \cdot e^{-\lambda(1-p)} \frac{\lambda(1+p)^j}{j!} =$

$= e^{-906 \cdot 0,64} \frac{(906 \cdot 0,64)^i}{i!} \cdot e^{-906 \cdot 0,36} \cdot \frac{906 \cdot 0,36}{j!}$

Rep. marginale:

$P(x=i) = \sum_{j \geq 0} P(x=i, y=j) = \sum_{j \geq 0} e^{-\lambda p} \frac{(\lambda p)^i}{i!} \cdot e^{-\lambda(1-p)} \cdot \frac{\lambda(1-p)^j}{j!}$

$= e^{-\lambda p} \cdot \frac{(\lambda p)^i}{i!} \Rightarrow x \sim Pois(\lambda p) \Leftrightarrow x \sim Pois(906 \cdot 0,64)$

Analog $y \sim Pois(\lambda(1-p)) \Leftrightarrow y \sim Pois(906 \cdot 0,36)$

b) $P(x=i, y=j) = P(x=i) P(y=j) \Rightarrow x \perp\!\!\!\perp y$

c) $E[V] = \lambda = 906$, $Var(V) = \lambda = 906$

⑤ x, y v.a. indep. $x, y > 0$, ...

- $E[\log(x)] \leq \log(E[x])$ (concavă din Jensen)
- $E[\sqrt{x}] \leq \sqrt{E[x]}$ (concavă din Jensen)
- $E[\sin^2(x)] + E[\cos^2(x)] = 1$ $(E[\sin^2(x) + \cos^2(x)])$
- $P(x > c) \leq \frac{E[x^3]}{c^3}$ (din Markov)
- $P(x \leq y) = P(x \geq y)$ ($x \perp\!\!\!\perp y$, nu avem alte info)
- $P(|x+y| > 2) \leq \frac{E[(x+y)^4]}{2^4}$ (din Markov)
- $E[x^4] \; ? \; E[x]^{E[y]}$
- $P(x+y < 10) \leq P(x > 5 \text{ sau } y > 5)$ (evenimentul $x+y>0$ e inclus în $x>5$ sau $y>5$)
- $E[\frac{x}{y}] = \frac{E[x]}{E[y]}$ (v.a. indep => liniaritate)
- $E[x^2(x^2+1)] = E[x^2(y^2+1)]$ ($x, y \perp\!\!\!\perp$ => sunt identice)
- $E[\frac{1}{x}] \geq \frac{1}{E[x]}$ (Jensen)
- $\frac{E[x^3]}{E[x^2]} \; ? \; \frac{E[y^4]}{E[x^3]}$ (x poate fi subunitar sau supraunitar)
- $P(|x-y| > 0) \leq \frac{Var(x)}{2}$ (Chebyshev)

⑥ Un jucător are zar și monedă cu $p = 0.75$ să cadă H

a) probabilitatea ca nr. de capete să fie 4 știind că nr. de pete pe fața superioară a fost 5

b) prob. ca nr. de capete să fie 1 sau 5

c) Știind că nu am obț. niciun cap. în cursul jocului care e P ca rezultatul zarului să fi fost 2?

sol.: x - nr. pete de pe zar, y = nr. capete monedă
$p = 0.75$ = prob. să pice cap

a) $P(y=4 \mid x=5) = ?$

$P(y=4 \mid x=5) = \frac{1}{6} \cdot C_5^4 (0.75)^4 \cdot (0.25)^1 = \frac{1}{6} \cdot \frac{5!}{4!} \cdot (\frac{3}{4})^4 \cdot \frac{1}{4} = \frac{5}{6} \cdot \frac{81}{256} \cdot \frac{1}{4} = \frac{135}{2048} = 0.0659$

b) $P(y=1 \cup y=5) = ?$

$P(y=1) + P(y=5) = P(y=1 \cup y=5)$, $P(y=1) = \sum_{x=1}^{6} C_x^1 (0.75)^1 (0.25)^{x-1}$, x - nr. pet. zar

$P(y=1) = C_1^1 \ldots$ (desfacem) $\ldots$

$P(y=5) = \sum_{x=5} C_x^5 \cdot (0.75)^5 \cdot (0,25)^{x-5}$, $x \geq 5$ deoarece e imposibil să pice 5 capete pt $x < 5$

...

c) $P(x=2 \mid y=0) = P(y=0 \mid x=2) \cdot P(x=2) / P(y=0)$

$P(y=0 \mid x=2) = C_2^0 (0.75)^0 \cdot (0.25)^2 = 1 \cdot 1 \cdot 0.0625 = 0.0625$

$P(x=2) = \frac{1}{6}$

$P(y=0) = \sum_{x=1}^{6} C_x^0 (0.75)^0 \cdot (0.25)^{x-0} = \sum_{x=1}^{6} (0.25)^x = \ldots$

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⑦ două zaruri echilibrate, unul alb, unul negru, fiecare 6 fețe, dintre care 2 cu 1p, 2 cu 2p, 2 cu 3p.
Considerăm urm. v.a: X_n = nr. pct. zar negru, X_a = -"- alb, $R = |X_n - X_a|$, $S = X_n + X_a$, $V = \min(X_n, X_a)$, $W = \max(X_n, X_a)$

a) rep. comună (X_n, X_a)
b) rep. V și W
c) rep. comună (V, W)
d) $E[9V]$, $E[6W]$ fără rep.
e) $E[10R]$, $E[8S]$, $Var(3R-11S+2)$

sol.: a)

$X_n \backslash X_a$	1	2	3	Σ
1	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{3}$
2	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{3}$
3	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{3}$
Σ	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	

$P(X_n=1, X_a=1) = p_n \cdot p_a = \frac{1}{3}\cdot\frac{1}{3} = \frac{1}{9}$
analog pt. restul

b) $V \sim \begin{pmatrix} 1 & 2 & 3 \\ \frac{5}{9} & \frac{3}{9} & \frac{1}{9} \end{pmatrix}$ $W \sim \begin{pmatrix} 1 & 2 & 3 \\ \frac{1}{9} & \frac{3}{9} & \frac{5}{9} \end{pmatrix}$

c)

$V \backslash W$	1	2	3	Σ
1	$\frac{5}{81}$	$\frac{15}{81}$	$\frac{25}{81}$	$\frac{5}{9}$
2	$\frac{3}{81}$	$\frac{9}{81}$	$\frac{15}{81}$	$\frac{3}{9}$
3	$\frac{1}{81}$	$\frac{3}{81}$	$\frac{5}{81}$	$\frac{1}{9}$
Σ	$\frac{1}{9}$	$\frac{3}{9}$	$\frac{5}{9}$	

d) $E(9V) = 9\cdot E(V) = 9\cdot\frac{14}{9} = 14$

$E(V) = \min(1,1)\cdot\frac{1}{9} + \min(1,2)\cdot\frac{1}{9} + \dots + \min(3,3)\cdot\frac{1}{9} =$
$= \frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{2}{9}+\frac{2}{9}+\frac{1}{9}+\frac{2}{9}+\frac{3}{9} = \frac{14}{9}$

$E(6W) = 6\cdot E(W) = 6\cdot\frac{22}{9} = \frac{44}{3}$

$E(W) = \max(1,1)\cdot\frac{1}{9} + \max(1,2)\cdot\frac{1}{9} + \dots + \max(3,3)\cdot\frac{1}{9} = \frac{1}{9}+\frac{2}{9}+\frac{3}{9}+\frac{2}{9}+\frac{2}{9}+\frac{3}{9}+\frac{3}{9}+\frac{3}{9}+\frac{3}{9} = \frac{22}{9}$

e) $R \sim \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ \frac{1}{9} & \frac{2}{9} & \frac{3}{9} & \frac{2}{9} & \frac{1}{9} \end{pmatrix}$ $S \sim \begin{pmatrix} 2 & 3 & 4 & 5 & 6 \\ \frac{1}{9} & \frac{2}{9} & \frac{3}{9} & \frac{2}{9} & \frac{1}{9} \end{pmatrix}$

$E[10R] = 10\cdot E(R) = 10\left((-2)\cdot\frac{1}{9} + (-1)\cdot\frac{2}{9} + 0\cdot\frac{3}{9} + 1\cdot\frac{2}{9} + 2\cdot\frac{1}{9}\right) = 10\cdot 0 = 0$

$E[8S] = 8\cdot E[S] = 8\cdot\left(2\cdot\frac{1}{9} + 3\cdot\frac{2}{9} + 4\cdot\frac{3}{9} + 5\cdot\frac{2}{9} + 6\cdot\frac{1}{9}\right) = 8\cdot 4 = 32$

$Var(3R-11S+2) = Var(3R-11S) = Var(3R) + Var(-11S) + 2Cov(3R, -11S)$
$= 9Var(R) + 121Var(S) + 2Cov(3R, -11S) = \dots$

⑧ $X \sim \begin{pmatrix} -5 & 1 \\ 0{,}24 & 0{,}76 \end{pmatrix}$ $Y \sim \begin{pmatrix} 0 & 6 \\ p_1 & p_2 \end{pmatrix}$ $p_1, p_2 \in (0,1)$ (v.a. neindep.!)

a) $P(X=-5, Y=6 = 0{,}12)$
$E[X|Y=6] = -2$

$X \backslash Y$	0	6	Σ
-5	0,12	0,12	0,24
1	$p_1-0{,}12$	$p_2-0{,}12$	0,76
Σ	p_1	p_2	

$X|Y=6 \sim \begin{pmatrix} -5 & 1 \\ \frac{0{,}12}{p_2} & \frac{p_2-0{,}12}{p_2} \end{pmatrix}$

$E[X|Y=6] = \cancel{0{,}24} -5\cdot\frac{0{,}12}{p_2} + \frac{p_2-0{,}12}{p_2} = -2 \dots$

b) Nu avem voie $Var(X+Y) = Var(X) + Var(Y)$

Binomiala $B(n,p)$ $E[X] = np$ $Var(X) = np(1-p)$ $P(X=k) = C_n^k\cdot(1-p)^{n-k}p^k$

Covarianța și corelația

X și Y necorelate => $Cov(X,Y) = 0$
$Cov = E[XY] - E[X]\cdot E[Y]$
$Var(X,Y) = Var(X) + Var(Y) + 2Cov(X,Y)$
cor: $\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$

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v.a. discrete:

$E[x] = \sum_{x} x P(x=x)$

$E[ax+by] = aE[x] + bE[y]$, $E[g(x)] = \sum_{x} g(x) P(x=x)$

x și y indep $\Rightarrow E[xy] = E[x] \cdot E[y]$

momentul de ordin k : $E[x^k]$

mom. de ordin k centrat în a : $E[(x-a)^k]$

varianța : $Var(x) = E[x^2] - E[x]^2$

$Var(ax+b) = a^2 Var(x)$

dacă x și y indep. $Var(x+y) = Var(x) + Var(y)$

abatere standard : $SD(x) = \sqrt{Var(x)}$

Repartiții comune, marginale și condiționate discret

f. de masă a vectorului (x,y) - rep. comună : $P_{x,y}(x,y) = P(x=x, y=y) = f_{x,y}(x,y)$

f. de rep. a vec. (x,y): $F_{x,y}(x,y) = P(x \leq x, y \leq y)$

$P((x,y) \in A \times B) = \sum P(x=x, y=y) \; x \in A \cap x(\Omega), y \in B \cap y(\Omega)$

rep. marginală x : $P(x=x) = P_x(x) = \sum_{y} P(x=x, y=y)$

rep. cond. a lui x la y=y : $P_{x|y}(x|y=y) = P(x=x | y=y) = \frac{P(x=x, y=y)}{P(y=y)} =$

$$= \frac{P_{x,y}(x,y)}{P_y(y)}$$

rep. cond. a lui x la A : $P_{x|A}(x) = P(x=x|A) = \frac{P(\{x=x\} \cap A)}{P(A)}$

media : $E[g(x,y)] = \sum_{x} \sum_{y} g(x,y) P_{x,y}(x|y)$

$E[x] = \sum_{y} E[x|y=y] P(y=y)$

Prob. cond.:

$Q(A|B) = \frac{Q(B|A) \cdot Q(A)}{Q(B)}$

$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A) \cdot P(A)}{P(B|A) \cdot P(A) + P(B|A^c) P(A^c)}$

$P(A) = \sum_{i=1}^{n} P(A|B_i) \cdot P(B_i)$

A și B indep. $\Leftrightarrow P(B|A) = P(B) \Leftrightarrow \frac{P(A \cap B)}{P(B)} = P(A) \Leftrightarrow P(A \cap B) = P(A) \cdot P(B)$

A și B indep. cond. la c. dacă $P(A \cap B | C) = P(A|C) \cdot P(B|C) \Leftrightarrow$

$$\Leftrightarrow P(A \cap B) = P(A) \cdot P(B)$$

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