• Pb examen

→ Geometrie

1) $A=(1,2,3)$; $B=(4,5,6) \in \mathbb{R}^3$

a) $C=(a,4,8)$. $\exists$ a a.î. A, B, C - col

determ $r(A,B,C)$

A, B, C - col $\longrightarrow$ $\overrightarrow{AB}$ și $\overrightarrow{BC}$ coliniare

$\overrightarrow{AB} = B-A = (3,3,3)$ $\quad \overrightarrow{BC} = C-B = (a-4, 2, 4)$

proporționali $\Leftrightarrow a-4=2 \Rightarrow a=6$

$\overrightarrow{AB} = (3,3,3)$ $\Rightarrow \overrightarrow{AB} = \frac{3}{2}\overrightarrow{BC}$

$\overrightarrow{BC} = (2,2,2)$

$r(A,B,C) = \frac{3}{2}$ [raportul = valoarea pt care $\overrightarrow{AB} = r\overrightarrow{BC}$]

b) $r(A,P,B)=1 \Rightarrow \overrightarrow{AP} = \overrightarrow{PB}$

P - mijl lui AB

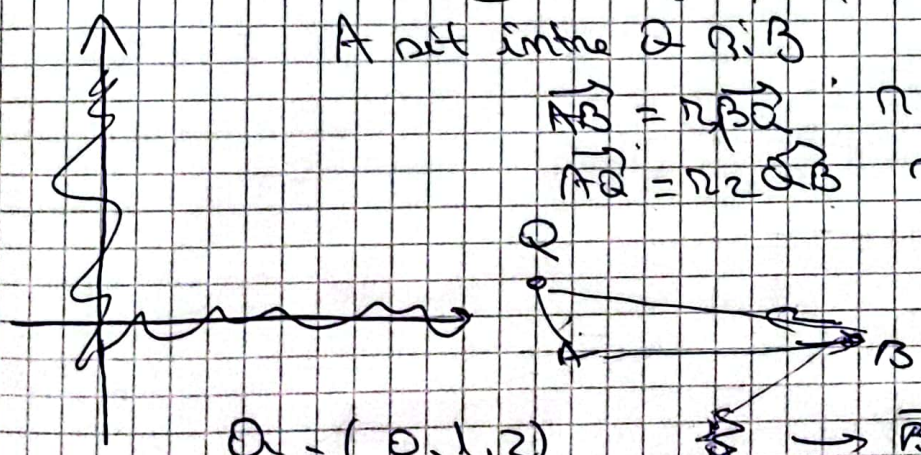
$P = \frac{1}{2}A + \frac{1}{2}B = \left(\frac{5}{2}, \frac{7}{2}, \frac{9}{2}\right)$

c) $Q = ?$ (punct) a.î. $r(A,B,Q)$ $\{r(A,Q,B) < 0\}$

A este între Q și B

$\overrightarrow{AB} = r_1\overrightarrow{BQ}$ $\quad r_1<0$

$\overrightarrow{AQ} = r_2\overrightarrow{QB}$ $\quad r_2<0$



$Q=(0,1,2)$ $\longrightarrow$ $\overrightarrow{BQ} = (-4,-4,-4)$

$\overrightarrow{AQ} = (-1,-1,-1)$

$\overrightarrow{QB} = (4,4,4)$

2) $P=(1,-1)$ $\quad Q=(3,3)$

a) val det = ? $\quad \overrightarrow{PQ}$ $\quad O=(0,0)$

$$D(P,Q,R) = \begin{vmatrix} 1 & 1 & 1 \\ P_1 & Q_1 & r_1 \\ P_2 & Q_2 & r_2 \end{vmatrix}$$

$$D(P,Q,R) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 0 \\ -1 & 3 & 0 \end{vmatrix} = 6$$

b) $R_\alpha = (\alpha, -1)$ $R_\alpha \to$ sit în dreapta $\overrightarrow{PQ}$

$\hookrightarrow \det < 0$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & \alpha \\ -1 & 3 & -2 \end{vmatrix} \qquad (\Delta(P, Q, R_\alpha) < 0)$$

$$\Delta = \begin{vmatrix} \alpha & 1 & 1 \\ \alpha & 3 & \alpha \\ -\alpha & 3 & -2 \end{vmatrix} = \begin{vmatrix} (\alpha-1) & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 3 & -2 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 3 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & 2 \end{vmatrix} = -3\alpha + 3 - 2 + 2 - 3\alpha + 3 = -6\alpha + 6 = 6(1-\alpha) < 0$$

$\Rightarrow \alpha > 1$

3) $r(A, P, B) \to \overrightarrow{AP} = r_1 \overrightarrow{PB}$

$r(B, P, A) \to \overrightarrow{BP} = r_2 \overrightarrow{PA}$

$r(P, A, B) \to \overrightarrow{PA} = r_3 \overrightarrow{AB}$

i) $A = (3, 3)$, $B = (2, 4)$, $P = (5, 1)$

$\overrightarrow{AP} = (2, -2)$

cac $\overrightarrow{AB} = (-1, -1)$, $\overrightarrow{BP} = (3, -3)$ sunt proporț. ⇓ sunt col

$r_1 = -\frac{2}{3}$

$r_2 = \frac{3}{2}$ la fel și r_3 $r_3 = 2$

$r(A, P, B) = 2$

4) $\alpha, \beta \to A, B, P$ col

$A = (6, 2)$ $P = (\alpha, \beta)$

$B = (2, -2)$ $\overrightarrow{AP} = 2\overrightarrow{PB}$

$\overrightarrow{AP} = (\alpha - 6, \beta - 2)$ $\overrightarrow{PB} = (2 - \alpha, -2 - \beta)$

$\alpha - 6 = 4 - 2\alpha$

$\Rightarrow 3\alpha = 10 \Rightarrow \alpha = \frac{10}{3}$

$\beta - 2 = -4 - 2\beta$ $3\beta = -2$ $\beta = -\frac{2}{3}$

6) poz punct R fata de $\overrightarrow{QP}$ — $D = \begin{vmatrix} 1 & 1 & x_1 \\ q_1 & p_1 & y_1 \\ q_2 & p_2 & y_2 \end{vmatrix}$

$D < 0 \Rightarrow R$ în stg $\overrightarrow{QP}$

$> 0 \Rightarrow R$ în stg $\overrightarrow{QP}$

$= 0 \Rightarrow R$ – pe dr $\overrightarrow{QP}$

Acoperiri convexe

6) $M = \{P_1, P_2 \dots P_9\}$

$P_1 = (-5, 0)$ $P_3 = (-3, -3)$ $P_5 = (1, -1)$

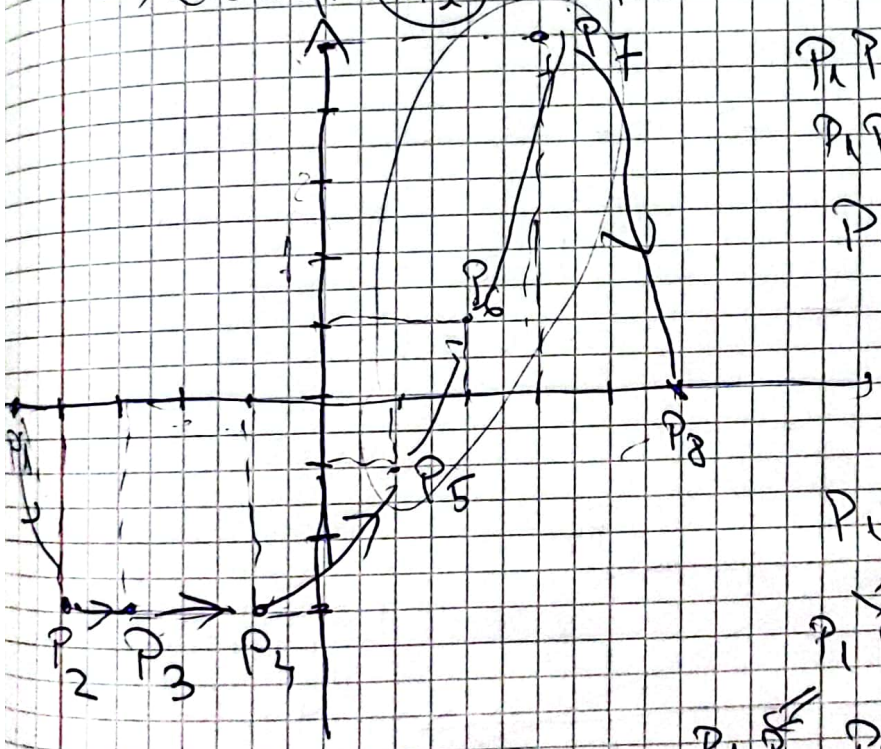
$P_2 = (-4, -3)$ $P_4 = (-1, -3)$ $P_6 = (2, 1)$

$P_7 = (3, 5)$

$P_8 = (5, 0)$

→ evoluția L_i a uf din marginea inf

fa inf



$P_1 P_2$

$P_1 P_2 P_3$

$P_1 P_2 \cancel{P_3} P_4$

coliniare

$P_1 P_2 P_4 P_5$

$P_1 P_2 P_4 P_5 P_6$

$\Downarrow$

$P_1 P_2 P_4 P_5 P_6 P_7$

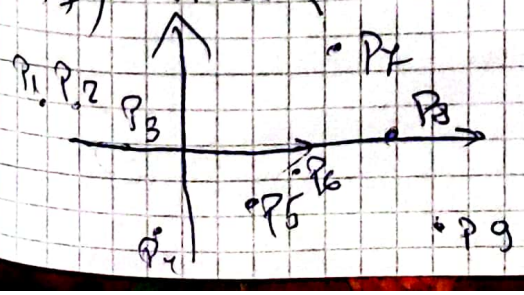
$P_1 P_2 P_4 P_5 P_6 P_7 P_8$

pct elim în ac ordine: $P_7 P_6$

$P_1 P_2 P_4 P_8$ → lista uf marginf

7) multi $M \in \mathbb{R}^2$ pt care L_i are 3 elem

max elem $L_i = 6$

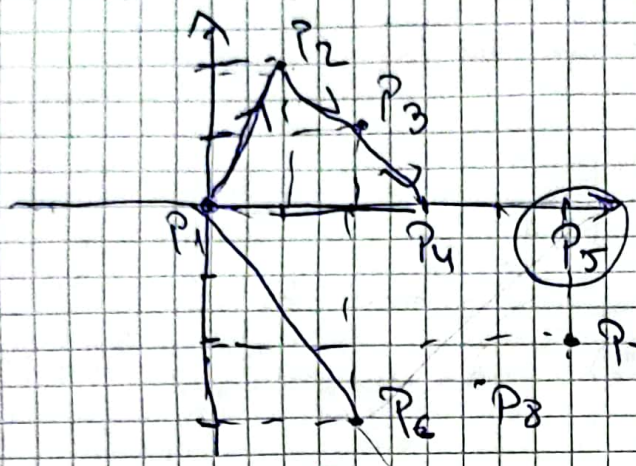


→ $P_1 P_4 P_5 P_6 P_7 P_8$

↳ $P = \{P_1, P_2 \dots P_7\}$ (convex)

$P_1 = (0,0)$ $P_3 = (2,1)$ $P_5 = (5,0)$ $P_7 = (5,$

$P_2 = (1,2)$ $P_4 = (3,0)$ $P_6 = (2,-3)$



$P_1 \to P_2$

P_3 este la dreapta P_1P_2

$\Rightarrow$ înlocuiesc P_2 cu P_3

P_5 nu e în dr $P(P_4)$ → pivot

P_7 nu e în dr P_1P_6 P_6: dr muchiei P_1P_4

$\Rightarrow P_1(P_6)$ → pivot

P_8: nu e în dr P_1P_6

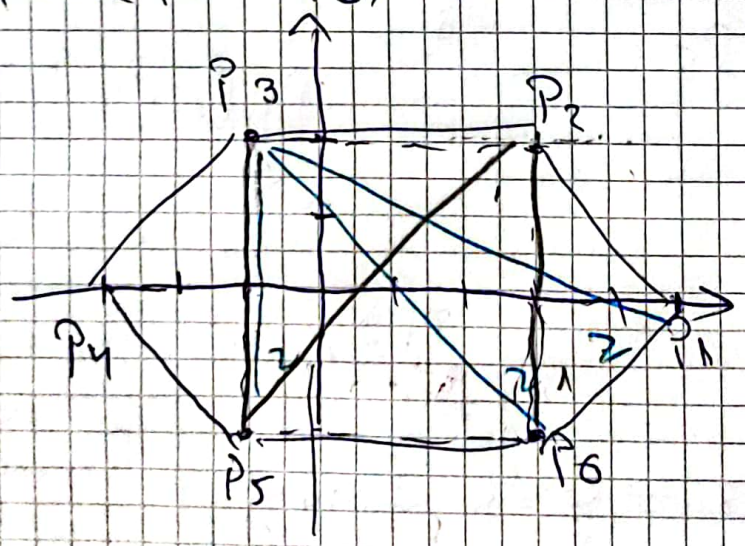
~ Triangulare → n = nr puncte din plan

h = nr puncte frontieră

↳ $\left\lceil \frac{n}{3} \right\rceil$ camere → $2n-h-2$ triunghiuri

necesare și suf $3n-h-3$ muchii

$P = (P_1 \dots P_6)$



$P_1 = (5,0)$

$P_2 = (3,2)$

$P_3 = (-1,2)$

$P_4 = (-3,0)$

$P_5 = (-1,-2)$

$P_6 = (3,-2)$

hexagon ⇒ $3 \cdot 6 - 6 - 3 = 9$ muchii

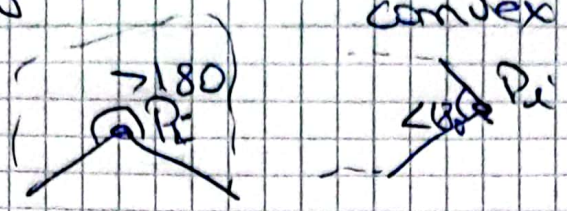
↳ 1) triangulare ⇒ 3 diag

2) punem culori

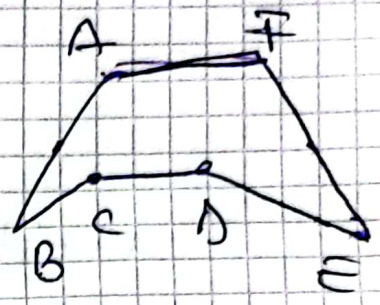
3) nr camere = max (nr-ap-culoare)

↳ **Tipuri de vf**

- principal → P_i prime de $\{P_{i-1}, P_{i+1}\}$ → diag
- concav — convex



- vf principal e convex ⇒ ear
  principal + concav ⇒ mouth.



- desenează poligon cu 6 vf ← convexe / concave

Graful asociat

↳ noduri → poz inițiale (n)

↳ muchiile → laturile Δ (nm)

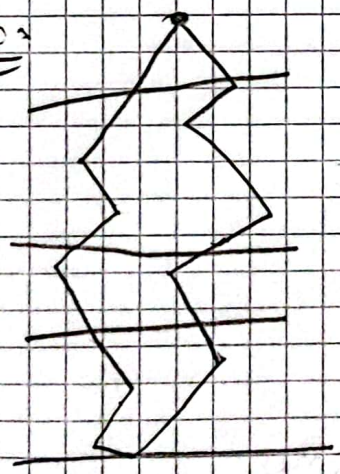
↳ fețe : fețele Δ + fața exterioară (nt)

- relații : Euler : $n - nm + (nt+1) = 2$

$$2nm = 3nt + \underbrace{k}_{\text{pe frontieră}}$$

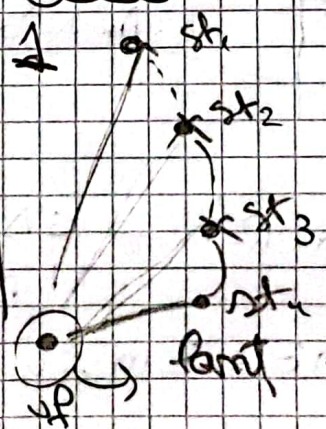
Poligon monoton ⇒ y-monoton: poate fi parcurs de sus în jos în 2 moduri, fără a me întoarce sus

ex:



Cazuri

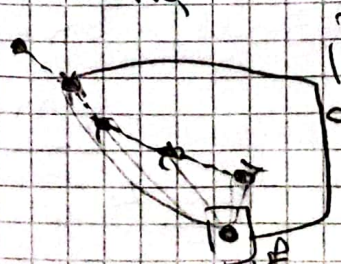
1



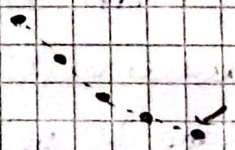
↳ lant pers cu st4 ⇒ intre vf și st4

2 ac lant

(a)



(b) adăug în stivă



vf nou → adăug st

dif la a) formă de

• Date niște puncte → verif monotonie

→ x-monotonie: P_1 (cel mai la stg) → x minim

↳ coordonata x crește o stg dată apoi x descrește sau invers (nu se permit 2 creșteri/descreșteri la fel și y necomsecutive)

pornim de la val minime x și y

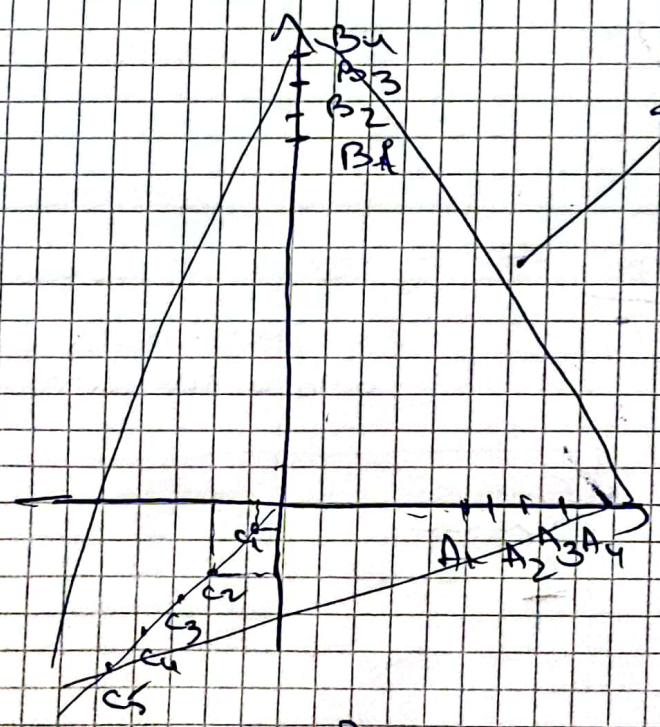
• $M = \{A_i \mid i=0,\dots,50\} \cup \{B_i \mid i=0,\dots,40\} \cup \{C_i \mid i=0..30\}$

↳ $A_i = (i \in 10, 0)$ $i = \overline{0,50}$

$B_i = (0, i \in 30)$, $i = \overline{0,40}$

$C_i = (-i, -i)$, $i = \overline{0,30}$

total = 120 puncte



acoperire convexă = 3 puncte

$m_t = 2m - 4 - 2 = 24 \wedge$

$m_m = 3m - 4 - 3 = 3 \times 3$

• mult din R^2 → 6 Δ

11 muchii

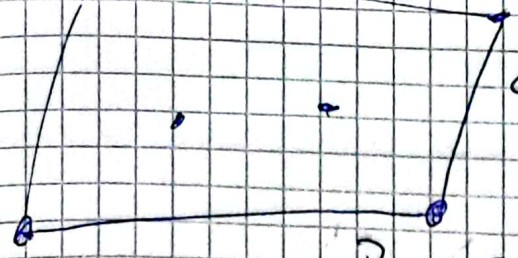
$m_t = 2m - 4 - 2 = 6 \rightarrow 2m - 4 = 8$

$m_m = 3m - 4 - 3 = 11$ $\quad 3m - 4 = 14$

⇒ avem 6 puncte dintre care 4 pe frontiera acoperirii convexe

$\Rightarrow \underline{m = 6}$

$\underline{\underline{4 = 4}}$



Structura punctelor

$\hookrightarrow$ $P_1 = (1,7)$ $\quad P_3 = (7,5)$ $\quad P_5 = (5,3)$

$P_2 = (5,7)$ $\quad P_4 = (1,3)$ $\quad P_6 = (\alpha-1, 5)$

- $\alpha - 1 \leq 1$ $\quad \alpha \in (-\infty, 2]$ $\rightarrow$ P_6 - ext sau pe poligonul $P_1P_2P_3P_4P_5$ $\Rightarrow n = 6$ $\quad k = 6$ $\rightarrow$ 4 fete 9 muchii

- $\alpha - 1 > 1$ $\quad \alpha - 1 < 7$ $\quad \alpha \in (2, 8)$ $\quad P_6$ - int în pentag $P_1P_2P_3P_4P_5$

  $n = 6$
  $k = 5$ $\Big\}$ 5 fete $\quad$ 10 muchii

- $\alpha - 1 = 7$ $\quad \alpha \in \{8\}$ $\quad P_6$ coincide cu P_3

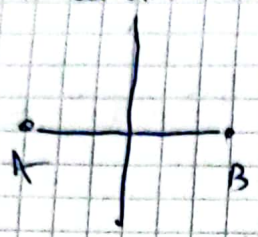
  $n = 5$ $\quad$ 3 fete
  $k = 5$ $\quad$ 7 muchii

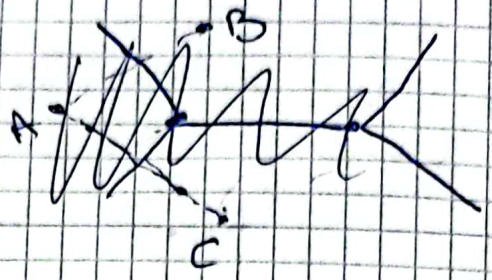
- $\alpha - 1 > 7$ $\rightarrow$ $\alpha \in (8, \infty)$ $\quad P_6$ - ext pentag $P_1P_2P_3P_4P_5$ $\quad n = 6$ $\quad k = 6$ $\quad$ 4 f 9 m

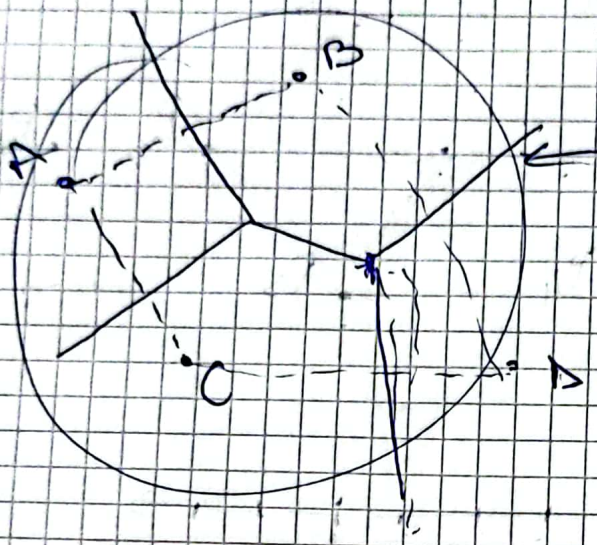
$$nm = \frac{3nf + k}{2}$$

→ Diagrame Voronoi ←

- în context 2D → drepte mediane la segm





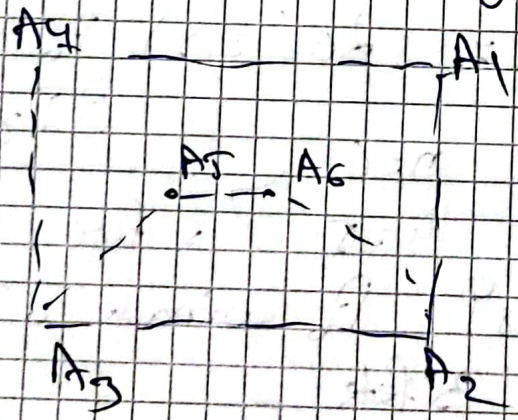


← diagrama Voronoi pt ABCD

↳ $M = \{A_1, A_2, A_3, A_4, A_5, A_6\}$

↳ diagrama Voronoi → 4 semidr

$M \setminus \{A_1\} \rightarrow$ 5 dr $\} \Rightarrow A_1 \rightarrow$ frontieră



↳ **Rezolvare**

1 $A = (1,2) \rightarrow d, g$ - trec prin A ⇒ dualele A^*, d^*, g^*

punct: $A(a_x, a_y) \xrightarrow{\text{dreapta}} y = a_x x - a_y$: A^*

dreapta: p $\quad y = ax + b \xrightarrow{\text{dreapta}} p^* = (a, b)$

↳ $A(1,2) \longrightarrow$ drepte $y = 2x$ $\quad y = -x + 3$

$A^*(y = x - 2)$ $\quad d^* = (2, 0)$ $\quad g^* = (-1, 3)$

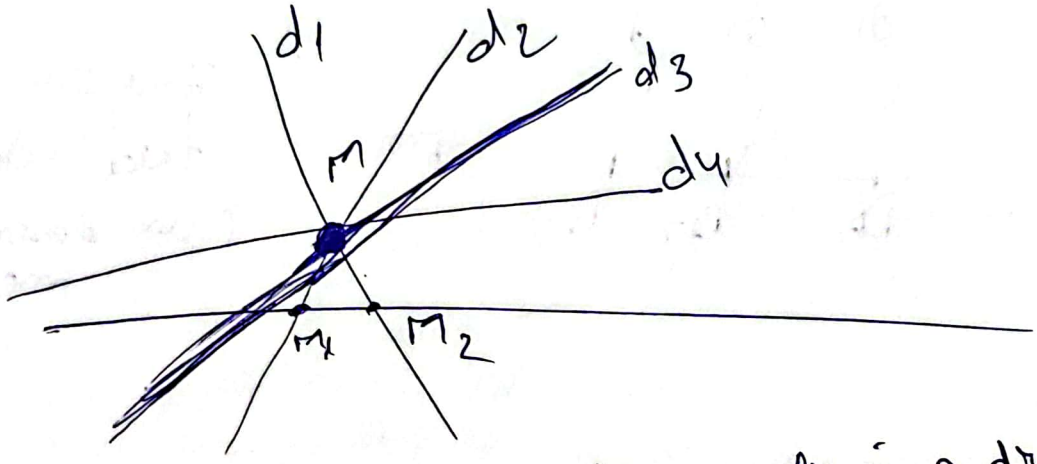
2. 4 drepte care trec prin ac punct

(d_1, d_2, d_3, d_4)

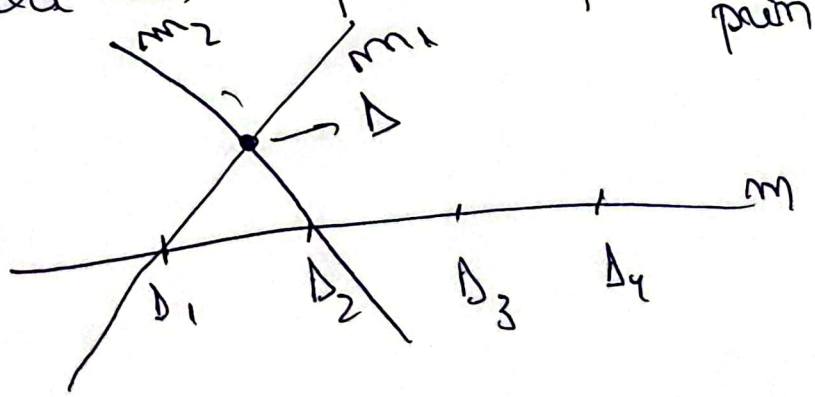
pe cons

$M_1, M_2 \neq M$

$\in d_1 \quad \in d_2$



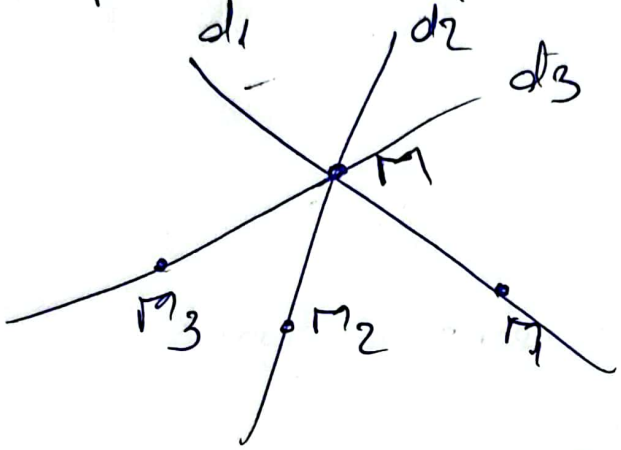
↳ dreapta $\longrightarrow$ 4 puncte pe dr și 2 dr care trec prin 2 puncte de pe m



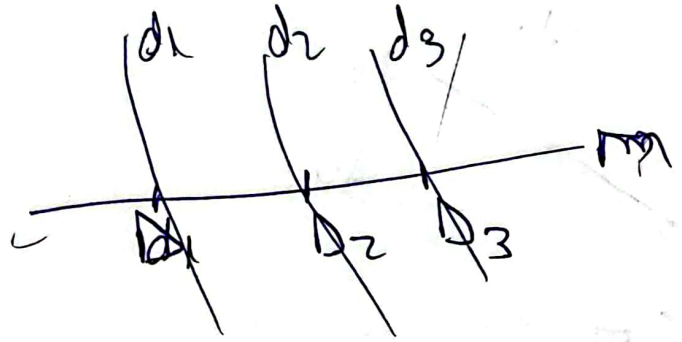
config autodrală = conf drală transf pta fi la fel ca cea init

↳ 4 drepte care trec printr-un punct
↳ determină 4 puncte, fiec de pe o dreaptă, a.î. toate ce să fie coliniare.

↳ 3 drepte care trec prin ac punct



⇓ config drală
↳ dr M cu D_1, D_2, D_3 puncte



prin fiec punct trec o dreaptă (în desenul meu sunt ||)

↳ Dacă am 3 drepte inters într-un punct M; pe fiec dreaptă câte un punct, iar acestea sunt col ⇒ config drală are o dr cu 3 puncte prin care trec 3 drepte ce se ∩ într-un pct
⇒ conf drală = conf init

↳ Diagrama Voronoi → pb 2)

Rezumat: - facem mediatoarele segm dintre puncte
- tăiem mediatoarele
- nr semidrepte = nr puncte acoperire convexă

1) a) n puncte necoliniare ⟶ muchii de tip segm + semidreaptă

$n_v \le 2n-5$

$n_m \le 3n-6$

$\boxed{(n_v+1) - n_m + n = 2}$ (rel Euler) (1)

- fiec muchie = incidentă cu 2 vârfuri
- fiec vârf = incident cu cel puțin 3 muchii
- $2n_m \ge 3(n_v+1)$

$n_m \ge \frac{3(n_v+1)}{2}$ înlocuim n_m în rel 1)

$n_v+1 - n_m + n + 2 \le$

$2 \le n_v+2 - \frac{3(n_v+1)}{2} + n$

$4 \le 2n_v+2 - 3n_v - 3 + 2n$

$n_v \le 2 - 3 - 4 + 2n$ $[n_v \le 2n-5]$

înlocuim n_v în rel 1)

$n_v+1 - n_m + n = 2 \le 2n-5 - n_m + n + 1$

$2 \le 3n-5-n_m+1$

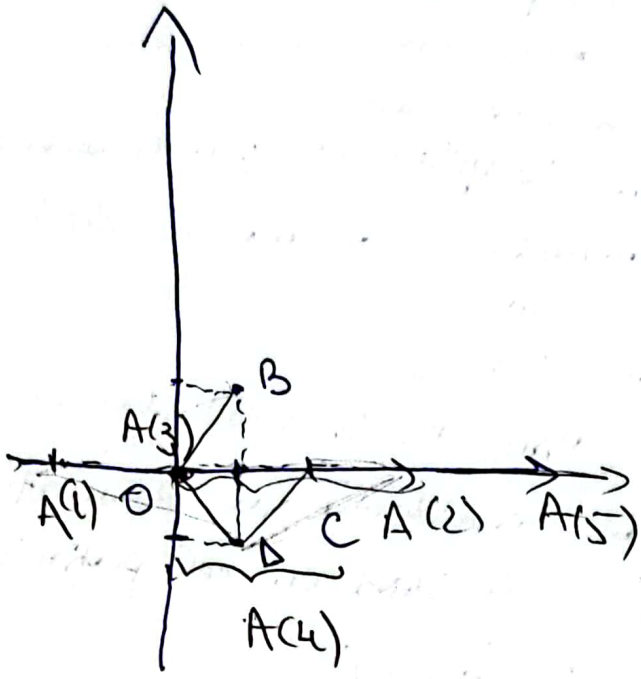
$n_m \le 3n-6$

b) D – 5 semidrepte ⇒ 5 puncte pe frontiera convexă

$n_v = 2n-5 = 5$

$n_v \le 3$ → 3 situații
- pot coincide $n_v=1$
- 4 pot coincide 1 afară $n_v=2$
- ∀ 4 pot merge cât

↳ punctele $O(0,0)$ $B=(1,1)$ $D=(1,-1)$
$A(2,0)$ $C=(2,0)$ $\{O, A, B, C, D\}$ → mulțimile semnifică



1) $\alpha < 0$
A → ext OBCD
4 puncte acoperă convexă
→ ABCD ⇒ 4m

2) $\alpha = 2$ $A = C$
$m = 4$

3) $\alpha = 0$ $A = O$ ⇒
$m = 4$

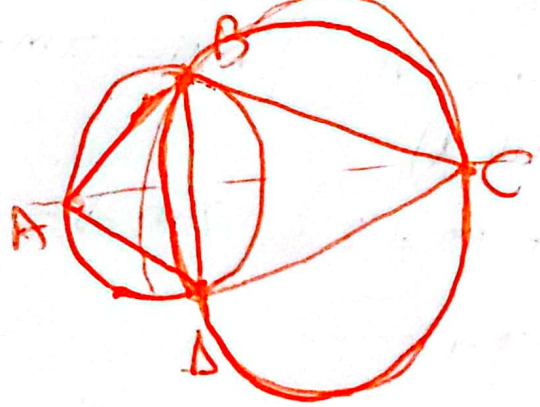
4) $0 < \alpha < 2$ ⇒ A int OBCD
front OBCD ⇒ m=4

5) $\alpha > 2$ ⇒ A exterior OBCD
- front ∈ ODAB ⇒ $m = 4$

→ Triangularea Delaunay

ABCD - patrulater
τ - triangulare P de $\forall \Delta \in \tau$ cercul circumscris nu conț în int alt punct al P

ex



ΔABD, ΔBCD → Δ Delaunay

ΔABC → cercul circ conț D

SEMIPLANE

1) Semiplanele → mulț. de ec. $ax+by+c \geq 0$ sau $ax+by+c \leq 0$

$H: x+y-3 \leq 0$ $\quad$ H''

$H': -2x+y+1 \leq 0$

alegem H'' a.î. $H \cap H' \cap H'' = \Delta$ dr

a.î. dr suport H'' $\perp$ dr suport H

$x-y-5 \leq 0$: H''

↳ n semiplanelor este un Δ (Geogebra)

↳ $H_1 \cap H_2 \cap H_3 \cap H_4$: H_1: $-y+1 \leq 0$ $\quad -y \leq -1$ $\quad y \geq 1$

H_2: $y-5 \leq 0$ $\quad y \leq 5$

H_3: $-x \leq 0$ $\quad x \geq 0$

$x-y+a \leq 0$

$x \leq y-a$

$a>5$ $\quad 0 \geq x-y+a > x-y+5 \geq 0 \Rightarrow \nexists$

$a=5$ $\quad x-y+5 \leq 0$

$x \leq y-5$ $\quad y \leq 5$ $\quad y \in [1,5]$

$x \in [0, \infty)$

$\Rightarrow \begin{matrix} y=5 \\ x=0 \end{matrix}$ $\Rightarrow$ pct de intersect $= (0,5)$

$1 \leq a < 5$ $\quad x=0$ $\quad x-y+a=0$

$(0,5)$ ↳ uf $\quad$ un uf aici

$y=5$ $\quad x-y+a=0$

$a=1 \Rightarrow (0,1)$ uf $\quad$ un uf aici

↳ $a<1$ ⇒ trapez dreptunghic isoscel ar ca dr suport alte semiplane (Geogebra)

↳ normalele exterioare sunt col cu vectorii

$$\begin{bmatrix} (0,1,-1) & (0,0,-1) \\ (0,1,0) & (0,-1,0) \\ & (0,-1,-1) \end{bmatrix}$$

↳ Dc normala ext a unei fete f → col cu un vector $\vec{v_f} = (a,b,c)$

Semiplanul feței f:

$ax+by+c \leq 0$

$y-1 \leq 0 \Rightarrow y \leq 1$

$y \leq 0$

$-1 \leq 0$

$\boxed{-y \leq 0} \Rightarrow y \geq 0$

$-y-1 \leq 0 \quad -y \leq 1$

$\Rightarrow y = 0$ sol