SERII DE PUTERI - CURS 4

① $\sum_{n\geq 1} (-1)^{n+1} \cdot \frac{x^n}{n}$ R și mulț. de conv.

$a_n = \frac{(-1)^{n+1}}{n}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+2}}{n+1} \cdot \frac{n}{(-1)^{n+1}} \right| =$

$= \lim_{n\to\infty} \frac{n}{n+1} = 1 \Rightarrow R = \frac{1}{L} \Rightarrow R = 1.$

Pt. $x = -1 \Rightarrow \sum_{n\geq 1} (-1)^{n+1} \cdot \frac{(-1)^n}{n} = \sum -\frac{1}{n}$ div.

serie armon. cu $\alpha = 1 \Rightarrow$ div.

Pt. $x = 1 \Rightarrow \sum_{n\geq 1} (-1)^{n+1} \cdot \frac{1}{n}$

C. Leibniz: $\sum_{n\geq 0} (-1)^n \cdot u_n$, $u_n \downarrow 0 \Rightarrow \sum (-1)^n u_n$ conv.

$u_n = \frac{1}{n} \to 0$

$\frac{u_{n+1}}{u_n} = \frac{n}{n+1} < 1 \Rightarrow u_n \downarrow$

$\Rightarrow \sum (-1)^{n+1} \frac{1}{n}$ conv.

$\Rightarrow$ Mulț. de conv. $(-1; 1]$

(4)

$$\sum_{n\geq 0} \frac{n+1}{\sqrt{n^4+n^3+1}} \left(\frac{x+1}{2x+3}\right)^n, \quad x \neq -\frac{3}{2}$$

Notam $y = \left(\frac{x+1}{2x+3}\right)^n \Rightarrow \sum \frac{n+1}{\sqrt{n^4+n^3+1}} y^n$

$$a_n = \frac{n+1}{\sqrt{n^4+n^3+1}}$$

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{n+2}{\sqrt{(n+1)^4+(n+1)^3+1}} \cdot \frac{\sqrt{n^4+n^3+1}}{n+1} =$$

$$= \lim \underbrace{\frac{n+2}{n+1}}_{\substack{\text{au ac grad}\\ \Rightarrow l=1}} \cdot \underbrace{\sqrt{\frac{n^4+n^3+1}{(n+1)^4+(n+1)^3+1}}}_{\substack{\text{au ac. grad}\\ \Rightarrow l=1}} = 1 \Rightarrow R = \frac{1}{L} = 1.$$

pt $y = 1 \Rightarrow \sum \frac{n+1}{\sqrt{n^4+n^3+1}}$

! C. Comp. la lim. $\lim \frac{u_n}{v_n} = L \neq 0; \infty \Rightarrow \sum u_n, \sum v_n$ au ac nat.

$$\lim \frac{n+1}{\sqrt{n^4+n^3+1}} \cdot \frac{n}{1} = \lim \frac{n^2+n}{n^2\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}} = 1 \neq 0. \Bigg| \Rightarrow$$

$v_n = \frac{1}{n} \Rightarrow \sum v_n$ - div. (serie armon. $\alpha = 1 \Rightarrow$ div.)

$\sum u_n$ div.

pt. $y = -1 \Rightarrow \sum (-1)^n \frac{n+1}{\sqrt{n^4+n^3+1}}$

! C. Leibniz $\sum (-1)^n u_n$; $u_n \searrow 0 \Rightarrow \sum (-1)^n u_n$ conv.

$$u_n = \frac{n+1}{\sqrt{n^4+n^3+1}} \to 0$$

$$\frac{u_{n+1}}{u_n} = \frac{n+2}{n+1} \cdot \sqrt{\frac{n^4+n^3+1}{n^4+4n^3+6n^2+4n+1+n^3+3n^2+3n+1+1}} =$$

$$= \sqrt{\frac{(n^2+4n+4)(n^4+n^3+1)}{(n^2+2n+1)(n^4+5n^3+9n^2+7n+3)}}$$

$$= \sqrt{\frac{n^6+n^5+n^2+4n^5+4n^4+4n+4n^4+4n^3+4}{n^6+5n^5+9n^2+7n^3+3n^2+2n^5+10n^4+18n^3+14n^2+6n+n^4+5n^3+9n^2+7n+3}}$$

$$= \sqrt{\frac{n^6+5n^5+8n^4+4n^3+n^2+4n+4}{n^6+7n^5+11n^4+30n^3+35n^2+13n+3}} < 1.$$

$\Rightarrow u_n \downarrow$

$\Rightarrow \sum (-1)^n u_n$ conv.

$\Rightarrow$ Mult. de conv. pt. seria în y este $[-1;1)$

$$\Rightarrow \quad -1 \le \frac{x+1}{2x+3} < 1$$

$$\begin{cases} \dfrac{x+1}{2x+3} - 1 < 0 \\ \dfrac{x+1}{2x+3} + 1 \ge 0. \end{cases} \Leftrightarrow \begin{cases} \dfrac{x+1-2x-3}{2x+3} < 0 \\ \dfrac{x+1+2x+3}{2x+3} \ge 0 \end{cases} \Leftrightarrow$$

$$\begin{cases} \dfrac{-x-2}{2x+3} < 0 \\ \dfrac{3x+4}{2x+3} \ge 0 \end{cases} \Leftrightarrow \begin{cases} \dfrac{x+2}{2x+3} \ge 0 \\ \dfrac{3x+4}{2x+3} \ge 0 \end{cases}$$

x		-2		$-\frac{3}{2}$	
$x+2$	$-$	0	$+$		$+$
$2x+3$	$-$	$-$	$-$	0	$+$
fc.	$+$	0	$-$	$\vert$	$+$

$x \in (-\infty; -2) \cup (-\frac{3}{2}; \infty)$

x			$-\frac{3}{2}$		$-4/3$		
$3x+4$	$-$	$-$	$-$	$-$	0	$+$	$+$
$2x+3$	$-$	$-$	0	$+$	$+$	$+$	$+$
fc	$+$		$\vert$	$-$	0	$+$	

$x \in (-\infty, -\frac{3}{2}) \cup [-\frac{4}{3}, \infty)$

"$\cap$" $\Rightarrow$ $\boxed{x \in (-\infty, -2) \cup [-\frac{4}{3}; \infty)}$

⑤ $$\sum_{n\geq 0} (-1)^n \frac{\sqrt{n^2+1}}{n^2+n+1}\left(\frac{4\cancel{x}-1}{x+3}\right)^n$$

Notăm $y = \frac{4x-1}{x+3} \Rightarrow \sum (-1)^n \frac{\sqrt{n^2+n}}{n^2+n+1} \cdot y^n$

$$a_n = (-1)^n \frac{\sqrt{n^2+n}}{n^2+n+1}.$$

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{\sqrt{(n+1)^2+(n+1)}}{(n+1)^2+(n+1)+1} \cdot \frac{n^2+n+1}{\sqrt{n^2+n}}$$

$$= \lim \sqrt{\frac{n^2+3n+2}{n^2+n}} \cdot \frac{n^2+n+1}{n^2+3n+3} = 1\cdot 1 = 1.$$

$$\Rightarrow R = \frac{1}{L} = 1.$$

Pt. $y=1 \Rightarrow \sum (-1)^n \frac{\sqrt{n^2+1}}{n^2+n+1}.$

! $\underline{\text{c. Leibniz}}$ $\sum (-1)^n u_n$, $u_n \downarrow 0 \Rightarrow \sum (-1)^n u_n$ conv.

$$u_n = \frac{n\sqrt{1+1/n^2}}{n^2\left(1+\frac{1}{n}+\frac{1}{n^2}\right)} \to 0$$

$$\frac{u_{n+1}}{u_n} = \sqrt{\frac{n^2+3n+2}{n^2+n}} \cdot \frac{n^2+n+1}{n^2+3n+3} = \sqrt{\frac{(n+\cancel{1})(n+2)}{n(n+\cancel{1})}} \cdot \frac{n^2+n+1}{n^2+3n+3}$$

$$= \sqrt{\frac{(n^2+n+1)^2(n+2)}{n(n^2+3n+3)^2}} = \sqrt{\frac{(n^4+n^2+1+2n^3+2n^2+2n)(n+2)}{n(n^4+9n^2+9+6n^3+18n+6n^2)}}$$

$$= \sqrt{\frac{(n^4+2n^3+3n^2+2n+1)(n+2)}{(n^4+6n^3+15n^2+18n+9)\,n}} = \sqrt{\frac{n^5+2n^4+2n^4+4n^3+3n^3+6n^2+2n^2+4n+n+2}{n^5+6n^4+15n^3+18n^2+9n}} < 1 \Rightarrow u_n \downarrow \Rightarrow \sum (-1)^n u_n \text{ conv.}$$

Pt. $y=-1$. $\Rightarrow \sum\limits_{n\geq 0} \frac{\sqrt{n^2+1}}{n^2+n+1}$

! C. comp. la lim. $\lim \frac{u_n}{v_n} \neq 0, \infty \Rightarrow \sum u_n, \sum v_n$ au ac. nat

$\lim \frac{u_n}{v_n} = \lim \frac{\sqrt{n^2+1}}{n^2+n+1} \cdot \frac{n}{1} = 1 \neq 0; \infty \Rightarrow$ seriile au ac. nat

$v_n = \frac{1}{n} \Rightarrow \sum \frac{1}{n}$ div. (serie arm. $\alpha=1 \Rightarrow$ div)

$\Rightarrow \sum \frac{\sqrt{n^2+1}}{n^2+n+1}$ div.

$\Rightarrow$ Mulț de conv. ~~pt~~. seria în y este $(-1; 1]$

$\Rightarrow \quad -1 < \frac{4x-1}{x+3} \leq 1.$

$\begin{cases} \frac{4x-1}{x+3} - 1 \leq 0 \\ \frac{4x-1}{x+3} + 1 > 0 \end{cases} \Leftrightarrow \begin{cases} \frac{4x-1-x-3}{x+3} \leq 0 \\ \frac{4x-1+x+3}{x+3} > 0 \end{cases} \Leftrightarrow \begin{cases} \frac{3x-4}{x+3} \leq 0 \\ \frac{5x+2}{x+3} > 0 \end{cases}$

x			-3		$\frac{4}{3}$	
$3x-4$	$-$	$-$		$-$	0	$+$
$x+3$	$-$	$-$	0	$+$	$+$	$+$
fr_1	$+$		$\mid$	$-$	0	$+$

$x \in (-3; \frac{4}{3}]$

x			-3		$-\frac{2}{5}$	
$5x+2$	$-$	$-$		$-$	0	$+$ $+$
$x+3$	$-$	$-$	0	$+$ $+$		$+$ $+$
fr_2	$+$		$\mid$	$-$	0	$+$

$x \in (-\infty, -3) \cup (-\frac{2}{5}; \infty)$

$\overset{\text{„}\cap\text{"}}{\Rightarrow} \boxed{x \in \left(-\frac{2}{5}; \frac{4}{3}\right]}$.

⑥ $\sum_{n\geq 0} (-1)^n \dfrac{1}{3^{\frac{n}{2}} \cdot \sqrt{1+n^2}} \cdot tg^n x \quad , x \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$

Notam

$y = tg\,x \Rightarrow \sum (-1)^n \dfrac{1}{3^{n/2}\sqrt{1+n^2}} y^n$ cu

$a_n = (-1)^n \cdot \dfrac{1}{3^{n/2}\sqrt{1+n^2}}$

$\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \dfrac{3^{n/2}\sqrt{1+n^2}}{3^{\frac{n+1}{2}} \underbrace{\sqrt{1+(n+1)^2}}_{\text{au ac. gr.} \Rightarrow 1}} \right| = \dfrac{1}{\sqrt{3}} \Rightarrow R = \dfrac{1}{\ell} = \sqrt{3}$

Pt. $\underline{y = \sqrt{3}} \Rightarrow \sum (-1)^n \dfrac{1}{\sqrt{1+n^2}}$

! $\underline{\text{C. Leibniz}}$ $\sum (-1)^n u_n$, $u_n \downarrow 0 \Rightarrow \sum (-1)^n u_n$ conv.

$u_n = \dfrac{1}{\sqrt{1+n^2}} \to 0$

$\dfrac{u_{n+1}}{u_n} = \dfrac{\sqrt{1+n^2}}{\sqrt{1+(n+1)^2}} < 1 \Rightarrow u_n \downarrow$

$\Rightarrow \sum (-1)^n u_n$ $\underline{\text{conv.}}$

Pt. $\underline{y = -\sqrt{3}} \Rightarrow \sum \dfrac{1}{\sqrt{1+n^2}}$

! $\underline{\text{C. comp. la lim.}}$ $\lim \dfrac{u_n}{v_n} \neq 0; \infty \Rightarrow \sum u_n, \sum v_n$ au ac nat.

$\lim \dfrac{u_n}{v_n} = \dfrac{n}{\sqrt{1+n^2}} = 1 \neq 0 \Rightarrow$ seriile au ac nat

$v_n = \dfrac{1}{n} \Rightarrow \sum \dfrac{1}{n}$ div (serie arm. cu $\alpha = 1 \Rightarrow$ div.)

$\Rightarrow$ $\sum u_n$ $\underline{\text{div.}}$

Mult. de conv. pt. seria în y este $(-\sqrt{3}; \sqrt{3}]$

$\Rightarrow -\sqrt{3} < tg\, x \le \sqrt{3} \mid arctg(\,)$ care este fct. $\uparrow$

$\Rightarrow -\frac{\pi}{3} < x \le \frac{\pi}{3} \Rightarrow \left(-\frac{\pi}{3}; \frac{\pi}{3}\right]$.

(7) $\sum_{n\ge 1} \frac{n^{n+\frac{1}{n}}}{\left(n+\frac{1}{n}\right)^n} x^n$

$a_n = \frac{n^{n+1/n}}{\left(n+\frac{1}{n}\right)^n} \Rightarrow \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{(n+1)^{n+1+\frac{1}{n+1}}}{\left(n+1+\frac{1}{n+1}\right)^{n+1}} \cdot$

$\frac{\left(n+\frac{1}{n}\right)^n}{n^{n+\frac{1}{n}}} = \frac{(n+1)^{\cancel{n+1}+\frac{1}{n+1}}}{\cancel{(n+1)^{n+1}}\left(1+\frac{1}{(n+1)^2}\right)^{(n+1)^2\cdot\frac{1}{(n+1)^2}\cdot\cancel{(n+1)}}} \cdot$

$\frac{\cancel{n^n}\left(1+\frac{1}{n^2}\right)^{n^2\cdot\frac{1}{n^2}\cdot n}}{n^{\cancel{n}+\frac{1}{n}}} = \frac{\sqrt[n+1]{n+1}}{\sqrt[n]{n}} \cdot \frac{\left[\left(1+\frac{1}{n^2}\right)^{n^2}\right]^{\frac{1}{n}}}{\left[\left(1+\frac{1}{(n+1)^2}\right)^{(n+1)^2}\right]^{\frac{1}{n+1}}}$

$= \frac{\sqrt[n+1]{n+1}}{\sqrt[n]{n}} \cdot \frac{e^0}{e^0} = \frac{\sqrt[n+1]{n+1}}{\sqrt[n]{n}} = \frac{1}{1}$.

$\lim \sqrt[n]{n} = \lim \frac{n+1}{n} = 1$.

! CRIT RADICALULUI pt. șiruri

$\lim \sqrt[n]{x_n} = \lim \frac{x_{n+1}}{x_n}$

$\Rightarrow R = 1$.

Pt. $x = -1$. $\Rightarrow \sum (-1)^n \frac{n^{n+\frac{1}{n}}}{\left(n+\frac{1}{n}\right)^n}$

C. Nec. conv. $\lim u_n \ne 0 \Rightarrow \sum$ div.

$\lim_{n\to\infty} \frac{n^{n+1/n}}{n^n\left(1+\frac{1}{n^2}\right)^n} = \frac{n^{1/n}}{\left[\left(1+\frac{1}{n^2}\right)^{n^2}\right]^{\frac{1}{n^2}\cdot n}} = \frac{n^{\frac{1}{n}}}{[\;]^{\frac{1}{n}}} =$

$$\sqrt[n]{\frac{n}{\left(1+\frac{1}{n^2}\right)^{n^2}}} = (*)$$

CRIT RAD. pt. șiruri

$$\boxed{\lim \sqrt[n]{u_n} = \lim \frac{u_{n+1}}{u_n}}$$

$$\lim \frac{n+1}{\left[1+\frac{1}{(n+1)^2}\right]^{(n+1)^2}} \cdot \frac{\left(1+\frac{1}{n^2}\right)^{n^2}}{n} =$$

$$\frac{n+1}{n} \cdot \frac{\left(1+\frac{1}{n^2}\right)^{n^2}}{\left[1+\frac{1}{(n+1)^2}\right]^{(n+1)^2}} = 1 \cdot \frac{e}{e} = 1.$$

$$\Rightarrow (*) = 1. \neq 0 \Rightarrow \underline{\sum u_n \text{ div.}}$$

Pt. $\underline{x=1.}$ $\Rightarrow \sum\limits_{n\geq 1} \frac{n^{n+\frac{1}{n}}}{\left(n+\frac{1}{n}\right)^n}$

C. Nec. conv

$\lim u_n \neq 0$

$\Rightarrow \sum$ div.

$$\lim \frac{n^{n+\frac{1}{n}}}{\cancel{n^n}\left(1+\frac{1}{n^2}\right)^{n^2 \cdot \frac{1}{n^2} \cdot n}} = \frac{\sqrt[n]{n}}{\left[\left(1+\frac{1}{n^2}\right)^{n}\right]^{\frac{1}{n}}} \longrightarrow \frac{1}{e^0} = 1 \neq 0$$

$\Rightarrow \sum u_n$ div.

$\Rightarrow$ Mult. de conv. $(-1;1)$.

⑧ $\sum_{n\geq 1} \frac{n+2}{n^2+1}(x-2)^n$

Not. $y = x-2 \Rightarrow \sum \frac{n+2}{n^2+1} y^n$

$\Rightarrow a_n = \frac{n+2}{n^2+1}$.

$\Rightarrow \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{n+3}{(n+1)^2+1} \cdot \frac{n^2+1}{n+2} =$

$\lim \frac{n+3}{n+2} \cdot \frac{n^2+1}{n^2+2n+2} = 1. \Rightarrow R = \frac{1}{L} = 1.$

Pt. $y=1 \Rightarrow \sum \frac{n+2}{n^2+1}$.

! Crit. comp. la lim. $\lim \frac{u_n}{v_n} \neq 0, \infty \Rightarrow$ seriile au ac. nat

$\lim_{n\to\infty} \frac{n+2}{n^2+1} \cdot \frac{n}{1} = 1 \neq 0 \Rightarrow$ seriile au ac nat

$v_n = \frac{1}{n} \Rightarrow \sum \frac{1}{n}$ div. (serie armonică gen. $\alpha = 1 \Rightarrow$ div.)

$\Rightarrow \sum \frac{n+2}{n^2+1}$ div.

Pt $y=-1 \Rightarrow \sum (-1)^n \frac{n+2}{n^2+1}$.

C. Leibniz $\sum (-1)^n u_n$, $u_n \searrow 0 \Rightarrow \sum (-1)^n u_n$ conv.

$u_n = \frac{n+2}{n^2+1} \to 0$

$\frac{u_{n+1}}{u_n} = \frac{n+3}{n^2+2n+2} \cdot \frac{n^2+1}{n+2} = \frac{n^3+3n^2+n+3}{n^3+2n^2+4n^2+4n+2n+4} < 1 \Rightarrow u_n \searrow$

$\Rightarrow \sum(-1)^n u_n$ conv.

$\Rightarrow$ Mult. de conv. pt. seria în y este $[-1;1)$

$\Rightarrow -1 \le y-2 < 1 \mid +2 \Rightarrow 1 \le y < 3$.

$\Rightarrow [1;3)$. mult. de conv.

(9) $\sum_{n\ge 0} (-1)^n \frac{x^{2n+1}}{2n+1}$ mult. de conv. pt fiecare serie

$a_n = (-1)^n \cdot \frac{1}{2n+1}$

$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{1}{2n+3} \cdot \frac{2n+1}{1} = 1 \Rightarrow R = 1$.

Pt. $x=1 \Rightarrow \sum(-1)^n \cdot \frac{1}{2n+1}$.

! C. Leibniz $\sum(-1)^n u_n$, $u_n \downarrow 0 \Rightarrow \sum(-1)^n u_n$ conv.

$u_n = \frac{1}{2n+1} \to 0$

$\frac{u_{n+1}}{u_n} = \frac{2n+1}{2n+3} < 1 \Rightarrow \downarrow$

$\Rightarrow \sum(-1)^n \frac{1}{2n+1}$ conv.

Pt. $x=-1 \Rightarrow \sum + \frac{(-1)^{n+1}}{2n+1} = \sum(-1)^{n+1} \cdot \frac{1}{2n+1}$.

C. Leibniz $\Rightarrow$ conv.

$\Rightarrow$ Mult. de conv. $[-1;1]$

(9) Suma seriei $f : [-1, 1] \to \mathbb{R}$

$f(x) = \sum_{n\geq 0} (-1)^n \cdot \frac{x^{2n+1}}{2n+1}$ se derivează termen cu termen $\Rightarrow f'(x) = \sum_{n\geq 0} (-1)^n \cdot x^{2n}$

$S_n = u_0 + u_1 + \dots + u_n$ şirul sumelor parţiale

$S = \lim S_n$. suma seriei

$$f'(x) = 1 - x^2 + x^4 - x^6 + \dots + (-1)^n x^{2n} + \dots = \frac{1}{1+x^2}$$

Integrăm termen cu termen

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + \dots = \operatorname{arctg} x$$

suma seriei

(10) $$\sum_{n\geq 1} \frac{(-1)^{n-1}}{n(2n-1)} \cdot \left(\frac{1-x}{1-2x}\right)^{2n-1}$$

Notăm $y = \frac{1-x}{1-2x} \Rightarrow \sum \frac{(-1)^{n-1}}{n(2n-1)} \cdot y^{2n-1}$.

$$a_n = \frac{(-1)^{n-1}}{n(2n-1)}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^n}{(n+1)(2n+1)} \cdot \frac{n(2n-1)}{(-1)^{n-1}}\right| =$$

$$\lim_{n\to\infty} \frac{n(2n-1)}{(n+1)(2n+1)} = 1. \Rightarrow R = 1.$$

- $y=1 \Rightarrow \sum \frac{(-1)^{n-1}}{n(2n-1)}$

! C. Leibniz $u_n \downarrow 0 \Rightarrow \sum (-1)^n u_n$ conv.

$u_n = \frac{1}{n(2n-1)} \to 0$.

$\frac{u_{n+1}}{u_n} = \frac{n(2n-1)}{(n+1)(2n+1)} < 1. \Rightarrow u_n \downarrow$

$\Rightarrow \sum (-1)^{n-1} \cdot u_n$ conv.

- $y=-1 \Rightarrow \sum (-1)^n \cdot \frac{1}{n(2n-1)}$ conv. (C.L.)

$\Rightarrow$ Mult. de conv. $[-1;1]$

$-1 \le \frac{1-x}{1-2x} \le 1$

$\begin{cases} \frac{1-x}{1-2x} + 1 \ge 0 \\ \frac{1-x}{1-2x} - 1 \le 0. \end{cases} \Leftrightarrow \begin{cases} \frac{2-3x}{1-2x} \ge 0 \\ \frac{x}{1-2x} \le 0 \end{cases}$

x		$\frac{1}{2}$		$\frac{2}{3}$	
$2-3x$	$+$	$+$	$+$	0	$-$
$1-2x$	$+$	0	$-$	$-$	$-$
fr.	$+$	$\mid$	$-$	0	$+$

$x \in (-\infty, \frac{1}{2}) \cup [\frac{2}{3}; \infty)$

x		0		$\frac{1}{2}$	
x	$-$	0	$+$		$+$
$1-2x$	$+$	$+$	$+$	0	$-$
$\frac{x}{1-2x}$	$-$	0	$+$	$\mid$	$-$

$\Rightarrow x \in (-\infty; 0] \cup (\frac{1}{2}; \infty)$

$\overset{\cap}{\Rightarrow} (-\infty; 0] \cup [\frac{2}{3}; \infty)$.

Suma seriei în y este $f: [-1, 1] \to \mathbb{R}$,

$f(y) = \sum_{n \ge 1} \frac{(-1)^{n-1}}{n(2n-1)} y^{2n-1}$

$f'(y) = \sum \frac{(-1)^{n-1}}{n} y^{2n-2} \quad | \cdot y^2 \Rightarrow y^2 f'(y) = \sum \frac{(-1)^{n-1}}{n} y^{2n}$

Derivăm din nou

$$2yf'(y) + y^2 f''(y) = \sum (-1)^{n+1} \cdot 2y^{2n-1}$$

$$[y^2 f'(y)]' = 2y \sum (-1)^{n+1} \cdot y^{2n-2}$$

$$— " — \quad = 2y \sum (-y^2)^{n-1}$$

$$— " — \quad = 2y \cdot \frac{1}{1-(-y^2)}$$

(serie geom. $S = \frac{1}{1-x}$, $x = -y^2$)

$$[y^2 f'(y)]' = \frac{2y}{1+y^2} \quad \Big| \int$$

$$\Rightarrow y^2 f'(y) = \ln(1+y^2) + \mathscr{C} \Rightarrow f'(y) = \frac{\ln(1+y^2)}{y^2}$$

pt. $y = 0 \Rightarrow \mathscr{C} = 0$.

$$\Rightarrow f(y) = \int \frac{\ln(1+y^2)}{y^2} dy = \int \left(-\frac{1}{y}\right)' \ln(1+y^2)\, dy =$$

$$-\frac{1}{y} \ln(1+y^2) + \int \frac{1}{\cancel{y}} \cdot \frac{2\cancel{y}}{1+y^2} \Rightarrow$$

$$f(y) = -\frac{1}{y} \ln(1+y^2) + 2 \operatorname{arctg} y + K$$

!? Se det $K \Rightarrow K = 0$.

$$\Rightarrow f(y) = -\frac{\ln(1+y^2)}{y} + 2 \operatorname{arctg} y$$

$$\Rightarrow f(x) = \frac{1-2x}{x-1} \cdot \ln\left[1 + \left(\frac{1-x}{1-2x}\right)^2\right] + 2 \operatorname{arctg} \frac{1-x}{1-2x}$$

suma seriei

$x \in (-\infty; 0] \cup [\frac{2}{3}; \infty)$

(11) $\sum \frac{2^n}{2n+1} \cdot x^n$

$a_n = \frac{2^n}{2n+1}$

$\lim \left| \frac{a_{n+1}}{a_n} \right| = \lim \left| \frac{2^{n+1}}{2n+3} \cdot \frac{2n+1}{2^n} \right| = 2.$

$\Rightarrow R = \frac{1}{2}$

$x = \frac{1}{2} \Rightarrow \sum \frac{1}{2n+1} \sim \sum \frac{1}{n}$ div.

c. comp. la lim $\frac{u_n}{v_n} = \frac{n}{2n+1} = \frac{1}{2} \neq 0, \infty$

$\Rightarrow$ seriile au ac. nat

$\Rightarrow$ div.

$x = -\frac{1}{2} \Rightarrow \sum (-1)^n \frac{1}{2n+1}$

c. Leibniz $u_n = \frac{1}{2n+1} \to 0$

$\frac{u_{n+1}}{u_n} = \frac{2n+1}{2n+3} < 1 \Rightarrow u_n \searrow$ $\Big| \Rightarrow \sum (-1)^n \frac{1}{2n+1}$ conv

Mult. de conv. $\left[-\frac{1}{2}; \frac{1}{2}\right)$

Suma seriei: $f : \left[-\frac{1}{2}; \frac{1}{2}\right) \to \mathbb{R}$

$f(x) = \sum_{n \geq 0} \frac{1}{2n+1} (2x)^n$

Pt. $x \geq 0$, not. $2x = t^2$, $t \in (0,1) \Rightarrow f(t) = \sum \frac{1}{2n+1} \cdot t^{2n} \Big/ \cdot t$

$$t f(t) = \sum \frac{1}{2n+1} t^{2n+1} \quad \Big| \; ( )'$$

$$[t f(t)]' = \underbrace{\sum t^{2n}}_{\text{serie geom.}} \qquad \left(S = \frac{1}{1-x}, \; x = t^2 \right)$$

$$[t f(t)]' = \frac{1}{1-t^2} \quad \Big| \int$$

$$t f(t) = \frac{1}{2} \ln \frac{1+t}{1-t} \Rightarrow f(t) = \frac{1}{2t} \ln \frac{1+t}{1-t}$$

$$t \in (0,1)$$

$$\Rightarrow f(x) = \frac{1}{2\sqrt{2x}} \ln \frac{1+\sqrt{2x}}{1-\sqrt{2x}}, \; x \in \left[0, \frac{1}{2}\right)$$

pt. $x < 0$, not. $2x = -t^2$, $t \in (-1, 0)$

$$f(t) = \sum \frac{(-1)^n}{2n+1} \cdot t^{2n} \quad \Big| \cdot t$$

$$t f(t) = \sum \frac{(-1)^n t^{2n+1}}{2n+1} \quad \Big| \; ( )'$$

$$[t f(t)]' = \sum (-1)^n t^{2n}$$

$$[t f(t)]' = \frac{1}{1+t^2} \quad \Big| \int.$$

$$t f(t) = \operatorname{arctg} t \Rightarrow f(t) = \frac{1}{t} \operatorname{arctg} t, \; t \in (-1, 0)$$

$$f(x) = \frac{1}{\sqrt{-2x}} \operatorname{arctg} \sqrt{-2x}, \quad x \in \left[-\frac{1}{2}; 0\right)$$

(12) $\sum_{n\geq 1} \frac{n}{n+1} \cdot x^n$

$a_n = \frac{n}{n+1}.$

$\lim \left| \frac{a_{n+1}}{a_n} \right| = \lim \frac{n+1}{n+2} \cdot \frac{n+1}{n} = 1. \Rightarrow R=1$

$x=1 \Rightarrow \sum \frac{n}{n+1}$

C. Nec. conv. $\lim u_n = 1 \neq 0 \Rightarrow$ div.

$x=-1 \Rightarrow \sum (-1)^n \frac{n+1-1}{n+1} = \underbrace{\underbrace{\sum(-1)^n}_{div.} - \underbrace{\sum(-1)^n \frac{1}{n+1}}_{conv.}}_{div.}$

C. Leibniz $u_n = \frac{1}{n+1} \Rightarrow 0.$

$\frac{u_{n+1}}{u_n} = \frac{n+1}{n+2} < 1 \Rightarrow \downarrow$ $\Bigg| \Rightarrow$ conv.

$\Rightarrow$ Mult. de conv. $(-1;1)$

Suma seriei $f:(-1;1) \to \mathbb{R}$

$f(x) = \sum_{n\geq 1} \frac{n}{n+1} x^n = \sum x^n - \sum \frac{1}{n+1} x^n \quad \Big| \cdot x$

$x f(x) = \underbrace{\sum x^{n+1}}_{\frac{x}{1-x}} - \sum \frac{1}{n+1} x^{n+1} \quad \Big| ()'$

$[x f(x)]' = \left(\sum x^{n+1}\right)' - \underbrace{\sum x^n}_{\text{serie geom.}}$ $\quad S = \frac{1}{1-x}.$

$[x f(x)]' = \frac{1-x+x}{(1-x)^2} - \frac{1}{1-x} \quad \Big| \int$

$$x f(x) = +\frac{1}{1-x} + \ln(1-x) + \mathscr{C}.$$

$$f(x) = \frac{1}{x(1-x)} + \frac{1}{x}\ln(1-x) + \frac{1}{x}\mathscr{C}.$$

$$x=0 \Rightarrow 0 = 1 + \mathscr{C} \Rightarrow \mathscr{C} = -1.$$

$$f(x) = \begin{cases} \frac{1}{x(1-x)} + \frac{1}{x}\left[\ln(1-x) - 1\right], & x \in (-1,1)\setminus\{0\} \\ 0, & x = 0 \end{cases}$$