SERII - CURS 2

Natura seriilor:

① $\sum_{n=0}^{\infty} \left(\frac{3n}{3n+1}\right)^n$

$u_n = \left(\frac{3n}{3n+1}\right)^n$ și $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(\frac{3n}{3n+1}\right)^n =$

$= \lim_{n\to\infty} \left(1 + \frac{3n}{3n+1} - 1\right)^n = \lim_{n\to\infty} \left(1 + \frac{3n-3n-1}{3n+1}\right)^n =$

$= \lim_{n\to\infty} \left[\left(1 + \frac{-1}{3n+1}\right)^{\frac{3n+1}{-1}}\right]^{\frac{-1}{3n+1}\cdot n} = e^{\lim_{n\to\infty} \frac{-n}{n(3+\frac{1}{n})}}$

$= e^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{e}} \neq 0 \Rightarrow$ seria este divergentă

Am aplicat: Crit. necesar de conv.

$\sum u_n$ conv. $\Rightarrow \lim_{n\to\infty} u_n = 0$.

sau

$\lim u_n \neq 0 \Rightarrow \sum u_n$ divergentă.

② $\sum_{n=1}^{\infty} n^a(\sqrt{n+1}-\sqrt{n})$, $a\in\mathbb{R}$

crit. comparației la limită

$$\lim \frac{u_n}{v_n} = \ell \begin{cases} \ell=0, \ \sum v_n \text{ conv.} \Rightarrow \sum u_n \text{ conv.} \\ \ell=\infty, \ \sum v_n \text{ div.} \Rightarrow \sum u_n \text{ div.} \\ \ell\neq 0 \text{ și } \ell\neq\infty \Rightarrow \sum u_n, \sum v_n \text{ au aceeași natură} \end{cases}$$

$$\sum_{n=1}^{\infty} n^a \frac{\sqrt{n+1}+\sqrt{n}}{}(\sqrt{n+1}-\sqrt{n}) = \sum_{n=1}^{\infty}\frac{n^a}{\sqrt{n+1}+\sqrt{n}}$$

$$u_n = \frac{n^a}{\sqrt{n+1}+\sqrt{n}} \ ; \quad v_n = n^{a-\frac{1}{2}} = \frac{n^a}{\sqrt{n}}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty}\frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}\left(\sqrt{1+\frac{1}{n}}+1\right)} = \frac{1}{2}$$

$\Rightarrow \ell=\frac{1}{2} \Rightarrow \sum u_n$ și $\sum v_n$ au aceeași natură

$$\boxed{\text{seria armonică generalizată } \sum \frac{1}{n^\alpha} \text{ este } \begin{cases} \text{conv. } \alpha>1 \\ \text{div } \alpha\leq 1 \end{cases}}$$

$$-\left(a-\frac{1}{2}\right)>1 \Rightarrow a-\frac{1}{2}<-1 \Rightarrow a<-\frac{1}{2} \Rightarrow \text{conv.}$$

$$-\left(a-\frac{1}{2}\right)\leq 1 \Rightarrow a-\frac{1}{2}\geq -1 \Rightarrow a\geq -\frac{1}{2} \Rightarrow \text{div.}$$

⑤ $\sum_{n=1}^{\infty} \arcsin \frac{n+1}{2n+3}$

! <u>Crit. necesar de conv.</u>

$\lim u_n \neq 0 \Rightarrow \sum u_n$ div.

$\lim_{n\to\infty} \arcsin \frac{n+1}{2n+3} = \arcsin\left(\lim_{n\to\infty} \frac{n(1+\frac{1}{2})}{n(2+\frac{3}{n})}\right) = \arcsin \frac{1}{2} =$

$= \frac{\pi}{6} \neq 0 \Rightarrow \sum u_n$ div.

⑥ $\sum_{n=1}^{\infty} (\operatorname{arctg} 1)^n = \sum_{n=1}^{\infty} \left(\frac{\pi}{4}\right)^n$

! serie geometrică $\sum q^n \Rightarrow \begin{cases} |q| < 1 \Rightarrow \text{conv.} \\ |q| \geq 1 \Rightarrow \text{div.} \end{cases}$

$q = \frac{\pi}{4} < 1 \Rightarrow$ conv.

⑦ $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sin \frac{1}{n}}$

! <u>Crit necesar de conv.</u>

$\lim u_n \neq 0 \Rightarrow \sum u_n$ div.

$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \left|\frac{(-1)^n}{\sin \frac{1}{n}}\right| = \left|\frac{1}{0}\right| = \infty \neq 0$

$\Rightarrow \sum u_n$ div.

③ $\sum_{n=0}^{\infty} \frac{1}{n!}$

Crit. raportului

$\sum_{n=0}^{\infty} u_n$ cu term. poz. și $\exists \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \ell$

- $\ell < 1 \Rightarrow$ conv.
- $\ell > 1 \Rightarrow$ div.
- $\ell = 1 \Rightarrow$ nu se poate decide

$u_n = \frac{1}{n!} \Rightarrow \lim \frac{u_{n+1}}{u_n} = \lim \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim \frac{n!}{(n+1)!} =$

$= \lim \frac{1}{n+1} = 0$

$\ell = 0 < 1 \Rightarrow$ conv.

④. $\sum_{n=0}^{\infty} \frac{1}{n\sqrt{n+1}}$

seria armonică generalizată $\sum \frac{1}{n^{\alpha}}$

$u_n = \frac{1}{n\sqrt{n+1}}$ $\quad v_n = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$ $\left(\alpha = \frac{3}{2} > 1 \Rightarrow \text{conv.}\right)$

Crit. comp. la lim.

$\lim \frac{u_n}{v_n} = \ell$

- $\ell = 0$, $\sum v_n$ conv $\Rightarrow \sum u_n$ conv.
- $\ell = \infty$, $\sum v_n$ div $\Rightarrow \sum u_n$ div.
- $\ell \neq 0$ și $\neq \infty \Rightarrow$ au ac. natură

$\ell = \lim\left(\frac{1}{n\sqrt{n+1}} : \frac{1}{n\sqrt{n}}\right) = 1 \neq 0 \Rightarrow$ au ac. nat.

$\sum v_n$ conv $\Rightarrow \sum u_n$ conv.

(10) $\sum_{n=1}^{\infty} \frac{n!\,3^n}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}$

CRIT. RAPORTULUI

$\sum_{n=0}^{\infty} u_n$, $\exists \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \ell$
- $\ell<1 \Rightarrow \sum u_n$ conv.
- $\ell>1 \Rightarrow$ div.
- $\ell=1 \Rightarrow$ nu se poate

$u_n = \frac{n!\,3^n}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!\,3^{n+1}}{1\cdot 3\cdot \ldots \cdot (2n+1)}}{\frac{n!\,3^n}{1\cdot 3\cdot \ldots \cdot (2n-1)}} =$$

$$= \lim_{n\to\infty} \frac{(n+1)\cdot 3}{(2n+1)} = \frac{3}{2} > 1 \Rightarrow \text{div.}$$

(11) $\sum_{n=1}^{\infty} \frac{1}{7^n+3^n}$

$u_n = \frac{1}{7^n+3^n} < \frac{1}{7^n} = v_n$

Crit. comp. cu ineg. $u_n \le v_n$
- $\sum v_n$ conv. $\Rightarrow \sum u_n$ conv.
- $\sum u_n$ div $\Rightarrow \sum v_n$ div.

$\sum v_n = \sum \left(\frac{1}{7}\right)^n$ serie geom.

$q = \frac{1}{7} \in (-1,1) \Rightarrow$ conv.

$\sum v_n$ conv. $\Rightarrow \sum u_n$ conv.

⑧ $\sum_{n=1}^{\infty}\left(\frac{2}{n}-\frac{1}{2^n}\right)$

$\sum \frac{2}{n} = 2\underbrace{\sum \frac{1}{n}}$

seria armonică

pt. $\alpha = 1 \Rightarrow$ div.

$\sum \frac{1}{2^n}$ serie geom.

$q = \frac{1}{2} \in (-1;1) \Rightarrow$ conv.

Operații cu serii

$\left.\begin{array}{l}\sum u_n \text{ div.}\\ \sum v_n \text{ conv.}\end{array}\right| \Rightarrow u_n - v_n \to$ divergentă

$$\boxed{\left.\begin{array}{l}\sum u_n \text{ conv.}\\ \sum v_n \text{ div.}\end{array}\right| \Rightarrow \sum(\alpha \cdot u_n + \beta v_n) \text{ div.}}$$

⑨ $\sum_{n=1}^{\infty}\left(\frac{n+1}{3n+1}\right)^n$

Crit. radicalului

$\sum u_n$ și $\lim \sqrt[n]{u_n} = l$ $\begin{cases} l<1 \Rightarrow \text{conv.} \\ l>1 \Rightarrow \text{div} \\ l=1 \Rightarrow \text{nu se poate sp.}\end{cases}$

$u_n = \left(\frac{n+1}{3n+1}\right)^n \Rightarrow \lim_{n\to\infty} \sqrt[n]{\left(\frac{n+1}{3n+1}\right)^n} = \lim \frac{\cancel{n}(1+\frac{1}{n})}{\cancel{n}(3+\frac{1}{n})} = \frac{1}{3} < 1$

$\Rightarrow$ conv.

(12) $\sum_{n=1}^{\infty} \frac{1+2^n+5^n}{3^n} = \sum\left(\frac{1}{3}\right)^n + \sum\left(\frac{2}{3}\right)^n + \sum\left(\frac{5}{3}\right)^n$

serii geometrice

$\underbrace{q_1=\frac{1}{3} \quad q_2=\frac{2}{3}}_{q_1, q_2 \in (-1,1)} \quad \underbrace{q_3=\frac{5}{3}>1}_{\text{div.}}$

$\Rightarrow$ conv.

! Op. cu serii

$\sum u_n$ conv., $\sum x_n$ div $\Rightarrow \sum \alpha u_n + \beta x_n$ div.

$\Rightarrow \sum_{n=1}^{\infty} \frac{1+2^n+5^n}{3^n}$ div.

(13) $\sum \frac{1}{n} \sqrt[n]{(n+1)(n+2)\dots(n+n)}$

$\lim u_n = \lim_{n\to\infty} \sqrt[n]{\frac{(n+1)(n+2)\dots(n+n)}{n\cdot n\cdot \dots n}} = \sqrt[n]{\frac{(2n)!}{n!\; n^n}}$

! $\left[\begin{array}{l}\text{CRIT. RĂDĂCINII} \\ x_n>0,\ \frac{x_{n+1}}{x_n}\to \ell\end{array}\right| \Rightarrow \lim \sqrt[n]{x_n} = \lim \frac{x_{n+1}}{x_n} = \ell.$

! CRIT necesar de conv.

$\lim_{n\to\infty} u_n \neq 0 \Rightarrow \sum u_n$ div.

$\lim \frac{(2n+2)!}{(n+1)!\,(n+1)^{n+1}} \cdot \frac{n!\,n^n}{(2n)!}$

$= \lim \frac{(2n+1)(2n+2)}{(n+1)^2}\left(\frac{n}{n+1}\right)^n$

$= \lim \frac{2(2n+1)}{n+1} \cdot \left[\left(1+\frac{-1}{n+1}\right)^{\frac{n+1}{-1}}\right]^{\frac{-1}{n+1}\cdot n} = 4e^{-1} = \frac{4}{e}$

$\Rightarrow \lim u_n = \frac{4}{e} \neq 0.$

$\Rightarrow \sum u_n$ div.

(14) $\sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{e}\right)^n = \sum\left(-\frac{3}{e}\right)^n$

serie geom. $q = -\frac{3}{e} < -1 \Rightarrow$ div.

(15) $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^n$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n = \lim_{n\to\infty} \left[\left(1+\frac{-1}{n}\right)^{\frac{n}{-1}}\right]^{\cdot\frac{-1}{n}\cdot n}$

$= e^{-1} = \frac{1}{e} \neq 0 \Rightarrow$ div.

CRIT. necesar de conv.

$\lim_{n\to\infty} u_n \neq 0 \Rightarrow \sum u_n$ div.

(16) $\sum_{n=1}^{\infty} \frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{(n!)^2}\cdot\left(\frac{n}{6}\right)^{n+a}$, $a>0$

CRIT. RAPORTULUI

$\sum u_n$ şi $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \ell$ — $\ell<1 \Rightarrow$ conv.; $\ell>1 \Rightarrow$ div; $\ell=1 \Rightarrow$ nu se poate

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\frac{1\cdot3\cdot5\cdot\ldots\cdot(2n+1)}{(n+1)!^2}\cdot\left(\frac{n+1}{6}\right)^{n+1+a}}{\frac{1\cdot3\cdot\ldots\cdot(2n-1)}{(n!)^2}\cdot\left(\frac{n}{6}\right)^{n+a}} =$$

$$= \lim \frac{(2n+1)}{(n+1)^2}\cdot\frac{n+1}{6}\cdot\left(\frac{n+1}{n}\right)^{n+a} = \frac{1}{3}\cdot\lim\left[\left(1+\frac{1}{n}\right)^n\right]^{\cdot\frac{n+a}{n}}$$

$= \frac{e}{3} < 1 \Rightarrow$ conv.

(18) $\sum_{n=1}^{\infty} \sqrt{n} \ln\left(1+\frac{1}{n}\right)$

$u_n = \sqrt{n} \ln\left(1+\frac{1}{n}\right)$

CRIT. COMP. LA LIM. $\lim \frac{u_n}{v_n} = \ell$.

$\ell = 0$, $\sum v_n$ - conv $\Rightarrow \sum u_n$ conv.

$\ell = \infty$, $\sum v_n$ - div $\Rightarrow \sum u_n$ div

$\ell \neq 0, \infty \Rightarrow \sum v_n, \sum u_n$ au ac nat

Fie $v_n = \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$ $\quad \alpha = \frac{1}{2} \leq 1 \Rightarrow \sum v_n$ div.

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sqrt{n} \ln\left(1+\frac{1}{n}\right)}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} n \ln\left(1+\frac{1}{n}\right) =$$

$$= \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = 1 \neq 0. \Rightarrow \sum u_n, \sum v_n \text{ au ac natură}$$

$\Rightarrow \sum u_n$ divergentă

(17) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

CRIT. INTEGRAL

$f:(0,\infty) \to [0,\infty)$ ↘ și $a_n = \int_1^n f(t)\,dt$

$\Rightarrow \sum f(n)$ conv. $\Leftrightarrow a_n$ conv.

$f:(2,\infty) \to [0,\infty)$ $f(t) = \frac{1}{t \ln t}$.

$$f'(t) = -\frac{(t\ln t)'}{(t\ln t)^2} = -\frac{\ln t + 1}{(t\ln t)^2}$$

$t > 2 \Rightarrow \ln t > 0. \Rightarrow f'(t) < 0.$

$\Rightarrow f$ ↘

$$a_n = \int_2^n \frac{1}{t\ln t}\,dt = \int_{\ln 2}^{\ln n} \frac{1}{x}\,dx$$

$\ln t = x$

$\frac{1}{t}\,dt = dx$

$t = 1 \Rightarrow x = 0$

$t = n \Rightarrow x = \ln n$

$$a_n = \ln x \Big|_{\ln 2}^{\ln n} = \ln(\ln n) - \ln(\ln 2) =$$

$= \ln\left(\frac{\ln n}{\ln 2}\right) \longrightarrow \infty$ $\Rightarrow$ divergentă

$\Rightarrow \sum f(n)$ divergentă

19 $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{\ln(n+1)}$

CRIT necesar de conv.

! $\lim u_n \neq 0 \Rightarrow \sum u_n$ div.

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\ln(n+1)} = \lim \frac{1}{\frac{\ln(n+1)}{\sqrt{n}}} = \lim \frac{1}{\frac{\ln\left(n\left(1+\frac{1}{n}\right)\right)}{\sqrt{n}}} =$$

$$= \lim \frac{1}{\frac{\ln n}{\sqrt{n}} + \frac{1}{\sqrt{n}}\ln\left(1+\frac{1}{n}\right)} = \lim \frac{1}{\frac{\ln \sqrt{n}^2}{\sqrt{n}} + \ln\left(1+\frac{1}{n}\right)^{\frac{1}{\sqrt{n}}}} =$$

$$= \lim \frac{1}{2\frac{\ln\sqrt{n}}{\sqrt{n}} + \ln\left[\left(1+\frac{1}{n}\right)^{n}\right]^{\frac{1}{n}\cdot\frac{1}{\sqrt{n}}}} =$$

$$= \frac{1}{2\cdot 0 + \ln e^{0}} = \frac{1}{0} = \infty \neq 0 \Rightarrow \text{div.}$$

20 $\sum_{n=1}^{\infty} \frac{2^{\frac{1}{n}}}{n^2}$

! Crit. comp. la lim

$\sum u_n$ și $\sum v_n$; $\lim \frac{u_n}{v_n} = \ell$

- $\ell = 0$, $\sum v_n$ conv $\Rightarrow \sum u_n$ conv.
- $\ell = \infty$, $\sum v_n$ div $\Rightarrow \sum u_n$ div
- $\ell \neq 0$, $\ell \neq \infty$: $\sum u_n, \sum v_n$ au ac. nat.

$u_n = \frac{2^{\frac{1}{n}}}{n^2}$ și $v_n = \frac{1}{n^2}$

$\lim \frac{u_n}{v_n} = \lim 2^{\frac{1}{n}} = 2^{\frac{1}{\infty}} = 2^0 = 1 \neq 0 \Rightarrow$ seriile au ac. natură

$\sum v_n = \sum \frac{1}{n^2}$, serie armonică $\alpha = 2 > 1 \Rightarrow \sum v_n$ conv. $\Rightarrow \sum u_n$ conv.

$\sum \frac{2^{\frac{1}{n}}}{n^2}$

CRIT integral $f:(0,\infty) \to [0,\infty)$ $\searrow$, fie

$a_n = \int_1^n f(t)\,dt$

$\sum f(n)$ conv. $\Leftrightarrow$ a_n conv.

$f:(1,\infty) \to [0,\infty)$ $\quad f(t) = \frac{2^{\frac{1}{t}}}{t^2}$

$$f' = \frac{2^{\frac{1}{t}} \cdot \left(-\frac{1}{t^2}\right) \cdot \ln 2 \cdot t^2 - 2^{\frac{1}{t}} \cdot 2t}{t^4} = -\frac{2^{\frac{1}{t}}(\ln 2 + t)}{t^4} < 0$$

$\Rightarrow f \searrow$

$$a_n = \int_1^n 2^{\frac{1}{t}} \cdot \frac{1}{t^2}\,dt = -\int_1^n 2^{\frac{1}{t}} \left(+\frac{1}{t}\right)' dt$$

$\frac{1}{t} = x \Rightarrow \left(\frac{1}{t}\right)' dt = dx$

$t = 1 \Rightarrow x = 1$

$t = n \Rightarrow x = \frac{1}{n}$

$$a_n = -\int_1^{1/n} 2^x\,dx = -\frac{2^x}{\ln 2}\Bigg|_1^{1/n} = -\frac{2^{\frac{1}{n}}}{\ln 2} + \frac{2}{\ln 2}$$

$\to -\frac{1}{\ln 2} + \frac{2}{\ln 2} = \frac{1}{\ln 2} \Rightarrow$ conv. $\Rightarrow \sum f(n)$ conv.

(21) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \ln\left(1+\frac{1}{\sqrt{n^3+1}}\right)$

Crit. la comp la lim.

$\sum u_n$, $\sum v_n$; $\lim \frac{u_n}{v_n} = l$

$l=0$, $\sum v_n$ - conv $\Rightarrow$ $\sum u_n$ conv.

$l=\infty$, $\sum v_n$ - div. $\Rightarrow$ $\sum u_n$ div.

$l\neq 0, l\neq\infty \Rightarrow$ $\sum u_n$, $\sum v_n$ au ac nat

Fie $\sum v_n = \sum \frac{1}{n^2}$ conv. ($\alpha=2>1$) serie armonică

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \ln\left(1+\frac{1}{\sqrt{n^3+1}}\right) \cdot n^2 =$$

$$= \lim_{n\to\infty} \ln\left[\left(1+\frac{1}{\sqrt{n^3+1}}\right)^{\sqrt{n^3+1}}\right] \cdot \frac{1}{\sqrt{n^3+1}} \cdot \frac{n^2}{\sqrt{n}} =$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2\sqrt{1+\frac{1}{n^3}}} = 1 \neq 0 \Rightarrow$$ serii au ac natură

$\sum v_n$ conv. $\Rightarrow \sum u_n$ conv.

(22) $\sum_{n=1}^{\infty} \left[3+\left(1+\frac{1}{n}\right)^n\right]$

Crit necesar de conv.

$\lim u_n \neq 0 \Rightarrow \sum u_n$ div.

- $\lim_{n\to\infty} \left[3+\left(1+\frac{1}{n}\right)^n\right] = 3+e \neq 0 \Rightarrow \sum u_n$ div.

(23) $$\sum_{n=1}^{\infty} \frac{\sqrt[3]{n+1}-\sqrt[3]{n}}{n^{\alpha}} = \sum \frac{1}{n^{\alpha}\left(\sqrt[3]{(n+1)^2}+\sqrt[3]{n^2+n}+\sqrt[3]{n^2}\right)}$$

Fie $v_n = \frac{1}{n^{\alpha+2/3}}$

CRIT comp. la lim

$\sum u_n, \sum v_n \quad \lim \frac{u_n}{v_n} = \ell$

$\ell = 0, \sum v_n$ conv $\Rightarrow \sum u_n$ conv.

$\ell = \infty, \sum v_n$ div. $\Rightarrow \sum u_n$ div.

$\ell \neq 0, \ell \neq \infty \Rightarrow \sum u_n, \sum v_n$ au ac. nat.

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim \frac{n^{2/3}}{\sqrt[3]{n^2}\left(\sqrt[3]{\left(1+\frac{1}{n}\right)^2}+\sqrt[3]{1+\frac{1}{n}}+1\right)} = \frac{1}{3} \neq 0$$

$\Rightarrow$ seriile au ac. nat

$\alpha + \frac{2}{3} > 1 \Rightarrow \alpha > \frac{1}{3} \Rightarrow \sum v_n$ conv. $\Rightarrow \sum u_n$ conv.

$\alpha + \frac{2}{3} \leq 1 \Rightarrow \alpha \leq \frac{1}{3} \Rightarrow \sum v_n$ div. $\Rightarrow \sum u_n$ div.

(24) $$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt[3]{n^2}} = \sum \frac{1}{\sqrt[3]{n^2}\left(\sqrt{n+1}+\sqrt{n}\right)}$$

$v_n = \frac{1}{n^{2/3} \cdot n^{1/2}} = \frac{1}{n^{7/6}}$ serie armonică

$\alpha = \frac{7}{6} > 1 \Rightarrow \sum v_n$ conv.

CRIT comp. $\sum u_n, \sum v_n$, $\lim \frac{u_n}{v_n} = \ell$
- $\ell = 0, \sum v_n$ conv $\Rightarrow \sum u_n$ conv
- $\ell = \infty, \sum v_n$ div $\Rightarrow \sum u_n$ div
- $\ell \neq 0, \infty \Rightarrow$ au ac nat

$\lim \frac{u_n}{v_n} = \lim \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{2} \neq 0 \Rightarrow$ au ac. nat.

$\Rightarrow \sum u_n$ conv.

(27) $\sum_{n=1}^{\infty} \frac{\sqrt{7n}}{n^2+3n+5}$

$v_n = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$

serie armonică

$\alpha = \frac{3}{2} > 1 \Rightarrow$ conv.

$\lim \frac{u_n}{v_n} = \lim \frac{7n^2}{n^2+3n+5} = 7 \neq 0.$

$\Rightarrow \sum u_n, \sum v_n$ au ac. natură $\Rightarrow \sum u_n$ conv.

CRIT. comp. la lim. $\lim_{n\to\infty} \frac{u_n}{v_n} = l$

$l = 0, \sum v_n$ conv $\Rightarrow \sum u_n$ conv

$l = \infty, \sum v_n$ div. $\Rightarrow \sum u_n$ div.

$l \neq 0, l \neq \infty \Rightarrow \sum u_n, \sum v_n$ ac. nat.

(28) $\sum_{n=1}^{\infty} \frac{1}{n}\left(\sqrt{n^2+n+1} - \sqrt{n^2-n+1}\right) = \sum \frac{1}{\cancel{n}} \cdot \frac{2\cancel{n}}{\sqrt{} + \sqrt{}}$

$v_n = \frac{1}{n}$ div. $(\alpha = 1 \leq 1)$

seria armonică

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left(\sqrt{n^2+n+1} - \sqrt{n^2-n+1}\right) = \lim \frac{2n}{\sqrt{n^2+n+1} + \sqrt{n^2-n+1}}$

$= \lim_{n\to\infty} \frac{2\cancel{n}}{\cancel{n}\left(\sqrt{1+\frac{1}{n}+\frac{1}{n^2}} + \sqrt{1-\frac{1}{n}+\frac{1}{n^2}}\right)} = 1 \neq 0 \Rightarrow \sum u_n, \sum v_n$ au ac nat

$\Rightarrow \sum u_n$ divergentă

(25) $\sum_{n=1}^{\infty} \left(\sqrt{n^2+1+an} - n\right)^n$, $a \geq 0$.

Crit radicalului

$\sum u_n$ și $\lim \sqrt[n]{u_n} = \ell \begin{cases} < 1 \Rightarrow \text{conv.} \\ > 1 \Rightarrow \text{div} \\ 1 \Rightarrow \text{nu se poate} \end{cases}$

$\lim \sqrt[n]{u_n} = \lim \left(\sqrt{n^2+1+an} - n\right) = \lim \frac{an+1}{\sqrt{n^2+1+an}+n} \Rightarrow$

$\ell = \frac{a}{2}$

Dacă $\frac{a}{2} < 1 \Rightarrow a < 2 \Rightarrow \sum u_n$ conv.

Dacă $\frac{a}{2} > 1 \Rightarrow a > 2 \Rightarrow \sum u_n$ div.

Dacă $a = 2 \Rightarrow \sum \left(\sqrt{n^2+2n+1} - n\right)^n =$

$\sum (n+1-n)^n = \sum 1 = \sum \frac{1}{n^0}$ serie armonică $\alpha = 0 \leq 1$

$\Rightarrow$ divergentă.

(26) $\sum_{n=1}^{\infty} \frac{\ln n}{2n^3-1}$

$u_n = \frac{\ln n}{2n^3-1}$ $\quad v_n = \frac{1}{n^2}$

serie armonică

$\alpha = 2 > 1 \Rightarrow$ conv.

$u_n \leq v_n \iff \frac{\ln n}{2n^3-1} \leq \frac{n}{2n^3-1} \leq \frac{1}{n^2}$

$n^3 \leq 2n^3 - 1$

$1 \leq n^3$

Crit comp.

$u_n \leq v_n$, $\sum v_n$ conv. $\Rightarrow \sum u_n$ conv

$\sum u_n$ conv. $\Rightarrow \sum v_n$ conv.

(29) $\sum_{n=1}^{\infty} \frac{1}{n^a} \sin \frac{\pi}{n}$ $\quad v_n = \frac{1}{n^{a+1}}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sin \frac{\pi}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sin \frac{\pi}{n}}{\frac{\pi}{n}} \cdot \frac{1}{\pi} = \frac{1}{\pi} \neq 0 \Rightarrow$

$\sum u_n, \sum v_n$ au ac nat.

$\Rightarrow a+1 > 1 \Leftrightarrow a > 0 \Rightarrow \sum v_n$ conv. $\Rightarrow \sum u_n$ conv.
(serie armonică)

$a+1 \leq 1 \Leftrightarrow a \leq 0 \Rightarrow \sum v_n$ div. $\Rightarrow \sum u_n$ div.

CRIT comp. la lim $\lim_{n\to\infty} \frac{u_n}{v_n} = \ell$

$\ell = 0$, $\sum v_n$ conv $\Rightarrow \sum u_n$ conv.
$\ell = \infty$, $\sum v_n$ div $\Rightarrow \sum u_n$ div
$\ell \neq 0, \infty \Rightarrow \sum u_n, \sum v_n$ au ac. nat

(30) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{3^n}$

CRIT LEIBNIZ
$u_n \geq 0$, $\sum_{n=0}^{\infty} (-1)^n \cdot u_n$
$u_n \searrow$, $\lim u_n = 0$
$\Rightarrow \sum_{n=0}^{n} (-1)^n u_n$ conv.

$u_n = \frac{2n+1}{3^n}$

$\frac{u_{n+1}}{u_n} = \frac{2n+3}{3^{n+1}} \cdot \frac{3^n}{2n+1} = \frac{2n+3}{3(2n+1)} < 1 \Rightarrow u_n \searrow$

$\lim u_n = 0. \Rightarrow \sum (-1)^n u_n$ conv

(31) $\sum_{n=1}^{\infty} \frac{1\cdot 6\cdot 11\cdot \ldots \cdot(5n-4)}{1\cdot 7\cdot 13\cdot \ldots \cdot(6n-5)}\cdot a^n,\ a>0$

CRIT. RAPORTULUI $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \ell$

$\ell<1 \Rightarrow \sum u_n$ conv.

$\ell>1 \Rightarrow \sum u_n$ div.

$\ell=1 \Rightarrow$ nu se poate

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim \frac{5n+1}{6n+1}\cdot a = \frac{5}{6}a$$

$\frac{5a}{6}<1 \Rightarrow a<\frac{6}{5} \Rightarrow \sum u_n$ conv.

$\Rightarrow a>\frac{6}{5} \Rightarrow \sum u_n$ div.

$a=\frac{6}{5}$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1\cdot 6\cdot 11\ldots(5n-4)}{1\cdot 7\cdot 13\ldots(6n-5)}\cdot\left(\frac{6}{5}\right)^n$$

CRIT. RAABE-DUHAMEL

$\lim n\left(\frac{u_n}{u_{n+1}}-1\right)>1 \Rightarrow \sum u_n$ conv.

$\lim n\left(\frac{u_n}{u_{n+1}}-1\right)<1 \Rightarrow \sum u_n$ div.

$=1 \Rightarrow$ nu se poate

$$\lim_{n\to\infty} n\left(\frac{6n+1}{(5n+1)}\cdot\frac{5}{6}-1\right) = \lim n\left(\frac{30n+5-30n-6}{6(5n+1)}\right) = \frac{-1}{6\cdot 5}<1$$

$\Rightarrow \sum u_n$ div.

(32) $\sum_{n=1}^{\infty} \frac{1}{n^a} \ln\left(1+\frac{1}{n^b}\right)$ $a, b \in \mathbb{R}^+$

CRIT. COMP la lim. $\lim_{n\to\infty} \frac{u_n}{v_n} = l$

$l=0$, $\sum v_n$ conv $\Rightarrow \sum u_n$ conv.
$l=\infty$, $\sum v_n$ div $\Rightarrow \sum u_n$ div.
$l\neq 0$, $l \neq \infty \Rightarrow \sum u_n$, $\sum v_n$ au ac nat

$v_n = \frac{1}{n^{a+b}}$ $\left\langle \begin{array}{l} a+b>1 \Rightarrow \sum v_n \text{ conv.} \\ a+b\leq 1 \Rightarrow \sum v_n \text{ div.} \end{array} \right.$

$\lim \frac{u_n}{v_n} = \lim \frac{\ln\left(1+\frac{1}{n^b}\right)}{\frac{1}{n^b}} = n^b \ln\left(1+\frac{1}{n^b}\right) = \ln e = 1$ $\neq 0$

$\Rightarrow \sum u_n$, $\sum v_n$ au ac nat.

$\Rightarrow$ $a+b>1 \Rightarrow \sum u_n$ conv.
$a+b\leq 1 \Rightarrow \sum v_n$ div.

(33) $\sum_{n=1}^{\infty} \left(\frac{an+1}{bn+1}\right)^n$ $a, b \geq 0$.

CRIT. RADICALULUI $\lim_{n\to\infty} \sqrt[n]{u_n} = l$

$l<1 \Rightarrow \sum u_n$ conv.
$l>1 \Rightarrow \sum u_n$ div.
$l=1 \Rightarrow$ nu se poate

$\lim_{n\to\infty} \sqrt[n]{u_n} = \lim_{n\to\infty} \frac{an+1}{bn+1} = \frac{a}{b}$ $\left\langle \begin{array}{l} \frac{a}{b}<1 \Rightarrow a<b \Rightarrow \sum u_n \text{ conv.} \\ >1 \Rightarrow a>b \Rightarrow \sum u_n \text{ div.} \\ a=b. \Rightarrow \sum 1 \Rightarrow \text{div.} \end{array} \right.$

seria arm.
$\alpha=0<1 \Rightarrow$ div.

(34) $\sum_{n=1}^{\infty} \frac{\sqrt{1-\cos\frac{\pi}{n}}}{n\ln(n+1)}$

$v_n = \frac{\sqrt{2}\pi}{2n^2}$ $\frac{\sqrt{2}\pi}{2} \cdot \sum \frac{1}{n^2}$

$\alpha = 2 > 1 \Rightarrow$ conv

serie armonică.

$\frac{\sqrt{1-\cos\frac{\pi}{n}}}{n\ln(n+1)} < \cancel{\frac{1}{n\ln(n+1)}} < \cancel{\frac{1}{n}}$

$1-\cos\frac{\pi}{n} =$
$1-\left(1-2\sin^2\frac{\pi}{2n}\right) =$
$2\sin^2\frac{\pi}{2n}$

$\Rightarrow \frac{\sqrt{2}\sin\frac{\pi}{2n}}{n\ln(n+1)} < \frac{\frac{\pi}{2n}\cdot\sqrt{2}}{n\ln(n+1)} < \frac{\pi\sqrt{2}}{2n^2}$

Am folosit $\sin x < x$.

și $\frac{1}{\ln(n+1)} < 1$.

$u_n \le v_n$
$\sum v_n$ conv. $\Big| \Rightarrow \sum u_n$ conv.

CRIT COMP cu ineg. $u_n \le v_n$

i) $\sum v_n$ conv. $\Rightarrow \sum u_n$ conv.

ii) $\sum u_n$ div $\Rightarrow \sum v_n$ div.