Rational Parking Functions

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Abstract

This is an abstract about Rational Parking Functions.

Contents

Chapter 1

The integer case

1.1 Parking Functions

Definition 1 (Parking Function)**.** *A* parking function *is a sequence of positive integers* $(a_1, a_2, \ldots, a_n)$ *such that its non-decreasing reordering* $(b_1, b_2, \ldots, b_n)$ *has* $b_i \leqslant i$ *for all* i.
We denote by $\mathcal{PF}_n$ *the set of parking functions of length* n.

$$\mathcal{PF} = \bigcup_{n>0} \mathcal{PF}_n$$

.

Example.

$$f_1 = (7, 3, 1, 4, 2, 5, 2) \in \mathcal{PF}_7$$
$$f_2 = (7, 3, 1, 4, 2, 5, 4) \notin \mathcal{PF}_7$$

Theorem 1. *Let* pf_n *be the cardinal of* $\mathcal{PF}_n$*. We have*

$$pf_n = (n+1)^{n-1}$$

.

Example $(n = 1, 2, 3)$**.**
• $n = 1 \ : \ pf_1 = 1$
(1)

• $n = 2 \ : \ pf_2 = 3$
$(1,1) \quad (1,2) \quad (2,1)$
• $n = 3 \ : \ pf_3 = 16$
$(1,1,1) \quad (1,1,2) \quad (1,1,3) \quad (1,2,1) \quad (1,2,2) \quad (1,2,3) \quad (1,3,1)$
$(1,3,2) \quad (2,1,1) \quad (2,1,2) \quad (2,1,3) \quad (2,2,1) \quad (2,3,1) \quad (3,1,1)$
$(3,1,2) \quad (3,2,1)$

1.1.1 Primitive parking functions

Definition 2 (Primitive). *A parking function $(a_1, a_2, \ldots, a_n)$ is said* primitive *if it is already in non-decreasing order.*
We denote by ${\mathcal{PF}'}_n$ the set of primitive parking functions of length n.

$$\mathcal{PF}' = \bigcup_{n>0} {\mathcal{PF}'}_n$$

Example.

$$f_1 = (1,2,2,3) \in {\mathcal{PF}'}_4$$
$$f_2 = (1,2,3,2) \notin {\mathcal{PF}'}_4, \textit{ even though } f_2 \in \mathcal{PF}_4$$

Theorem 2. *Let pf'_n be the cardinal of ${\mathcal{PF}'}_n$. We have*

$$pf'_n = \frac{1}{n+1}\binom{2n}{n}$$

which is the n^{th} Catalan number $Cat(n)$.

Example $(n = 1, 2, 3)$**.**
• $n = 1 \ : \ pf'_1 = 1$
(1)
• $n = 2 \ : \ pf'_2 = 2$
$(1,1) \quad (1,2)$
• $n = 3 \ : \ pf'_3 = 5$
$(1,1,1) \quad (1,1,2) \quad (1,1,3) \quad (1,2,2) \quad (1,2,3)$

1.2 Non-crossing Partitions

Definition 3 (Non-crossing Partition)**.** *A* non-crossing partition *of a* totally ordered *set E is a set partition $P = \{E_1, E_2, \ldots, E_k\}$ such that if $a, c \in E_i$, $b, d \in E_j$, and $i \neq j$, then we do* not *have $a < b < c < d$, nor $a > b > c > d$. We denote by $\mathcal{NC}_n$ the set of non-crossing partitions of $\{1, 2, \ldots, n\}$.*

$$\mathcal{NC} = \bigcup_{n>0} \mathcal{NC}_n$$

From this point, we assume that every partition $P = \{B_1, \ldots, B_l\}$ is *sorted* such that :
• For each block $B_i = \{b_1, \ldots, b_k\} \in P$, $b_1 < \ldots < b_k$
• $min(B_1) < \ldots < min(B_k)$

Notation. $[n] = \{1, 2, \ldots, n\}$

Example ($E = [6]$)**.**

$$P_1 = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\} \in \mathcal{NC}_6$$
$$P_2 = \{\{1, 2, 4\}, \{3, 5\}, \{6\}\} \notin \mathcal{NC}_6$$

Theorem 3. *Let nc_n be the cardinal of $\mathcal{NC}_n$. We have*

$$nc_n = \frac{1}{n+1}\binom{2n}{n}$$

which is again the n^{th} Catalan number $Cat(n)$.

Example ($n = 1, 2, 3$)**.**
• $n = 1 \ : \ nc_1 = 1$
  $\{\{1\}\}$
• $n = 2 \ : \ nc_2 = 2$
  $\{\{1, 2\}\}$ $\{\{1\}, \{2\}\}$
• $n = 3 \ : \ nc_3 = 5$
  $\{\{1, 2, 3\}\}$ $\{\{1\}, \{2, 3\}\}$ $\{\{1, 3\}, \{2\}\}$ $\{\{1, 2\}, \{3\}\}$ $\{\{1\}, \{2\}, \{3\}\}$

Proposition. *This means we can create a* bijection *between $\mathcal{PF}'_n$ and $\mathcal{NC}_n$.*

Proof.

- $\mathcal{NC}_n \to \mathcal{PF'}_n$: For each block B in the non-crossing partition, take $i = min(B)$, and let $k_i = size(B)$.
  $k_i = 0$ if i is not the minimum of a block.
  The corresponding parking function is $(\underbrace{1,\ldots,1}_{k_1},\underbrace{2,\ldots,2}_{k_2},\ldots,\underbrace{n,\ldots,n}_{k_n})$.

- $\mathcal{PF'}_n \to \mathcal{NC}_n$: For each i in $[n]$, if i appears n_i times in the parking function, B_i will be of size n_i with minimum element i. There is a unique set partition $P = \bigcup_i B_i$ of $[n]$ respecting these conditions that is non-crossing : for each minimum i in *decreasing order*, add the n_i first free elements of $[i+1, i+2, \ldots, n, 1, \ldots, i-1]$ to B_i.

$\square$

Example $(n = 6)$**.**

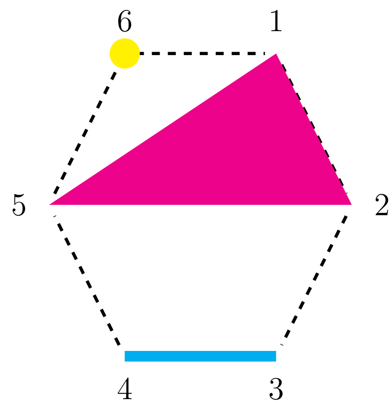
$$P = \{\{1,2,5\},\{3,4\},\{6\}\} \qquad\qquad f = (1,1,1,3,3,6)$$

Corollary. *A non-crossing partition can be represented by the minimums and sizes of its blocks.*

Example. $\{\{1,2,5\},\{3,4\},\{6\}\}$ *can be represented by the following dictionnary :*
• *1 : 3*
• *3 : 2*
• *6 : 1*

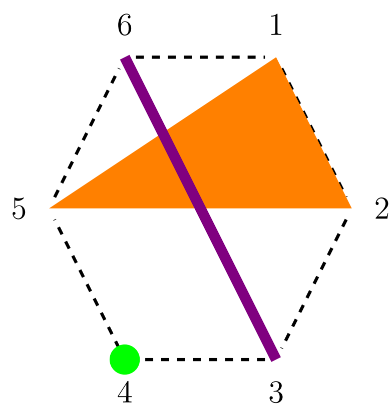
A non-crossing partition of $[n]$ can be represented graphically on a regular n-vertices polygon, with vertices labeled from 1 to n clockwise. We then represent each block $B = \{b_1, \ldots, b_k\}$ by the convex hull of $\{b_1, \ldots, b_k\}$.

Example $(P = \{\{1,2,5\},\{3,4\},\{6\}\})$**.**



Thus non-crossing meaning the hulls are disjoint*.*

Example (Counter-example : $P = \{\{1,5,2\},\{3,6\},\{4\}\}$)**.**



This partition is not *non-crossing, as the convex hulls of* $\{1,2,5\}$ *and* $\{3,6\}$ *are* not *disjoint.*

1.2.1 The non-crossing partitions poset

Definition 4 ($\succ$)**.** *We say that P covers Q, written $P \succ Q$, if $\exists B_i, B_j \in P$ such that $Q = P - \{B_i, B_j\} \cup \{B_i \cup B_j\}$*

Example. $\{\{1,6\},\{2,3\},\{4,5\}\} \succ \{\{1,2,3,6\},\{4,5\}\}$

- $B_i = \{1,6\}$

- $B_j = \{2, 3\}$

Proposition. *This covering relation defines the* poset *of* $\mathcal{NC}_n$. *We denote by* $\mathcal{NCC}_n$ *the set of* maximal chains *in the poset of* $\mathcal{NC}_n$.

$$\mathcal{NCC} = \bigcup_{n>0} \mathcal{NCC}_n$$

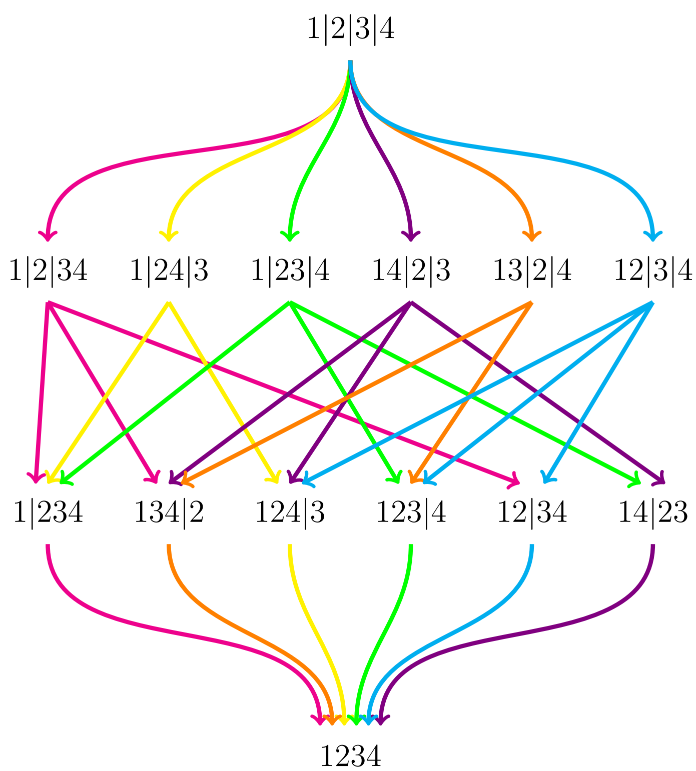
Remark. *The bottom element of this poset is* $\{\{1, \ldots, n\}\}$, *and the top element is* $\{\{1\}, \ldots, \{n\}\}$.

Theorem 4. *Let* ncc_n *be the cardinal of* $\mathcal{NCC}_n$. *We have*

$$ncc_n = n^{n-2}$$

.

Example (The poset of $\mathcal{NC}_4$)**.**
To shorten labels, we represent $\{\{1\}, \{2, 3\}, \{4\}\}$ *by* 1|23|4.



There are $4^2 = 16$ different maximal chains, and $\frac{1}{5}\binom{8}{4} = \frac{70}{5} = 14$ elements in this poset.

1.2.2 Kreweras complement

Definition 5 (Associated Permutation)**.** *The* permutation σ *associated to a non-crossing partition has a cycle* $(b_1, \dots, b_k)$ *for each block* $B = \{b_1, \dots, b_k\}$ *of the partition.*

Example. *The permutation associated to* $\{\{1,2,5\},\{3,4\},\{6\}\}$ *is* $(1\ 2\ 5)\ (3\ 4)\ (6) = 254316$.

Definition 6 (Kreweras Complement)**.** *The* Kreweras complement $K(P)$ *of a non-crossing partition* P *is defined as follows :*
- *Let* σ *be the permutation associated to* P
- *Let* π *be the permutation* $(n\ n-1\ n-2\ \dots\ 3\ 2\ 1) = n123\dots n-1$
- $K(P)$ *is the* non-crossing partition *associated to* $\pi\sigma$.

Example ($P = \{\{1,2,5\},\{3,4\},\{6\}\}$)**.**
- $\sigma = (1\ 2\ 5)\ (3\ 4)\ (6) = 254316$
- $\pi = (6\ 5\ 4\ 3\ 2\ 1) = 612345$
- $\pi\sigma = 143265 = (1)\ (2\ 4)\ (3)\ (5\ 6)$
- $K(P) = \{\{1\},\{2,4\},\{3\},\{5,6\}\}$

Proposition (Kreweras minimums)**.** *Let* $P = \{B_1, \dots, B_k\}$ *be a non-crossing partition. Let* $K(P) = \{B'_1, \dots, B'_l\}$ *be its Kreweras complement. Then*

$$\bigcup_{1 \le i \le l} min(B'_i) = B_1 \cup \bigcup_{1 < j \le k} B_i - max(B_i)$$

Example ($P = \{\{1,2,5\},\{3,4\},\{6\}\}$)**.**

- $K(P) = \{\{1\},\{2,4\},\{3\},\{5,6\}\}$
- $\bigcup min(B'_i) = \{1,2,3,5\}$

- $B_1 \cup \bigcup B_i - max(B_i) = \{1,2,5\} \cup \{3,4\} - \{4\} \cup \{6\} - \{6\} = \{1,2,5\} \cup \{3\} \cup \emptyset = \{1,2,3,5\}$

Notation. $B_{[i]}$ = *block containing* i.

Proposition (Kreweras block sizes)**.** *Let* $P = \{B_1, \ldots, B_k\}$ *be a non-crossing partition. Let* $K(P) = \{B'_1, \ldots, B'_l\}$ *be its Kreweras complement. Then the size of the block* B'_i *is defined as follows :*

- *Let* m_i *be the the* i^{th} *minimum of* $K(P)$
- *Define a* transition $\phi(e)$ *as*

  Let $j = e + 1$ *(or* 1 *if* $e = n$*)*

  $\phi(e) = max(B_{[j]})$
- *The size of* B'_i *is* k_{min} *such that* $k_{min} = min\{k > 0 \mid \phi^k(m_i) \in B_{[m_i]}\}$.

Example ($P = \{\{1,2,5\},\{3,4\},\{6\}\}$)**.**

- $mins = \{1,2,3,5\}$
- $m_1 = 1$

  $B_{[1]} = B_1$

  $max(B_{[2]} = max(B_1) = 5$

  The size for m_1 *is* 1.
- m_2

  $B_{[2]} = B_1$

  $max(B_{[3]}) = max(B_2) = 4$

  $max(B_{[5]}) = max(B_1) = 5$

  The size for m_2 *is* 2.
- $m_3 = 3$

  $B_{[3]} = B_2$

  $max(B_{[4]}) = max(B_2) = 4$

  The size for m_3 *is* 1.

- $m_4 = 5$

  $B_{[5]} = B_1$

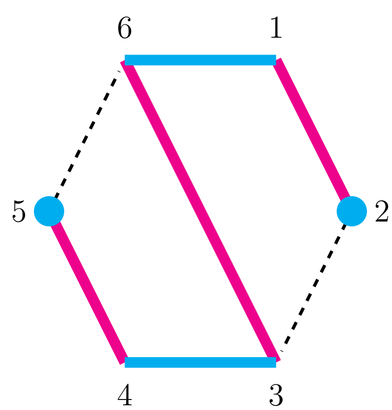
  $max(B_{[6]}) = max(B_3) = 6$

  $max(B_{[1]}) = max(B_1) = 5$

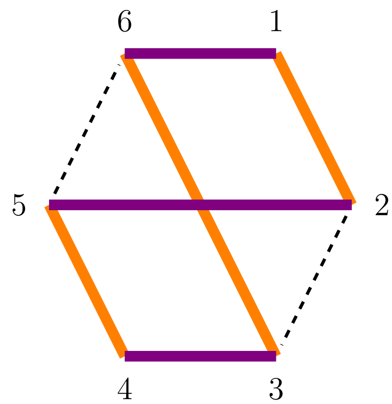
  The size for m_4 *is* 2*.*

Definition 7 (Mutually Non-crossing Partitions)**.** *2 partitions* P *and* Q *are said* mutually non-crossing *if :*
• P *is non-crossing*
• Q *is non-crossing*
• *For every block* B_i *of* P *and every block* B_j *of* Q*, if* $a, c \in B_i$ *and* $b, d \in B_j$*, then we can* not *have* $a < b < c < d$*, nor* $a > b > c > d$*.*

Example $(P = \{\{1, 2\}, \{3, 6\},\ \{4, 5\}\}, Q = \{\{1, 6\}, \{2\}, \{3, 4\}, \{5\}\})$**.**



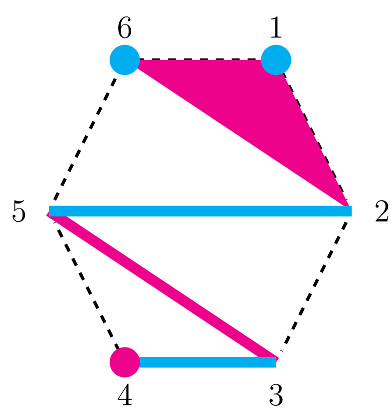
Example (Counter-example : $P = \{\{1, 2\}, \{3, 6\},\ \{4, 5\}\}, Q = \{\{1, 6\}, \{2, 5\}, \{3, 4\}\}$)**.**



Remark. *Note that vertices* can *touch, but the edges of the convex hulls can* not *cross.*

Proposition. *For any non-crossing partition P, P and $K(P)$ are mutually non-crossing.*
Furthermore, $K(P)$ is a densest *partition that is mutually non-crossing with P. That is,* no *partition Q that is mutually non-crossing with P has less blocks than $K(P)$.*

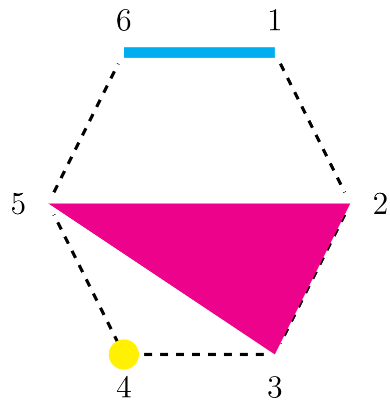
Example ($P = \{1, 2, 6\}, \{3, 5\}, \{4\}\}$)**.** $Q = \{\{1\}, \{2, 5\}, \{3, 4\}, \{6\}\}$



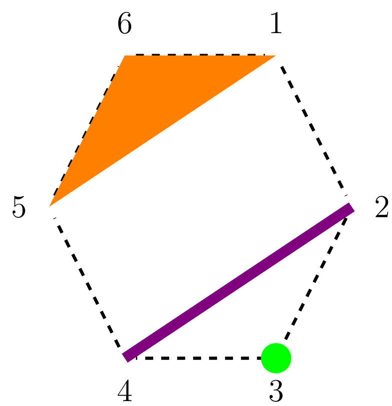
1.2.3 Action of $\mathfrak{S}_n$ on partitions of $[n]$

Definition 8 (Action of $\mathfrak{S}_n$)**.** *The action of $\mathfrak{S}_n$ on a partition $P = \{B_1, \ldots, B_l\}$ of $[n]$ is defined by :*
- *For each block $B_i = \{b_1, \ldots, b_k\}$: $\sigma(Bi) = \{\sigma(b_1), \ldots, \sigma(b_k)\}$*
- *When $P \in \mathcal{NC}_n$, we denote $\rho = \sigma(P) = \{\sigma(B_1), \ldots, \sigma(B_l)\}$*

Example ($\sigma = 415362, P = \{\{1,6\},\{2,3,5\},\{4\}\}$)**.**
$\sigma(P) = \{\{1,5,6\},\{2,4\},\{3\}\}$



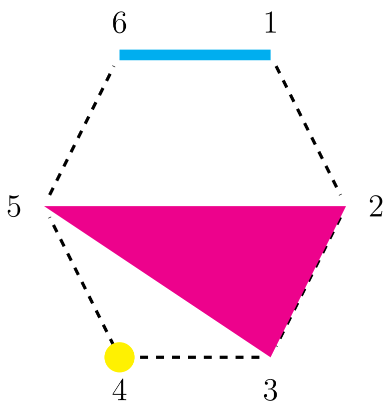
P



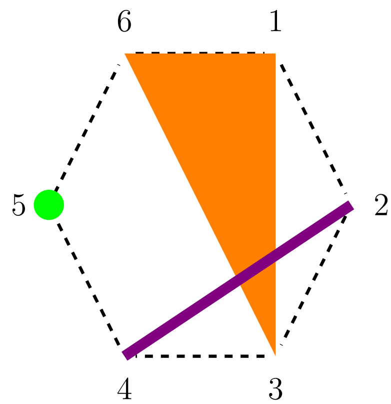
$\sigma(\mathbf{P})$

Remark. *Note that $\mathcal{NC}_n$ is* not *stable under the action of $\mathfrak{S}_n$. That is, even if P is non-crossing, $\sigma(P)$ is* not *necessarily non-crossing.*

Example (Counter-example : $\sigma = 413562, P = \{\{1,6\},\{2,3,5\},\{4\}\}$)**.**
$\sigma(P) = \{\{1,3,6\},\{2,4\},\{5\}\}$



P

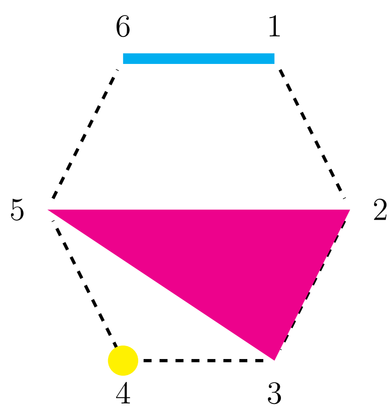


$\sigma(\mathbf{P})$

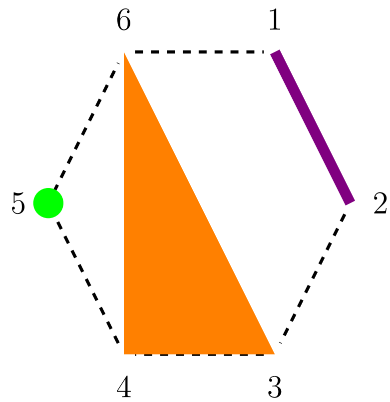
Definition 9 (Rotation)**.** *We define the* rotation operator *rot of* $P \in \mathcal{NC}_n$ *as* $rot(P) = (1\ 2\ 3\ \ \ldots\ n)(P) = 23\ldots n1(P)$.
Conversely, we define rot^{-1} *of* P *as* $rot^{-1}(P) = (n\ n-1\ \ \ldots 3\ 2\ 1)(P) = n12\ldots n-1(P)$.

Example ($P = \{\{1,6\},\{2,3,5\},\{4\}\}$)**.**

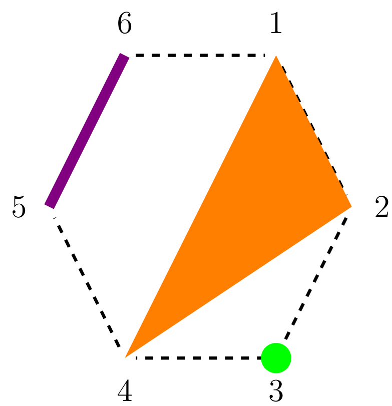
- $rot(P) = \{\{1,2\},\{3,4,6\},\{5\}\}$
- $rot^{-1}(P) = \{\{1,2,4\},\{3\},\{5,6\}\}$



P



$\mathbf{rot(P)}$



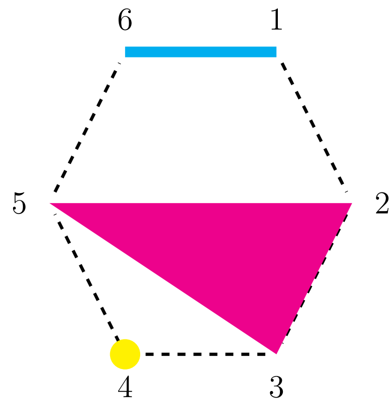
$\mathbf{rot^{-1}(P)}$

Remark.

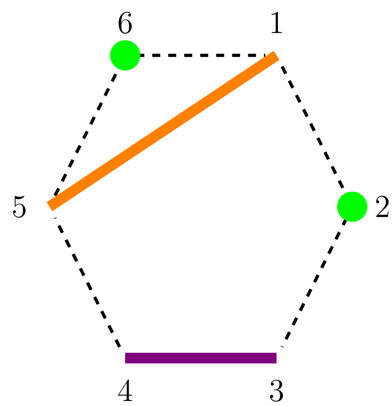
- $rot(rot^{-1}(P)) = rot^{-1}(rot(P)) = P$
- *$rot(P)$ and $rot^{-1}(P)$ are always non-crossing partitions.*
- *If $P \in \mathcal{NC}_n$, then $rot^n(P) = rot^{-n}(P) = P$.*

Proposition. $K(K(P)) = rot^{-1}(P)$.

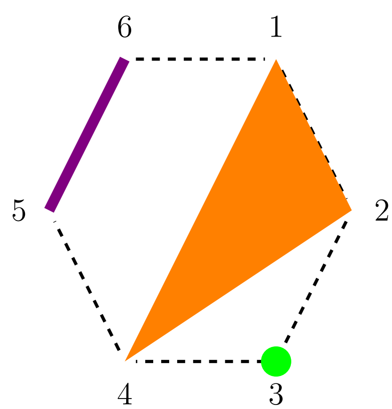
Example ($P = \{\{1,6\},\{2,3,5\},\{4\}\}$)**.**



$\mathbf{P}$



$\mathbf{K}(\mathbf{P})$



$\mathbf{K}(\mathbf{K}(\mathbf{P}))$

1.3 Non-crossing 2-partitions

Definition 10 (Non-crossing 2-partition)**.** *A* non-crossing 2-partition *of a* totally ordered *set E is a pair (P, σ) where :*
- *P is a non-crossing partition of E*
- *σ is a permutation of the elements of E*
- *For each* sorted *block $B_i = \{b_1, \ldots, b_k\} \in P$, we have $\sigma(b_i) < \ldots < \sigma(b_k)$*

We denote by $\mathcal{NC}_n^2$ the set of non-crossing 2-partitions of $[n]$.

$$\mathcal{NC}^2 = \bigcup_{n>0} \mathcal{NC}_n^2$$

.

Example ($\mathcal{NC}_6^2$)**.** $P = \{\{1,6\},\{2,3,5\},\{4\}\}$ $\sigma = 413265$
$\rho = \{\{1,3,6\},\{2\},\{4,5\}\}$

Theorem 5. *Let nc_n^2 be the cardinal of $\mathcal{NC}_n^2$. We have*

$$nc_n^2 = (n+1)^{n-1}$$

Example ($n = 1, 2, 3$)**.**

- $n = 1$: $nc_1^2 = 1$

  $\{\{1\}\}$ 1 $\rho = P$

- $n = 2$: $nc_2^2 = 3$

  $\{\{1\},\{2\}\}$ 12 $\rho = P$

  $\{\{1\},\{2\}\}$ 21 $\rho = P$

  $\{\{1,2\}\}$ 12 $\rho = P$

- $n = 3$: $nc_3^2 = 16$

  $\{\{1\},\{2\},\{3\}\}$ 123 $\rho = P$

  $\{\{1\},\{2\},\{3\}\}$ 132 $\rho = P$

  $\{\{1\},\{2\},\{3\}\}$ 213 $\rho = P$

  $\{\{1\},\{2\},\{3\}\}$ 231 $\rho = P$

$\{\{1\},\{2\},\{3\}\}$	312	$\rho = P$
$\{\{1\},\{2\},\{3\}\}$	321	$\rho = P$
$\{\{1,2\},\{3\}\}$	123	$\rho = P$
$\{\{1,2\},\{3\}\}$	132	$\rho = \{\{1,3\},\{2\}\}$
$\{\{1,2\},\{3\}\}$	231	$\rho = \{\{1\},\{2,3\}\}$
$\{\{1\},\{2,3\}\}$	123	$\rho = P$
$\{\{1\},\{2,3\}\}$	213	$\rho = \{\{1,3\},\{2\}\}$
$\{\{1\},\{2,3\}\}$	312	$\rho = \{\{1,2\},\{3\}\}$
$\{\{1,3\},\{2\}\}$	123	$\rho = P$
$\{\{1,3\},\{2\}\}$	132	$\rho = \{\{1,2\},\{3\}\}$
$\{\{1,3\},\{2\}\}$	213	$\rho = \{\{1\},\{2,3\}\}$
$\{\{1,2,3\}\}$	123	$\rho = P$

Proposition. *This means we can create a* bijection *between $\mathcal{PF}_n$ and $\mathcal{NC}_n^2$.*

Proof.

- $\mathcal{PF}_n \to \mathcal{NC}_n^2$: Let $f = (a_1, \ldots, a_n) \in \mathcal{PF}_n$ be our parking function. For $i \in \{1, \ldots, n\}$, we define :

  l_i : the number of occurences of i in f.

  im_i : $\{j \mid a_j = i\}$

  The corresponding non-crossing partition will have the following constraints :

  For each $i \in \{1, \ldots, n\}$, if $l_i > 0$, then there is a block $B_{[i]}$ of length l_i with minimum element i.

  $\sigma(B_{[i]}) = im_i$

  There is a unique set partition $P = \bigcup_i B_{[i]}$ of $[n]$ and a unique permutation σ respecting these conditions such that $(P, \sigma) \in \mathcal{NC}_n^2$: for each minimum i in *decreasing order*, add the n_i first free elements of $[i+1, i+2, \ldots, n, 1, \ldots, i-1]$ to B_i. σ is then trivially obtained by the second constraint.

- $\mathcal{NC}_n^2 \to \mathcal{PF}_n$: Let (P, σ) with $P = \{B_1, \dots, B_l\}$ be our non-crossing 2-partition. For each block $B_i = \{b_1, \dots, b_k\} \in P$:

  $m_i = min(B_i) = b_1$

  $pos_i = \sigma(B_i)$

  For each $j \in pos_i$, we define $a_j = m_i$

  The corresponding parking function is $(a_1, \dots, a_n)$.

□

Example $(n = 8)$**.**

$$P = \{\{1, 2, 5\}, \{3, 4\}, \{6, 8\}, \{7\}\}$$
$$\sigma = 36187245$$
$$f = (3, 6, 1, 7, 6, 1, 1, 3)$$

1.3.1 The non-crossing 2-partitions poset

Definition 11 $(\succ^2)$**.** *We say that (P, σ) covers (Q, τ), written $(P, \sigma) \succ^2 (Q, \tau)$, if $\exists B_i, B_j \in P$ such that*
- $Q = P - \{B_i, B_j\} \cup \{B_i \cup B_j\}$
- $l \neq i, jb \in B_l \to \tau(b) = \sigma(b)$
- *Let* $B_i \cup B_j = \{b_1, \dots, b_k\}$:

  $\tau(B_i \cup B_j) = \sigma(B_i \cup B_j)$

  $\tau(b_1) < \dots < \tau(b_k)$

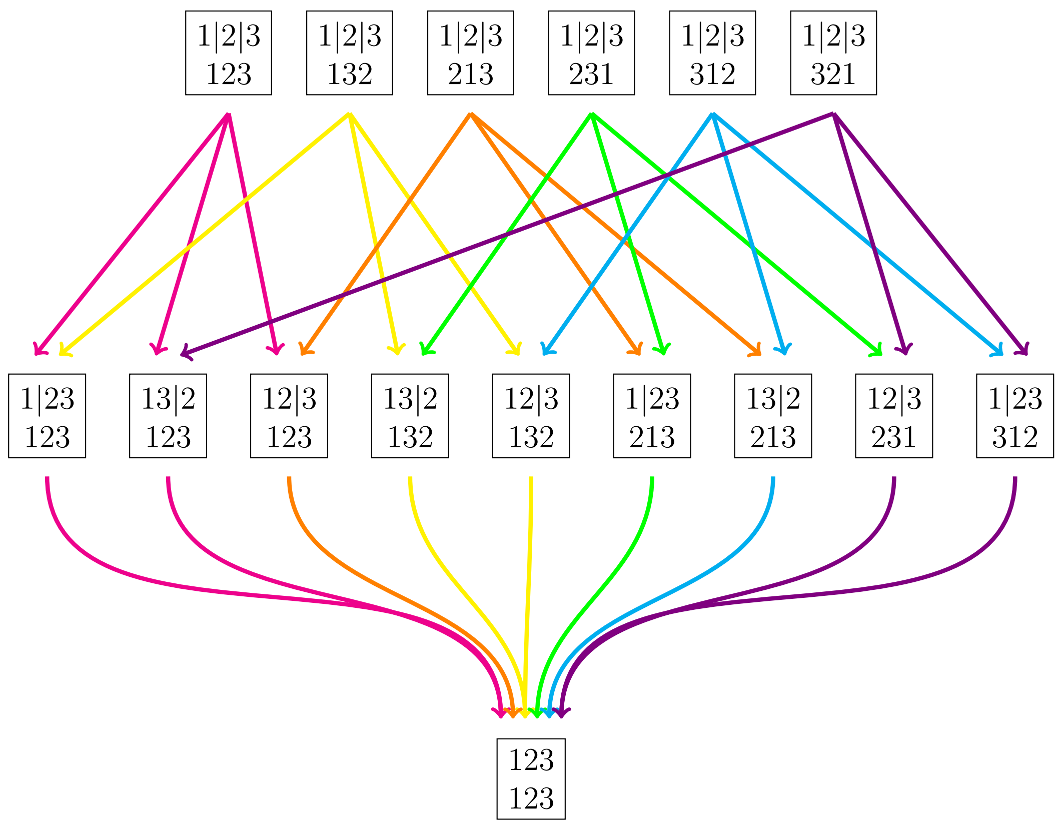
Example.
- $P = \{\{1, 6\}, \{2, 3\}, \{4\}, \{5\}\}$
- $\sigma = 236154$
- $Q = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$
- $\tau = 235164$
- $(P, \sigma) \succ^2 (Q, \tau)$
- $(P, \sigma) \not\succ^2 (Q, \sigma)$, *because* $\sigma(\{2, 3, 5\}) = \{3, 6, 5\}$ *is* not *ordemagenta.*

Proposition. *This covering relation defines the* poset *of* $\mathcal{NC}_n^2$.

Remark. *The bottom element of this poset is* $(\{\{1, \dots, n\}\}, 12 \dots n)$*, and the top elements are* $\{(\{\{1\}, \dots, \{n\}\}, \sigma) \mid \sigma \in \mathfrak{S}_n\}$.

Example (The poset of $\mathcal{NC}_3^2$)**.**
To shorten labels, we represent $(\{\{1,3\},\{2\}\},213)$ *by*

$$\boxed{\begin{array}{c}13|2\\213\end{array}}$$



There are $4^2 = 16$ *elements in this poset.*

1.3.2 The parking functions poset

Definition 12 (Rank)**.** *Given* $f = (a_1, \ldots, a_n) \in \mathcal{PF}_n$*, let*

$$b_i = \begin{cases} 1 \text{ } \textit{if} \ \exists j \mid a_j = i \\ 0 \ \textit{otherwise} \end{cases}$$

We define the rank *of f, noted $rk(f)$, as*

$$\sum_{1 \leq i \leq n} b_i$$

Example.

$$rk((1,5,4,2,3,3,1)) = 5$$
$$rk((4,7,1,1,3,2,2,8)) = 6$$

Definition 13 ($\succ_{pf}$)**.** *Since $\mathcal{PF}_n$ and $\mathcal{NC}_n^2$ are in bijection, we can define a* covering relation $\succ_{pf}$ *for $\mathcal{PF}_n$ as follows :*
$f \in \mathcal{PF}_n \succ_{pf} g \in \mathcal{PF}_n$ if and only if :

- *(P, σ) is the non-crossing 2-partition associated to f*
- *(Q, τ) is the non-crossing 2-partition associated to g*
- *$(P, \sigma) \succ^2 (Q, \tau)$*

Example.
- $P = \{\{1,6\},\{2,3\},\{4\},\{5\}\}$
- $\sigma = 236154$
- $Q = \{\{1,6\},\{2,3,5\},\{4\}\}$
- $\tau = 235164$
- $f = (4,1,2,1,5,2) \succ_{pf} g = (4,1,2,1,2,2)$

Remark. *If $f \succ_{pf} g$, then $rk(f) = rk(g) + 1$, and there exists i and j such that :*

- *$i < j$*
- *There is at least 1 occurence of i in f*
- *There is at least 1 occurence of j in f*

$$b_k = \begin{cases} i \text{ if } a_k = j \\ a_k \text{ otherwise} \end{cases}$$

Proposition. *This covering relation defines the* poset *of* $\mathcal{PF}_n$.

Remark. *The bottom element of this poset is* $(\underbrace{1,\dots,1}_{n})$*, and the top elements are the* permutations *of* $\{1,\dots,n\}$.

Example (The poset of $\mathcal{PF}_3$)**.**

1.4 A direct poset linked to Dyck paths

1.4.1 Dyck Paths

Notation. *We denote the number of occurences of a symbol* s *in a word* w *by* $|w|_s$.

Definition 14 (Dyck path)**.** *A* Dyck word *is a word* $w \in \{0,1\}^*$ *such that :*

- *for each* suffix w' *of* w, $|w'|_1 \geqslant |w'|_0$.
- $|w|_0 = |w|_1$.

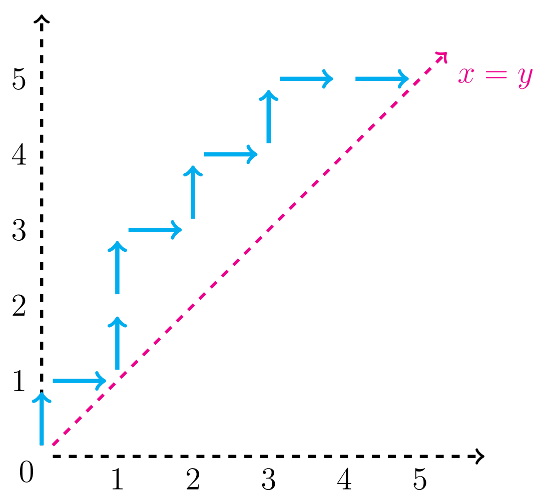
A Dyck word of length $2n$ *can be represented as a* path *from* $(0,0)$ *to* (n,n) *that stays over* $x=y$*, called a* Dyck path *:*

- *Each* 1 *corresponds to a* North step $\uparrow$.
- *Each* 0 *corresponds to an* East step $\rightarrow$.

We denote by $\mathcal{D}_n$ *the set of Dyck words of length* $2n$*.*

Example $(n=5)$**.**

$w_1 = 1011000110$ *is* not *a Dyck word, because* $|1011000|_0 > |1011000|_1$.
$w_2 = 1011010101$ *is* not *a Dyck word, because* $|w_2|_0 \neq |w_2|_1$.
$w_3 = 1011010100$ is *a Dyck word :*

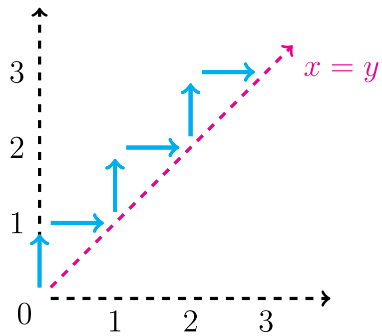


Theorem 6. *Let* d_n *be the cardinal of* $\mathcal{D}_n$*. We have*

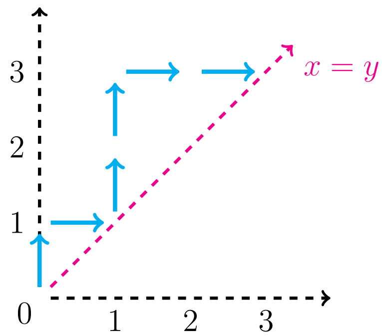
$$d_n = \frac{1}{n+1}\binom{2n}{n}$$

which is the n^{th} *Catalan number.*

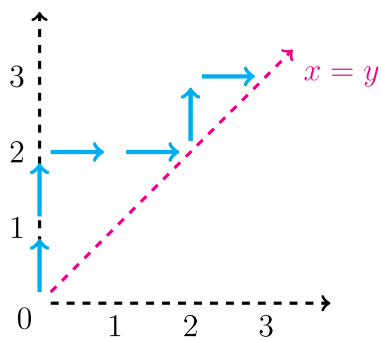
Example $(n=3)$**.** $d_n = 5$.



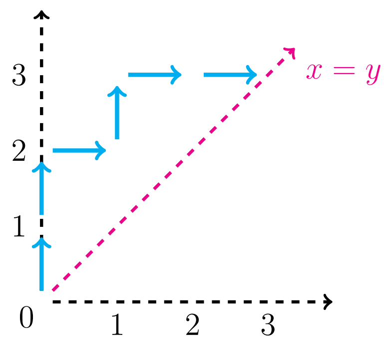
101010



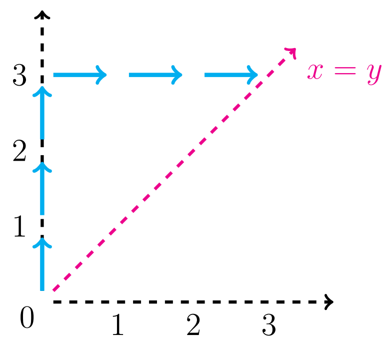
101100



110010



110100



111000

Proposition. *This means we can create a* bijection *between $\mathcal{PF'}_n$ and $\mathcal{D}_n$.*

Proof.

- $\mathcal{PF'}_n \to \mathcal{D}_n$: Let $f = (a_1, \ldots, a_n) \in \mathcal{PF'}_n$ be our primitive parking function. For $i \in \{1, \ldots, n\}$, we define l_i the number of occurences of i in f.
  The corresponding Dyck word will be $\underbrace{1\cdots1}_{l_1}0\underbrace{1\cdots1}_{l_2}0\cdots\underbrace{1\cdots1}_{l_n}0$.

- $\mathcal{D}_n \to \mathcal{PF'}_n$: Let w be our Dyck word, and consider its path representation. We define s_i to be the distance between the segment from $(0, i-1)$ to $(0, i)$ and the i^{th} North step. Then, let $a_i = s_i + 1$.
  The corresponding primitive parking function is $(a_1, \ldots, a_n)$.

$\square$

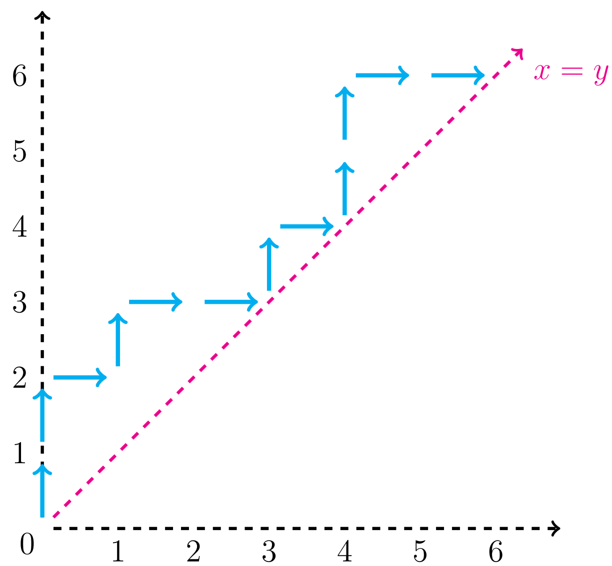
Example $(n = 6, \mathcal{PF'}_n \to \mathcal{D}_n)$**.**

- $f = (1, 1, 2, 4, 5, 5)$

  $l_1 = 2$ $\quad$ $l_2 = 1$ $\quad$ $l_3 = 0$

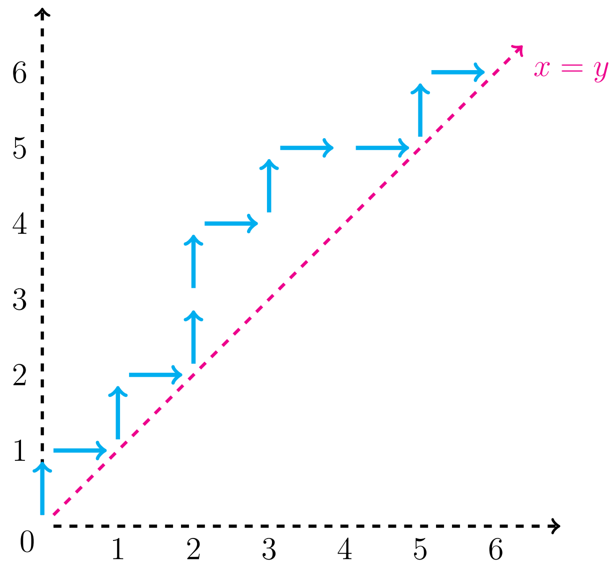
  $l_4 = 1$ $\quad$ $l_5 = 2$ $\quad$ $l_6 = 0$

- $w = (110100101100)$



Example $(n = 6, \mathcal{D}_n \to \mathcal{PF}'_n)$**.**

- $w = 101011010010$



- *Distances :*

$s_1 = 0$	$a_1 = 1$
$s_2 = 1$	$a_2 = 2$
$s_3 = 2$	$a_3 = 3$
$s_4 = 2$	$a_4 = 3$
$s_5 = 3$	$a_5 = 4$
$s_6 = 5$	$a_6 = 6$

- $f = (1, 2, 3, 3, 4, 6)$

1.4.2 Labeled Dyck Paths

Definition 15 (Labeled Dyck Path)**.** *A* labeled Dyck word *is a word* $w \in \{0, \ldots, n\}^*$ *such that :*

- *for each suffix* w' *of* w, $|w'|_{\neq 0} \geqslant |w'|_0$.
- $|w|_0 = |w|_{\neq 0}$.
- *for each* $i \in \{1, \ldots, n\}$, w *has exactly one occurence of* i.

- *if $w_i \neq 0$ and $w_{i+1} \neq 0$, then $w_i < w_{i+1}$. That is, consecutive North steps have increasing labels.*

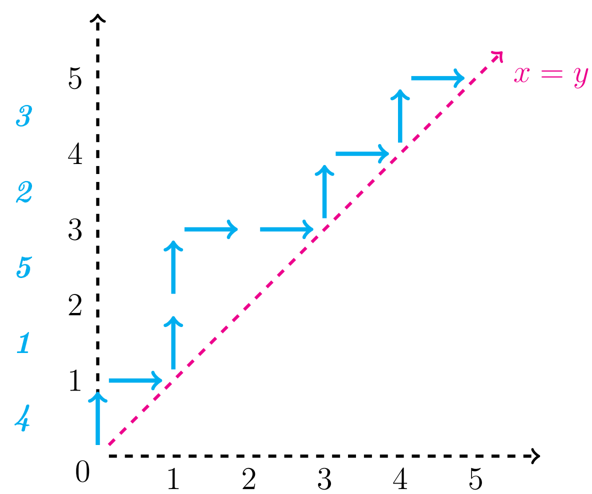
A labeled Dyck word of length $2n$ can be represented as a path from $(0,0)$ to (n,n), where each North step is associated to a label :

- *Each $i \neq 0$ corresponds to a* North step $\uparrow$ *labeled i.*
- *Each 0 corresponds to an* East step $\rightarrow$.

Those paths are called labeled Dyck paths.
We denote by $\mathcal{LD}_n$ the set of labeled Dyck words of length $2n$.

Example $(n = 5)$**.**

$w_1 = 4051002030$ *is* not *a labeled Dyck word, because* $5 > 1$.
$w_2 = 4015002030$ is *a labeled Dyck word :*



Theorem 7. *Let ld_n be the cardinal of $\mathcal{LD}_n$. We have*

$$ld_n = (n+1)^{n-1}$$

.

Example $(n = 3)$**.** $ld_n = 4^2 = 16$

- *Word of shape $XXX000$:*

  123000

- *Words of shape $XX0X00$:*

  120300 130200 230100

- *Words of shape $XX00X0$:*

  120030 130020 230010

- *Words of shape $X0XX00$:*

  102300 201300 301200

- *Words of shape $X0X0X0$:*

  102030 103020 201030

  203010 301020 302010

Proposition. *This means we can create a* bijection *between $\mathcal{PF}_n$ and $\mathcal{LD}_n$.*

Proof.

- $\mathcal{PF}_n \to \mathcal{LD}_n$: Let $f = (a_1, \ldots, a_n) \in \mathcal{PF}_n$ be our parking function. For $i \in \{1, \ldots, n\}$, we define $im_i : \{j \mid a_j = i\}$.
  We then define $im_{i,1}, \ldots, im_{i,k_i}$ to be the elements of im_i in increasing order.
  The corresponding labeled Dyck word will be
  $\underbrace{im_{1,1} \cdots im_{1,k_1}}_{im_1} 0 \underbrace{im_{2,1} \cdots im_{2,k_2}}_{im_2} 0 \cdots \underbrace{im_{n,1} \cdots im_{n,k_n}}_{im_n} 0.$

- $\mathcal{LD}_n \to \mathcal{PF}_n$: Let w be our labeled Dyck word, and consider its path representation. We define s_i to be the distance between the segment from $(0, i-1)$ to $(0, i)$ and the i^{th} North step.
  Then, let $label(i)$ be the label of the i^{th} North step, and $dist_i = \{label(j) | s_j = i\}$ be the set of the labels of all North steps at distance i.
  Then, if $j \in dist_i$, let $a_j = i + 1$.
  The corresponding parking function is $(a_1, \ldots, a_n)$.

$\square$

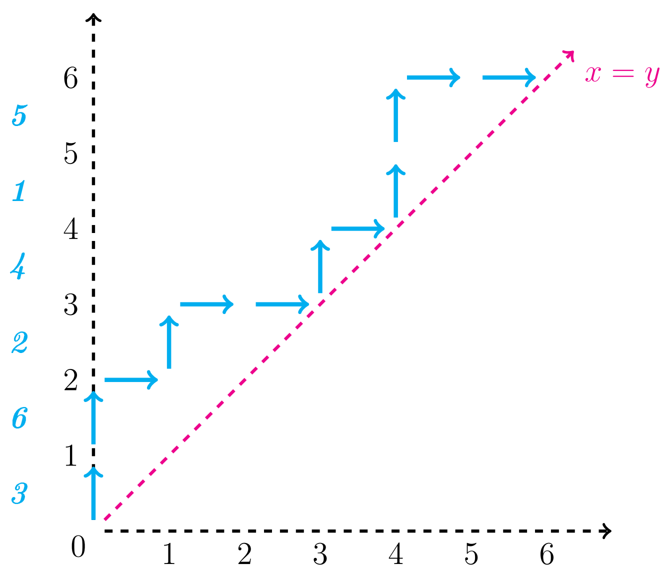
Example $(n = 6, \mathcal{PF}_n \to \mathcal{LD}_n)$**.**

- $f = (5, 2, 1, 4, 5, 1)$

  $im_1 = \{3, 6\}$ $\quad$ $im_2 = \{2\}$ $\quad$ $im_3 = \emptyset$

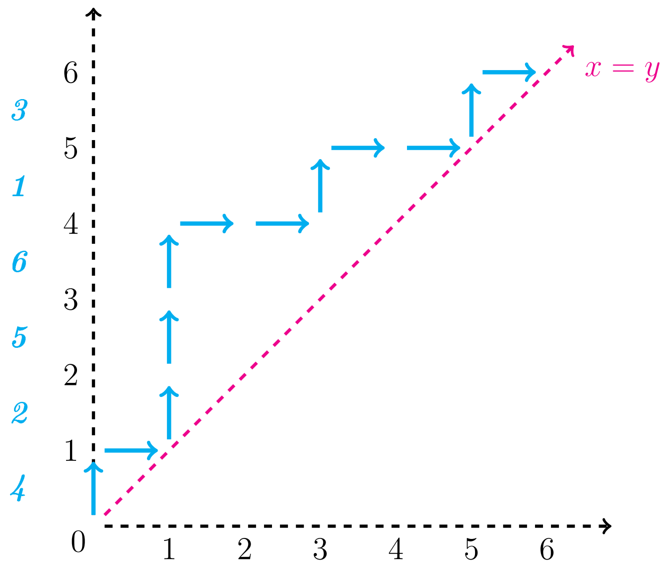
  $im_4 = \{4\}$ $\quad$ $im_5 = \{1, 5\}$ $\quad$ $im_6 = \emptyset$

- $w = 360200401500$



Example $(n = 6, \mathcal{LD}_n \to \mathcal{PF}_n)$**.**

- $w = 402560010030$



- *Distances :*

  $s_1 = 0$ $\quad s_2 = 1$ $\quad s_3 = 1$

  $s_4 = 1$ $\quad s_5 = 3$ $\quad s_6 = 5$

- *Labels :*

  $dist_0 = \{4\}$ $\quad dist_1 = \{2,5,6\}$ $\quad dist_2 = \emptyset$

  $dist_3 = \{1\}$ $\quad dist_4 = \emptyset$ $\quad dist_5 = \{3\}$

- $f = (4, 2, 6, 1, 2, 2)$

Remark. *The primitive parking functions are exactly the parking functions corresponding to labeled Dyck paths where the i^{th} North step is labeled i.*

1.4.3 Dyck - Parking Posets

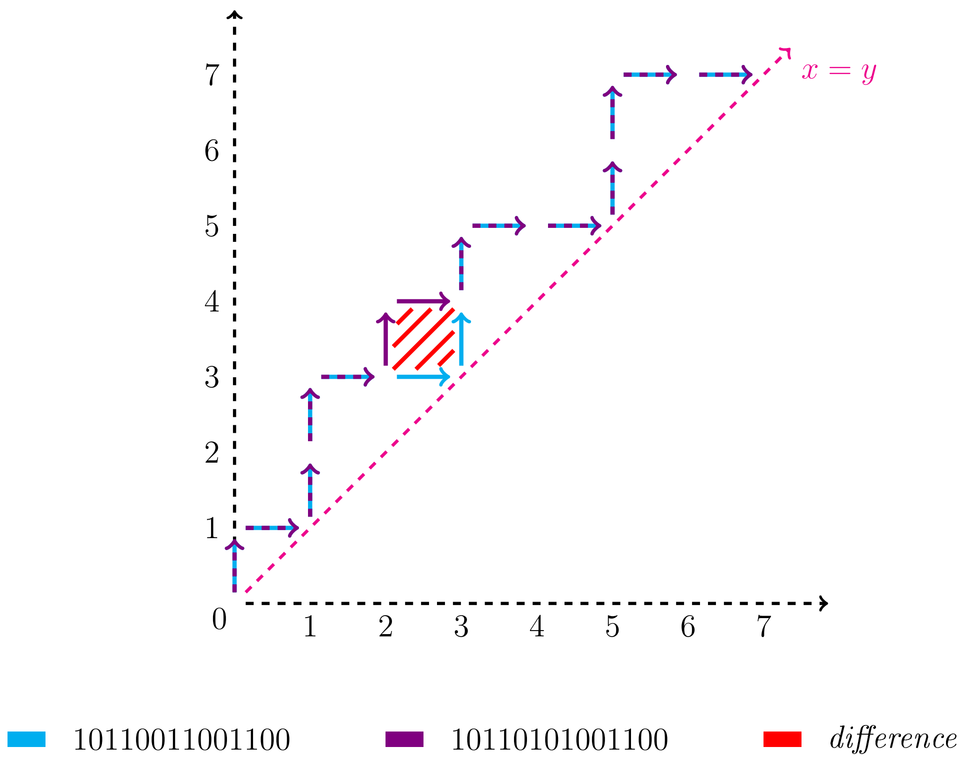
Primitive Dyck - Parking Posets

Definition 16 ($\gtrdot_d$). *For w and w' two Dyck words, we say that w covers w', written $w \gtrdot_d w'$, if $\exists w_1, w_2$ such that :*

- $w = w_1 01 w_2$
- $w' = w_1 10 w_2$

Example ($n = 7$)**.** 10110011001100 $\gtrdot_d$ 10110101001100

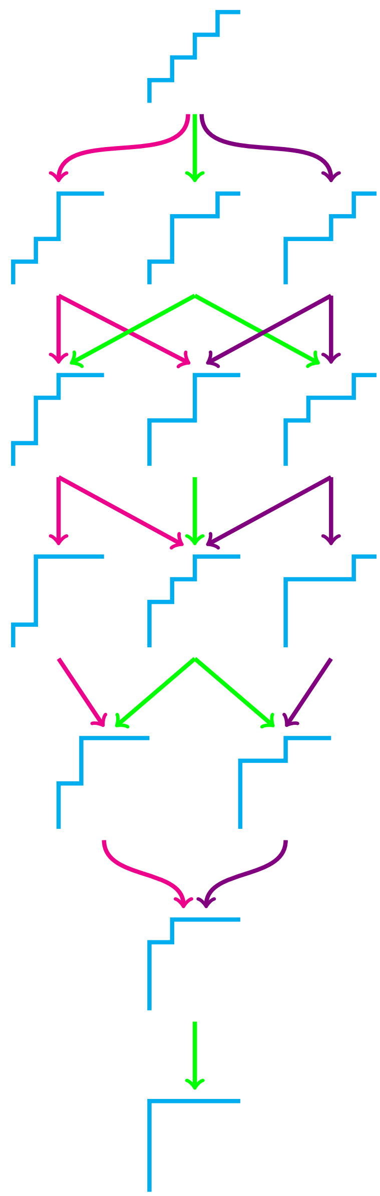
- $w_1 = 10110$
- $w_2 = 1001100$



Remark. *If $w_1 \gtrdot_d w_2$, then the path corresponding to w_2 is* over *the path corresponding to w_1, and the* difference *between the two paths is a square of size 1 by 1.*

Proposition. *This covering relation defines a* poset *for $\mathcal{D}_n$.*

Example (The poset of $\mathcal{D}_4$)**.**



There are $\frac{1}{5}\binom{8}{4} = \frac{70}{5} = 14$ *elements in this poset.*

Definition 17 (Nested Dyck paths)**.** *Two Dyck Paths* w_1 *and* w_2 *are said* nested *if* w_1 *is equal to* w_2 *or over* w_2*.*

Proposition. *If there exists a sequence* $w_1 \gtrdot_d w_2 \gtrdot_d w_3 \gtrdot_d \cdots \gtrdot_d w_k$ *with* $k \geqslant 0$*, then* w_1 *and* w_k *are* nested*.*

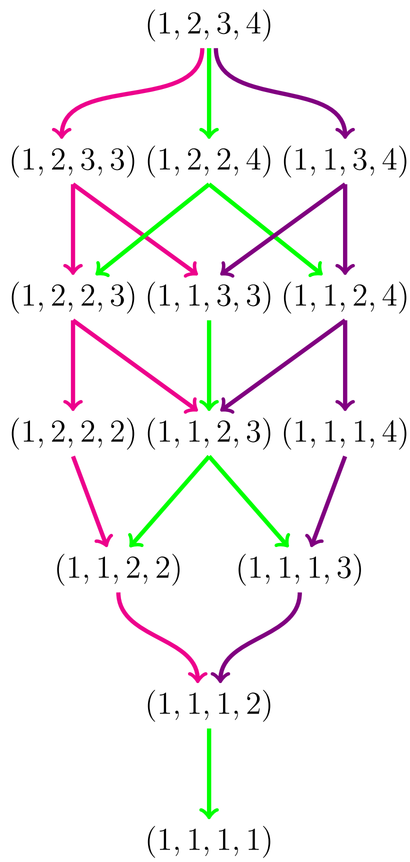
Definition 18 ($\gtrdot'$)**.** *For f and g two primitive parking functions, we say that f covers g, written $f \gtrdot' g$, if $\exists i$ such that :*

- $f = (a_1, \ldots, a_{i-1}, a_i, \quad a_{i+1}, \ldots, a_n)$
- $g = (a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_n)$

Example ($n = 6$)**.** $(1, 1, 2, 3, 4, 5) \gtrdot' (1, 1, 2, 3, 3, 5)$

Proposition. *This covering relation defines a* poset *for $\mathcal{PF'}_n$.*

Example (The poset of $\mathcal{PF'}_4$)**.**



There are $\frac{1}{5}\binom{8}{4} = \frac{70}{5} = 14$ elements in this poset.

Remark. *The two posets are isomorphic, and one can be obtained by applying the aforementioned bijection to the other.*

Theorem 8. *The number of intervals in those posets is equal to the* $n+1^{th}$ *term of the integer sequence defined by https://oeis.org/A005700.*
The first terms of this sequence are $1, 1, 3, 14, 84, 594, 4719, 40898, 379236, 3711916, ...$
Alec Mihailovs proved this sequence to be equal to

$$\frac{6(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!}$$

.

Proof. As the number of intervals in the poset for Dyck words can be seen as the number of pairs (w_1, w_k) such that $w_1 \gtrdot_d w_2 \gtrdot_d \cdots \gtrdot_d w_k$, we can define it as the number of *nested pairs of Dyck paths.* This has been proved to be equal to this integer sequence by Bruce Westbury in 2013. □

Classical Dyck - Parking Posets

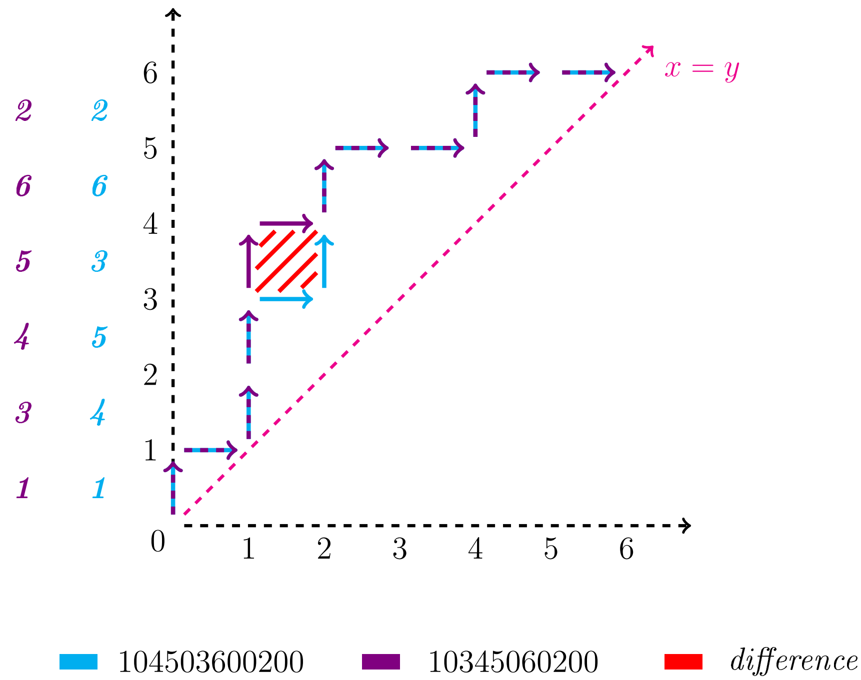
Definition 19 ($\gtrdot_{ld}$)**.** *For* w *and* w' *two labeled Dyck words, we say that* w *covers* w'*, written* $w \gtrdot_{ld} w'$*, if* $\exists l, r, x, x', y, z, z'$ *such that :*

- l *is either empty or ends with* 0
- r *is either empty or starts with* 0
- $x = x_1x_2\cdots$ *has all its digits* > 0
- $z = z_1z_2\cdots$ *has all its digits* > 0
- $x' = x$ *where* y *is correctly inserted regarding the order condition*
- y *is in* z*, and* $z' = z$ *where* y *is removed*
- $w = lx0zr$
- $w' = lx'0z'r$

Example ($n = 5$)**.** $104503600200 \gtrdot_{ld} 10345060200$

- $l = 10$
- $r = 0200$
- $x = 45$

- $x' = 345$
- $y = 3$
- $z = 36$
- $z' = 6$



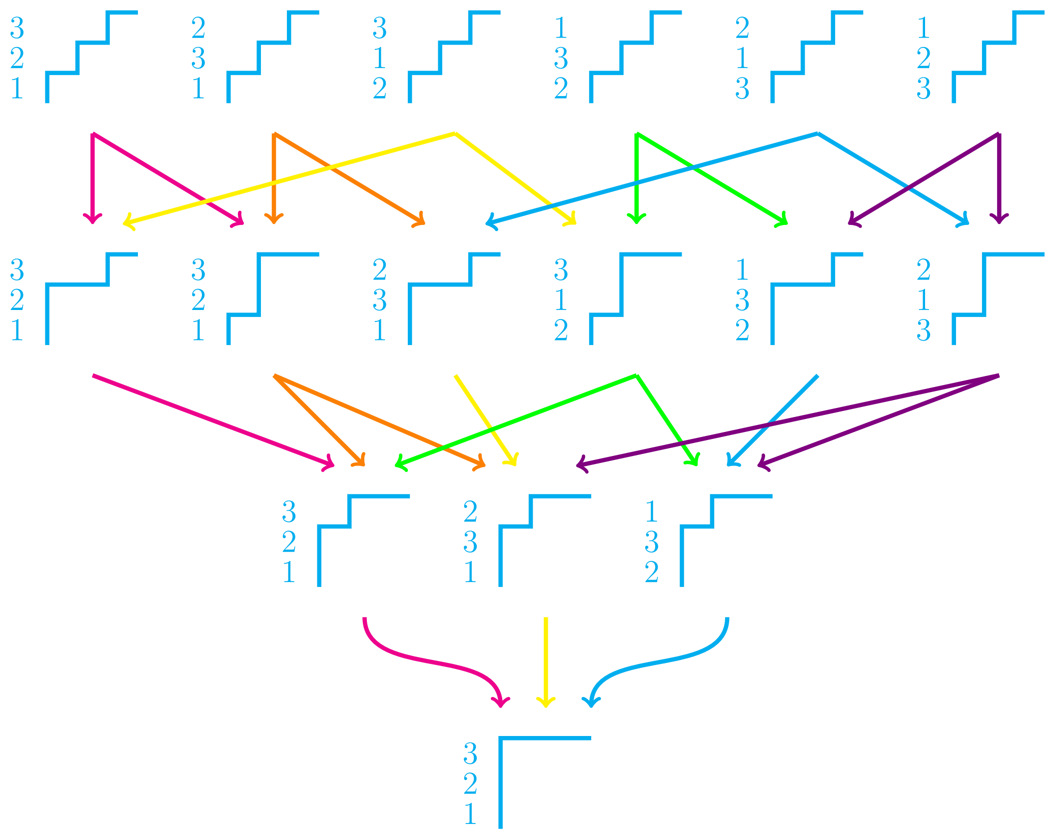
Definition 20 (Rise)**.** *A* rise *of a decorated Dyck word is a maximal sequence of non-zero digits preceding a zero.*

Example ($n = 5$)**.** *In order, the rises of* 104503600200 *are :*
• 1 • 45 • 36 • ∅ • 2 • ∅

Remark. *If $w_1 \gtrdot_{ld} w_2$, then the path corresponding to w_2 is* over *the path corresponding to w_1, and the* difference *between the two paths is a square of size 1 by 1.*
Furthermore, the covering relation can be seen as follows : w_1 covers w_2 if we can obtain w_2 by taking a digit from the $i+1^{th}$ rise of w_1, and inserting it into the i^{th} rise of w_1 in increasing order.

Proposition. *This covering relation defines a* poset *for $\mathcal{LD}_n$.*

Example (The poset of $\mathcal{LD}_3$)**.**



There are $4^2 = 16$ elements in this poset.

Definition 21 ($\gtrdot$)**.** *For f and g two parking functions, we say that f covers g, written $f \gtrdot g$, if $\exists i$ such that :*

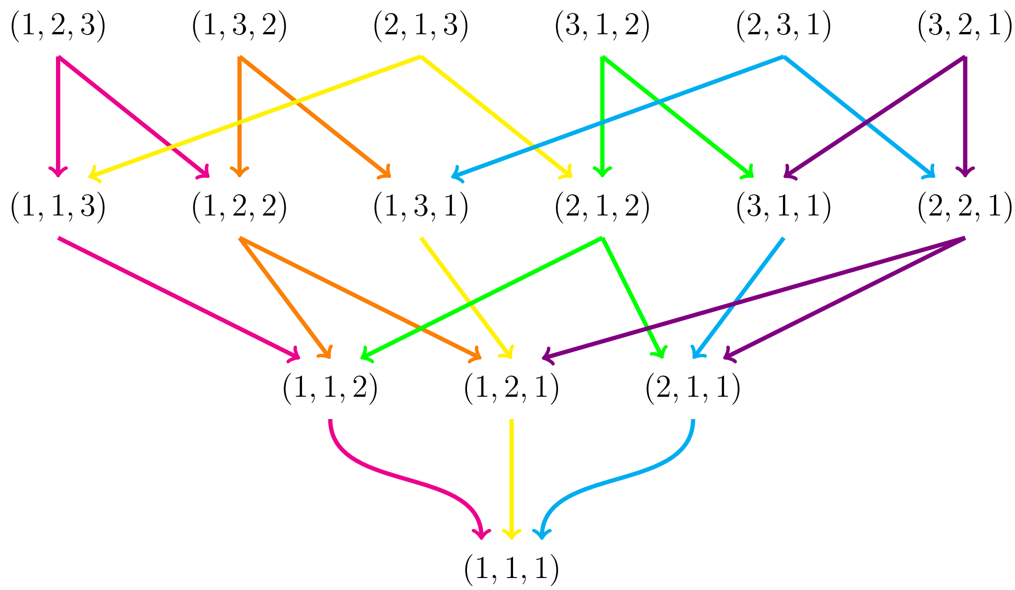
- $f = (a_1, \ldots, a_{i-1}, a_i, \quad a_{i+1}, \ldots, a_n)$
- $g = (a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_n)$

That is, the same relation as for primitive parking functions.

Example ($n = 6$)**.** $(2, 1, 5, 3, 1, 4) \gtrdot (2, 1, 5, 2, 1, 4)$

Proposition. *This covering relation defines a* poset *for* $\mathcal{PF}_n$*.*

Example (The poset of $\mathcal{PF}_3$)**.**



There are $4^2 = 16$ *elements in this poset.*

Remark. *The two posets are isomorphic, and one can be obtained by applying the aforementioned bijection to the other.*

Theorem 9. *The number of intervals in those posets is equal to the* $n + 1^{th}$ *term of the integer sequence defined by https://oeis.org/A196304.*
The first terms of this sequence are $1, 1, 5, 64, 1587, 65421, 4071178, 357962760, 4237910716, \ldots$

Chapter 2

The rational case

For the whole chapter, we will consider 2 *coprime* integers a and b (meaning a and b have 1 as their greatest common divisor).

2.1 Rational Parking Functions

Definition 22 (a, b - Parking Function)**.** *An* a, b - parking function *is a sequence* $(a_1, a_2, \ldots, a_n)$ *such that :*
• $n = a$
• *its non-decreasing reordering* $(b_1, b_2, \ldots, b_n)$ *has* $b_i \leqslant \frac{b}{a}(i-1)+1$ *for all* i*.*

We denote by $\mathcal{PF}_{a,b}$ *the set of a, b - parking functions.*

Example.

- *Ex. 1 :* $a > b$

  $a = 7$

  $b = 3$

  Limits of the non-decreasing reordering of any $f \in \mathcal{PF}_{7,3}$ *:*

  $[1,\ 1\frac{3}{7},\ 1\frac{6}{7},\ 2\frac{2}{7},\ 2\frac{5}{7},\ 3\frac{1}{7},\ 3\frac{4}{7}]$

  $f_1 = (2,1,1,3,2,3,1) \in \mathcal{PF}_{7,3}$

  $f_2 = (2,1,2,3,2,3,1) \notin \mathcal{PF}_{7,3},$ *even though* $f_2 \in \mathcal{PF}_7$

- *Ex. 2 :* $a < b$

  $a = 5$

  $b = 7$

  Limits of the non-decreasing reordering of any $f \in \mathcal{PF}_{5,7}$ *:*

  $[1,\ 2\frac{2}{5},\ 3\frac{4}{5},\ 5\frac{1}{5},\ 6\frac{3}{5}]$

  $f_3 = (6,3,5,1,2) \in \mathcal{PF}_{5,7}$, *even though* $f_3 \notin \mathcal{PF}_5$

  $f_4 = (6,3,5,1,3) \notin \mathcal{PF}_{5,7}$

Theorem 10. *Let* $pf_{a,b}$ *be the cardinal of* $\mathcal{PF}_{a,b}$*. We have*

$$pf_{a,b} = b^{a-1}$$

Example $(a = 3, b = 5)$**.**
• $pf_{a,b} = 25$ • *Limits :* $[1,\ 2\frac{2}{3},\ 4\frac{1}{3}]$

$(1,1,1)$	$(1,1,2)$	$(1,1,3)$	$(1,1,4)$	$(1,2,1)$	$(1,2,2)$	$(1,2,3)$
$(1,2,4)$	$(1,3,1)$	$(1,3,2)$	$(1,4,1)$	$(1,4,2)$	$(2,1,1)$	$(2,1,2)$
$(2,1,3)$	$(2,1,4)$	$(2,2,1)$	$(2,3,1)$	$(2,4,1)$	$(3,1,1)$	$(3,1,2)$
$(3,2,1)$	$(4,1,1)$	$(4,1,2)$	$(4,2,1)$			

Remark. $\mathcal{PF}_{n,n+1} = \mathcal{PF}_n$. *In fact, we do have* $b^{a-1} = (n+1)^{n-1}$.

2.1.1 Rational primitive parking functions

Definition 23 (Rational Primitive)**.** *A rational parking function* f *is said* primitive *if it is already in non-decreasing order.*
We denote by $\mathcal{PF}'_{a,b}$ *the set of primitive a, b - parking functions.*

Example $(a = 4, b = 3)$**.** *Limits :* $[1,\ 1\frac{3}{4},\ 2\frac{1}{2},\ 3\frac{1}{4}]$

$f_1 = (1,1,2,2) \in \mathcal{PF}'_{4,3}$

$f_2 = (1,1,2,1) \notin \mathcal{PF}'_{4,3}$, *even though* $f_2 \in \mathcal{PF}_{4,3}$.

Theorem 11. *Let* $pf'_{a,b}$ *be the cardinal of* $\mathcal{PF}'_{a,b}$*. We have*

$$pf'_{a,b} = \frac{1}{a+b}\binom{a+b}{b}$$

which is the rational Catalan number $Cat(a,b)$.

Example $(a = 3, b = 5)$**.**
• $pf'_{a,b} = 7$ • *Limits :* $[1,\ 2\frac{2}{3},\ 4\frac{1}{3}]$

$$(1,1,1) \qquad (1,1,2) \qquad (1,1,3) \qquad (1,1,4) \qquad (1,2,2) \qquad (1,2,3) \qquad (1,2,4)$$

Remark. $\mathcal{PF'}_{n,n+1} = \mathcal{PF'}_n$. *In fact, we do have*

$$\frac{1}{n+n+1}\binom{n+n+1}{n+1} = \frac{1}{2n+1}\binom{2n+1}{n+1} = \frac{1}{2n+1}\frac{(2n+1)!}{n!(n+1)!}$$

$$= \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1}\frac{(2n)!}{n!n!} = \frac{1}{n+1}\binom{2n}{n}$$

2.2 Rational Non-crossing Partitions

Definition 24 (a, b - Non-crossing Partition)**.** *An* a, b - non-crossing partition *is TODO*
We denote by $\mathcal{NC}_{a,b}$ *the set of a, b - non-crossing partitions.*

Example. *An abncp*

Theorem 12. *Let* $nc_{a,b}$ *be the cardinal of* $\mathcal{NC}_{a,b}$*. We have*

$$nc_{a,b} = \frac{1}{a+b}\binom{a+b}{a} = \frac{(a+b-1)!}{a!b!}$$

, which is the rational Catalan number.

Example. *all abncp for some a b*

Proposition. *This means we can create a* bijection *between* $\mathcal{PF'}_{a,b}$ *and* $\mathcal{NC}_{a,b}$.

Proof.

- $\mathcal{NC}_{a,b} \to \mathcal{PF'}_{a,b}$:
- $\mathcal{PF'}_{a,b} \to \mathcal{NC}_{a,b}$:

□

Definition 25 (a, b - Non-crossing 2-Partition)**.** *An* a, b - Non-crossing 2-partition *is TODO*
We denote by $\mathcal{NC}^2_{a,b}$ *the set of a, b - non-crossing 2-partitions.*

Example. *some ncab2*

Theorem 13. *Let* $nc^2_{a,b}$ *be the cardinal of* $\mathcal{NC}^2_{a,b}$*. We have*

$$nc^2_{a,b} = b^{a-1}$$

.

Example. *all ncab2 for some a b*

Proposition. *bijection*

Proof. bijection proof □

2.3 A direct poset linked to Rational Dyck paths

2.3.1 Rational Dyck Paths

Definition 26 (a, b - Dyck path)**.** *An* a, b - Dyck word *is a word* $w \in \{0,1\}^*$ *such that :*

- *for each* suffix w' *of* w*,*
$$|w'|_1 \geqslant \frac{a}{b}|w'|_0$$
.
- $|w|_0 = b$.
- $|w|_1 = a$.

An a, b - Dyck word can be represented as a path *from* $(0,0)$ *to* (b,a) *that stays over* $y = \frac{a}{b}x$*, called an* a, b - Dyck path *:*

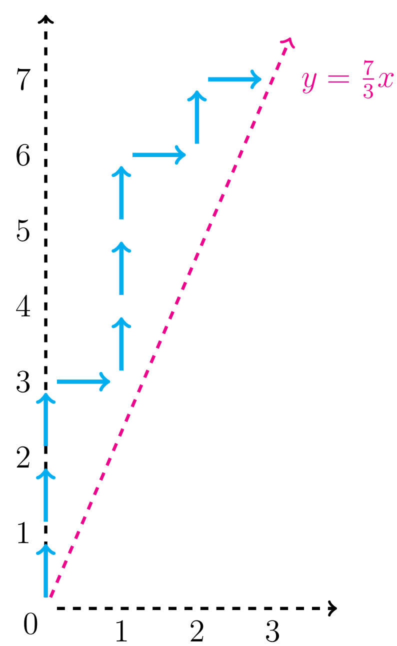
- *Each* 1 *corresponds to a* North step $\uparrow$.
- *Each* 0 *corresponds to an* East step $\rightarrow$.

We denote by $\mathcal{R}_{a,b}$ *the set of a, b - Dyck words.*

Example $(a > b : a = 7, b = 3)$**.**

$$w_1 = 1110011110 \textit{ is } \text{not } \textit{a 7, 3 - Dyck word, because } |11100|_1 = 3 < \frac{7}{3}|11100|_0 = \frac{14}{3} = 4\frac{1}{3}.$$

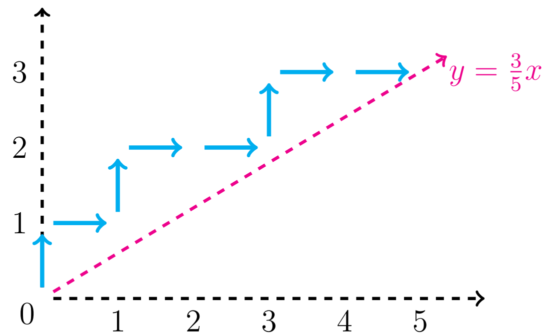
$w_2 = 1110111010$ is *a 7, 3 - Dyck word :*



Example $(a < b : a = 3, b = 5)$**.**

$$w_1 = 10100010 \textit{ is } \text{not } \textit{a 3, 5 - Dyck word, because } |101000|_1 = 2 < \frac{3}{5}|101000|_0 = \frac{12}{5} = 2\frac{2}{5}.$$

$w_2 = 10100100$ is *a 3, 5 - Dyck word :*

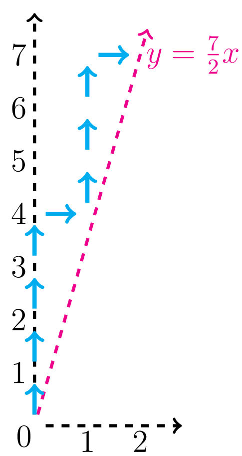


Theorem 14. *Let $r_{a,b}$ be the cardinal of $\mathcal{R}_{a,b}$. We have*

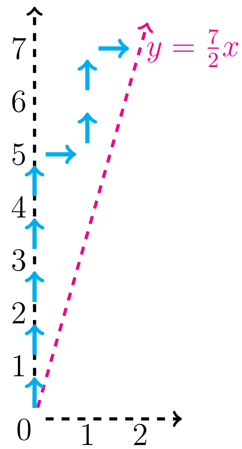
$$r_{a,b} = \frac{1}{a+b}\binom{a+b}{a} = \frac{(a+b-1)!}{a!b!}$$

, which is again the rational Catalan number.

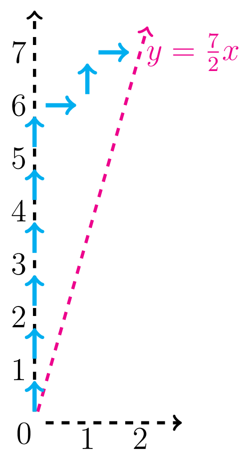
Example $(a = 7, b = 2)$**.** $r_n = 4$.



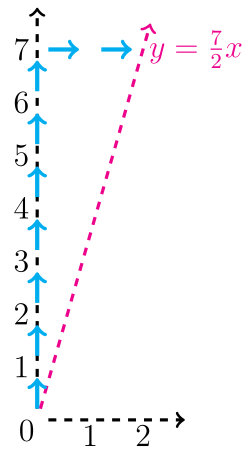
111101110



111110110



111111010



111111100

Proposition. *This means we can create a* bijection *between $\mathcal{PF}'_{a,b}$ and $\mathcal{R}_{a,b}$.*

Proof.

- $\mathcal{PF}'_{a,b} \to \mathcal{R}_{a,b}$: Let $f = (a_1, \ldots, a_n) \in \mathcal{PF}'_{a,b}$ be our rational primitive parking function. For $i \in \{1, \ldots, b\}$, we define l_i the number of occurences of i in f.
  The corresponding rational Dyck word will be $\underbrace{1\cdots1}_{l_1}0\underbrace{1\cdots1}_{l_2}0\cdots\underbrace{1\cdots1}_{l_b}0$.

- $\mathcal{R}_{a,b} \to \mathcal{PF}'_{a,b}$: Let w be our rational Dyck word, and consider its path representation. We define s_i to be the distance between the segment from $(0, i-1)$ to $(0, i)$ and the i^{th} North step. Then, let $a_i = s_i + 1$.
  The corresponding rational primitive parking function is $(a_1, \ldots, a_a)$.

$\square$

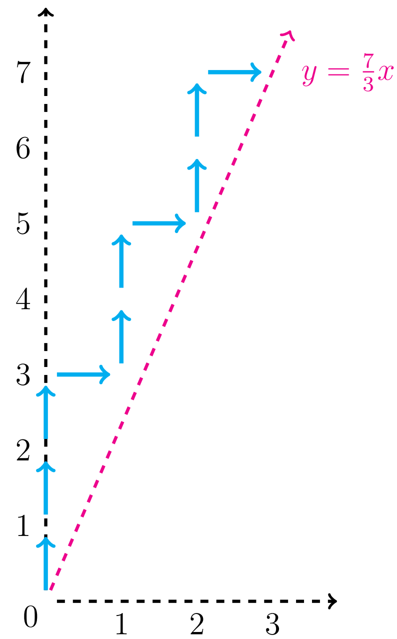
Remark. *This bijection is exactly the same as the one between classical primitive parking functions and Dyck paths.*

Example $(a > b : a = 7, b = 3, \mathcal{PF}'_{a,b} \to \mathcal{R}_{a,b})$**.**

- $f = (1, 1, 1, 2, 2, 3, 3)$

  $l_1 = 3$ $\qquad$ $l_2 = 2$ $\qquad$ $l_3 = 2$

- $w = (1110110110)$



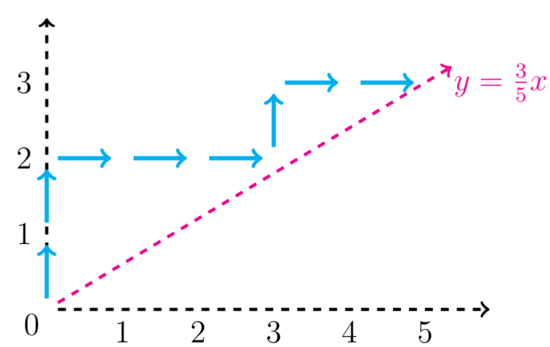
Example $(a < b : a = 3, b = 5, \mathcal{PF'}_{a,b} \to \mathcal{R}_{a,b})$**.**

- $f = (1, 1, 4)$

  $l_1 = 2$ $\quad$ l_2 $\quad$ $l_3 = 0$

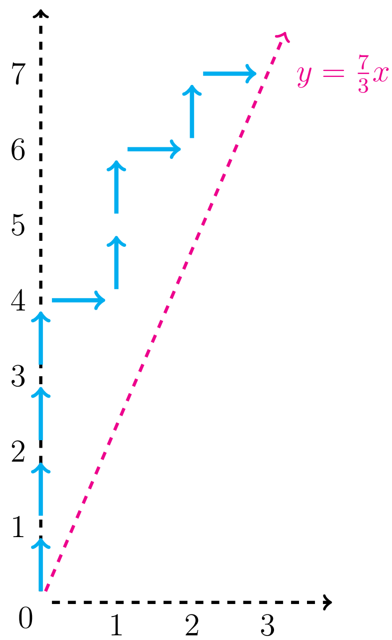
  $l_4 = 1$ $\quad$ $l_5 = 0$

- $w = 11000100$



Example $(a > b : a = 7, b = 3, \mathcal{R}_{a,b} \to \mathcal{PF'}_{a,b})$**.**

- $w = 1111011010$



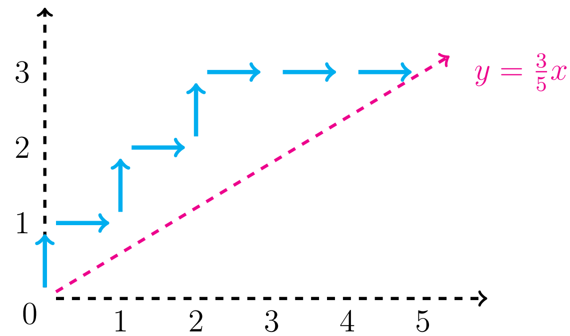
- *Distances :*

$s_1 = 0$	$a_1 = 1$
$s_2 = 0$	$a_2 = 1$
$s_3 = 0$	$a_3 = 1$
$s_4 = 0$	$a_4 = 1$
$s_5 = 1$	$a_5 = 2$
$s_6 = 1$	$a_6 = 2$
$s_7 = 2$	$a_7 = 3$

- $f = (1, 1, 1, 1, 2, 2, 3)$

Example $(a < b : a = 3, b = 5, \mathcal{R}_{a,b} \to \mathcal{PF}'_{a,b})$**.**

- $w = 10101000$



- *Distances :*

$$s_1 = 0 \qquad a_1 = 1$$
$$s_2 = 1 \qquad a_2 = 2$$
$$s_3 = 2 \qquad a_3 = 3$$

- $f = (1, 2, 3)$

2.3.2 Rational Labeled Dyck Paths

Definition 27 (Labeled a, b - Dyck Path)**.** *A* labeled a, b - Dyck word *is a word $w \in \{0, \ldots, n\}^*$ such that :*

- *for each suffix w' of w,*

$$|w'|_{\neq 0} \geqslant \frac{a}{b}|w'|_0$$

.

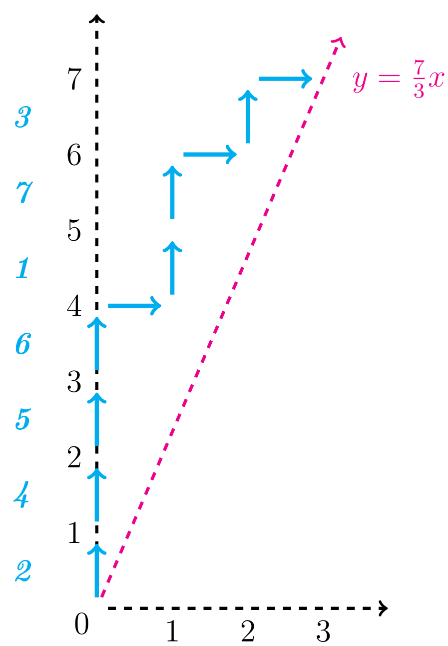
- $|w|_0 = b$.
- $|w|_{\neq 0} = a$.
- *for each $i \in \{1, \ldots, a\}$, w has exactly one occurence of i.*
- *if $w_i \neq 0$ and $w_{i+1} \neq 0$, then $w_i < w_{i+1}$. That is, consecutive North steps have increasing labels.*

A labeled a, b - Dyck word can be represented as a path from $(0, 0)$ to (b, a), where each North step is associated to a label :

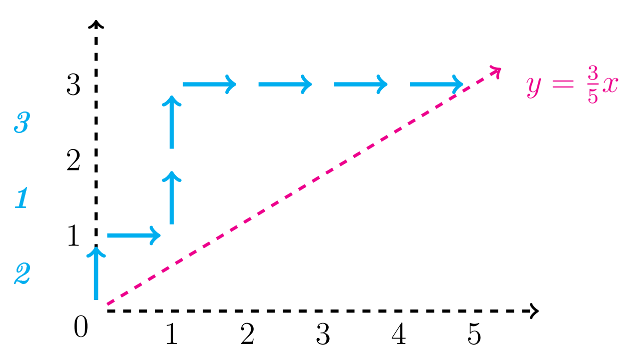
- *Each $i \neq 0$ corresponds to a* North step $\uparrow$ *labeled i.*
- *Each 0 corresponds to an* East step $\rightarrow$.

Those paths are called labeled a, b - Dyck paths.
We denote by $\mathcal{LR}_{a,b}$ the set of labeled a, b - Dyck words.

Example $(a > b : a = 7, b = 3)$**.** $w_2 = 2456017030$ *:*



Example $(a < b : a = 3, b = 5)$**.** $w = 20130000$ *:*



Theorem 15. *Let $lr_{a,b}$ be the cardinal of $\mathcal{LR}_{a,b}$. We have*

$$lr_{a,b} = b^{a-1}$$

.

Example ($a > b : a = 4, b = 3$)**.** $lr_{a,b} = 3^3 = 27$

- *Word of shape $XXXX000$:*

1234000		

- *Words of shape $XXX0X00$:*

1230400	1240300	1340200
2340100		

- *Words of shape $XX0XX00$:*

1203400	1302400	1402300
2301400	2401300	3401200

- *Words of shape $XXX00X0$:*

1230040	1240030	1340020
2340010		

- *Words of shape $XX0X0X0$:*

1203040	1204030	1302040
1304020	1402030	1403020
2301040	2304010	2401030
2403010	3401020	3402010

Proposition. *This means we can create a* bijection *between $\mathcal{PF}_{a,b}$ and $\mathcal{LR}_{a,b}$.*

Proof.

- $\mathcal{PF}_{a,b} \to \mathcal{LR}_{a,b}$: Let $f = (a_1, \ldots, a_n) \in \mathcal{PF}_{a,b}$ be our a, b - parking function. For $i \in \{1, \ldots, b\}$, we define $im_i : \{j \mid a_j = i\}$.
  We then define $im_{i,1}, \ldots, im_{i,k_i}$ to be the elements of im_i in increasing order.
  The corresponding labeled a, b - Dyck word will be
  $\underbrace{im_{1,1} \cdots im_{1,k_1}}_{im_1} 0 \underbrace{im_{2,1} \cdots im_{2,k_2}}_{im_2} 0 \cdots \underbrace{im_{n,1} \cdots im_{b,k_b}}_{im_b} 0.$

- $\mathcal{LR}_{a,b} \to \mathcal{PF}_n$: Let w be our labeled a, b - Dyck word, and consider its path representation. We define s_i to be the distance between the segment from $(0, i-1)$ to $(0, i)$ and the i^{th} North step.
  Then, let $label(i)$ be the label of the i^{th} North step, and $dist_i = \{label(j) | s_j = i\}$ be the set of the labels of all North steps at distance i.
  Then, if $j \in dist_i$, let $a_j = i + 1$.
  The corresponding parking function is $(a_1, \ldots, a_a)$.

$\square$

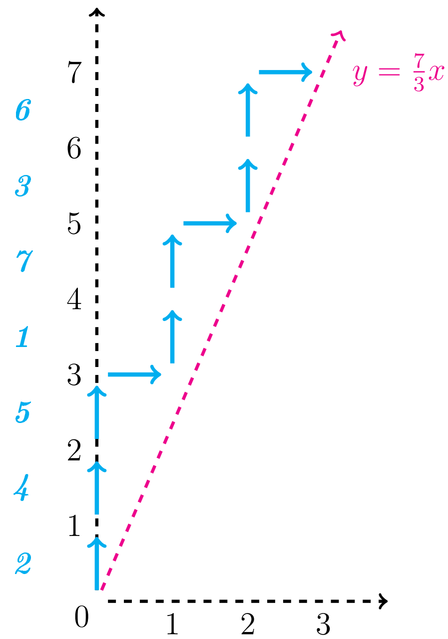
Remark. *This bijection is exactly the same as the one between classical parking functions and labeled Dyck paths.*

Example $(a > b : a = 7, b = 3, \mathcal{PF}_{a,b} \to \mathcal{LR}_{a,b})$**.**

- $f = (2, 1, 3, 1, 1, 3, 2)$

  $im_1 = \{2, 4, 5\}$ $\quad$ $im_2 = \{1, 7\}$ $\quad$ $im_3 = \{3, 6\}$

- $w = 2450170360$



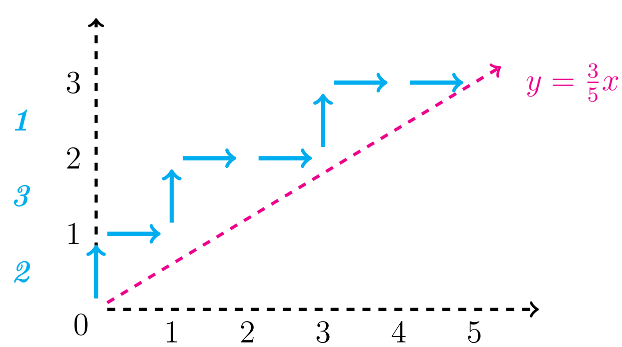
Example $(a < b : a = 3, b = 5, \mathcal{PF}_{a,b} \to \mathcal{LR}_{a,b})$**.**

- $f = (4, 1, 2)$

  $im_1 = \{2\}$ $\quad$ $im_2 = \{3\}$ $\quad$ $im_3 = \emptyset$

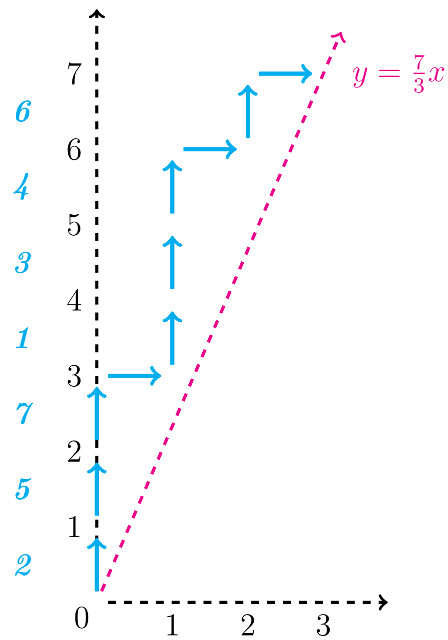
  $im_4 = \{1\}$ $\quad$ $im_5 = \emptyset$

- $w = 20300100$



Example $(a > b : a = 7, b = 3, \mathcal{LR}_{a,b} \to \mathcal{PF}_{a,b})$**.**

- $w = 2570134060$



- *Distances :*

  $s_1 = 0$ $s_2 = 0$ $s_3 = 0$
  $s_4 = 1$ $s_5 = 1$ $s_6 = 1$
  $s_7 = 2$

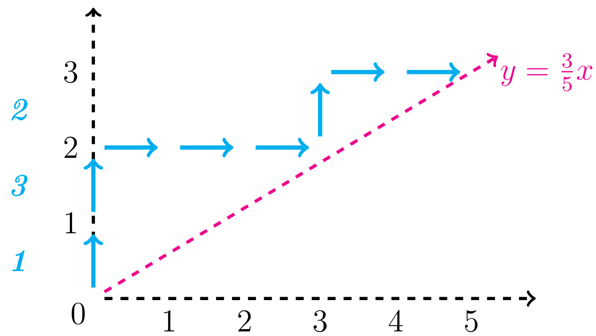
- *Labels :*

  $dist_0 = \{2, 5, 7\}$ $dist_1 = \{1, 3, 4\}$ $dist_2 = \{6\}$

- $f = (2, 1, 2, 2, 1, 3, 1)$

Example $(a < b : a = 3, b = 5, \mathcal{LR}_{a,b} \to \mathcal{PF}_{a,b})$**.**

- $w = 13000200$



- *Distances :*

$$s_1 = 0 \qquad s_2 = 0 \qquad s_3 = 3$$

- *Labels :*

$$dist_0 = \{1, 3\} \qquad dist_1 = \emptyset \qquad dist_2 = \emptyset$$
$$dist_3 = \{2\} \qquad dist_4 = \emptyset$$

- $f = (1, 4, 1)$

Remark. *The rational primitive parking functions are exactly the rational parking functions corresponding to rational labeled Dyck paths where the i^{th} North step is labeled i.*

2.3.3 Rational Dyck - Parking Posets

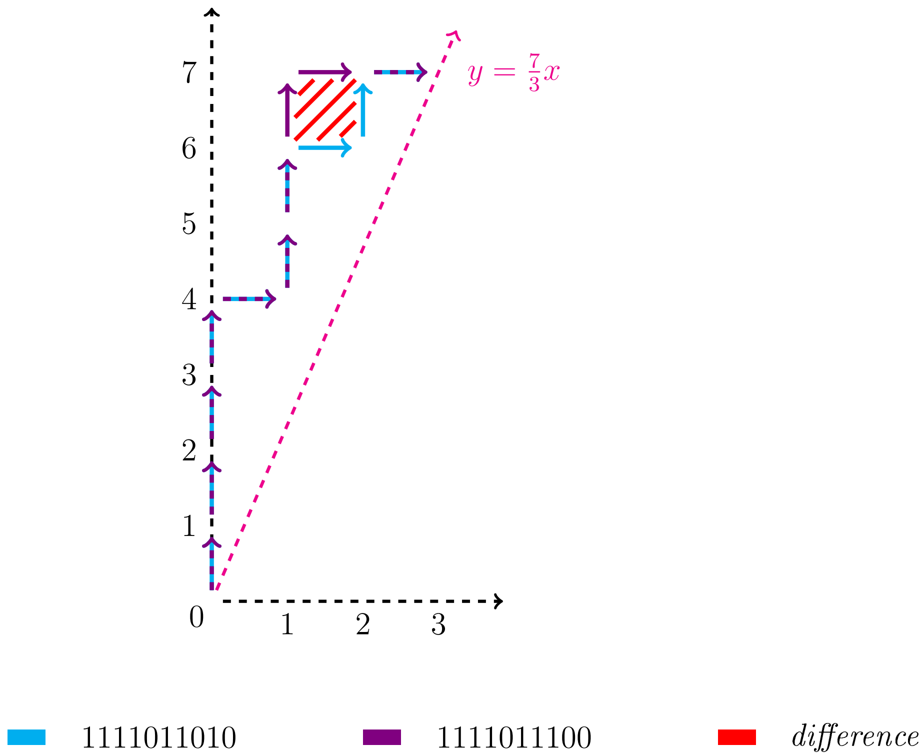
Rational Primitive Dyck - Parking Posets

Definition 28 ($\gtrdot_r$)**.** *For w and w' two a, b - Dyck words, we say that w covers w', written $w \gtrdot_r w'$, if $\exists w_1, w_2$ such that :*

- $w = w_1 01 w_2$
- $w' = w_1 10 w_2$

Example ($a = 7, b = 3$)**.** $1111011010 \gtrdot_r 1111011100$

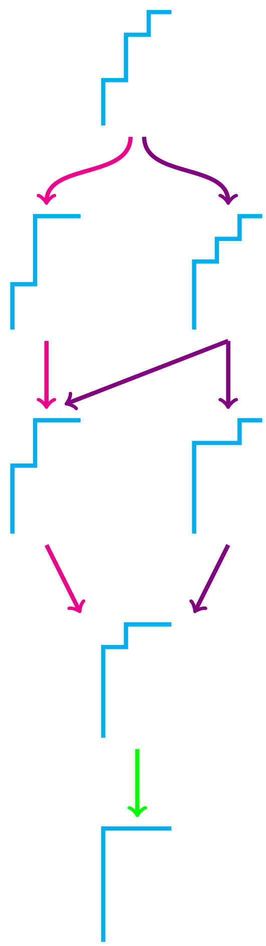
- $w_1 = 1111011$
- $w_2 = 0$



Remark. *If $w_1 \gtrdot_r w_2$, then the path corresponding to w_2 is* over *the path corresponding to w_1, and the* difference *between the two paths is a square of size 1 by 1.*

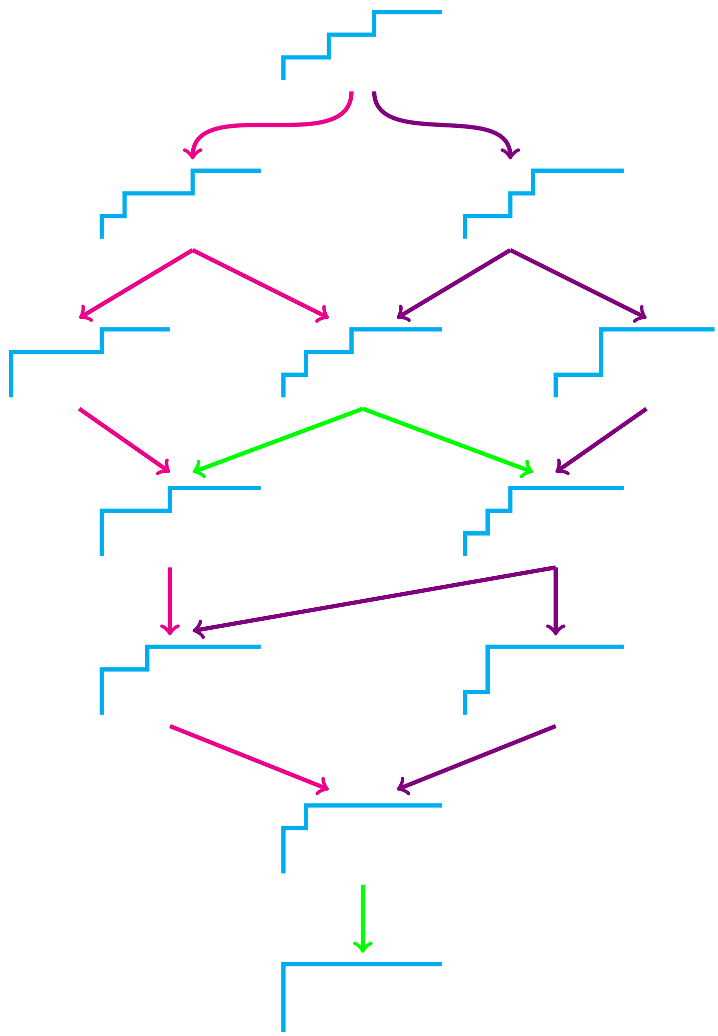
Proposition. *This covering relation defines a* poset *for $\mathcal{R}_{a,b}$.*

Example ($a > b$: The poset of $\mathcal{R}_{5,3}$)**.**



There are $\frac{1}{8}\binom{8}{5} = \frac{42}{6} = 7$ *elements in this poset.*

Example ($a < b$: The poset of $\mathcal{R}_{3,7}$)**.**



There are $\frac{1}{10}\binom{10}{3} = \frac{72}{6} = 12$ *elements in this poset.*

Definition 29 ($\gtrdot'$)**.** *For f and g two rational primitive a, b - parking functions, we say that f covers g, written $f \gtrdot' g$, if $\exists i$ such that :*

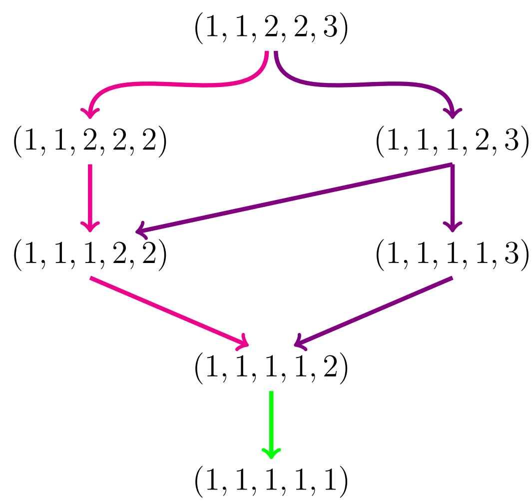
- $f = (a_1, \ldots, a_{i-1}, a_i, \quad a_{i+1}, \ldots, a_n)$
- $g = (a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_n)$

Example ($a > b : a = 7, b = 3$)**.** $(1, 1, 1, 2, 2, 2, 3) \gtrdot' (1, 1, 1, 1, 2, 2, 3)$

Example ($a < b : a = 3, b = 5$)**.** $(1, 2, 4) \gtrdot' (1, 1, 4)$

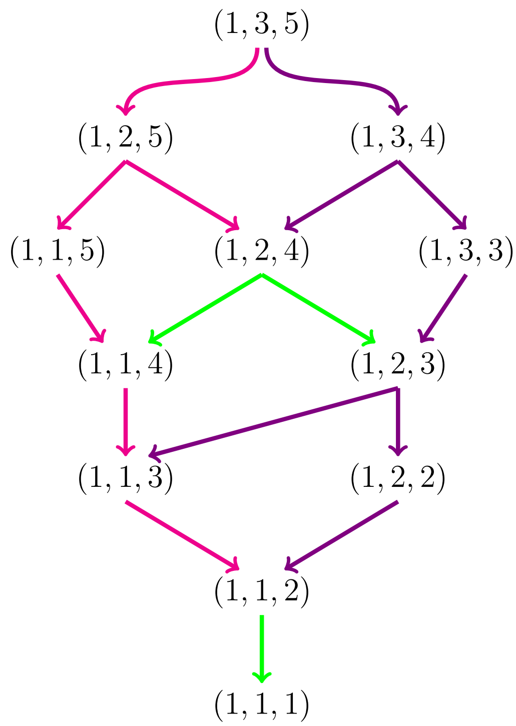
Proposition. *This covering relation defines a* poset *for* $\mathcal{PF}'_{a,b}$.

Example $(a > b :$ The poset of $\mathcal{PF}'_{5,3})$**.**



There are $\frac{1}{8}\binom{8}{5} = \frac{42}{6} = 7$ *elements in this poset.*

Example $(a < b :$ The poset of $\mathcal{PF}'_{3,7})$**.**



There are $\frac{1}{10}\binom{10}{3} = \frac{72}{6} = 12$ *elements in this poset.*

Remark. *The posets of* $\mathcal{PF}'_{a,b}$ *and* $\mathcal{R}_{a,b}$ *are isomorphic, and one can be obtained by applying the aforementioned bijection to the other.*

Rational Dyck - Parking Posets

Definition 30 ($\gtrdot_{lr}$)**.** *For* w *and* w' *two labeled* a, b *- Dyck words, we say that* w *covers* w'*, written* $w \gtrdot_{lr} w'$*, if* $\exists l, r, x, x', y, z, z'$ *such that :*

- l *is either empty or ends with* 0
- r *is either empty or starts with* 0
- $x = x_1 x_2 \cdots$ *has all its digits* > 0
- $z = z_1 z_2 \cdots$ *has all its digits* > 0
- $x' = x$ *where* y *is correctly inserted regarding the order condition*

- *y is in z, and $z' = z$ where y is removed*
- $w = lx0zr$
- $w' = lx'0z'r$

Example $(a > b : a = 7, b = 3)$**.** $2460150370 \gtrdot_{lr} 2460135070$

- $l = 2460$
- $r = 0$
- $x = 15$
- $x' = 135$
- $y = 3$
- $z = 37$
- $z' = 7$

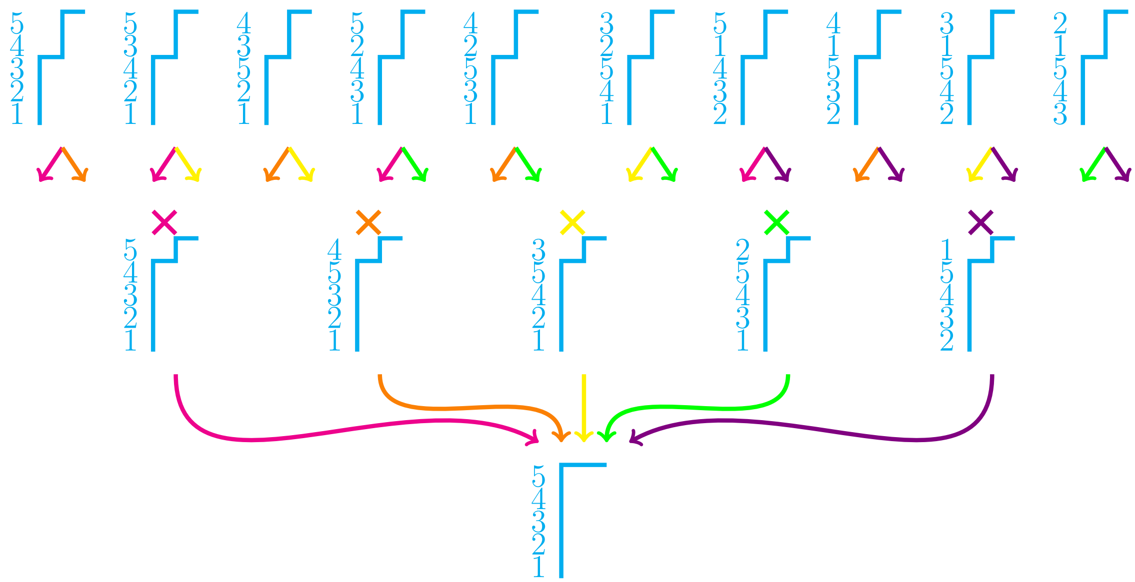
Example $(a < b : a = 3, b = 5)$**.** $20301000 \gtrdot_{lr} 20130000$

- $l = 20$
- $r = 000$
- $x = 3$
- $x' = 13$
- $y = 1$
- $z = 1$
- $z' = \emptyset$

Remark. *If $w_1 \gtrdot_{lr} w_2$, then the path corresponding to w_2 is* over *the path corresponding to w_1, and the* difference *between the two paths is a square of size 1 by 1.*
Furthermore, the covering relation can be seen as follows : w_1 covers w_2 if we can obtain w_2 by taking a digit from the $i + 1^{th}$ rise of w_1, and inserting it into the i^{th} rise of w_1 in increasing order.

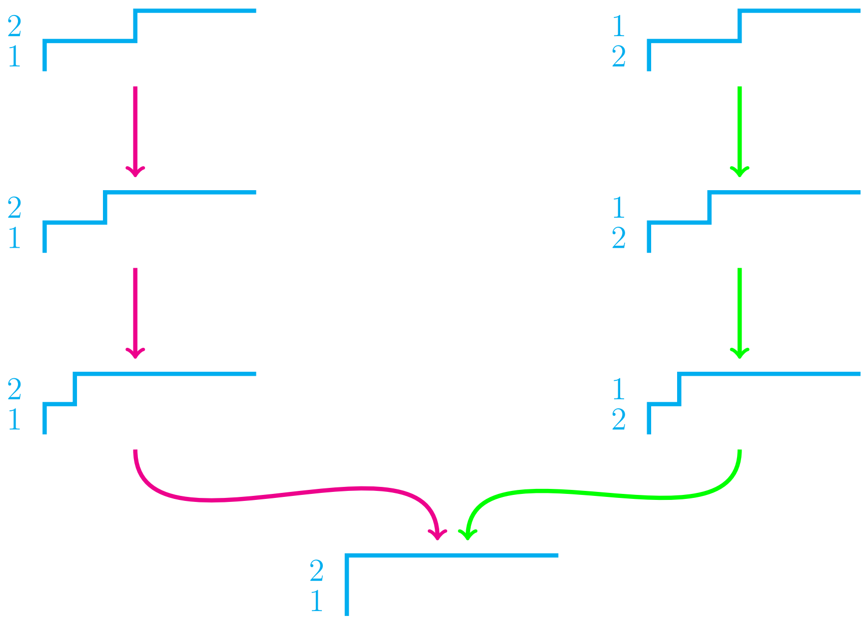
Proposition. *This covering relation defines a* poset *for* $\mathcal{LR}_{a,b}$*.*

Example ($a > b$: The poset of $\mathcal{LR}_{5,2}$)**.**



Arrows have been simplified for readability.
Arrows between the top 2 levels are to be read as ending at the cross of the corresponding color.
There are $2^4 = 16$ *elements in this poset.*

Example ($a < b$: The poset of $\mathcal{LR}_{2,7}$)**.**



There are $7^1 = 7$ elements in this poset.

Definition 31 ($\gtrdot$)**.** *For f and g two rational parking functions, we say that f covers g, written $f \gtrdot g$, if $\exists i$ such that :*

- $f = (a_1, \ldots, a_{i-1}, a_i, \quad a_{i+1}, \ldots, a_n)$
- $g = (a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_n)$

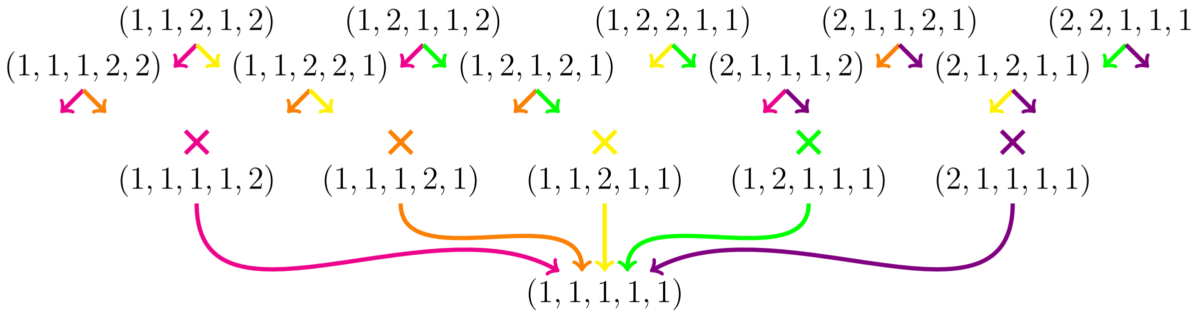
That is, the same relation as for rational primitive parking functions.

Example ($a > b : a = 7, b = 3$)**.** $(2, 3, 1, 1, 2, 1, 3) \gtrdot (2, 3, 1, 1, 1, 1, 3)$

Example ($a < b : a = 3, b = 5$)**.** $(4, 1, 2) \gtrdot (3, 1, 2)$

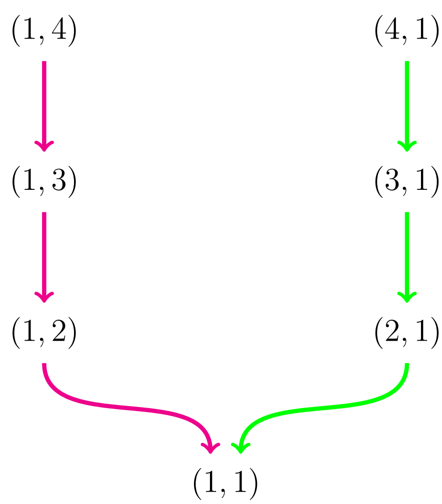
Proposition. *This covering relation defines a* poset *for $\mathcal{PF}_{a,b}$.*

Example ($a > b :$ The poset of $\mathcal{PF}_{5,2}$)**.**



Arrows have been simplified for readability.
Arrows between the top 2 levels are to be read as ending at the cross of the corresponding color.
There are $2^4 = 16$ elements in this poset.

Example ($a < b$: The poset of $\mathcal{PF}_{2,7}$)**.**



There are $7^1 = 7$ elements in this poset.

Remark. *The posets of $\mathcal{PF}_{a,b}$ and $\mathcal{LR}_{a,b}$ are isomorphic, and one can be obtained by applying the aforementioned bijection to the other.*

Chapter 3

Trees

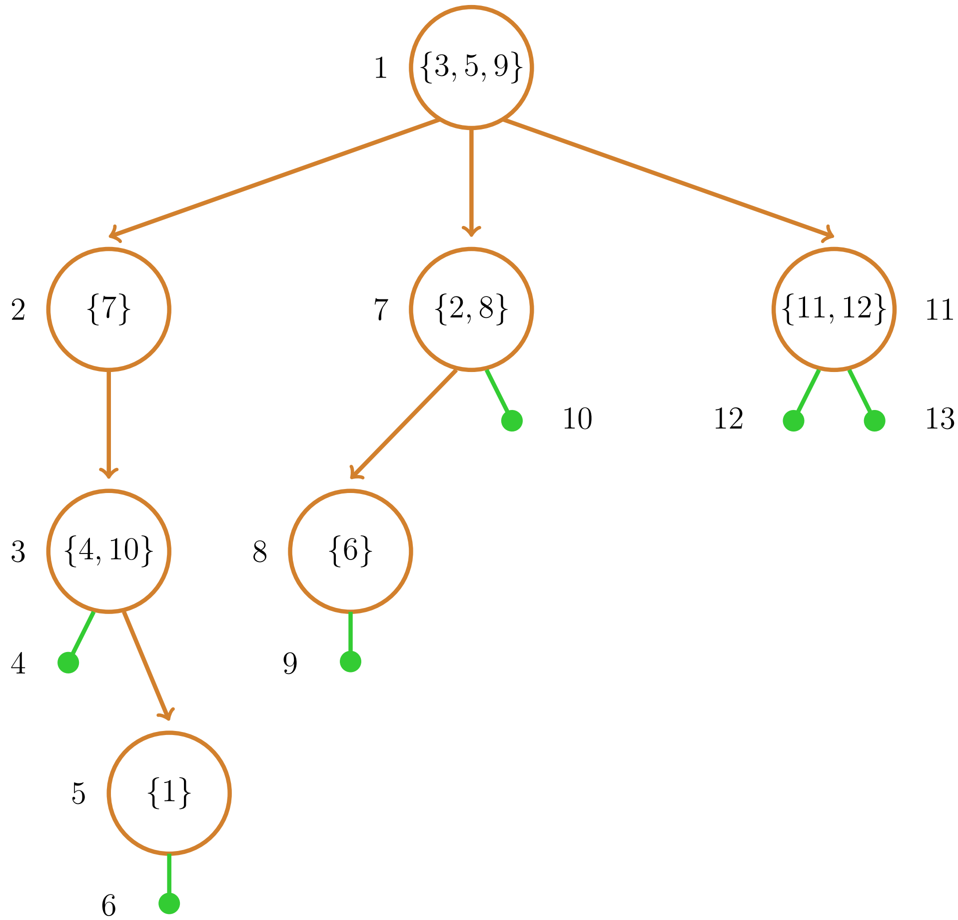
3.1 Parking Trees

Definition 32 (Parking Trees). *A* parking tree *is defined from a parking function $f = (a_1, \ldots, a_n) \in \mathcal{PF}_n$ as follows :*

- *For $1 \leqslant i \leqslant n+1$, we define s_i as $\{j \mid a_j = i\}$*
- *$[s_1, \ldots, s_{n+1}]$ describes the pre-order depth-first traversal of the tree.*
- *Each node labeled by a set of size k has k children.*

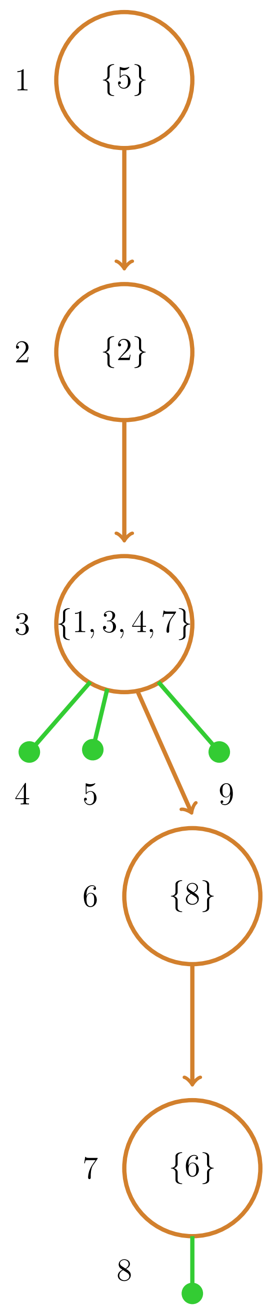
Remark. *The leaves of the tree are those corresponding to an element i such that $1 \leqslant i \leqslant n+1$, and i is* not *in f.*
Furthermore, as we will have a total edges by definition, the presence of a node corresponding to $n+1$ is necessary, even though it will always be empty.

Example ($n = 12$)**.**
• $f = (5, 7, 1, 3, 1, 8, 2, 7, 1, 3, 11, 11)$
• *Labels :* $[\{3, 5, 9\},\ \{7\},\ \{4, 10\},\ \emptyset,\ \{1\},\ \emptyset,\ \{2, 8\},\ \{6\},\ \emptyset,\ \emptyset,\ \{11, 12\},\ \emptyset, \emptyset]$



Conversely, by reading the labels of a parking tree depth-first in pre-order, we get the list of positions of each number in the corresponding parking function, thus creating a *bijection*.

Example (From parking tree to parking function)**.**



- *The labels are* $[\{5\},\ \{2\},\ \{1,3,4,7\},\ \emptyset,\ \emptyset,\ \{8\},\ \{6\},\ \emptyset,\ \emptyset]$.
- *Thus the corresponding parking function is* $(3,2,3,3,1,7,3,6) \in \mathcal{PF}_8$.

3.2 Rational Parking Trees

Definition 33 (Rational Parking Trees)**.** *A* rational parking tree *is defined from a rational parking function $f = (a_1, \ldots, a_a) \in \mathcal{PF}_{a,b}$ as follows :*

- *For $1 \leqslant i \leqslant n+1$, we define the limit l_i as the* integer portion *of $\frac{b}{a}(i-1)+1$.*

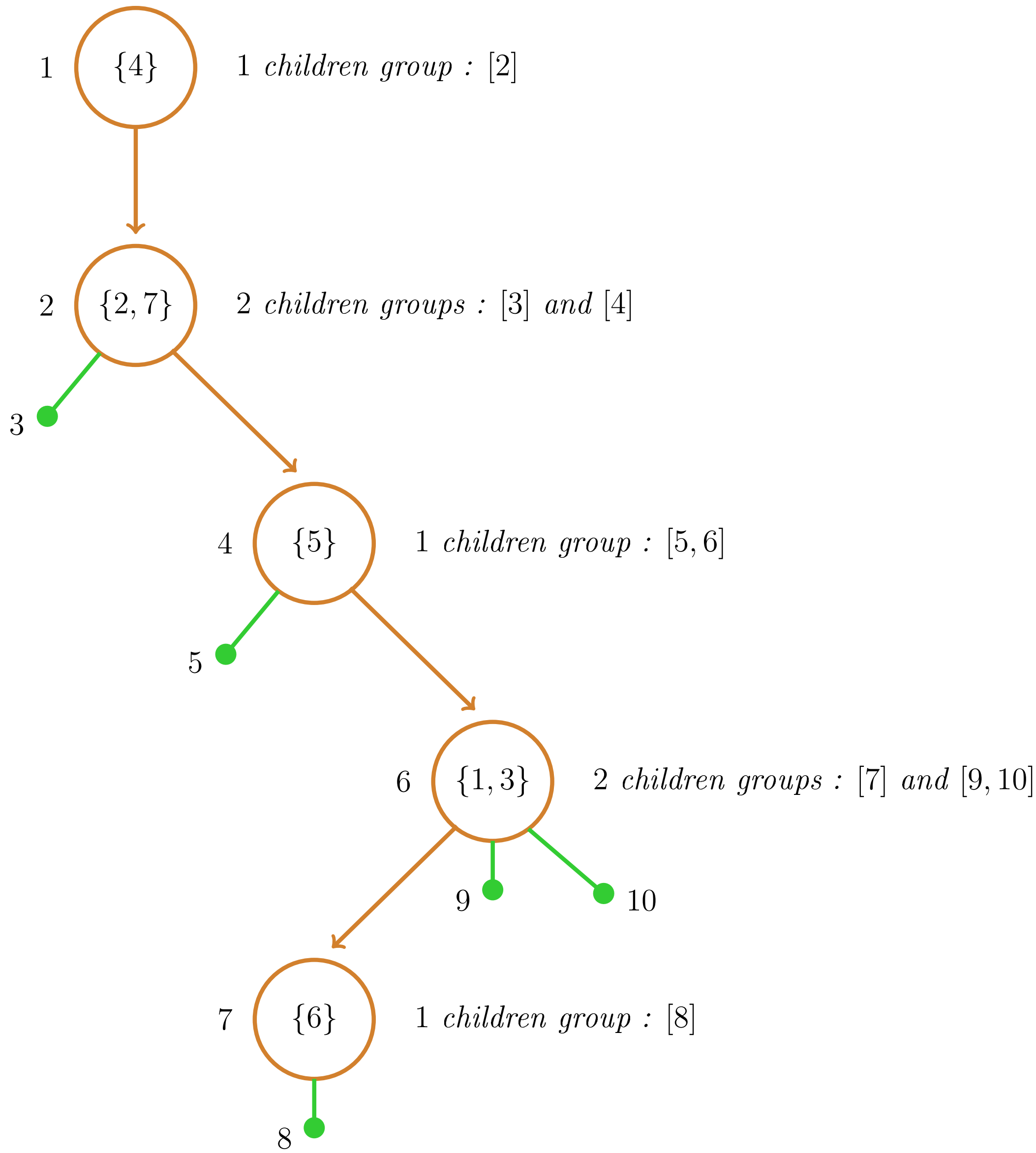
  Let $l_0 = 0$.
- *From these limits, we deduce the intervals $itv_i =]l_{i-1}, l_i]$ for $1 \leqslant i \leqslant a+1$.*
- *For $1 \leqslant i \leqslant b+1$, define s_i as $\{j \mid a_j = i\}$.*
- *$[s_1, \ldots, s_{b+1}]$ describes the pre-order depth-first traversal of the tree.*
- *Each node labeled by a set of size k has k* groups *of children, which are defined by the intervals.*

Example ($a < b$)**.**

- $a = 7$
- $b = 9$
- *Limits :* $[1,\ 2\frac{2}{7},\ 3\frac{4}{7},\ 4\frac{6}{7},\ 6\frac{1}{7},\ 7\frac{3}{7},\ 8\frac{5}{7},\ 10]$
- *Integral limits :* $[0, 1, 2, 3, 4, 6, 7, 8, 10]$
- *Intervals :*

  $]0,1] \quad]1,2] \quad]2,3] \quad]3,4] \quad]4,6] \quad]6,7] \quad]7,8] \quad]8,10]$
- *Children groups :*

  $[1] \quad [2] \quad [3] \quad [4] \quad [5,6] \quad [7] \quad [8]$
- $f = (6, 2, 6, 1, 4, 7, 2)$
- *Labels :* $\{\{4\},\ \{2,7\},\ \emptyset,\ \{5\},\ \emptyset,\ \{1,3\},\ \{6\},\ \emptyset,\ \emptyset,\ \emptyset\}$



Example $(a > b)$**.**

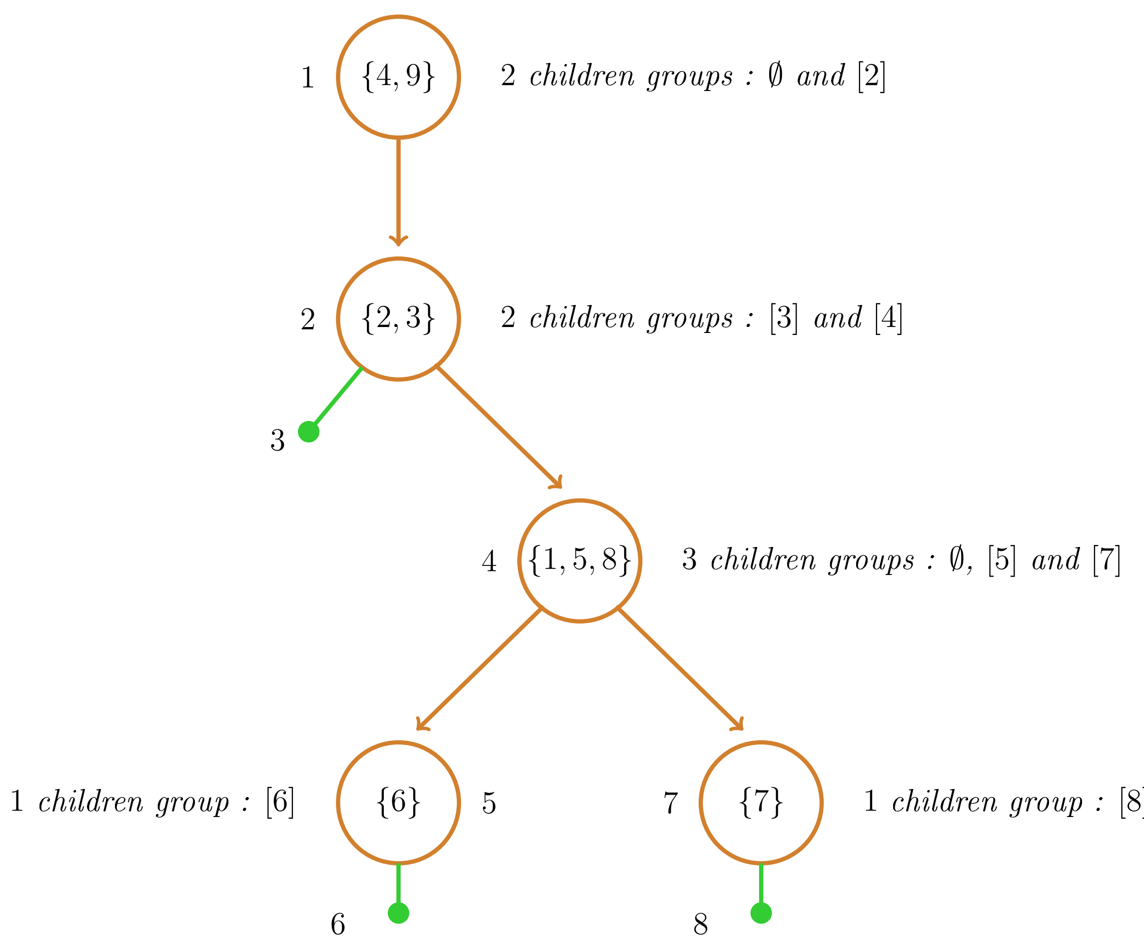
- $a = 9$
- $b = 7$
- *Limits :* $[1,\ 1\frac{7}{9},\ 2\frac{5}{9},\ 3\frac{3}{9},\ 4\frac{1}{9},\ 4\frac{8}{9},\ 5\frac{6}{9},\ 6\frac{4}{9},\ 7\frac{2}{9},\ 8]$

- *Integral limits :* $[0, 1, 1, 2, 3, 4, 4, 5, 6, 7, 8]$
- *Intervals :*

  $]0,1]$ $]1,1]$ $]1,2]$ $]2,3]$ $]3,4]$

  $]4,4]$ $[4,5]$ $]5,6]$ $]6,7]$ $]7,8]$
- *Children groups :*

  $[1]$ $\emptyset$ $[2]$ $[3]$ $[4]$ $\emptyset$ $[5]$ $[6]$ $[7]$ $[8]$
- $f = (4, 2, 2, 1, 4, 5, 7, 4, 1)$
- *Labels :* $\{\{4,9\},\ \{2,3\},\ \emptyset,\ \{1,5,8\}, \{6\},\ \emptyset,\ \{7\},\ \emptyset\}$



In both cases, the converse direction of the *bijection* is obtained with the same method as for classical parking trees.