SERII DE FUNCȚII - CURS 3

① $f_n : (0,1) \to \mathbb{R}, \quad f_n(x) = \frac{1}{nx+1}, \quad n \geq 0$

conv. simplă și unif.

CS: $\lim\limits_{n\to\infty} f_n(x) = 0 \Rightarrow f_n(x) \xrightarrow{s} 0$

C.U. Fie $g(x) = |f_n(x) - \underset{\lim\limits_{n\to\infty} f_n(x)}{\underset{\shortparallel}{f(x)}}| = |f_n(x) - 0| = \frac{1}{nx+1}$

$\Rightarrow \sup\limits_{x\in(0,1)} g(x) = g(0) = 1 \neq 0$

$g'(x) = -\frac{n}{(nx+1)^2} < 0 \Rightarrow g \searrow$

$\left.\begin{array}{l} 0 < x < 1 \\ g \searrow \end{array}\right| \Rightarrow g(0) > g(x) > g(1)$

$1 > g(x) > \frac{1}{n+1}.$

$\Rightarrow f_n(x) \not\xrightarrow{u} 0.$

② $f_n : [0,1] \to \mathbb{R}, \quad f_n(x) = nx(1-x)^n, \quad n \geq 0$

CS: $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} nx(1-x)^n$

crit. rap. pt. șiruri: $\left.\begin{array}{l} x_n > 0. \\ \lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = \ell \in [0,1) \end{array}\right| \Rightarrow$

$\lim\limits_{n\to\infty} x_n = 0$

$\Rightarrow \lim\limits_{n\to\infty} \frac{(n+1)\cancel{x}(1-x)^{n+1}}{n\cancel{x}(1-x)^n} = 1-x$

$x \in (0,1] \Rightarrow 1-x \in [0,1)$

pt. $x=0 \Rightarrow f_n(0) = 0.$ $\quad \overset{\text{C.Rap.}}{\Rightarrow} \lim\limits_{n} f(x) = 0. \Rightarrow f_n(x) \xrightarrow{s} 0.$

C.U. Pt. $x \in (0,1]$

Fie $g(x) = |f_n(x) - \underset{\substack{\parallel \\ 0 \text{ (din C.S.)}}}{f(x)}| = |f_n(x)| = n x (1-x)^n$

$$g' = n(1-x)^n + n^2 x (1-x)^{n-1} \cdot (-1) =$$
$$n(1-x)^{n-1}(1 - x - nx) = 0$$

$\Downarrow$

$x = 1$ $\quad$ $1 - x - nx = 0$

$1 = x + nx$

$x(n+1) = 1 \Rightarrow x = \frac{1}{n+1}.$

x	0		$\frac{1}{n+1}$		1
$g'(x)$		$+$	0	$-$	0
$g(x)$		$\nearrow$	$g\left(\frac{1}{n+1}\right)$ max.	$\searrow$	

$$\Rightarrow \sup_{x \in (0,1]} g(x) \geq g\left(\frac{1}{n+1}\right) = n \cdot \frac{1}{n+1}\left(1 - \frac{1}{n+1}\right)^n$$
$$= \frac{n}{n+1} \cdot \left(\frac{n}{n+1}\right)^n = \left(\frac{n}{n+1}\right)^{n+1}$$

$$\lim\left(\sup_{x \in (0,1]} |f_n(x) - 0|\right) = \lim g\left(\frac{1}{n+1}\right) = \lim \frac{n}{n+1} \cdot \left(1 + \frac{-1}{n+1}\right)^{-(n+1) \cdot \frac{n}{-(n+1)}}$$

$$= 1 \cdot e^{-1} = \frac{1}{e} \neq 0 \Rightarrow f_n(x) \not\xrightarrow{u} 0.$$

Pt. $x = 0 \Rightarrow f_n(0) = 0$ $\quad$ fct. constantă

③ $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n - x^{2n}$, $n \geq 0$

CS: $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} (x^n - x^{2n}) = 0$

$\left(x \in [0,1] \Rightarrow x^n \to 0 ; x^{2n} \to 0 \right)$

$\Rightarrow f_n \xrightarrow{S} 0$.

C.U. $\lim\limits_{n\to\infty} \sup\limits_{x\in[0,1]} |f_n(x) - f(x)| = \lim\limits_{n\to\infty} \sup\limits_{x\in[0,1]} (x^n - x^{2n})$

Fie $g : [0,1] \to \mathbb{R}$, $g(x) = |f_n(x) - \underset{0}{\overset{\shortparallel}{f(x)}}| = |f_n(x)|$

$\Rightarrow g(x) = x^n - x^{2n}$

$g'(x) = nx^{n-1} - 2nx^{2n-1} = nx^{n-1}(1 - 2x^n) = 0$

$x = 0$ sau $1 - 2x^n = 0 \Rightarrow 2x^n = 1$.

$x = \frac{1}{\sqrt[n]{2}}$

x	0		$\frac{1}{\sqrt[4]{2}}$		1
$g'(x)$	0	$+ + +$	0	$- - -$	
$g(x)$		$\nearrow$	$g\left(\frac{1}{\sqrt[4]{2}}\right)$ Max.	$\searrow$	

$M = g\left(\frac{1}{\sqrt[n]{2}}\right) = \left(\frac{1}{\sqrt[n]{2}}\right)^n - \left(\frac{1}{\sqrt[4]{2}}\right)^{2n} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$\Rightarrow \sup g = \frac{1}{4}$.

$\lim\limits_{n\to\infty} \sup\limits_{x\in[0,1]} g(x) = \frac{1}{4} \neq 0 \Rightarrow f_n \overset{u}{\not\to} f$

4. $f_n:\mathbb{R}\to\mathbb{R}$, $f_n(x)=\sqrt{x^2+\frac{1}{n^2}}$, $n>0$

CS: $\lim\limits_{n\to\infty} f_n(x) = \lim \sqrt{x^2+\frac{1}{n^2}} = \sqrt{x^2} = |x|$

$$f_n(x) \xrightarrow{s} |x|.$$

C.U. $\sup\limits_{x\in\mathbb{R}} |f_n(x)-f(x)| = \sup\limits_{x\in\mathbb{R}} \left| f_n(x) - |x| \right| =$

$$= \sup_{x\in\mathbb{R}} \left| \sqrt{x^2+\frac{1}{n^2}} - \sqrt{x^2} \right| = \sup \left| \frac{\frac{1}{n^2}}{\sqrt{x^2+\frac{1}{n^2}}+\sqrt{x^2}} \right|$$

Fie $g:\mathbb{R}\to\mathbb{R}$, $g(x) = \dfrac{1}{n^2\left(\sqrt{x^2+\frac{1}{n^2}}+\sqrt{x^2}\right)}$

$$g'(x) = -\frac{n^2\left(\frac{2x}{2\sqrt{x^2+1/n^2}} + \frac{2x}{2\sqrt{x^2}}\right)}{n^4\left(\sqrt{x^2+\frac{1}{n^2}}+\sqrt{x^2}\right)^2} = -\frac{\frac{x}{\sqrt{x^2+1/n^2}} + \frac{x}{\sqrt{x^2}}}{n^2\left(\sqrt{x^2+\frac{1}{n^2}}+\sqrt{x^2}\right)^2}$$

$g'(x)=0 \Rightarrow x=0. \Rightarrow g'(x)<0. \Rightarrow g \downarrow$

$\Rightarrow x=0$ pct. de max.

$g(0)=\frac{1}{n}$

$$\Rightarrow \lim_{n\to\infty}\left(\sup \left|f_n(x)-|x|\right|\right) = \lim_{n\to\infty}\frac{1}{n} = 0.$$

$$f_n(x) \xrightarrow{u} |x|.$$

⑤ $f_n : (-\infty; 0) \to \mathbb{R}$, $f_n(x) = \dfrac{e^{nx}-1}{e^{nx}+1}$, $n \geq 0$

C.S.: $\lim\limits_{n\to\infty} f_n(x) = \lim \dfrac{e^{nx}-1}{e^{nx}+1} = -1 \Rightarrow f_n(x) \xrightarrow{s} -1.$

$\left(x \in (-\infty, 0) \Rightarrow \underset{(e^{-\infty})}{e^{nx}} \to 0 \right)$

C.U. $\sup\limits_{x\in(-\infty,0)} |f_n(x) - f(x)| = \sup\limits_{x\in(-\infty,0)} \left| \dfrac{e^{nx}-1}{e^{nx}+1} + 1 \right| =$

$= \sup\limits_{x\in(-\infty,0)} \dfrac{2e^{nx}}{e^{nx}+1}$

Fie $g : (-\infty, 0) \to \mathbb{R}$, $g(x) = \dfrac{2e^{nx}}{e^{nx}+1}$.

$g'(x) = \dfrac{2ne^{nx}(e^{nx}+1) - 2e^{nx} \cdot n \cdot e^{nx}}{(e^{nx}+1)^2} = \dfrac{2ne^{nx}}{(e^{nx}+1)^2} > 0$

$\Rightarrow g \nearrow$

$x \in (-\infty, 0) \Rightarrow \left.\begin{matrix} x < 0 \\ g \nearrow \end{matrix}\right| \Rightarrow g(x) < g(0)$

$\Rightarrow M = g(0) = 1.$

$\Rightarrow \lim\limits_{n\to\infty} \left(\sup\limits_{x\in(-\infty,0)} |f_n(x) + 1| \right) = 1 \neq 0 \Rightarrow f_n(x) \overset{u}{\not\to} -1.$

⑥ $f_n : [-1, 1] \to \mathbb{R}$, $f_n(x) = \frac{x}{nx^2+1}$, $n \geq 0$

C.S. $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{nx^2+1} = 0$, $x \in [-1, 1]$

$\Rightarrow f_n(x) \xrightarrow{s} 0.$

C.U. $\sup_{x\in[-1,1]} |f_n(x) - f(x)| = \sup_{x\in[-1,1]} |f_n(x) - 0| = \sup \frac{|x|}{nx^2+1}.$

Fie $g : [-1, 1] \to \mathbb{R}$, $g(x) = \frac{|x|}{nx^2+1}.$

$$g(x) = \begin{cases} \frac{-x}{nx^2+1} , & x \in [-1, 0) \\ \frac{x}{nx^2+1} , & x \in [0, 1] \end{cases}$$

$$g'_s(x) = -\frac{1\cdot(nx^2+1) - x\cdot 2nx}{(nx^2+1)^2} = -\frac{-nx^2+1}{(nx^2+1)^2} = \frac{nx^2-1}{(nx^2+1)^2}$$

$$g'(x) = \begin{cases} \frac{nx^2-1}{(nx^2+1)^2} , & x \in [-1, 0) \\ \frac{1-nx^2}{(nx^2+1)^2} , & x \in (0, 1] \end{cases}$$

$g'_s(0) = -1$; $g'_d(0) = 1$ $\Rightarrow g$ nu e deriv. în $x=0$

$g'_s = 0 \Rightarrow nx^2 = 1 \Rightarrow \underset{x\in[-1,0)}{x^2 = \frac{1}{n}} \Rightarrow x = -\frac{1}{\sqrt{n}}$

$g'_d = 0 \Rightarrow nx^2 = 1 \Rightarrow \underset{x\in(0,1]}{x^2 = \frac{1}{n}} \Rightarrow x = \frac{1}{\sqrt{n}}$

x	-1		$-\frac{1}{\sqrt{n}}$		0		$\frac{1}{\sqrt{n}}$		1
$g'(x)$		$+$	0	$-$	$\vert$	$+$	0	$-$	
$g(x)$		$\nearrow$	$g\left(-\frac{1}{\sqrt{n}}\right)$ M	$\searrow$	m	$\nearrow$	$g\left(\frac{1}{\sqrt{n}}\right)$ M	$\searrow$	

$$g\left(-\frac{1}{\sqrt{n}}\right) = \frac{\frac{1}{\sqrt{n}}}{n \cdot \frac{1}{n} + 1} = \frac{1}{2\sqrt{n}} \qquad \Rightarrow g(x) < \frac{1}{2\sqrt{n}}$$

$$g\left(\frac{1}{\sqrt{n}}\right) = \frac{\frac{1}{\sqrt{n}}}{n \cdot \frac{1}{\sqrt{n}}^2 + 1} = \frac{1}{2\sqrt{n}}$$

$$\lim_{n\to\infty} \left(\sup \left| f_n(x) - 0 \right| \right) = \lim_{n\to\infty} \frac{1}{2\sqrt{n}} = 0.$$

$$\Rightarrow f_n(x) \xrightarrow{u} 0.$$

(7) $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \operatorname{arctg} \frac{x}{1 + n(n+1)x^2}$, $n > 0$

C.S.: $\lim_{n\to\infty} f_n(x) = 0$, $x \in \mathbb{R}$ $\Rightarrow f_n(x) \xrightarrow{\wedge} 0$.

C.U.: Fie $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \left| f_n(x) - f(x) \right| = \operatorname{arctg} \frac{x}{1 + n(n+1)x^2}$

$$g'(x) = \frac{1}{\left(\frac{x}{1 + n(n+1)x^2} \right)^2 + 1} \cdot \frac{1 + n(n+1)x^2 - x \cdot 2x\, n(n+1)}{\left[1 + n(n+1)x^2 \right]^2}$$

$$g'(x) = \frac{1 - x^2 n(n+1)}{x^2 + \left[1 + n(n+1)x^2 \right]^2} = 0 \quad \Rightarrow x = \pm \frac{1}{\sqrt{n(n+1)}}$$

x	$-\infty$		$-\frac{1}{\sqrt{n(n+1)}}$		$\frac{1}{\sqrt{n(n+1)}}$		$+\infty$
$g'(x)$		$-$	0	$+$	0	$-$	
$g(x)$		$\searrow$	m	$\nearrow$	M	$\searrow$	

$$\lim_{n\to\infty} \left(\sup \left| f_n(x) - 0 \right| \right) = \lim_{n\to\infty} g\left(\frac{1}{\sqrt{n(n+1)}} \right) = \operatorname{arctg} \left(\frac{1}{2\sqrt{n(n+1)}} \right) \to 0$$

$\Rightarrow f_n(x) \xrightarrow{u} 0$

$$x^2+2x(n+1)+(n+1)=0$$

$$\Delta = 4(n+1)^2-4(n+1) = 4(n^2+2n+1-n-1) = 4(n^2+n)$$

$$x_{1,2} = \frac{-2(n+1) \pm 2\sqrt{n^2+n}}{2} = -(n+1) \pm \sqrt{n^2+n}$$

x	x_1		0	x_2		1	
$g'(x)$	$+$ 0	$-$	$-$	$-$ 0	$+$ $+$	$+$	$+$
$g(x)$			0 $\searrow$	m	$\nearrow$	M	

$$\Rightarrow M = g(1) = \frac{2}{n+2}. \quad \Rightarrow \sup_{x\in[0,1]} g(x) \geq g(1)$$

$$x_1 = -n-1-\sqrt{n^2+n} < 0.$$

$$x_2 = -n-1+\sqrt{n^2+n} < 1 \Leftrightarrow \sqrt{n^2+n} < n+2$$

$$n^2+n < n^2+4n+4. \text{ (A)}$$

$$\Rightarrow \lim_{n\to\infty}\left(\sup_{x\in[0,1]} |f_n(x)-x|\right) = \lim_{n\to\infty} \frac{2}{n+2} = 0.$$

$$\Rightarrow f_n(x) \xrightarrow{u} x.$$

(10) $f_n : \mathbb{R}\to\mathbb{R}$, $f_n(x) = x+\frac{1}{n}$, $n>0$.

conv. simplă și unif pt. $f_n(x)$ și $f_n^2(x)$.

c.s. $\lim_{n\to\infty} f_n(x) = x \Rightarrow f_n(x) \xrightarrow{s} x$

c.u. Fie $g : \mathbb{R}\to\mathbb{R}$, $g(x) = |f_n(x)-x| = \frac{1}{n}$

$$\lim_{n\to\infty}\left(\sup |f_n(x)-x|\right) = \lim_{n\to\infty}\frac{1}{n} = 0 \Rightarrow f_n(x) \xrightarrow{u} x$$

$$f_n^2(x) = \left(x + \frac{1}{n}\right)^2$$

C.S: $\lim\limits_{n\to\infty} f_n^2(x) = x^2 \Rightarrow f_n(x) \xrightarrow{s} x^2$

C.U. Fie $g:\mathbb{R}\to\mathbb{R}$, $g(x) = \left|f_n(x) - x^2\right| =$

$$= \left|\frac{2x}{n} + \frac{1}{n^2}\right| = \frac{|2nx+1|}{n^2}$$

$$g(x) = \begin{cases} \dfrac{2nx+1}{n^2}, & x \geq -\dfrac{1}{2n} \\ -\dfrac{2nx+1}{n^2}, & x < -\dfrac{1}{2n} \end{cases}$$

$g_d'(x) = \frac{2n}{n^2} = \frac{2}{n}$

$g_s'(x) = -\frac{2}{n}$

x	$-\infty$		$-\frac{1}{2n}$		$+\infty$
$g'(x)$		$-$	|	$+$	
$g(x)$		$\searrow$	m	$\nearrow$	M

$\Rightarrow \sup\limits_{x\in\mathbb{R}}(g(x)) \geq g(\infty) = \infty$

$$\Rightarrow f_n^2(x) \not\xrightarrow{u} x^2$$

(11) $\sum\limits_{n\geq 1} n^{-x}$, $x\in\mathbb{R}$ conv. simplă și C.U. și decideți dacă se poate deriva termen cu termen.

$f_n(x) = \frac{1}{n^x}$

C.S. $\sum \frac{1}{n^x}$ pt. $x > 1 \Rightarrow$ serie armonică gen. $\Rightarrow$ conv.

C.U. ! Crit. Weierstrass pt. $x>1$

$\sum f_n$ u.c, dacă $\exists \sum u_n$ conv. a.î. $|f_n(x)| \le u_n$

$\left|\frac{1}{n^x}\right| \le u_n = \frac{1}{n^\alpha}$, $\alpha > 1$

u_n conv. pe $[\alpha, \infty)$, $\alpha > 1$

$\Rightarrow \sum f_n$ u.c.

Pt. ca seria de fct. să se poată deriva termen cu termen se verifică dacă $f_n(x)$ este de clasă $\mathcal{C}^1$, adică

$\sum f_n \xrightarrow{u} f$ și $\sum f_n' \xrightarrow{u} g$ $\Rightarrow f' = g$

$\lim f_n(x) = f(x)$

Seria derivatelor $\sum\limits_{n\ge 0} f_n' = n^{-x} \ln n$

Crit. Weierstrass. $|n^{-x} \ln x| \le n^{-x} \cdot n = \frac{1}{n^{x-1}} = u_n$

pt. $x-1>1 \Rightarrow x>2$ u_n conv.

$\Rightarrow \sum\limits_{n\ge 0} f_n' \xrightarrow{u} g(x)$

C. Weierstrass $\sum f_n$ u.c. dacă $\sum u_n$ conv.

și $|f_n(x)| \le u_n$, $\forall x \in I$.

⑧ $f_n : [0, \infty) \to \mathbb{R}, \quad f_n(x) = x^n \cdot e^{-n}$

C.S. $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \dfrac{x^n}{e^n} = 0 \Rightarrow f_n(x) \xrightarrow{s} 0$

(fct. exp. este mult mai mare decât fct. polinomiale)

C.U. Fie $g : [0, \infty) \to \mathbb{R}$

$g(x) = |f_n(x) - 0| = x^n e^{-nx}$

$g'(x) = n x^{n-1} e^{-nx} + x^n \cdot (-n) e^{-nx} =$

$n x^{n-1} e^{-nx} (1 - x) = 0 \Rightarrow x = 0$

$1 - x = 0 \Rightarrow x = 1.$

x	0		1		∞
$g'(x)$	0	$+$	0	$-$	
$g(x)$		$\nearrow$	M	$\searrow$	

$g(1) = e^{-n}$

$\lim\limits_{n\to\infty} \left(\sup\limits_{x\in[0,\infty)} |f_n(x) - 0| \right) = \lim\limits_{n\to\infty} \dfrac{1}{e^n} = 0 \Rightarrow f_n(x) \xrightarrow{u} 0.$

⑨ $f_n : [0, 1] \to \mathbb{R}, \quad f_n(x) = \dfrac{nx}{1+n+x}, \quad n \geq 0.$

C.S. $\lim\limits_{n\to\infty} f_n(x) = x \Rightarrow f_n(x) \xrightarrow{s} x$

C.U. Fie $g : [0, 1] \to \mathbb{R}, \quad g(x) = |f_n(x) - x| =$

$\left| \dfrac{nx}{1+n+x} - x \right| = \left| \dfrac{-x - x^2}{1+n+x} \right| = \dfrac{x + x^2}{1+n+x}$

$g'(x) = \dfrac{(1+2x)(1+n+x) - (x+x^2)\cdot 1}{(1+n+x)^2} = \dfrac{1+n+\cancel{x}+2x+2xn+2x^2-\cancel{x}-x^2}{(1+n+x)^2}$

$= \dfrac{x^2 + 2x(n+1) + n+1}{(1+n+x)^2} = 0$

(13) $\sum_{n\geq 1} \frac{\sin(nx)}{n^3}$, $x \in \mathbb{R}$. C.U., propr. de transfer a cont.; se poate deriva term. cu termen?

<u>C.U.</u> | <u>Crit. Weierstrass.</u>

$\sum f_n$ u.c., dacă $\exists \sum u_n$ conv. a.î. $|f_n(x)| \leq u_n$, $\forall x \in I$.

$$\left|\frac{\sin(nx)}{n^3}\right| \leq \underbrace{\frac{1}{n^3} = u_n}$$

$\sum u_n$ serie armonică gen. $\alpha = 3 > 1 \Rightarrow$ conv.

$$\Rightarrow \sum_{n\geq 0} f_n \xrightarrow{u} f \Rightarrow \sum f_n \xrightarrow{s} f$$

<u>Transfer de cont</u> $f_n(x)$ cont. și $\sum_{n\geq 0} f_n \xrightarrow{u} f \Rightarrow f$ cont

$$\left.\begin{array}{l} f_n \text{ cont. ca fct. elem.} \\ \sum_{n\geq 0} f_n \xrightarrow{u} f \end{array}\right| \Rightarrow f \text{ cont.}$$

<u>Derivarea termen cu termen</u>

! $\sum f_n \xrightarrow{s} f$; $\sum f_n' \xrightarrow{u} g \Rightarrow f$ de clasă C_1 și $f' = g$

$\sum f_n \xrightarrow{s} f$ (am ar mai sus)

$\sum f_n' = \sum \frac{\cos nx}{n^2}$ Ar. cu <u>Crit. Weierstrass</u>

$\left|\frac{\cos nx}{n^2}\right| \leq \frac{1}{n^2} = u_n$ $\sum u_n$ conv. (serie arm. $\alpha = 2 > 1$)

$\Rightarrow \sum f_n' \xrightarrow{u} g$

(14) $\sum_{n\geq 1}(1-x)x^n$, $x\in(0,1)$ C.S și C.U.

C.S. | <u>CRIT. Rap.</u> $\sum_{n=0}^{\infty} u_n$ și $\exists \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \begin{cases} l<1 \Rightarrow \text{conv.} \\ l>1 \Rightarrow \text{div.} \\ l=1 \Rightarrow \text{nu se poate} \end{cases}$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{(1-x)x^{n+1}}{(1-x)x^n} = x\in(0,1) \Rightarrow \text{conv.}$$

$$\Rightarrow \sum_{n\geq 1} f_n \xrightarrow{s} f.$$

C.U. | <u>CRIT. Weierstrass.</u>

$\sum f_n$ u.c., dacă $\exists \sum_{n\geq 0} u_n$ conv. a.î.

$|f_n(x)| \leq u_n$, $\forall x\in I.$

$$|f_n(x)| = |(1-x)x^n| = (1-x)x^n \leq x^n = u_n$$

$\sum_{n\geq 1} u_n$ conv. (serie geom. cu $q = x\in(0,1)$)

$$\Rightarrow \sum f_n \text{ u.c.} \Rightarrow \sum_{n\geq 0} f_n \xrightarrow{u} f$$

(12) $\sum (-1)^n \frac{e^{-nx}+\sqrt{n}}{n}$ $x \in \mathbb{R}_+$ CS, Abs. conv.

CS: crit. Leibniz serii numerice

$\sum(-1)^n u_n$ $u_n \downarrow 0 \Rightarrow \sum(-1)^n u_n$ conv.

$$u_n = \frac{e^{-nx}+\sqrt{n}}{n} = \frac{1}{ne^{nx}} + \frac{1}{\sqrt{n}} \to 0, \quad x \geq 0.$$

$$u_{n+1} - u_n = \underbrace{\frac{1}{(n+1)e^{(n+1)x}} - \frac{1}{ne^{nx}}}_{<0} + \underbrace{\frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}}}_{<0} < 0$$

$\Rightarrow u_n \downarrow$

$\Rightarrow \sum(-1)^n \cdot \frac{e^{-nx}+\sqrt{n}}{n}$ conv.

$\Rightarrow \sum f_n \xrightarrow{s} f$

Abs. conv. $|f_n(x)| = \frac{1}{ne^{nx}} + \frac{1}{\sqrt{n}} > \underbrace{\frac{1}{\sqrt{n}}}_{\text{serie armonică divergentă}} = u_n$

$\alpha = 1/2$.

$\Rightarrow \nexists\, x \in \mathbb{R}$ pt. care $\sum f_n(x)$ să fie abs. conv.

Teorie

Abs. conv.: 1) $\sum u_n$ abs. conv., dc. $\sum |u_n|$ conv.

2) $\sum |u_n|$ div. $\Rightarrow \sum u_n$ nu e abs. conv.

C.U. | Crit. Weierstrass

$\sum f_n$ u.c. , dacă $\sum_{n\geq 0} u_n$ conv. a.î.

$|f_n(x)| \leq u_n$, $\forall x \in I$.

$|f_n(x)| = \frac{1}{n^2+n+1} \cdot x^n \leq x^n = u_n$

$\sum u_n$ conv. (serie geom. pt. $x \in (-1,1)$)

$\Rightarrow \sum f_n \xrightarrow{u} f$ pt. $x \in (-1,1)$.

(15) $\sum_{n\geq 1} \frac{1}{n^2+n+1} x^n$, $x \in [-1,1]$ CS și U.C.

C.S. | Crit. comp. cu ineg. $u_n \leq v_n$

i) $\sum v_n$ conv. $\Rightarrow \sum u_n$ conv.

ii) $\sum u_n$ div. $\Rightarrow \sum v_n$ div.

$\boxed{x \in (-1,1)}$ $u_n = \frac{1}{n^2+n+1} \cdot x^n \leq x^n = v_n$.

$\sum v_n = x^n$ conv. (serie geom. cu $q = x \in (-1,1)$, deci conv.)

$\boxed{x=-1}$ $\Rightarrow$ serie alternantă $\sum (-1)^n \cdot \frac{1}{n^2+n+1}$.

Crit. Leibniz

$\sum_{n\geq 0} (-1)^n u_n$; $u_n \downarrow 0 \Rightarrow \sum (-1)^n u_n$ conv.

$u_n = \frac{1}{n^2+n+1} \to 0$.

$\frac{u_{n+1}}{u_n} = \frac{n^2+n+1}{(n+1)^2+(n+1)+1} < 1 \Rightarrow u_n \downarrow$

$\Rightarrow \sum (-1)^n u_n$ conv

$\Rightarrow \sum (-1)^n \frac{1}{n^2+n+1}$ conv.

$\boxed{x=1}$ $\Rightarrow \sum_{n\geq 0} \frac{1}{n^2+n+1}$.

! CRIT comp. la lim. $\lim \frac{u_n}{v_n} = \ell \neq 0, \infty$ $\Rightarrow \sum u_n, \sum v_n$ au ac. nat

Fie $v_n = \frac{1}{n^2}$; $\sum v_n$ - conv. (serie armonică $\alpha = 2 > 1 \Rightarrow$ conv.)

$\Rightarrow \lim \frac{u_n}{v_n} = \lim \frac{n^2}{n^2+n+1} = 1 \neq 0 \Rightarrow \sum u_n, \sum v_n$ au ac nat. $\Rightarrow \sum u_n$ conv.

$\Rightarrow \sum f_n \xrightarrow{u} f$