- Ω = mp tuturor stărilor pos, $P(\Omega) = \{A \mid A \subseteq \Omega\}$ → mulț ev pos = A

$F \subseteq P(\Omega)$: mulț ev pos ; $N(A)$ = nr de ap ev A în N exp

$P(A) = \lim_{n \to \infty} \frac{N(A)}{N} \in [0,1]$ → frecv relativă a ap A

$N(A \cup B) = N(A) + N(B)$; $P(A \cup B) = P(A) + P(B)$; $A \cap B = \emptyset$

→ $P(A^c) = 1 - P(A)$; $A \subseteq B \Rightarrow P(A) \le P(B)$; $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \Delta B) = (A \cup B) \setminus (A \cap B)$; $P(A \cap B^c) = P(A \setminus (A \cap B)) = P(A) - P(A \cap B)$

$P(B \cap A^c) = P(B \setminus (A \cap B)) = P(B) - P(A \cap B)$

$P(A \Delta B) = P(A \cap B^c) + P(B \cap A^c)$

→ Formula lui Poincaré: $P(\bigcup_{i=1}^{m} A_i) = \sum_{i=1}^{m} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) + \dots + (-1)^{n-1} P(A_1 \cap \dots \cap A_m)$

$P(A) = \frac{\text{nr caz fav}}{\text{nr caz nefav(pos)}} = \frac{|A|}{|\Omega|}$; Formula prod: $|A \times B| = |A| \cdot |B|$

→ extragere cu/fără revenire: Nr părți de card k ale unei mulțimi card $m \Rightarrow C_m^k$

$\frac{m!}{m_1! \dots m_k!}$ → coef multinomial m_1 - elem tipul 1 ... m_k - elem tip k $\sum_{i=1}^{k} n_i = n$

- Probabilități condiționale: $P(A|B) = \frac{P(A \cap B)}{P(B)}$; $P(A|B \cap C) = \frac{P(B|A \cap C) \cdot P(A|C)}{P(B|C)} = Q(A|B)$

$P(A \cap B) = P(B) \cdot P(A|B) = P(A) \cdot P(B|A)$; $P(A_1 \cap \dots \cap A_m) = P(A_1) \cdot P(A_2|A_1) \dots P(A_m | A_1 \cap A_2 \dots \cap A_{m-1})$; $P(A) = P(A|B) \cdot P(B) + P(A|B^c) \cdot P(B^c)$

$P(A) = \sum_{i=1}^{m} P(A|B_i) \cdot P(B_i)$; Form Bayes: $P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A|B) \cdot P(B) + P(A|B^c) \cdot P(B^c)}$

- variabile aleatoare : $X : \Omega \to \mathbb{R}$ var aleat dc $\{\omega \in \Omega \mid X(\omega) \le x\} = F$; discretă $X(\Omega)$ = cel mult num
- Repartiția lui X : $P_X(A) = P(X \in A) = P(X^{-1}(A)) = P \cdot X^{-1}$
- Fcția de rep a lui X : $F(x) = P(X \le x)$; F cresc și continuă la dr $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$
- Propr : $P(X > x) = 1 - P(X \le x) = 1 - F(x)$

$P(x < X \le y) = P(X \le y) - P(X \le x) = F(y) - F(x)$

$P(X = x) = P(X \le x) - P(X < x) = F(x) - \lim_{\substack{y \to x \\ y < x}} F(y)$

- Fcția de masă : $p_X(x) = P(X = x)$

$p_X(x) \ge 0 \quad \sum_{x \in X(\Omega)} p_X(x) = 1$

$P(X \in A) = \sum_{x \in A \cap X(\Omega)} p_X(x) \qquad F(x) = \sum_{\substack{y \le x \\ y \in X(\Omega)}} p_X(y)$

→ var aleatoare Bernoulli: $X(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$

$p_X(x) = p^x (1-p)^{1-x} \qquad F(x) = \begin{cases} 0 & x < 0 \\ 1-p & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases} \quad X \sim B(p)$

→ var aleat binomială : $X \sim B(m, p)$, $p_X(k) = \binom{m}{k} p^k (1-p)^{m-k}$ (extragere cu întoarcere); nr total de realiz ale ev A în m exp $P(A) = p$

• variab aleat repartiz hipergeom (fără întoarcere): $X \sim HG(m, N, M)$ nr bile ext / nr extr / bile total / deci dat test

$$P(X=k) = \frac{\binom{M}{k}\binom{N-M}{m-k}}{\binom{N}{m}}$$ f de masă

• v. a discretă unif: $X \sim U(D)$ val aa a j-a extragere ; $p_X(x) = \frac{1}{|D|}$ $P(x \in A) = \frac{|A|}{|D|}$

• v. a rep geom $X \sim Geom(p)$ nr încercări până la succes $p_X(k) = (1-p)^{k-1} p$; $P(X = m+k | X > m) =$ $P(X=k)$

• v.a repartiz neg binomial $X \sim NB(n, p)$ <-> nr încerc până la $n \geq 1$ succese $p_X(k) = \binom{k-1}{n-1}(1-p)^{k-n} p^n$

• v. a Poisson $X \sim P(\lambda)$ $p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}$ (șansa de realiz a unui ev. f mică, dar nr de ev. într-un interval de timp/spațiu e f mare)

• X și Y sunt v.a discrete indep ($X \perp Y$) dc:
$P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$

• Media: $E[X] = \sum_x x f(x)$; $\sum_x x f(x)$ diverg => media nu e def
$E[aX+bY] = aE[X] + bE[Y]$; $X \perp Y \Rightarrow E[XY] = E[X]E[Y]$; $X = ct \Rightarrow E[X] = c$
X = v.a discr $g: \mathbb{R} \to \mathbb{R}$ $Y = g(X)$; $E[Y] = E[g(X)] = \sum_x g(x) P(X=x)$

• Varianța
$Var(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$; $Var(X+a) = Var(X)$
$Var(aX+b) = a^2 Var(X) = \sigma^2$ – abatere standard; $Var(aX) = a^2 Var(X)$
$X \perp Y \Rightarrow Var(X \pm Y) = Var(X) + Var(Y)$

• Mom de ordin k : $E[X^k] = \sum x^k f(x)$; $E[(X-a)^k]$ centrat în a

• covarianța : $cov(X,Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y]$
$cov(Y,X)$
$cov(X,X) = Var(X)$ $X \perp Y \Rightarrow cov(X,Y) = 0$ (sunt necorelate)
$cov(aX+b, Y) = a\,cov(X,Y)$; $cov(X+Y, Z) = cov(X,Z) + cov(Y,Z)$
$Var(X \pm Y) = Var(X) + Var(Y) \pm 2cov(X,Y)$;

• $\boxed{\text{Variabile aleatoare continue}}$: $P(X \in A) = \int_A f(x)\,dx \quad \forall A \subseteq \mathbb{R}$

– $P(a \leq X \leq b) = \int_a^b f(x)\,dx$ $P(X \in \mathbb{R}) = \int_{-\infty}^{\infty} f(x)\,dx = 1$

– Densit de repartiție: $f: \mathbb{R} \to \mathbb{R}$ (nu e o probabilit)
$f(x) \geq 0$ și $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$P(X=a) = \int_a^a f(x) = 0$
$P(a \leq X \leq b) = \dots = P(a < X < b)$

– Fcția de rep : $F: \mathbb{R} \to [0,1]$ $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$
f cont $\Rightarrow F'(x) = f(x)$
$P(X > a) = 1 - P(X \leq a) = 1 - F(a) = \int_a^{\infty} f(x)$

– Media : $E[X] = \int_{-\infty}^{\infty} x f(x)\,dx$; $\int_{-\infty}^{\infty} |x| f(x)\,dx \to +\infty \Rightarrow$ media ∄
$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx$

– Varianța : $Var(X) = E[X^2] - E[X]^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left(\int_{-\infty}^{\infty} x f(x)\,dx\right)^2$

– Mom de ordin k : $E[X^k] = \int_{-\infty}^{\infty} x^k f(x)\,dx$

– v.a repartizată unif : $U \sim U(a,b)$ $f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0 & \text{altfel} \end{cases}$
$Y = a + (b-a)U$; $E[U] = \frac{a^2 + ab + b^2}{2}$

$var(U) = \frac{(b-a)^2}{12}$ $F(x) = \begin{cases} 0, & x \le a \\ \frac{x-a}{b-a}, & a < x < b \\ 0, & x \ge b \end{cases}$ $E[X] = \int_a^b \frac{x}{b-a} dx$

- variab rep exponentiale : $X \sim Exp(\lambda)$ $E[X] = \frac{1}{\lambda}$ $f(x) = \lambda \cdot e^{-\lambda x}, x > 0$
$var(X) = \frac{1}{\lambda^2}$; $P(X \ge a) = e^{-\lambda a}$ $F(x) = \begin{cases} 1 - e^{-\lambda x}, & x > 0 \\ 0, & x \le 0 \end{cases}$

- Rep comune, marg și condiț - discrete → f de masă (X,Y) - rep comună
↳ f rep a (X,Y) : $F_{X,Y}(x,y) = P(X \le x, Y \le y)$, $P(X,Y)(x,y) = P(X=x, Y=y) = f_{X,Y}(x,y)$
↳ rep marg X : $P(X=x) = p_X(x) = \sum_y P(X=x, Y=y)$ $P((X,Y) \in A \times B) = \sum_{\substack{x \in A \cap X(\Omega) \\ y \in B \cap Y(\Omega)}} P(X=x, Y=y)$
↳ rep cond a lui X la $Y=y$ → $p_{X|Y}(x|y) = P(X=x | Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}$
↳ formula prob totale : $p_X(x) = \sum_{i=1}^{m} p_{X|A_i}(x) \cdot P(A_i)$; $p_X(x) = \sum_{i=1}^{m} p_{X|Y}(x|y_i) \cdot P(Y=y_i)$
↳ form lui Bayes
$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} = \frac{p_X(x) \cdot p_{Y|X}(y|x)}{\sum_{x'} p_X(x') p_{Y|X}(y|x')}$
↳ media : $E[g(X,Y)] = \sum_x \sum_y g(x,y) p_{X,Y}(x,y)$
↳ media condiț : $E[X|Y=y] = \sum_x x \, p_{X|Y}(x|y)$, $E[X] = \sum_y E[X|Y=y] P(Y=y)$

- Rep comune → cont
↳ $f(x,y)$: densit comună (X,Y) ⇔ $f(x,y) \ge 0$ $\int\int_{-\infty}^{+\infty} f(x,y) = 1$
↳ $P((X,Y) \in A) = P(a \le X \le b, c \le Y \le d) = \int_a^b \int_c^d f_{X,Y}(x,y) \, dy dx$
↳ densit marginală X : $f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x,y) dy$
↳ f rep $F_{X,Y}(x,y) = P(X \le x, Y \le y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v) \, du dv$
↳ rep condiționată : $P(X \in B | A) = \int_B f_{X|A}(x) dx$, $\forall B \subseteq \mathbb{R}$ → $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
↳ probab totală $f_X(x) = \sum_{i=1}^{m} f_{X|A_i}(x) P(A_i)$ → $f_{X,Y}(x,y) = f_{X|Y}(x|y) \cdot f_Y(y)$
↳ indep var cont : $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$ ⇒ $f_{X|Y}(x|y) = f_X(x)$
↳ form lui Bayes : $f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) \cdot f_X(x)}{\int_{-\infty}^{\infty} f_{Y|X}(y|x') f_X(x') dx'}$

→ Inegalități :
- Cauchy-Schwarz : $E[XY] \le \sqrt{E[X^2] E[Y^2]}$
- Jensen
  - g convexă $E[g(X)] \ge g(E[X])$
  - g concavă $E[g(X)] \le g(E[X])$
  - g afină ⇒ $E[g(X)] = g(E[X])$
- Markov : $P(X \ge a) \le \frac{E[X]}{a}$, $\forall X \ge 0, a > 0$
- Chebyshev $P(|X - \mu| \ge a) \le \frac{\sigma^2}{a^2}$, $\forall a > 0$, $\mu = E[X]$ $\sigma^2 = var(X)$
- Chernoff : $P(X \ge a) \le E[e^{tX}] / e^{ta}$, $\forall a > 0, t > 0$

→ Legea nr mari : $X_1, \dots X_n$ v.a. iid $E[X_i] = \mu$, $var(X_i) = \sigma^2$, $var(\bar{X}_n) = \frac{\sigma^2}{n}$
$\bar{X}_n = \frac{X_1 + \dots + X_n}{n}$ → media eșantionului $E[\bar{X}_n] = \mu$, $\forall \omega \in \Omega_0$, $P(\Omega_0) = 1$
↳ Legea tare a nr mari : $P(\bar{X}_n \to \mu) = 1$ $\bar{X}_n(\omega) \xrightarrow[n \to \infty]{} \mu$

• Legea slabă a nr. mari: $P(|\bar{X}_n - \mu| > \varepsilon) \xrightarrow[n \to \infty]{} 0$, $\varepsilon > 0$, $(X_n)_n$, $X_n \xrightarrow{P} X$ $(\Leftarrow)$

$\Downarrow$ convergența în probabilitate $P(|X_n - X| > \varepsilon) \xrightarrow[n\to\infty]{} 0$

• Teorema limită centrală: $Z_n = \dfrac{(X_1 + \dots + X_n) - E[X_1 + \dots + X_n]}{\sqrt{Var(X_1 + \dots + X_n)}}$

var. normalizată $\sim N(0,1)$

$S_n = X_1 + \dots + X_n \Rightarrow Z_n = \dfrac{S_n - E[S_n]}{\sqrt{Var(S_n)}} = \dfrac{S_n - \mu \cdot n}{\sigma \sqrt{n}}$, $E[Z_n] = 0$, $Var(Z_n) = 1$

$\lim\limits_{n \to \infty} P(Z_n \le z) = \Phi(z)$, $\forall z \in \mathbb{R}$, $P(S_n \le c) \simeq P\left(\dfrac{S_n - n\mu}{\sigma\sqrt{n}} \le \dfrac{c - n\mu}{\sigma\sqrt{n}}\right) =$

$= P\left(Z_n \le \dfrac{c - n\mu}{\sigma\sqrt{n}}\right) = \Phi\left(\dfrac{c - n\mu}{\sigma\sqrt{n}}\right)$

• Aprox. Laplace a bin.: $S_n \sim B(n, p)$, $S = X_1 + \dots + X_n$, $X_i \sim B(p)$ $\perp$, $\mu = E[X_i] = p$, $\sigma^2 = Var(X_i) = p(1-p)$, $P(k \le S_n \le l) = \Phi\left(\dfrac{l + \frac{1}{2} - np}{\sqrt{np(1-p)}}\right) - \Phi\left(\dfrac{k - \frac{1}{2} - np}{\sqrt{np(1-p)}}\right)$