

Probabilități și Statistică
Enăscută Roxane - Adele
Grupe 26/27
Examen - 30.05.2022

Exercițiu 1

a) $P(X=-1, Y=7) = 0.045$

$$\Rightarrow P(X=-1) \cdot P(Y=7 | X=-1) = 0.045$$

$$0.18 \cdot P(Y=7 | X=-1) = 0.045$$

$$0.18 \cdot P_2 = 0.045 \Leftrightarrow P_2 = 0.25$$

$$P_1 = 1 - P_2 = 0.75$$

$$Y \sim \begin{pmatrix} -5 & 7 \\ 0.45 & 0.25 \end{pmatrix}$$

• $X+Y \sim \begin{pmatrix} -6 & 6 & 2 & 14 \\ 0.135 & 0.045 & 0.615 & 0.205 \end{pmatrix}$

$$P(X=x, Y=y) = P_x \cdot P_y$$

• $X-Y \sim \begin{pmatrix} 4 & -8 & 12 & 0 \\ 0.135 & 0.045 & 0.615 & 0.205 \end{pmatrix}$

• $X^2 \sim \begin{pmatrix} 1 & 49 \\ 0.18 & 0.82 \end{pmatrix} \quad Y^2 \sim \begin{pmatrix} 25 & 49 \\ 0.75 & 0.25 \end{pmatrix}$

• $4X^2 + 7Y^2 \sim \begin{pmatrix} 149 & 347 & 371 & 539 \\ 0.135 & 0.045 & 0.615 & 0.205 \end{pmatrix}$

• $E[X] = (-1) \cdot 0.18 + 7 \cdot 0.82 = 5.56$

• $E[Y] = (-5) \cdot 0.75 + 7 \cdot 0.25 = -2$

$$\bullet \text{Var}(X) = E[X^2] - E[X]^2 = 9.4464$$

$$\bullet \text{Var}(Y) = 27$$

$$\bullet \text{Var}(7X - 2Y + 8) = \text{Var}(7X) + \text{Var}(-2Y) + \text{Var}(8)$$

$$= 49\text{Var}(X) + 4\text{Var}(Y)$$

$$= 570.8736$$

$$\bullet \rho = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)} \cdot \sqrt{\text{Var}(Y)}}$$

$$\text{Cov}(X, Y) = 0 \Rightarrow \rho(X, Y) = 0.$$

Exercitie 2

$$1. E[\log(X)] \neq \log(E[X])$$

~~\log este definit (X pozitiv)~~

$$\cancel{X > 0 \Rightarrow E[X] > 0}$$

$$\cancel{\text{Obtin } X - E[X] \Rightarrow X > \log X}$$

$\log(X)$ -concavă
din ineq lui Jensen
 $\Rightarrow E[\log X] \leq \log E[X]$

$$2. E[X] \geq \sqrt{E[X]}$$

$$\cancel{X > 0 \Rightarrow E[X] > 0 \Rightarrow E[X] \text{ definit}}$$

$$\cancel{\text{Obtin } X - E[X] = \cancel{X} \geq \sqrt{X}, \forall X > 0}$$

$\sqrt{}$ - convex pt $E[X] \geq 1$

din ineq lui Jensen $\Rightarrow E[X] \geq \sqrt{E[X]}$

$$3. \mathbb{E}[\sin^2(x)] + \mathbb{E}[\cos^2(x)] = 1$$

$$\mathbb{E}[\sin^2(x)] + \mathbb{E}[\cos^2(x)] = \mathbb{E}[\sin^2(x) + \cos^2(x)] = \mathbb{E}[1] = 1$$

$$4. P(X > c) \leq \frac{\mathbb{E}(X^3)}{c^3}$$

din uinegalitățile lui Markov

$$5. P(X \leq Y) = P(X \geq Y)$$

$X \perp\!\!\! \perp Y$, și nu avem altă informație
înțelesă, deci probabilitățile sunt aceeași

$$6. P(X+Y > 10) \leq P(X > 5 \text{ sau } Y > 5)$$

Există cazuri în care $X > 5$ sau $Y > 5$,
dar $X+Y$ nu e mai mare decât 10

$$7. \mathbb{E}[\min(X, Y)] \quad ? \quad \min(\mathbb{E}(X), \mathbb{E}(Y))$$

depinde de $P(X), P(Y)$

$$8. \mathbb{E}\left[\frac{X}{Y}\right] \geq \frac{\mathbb{E}(X)}{\mathbb{E}(Y)}$$

$$\frac{\mathbb{E}(X)}{\mathbb{E}(Y)} = \frac{x \cdot P(X)}{Y \cdot P(Y)} + \frac{X}{Y} \cdot \frac{P(X)}{P(Y)} \quad | \cdot \mathbb{E} \Rightarrow$$

$$\Rightarrow \mathbb{E}\left[\frac{X}{Y}\right] \cdot E\left[\frac{P(X)}{P(Y)}\right] = \mathbb{E}\left[\frac{X}{Y}\right] \cdot \frac{P(X)}{P(Y)} \underset{=const}{\leq} 0 \geq \mathbb{E}\left[\frac{X}{Y}\right]$$

$$9. \mathbb{E}[x^2(x^2+1)] ? \mathbb{E}[x^2(y^2+1)]$$

$$\mathbb{E}[x^2(x^2+1)] = \mathbb{E}[x^2] \cdot \mathbb{E}[x^2+1]$$

$$\mathbb{E}[x^2(y^2+1)] = \mathbb{E}[x^2] \mathbb{E}[y^2+1]$$

$\mathbb{E}[x^2]$ nu reduce în răsuflare de comparat
 $\mathbb{E}[x^2+1]$ și $\mathbb{E}[y^2+1]$, dar nu suntem relațiile
 dintre x și y deci nu le putem compara.

$$10. \mathbb{E}\left[\frac{1}{x}\right] ? \frac{1}{\mathbb{E}[x]}$$

$$\mathbb{E}[x^{-1}] ? \mathbb{E}[x]^{-1}$$

"moment ~~există~~ de ordin -1" - nu ~~există~~

Exercițiu 3

$$a) 2 \leq X+Y \leq 6$$

Hui - telefoanele sunt defecte

$$P(X=1, Y=1) = \frac{2}{7} \cdot \frac{1}{6} = \frac{2}{42} = \frac{1}{21}$$

$$P(X=1, Y=2) = \frac{2}{7} \cdot \frac{8}{6} \cdot \frac{1}{8} = \frac{2}{42} = \frac{1}{21}$$

$$P(X=1, Y=3) = \frac{2}{7} \cdot \frac{8}{6} \cdot \frac{21}{8} = \frac{21}{42} = \frac{1}{21}$$

$$P(X=1, Y=4) = \frac{2}{7} \cdot \frac{8}{6} \cdot \frac{31}{8} = \frac{31}{42} = \frac{1}{21}$$

$$P(X=1, Y=5) = \frac{2}{7} \cdot \frac{8}{6} \cdot \frac{4}{8} = \frac{8}{42} = \frac{2}{21}.$$

Restul rezultatelor: (rep comună):

$X \setminus Y$	0	1	2	3	4	5
1	0	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{2}{21}$
2	0	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{2}{21}$	0
3	0	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{2}{21}$	0	0
4	0	$\frac{1}{21}$	$\frac{2}{21}$	0	0	0
5	$\frac{1}{21}$	$\frac{2}{21}$	0	0	0	0

Rep marginale:

$$X \sim \left(\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \frac{6}{21} & \frac{5}{21} & \frac{4}{21} & \frac{3}{21} & \frac{3}{21} \end{array} \right)$$

$$Y \sim \left(\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \\ \frac{1}{21} & \frac{6}{21} & \frac{5}{21} & \frac{4}{21} & \frac{3}{21} & \frac{2}{21} \end{array} \right)$$

$$\text{a)} E[X] = \frac{6+10+12+12+15}{21} = \frac{55}{21} = 2,61$$

$$E[Y] = \frac{6+10+12+12+10}{21} = \frac{50}{21} = 2,38$$

$$X^2 \sim \left(\begin{array}{ccccc} 1 & 4 & 9 & 16 & 25 \\ \frac{6}{21} & \frac{5}{21} & \frac{4}{21} & \frac{3}{21} & \frac{3}{21} \end{array} \right)$$

$$Y^2 \sim \left(\begin{array}{ccccc} 0 & 1 & 4 & 9 & 16 & 25 \\ \frac{1}{21} & \frac{6}{21} & \frac{5}{21} & \frac{4}{21} & \frac{3}{21} & \frac{2}{21} \end{array} \right)$$

$$\mathbb{E}[X^2] = \frac{6+20+36+48+75}{21} = 8,20$$

$$\mathbb{E}[Y^2] = \frac{6+20+36+48+50}{21} = 7,61$$

$$\text{Var}(X) = E[X^2] - \mathbb{E}[X]^2 = 1,99$$

$$\text{Var}(Y) = 1,95.$$

$$S(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}}$$

$$\mathbb{E}[XY] = \sum xy \cdot P(X=x, Y=y)$$

$$= 1 \cdot 0 \cdot 0 + 1 \cdot 1 \cdot \frac{1}{21} + 1 \cdot 2 \cdot \frac{1}{21} + 1 \cdot 3 \cdot \frac{1}{21} + \dots$$

$$= 5$$

$$\text{Cov}(X,Y) = 5 - 1,99 \cdot 1,95 = 1,12$$

$$S(X,Y) = \frac{1,12}{\sqrt{1,99} \cdot \sqrt{1,95}} = 0,57$$

c) $\mathbb{E}[X|Y=2] =$

$$X|Y=2 \sim \left(\begin{matrix} 1 \\ \frac{1}{5/21} \end{matrix} \quad \begin{matrix} 2 \\ \frac{1}{5/21} \end{matrix} \quad \begin{matrix} 3 \\ \frac{1}{5/21} \end{matrix} \quad \begin{matrix} 4 \\ \frac{2}{5/21} \end{matrix} \quad \begin{matrix} 5 \\ \frac{0}{5/21} \end{matrix} \right) =$$

$$= \left(\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} & 0 \end{matrix} \right)$$

$$\mathbb{E}[X | Y=2] = \frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{8}{5} + 0 = \frac{14}{5} = 2,8$$

$$\text{Var}(X | Y=2) = E[(X | Y=2)^2] - \mathbb{E}[X | Y=2]^2$$

$$(X | Y=2)^2 \sim \begin{pmatrix} 1 & 4 & 9 & 16 & 25 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} & 0 \end{pmatrix}$$

$$\mathbb{E}[(X | Y=2)^2] = \frac{1+4+9+32}{5} = 9,2$$

$$\text{Var}(X | Y=2) = 9,2 - 2,8^2 = 1,36$$

Exercitiul 4

$$f(x) = \frac{x}{64} e^{-\frac{x^2}{128}} \quad \forall x \geq 0$$

$$\begin{aligned} F(x) &= \int_{-\infty}^x f(t) dt \quad \Rightarrow \quad F(x) = \int_0^x f(t) dt = \\ &= \int_0^x \frac{t}{64} e^{-\frac{t^2}{128}} dt = -e^{-\frac{t^2}{128}} \Big|_0^x \\ &= -e^{-\frac{x^2}{128}} + 1 = 1 - e^{-\frac{x^2}{128}} - \frac{y^2}{128} \end{aligned}$$

$$F^{-1}(x) = y \quad | F \Rightarrow x = f(y) = 1 - e^{-\frac{y^2}{128}}$$

$$e^{-\frac{y^2}{128}} = 1-x$$

$$-\frac{y^2}{128} = \ln(1-x)$$

$$y = \sqrt{128 \ln\left(\frac{1}{1-x}\right)} = \sqrt{2 \ln\left(\frac{1}{1-x}\right)} \Rightarrow$$

$$\Rightarrow F^{-1}(x) = 8\sqrt{2 \ln \left(\frac{1}{1-x}\right)}$$

$$F^{-1}\left(\frac{3}{5}\right) = 8\sqrt{2 \ln \frac{1}{2}}, \quad F^{-1}\left(\frac{1}{3}\right) = 8\sqrt{2 \ln \frac{1}{2}}$$

$$\text{Var}(x) = E[x^2] - E[x]^2$$

$$E[x] = \int_0^\infty x f(x) dx = \int_0^\infty x \frac{x}{64} e^{-\frac{x^2}{128}} dx$$

$$= \int_0^\infty (-x) \left(e^{-\frac{x^2}{128}}\right)' dx = \frac{-xe^{-\frac{x^2}{128}}}{0} \Big|_0^\infty + \int_0^\infty xe^{-\frac{x^2}{128}} dx$$

$$= \int_0^\infty (-64) \left(e^{-\frac{x^2}{128}}\right)' dx = -64 \left[e^{-\frac{x^2}{128}}\right]_0^\infty + 64 \int_0^\infty e^{-\frac{x^2}{128}} dx$$

$$= -64 + 64 \cdot 8 \cdot \frac{\int_0^\infty e^{-\frac{y^2}{2}} dy}{\sqrt{\frac{2\pi}{2}}}$$

$$= -64 + \frac{512\sqrt{2\pi}}{2}$$

$$E[x^2] = \int_0^\infty x^2 f(x) dx = \int_0^\infty (-x^2) \cdot \left(e^{-\frac{x^2}{128}}\right)' dx$$

$$= -x^2 \cdot \left.e^{-\frac{x^2}{128}}\right|_0^\infty + 64^2 \int_0^\infty (-1) \left(e^{-\frac{x^2}{128}}\right)' dx$$

$$= 64^2 \cdot (-1) \cdot \left.e^{-\frac{x^2}{128}}\right|_0^\infty + 64^2 \int_0^\infty e^{-\frac{x^2}{128}} dx$$

$$= -64^2 + 64^2 \cdot 8 \cdot \frac{\sqrt{2\pi}}{2}$$

$$\text{Var}(X) = -64^2 + 64^2 \cdot 8 \cdot \frac{\sqrt{2\pi}}{2} + 64^2 - 64 \cdot 8 \cdot \frac{\sqrt{2\pi}}{2} +$$

$$+ \frac{2 \cdot 64^2 \cdot 8 \cdot \sqrt{2\pi}}{2}$$

$$\text{Var}(X) = -2 \cdot 64^2 + \frac{128}{2} (64 \cdot 8 - 512 + 2 \cdot 64^2 \cdot 8)$$

Înlocuim și calculăm raportul de către

Exercițiu 5

N - nr alegătorilor, $N \sim \text{Pois}(906)$

$$906 = 9$$

X - alegătorii care votază Debanu

Y - alegătorii care votază Cițu

p - probabilitatea ca un aleg să voteze Cițu

$$a) P(X=i, Y=j) = P(X=i | Y=j) P(Y=j | N=i+j) P(N=i+j)$$

$$P(X=i, Y=j | N=i+j) = P(X=i | N=i+j) P(Y=j | N=i+j)$$

$$P(X=i, Y=j) = \binom{i+j}{i} p^i (1-p)^j e^{-9} \frac{9^{i+j}}{(i+j)!} =$$

$$= \frac{(i+j)!}{i! j!} p^i (1-p)^j e^{-9} \frac{9^{i+j}}{(i+j)!} =$$

$$= \frac{e^{-\lambda p} (\lambda p)^i}{i!} \cdot e^{-\lambda(1-p)} \cdot \frac{\lambda(1-p)^j}{j!}$$

$$= \frac{e^{-906 \cdot 0.64} (906 \cdot 0.64)^i}{i!} \cdot e^{-906 \cdot 0.36} \cdot \frac{906 \cdot 0.36^j}{j!}$$

Repr marginalen:

$$P(X=i) = \sum_{j \geq 0} P(X=i, Y=j)$$

$$= \sum_{j \geq 0} e^{-\lambda p} \frac{(\lambda p)^i}{i!} \cdot e^{-\lambda(1-p)} \cdot \frac{\lambda(1-p)^j}{j!} = e^{-\lambda p} \frac{(\lambda p)^i}{i!}$$

$$\Rightarrow X \sim \text{Pois}(\lambda p) \quad (= X \sim \text{Pois}(906 \cdot 0.64))$$

$$\text{analog } Y \sim \text{Pois}(\lambda(1-p)) \quad (= Y \sim \text{Pois}(906 \cdot 0.36))$$

b) $P(X=i, Y=j) = P(X=i) P(Y=j) \Rightarrow X \perp\!\!\!\perp Y$

c) $E[X] = \lambda = 906$

$\text{Var}(Y) = \lambda = 906$

Exercițiu 6

X - nr de succese înainte de al 4-lea eșec

$$P(X=k) = \binom{k-1}{3} \cdot (1-p)^{k-4} \cdot p^4, k \geq 4$$

$$E[X] = \frac{0.33 \cdot 4}{0.67} = 1,97$$

$$E[15X - 9] = 15 \cdot 1,97 - 9 = 20,55$$

$$\text{Var}(X) = \frac{0,33 \cdot 4}{(0,67)^2} = 3$$

$$\text{Var}(2X + 7) = 4 \text{Var}(X) = 4 \cdot 3 = 12$$