(Rational) Parking Functions

Tessa Lelièvre-Osswald

Advisor: Matthieu Josuat-Vergès

Team: IRIF Combinatorics

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Abstract

Introduced in 1966 by Konheim and Weiss [1], a (classical) Parking Function is a combinatorial structure represented by an integer sequence $(a_1, \dots, a_n)$ such that $\#\{i \mid a_i \leqslant k\} \geqslant k \ \forall k$.

In this article, we will address bijections and posets on four types of parking functions : classical (also called *integer*), primitive classical, rational, and primitive rational. Those four types are defined throughout the article.

Each of these types has been proved to be counted by a well-known number – respectively $(n+1)^{n-1}$, Cat_n, b^{a-1}, and $Cat_{a,b}$. Nevertheless, to the best of our knowledge, the posets we will present here have not been exploited on parking functions specifically.
Furthermore, Theorem 8 and its equivalent Conjecture for classical parking functions reveal interesting results on the number of intervals in those posets.

This report goes hand in hand with a Sagemath implementation. Notions found in [2], [3], [4], [5], [6], [7], [8], and [9] are encoded alongside the aforementioned posets. All of the source code and premade tests can be found here [1].

Finally, The notion of Parking Trees defined in [9] is extended to the rational case, hence creating a bijection between rational parking functions and Rational Parking Trees.

[1]github.com/tessalsifi/ParkingFunctions

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Chapter 1

The integer case

1.1 Parking Functions

Definition 1 (Parking Function)**.** *A* parking function *is a sequence of positive integers* $(a_1, a_2, \ldots, a_n)$ *such that its non-decreasing reordering* $(b_1, b_2, \ldots, b_n)$ *has* $b_i \leqslant i$ *for all* i*. In other words,* $\#\{i \mid a_i \leqslant k\} \geqslant k \ \forall k$*.*

We denote by $\mathcal{PF}_n$ the set of parking functions of length n.

$$\mathcal{PF} = \bigcup_{n>0} \mathcal{PF}_n.$$

.

Example.

$$f_1 = (7,3,1,4,2,5,2) \in \mathcal{PF}_7$$
$$f_2 = (7,3,1,4,2,5,4) \notin \mathcal{PF}_7$$

Theorem 1 (Konheim and Weiss, 1966)**.** *Let* pf_n *be the cardinal of* $\mathcal{PF}_n$*. We have*

$$pf_n = (n+1)^{n-1}.$$

Example $(n = 1, 2, 3)$**.**
• $n = 1 \ : \ pf_1 = 1$
  (1)
• $n = 2 \ : \ pf_2 = 3$

$(1,1) \quad (1,2) \quad (2,1)$

- $n=3 \; : \; pf_3 = 16$

  $(1,1,1) \quad (1,1,2) \quad (1,1,3) \quad (1,2,1) \quad (1,2,2) \quad (1,2,3) \quad (1,3,1)$
  $(1,3,2) \quad (2,1,1) \quad (2,1,2) \quad (2,1,3) \quad (2,2,1) \quad (2,3,1) \quad (3,1,1)$
  $(3,1,2) \quad (3,2,1)$

1.1.1 Primitive parking functions

Definition 2 (Primitive)**.** *A parking function* $(a_1, a_2, \ldots, a_n)$ *is said* primitive *if it is already in non-decreasing order.*

We denote by $\mathcal{PF'}_n$ the set of primitive parking functions of length n.

$$\mathcal{PF'} = \bigcup_{n>0} \mathcal{PF'}_n.$$

Example.

$$f_1 = (1,2,2,3) \in \mathcal{PF'}_4$$
$$f_2 = (1,2,3,2) \notin \mathcal{PF'}_4, \text{ even though } f_2 \in \mathcal{PF}_4$$

Theorem 2 (Stanley, 1999)**.** *Let* pf'_n *be the cardinal of* $\mathcal{PF'}_n$*. We have*

$$pf'_n = \frac{1}{n+1}\binom{2n}{n},$$

which is the n^{th} Catalan number $Cat(n)$.

Example ($n = 1,2,3$)**.**

- $n=1 \; : \; pf'_1 = 1$

  (1)
- $n=2 \; : \; pf'_2 = 2$

  $(1,1) \quad (1,2)$
- $n=3 \; : \; pf'_3 = 5$

  $(1,1,1) \quad (1,1,2) \quad (1,1,3) \quad (1,2,2) \quad (1,2,3)$

As comparing two parking functions can be tricky, one could wonder whether there exists an other combinatorial structure with a well-defined order that is in bijection with $\mathcal{PF}_n$ or $\mathcal{PF'}_n$. The next section will present such a candidate.

1.2 Non-crossing Partitions

Definition 3 (Non-crossing Partition)**.** *A* non-crossing partition *of a* totally ordered *set E is a set partition $P = \{E_1, E_2, \ldots, E_k\}$ such that if $a, c \in E_i$, $b, d \in E_j$, and $i \neq j$, then we do* not *have $a < b < c < d$, nor $a > b > c > d$.*

We denote by $\mathcal{NC}_n$ the set of non-crossing partitions of $\{1, 2, \ldots, n\}$.

$$\mathcal{NC} = \bigcup_{n>0} \mathcal{NC}_n.$$

From this point, we assume that every partition $P = \{B_1, \ldots, B_l\}$ is *sorted* such that :
• For each block $B_i = \{b_1, \ldots, b_k\} \in P$, $b_1 < \ldots < b_k$.
• $min(B_1) < \ldots < min(B_k)$.

Notation. $[n] = \{1, 2, \ldots, n\}$.

Example ($E = [6]$)**.**

$$P_1 = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\} \in \mathcal{NC}_6$$
$$P_2 = \{\{1, 2, 4\}, \{3, 5\}, \{6\}\} \notin \mathcal{NC}_6$$

Theorem 3 (Kreweras, 1972)**.** *Let nc_n be the cardinal of $\mathcal{NC}_n$. We have*

$$nc_n = \frac{1}{n+1}\binom{2n}{n}.$$

which is, again, the n^{th} Catalan number $Cat(n)$.

Example ($n = 1, 2, 3$)**.**
• $n = 1 \ : \ nc_1 = 1$
$\{\{1\}\}$
• $n = 2 \ : \ nc_2 = 2$
$\{\{1, 2\}\}$ $\{\{1\}, \{2\}\}$
• $n = 3 \ : \ nc_3 = 5$
$\{\{1, 2, 3\}\}$ $\{\{1\}, \{2, 3\}\}$ $\{\{1, 3\}, \{2\}\}$ $\{\{1, 2\}, \{3\}\}$ $\{\{1\}, \{2\}, \{3\}\}$

Proposition. *This means we can create a* bijection *between $\mathcal{PF}'_n$ and $\mathcal{NC}_n$.*

Proof.

- $\mathcal{NC}_n \to \mathcal{PF'}_n$: For each block B in the non-crossing partition, take $i = min(B)$, and let $k_i = size(B)$.
  $k_i = 0$ if i is not the minimum of a block.
  The corresponding parking function is $(\underbrace{1,\ldots,1}_{k_1},\underbrace{2,\ldots,2}_{k_2},\ldots,\underbrace{n,\ldots,n}_{k_n})$.

- $\mathcal{PF'}_n \to \mathcal{NC}_n$: For each i in $[n]$, if i appears n_i times in the parking function, B_i will be of size n_i with minimum element i. There is a unique set partition $P = \bigcup_i B_i$ of $[n]$ respecting these conditions that is non-crossing : for each minimum i in *decreasing order*, add the n_i first free elements of $[i+1, i+2, \ldots, n, 1, \ldots, i-1]$ to B_i.

□

Example $(n = 6)$**.**

$$P = \{\{1,2,5\},\{3,4\},\{6\}\} \qquad\qquad f = (1,1,1,3,3,6)$$

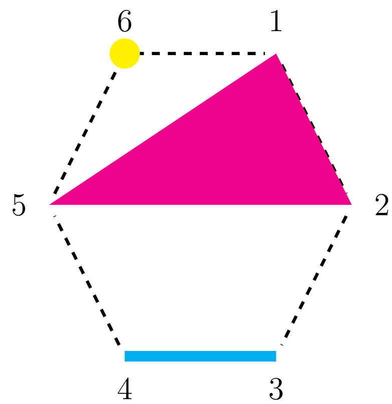
From the construction of the proof, we can deduce the following corollary:

Corollary. *A non-crossing partition can be represented by the minimums and sizes of its blocks.*

Example. $\{\{1,2,5\},\{3,4\},\{6\}\}$ *can be represented by the following dictionnary :*
- *1 : 3*
- *3 : 2*
- *6 : 1*

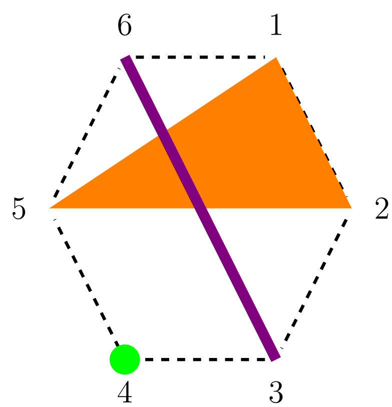
A non-crossing partition of $[n]$ can be represented graphically on a regular n-vertices polygon, with vertices labeled from 1 to n clockwise. We then represent each block $B = \{b_1, \ldots, b_k\}$ by the convex hull of $\{b_1, \ldots, b_k\}$.

Example $(P = \{\{1,2,5\},\{3,4\},\{6\}\})$**.**



Thus non-crossing meaning the hulls are disjoint.

Example (Counter-example : $P = \{\{1,5,2\},\{3,6\},\{4\}\}$)**.**



This partition is not *non-crossing, as the convex hulls of* $\{1,2,5\}$ *and* $\{3,6\}$ *are* not *disjoint.*

Now that we have a bijection between the two combinatorial structures, we wish to use the well-known order on $\mathcal{NC}_n$ and apply it to $\mathcal{PF}_n$.

1.2.1 The non-crossing partitions poset

Definition 4 ($\succ$)**.** *We say that P covers Q, written $P \succ Q$, if $\exists B_i, B_j \in P$ such that $Q = P - \{B_i, B_j\} \cup \{B_i \cup B_j\}$.*

Example. $\{\{1,6\},\{2,3\},\{4,5\}\} \succ \{\{1,2,3,6\},\{4,5\}\}$
- $B_i = \{1,6\}$
- $B_j = \{2,3\}$

Proposition. *This covering relation defines the* poset *of* $\mathcal{NC}_n$*.*

We denote by $\mathcal{NCC}_n$ the set of *maximal chains* in the poset of $\mathcal{NC}_n$.

$$\mathcal{NCC} = \bigcup_{n>0} \mathcal{NCC}_n.$$

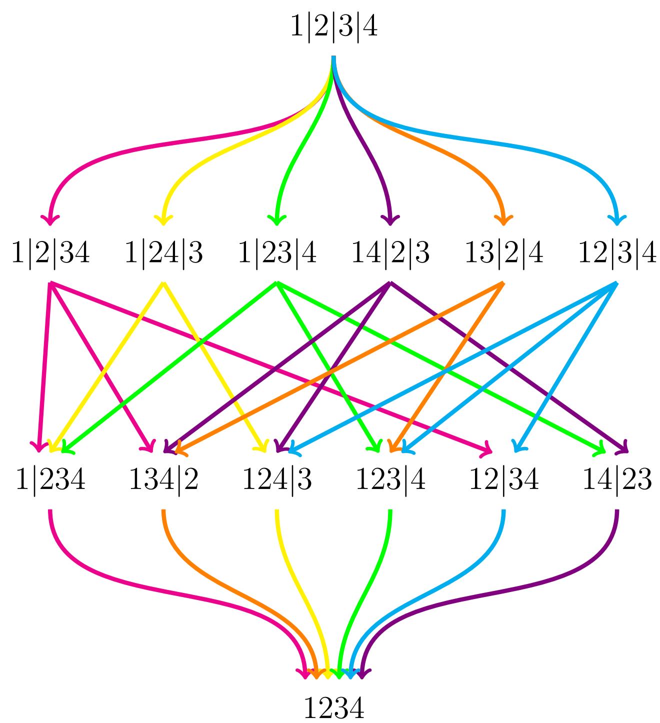
Remark. *The bottom element of this poset is* $\{\{1,\ldots,n\}\}$*, and the top element is* $\{\{1\},\ldots,\{n\}\}$*.*

Theorem 4 (Kreweras, 1972)**.** *Let* ncc_n *be the cardinal of* $\mathcal{NCC}_n$*. We have*

$$ncc_n = n^{n-2}.$$

.

Example (The poset of $\mathcal{NC}_4$)**.**
To shorten labels, we represent $\{\{1\},\{2,3\},\{4\}\}$ *by* $1|23|4$.



There are $4^2 = 16$ different maximal chains, and $\frac{1}{5}\binom{8}{4} = \frac{70}{5} = 14$ elements in this poset.

One of the main constructions on non-crossing partitions is the *complement*, defined by Kreweras. Not only does it posess multiple interesting properties that will be presented in the next part, but will also be helpful once we attempt to generalize the aforementioned bijection to the rational case, in the second chapter.

1.2.2 Kreweras complement

Definition 5 (Associated Permutation)**.** *The* permutation σ *associated to a non-crossing partition has a cycle $(b_1, \ldots, b_k)$ for each block $B = \{b_1, \ldots, b_k\}$ of the partition.*

Example. *The permutation associated to* $\{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$ *is* $(1\ 2\ 5)\ (3\ 4)\ (6) = 254316$.

Definition 6 (Kreweras Complement)**.** *The* Kreweras complement $K(P)$ *of a non-crossing partition P is defined as follows :*
- *Let σ be the permutation associated to P*
- *Let π be the permutation $(n\ n-1\ n-2\ \ldots\ 3\ 2\ 1) = n123 \ldots n-1$*
- *$K(P)$ is the* non-crossing partition *associated to $\pi\sigma$.*

Example $(P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\})$**.**
- $\sigma = (1\ 2\ 5)\ (3\ 4)\ (6) = 254316$
- $\pi = (6\ 5\ 4\ 3\ 2\ 1) = 612345$
- $\pi\sigma = 143265 = (1)\ (2\ 4)\ (3)\ (5\ 6)$
- $K(P) = \{\{1\}, \{2, 4\}, \{3\}, \{5, 6\}\}$

Proposition (Kreweras minimums)**.** *Let $P = \{B_1, \ldots, B_k\}$ be a non-crossing partition. Let $K(P) = \{B'_1, \ldots, B'_l\}$ be its Kreweras complement. Then*

$$\bigcup_{1 \leq i \leq l} min(B'_i) = B_1 \cup \bigcup_{1 < j \leq k} B_i - max(B_i).$$

Example ($P = \{\{1,2,5\},\{3,4\},\{6\}\}$)**.**

- $K(P) = \{\{1\},\{2,4\},\{3\},\{5,6\}\}$
- $\bigcup min(B'_i) = \{1,2,3,5\}$
- $B_1 \cup \bigcup B_i - max(B_i) = \{1,2,5\} \cup \{3,4\} - \{4\} \cup \{6\} - \{6\} = \{1,2,5\} \cup \{3\} \cup \emptyset = \{1,2,3,5\}$

Notation. $B_{[i]}$ *= block containing* i*.*

Proposition (Kreweras block sizes)**.** *Let* $P = \{B_1, \ldots, B_k\}$ *be a non-crossing partition. Let* $K(P) = \{B'_1, \ldots, B'_l\}$ *be its Kreweras complement. Then the size of the block* B'_i *is defined as follows :*

- *Let* m_i *be the the* i^{th} *minimum of* $K(P)$*.*
- *Define a* transition $\phi(e)$ *as*

  Let $j = e + 1$ *(or* 1 *if* $e = n$*)*

  $\phi(e) = max(B_{[j]})$
- *The size of* B'_i *is* k_{min} *such that* $k_{min} = min\{k > 0 \mid \phi^k(m_i) \in B_{[m_i]}\}$*.*

Example ($P = \{\{1,2,5\},\{3,4\},\{6\}\}$)**.**

- $mins = \{1,2,3,5\}$
- $m_1 = 1$

  $B_{[1]} = B_1$

  $max(B_{[2]} = max(B_1) = 5$

  The size for m_1 *is* 1*.*
- m_2

  $B_{[2]} = B_1$

  $max(B_{[3]}) = max(B_2) = 4$

  $max(B_{[5]}) = max(B_1) = 5$

  The size for m_2 *is* 2*.*

- $m_3 = 3$

  $B_{[3]} = B_2$

  $max(B_{[4]}) = max(B_2) = 4$

  The size for m_3 is 1.

- $m_4 = 5$

  $B_{[5]} = B_1$

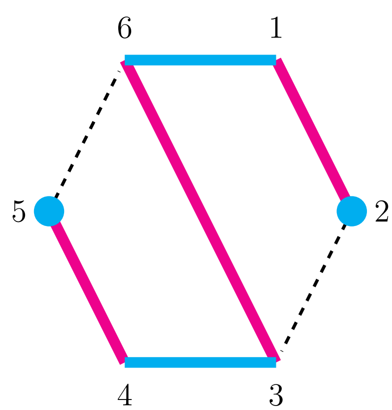
  $max(B_{[6]}) = max(B_3) = 6$

  $max(B_{[1]}) = max(B_1) = 5$

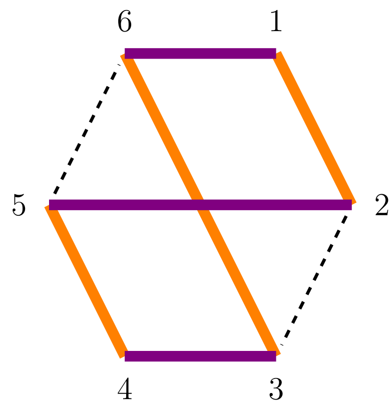
  The size for m_4 is 2.

Definition 7 (Mutually Non-crossing Partitions)**.** *2 partitions P and Q are said* mutually non-crossing *if :*
• *P is non-crossing*
• *Q is non-crossing*
• *For every block B_i of P and every block B_j of Q, if $a, c \in B_i$ and $b, d \in B_j$, then we can* not *have $a < b < c < d$, nor $a > b > c > d$.*

Example ($P = \{\{1, 2\}, \{3, 6\},\ \{4, 5\}\}, Q = \{\{1, 6\}, \{2\}, \{3, 4\}, \{5\}\}$)**.**



Example (Counter-example : $P = \{\{1, 2\}, \{3, 6\},\ \{4, 5\}\}, Q = \{\{1, 6\}, \{2, 5\}, \{3, 4\}\}$)**.**

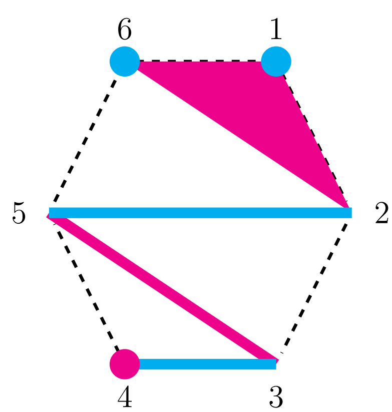


Remark. *Note that vertices* can *touch, but the edges of the convex hulls can* not *cross.*

Proposition (Bodnar, 2017)**.** *For any non-crossing partition P, P and $K(P)$ are mutually non-crossing.*

Furthermore, $K(P)$ is a *densest* partition that is mutually non-crossing with P. That is, *no* partition Q that is mutually non-crossing with P has less blocks than $K(P)$.

Example ($P = \{1, 2, 6\}, \{3, 5\}, \{4\}\}$)**.** $Q = \{\{1\}, \{2, 5\}, \{3, 4\}, \{6\}\}$

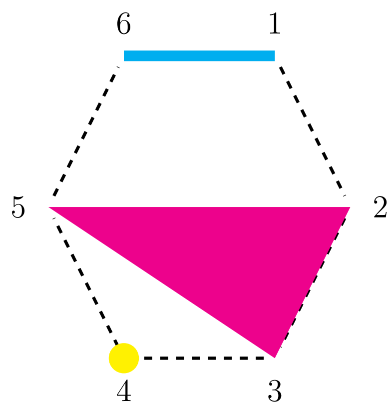


By construction, the symmetric group of order n carries a natural action on $\mathcal{NC}_n$. The following section presents some of the constructions and properties that emerge from this action.

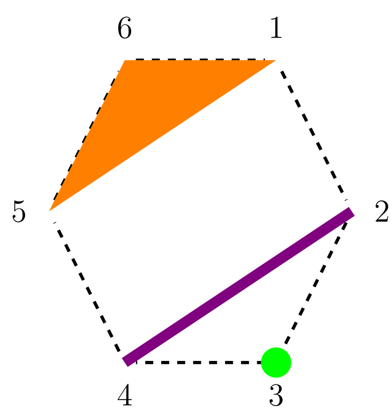
1.2.3 Action of $\mathfrak{S}_n$ on partitions of $[n]$

Definition 8 (Action of $\mathfrak{S}_n$)**.** *The action of $\mathfrak{S}_n$ on a partition $P = \{B_1, \ldots, B_l\}$ of $[n]$ is defined by :*
- *For each block $B_i = \{b_1, \ldots, b_k\}$: $\sigma(Bi) = \{\sigma(b_1), \ldots, \sigma(b_k)\}$.*
- *When $P \in \mathcal{NC}_n$, we denote $\rho = \sigma(P) = \{\sigma(B_1), \ldots, \sigma(B_l)\}$.*

Example ($\sigma = 415362, P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$)**.**
$\sigma(P) = \{\{1, 5, 6\}, \{2, 4\}, \{3\}\}$



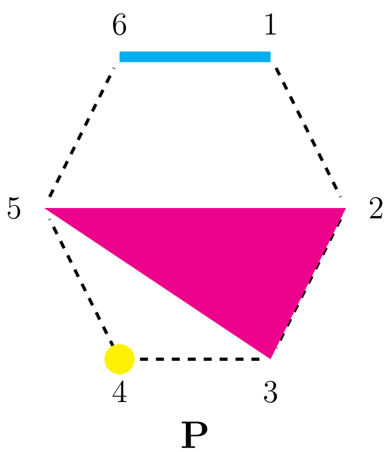
P

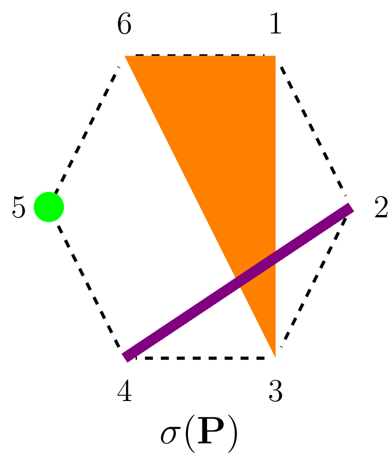


$\sigma(\mathbf{P})$

Remark. *Note that $\mathcal{NC}_n$ is* not *stable under the action of $\mathfrak{S}_n$. That is, even if P is non-crossing, $\sigma(P)$ is* not *necessarily non-crossing.*

Example (Counter-example : $\sigma = 413562, P = \{\{1,6\},\{2,3,5\},\{4\}\}$)**.** $\sigma(P) = \{\{1,3,6\},\{2,4\},\{5\}\}$



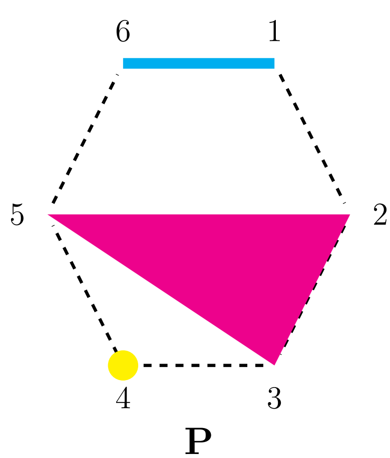


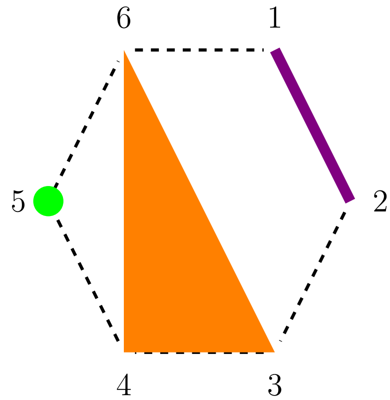
Definition 9 (Rotation)**.** *We define the* rotation operator *rot of $P \in \mathcal{NC}_n$ as $rot(P) = (1\ 2\ 3\ \ \ldots\ n)(P) = 23\ldots n1(P)$.*

Conversely, we define rot^{-1} of P as $rot^{-1}(P) = (n\ n-1\ \ \ldots 3\ 2\ 1)(P) = n12\ldots n-1(P)$.

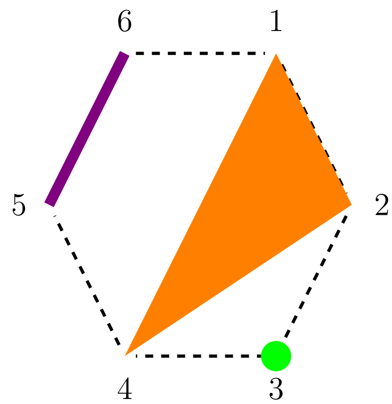
Example ($P = \{\{1,6\},\{2,3,5\},\{4\}\}$)**.**

- $rot(P) = \{\{1,2\},\{3,4,6\},\{5\}\}$
- $rot^{-1}(P) = \{\{1,2,4\},\{3\},\{5,6\}\}$





$\mathbf{rot(P)}$



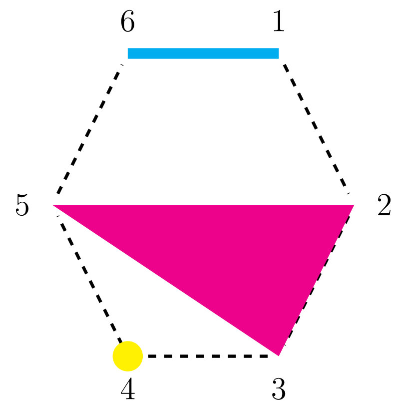
$\mathbf{rot^{-1}(P)}$

Remark.

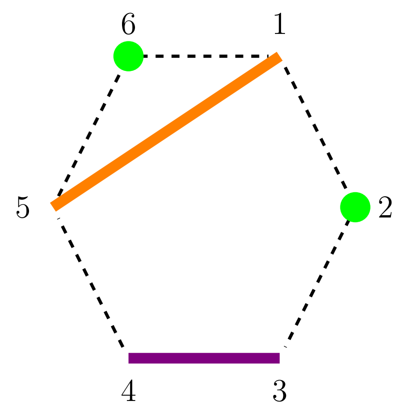
- *$rot(rot^{-1}(P)) = rot^{-1}(rot(P)) = P$.*
- *$rot(P)$ and $rot^{-1}(P)$ are always non-crossing partitions.*
- *If $P \in \mathcal{NC}_n$, then $rot^n(P) = rot^{-n}(P) = P$.*

Proposition (Bodnar, 2017)**.** *$K(K(P)) = rot^{-1}(P)$.*

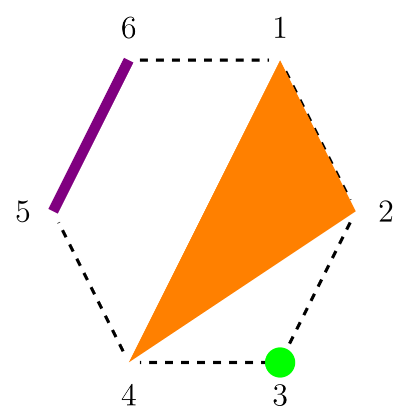
Example ($P = \{\{1,6\},\{2,3,5\},\{4\}\}$)**.**



P



K(**P**)



K(**K**(**P**))

Now that we assessed the link between primitive parking functions and non-crossing partitions, we wish to find a similar structure for classical parking functions. Hence, we introduce an other combinatorial object : the *non-crossing 2-partition.*

1.3 Non-crossing 2-partitions

Definition 10 (Non-crossing 2-partition)**.** *A* non-crossing 2-partition *of a* totally ordered *set E is a pair (P, σ) where :*
- *P is a non-crossing partition of E.*
- *σ is a permutation of the elements of E.*
- *For each* sorted *block $B_i = \{b_1, \ldots, b_k\} \in P$, we have $\sigma(b_i) < \ldots < \sigma(b_k)$.*

We denote by $\mathcal{NC}_n^2$ the set of non-crossing 2-partitions of $[n]$.

$$\mathcal{NC}^2 = \bigcup_{n>0} \mathcal{NC}_n^2.$$

.

Example ($\mathcal{NC}_6^2$)**.** $P = \{\{1,6\},\{2,3,5\},\{4\}\}$ $\sigma = 413265$
$\rho = \{\{1,3,6\},\{2\},\{4,5\}\}$

Theorem 5 (Edelman, 1979)**.** *Let nc_n^2 be the cardinal of $\mathcal{NC}_n^2$. We have*

$$nc_n^2 = (n+1)^{n-1}.$$

Example ($n = 1, 2, 3$)**.**

- $n = 1$: $nc_1^2 = 1$

  $\{\{1\}\}$ 1 $\rho = P$

- $n = 2$: $nc_2^2 = 3$

  $\{\{1\},\{2\}\}$ 12 $\rho = P$

  $\{\{1\},\{2\}\}$ 21 $\rho = P$

  $\{\{1,2\}\}$ 12 $\rho = P$

- $n = 3$: $nc_3^2 = 16$

$\{\{1\},\{2\},\{3\}\}$	123	$\rho = P$
$\{\{1\},\{2\},\{3\}\}$	132	$\rho = P$
$\{\{1\},\{2\},\{3\}\}$	213	$\rho = P$
$\{\{1\},\{2\},\{3\}\}$	231	$\rho = P$
$\{\{1\},\{2\},\{3\}\}$	312	$\rho = P$
$\{\{1\},\{2\},\{3\}\}$	321	$\rho = P$
$\{\{1,2\},\{3\}\}$	123	$\rho = P$
$\{\{1,2\},\{3\}\}$	132	$\rho = \{\{1,3\},\{2\}\}$
$\{\{1,2\},\{3\}\}$	231	$\rho = \{\{1\},\{2,3\}\}$
$\{\{1\},\{2,3\}\}$	123	$\rho = P$
$\{\{1\},\{2,3\}\}$	213	$\rho = \{\{1,3\},\{2\}\}$
$\{\{1\},\{2,3\}\}$	312	$\rho = \{\{1,2\},\{3\}\}$
$\{\{1,3\},\{2\}\}$	123	$\rho = P$
$\{\{1,3\},\{2\}\}$	132	$\rho = \{\{1,2\},\{3\}\}$
$\{\{1,3\},\{2\}\}$	213	$\rho = \{\{1\},\{2,3\}\}$
$\{\{1,2,3\}\}$	123	$\rho = P$

Proposition. *This means we can create a* bijection *between* $\mathcal{PF}_n$ *and* $\mathcal{NC}_n^2$.

Proof.

- $\mathcal{PF}_n \to \mathcal{NC}_n^2$: Let $f = (a_1, \ldots, a_n) \in \mathcal{PF}_n$ be our parking function. For $i \in \{1, \ldots, n\}$, we define :

  l_i : the number of occurences of i in f.

  im_i : $\{j \mid a_j = i\}$

  The corresponding non-crossing partition will have the following constraints :

  For each $i \in \{1, \ldots, n\}$, if $l_i > 0$, then there is a block $B_{[i]}$ of length l_i with minimum element i.

$\sigma(B_{[i]}) = im_i$

There is a unique set partition $P = \bigcup_i B_{[i]}$ of $[n]$ and a unique permutation σ respecting these conditions such that $(P,\sigma) \in \mathcal{NC}_n^2$: for each minimum i in *decreasing order*, add the n_i first free elements of $[i+1, i+2, \ldots, n, 1, \ldots, i-1]$ to B_i. σ is then trivially obtained by the second constraint.

- $\mathcal{NC}_n^2 \to \mathcal{PF}_n$: Let (P,σ) with $P = \{B_1, \ldots, B_l\}$ be our non-crossing 2-partition. For each block $B_i = \{b_1, \ldots, b_k\} \in P$:

  $m_i = min(B_i) = b_1$

  $pos_i = \sigma(B_i)$

  For each $j \in pos_i$, we define $a_j = m_i$

  The corresponding parking function is $(a_1, \ldots, a_n)$.

□

Example $(n = 8)$**.**

$$P = \{\{1,2,5\},\{3,4\},\{6,8\},\{7\}\}$$
$$\sigma = 36187245$$
$$f = (3,6,1,7,6,1,1,3)$$

Following the path of the classical primitive case, we recall the cover relation defined in [9] in order to deduce a poset for $\mathcal{PF}_n$.

1.3.1 The non-crossing 2-partitions poset

Definition 11 ($\succ^2$)**.** *We say that (P,σ) covers (Q,τ), written $(P,\sigma) \succ^2 (Q,\tau)$, if $\exists B_i, B_j \in P$ such that*

- $Q = P - \{B_i, B_j\} \cup \{B_i \cup B_j\}$
- $l \neq i, jb \in B_l \to \tau(b) = \sigma(b)$
- *Let* $B_i \cup B_j = \{b_1, \ldots, b_k\}$ *:*

  $\tau(B_i \cup B_j) = \sigma(B_i \cup B_j)$

  $\tau(b_1) < \ldots < \tau(b_k)$

Example.

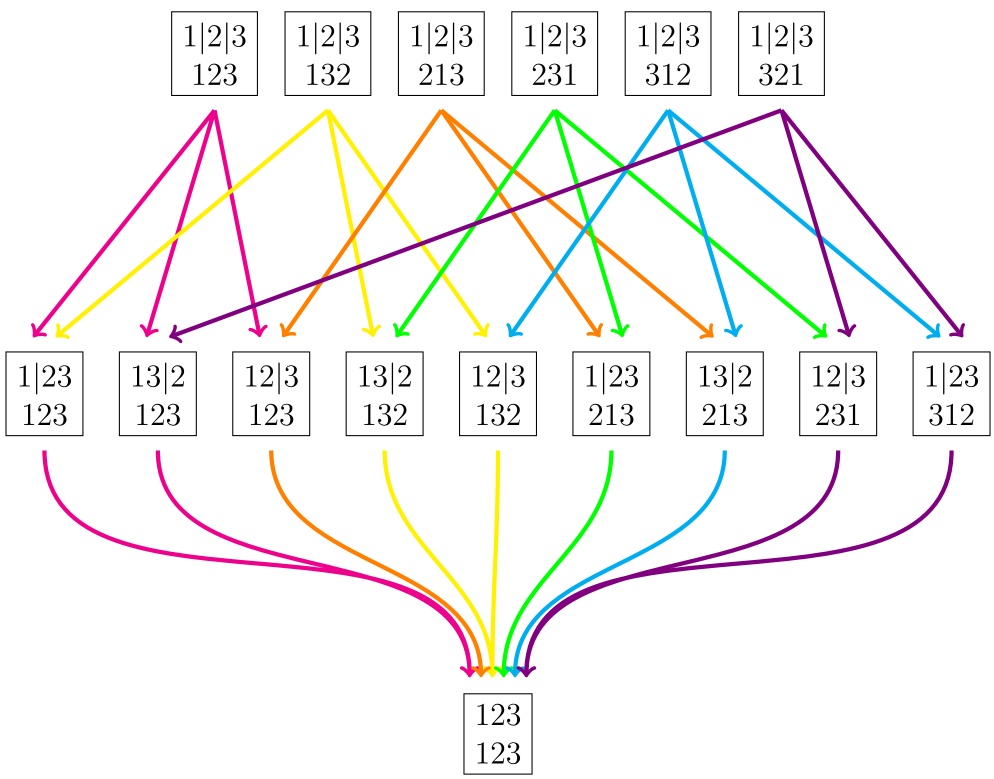
- $P = \{\{1,6\},\{2,3\},\{4\},\{5\}\}$
- $\sigma = 236154$
- $Q = \{\{1,6\},\{2,3,5\},\{4\}\}$
- $\tau = 235164$
- $(P,\sigma) \succ^2 (Q,\tau)$
- $(P,\sigma) \not\succ^2 (Q,\sigma)$, *because* $\sigma(\{2,3,5\}) = \{3,6,5\}$ *is* not *ordered.*

Proposition. *This covering relation defines the* poset *of* $\mathcal{NC}_n^2$.

Remark. *The bottom element of this poset is* $(\{\{1,\ldots,n\}\}, 12\ldots n)$, *and the top elements are* $\{(\{\{1\},\ldots,\{n\}\},\sigma) \mid \sigma \in \mathfrak{S}_n\}$.

Example (The poset of $\mathcal{NC}_3^2$)**.**
To shorten labels, we represent $(\{\{1,3\},\{2\}\}, 213)$ *by*

$$\begin{array}{|c|}\hline 13|2 \\ 213 \\ \hline\end{array}$$



There are $4^2 = 16$ *elements in this poset.*

We can now define a poset for $\mathcal{PF}_n$, in which the *height* of any parking function will be defined by its *rank*.

1.3.2 The parking functions poset

Definition 12 (Rank). *Given $f = (a_1, \ldots, a_n) \in \mathcal{PF}_n$, let*

$$b_i = \begin{cases} 1 \text{ if } \exists j \mid a_j = i \\ 0 \text{ otherwise} \end{cases}$$

We define the rank *of f, noted $rk(f)$, as*

$$\sum_{1 \le i \le n} b_i.$$

Example.

$$rk((1,5,4,2,3,3,1)) = 5$$
$$rk((4,7,1,1,3,2,2,8)) = 6$$

Definition 13 ($\succ_{pf}$). *Since $\mathcal{PF}_n$ and $\mathcal{NC}_n^2$ are in bijection, we can define a* covering relation *$\succ_{pf}$ for $\mathcal{PF}_n$ as follows :*
$f \in \mathcal{PF}_n \succ_{pf} g \in \mathcal{PF}_n$ if and only if :

- *(P, σ) is the non-crossing 2-partition associated to f.*
- *(Q, τ) is the non-crossing 2-partition associated to g.*
- *$(P, \sigma) \succ^2 (Q, \tau)$*

Example.
- $P = \{\{1,6\}, \{2,3\}, \{4\}, \{5\}\}$
- $\sigma = 236154$
- $Q = \{\{1,6\}, \{2,3,5\}, \{4\}\}$
- $\tau = 235164$
- $f = (4,1,2,1,5,2) \succ_{pf} g = (4,1,2,1,2,2)$

Remark. *If $f \succ_{pf} g$, then $rk(f) = rk(g) + 1$, and there exists i and j such that :*

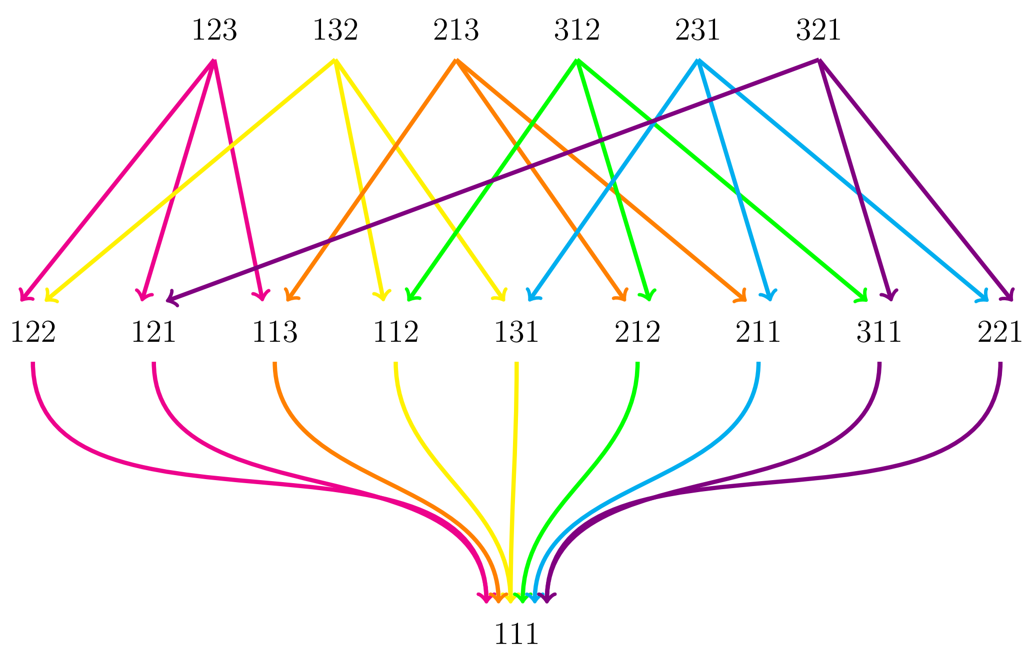
- $i < j$
- *There is at least* 1 *occurence of* i *in* f.
- *There is at least* 1 *occurence of* j *in* f.

$$b_k = \begin{cases} i \text{ if } a_k = j \\ a_k \text{ otherwise} \end{cases}$$

Proposition. *This covering relation defines the* poset *of* $\mathcal{PF}_n$.

Remark. *The bottom element of this poset is* $(\underbrace{1, \ldots, 1}_{n})$, *and the top elements are the* permutations *of* $\{1, \ldots, n\}$.

Example (The poset of $\mathcal{PF}_3$)**.**



While non-crossing (2-)partitions are frequently used to define a poset for (primitive) parking functions, it is rather unpractical that using a bijection to define the cover relation is necessary.
Thereby, the following section will present our tentative at finding a more

direct definition for the cover relation of $\mathcal{PF}_n$ and $\mathcal{PF'}_n$, this time using a bijection with (decorated) Dyck Paths. A main benefit of this solution is that we define a cover relation for both structures, and the poset of one can be obtained by applying the given bijection to the poset of the other.
Furthermore, the two main results of this article – Theorem 8 and the corresponding Conjecture for the classical case – rise from the number of intervals in those posets.

1.4 A direct poset linked to Dyck paths

1.4.1 Dyck Paths

Notation. *We denote the number of occurences of a symbol s in a word w by $|w|_s$.*

Definition 14 (Dyck path)**.** *A* Dyck word *is a word $w \in \{0,1\}^*$ such that :*

- *for each* prefix *w' of w, $|w'|_1 \geqslant |w'|_0$.*
- *$|w|_0 = |w|_1$.*

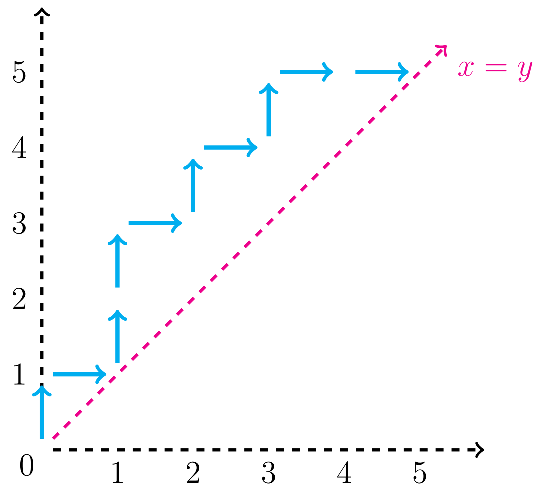
A Dyck word of length $2n$ can be represented as a path *from $(0,0)$ to (n,n) that stays over $x = y$, called a* Dyck path *:*

- *Each* 1 *corresponds to a* North step $\uparrow$.
- *Each* 0 *corresponds to an* East step $\rightarrow$.

We denote by $\mathcal{D}_n$ the set of Dyck words of length $2n$.

Example $(n = 5)$**.**

$w_1 = 1011000110$ *is* not *a Dyck word, because* $|1011000|_0 > |1011000|_1$.
$w_2 = 1011010101$ *is* not *a Dyck word, because* $|w_2|_0 \neq |w_2|_1$.
$w_3 = 1011010100$ is *a Dyck word :*

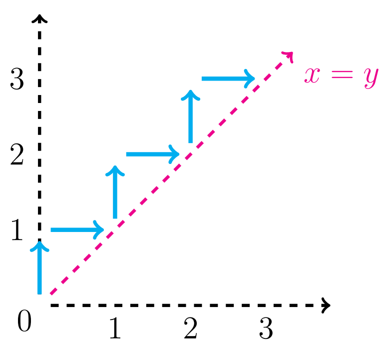


Theorem 6 (André, 1887)**.** *Let d_n be the cardinal of $\mathcal{D}_n$. We have*

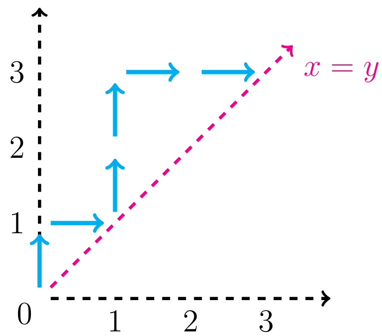
$$d_n = \frac{1}{n+1}\binom{2n}{n}.$$

which is the n^{th} Catalan number.

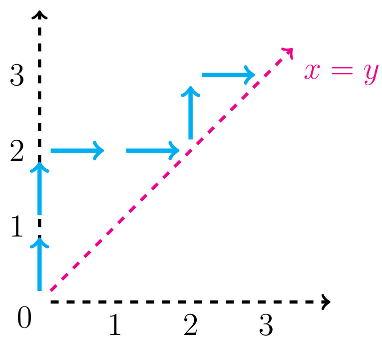
Example ($n = 3$)**.** $d_n = 5$.



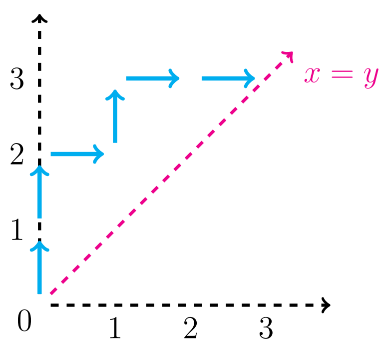
101010



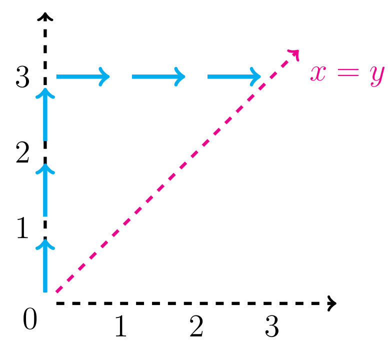
101100



110010



110100



111000

Proposition. *This means we can create a* bijection *between* $\mathcal{PF}'_n$ *and* $\mathcal{D}_n$*.*

Proof.

- $\mathcal{PF}'_n \to \mathcal{D}_n$: Let $f = (a_1, \dots, a_n) \in \mathcal{PF}'_n$ be our primitive parking function. For $i \in \{1, \dots, n\}$, we define l_i the number of occurences of i in f.
  The corresponding Dyck word will be $\underbrace{1 \cdots 1}_{l_1} 0 \underbrace{1 \cdots 1}_{l_2} 0 \cdots \underbrace{1 \cdots 1}_{l_n} 0$.

- $\mathcal{D}_n \to \mathcal{PF}'_n$: Let w be our Dyck word, and consider its path representation. We define s_i to be the distance between the segment from $(0, i-1)$ to $(0, i)$ and the i^{th} North step. Then, let $a_i = s_i + 1$.
  The corresponding primitive parking function is $(a_1, \dots, a_n)$.

□

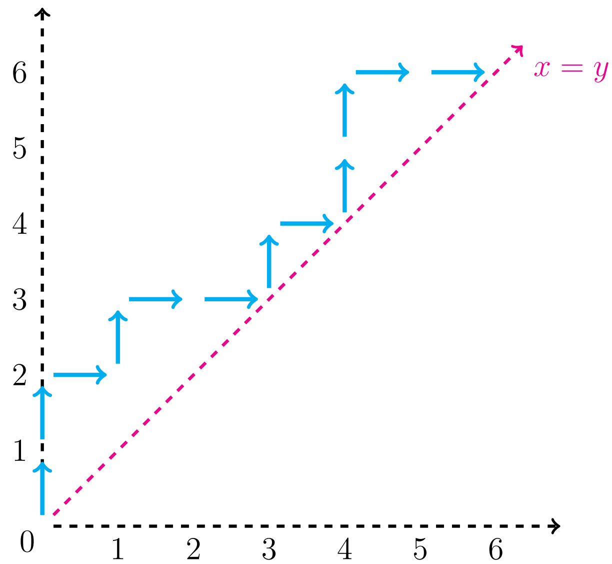
Example $(n = 6, \mathcal{PF}'_n \to \mathcal{D}_n)$**.**

- $f = (1, 1, 2, 4, 5, 5)$

  $l_1 = 2$ $\quad$ $l_2 = 1$ $\quad$ $l_3 = 0$

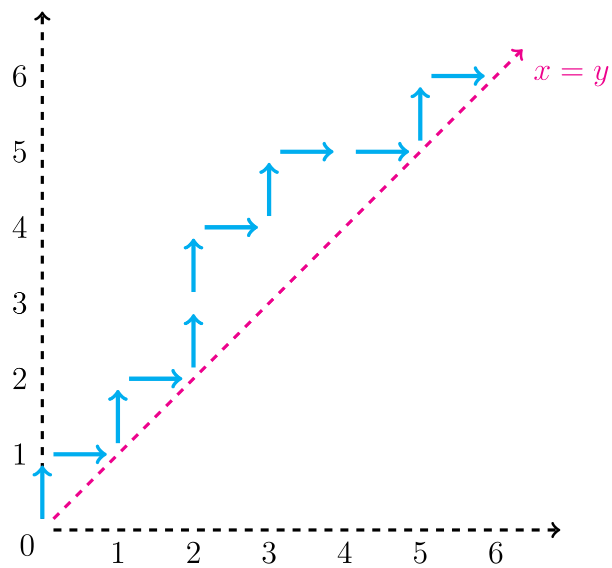
  $l_4 = 1$ $\quad$ $l_5 = 2$ $\quad$ $l_6 = 0$

- $w = (110100101100)$



Example $(n = 6, \mathcal{D}_n \to \mathcal{PF'}_n)$**.**

- $w = 101011010010$



- *Distances :*

$s_1 = 0 \qquad\qquad\qquad a_1 = 1$

$$s_2 = 1 \qquad a_2 = 2$$
$$s_3 = 2 \qquad a_3 = 3$$
$$s_4 = 2 \qquad a_4 = 3$$
$$s_5 = 3 \qquad a_5 = 4$$
$$s_6 = 5 \qquad a_6 = 6$$

- $f = (1, 2, 3, 3, 4, 6)$

To apply a similar bijection to non-primitive parking functions, we will need an upgraded version of Dyck paths, that is, Labeled Dyck Paths.

1.4.2 Labeled Dyck Paths

Definition 15 (Labeled Dyck Word)**.** *A* labeled Dyck word *is a word* $w \in \{0, \ldots, n\}^*$ *such that :*

- *for each prefix* w' *of* w, $|w'|_{\neq 0} \geqslant |w'|_0$.
- $|w|_0 = |w|_{\neq 0}$.
- *for each* $i \in \{1, \ldots, n\}$, *w has exactly one occurence of* i.
- *if* $w_i \neq 0$ *and* $w_{i+1} \neq 0$, *then* $w_i < w_{i+1}$. *That is, consecutive North steps have increasing labels.*

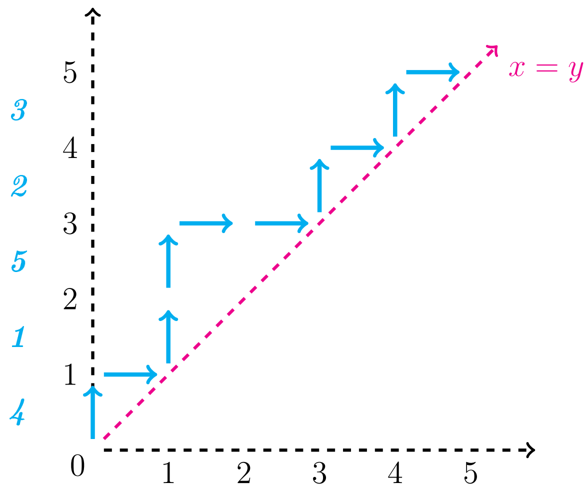
A labeled Dyck word of length $2n$ can be represented as a *path* from $(0, 0)$ to (n, n), where each North step is associated to a label :

- Each $i \neq 0$ corresponds to a *North step* $\uparrow$ labeled i.
- Each 0 corresponds to an *East step* $\rightarrow$.

Those paths are called *labeled Dyck paths.*
We denote by $\mathcal{LD}_n$ the set of labeled Dyck words of length $2n$.

Example $(n = 5)$**.**

$w_1 = 4051002030$ *is* not *a labeled Dyck word, because* $5 > 1$.
$w_2 = 4015002030$ is *a labeled Dyck word :*



Theorem 7. *Let ld_n be the cardinal of $\mathcal{LD}_n$. We have*

$$ld_n = (n+1)^{n-1}$$

.

Example $(n = 3)$**.** $ld_n = 4^2 = 16$

- *Word of shape $XXX000$:*

  123000

- *Words of shape $XX0X00$:*

  120300 130200 230100

- *Words of shape $XX00X0$:*

  120030 130020 230010

- *Words of shape $X0XX00$:*

  102300 201300 301200

- *Words of shape $X0X0X0$:*

  102030 103020 201030
  203010 301020 302010

Proposition. *This means we can create a* bijection *between $\mathcal{PF}_n$ and $\mathcal{LD}_n$.*

Proof.

- $\mathcal{PF}_n \to \mathcal{LD}_n$: Let $f = (a_1, \dots, a_n) \in \mathcal{PF}_n$ be our parking function. For $i \in \{1, \dots, n\}$, we define $im_i : \{j \mid a_j = i\}$.
  We then define $im_{i,1}, \dots, im_{i,k_i}$ to be the elements of im_i in increasing order.
  The corresponding labeled Dyck word will be
  $\underbrace{im_{1,1} \cdots im_{1,k_1}}_{im_1} 0 \underbrace{im_{2,1} \cdots im_{2,k_2}}_{im_2} 0 \cdots \underbrace{im_{n,1} \cdots im_{n,k_n}}_{im_n} 0.$

- $\mathcal{LD}_n \to \mathcal{PF}_n$: Let w be our labeled Dyck word, and consider its path representation. We define s_i to be the distance between the segment from $(0, i-1)$ to $(0, i)$ and the i^{th} North step.
  Then, let $label(i)$ be the label of the i^{th} North step, and $dist_i = \{label(j) | s_j = i\}$ be the set of the labels of all North steps at distance i.
  Then, if $j \in dist_i$, let $a_j = i + 1$.
  The corresponding parking function is $(a_1, \dots, a_n)$.

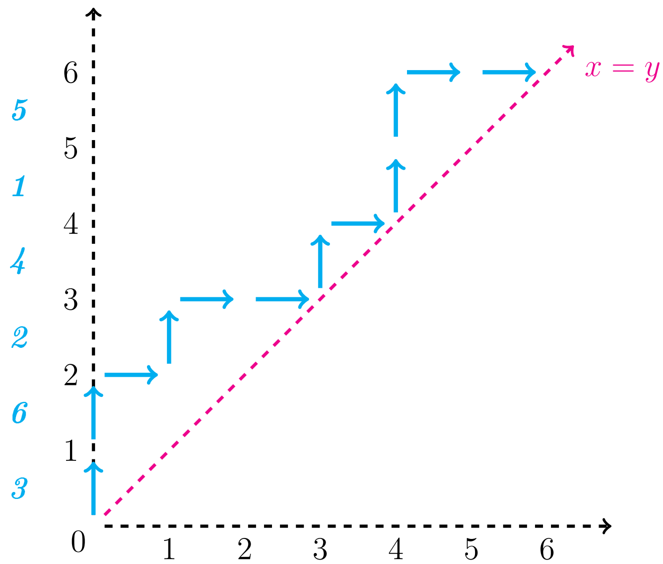
□

Example ($n = 6, \mathcal{PF}_n \to \mathcal{LD}_n$)**.**

- $f = (5, 2, 1, 4, 5, 1)$

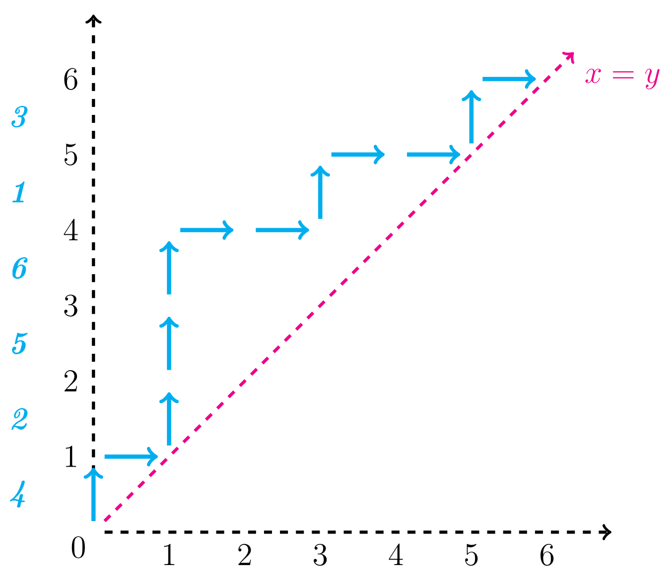
$im_1 = \{3, 6\}$	$im_2 = \{2\}$	$im_3 = \emptyset$
$im_4 = \{4\}$	$im_5 = \{1, 5\}$	$im_6 = \emptyset$

- $w = 360200401500$



Example $(n = 6, \mathcal{LD}_n \to \mathcal{PF}_n)$.

- $w = 402560010030$



- *Distances :*

$s_1 = 0$	$s_2 = 1$	$s_3 = 1$
$s_4 = 1$	$s_5 = 3$	$s_6 = 5$

- *Labels :*

$dist_0 = \{4\}$ $\quad$ $dist_1 = \{2,5,6\}$ $\quad$ $dist_2 = \emptyset$

$dist_3 = \{1\}$ $\quad$ $dist_4 = \emptyset$ $\quad$ $dist_5 = \{3\}$

- $f = (4, 2, 6, 1, 2, 2)$

Remark. *The primitive parking functions are exactly the parking functions corresponding to labeled Dyck paths where the i^{th} North step is labeled i.*

With those bijections in mind, we can now define cover relations that will issue in the expected bijective posets.

1.4.3 Dyck - Parking Posets

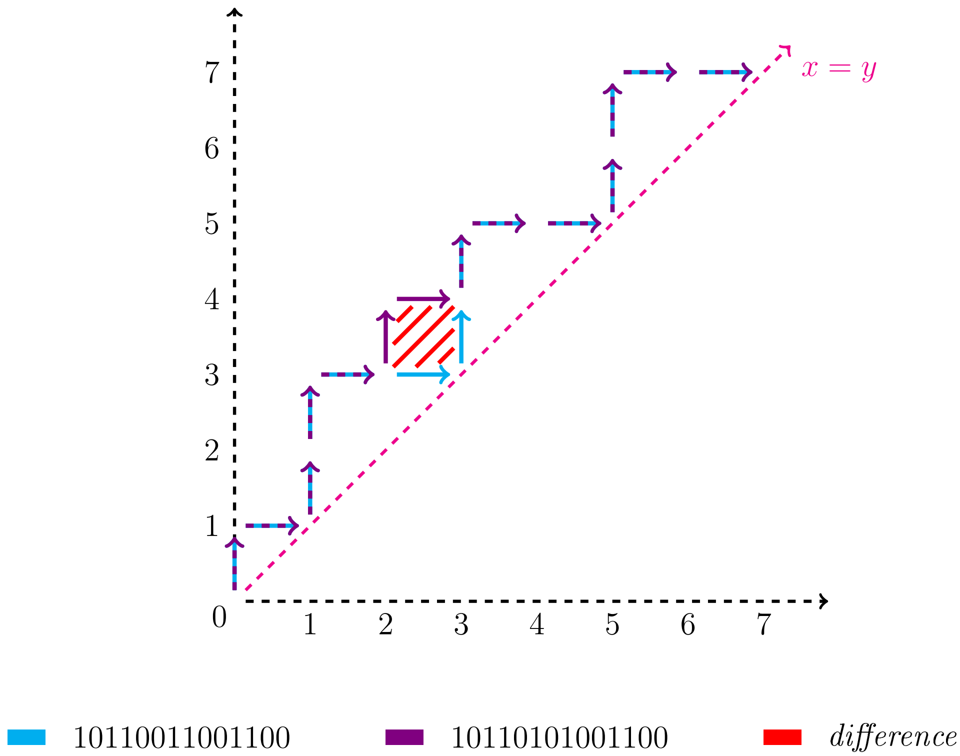
Primitive Dyck - Parking Posets

Definition 16 ($\gtrdot_d$)**.** *For w and w' two Dyck words, we say that w covers w', written $w \gtrdot_d w'$, if $\exists w_1, w_2$ such that :*

- $w = w_1 01 w_2$
- $w' = w_1 10 w_2$

Example ($n = 7$)**.** $10110011001100 \gtrdot_d 10110101001100$

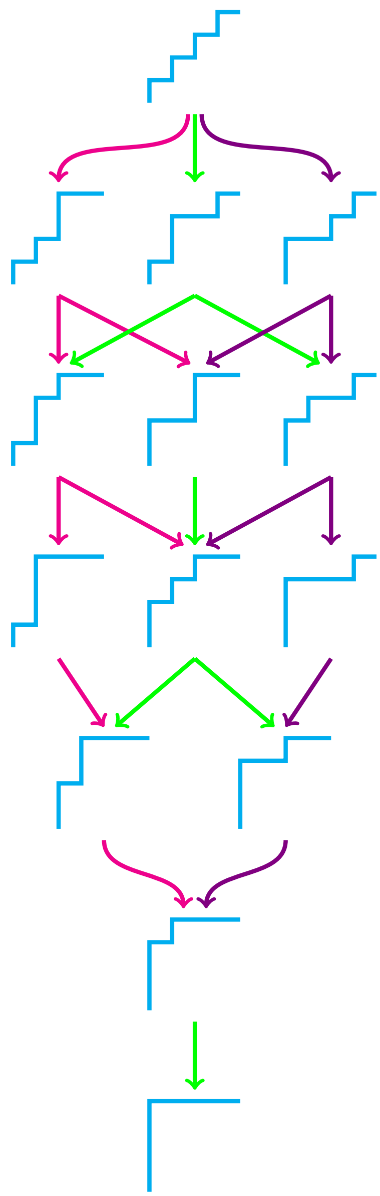
- $w_1 = 10110$
- $w_2 = 1001100$



Remark. *If $w_1 \gtrdot_d w_2$, then the path corresponding to w_2 is* over *the path corresponding to w_1, and the* difference *between the two paths is a square of size 1 by 1.*

Proposition. *This covering relation defines a* poset *for $\mathcal{D}_n$.*

Example (The poset of $\mathcal{D}_4$)**.**



There are $\frac{1}{5}\binom{8}{4} = \frac{70}{5} = 14$ *elements in this poset.*

Definition 17 (Nested Dyck paths)**.** *Two Dyck Paths* w_1 *and* w_2 *are said* nested *if* w_1 *is equal to* w_2 *or over* w_2*.*

We can thus easily deduce the following proposition from the preceding remark.

Proposition. *If there exists a sequence* $w_1 \gtrdot_d w_2 \gtrdot_d w_3 \gtrdot_d \cdots \gtrdot_d w_k$ *with* $k \geqslant 0$*, then* w_1 *and* w_k *are* nested.

Now that we have defined the cover relation for Dyck paths, we have to define the corresponding relation for primitive parking functions, that will allow us to create the wanted bijective posets.

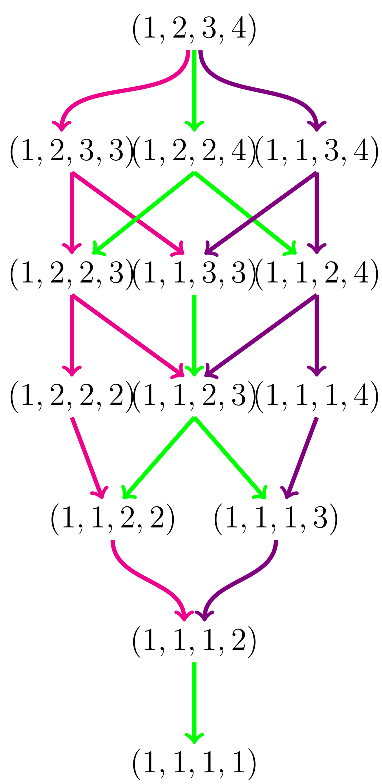
Definition 18 ($\gtrdot'$)**.** *For* f *and* g *two primitive parking functions, we say that* f *covers* g*, written* $f \gtrdot' g$*, if* $\exists i$ *such that :*

- $f = (a_1, \ldots, a_{i-1}, a_i, \quad a_{i+1}, \ldots, a_n)$
- $g = (a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_n)$

Example ($n = 6$)**.** $(1, 1, 2, 3, 4, 5) \gtrdot' (1, 1, 2, 3, 3, 5)$

Proposition. *This covering relation defines a* poset *for* $\mathcal{PF'}_n$*.*

Example (The poset of $\mathcal{PF'}_4$)**.**



There are $\frac{1}{5}\binom{8}{4} = \frac{70}{5} = 14$ *elements in this poset.*

Remark. *The two posets are isomorphic, and one can be obtained by applying the aforementioned bijection to the other.*

Theorem 8 (Main Theorem)**.** *The number of intervals in those posets is equal to the* $n{+}1^{th}$ *term of the integer sequence defined by https://oeis.org/A005700. The first terms of this sequence are* $1, 1, 3, 14, 84, 594, 4719, 40898, 379236, 3711916, ...$ *Alec Mihailovs proved this sequence to be equal to*

$$\frac{6(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!}$$

.

Proof. As the number of intervals in the poset of $\mathcal{D}_n$ can be seen as the number of pairs (w_1, w_k) such that $w_1 \gtrdot_d w_2 \gtrdot_d \cdots \gtrdot_d w_k$, we can describe the number of intervals as the number of *pairs of nested Dyck paths.*
We thus introduce the notion of a*marked* Dyck path, which is a Dyck path in which some steps are marked by a star *. Those paths are a representation of the pairs of nested Dyck paths : ignoring the stars, we obtain the "below" path. Then, to deduce the "over" path :

- Unmarked steps stay unchanged.
- Marked North steps become East steps.
- Marked East steps become North steps.

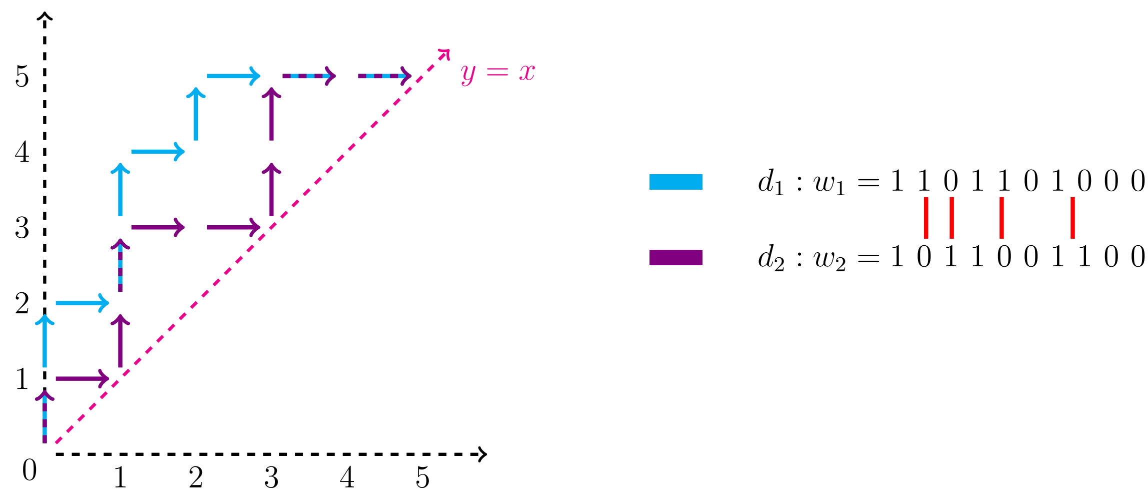
We now create a bijection between marked Dyck paths of length $2n$ that correspond to pairs of nested Dyck paths of length $2n$, and paths from (0,0) to (0,0) composed of $2n$ North, East, South, and West steps that stay in the first octant. To do so, we use the following transformation :

- Unmarked North step $\longleftrightarrow$ North step
- Marked North step $\longleftrightarrow$ West step
- Unmarked East step $\longleftrightarrow$ South step
- Marked East step $\longleftrightarrow$ East step

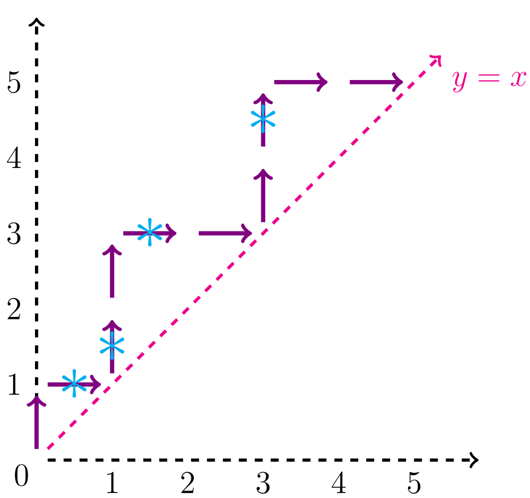
Thus, since $\mathcal{D}_n \longleftrightarrow$ {Marked Dyck paths of length $2n$} $\longleftrightarrow$ { NESW paths of length $2n$}, we know that the number of pairs of nested Dyck paths of length d$2n$ is equal to the number of NESW paths of length$2n$.

But, by definition, this number is equal to the $n+1^{th}$ term of the aforementioned integer sequence (which joins the comment made by Bruce Westbury). Therefore, we do have the aforementioned result. □

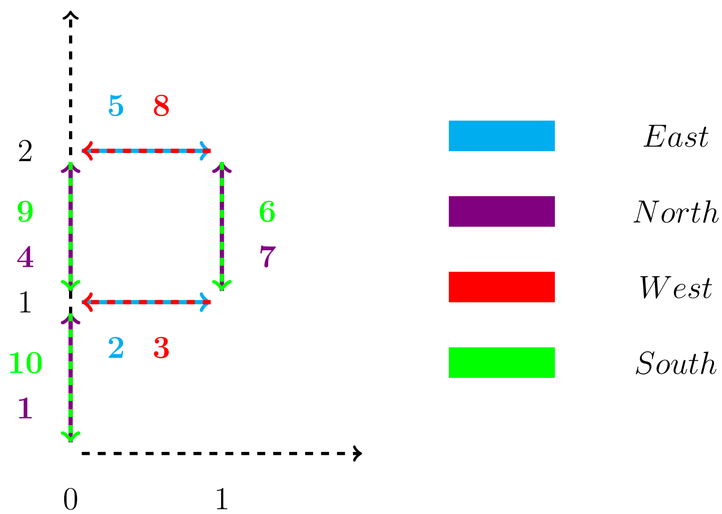
Example $(n = 5)$**.** *We define $d_1, d_2 \in \mathcal{D}_5$ to be two nested Dyck paths :*



The corresponding marked Dyck path is the following :



The NESW path inn bijection is thus NEWNESNWSS : ↑→←↑→↓↑←→→ *:*



We now extend this construction to the non-primitive case.

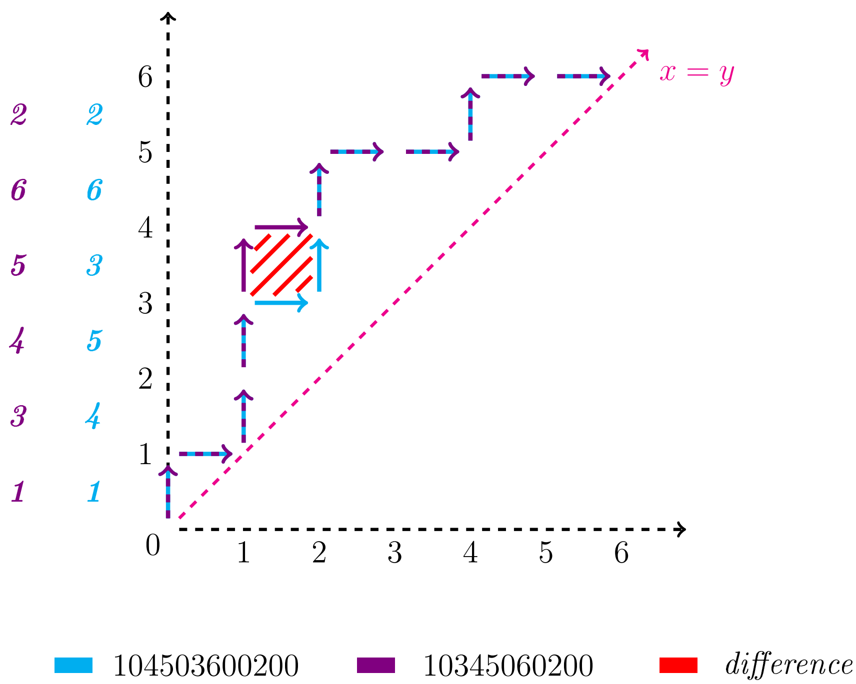
Classical Dyck - Parking Posets

Definition 19 ($\gtrdot_{ld}$)**.** *For w and w' two labeled Dyck words, we say that w covers w', written $w \gtrdot_{ld} w'$, if $\exists l, r, x, x', y, z, z'$ such that :*

- *l is either empty or ends with* 0.
- *r is either empty or starts with* 0.
- *$x = x_1 x_2 \cdots$ has all its digits* > 0.
- *$z = z_1 z_2 \cdots$ has all its digits* > 0.
- *$x' = x$ where y is correctly inserted regarding the order condition.*
- *y is in z, and $z' = z$ where y is removed.*
- $w = lx0zr$
- $w' = lx'0z'r$

Example ($n = 5$)**.** $104503600200 \gtrdot_{ld} 10345060200$

- $l = 10$
- $r = 0200$
- $x = 45$
- $x' = 345$
- $y = 3$
- $z = 36$
- $z' = 6$



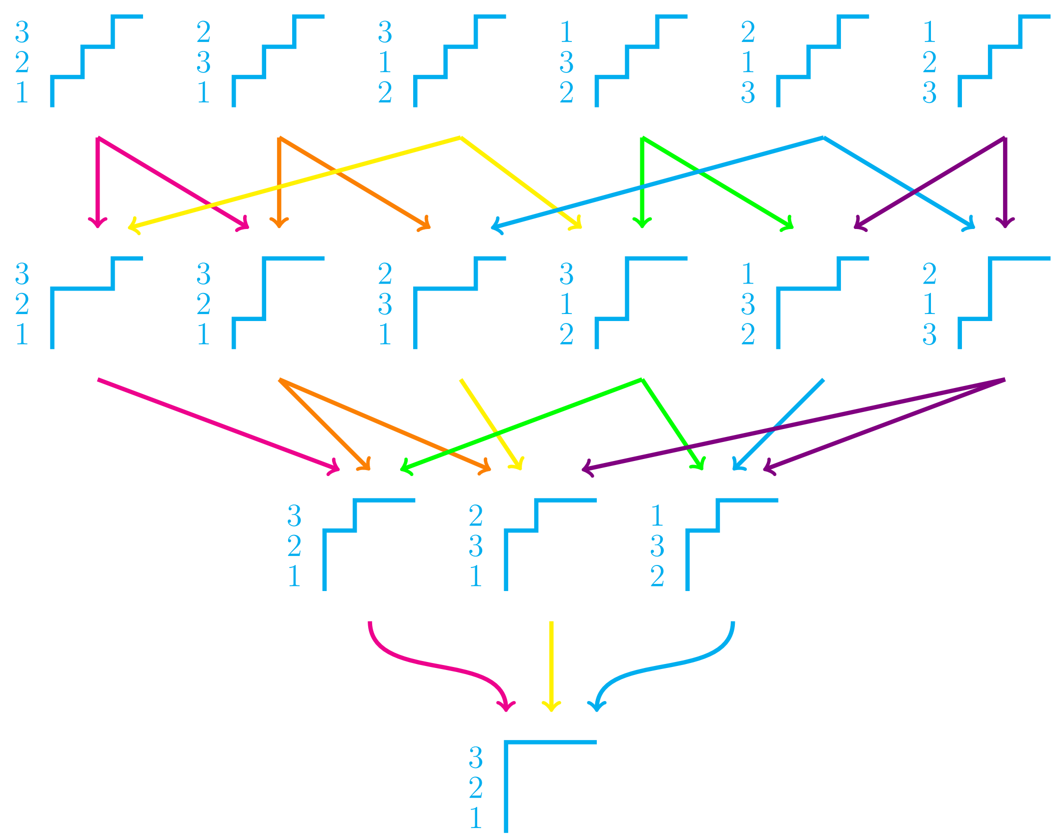
Definition 20 (Rise)**.** *A* rise *of a decorated Dyck word is a maximal sequence of non-zero digits preceding a zero.*

Example ($n = 5$)**.** *In order, the rises of* 104503600200 *are :*
• 1 • 45 • 36 • $\emptyset$ • 2 • $\emptyset$

Remark. *If $w_1 \gtrdot_{ld} w_2$, then the path corresponding to w_2 is* over *the path corresponding to w_1, and the* difference *between the two paths is a square of size 1 by 1.*
Furthermore, the covering relation can be seen as follows : w_1 covers w_2 if we can obtain w_2 by taking a digit from the $i+1^{th}$ rise of w_1, and inserting it into the i^{th} rise of w_1 in increasing order.

Proposition. *This covering relation defines a* poset *for $\mathcal{LD}_n$.*

Example (The poset of $\mathcal{LD}_3$)**.**



There are $4^2 = 16$ elements in this poset.

Definition 21 ($\gtrdot$)**.** *For f and g two parking functions, we say that f covers g, written $f \gtrdot g$, if $\exists i$ such that :*

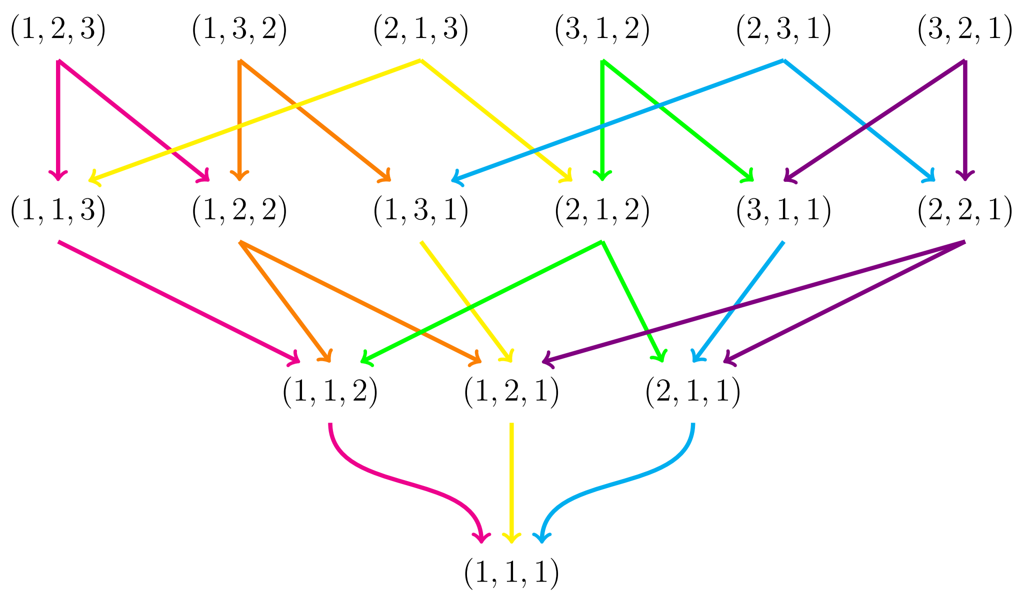
- $f = (a_1, \ldots, a_{i-1}, a_i, \quad a_{i+1}, \ldots, a_n)$
- $g = (a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_n)$

That is, the same relation as for primitive parking functions.

Example ($n = 6$)**.** $(2, 1, 5, 3, 1, 4) \gtrdot (2, 1, 5, 2, 1, 4)$

Proposition. *This covering relation defines a* poset *for* $\mathcal{PF}_n$*.*

Example (The poset of $\mathcal{PF}_3$)**.**



There are $4^2 = 16$ *elements in this poset.*

Remark. *The two posets are isomorphic, and one can be obtained by applying the aforementioned bijection to the other.*

Conjecture (Main Conjecture)**.** *The number of intervals in those posets is equal to the* $n+1^{th}$ *term of the integer sequence defined by https://oeis.org/A196304.* *The first terms of this sequence are* $1, 1, 5, 64, 1587, 65421, 4071178, 357962760, 4237910716, \ldots.$

While, to the best of our knowledge, there is no combinatorial structure proved to follow this integer sequence, tests on $n = 1, 2, \cdots, 8$ suggest that the number of intervals in those posets might be one.

To go further, the next chapter will tackle a generalization of classical parking functions called *Rational Parking Functions*. The upgrade can be seen as such : Let $(a_1, \dots, a_n)$ be a sequence of positive integers, and $(b_1, \dots, b_n)$ its non-decreasing rearrangement. In the classical case, the bounds for $(b_1, \dots, b_n)$ were $(1, \dots, n)$, thus simply depending on the integer n. In the rational case, the bounds will depend on *two coprime integers* a and b, namely $(1, 1 + \frac{b}{a}, 1 + \frac{2b}{a}, 1 + \frac{3b}{a}, \dots)$, with $a = n$.

Chapter 2

The rational case

For the whole chapter, we will consider 2 *coprime* integers a and b (meaning a and b have 1 as their greatest common divisor).

2.1 Rational Parking Functions

Definition 22 (a, b - Parking Function). *An* a, b - parking function *is a sequence* $(a_1, a_2, \ldots, a_n)$ *such that :*
• $n = a$
• *its non-decreasing reordering* $(b_1, b_2, \ldots, b_n)$ *has* $b_i \leqslant \frac{b}{a}(i-1)+1$ *for all* i.

We denote by $\mathcal{PF}_{a,b}$ the set of a, b - parking functions.

Example.

- *Ex. 1 :* $a > b$

  $a = 7$

  $b = 3$

  Limits of the non-decreasing reordering of any $f \in \mathcal{PF}_{7,3}$ *:*

  $[1,\ 1\frac{3}{7},\ 1\frac{6}{7},\ 2\frac{2}{7},\ 2\frac{5}{7},\ 3\frac{1}{7},\ 3\frac{4}{7}]$

  $f_1 = (2,1,1,3,2,3,1) \in \mathcal{PF}_{7,3}$

  $f_2 = (2,1,2,3,2,3,1) \notin \mathcal{PF}_{7,3}$, *even though* $f_2 \in \mathcal{PF}_7$

- *Ex. 2 :* $a < b$

  $a = 5$

  $b = 7$

  Limits of the non-decreasing reordering of any $f \in \mathcal{PF}_{5,7}$ *:*

  $[1,\ 2\frac{2}{5},\ 3\frac{4}{5},\ 5\frac{1}{5},\ 6\frac{3}{5}]$

  $f_3 = (6, 3, 5, 1, 2) \in \mathcal{PF}_{5,7}$*, even though* $f_3 \notin \mathcal{PF}_5$

  $f_4 = (6, 3, 5, 1, 3) \notin \mathcal{PF}_{5,7}$

Theorem 9 (Armstrong, Loehr and Warrington, 2014)**.** *Let* $pf_{a,b}$ *be the cardinal of* $\mathcal{PF}_{a,b}$*. We have*

$$pf_{a,b} = b^{a-1}.$$

Example $(a = 3, b = 5)$**.**
• $pf_{a,b} = 25$ • *Limits :* $[1,\ 2\frac{2}{3},\ 4\frac{1}{3}]$

$(1,1,1)$	$(1,1,2)$	$(1,1,3)$	$(1,1,4)$	$(1,2,1)$	$(1,2,2)$	$(1,2,3)$
$(1,2,4)$	$(1,3,1)$	$(1,3,2)$	$(1,4,1)$	$(1,4,2)$	$(2,1,1)$	$(2,1,2)$
$(2,1,3)$	$(2,1,4)$	$(2,2,1)$	$(2,3,1)$	$(2,4,1)$	$(3,1,1)$	$(3,1,2)$
$(3,2,1)$	$(4,1,1)$	$(4,1,2)$	$(4,2,1)$			

Remark. $\mathcal{PF}_{n,n+1} = \mathcal{PF}_n$*. In fact, we do have* $b^{a-1} = (n+1)^{n-1}$.

Similarly to the integer case, we can define a notion of *primitivity* for rational parking functions.

2.1.1 Rational primitive parking functions

Definition 23 (Rational Primitive)**.** *A rational parking function* f *is said* primitive *if it is already in non-decreasing order.*

We denote by $\mathcal{PF}'_{a,b}$ the set of primitive a, b - parking functions.

Example $(a = 4, b = 3)$**.** *Limits :* $[1,\ 1\frac{3}{4},\ 2\frac{1}{2},\ 3\frac{1}{4}]$

$$f_1 = (1, 1, 2, 2) \in \mathcal{PF}'_{4,3}$$
$$f_2 = (1, 1, 2, 1) \notin \mathcal{PF}'_{4,3},\ \textit{even though}\ f_2 \in \mathcal{PF}_{4,3}.$$

The following theorem can be seen as an extension of the main result of [10], as we will see later that rational primitive parking functions are in bijection with rational Dyck paths.

Theorem 10. *Let $pf'_{a,b}$ be the cardinal of $\mathcal{PF}'_{a,b}$. We have*

$$pf'_{a,b} = \frac{1}{a+b}\binom{a+b}{b}.$$

which is the *rational Catalan number $Cat(a,b)$.*

Example ($a = 3, b = 5$)**.**
• $pf'_{a,b} = 7$ • *Limits :* $[1,\ 2\frac{2}{3},\ 4\frac{1}{3}]$

$$(1,1,1) \qquad (1,1,2) \qquad (1,1,3) \qquad (1,1,4) \qquad (1,2,2) \qquad (1,2,3) \qquad (1,2,4)$$

Remark. $\mathcal{PF}'_{n,n+1} = \mathcal{PF}'_n$. *In fact, we do have*

$$\frac{1}{n+n+1}\binom{n+n+1}{n+1} = \frac{1}{2n+1}\binom{2n+1}{n+1} = \frac{1}{2n+1}\frac{(2n+1)!}{n!(n+1)!}$$

$$= \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1}\frac{(2n)!}{n!n!} = \frac{1}{n+1}\binom{2n}{n}.$$

In the same fashion as for classical parking functions, one can define *Rational Non-crossing Partitions* as a bijecting combinatorial structure. Defined by Michelle Bodnar in [8] for any coprime a and b, the construction presented depends on a heavy mechanism relying on rational Dyck paths. As there is – to the best of our knowledge – no easier way to define rational non-crossing partitions, the following section has for only purpose to give an idea of what such a partition looks like, and recall some of the main results from [8].

2.2 Rational Non-crossing Partitions

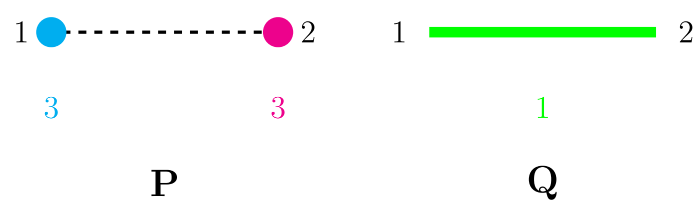
Definition 24 (a, b - Non-crossing Partition)**.** *An* a, b - non-crossing partition *is a tuple (P, Q, f_P, f_Q) such that :*

- $P \in \mathcal{NC}_{b-1}$

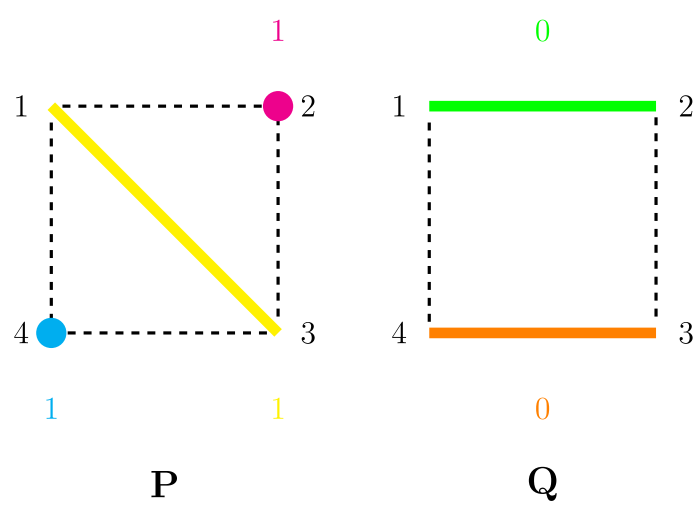
- *Q is the Kreweras complement of P : $K(P)$.*
- $\displaystyle\sum_{B\in P} f_P(B) + \sum_{B\in Q} f_Q(B) = a.$
- *$f_P(B) > \frac{a}{b}$ for each block B of P.*
- *$f_Q(B) < \frac{a}{b}$ for each block B of Q.*
- *The* rank condition *defined in [8] holds.*

We denote by $\mathcal{NC}_{a,b}$ the set of a, b - non-crossing partitions.

Example ($a > b : a = 7, b = 3$)**.**



Example ($a < b : a = 3, b = 5$)**.**



Theorem 11 (Bodnar, 2017)**.** *Let $nc_{a,b}$ be the cardinal of $\mathcal{NC}_{a,b}$. We have*

$$nc_{a,b} = \frac{1}{a+b}\binom{a+b}{a} = \frac{(a+b-1)!}{a!b!}.$$

which is the rational Catalan number $Cat(a, b)$.

Proposition. *This means we can create a* bijection *between* $\mathcal{PF}'_{a,b}$ *and* $\mathcal{NC}_{a,b}$.

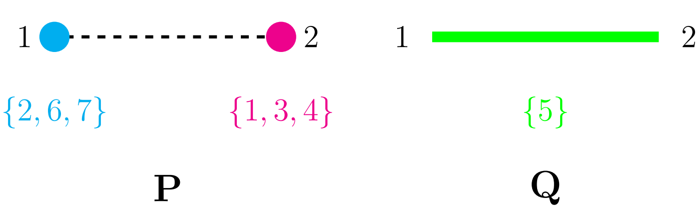
Proof. Following the proof for the non-primitive case in [8], only consider rational Dyck paths where the i^{th} North step is labeled i. $\square$

Definition 25 (a, b - Non-crossing 2-Partition)**.** *An* a, b - Non-crossing 2-partition *is a tuple* (P, Q, f_P, f_Q) *such that :*

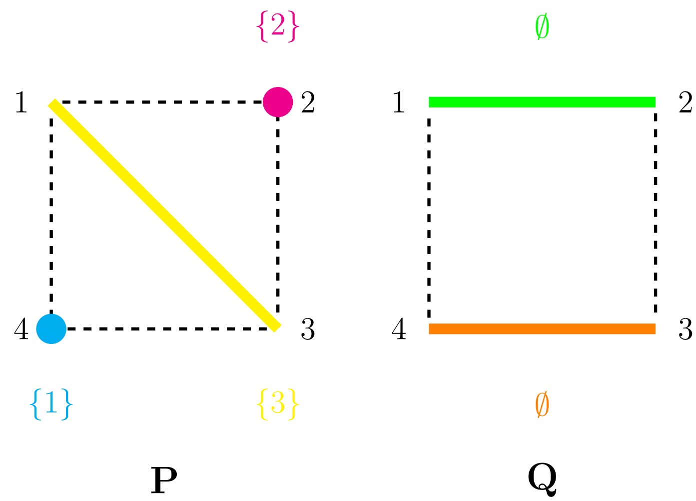
- $P \in \mathcal{NC}_{b-1}$.
- *Q is the Kreweras complement of P : $K(P)$.*
- $\displaystyle\bigcup_{B \in P} f_P(B) \cup \bigcup_{B \in Q} f_Q(B) = [a]$.
- $|f_P(B)| > \frac{a}{b}$ *for each block B of P.*
- $|f_Q(B)| < \frac{a}{b}$ *for each block B of Q.*
- *The* rank condition *defined in [8] holds.*

This can be seen as a *labeling* of the blocks of P and Q by $[a]$.
We denote by $\mathcal{NC}^2_{a,b}$ the set of a, b - non-crossing 2-partitions.

Example ($a > b : a = 7, b = 3$)**.**



Example ($a < b : a = 3, b = 5$)**.**



Theorem 12 (Bodnar, 2017)**.** *Let $nc^2_{a,b}$ be the cardinal of $\mathcal{NC}^2_{a,b}$. We have*

$$nc^2_{a,b} = b^{a-1}.$$

Proposition. *This means we can create a* bijection *between $\mathcal{PF}_{a,b}$ and $\mathcal{NC}^2_{a,b}$.*

Proof. See [8]. □

While this is an elegant solution, and seems to be the first to generalize to *all* coprime a and b – and not just those where $a < b$ as studied by Armstrong and others in the past –, we still wish to define a cover relation for rational parking functions without having to refer to an other structure – especially since this construction makes it even heavier by having to use rational Dyck paths to verify the *rank condition* defined in [8].

Therefore, in the next section, we define cover relations for $\mathcal{PF'}_{a,b}$ and $\mathcal{PF}_{a,b}$ through (labeled) Rational Dyck Paths, generalizing the classical case.

2.3 A direct poset linked to Rational Dyck paths

2.3.1 Rational Dyck Paths

Definition 26 (a, b - Dyck Word)**.** *An* a, b - Dyck word *is a word $w \in \{0, 1\}^*$ such that :*

- *for each* prefix w' *of* w,

$$|w'|_1 \geqslant \frac{a}{b}|w'|_0$$

.

- $|w|_0 = b$.
- $|w|_1 = a$.

An a, b - Dyck word can be represented as a *path* from $(0,0)$ to (b,a) that stays over $y = \frac{a}{b}x$, called an *a, b - Dyck path* :

- Each 1 corresponds to a *North step* $\uparrow$.
- Each 0 corresponds to an *East step* $\rightarrow$.

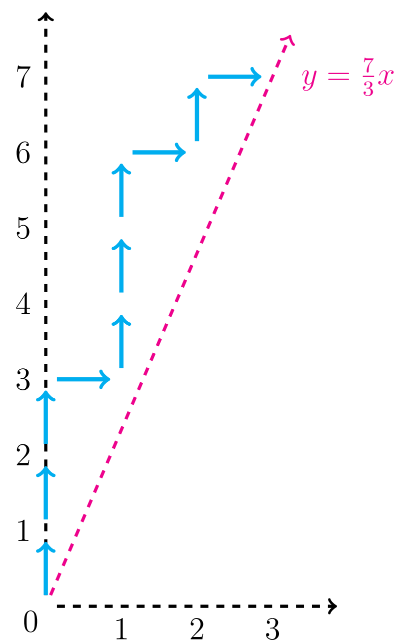
We denote by $\mathcal{R}_{a,b}$ the set of a, b - Dyck words.

Example $(a > b : a = 7, b = 3)$**.**

$w_1 = 1110011110$ *is* not *a 7, 3 - Dyck word, because* $|11100|_1 = 3$

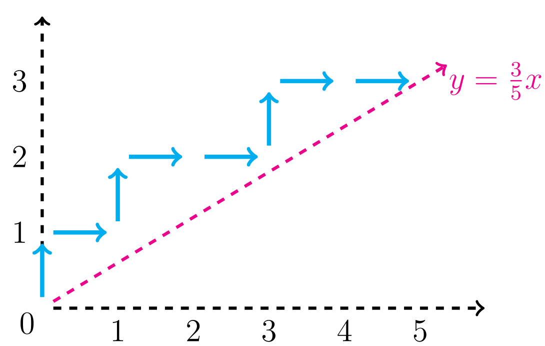
$$< \frac{7}{3}|11100|_0 = \frac{14}{3} = 4\frac{1}{3}.$$

$w_2 = 1110111010$ is *a 7, 3 - Dyck word :*



Example $(a < b : a = 3, b = 5)$**.**

$w_1 = 10100010$ *is* not *a 3, 5 - Dyck word, because* $|101000|_1 = 2$
$$< \frac{3}{5}|101000|_0 = \frac{12}{5} = 2\frac{2}{5}.$$
$w_2 = 10100100$ is *a 3, 5 - Dyck word :*

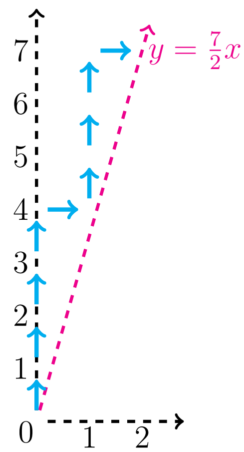


Theorem 13 (Bizley, 1954)**.** *Let $r_{a,b}$ be the cardinal of $\mathcal{R}_{a,b}$. We have*

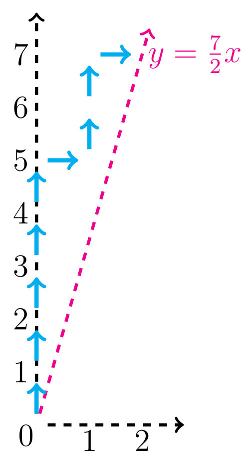
$$r_{a,b} = \frac{1}{a+b}\binom{a+b}{a} = \frac{(a+b-1)!}{a!b!}.$$

which is, again, the rational Catalan number $Cat(a,b)$.

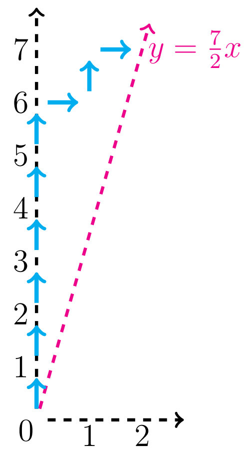
Example $(a = 7, b = 2)$**.** $r_n = 4$.



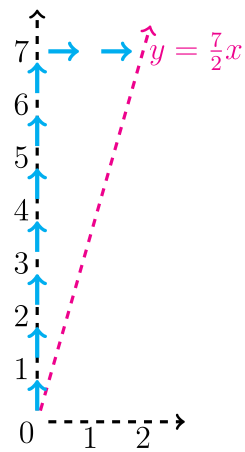
111101110



111110110



111111010



111111100

Proposition. *This means we can create a* bijection *between* $\mathcal{PF}'_{a,b}$ *and* $\mathcal{R}_{a,b}$.

Proof.

- $\mathcal{PF}'_{a,b} \to \mathcal{R}_{a,b}$: Let $f = (a_1, \ldots, a_n) \in \mathcal{PF}'_{a,b}$ be our rational primitive parking function. For $i \in \{1, \ldots, b\}$, we define l_i the number of occurences of i in f.
  The corresponding rational Dyck word will be $\underbrace{1 \cdots 1}_{l_1} 0 \underbrace{1 \cdots 1}_{l_2} 0 \cdots \underbrace{1 \cdots 1}_{l_b} 0$.

- $\mathcal{R}_{a,b} \to \mathcal{PF}'_{a,b}$: Let w be our rational Dyck word, and consider its path representation. We define s_i to be the distance between the segment from $(0, i-1)$ to $(0, i)$ and the i^{th} North step. Then, let $a_i = s_i + 1$. The corresponding rational primitive parking function is $(a_1, \ldots, a_a)$.

$\square$

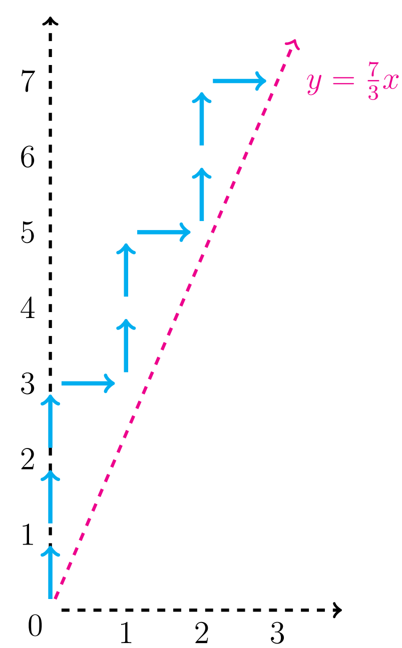
Remark. *This bijection is exactly the same as the one between classical primitive parking functions and Dyck paths.*

Example $(a > b : a = 7, b = 3, \mathcal{PF}'_{a,b} \to \mathcal{R}_{a,b})$**.**

- $f = (1, 1, 1, 2, 2, 3, 3)$

  $l_1 = 3$ $\quad l_2 = 2$ $\quad l_3 = 2$

- $w = (1110110110)$



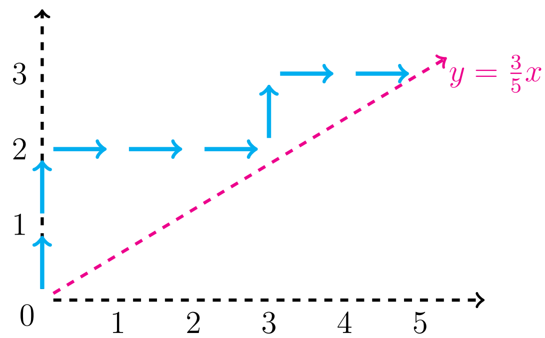
Example $(a < b : a = 3, b = 5, \mathcal{PF}'_{a,b} \to \mathcal{R}_{a,b})$**.**

- $f = (1, 1, 4)$

  $l_1 = 2$ $\quad l_2$ $\quad l_3 = 0$

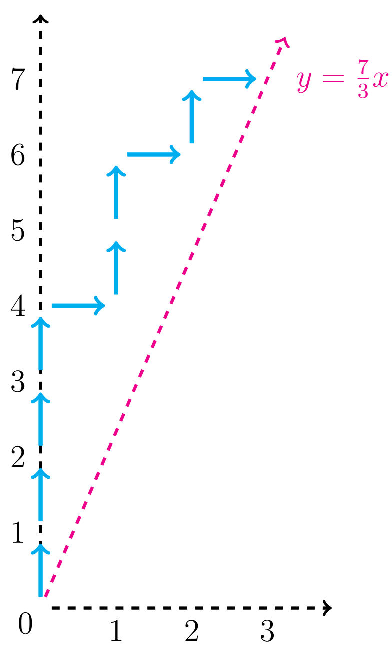
  $l_4 = 1$ $\quad l_5 = 0$

- $w = 11000100$



Example $(a > b : a = 7, b = 3, \mathcal{R}_{a,b} \to \mathcal{PF}'_{a,b})$.

- $w = 1111011010$



- *Distances :*

$s_1 = 0$ $\quad a_1 = 1$

$s_2 = 0$ $\quad a_2 = 1$

$s_3 = 0$ $a_3 = 1$

$s_4 = 0$ $a_4 = 1$

$s_5 = 1$ $a_5 = 2$

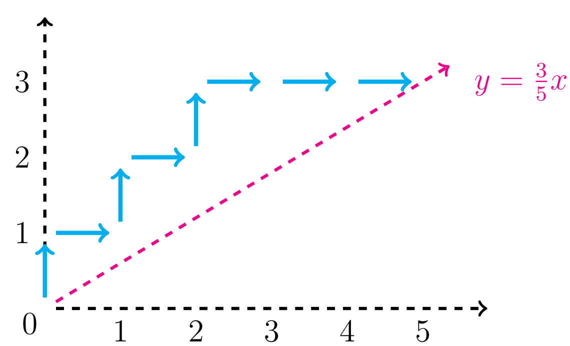
$s_6 = 1$ $a_6 = 2$

$s_7 = 2$ $a_7 = 3$

- $f = (1, 1, 1, 1, 2, 2, 3)$

Example $(a < b : a = 3, b = 5, \mathcal{R}_{a,b} \to \mathcal{PF'}_{a,b})$.

- $w = 10101000$



- *Distances :*

$s_1 = 0$ $a_1 = 1$

$s_2 = 1$ $a_2 = 2$

$s_3 = 2$ $a_3 = 3$

- $f = (1, 2, 3)$

Once again, we will generalize to the non-primitive case by adding a *labeling* to our rational Dyck paths.

2.3.2 Rational Labeled Dyck Paths

Definition 27 (Labeled a, b - Dyck Word)**.** *A* labeled a, b - Dyck word *is a word $w \in \{0, \ldots, n\}^*$ such that :*

- *for each prefix w' of w,*

$$|w'|_{\neq 0} \geqslant \frac{a}{b}|w'|_0.$$

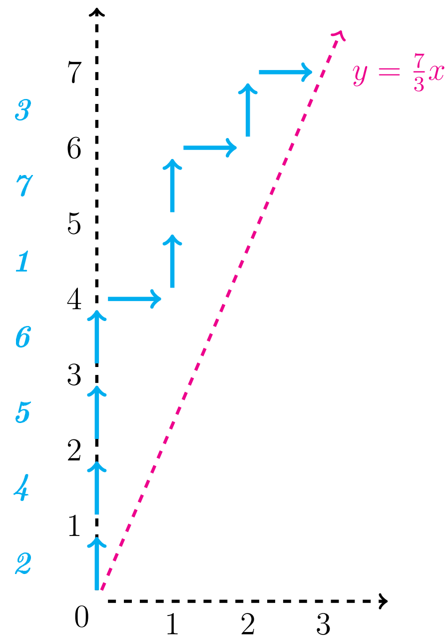
- $|w|_0 = b$.
- $|w|_{\neq 0} = a$.
- *for each $i \in \{1, \ldots, a\}$, w has exactly one occurence of i.*
- *if $w_i \neq 0$ and $w_{i+1} \neq 0$, then $w_i < w_{i+1}$. That is, consecutive North steps have increasing labels.*

A labeled a, b - Dyck word can be represented as a path from $(0, 0)$ to (b, a), where each North step is associated to a label :

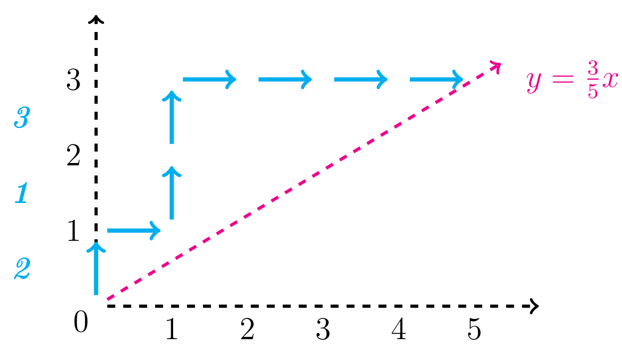
- Each $i \neq 0$ corresponds to a *North step* $\uparrow$ labeled i.
- Each 0 corresponds to an *East step* $\rightarrow$.

Those paths are called *labeled a, b - Dyck paths.*
We denote by $\mathcal{LR}_{a,b}$ the set of labeled a, b - Dyck words.

Example ($a > b : a = 7, b = 3$)**.** $w_2 = 2456017030$ *:*



Example $(a < b : a = 3, b = 5)$**.** *$w = 20130000$:*



Theorem 14. *Let $lr_{a,b}$ be the cardinal of $\mathcal{LR}_{a,b}$. We have*

$$lr_{a,b} = b^{a-1}$$

.

Example $(a > b : a = 4, b = 3)$**.** $lr_{a,b} = 3^3 = 27$

- *Word of shape $XXXX000$:*

  1234000

- *Words of shape $XXX0X00$:*

1230400	1240300	1340200
2340100		

- *Words of shape $XX0XX00$:*

1203400	1302400	1402300
2301400	2401300	3401200

- *Words of shape $XXX00X0$:*

1230040	1240030	1340020
2340010		

- *Words of shape $XX0X0X0$:*

1203040	1204030	1302040
1304020	1402030	1403020
2301040	2304010	2401030
2403010	3401020	3402010

Proposition. *This means we can create a* bijection *between $\mathcal{PF}_{a,b}$ and $\mathcal{LR}_{a,b}$.*

Proof.

- $\mathcal{PF}_{a,b} \to \mathcal{LR}_{a,b}$: Let $f = (a_1, \ldots, a_n) \in \mathcal{PF}_{a,b}$ be our a, b - parking function. For $i \in \{1, \ldots, b\}$, we define $im_i : \{j \mid a_j = i\}$.
  We then define $im_{i,1}, \ldots, im_{i,k_i}$ to be the elements of im_i in increasing order.
  The corresponding labeled a, b - Dyck word will be
  $\underbrace{im_{1,1} \cdots im_{1,k_1}}_{im_1} 0 \underbrace{im_{2,1} \cdots im_{2,k_2}}_{im_2} 0 \cdots \underbrace{im_{n,1} \cdots im_{b,k_b}}_{im_b} 0.$

- $\mathcal{LR}_{a,b} \to \mathcal{PF}_n$: Let w be our labeled a, b - Dyck word, and consider its path representation. We define s_i to be the distance between the segment from $(0, i-1)$ to $(0, i)$ and the i^{th} North step.
  Then, let $label(i)$ be the label of the i^{th} North step, and $dist_i = \{label(j) | s_j = i\}$ be the set of the labels of all North steps at distance i.
  Then, if $j \in dist_i$, let $a_j = i + 1$.
  The corresponding parking function is $(a_1, \ldots, a_a)$.

□

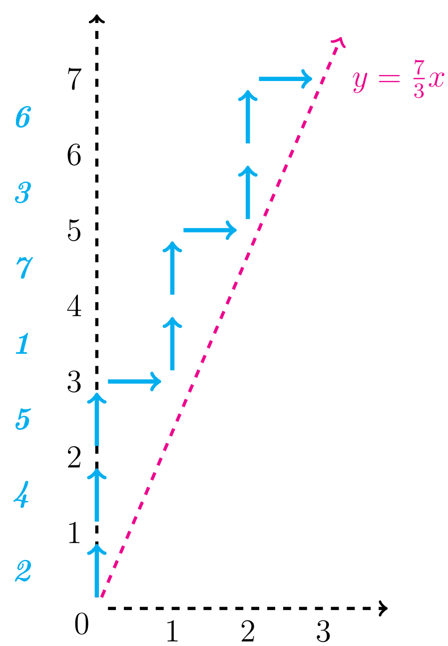
Remark. *This bijection is exactly the same as the one between classical parking functions and labeled Dyck paths.*

Example $(a > b : a = 7, b = 3, \mathcal{PF}_{a,b} \to \mathcal{LR}_{a,b})$**.**

- $f = (2, 1, 3, 1, 1, 3, 2)$

  $im_1 = \{2, 4, 5\}$ $\qquad im_2 = \{1, 7\}$ $\qquad im_3 = \{3, 6\}$

- $w = 2450170360$



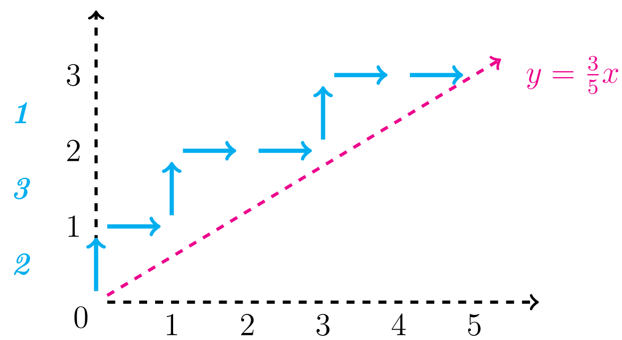
Example $(a < b : a = 3, b = 5, \mathcal{PF}_{a,b} \to \mathcal{LR}_{a,b})$**.**

- $f = (4, 1, 2)$

  $im_1 = \{2\}$ $\quad$ $im_2 = \{3\}$ $\quad$ $im_3 = \emptyset$

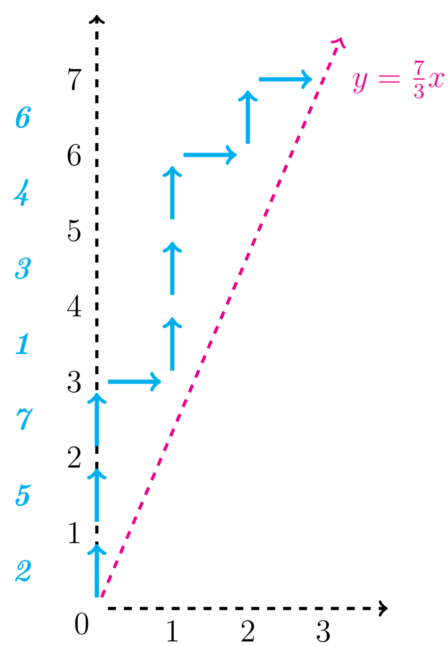
  $im_4 = \{1\}$ $\quad$ $im_5 = \emptyset$

- $w = 20300100$



Example $(a > b : a = 7, b = 3, \mathcal{LR}_{a,b} \to \mathcal{PF}_{a,b})$**.**

- $w = 2570134060$



- *Distances :*

  $s_1 = 0$ $\quad s_2 = 0$ $\quad s_3 = 0$

  $s_4 = 1$ $\quad s_5 = 1$ $\quad s_6 = 1$

  $s_7 = 2$

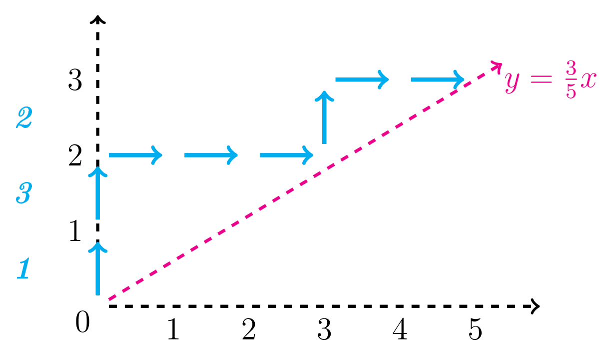
- *Labels :*

  $dist_0 = \{2,5,7\}$ $\quad dist_1 = \{1,3,4\}$ $\quad dist_2 = \{6\}$

- $f = (2,1,2,2,1,3,1)$

Example $(a < b : a = 3, b = 5, \mathcal{LR}_{a,b} \to \mathcal{PF}_{a,b})$**.**

- $w = 13000200$



- *Distances :*

  $s_1 = 0$ $\quad s_2 = 0$ $\quad s_3 = 3$

- *Labels :*

  $dist_0 = \{1,3\}$ $\quad dist_1 = \emptyset$ $\quad dist_2 = \emptyset$

  $dist_3 = \{2\}$ $\quad dist_4 = \emptyset$

- $f = (1,4,1)$

Remark. *The rational primitive parking functions are exactly the rational parking functions corresponding to rational labeled Dyck paths where the i^{th} North step is labeled i.*

Now that we have these four cover relations and the two appropriate bijections, we are able to create the corresponding bijective rational posets.

2.3.3 Rational Dyck - Parking Posets

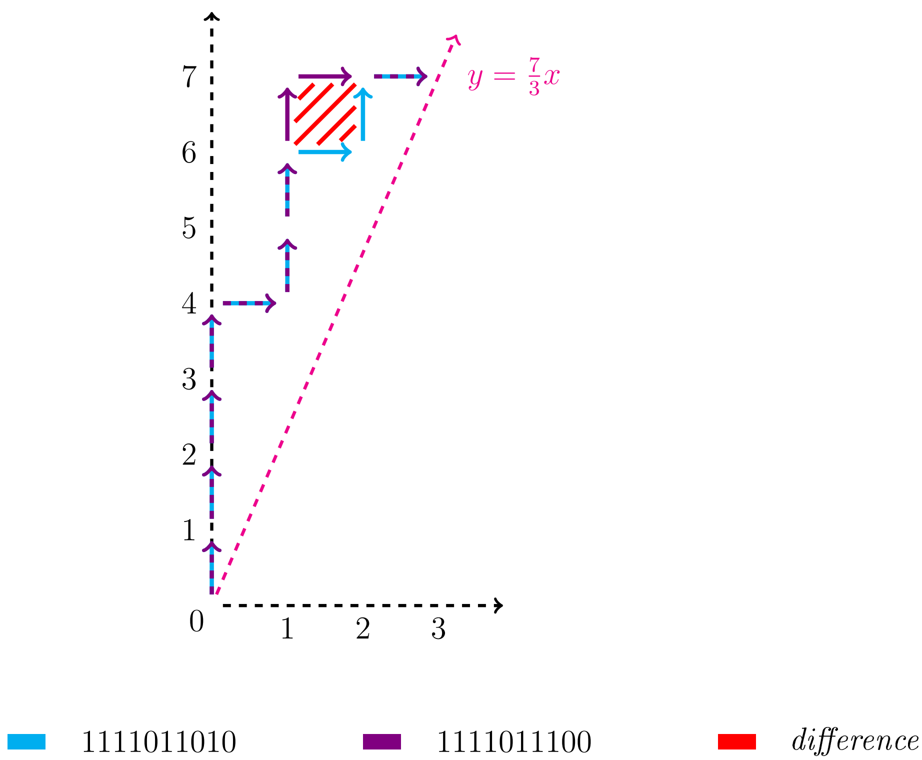
Rational Primitive Dyck - Parking Posets

Definition 28 ($\gtrdot_r$)**.** *For w and w' two a, b - Dyck words, we say that w covers w', written $w \gtrdot_r w'$, if $\exists w_1, w_2$ such that :*

- $w = w_1 01 w_2$
- $w' = w_1 10 w_2$

Example $(a = 7, b = 3)$**.** $1111011010 \gtrdot_r 1111011100$

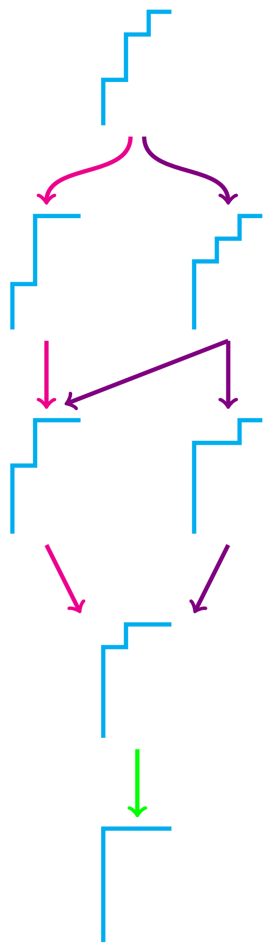
- $w_1 = 1111011$
- $w_2 = 0$



Remark. *If $w_1 \gtrdot_r w_2$, then the path corresponding to w_2 is* over *the path corresponding to w_1, and the* difference *between the two paths is a square of size 1 by 1.*

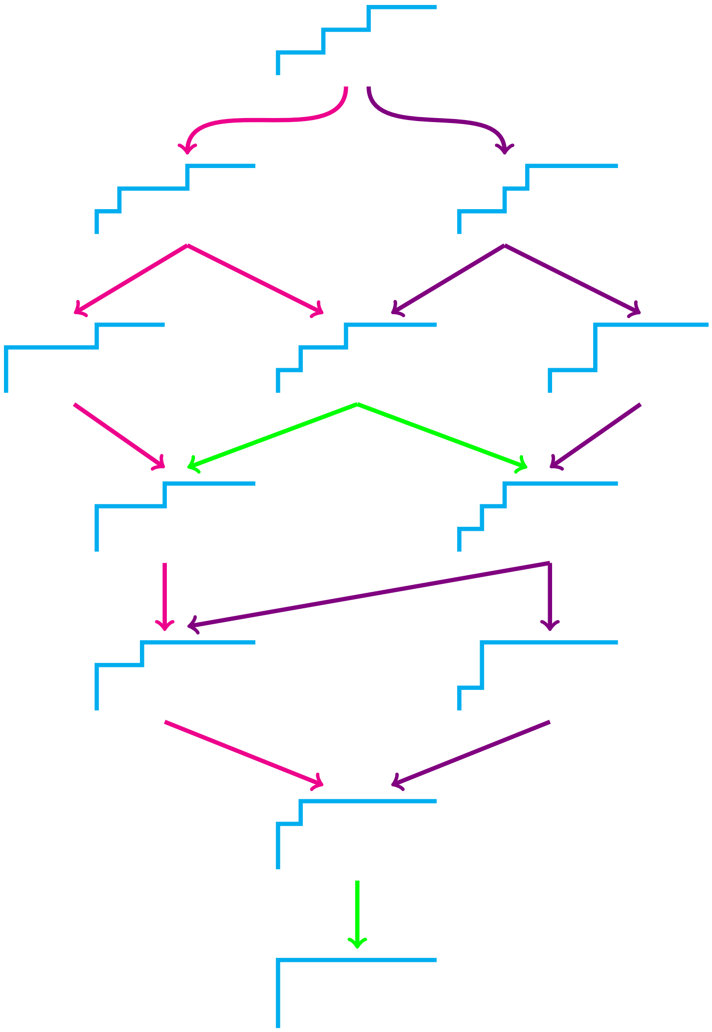
Proposition. *This covering relation defines a* poset *for* $\mathcal{R}_{a,b}$*.*

Example ($a > b$: The poset of $\mathcal{R}_{5,3}$)**.**



There are $\frac{1}{8}\binom{8}{5} = \frac{42}{6} = 7$ *elements in this poset.*

Example ($a < b$: The poset of $\mathcal{R}_{3,7}$)**.**



There are $\frac{1}{10}\binom{10}{3} = \frac{72}{6} = 12$ *elements in this poset.*

Definition 29 ($\gtrdot'$)**.** *For f and g two rational primitive a, b - parking functions, we say that f covers g, written $f \gtrdot' g$, if $\exists i$ such that :*

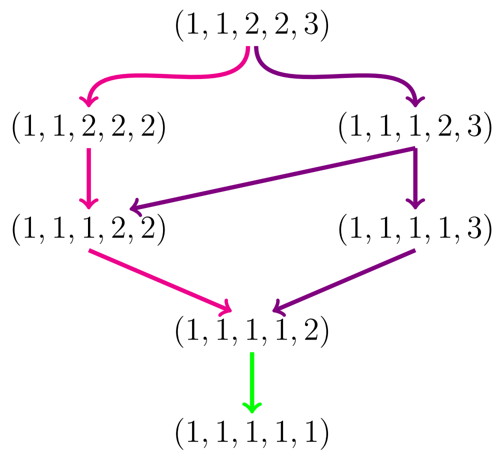
- $f = (a_1, \ldots, a_{i-1}, a_i, \quad a_{i+1}, \ldots, a_n)$
- $g = (a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_n)$

Example ($a > b : a = 7, b = 3$)**.** $(1, 1, 1, 2, 2, 2, 3) \gtrdot' (1, 1, 1, 1, 2, 2, 3)$

Example ($a < b : a = 3, b = 5$)**.** $(1, 2, 4) \gtrdot' (1, 1, 4)$

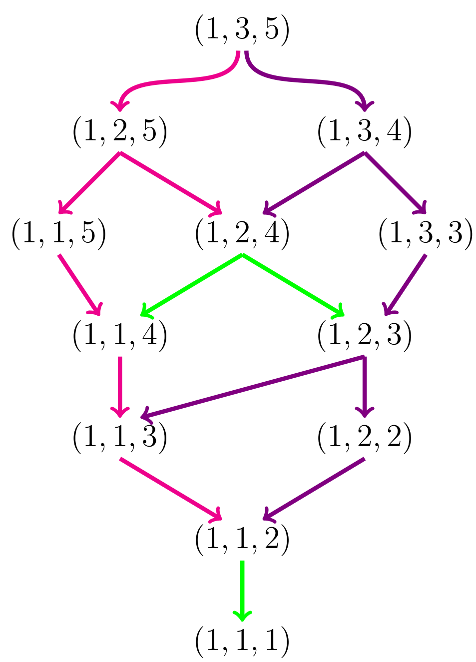
Proposition. *This covering relation defines a* poset *for* $\mathcal{PF'}_{a,b}$.

Example ($a > b$: The poset of $\mathcal{PF'}_{5,3}$)**.**



There are $\frac{1}{8}\binom{8}{5} = \frac{42}{6} = 7$ *elements in this poset.*

Example ($a < b$: The poset of $\mathcal{PF'}_{3,7}$)**.**



There are $\frac{1}{10}\binom{10}{3} = \frac{72}{6} = 12$ *elements in this poset.*

Remark. *The posets of* $\mathcal{PF'}_{a,b}$ *and* $\mathcal{R}_{a,b}$ *are isomorphic, and one can be obtained by applying the aforementioned bijection to the other.*

Rational Dyck - Parking Posets

Definition 30 ($\gtrdot_{lr}$)**.** *For* w *and* w' *two labeled* a*,* b *- Dyck words, we say that* w *covers* w'*, written* $w \gtrdot_{lr} w'$*, if* $\exists l, r, x, x', y, z, z'$ *such that :*

- l *is either empty or ends with* 0.
- r *is either empty or starts with* 0.
- $x = x_1x_2\cdots$ *has all its digits* > 0.
- $z = z_1z_2\cdots$ *has all its digits* > 0.
- $x' = x$ *where* y *is correctly inserted regarding the order condition.*
- y *is in* z*, and* $z' = z$ *where* y *is removed.*
- $w = lx0zr$
- $w' = lx'0z'r$

Example ($a > b : a = 7, b = 3$)**.** $2460150370 \gtrdot_{lr} 2460135070$

- $l = 2460$
- $r = 0$
- $x = 15$
- $x' = 135$
- $y = 3$
- $z = 37$
- $z' = 7$

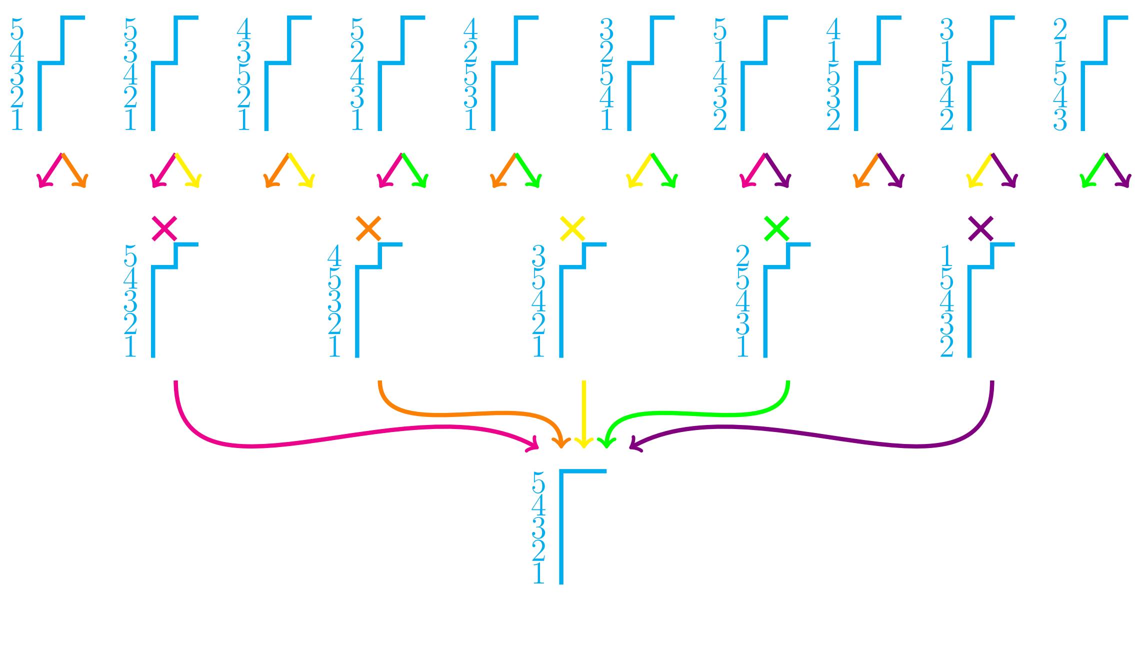
Example ($a < b : a = 3, b = 5$)**.** $20301000 \gtrdot_{lr} 20130000$

- $l = 20$
- $r = 000$
- $x = 3$
- $x' = 13$
- $y = 1$
- $z = 1$
- $z' = \emptyset$

Remark. *If $w_1 \gtrdot_{lr} w_2$, then the path corresponding to w_2 is* over *the path corresponding to w_1, and the* difference *between the two paths is a square of size 1 by 1.*
Furthermore, the covering relation can be seen as follows : w_1 covers w_2 if we can obtain w_2 by taking a digit from the $i+1^{th}$ rise of w_1, and inserting it into the i^{th} rise of w_1 in increasing order.

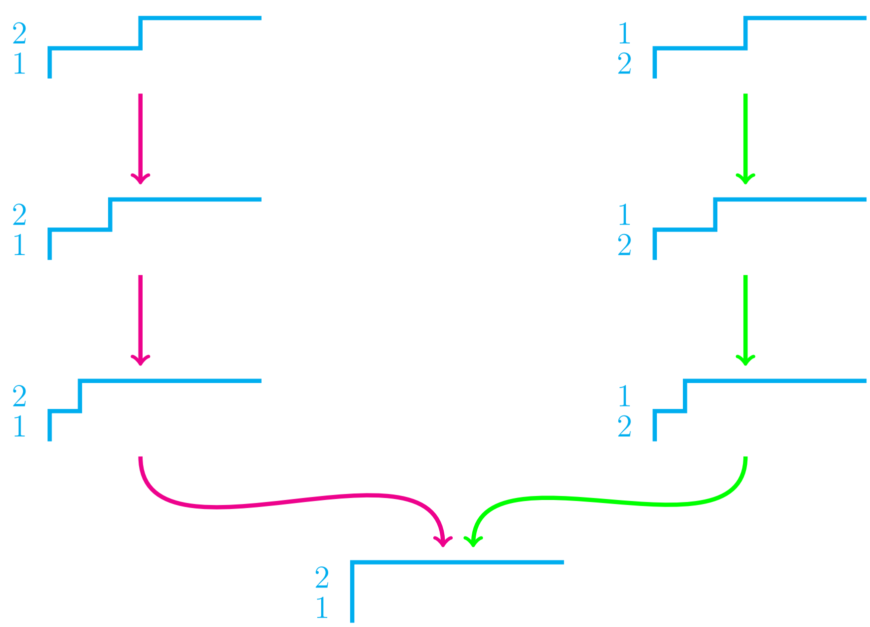
Proposition. *This covering relation defines a* poset *for $\mathcal{LR}_{a,b}$.*

Example ($a > b$: The poset of $\mathcal{LR}_{5,2}$)**.**



Arrows have been simplified for readability.
Arrows between the top 2 levels are to be read as ending at the cross of the corresponding color.
There are $2^4 = 16$ elements in this poset.

Example ($a < b$: The poset of $\mathcal{LR}_{2,7}$)**.**



There are $7^1 = 7$ elements in this poset.

Definition 31 ($\gtrdot$)**.** *For f and g two rational parking functions, we say that f covers g, written $f \gtrdot g$, if $\exists i$ such that :*

- $f = (a_1, \ldots, a_{i-1}, a_i, \quad\;\; a_{i+1}, \ldots, a_n)$
- $g = (a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_n)$

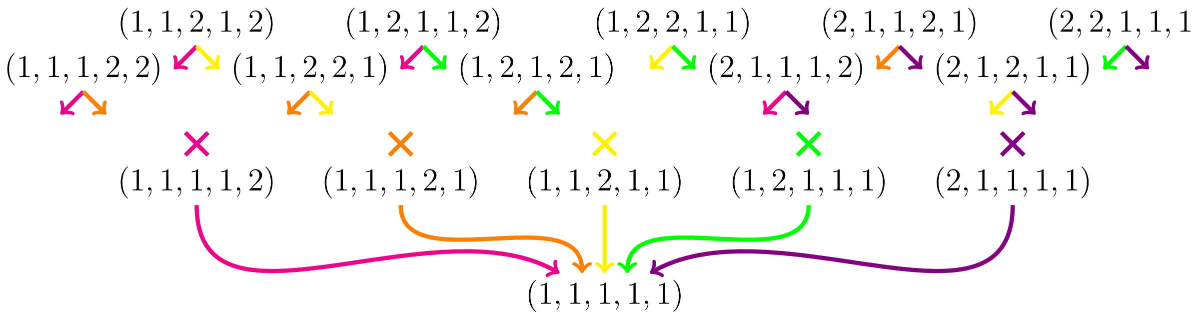
That is, the same relation as for rational primitive parking functions.

Example ($a > b : a = 7, b = 3$)**.** $(2, 3, 1, 1, 2, 1, 3) \gtrdot (2, 3, 1, 1, 1, 1, 3)$

Example ($a < b : a = 3, b = 5$)**.** $(4, 1, 2) \gtrdot (3, 1, 2)$

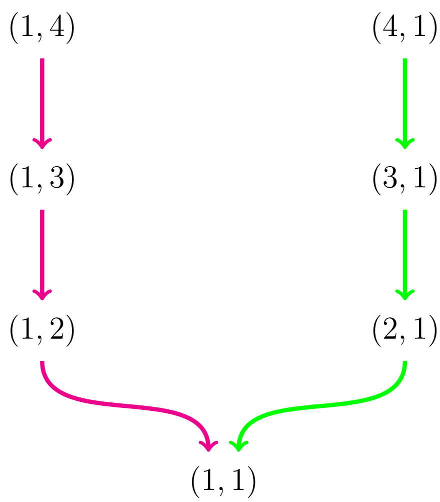
Proposition. *This covering relation defines a* poset *for* $\mathcal{PF}_{a,b}$.

Example ($a > b$: The poset of $\mathcal{PF}_{5,2}$)**.**



Arrows have been simplified for readability.
Arrows between the top 2 levels are to be read as ending at the cross of the corresponding color.
There are $2^4 = 16$ *elements in this poset.*

Example ($a < b$: The poset of $\mathcal{PF}_{2,7}$)**.**



There are $7^1 = 7$ *elements in this poset.*

Remark. *The posets of $\mathcal{PF}_{a,b}$ and $\mathcal{LR}_{a,b}$ are isomorphic, and one can be obtained by applying the aforementioned bijection to the other.*

As we finish this chapter, we now have *direct* ways to compare elements of the four types of parking functions. Furthermore, we also have four corresponding bijections with the fundamental combinatorial structure that is the Dyck path.

Furthermore, we have seen the two main results of this article in Theorem 8 and the corresponding Conjecture for the non-primitive case, regarding the number of relations in the aforementioned posets for the classical case.

We now jump to an other construction linked to parking functions named *Parking Trees.* Defined by Delcroix-Oger, Josuat-Vergès and Randazzo in [9], classical parking trees are in bijection with classical parking functions. In this third and final chapter, we recall the aforementioned notion, and extend it to the rational case.

Chapter 3

Trees

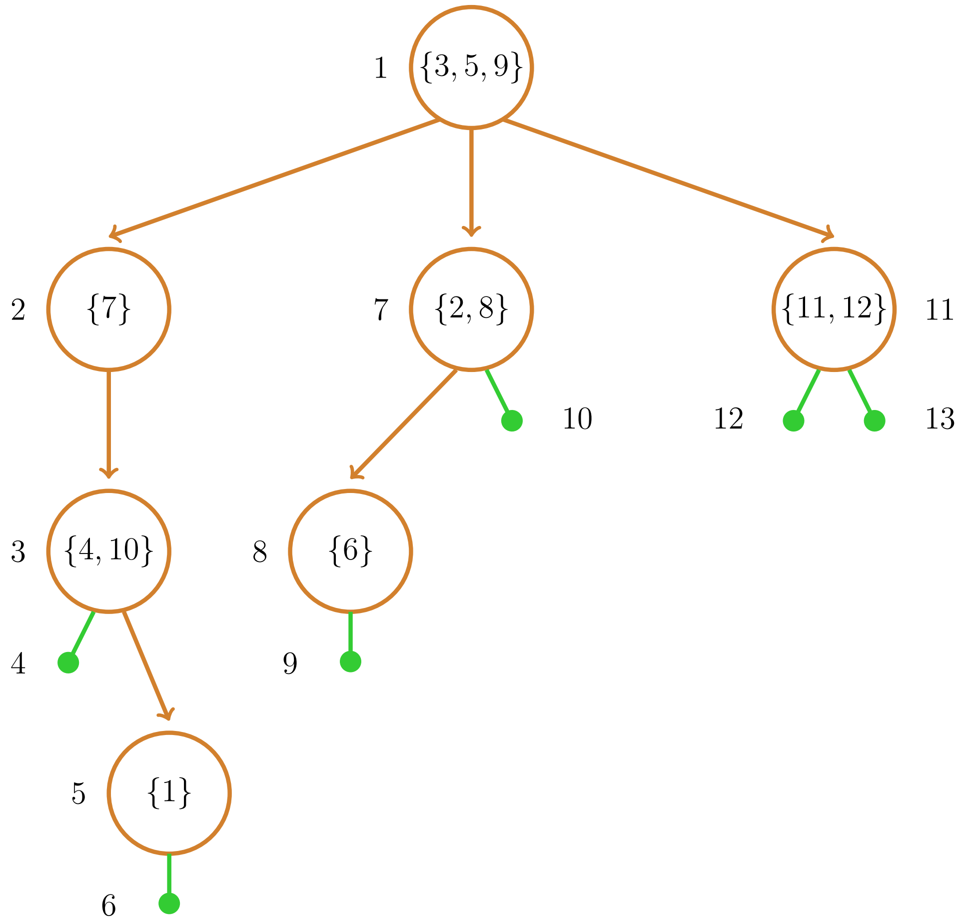
3.1 Parking Trees

Definition 32 (Parking Trees). *A* parking tree *is defined from a parking function* $f = (a_1, \ldots, a_n) \in \mathcal{PF}_n$ *as follows :*

- *For* $1 \leqslant i \leqslant n+1$*, we define* s_i *as* $\{j \mid a_j = i\}$.
- $[s_1, \ldots, s_{n+1}]$ *describes the pre-order depth-first traversal of the tree.*
- *Each node labeled by a set of size* k *has* k *children.*

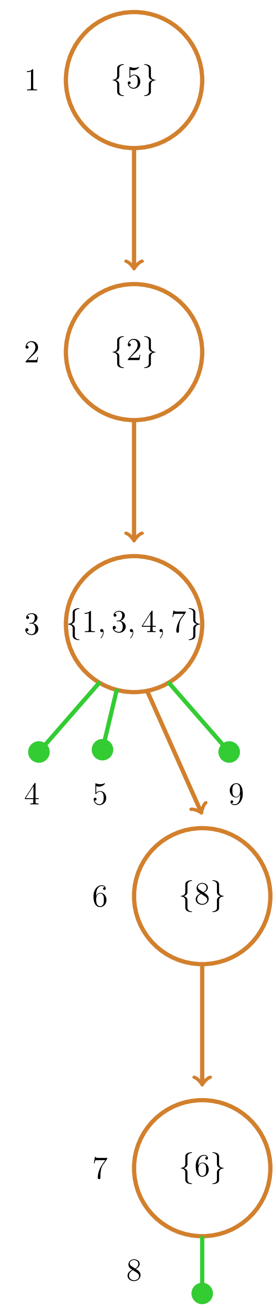
Remark. *The leaves of the tree are those corresponding to an element* i *such that* $1 \leqslant i \leqslant n+1$*, and* i *is* not *in* f*.*
Furthermore, as we will have a total edges by definition, the presence of a node corresponding to $n+1$ *is necessary, even though it will always be empty.*

Example ($n = 12$)**.**
• $f = (5, 7, 1, 3, 1, 8, 2, 7, 1, 3, 11, 11)$
• *Labels :* $[\{3, 5, 9\},\ \{7\},\ \{4, 10\},\ \emptyset,\ \{1\},\ \emptyset,\ \{2, 8\},\ \{6\},\ \emptyset,\ \emptyset,\ \{11, 12\},\ \emptyset, \emptyset]$



Conversely, by reading the labels of a parking tree depth-first in pre-order, we get the list of positions of each number in the corresponding parking function, thus creating a *bijection*.

Example (From parking tree to parking function)**.**



- *The labels are* $[\{5\},\ \{2\},\ \{1,3,4,7\},\ \emptyset,\ \emptyset,\ \{8\},\ \{6\},\ \emptyset,\ \emptyset]$.
- *Thus the corresponding parking function is* $(3,2,3,3,1,7,3,6) \in \mathcal{PF}_8$.

3.2 Rational Parking Trees

Definition 33 (Rational Parking Trees)**.** *A* rational parking tree *is defined from a rational parking function $f = (a_1, \dots, a_a) \in \mathcal{PF}_{a,b}$ as follows :*

- *For $1 \leqslant i \leqslant n+1$, we define the limit l_i as the* integer portion *of* $\dfrac{b}{a}(i-1)+1$.

  Let $l_0 = 0$.

- *From these limits, we deduce the intervals $itv_i =]l_{i-1}, l_i]$ for $1 \leqslant i \leqslant a+1$.*

- *For $1 \leqslant i \leqslant b+1$, define s_i as $\{j \mid a_j = i\}$.*

- *$[s_1, \dots, s_{b+1}]$ describes the pre-order depth-first traversal of the tree.*

- *Each node labeled by a set of size k has k* groups *of children, which are defined by the intervals.*

Example $(a < b)$**.**

- $a = 7$

- $b = 9$

- *Limits :* $[1,\ 2\frac{2}{7},\ 3\frac{4}{7},\ 4\frac{6}{7},\ 6\frac{1}{7},\ 7\frac{3}{7},\ 8\frac{5}{7},\ 10]$

- *Integral limits :* $[0, 1, 2, 3, 4, 6, 7, 8, 10]$

- *Intervals :*

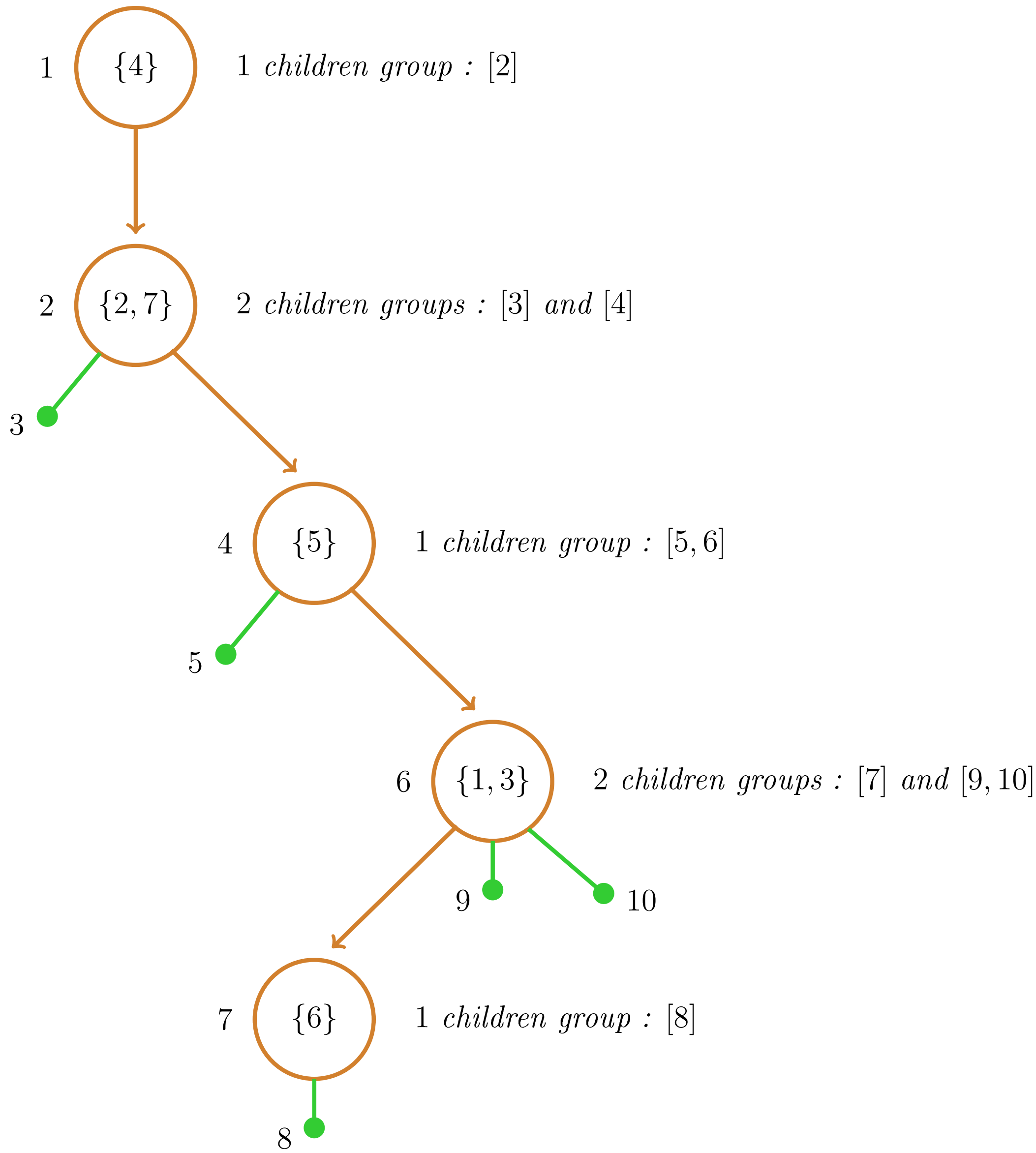
  $]0,1] \quad]1,2] \quad]2,3] \quad]3,4] \quad]4,6] \quad]6,7] \quad]7,8] \quad]8,10]$

- *Children groups :*

  $[1] \quad [2] \quad [3] \quad [4] \quad [5,6] \quad [7] \quad [8]$

- $f = (6, 2, 6, 1, 4, 7, 2)$

- *Labels :* $\{\{4\},\ \{2,7\},\ \emptyset,\ \{5\},\ \emptyset,\ \{1,3\},\ \{6\},\ \emptyset,\ \emptyset,\ \emptyset\}$



Example $(a > b)$**.**

- $a = 9$
- $b = 7$
- *Limits :* $[1,\ 1\frac{7}{9},\ 2\frac{5}{9},\ 3\frac{3}{9},\ 4\frac{1}{9},\ 4\frac{8}{9},\ 5\frac{6}{9},\ 6\frac{4}{9},\ 7\frac{2}{9},\ 8]$

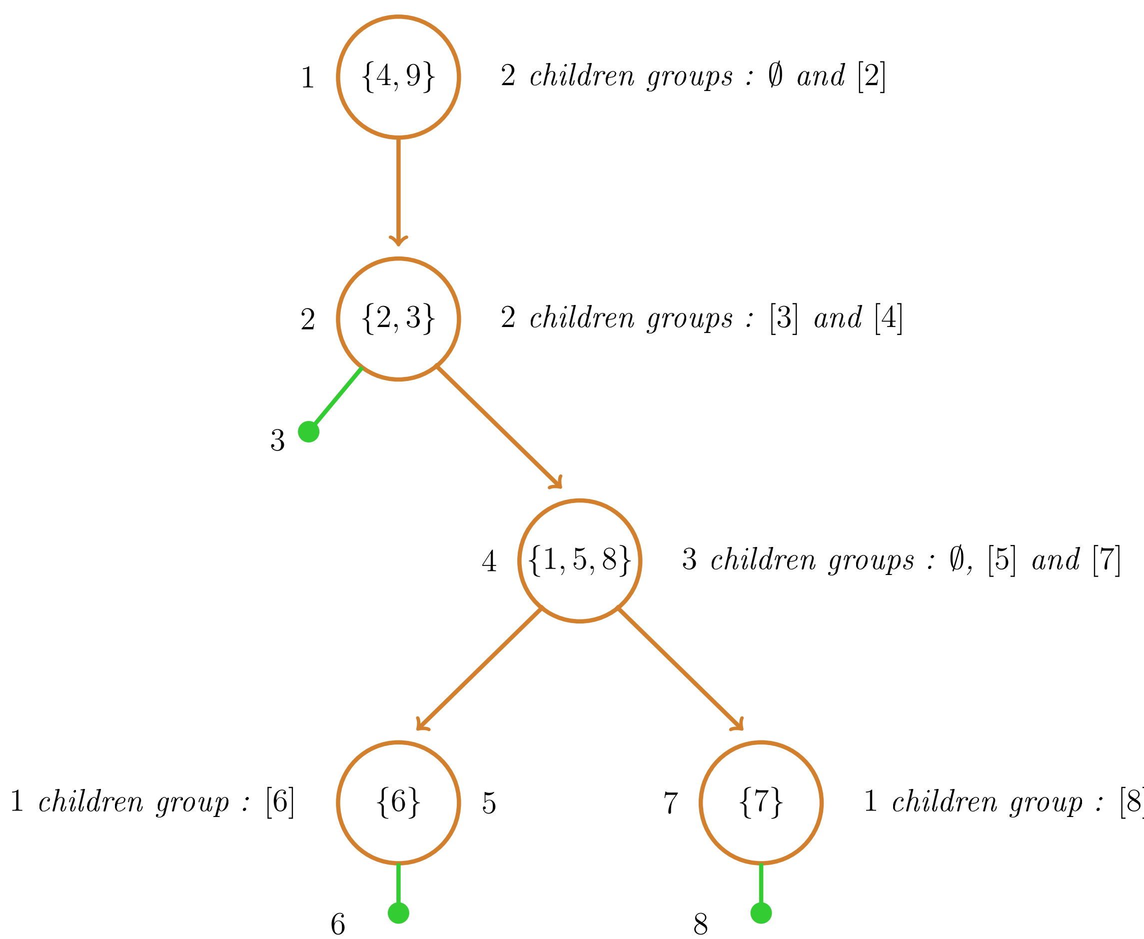
- *Integral limits :* $[0, 1, 1, 2, 3, 4, 4, 5, 6, 7, 8]$
- *Intervals :*

$$]0,1] \quad]1,1] \quad]1,2] \quad]2,3] \quad]3,4]$$
$$]4,4] \quad [4,5] \quad]5,6] \quad]6,7] \quad]7,8]$$

- *Children groups :*

$$[1] \quad \emptyset \quad [2] \quad [3] \quad [4] \quad \emptyset \quad [5] \quad [6] \quad [7] \quad [8]$$

- $f = (4, 2, 2, 1, 4, 5, 7, 4, 1)$
- *Labels :* $\{\{4,9\},\ \{2,3\},\ \emptyset,\ \{1,5,8\}, \{6\},\ \emptyset,\ \{7\},\ \emptyset\}$



In both cases, the converse direction of the *bijection* is obtained with the same method as for classical parking trees.

Future Work

A natural question that emerges from this work is a need for a proof of the main Conjecture. Furthermore, if a combinatorial proof arises, it could give the corresponding integer sequence a combinatorial formula and a combinatorial meaning that does not involve a generating series.

As for the rational case, future work might include the search for generalized formulae for the number of intervals in both posets defined in Section 2.3, whether it is expressed as a generating series or not.

Finally, following the work of [9] on parking trees, one could study the cover relations on rational parking trees.

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