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Anul 3, Seria 34, Grupa 343

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Calculabilitate si complexitate
~examen~

Undecidability

- **What is undecidability?**

Undecidability is ~~the~~ one of the most philosophically important theorems of the theory of computation. There is a specific problem that is algorithmically unsolvable. This theorem demonstrates that computers are limited in a fundamental way, even if they appear to be so powerful that all problems will eventually yield to them.

- **The diagonalization method** was discovered by Georg Cantor, who was concerned with the problem of measuring the sizes of infinite sets. He observed that two finite sets have the same size if the elements of one set can be paired with the elements of the other set. This idea can be extended for infinite sets.

Definition: Assume that we have sets A and B and a function f from A to B. We say that:

- f is one-to-one (injective) if $f(a) \neq f(b)$ $(\forall) a \neq b$
- f is onto (surjective) if $(\forall) b \in B$ $(\exists) a \in A$ such that $f(a) = b$
- f is a correspondence (bijective) if it is both one-to-one and onto.

In this last case, we say that sets A and B are the same size.

In a correspondence, every element of A maps to a unique element of B and each element of B has a unique element of A mapping to it.

Example: Let $\mathbb{N}$ be the set of natural numbers and $\mathcal{E}$ the set of even natural numbers

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$
$$\mathcal{E} = \{2, 4, 6, \ldots\}$$

Using previous definition of size, $\mathbb{N}$ and $\mathcal{E}$ have the same size because there is a correspondence f mapping $\mathbb{N}$ to $\mathcal{E}$: $f(n) = 2n$

n	$f(n)$
1	2
2	4
3	6
...	...

Even if $\mathcal{E}$ seems smaller than $\mathbb{N}$, cause $\mathcal{E}$ is a proper subset of $\mathbb{N}$, pairing each member of $\mathbb{N}$ ~~with~~ with a member of $\mathcal{E}$ is possible, so the two sets are the same size.

• (Un)countable sets

Definition: A set is countable if either it is finite or it has the same size ~~with~~ as $\mathbb{N}$.

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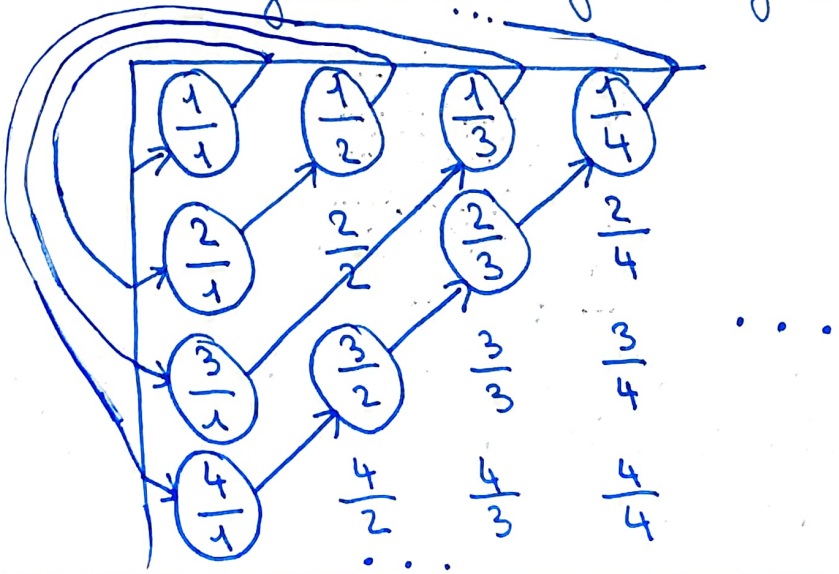
- $\mathbb{Q}$ is countable

Let $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{N}\}$ be the set of positive rational numbers.

At first, $\mathbb{Q}$ seems to be much larger than $\mathbb{N}$, which ~~However, the two sets are the same size~~ would make it seemingly uncountable. However, $\mathbb{Q}$ is ~~a~~ countable ~~set~~ because, according to definition, $\mathbb{Q}$ and $\mathbb{N}$ are the same size.

To show that, we create a matrix M, where $M[i,j] = \frac{i}{j}$ and turn it into a list. We list the elements on the diagonals (as shown in the diagram below) starting from the ~~top~~ left corner, skipping ~~on~~ elements which would cause repetitions (such as $\frac{2}{2}$, which equals $\frac{1}{1}$).

We list the elements in this way because if we would attempt to start with the first row, we would never get to the second one given that they are infinite.



Then, we pair the first element from this list with the first element from $\mathbb{N}$ (which is number 1), the second element on the list with number 2 from $\mathbb{N}$, and so on.

Using this approach, someone might think that any two infinite sets can be proved to have the same size, but that's not correct. Some sets are simply too big, and such sets are called uncountable.

- $\mathbb{R}$ is uncountable

An example of an uncountable set is the set of real numbers ($\mathbb{R}$).

To prove the theorem that $\mathbb{R}$ is uncountable, we show that no correspondence exists between $\mathbb{N}$ and $\mathbb{R}$. The proof is by contradiction: we suppose that a correspondence f exists between $\mathbb{N}$ and $\mathbb{R}$, then we show that f fails to work as it should.

To be a correspondence, f must pair all the members of $\mathbb{N}$ with all the members of $\mathbb{R}$. We must find an $x \in \mathbb{R}$ which is not paired with anything from $\mathbb{N}$.

Here is a hypothetical correspondence between $\mathbb{N}$ and $\mathbb{R}$:

n	$f(n)$
1	3.<u>1</u>4159...
2	55.5<u>5</u>555...
3	0.12<u>3</u>45...
4	0.500<u>0</u>0...
...	...

We construct x as a number between 0 and 1, so all its significant digits are fractional digits following the decimal point.

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To ensure that $x \neq f(n)$, we choose its digits so that the digit placed in the i^{th} position after the decimal point is different from the i^{th} digit of $f(i)$.

- for the first digit of x, we must choose anything different from the first fractional digit of $f(1)$, which is 1.

$$x = 0,7$$

- for the second digit, we choose anything different from the second fractional digit of $f(2)$, which is 5.

$$x = 0.72$$

- for the third digit, we choose anything different from the third fractional digit of $f(3)$, which is 3.

$$x = 0.724\ldots$$

* We avoid choosing digits 0 or 9 when constructing x because certain numbers are equal even if their decimal representation is different (for example: 0.2000 and 0.1999).

Continuing like this, we find a number x which we know that is not $f(n)$ $(\forall)$ $n \in \mathbb{N}$, because it differs from $f(n)$ in the n^{th} fractional digit.

Thus, $\mathbb{R}$ is uncountable because there is no correspondence between $\mathbb{N}$ and $\mathbb{R}$.

- Some languages are not Turing-recognizable

The previous proof has an important application in the theory of computation. A Turing Machine can recognize a single language and there is a finite number of Turing Machines, while languages are infinite. Therefore, some languages are not Turing-recognizable.

The set of all strings Σ^* is countable for any alphabet Σ because we can form a list of Σ^* by writing down all strings of length 0, 1, 2 etc.

The set of all Turing Machines is countable because each Turing Machine M has an encoding into a string $\langle M \rangle$. If we remove all illegal encodings, we can obtain a list of all Turing Machines.

To show that the set of all languages ($\mathcal{L}$) is uncountable, we first observe that the set of all infinite binary sequences (B) is uncountable. We can prove that using the diagonalization method, similar to the one used for $\mathbb{R}$.

We show that $\mathcal{L}$ is uncountable by giving a correspondence with B. Let $\Sigma^* = \{s_1, s_2, s_3, \ldots\}$. Each language $A \in \mathcal{L}$ has a unique sequence in B. The i^{th} bit of that sequence is a 1 if $s_i \in A$, else it's 0, and this sequence is called the characteristic sequence of A.

Example: A is the language of all strings which start with 0 over the alphabet $\{0,1\}$.

The characteristic sequence of A (χ_A) is:

$\Sigma^* = \{ \varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, \ldots \}$

$A = \{ \quad 0, \quad 00, 01, \quad 000, 001, \ldots \}$

$\chi_A = \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \ldots$

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The function $f: \mathcal{L} \to B$, where $f(A)$ equals the characteristic sequence of A, is bijective, and hence is a correspondence. Thus, as B is uncountable, $\mathcal{L}$ is, as well, uncountable. Therefore, some languages are not Turing recognizable.

- A_{TM} is undecidable

$$A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}$$

We assume that A_{TM} is decidable and search for a contradiction.

Let H a decider for A_{TM}. H is a TM where:

$$H(\langle M, w \rangle) = \begin{cases} \text{accept} & \text{if } M \text{ accepts } w \\ \text{reject} & \text{if } M \text{ does not accept } w \end{cases}$$

Let D another TM that has H as its subroutine. D takes another TM as input and it will return the ~~opp~~ opposite of what H returns.

$$D(\langle M \rangle) = \begin{cases} \text{accept} & \text{if } M \text{ does not accept } \langle M \rangle \\ \text{reject} & \text{if } M \text{ accepts } \langle M \rangle \end{cases}$$

$$D(\langle D \rangle) = \begin{cases} \text{accept} & \text{if } D \text{ does not accept } \langle D \rangle \\ \text{reject} & \text{if } D \text{ accepts } \langle D \rangle \end{cases}$$

No matter what D does, it's forced it to do the opposite => contradiction => Neither D or H can exist.