

Rational Parking Functions

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Abstract

This is an abstract about Rational Parking Functions.

Contents

1	The integer case	2
1.1	Parking Functions	2
1.1.1	Primitive parking functions	3
1.2	Non-crossing Partitions	4
1.2.1	The non-crossing partitions poset	6
1.2.2	Kreweras complement	8
1.2.3	Action of \mathfrak{S}_n on partitions of $[n]$	11
1.3	Non-crossing 2-partitions	14
1.3.1	The non-crossing 2-partitions poset	17
1.3.2	The parking functions poset	18
2	The rational case	21
2.1	Rational Parking Functions	21
2.1.1	Rational primitive parking functions	22
2.2	Rational Non-crossing Partitions	23
3	Trees	24
3.1	Parking Trees	24
3.2	Rational Parking Trees	27

Chapter 1

The integer case

1.1 Parking Functions

Definition 1 (Parking Function). *A parking function is a sequence of positive integers (a_1, a_2, \dots, a_n) such that its non-decreasing reordering (b_1, b_2, \dots, b_n) has $b_i \leq i$ for all i .*

We denote by \mathcal{PF}_n the set of parking functions of length n .

$$\mathcal{PF} = \bigcup_{n \geq 0} \mathcal{PF}_n$$

.

Example.

$$f_1 = (7, 3, 1, 4, 2, 5, 2) \in \mathcal{PF}_7$$

$$f_2 = (7, 3, 1, 4, 2, 5, 4) \notin \mathcal{PF}_7$$

Theorem 1. *Let pf_n be the cardinal of \mathcal{PF}_n . We have*

$$pf_n = (n + 1)^{n-1}$$

.

Example ($n = 1, 2, 3$).

- $n = 1$: $pf_1 = 1$
(1)

- $n = 2$: $pf_2 = 3$
 $(1, 1)$ $(1, 2)$ $(2, 1)$
- $n = 3$: $pf_3 = 16$
 $(1, 1, 1)$ $(1, 1, 2)$ $(1, 1, 3)$ $(1, 2, 1)$ $(1, 2, 2)$ $(1, 2, 3)$ $(1, 3, 1)$
 $(1, 3, 2)$ $(2, 1, 1)$ $(2, 1, 2)$ $(2, 1, 3)$ $(2, 2, 1)$ $(2, 3, 1)$ $(3, 1, 1)$
 $(3, 1, 2)$ $(3, 2, 1)$

1.1.1 Primitive parking functions

Definition 2 (Primitive). A parking function (a_1, a_2, \dots, a_n) is said primitive if it is already in non-decreasing order.

We denote by \mathcal{PF}'_n the set of primitive parking functions of length n .

$$\mathcal{PF}' = \bigcup_{n \geq 0} \mathcal{PF}'_n$$

Example.

$$\begin{aligned} f_1 &= (1, 2, 2, 3) \in \mathcal{PF}'_4 \\ f_2 &= (1, 2, 3, 2) \notin \mathcal{PF}'_4, \text{ even though } f_2 \in \mathcal{PF}_4 \end{aligned}$$

Theorem 2. Let pf'_n be the cardinal of \mathcal{PF}'_n . We have

$$pf'_n = \frac{1}{n+1} \binom{2n}{n}$$

which is the n^{th} Catalan number $Cat(n)$.

Example ($n = 1, 2, 3$).

- $n = 1$: $pf'_1 = 1$
 (1)
- $n = 2$: $pf'_2 = 2$
 $(1, 1)$ $(1, 2)$
- $n = 3$: $pf'_3 = 5$
 $(1, 1, 1)$ $(1, 1, 2)$ $(1, 1, 3)$ $(1, 2, 2)$ $(1, 2, 3)$

1.2 Non-crossing Partitions

Definition 3 (Non-crossing Partition). A non-crossing partition of a set E is a set partition $P = \{E_1, E_2, \dots, E_k\}$ such that if $a, c \in E_i$, $b, d \in E_j$, and $i \neq j$, then we do not have $a < b < c < d$, nor $a > b > c > d$.

We denote by \mathcal{NC}_n the set of non-crossing partitions of $\{1, 2, \dots, n\}$.

$$\mathcal{NC} = \bigcup_{n>0} \mathcal{NC}_n$$

From this point, we assume that every partition $P = \{B_1, \dots, B_l\}$ is sorted such that :

- For each block $B_i = \{b_1, \dots, b_k\} \in P$, $b_1 < \dots < b_k$
- $\min(B_1) < \dots < \min(B_k)$

Notation. $[n] = \{1, 2, \dots, n\}$

Example ($E = [6]$).

$$\begin{aligned} P_1 &= \{\{1, 2, 5\}, \{3, 4\}, \{6\}\} \in \mathcal{NC}_6 \\ P_2 &= \{\{1, 2, 4\}, \{3, 5\}, \{6\}\} \notin \mathcal{NC}_6 \end{aligned}$$

Theorem 3. Let nc_n be the cardinal of \mathcal{NC}_n . We have

$$nc_n = \frac{1}{n+1} \binom{2n}{n}$$

which is again the n^{th} Catalan number $Cat(n)$.

Example ($n = 1, 2, 3$).

- $n = 1$: $nc_1 = 1$
 $\{\{1\}\}$
- $n = 2$: $nc_2 = 2$
 $\{\{1, 2\}\} \quad \{\{1\}, \{2\}\}$
- $n = 3$: $nc_3 = 5$
 $\{\{1, 2, 3\}\} \quad \{\{1\}, \{2, 3\}\} \quad \{\{1, 3\}, \{2\}\} \quad \{\{1, 2\}, \{3\}\} \quad \{\{1\}, \{2\}, \{3\}\}$

Proposition. This means we can create a bijection between \mathcal{PF}'_n and \mathcal{NC}_n .

Proof.

- $\mathcal{NC}_n \rightarrow \mathcal{PF}'_n$: For each block B in the non-crossing partition, take $i = \min(B)$, and let $k_i = \text{size}(B)$.
 $k_i = 0$ if i is not the minimum of a block.
 The corresponding parking function is $(\underbrace{1, \dots, 1}_{k_1}, \underbrace{2, \dots, 2}_{k_2}, \dots, \underbrace{n, \dots, n}_{k_n})$.
- $\mathcal{PF}'_n \rightarrow \mathcal{NC}_n$: For each i in $[n]$, if i appears n_i times in the parking function, B_i will be of size n_i with minimum element i . There is a unique set partition $P = \bigcup_i B_i$ of $[n]$ respecting these conditions that is non-crossing : for each minimum i in *decreasing order*, add the n_i first free elements of $[i + 1, i + 2, \dots, n, 1, \dots, i - 1]$ to B_i .

□

Example ($n = 6$).

$$P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\} \quad f = (1, 1, 1, 3, 3, 6)$$

Corollary. *A non-crossing partition can be represented by the minimums and sizes of its blocks.*

Example. $\{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$ can be represented by the following dictionary :

- $1 : 3$
- $3 : 2$
- $6 : 1$

A non-crossing partition of $[n]$ can be represented graphically on a regular n -vertices polygon, with vertices labeled from 1 to n clockwise. We then represent each block $B = \{b_1, \dots, b_k\}$ by the convex hull of $\{b_1, \dots, b_k\}$.

Example ($P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$).



Thus non-crossing meaning the hulls are disjoint.

Example (Counter-example : $P = \{\{1, 5, 2\}, \{3, 6\}, \{4\}\}$).



This partition is not non-crossing, as the convex hulls of $\{1, 2, 5\}$ and $\{3, 6\}$ are not disjoint.

1.2.1 The non-crossing partitions poset

Definition 4 (\succ). We say that P covers Q , written $P \succ Q$, if $\exists B_i, B_j \in P$ such that $Q = P - \{B_i, B_j\} \cup \{B_i \cup B_j\}$

Example. $\{\{1, 6\}, \{2, 3\}, \{4, 5\}\} \succ \{\{1, 2, 3, 6\}, \{4, 5\}\}$

- $B_i = \{1, 6\}$

- $B_j = \{2, 3\}$

Proposition. *This covering relation defines the poset of \mathcal{NC}_n . We denote by \mathcal{NCC}_n the set of maximal chains in the poset of \mathcal{NC}_n .*

$$\mathcal{NCC} = \bigcup_{n>0} \mathcal{NCC}_n$$

Remark. *The bottom element of this poset is $\{\{1, \dots, n\}\}$, and the top element is $\{\{1\}, \dots, \{n\}\}$.*

Theorem 4. *Let ncc_n be the cardinal of \mathcal{NCC}_n . We have*

$$ncc_n = n^{n-2}$$

.

Example (The poset of \mathcal{NC}_4).

To shorten labels, we represent $\{\{1\}, \{2, 3\}, \{4\}\}$ by $1|23|4$.



There are $4^2 = 16$ different maximal chains, and $\frac{1}{5} \binom{8}{4} = \frac{70}{5} = 14$ elements in this poset.

1.2.2 Kreweras complement

Definition 5 (Associated Permutation). *The permutation σ associated to a non-crossing partition has a cycle (b_1, \dots, b_k) for each block $B = \{b_1, \dots, b_k\}$ of the partition.*

Example. *The permutation associated to $\{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$ is $(1\ 2\ 5)(3\ 4)(6) = 254316$.*

Definition 6 (Kreweras Complement). *The Kreweras complement $K(P)$ of a non-crossing partition P is defined as follows :*

- *Let σ be the permutation associated to P*
- *Let π be the permutation $(n\ n-1\ n-2\ \dots\ 3\ 2\ 1) = n123\dots n-1$*
- *$K(P)$ is the non-crossing partition associated to $\pi\sigma$.*

Example ($P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$).

- $\sigma = (1\ 2\ 5)(3\ 4)(6) = 254316$
- $\pi = (6\ 5\ 4\ 3\ 2\ 1) = 612345$
- $\pi\sigma = 143265 = (1)(2\ 4)(3)(5\ 6)$
- $K(P) = \{\{1\}, \{2, 4\}, \{3\}, \{5, 6\}\}$

Proposition (Kreweras minimums). *Let $P = \{B_1, \dots, B_k\}$ be a non-crossing partition. Let $K(P) = \{B'_1, \dots, B'_l\}$ be its Kreweras complement. Then*

$$\bigcup_{1 \leq i \leq l} \min(B'_i) = B_1 \cup \bigcup_{1 < j \leq k} B_j - \max(B_1)$$

Example ($P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$).

- $K(P) = \{\{1\}, \{2, 4\}, \{3\}, \{5, 6\}\}$
- $\bigcup \min(B'_i) = \{1, 2, 3, 5\}$

- $B_1 \cup \bigcup B_i - \max(B_i) = \{1, 2, 5\} \cup \{3, 4\} - \{4\} \cup \{6\} - \{6\} = \{1, 2, 5\} \cup \{3\} \cup \emptyset = \{1, 2, 3, 5\}$

Notation. $B_{[i]}$ = block containing i .

Proposition (Kreweras block sizes). *Let $P = \{B_1, \dots, B_k\}$ be a non-crossing partition. Let $K(P) = \{B'_1, \dots, B'_l\}$ be its Kreweras complement. Then the size of the block B'_i is defined as follows :*

- Let m_i be the i^{th} minimum of $K(P)$
- Define a transition $\phi(e)$ as

$$\text{Let } j = e + 1 \text{ (or 1 if } e = n)$$

$$\phi(e) = \max(B_{[j]})$$
- The size of B'_i is k_{\min} such that $k_{\min} = \min\{k > 0 \mid \phi^k(m_i) \in B_{[m_i]}\}$.

Example ($P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$).

- $\text{mins} = \{1, 2, 3, 5\}$
- $m_1 = 1$

$$B_{[1]} = B_1$$

$$\max(B_{[2]}) = \max(B_1) = 5$$

The size for m_1 is 1.
- m_2

$$B_{[2]} = B_1$$

$$\max(B_{[3]}) = \max(B_2) = 4$$

$$\max(B_{[5]}) = \max(B_1) = 5$$

The size for m_2 is 2.
- $m_3 = 3$

$$B_{[3]} = B_2$$

$$\max(B_{[4]}) = \max(B_2) = 4$$

The size for m_3 is 1.

- $m_4 = 5$

$$B_{[5]} = B_1$$

$$\max(B_{[6]}) = \max(B_3) = 6$$

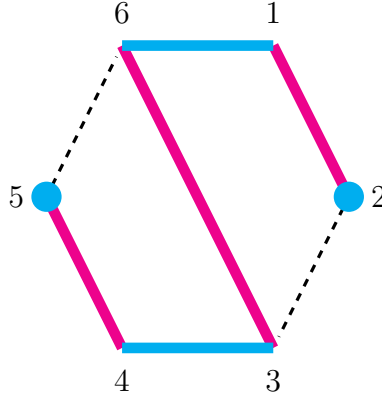
$$\max(B_{[1]}) = \max(B_1) = 5$$

The size for m_4 is 2.

Definition 7 (Mutually Non-crossing Partitions). 2 partitions P and Q are said mutually non-crossing if :

- P is non-crossing
- Q is non-crossing
- For every block B_i of P and every block B_j of Q , if $a, c \in B_i$ and $b, d \in B_j$, then we can not have $a < b < c < d$, nor $a > b > c > d$.

Example ($P = \{\{1, 2\}, \{3, 6\}, \{4, 5\}\}, Q = \{\{1, 6\}, \{2\}, \{3, 4\}, \{5\}\}$).



Example (Counter-example : $P = \{\{1, 2\}, \{3, 6\}, \{4, 5\}\}, Q = \{\{1, 6\}, \{2, 5\}, \{3, 4\}\}$).

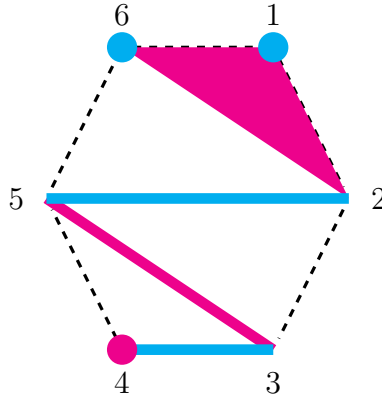


Remark. Note that vertices can touch, but the edges of the convex hulls can not cross.

Proposition. For any non-crossing partition P , P and $K(P)$ are mutually non-crossing.

Furthermore, $K(P)$ is a densest partition that is mutually non-crossing with P . That is, no partition Q that is mutually non-crossing with P has less blocks than $K(P)$.

Example ($P = \{1, 2, 6\}, \{3, 5\}, \{4\}\}$). $Q = \{\{1\}, \{2, 5\}, \{3, 4\}, \{6\}\}$

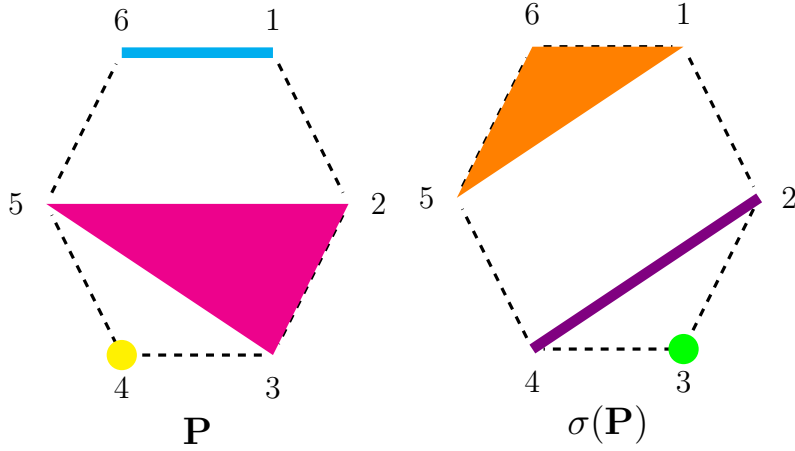


1.2.3 Action of \mathfrak{S}_n on partitions of $[n]$

Definition 8 (Action of \mathfrak{S}_n). The action of \mathfrak{S}_n on a partition $P = \{B_1, \dots, B_l\}$ of $[n]$ is defined by :

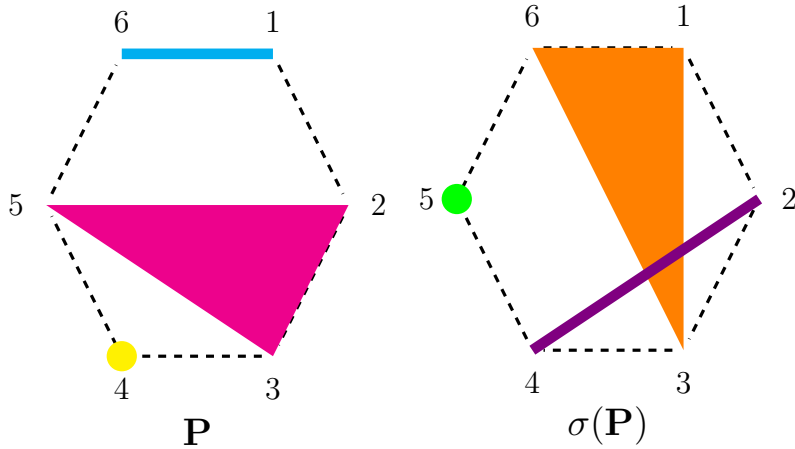
- For each block $B_i = \{b_1, \dots, b_k\} : \sigma(B_i) = \{\sigma(b_1), \dots, \sigma(b_k)\}$
- When $P \in \mathcal{NC}_n$, we denote $\rho = \sigma(P) = \{\sigma(B_1), \dots, \sigma(B_l)\}$

Example ($\sigma = 415362, P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$).
 $\sigma(P) = \{\{1, 5, 6\}, \{2, 4\}, \{3\}\}$



Remark. Note that \mathcal{NC}_n is not stable under the action of \mathfrak{S}_n . That is, even if P is non-crossing, $\sigma(P)$ is not necessarily non-crossing.

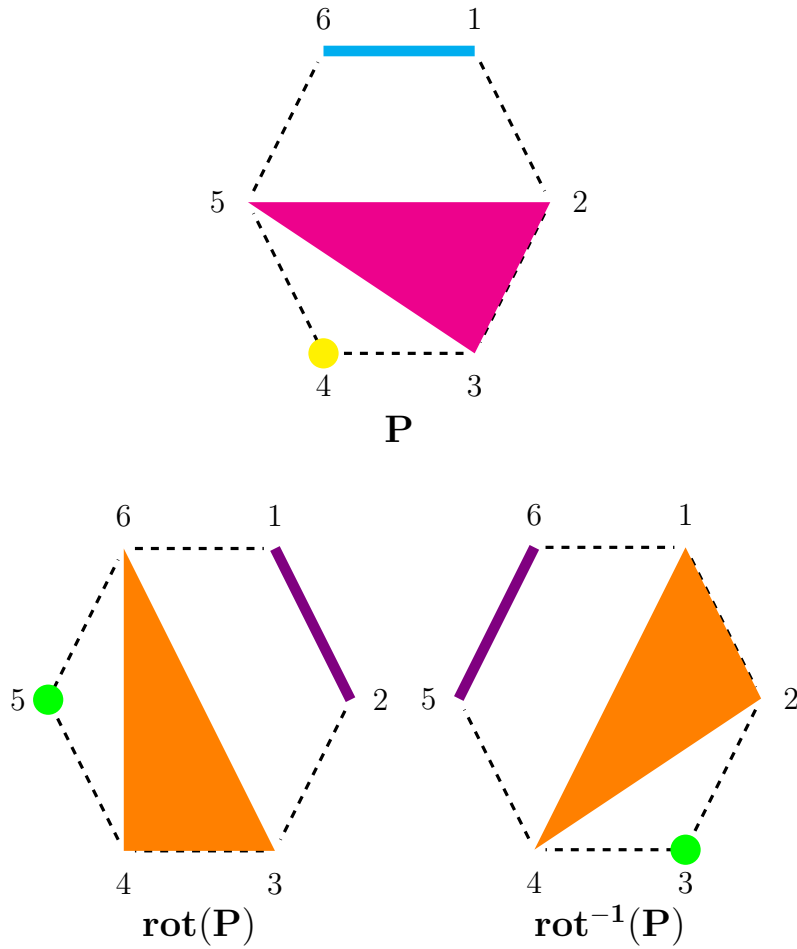
Example (Counter-example : $\sigma = 413562, P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$).
 $\sigma(P) = \{\{1, 3, 6\}, \{2, 4\}, \{5\}\}$



Definition 9 (Rotation). We define the rotation operator rot of $P \in \mathcal{NC}_n$ as $rot(P) = (1\ 2\ 3\ \dots\ n)(P) = 23\dots n1(P)$. Conversely, we define rot^{-1} of P as $rot^{-1}(P) = (n\ n-1\ \dots\ 3\ 2\ 1)(P) = n12\dots n-1(P)$.

Example ($P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$).

- $rot(P) = \{\{1, 2\}, \{3, 4, 6\}, \{5\}\}$
- $rot^{-1}(P) = \{\{1, 2, 4\}, \{3\}, \{5, 6\}\}$



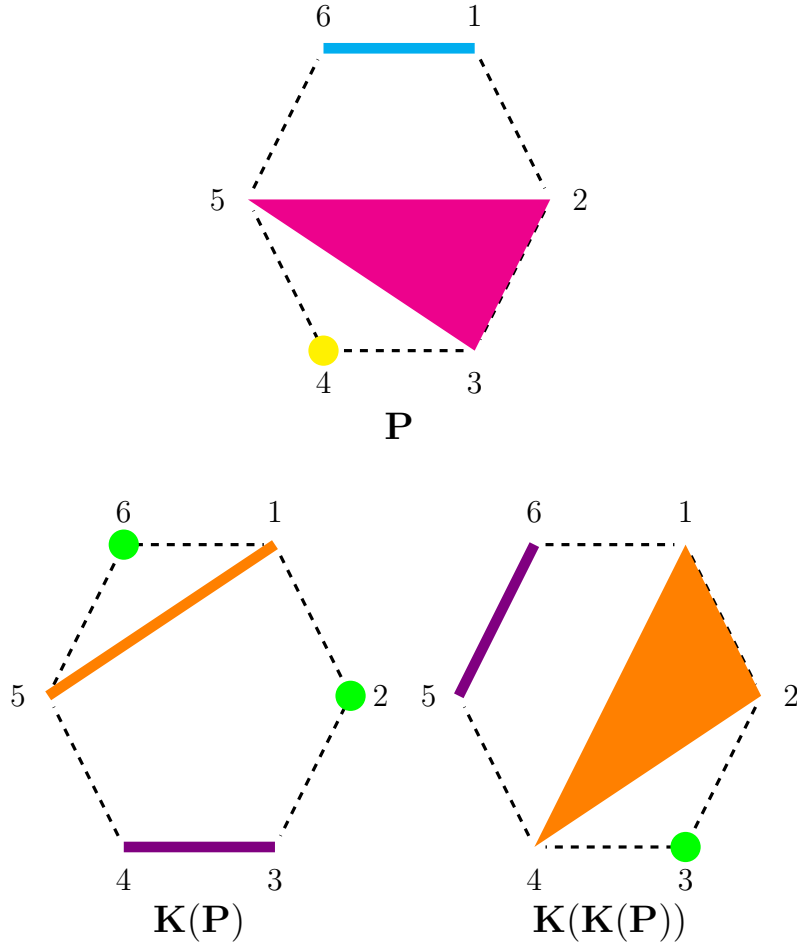
Remark.

- $rot(rot^{-1}(P)) = rot^{-1}(rot(P)) = P$

- $rot(P)$ and $rot^{-1}(P)$ are always non-crossing partitions.
- If $P \in \mathcal{NC}_n$, then $rot^n(P) = rot^{-n}(P) = P$.

Proposition. $K(K(P)) = rot^{-1}(P)$.

Example ($P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$).



1.3 Non-crossing 2-partitions

Definition 10 (Non-crossing 2-partition). A non-crossing 2-partition of a totally ordered set E is a pair (P, σ) where :

- P is a non-crossing partition of E
- σ is a permutation of the elements of E
- For each sorted block $B_i = \{b_1, \dots, b_k\} \in P$, we have $\sigma(b_i) < \dots < \sigma(b_k)$

We denote by \mathcal{NC}_n^2 the set of non-crossing 2-partitions of $[n]$.

$$\mathcal{NC}^2 = \bigcup_{n>0} \mathcal{NC}_n^2$$

Example (\mathcal{NC}_6^2). $P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$ $\sigma = 413265$
 $\rho = \{\{1, 3, 6\}, \{2\}, \{4, 5\}\}$

Theorem 5. Let nc_n^2 be the cardinal of \mathcal{NC}_n^2 . We have

$$nc_n^2 = (n+1)^{n-1}$$

Example ($n = 1, 2, 3$).

- $n = 1$: $nc_1^2 = 1$
 $\{\{1\}\} \quad 1 \quad \rho = P$
- $n = 2$: $nc_2^2 = 3$
 $\{\{1\}, \{2\}\} \quad 12 \quad \rho = P$
 $\{\{1\}, \{2\}\} \quad 21 \quad \rho = P$
 $\{\{1, 2\}\} \quad 12 \quad \rho = P$
- $n = 3$: $nc_3^2 = 16$
 $\{\{1\}, \{2\}, \{3\}\} \quad 123 \quad \rho = P$
 $\{\{1\}, \{2\}, \{3\}\} \quad 132 \quad \rho = P$
 $\{\{1\}, \{2\}, \{3\}\} \quad 213 \quad \rho = P$
 $\{\{1\}, \{2\}, \{3\}\} \quad 231 \quad \rho = P$
 $\{\{1\}, \{2\}, \{3\}\} \quad 312 \quad \rho = P$
 $\{\{1\}, \{2\}, \{3\}\} \quad 321 \quad \rho = P$
 $\{\{1, 2\}, \{3\}\} \quad 123 \quad \rho = P$

$\{\{1, 2\}, \{3\}\}$	132	$\rho = \{\{1, 3\}, \{2\}\}$
$\{\{1, 2\}, \{3\}\}$	231	$\rho = \{\{1\}, \{2, 3\}\}$
$\{\{1\}, \{2, 3\}\}$	123	$\rho = P$
$\{\{1\}, \{2, 3\}\}$	213	$\rho = \{\{1, 3\}, \{2\}\}$
$\{\{1\}, \{2, 3\}\}$	312	$\rho = \{\{1, 2\}, \{3\}\}$
$\{\{1, 3\}, \{2\}\}$	123	$\rho = P$
$\{\{1, 3\}, \{2\}\}$	132	$\rho = \{\{1, 2\}, \{3\}\}$
$\{\{1, 3\}, \{2\}\}$	213	$\rho = \{\{1\}, \{2, 3\}\}$
$\{\{1, 2, 3\}\}$	123	$\rho = P$

Proposition. *This means we can create a bijection between \mathcal{PF}_n and \mathcal{NC}_n^2 .*

Proof.

- $\mathcal{PF}_n \rightarrow \mathcal{NC}_n^2$: Let $f = (a_1, \dots, a_n) \in \mathcal{PF}_n$ be our parking function. For $i \in \{1, \dots, n\}$, we define :

l_i : the number of occurrences of i in f .

$im_i : \{j \mid a_j = i\}$

The corresponding non-crossing partition will have the following constraints :

For each $i \in \{1, \dots, n\}$, if $l_i > 0$, then there is a block $B_{[i]}$ of length l_i with minimum element i .

$\sigma(B_{[i]}) = im_i$

There is a unique set partition $P = \bigcup_i B_{[i]}$ of $[n]$ and a unique per-

mutation σ respecting these conditions such that $(P, \sigma) \in \mathcal{NC}_n^2$: for each minimum i in *decreasing order*, add the n_i first free elements of $[i+1, i+2, \dots, n, 1, \dots, i-1]$ to B_i . σ is then trivially obtained by the second constraint.

- $\mathcal{NC}_n^2 \rightarrow \mathcal{PF}_n$: Let (P, σ) with $P = \{B_1, \dots, B_l\}$ be our non-crossing 2-partition. For each block $B_i = \{b_1, \dots, b_k\} \in P$:

$$m_i = \min(B_i) = b_1$$

$$pos_i = \sigma(B_i)$$

For each $j \in pos_i$, we define $a_j = m_i$
The corresponding parking function is (a_1, \dots, a_n) .

□

Example ($n = 8$).

$$P = \{\{1, 2, 5\}, \{3, 4\}, \{6, 8\}, \{7\}\}$$

$$\sigma = 36187245$$

$$f = (3, 6, 1, 7, 6, 1, 1, 3)$$

1.3.1 The non-crossing 2-partitions poset

Definition 11 (\succ^2). We say that (P, σ) covers (Q, τ) , written $(P, \sigma) \succ^2 (Q, \tau)$, if $\exists B_i, B_j \in P$ such that

- $Q = P - \{B_i, B_j\} \cup \{B_i \cup B_j\}$
- $l \neq i, j, b \in B_l \rightarrow \tau(b) = \sigma(b)$
- Let $B_i \cup B_j = \{b_1, \dots, b_k\}$:
 $\tau(B_i \cup B_j) = \sigma(B_i \cup B_j)$
 $\tau(b_1) < \dots < \tau(b_k)$

Example.

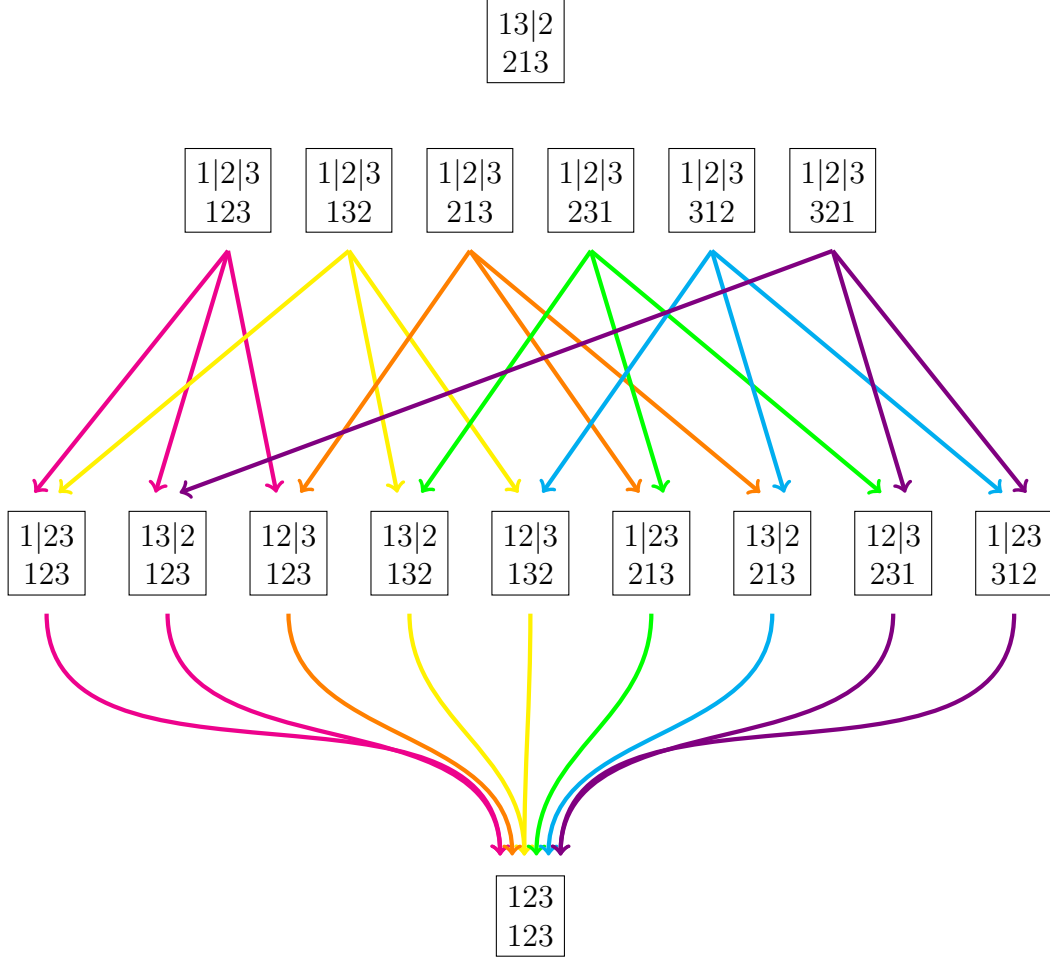
- $P = \{\{1, 6\}, \{2, 3\}, \{4\}, \{5\}\}$
- $\sigma = 236154$
- $Q = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$
- $\tau = 235164$
- $(P, \sigma) \succ^2 (Q, \tau)$
- $(P, \sigma) \not\succ^2 (Q, \sigma)$, because $\sigma(\{2, 3, 5\}) = \{3, 6, 5\}$ is not *ordemagenta*.

Proposition. This covering relation defines the poset of \mathcal{NC}_n^2 .

Remark. The bottom element of this poset is $(\{\{1, \dots, n\}\}, 12 \dots n)$, and the top elements are $\{(\{\{1\}, \dots, \{n\}\}, \sigma) \mid \sigma \in \mathfrak{S}_n\}$.

Example (The poset of \mathcal{NC}_3^2).

To shorten labels, we represent $(\{\{1, 3\}, \{2\}\}, 213)$ by



There are $4^2 = 16$ elements in this poset.

1.3.2 The parking functions poset

Definition 12 (Rank). Given $f = (a_1, \dots, a_n) \in \mathcal{PF}_n$, let

$$b_i = \begin{cases} 1 & \text{if } \exists j \mid a_j = i \\ 0 & \text{otherwise} \end{cases}$$

We define the rank of f , noted $rk(f)$, as

$$\sum_{1 \leq i \leq n} b_i$$

Example.

$$\begin{aligned} rk((1, 5, 4, 2, 3, 3, 1)) &= 5 \\ rk((4, 7, 1, 1, 3, 2, 2, 8)) &= 6 \end{aligned}$$

Definition 13 (\succ_{pf}). Since \mathcal{PF}_n and \mathcal{NC}_n^2 are in bijection, we can define a covering relation \succ_{pf} for \mathcal{PF}_n as follows :
 $f \in \mathcal{PF}_n \succ_{pf} g \in \mathcal{PF}_n$ if and only if :

- (P, σ) is the non-crossing 2-partition associated to f
- (Q, τ) is the non-crossing 2-partition associated to g
- $(P, \sigma) \succ^2 (Q, \tau)$

Example.

- $P = \{\{1, 6\}, \{2, 3\}, \{4\}, \{5\}\}$
- $\sigma = 236154$
- $Q = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$
- $\tau = 235164$
- $f = (4, 1, 2, 1, 5, 2) \succ_{pf} g = (4, 1, 2, 1, 2, 2)$

Remark. If $f \succ_{pf} g$, then $rk(f) = rk(g) + 1$, and there exists i and j such that :

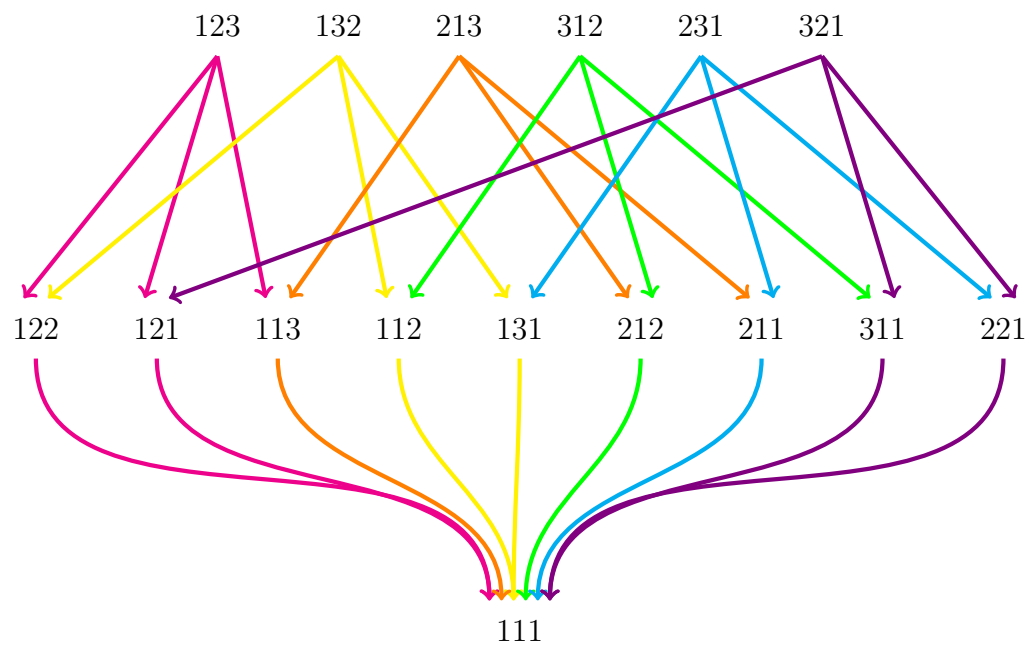
- $i < j$
- There is at least 1 occurrence of i in f
- There is at least 1 occurrence of j in f

$$b_k = \begin{cases} i & \text{if } a_k = j \\ a_k & \text{otherwise} \end{cases}$$

Proposition. This covering relation defines the poset of \mathcal{PF}_n .

Remark. The bottom element of this poset is $(\underbrace{1, \dots, 1}_n)$, and the top elements are the permutations of $\{1, \dots, n\}$.

Example (The poset of \mathcal{PF}_3).



Chapter 2

The rational case

For the whole chapter, we will consider 2 *coprime* integers a and b (meaning a and b have 1 as their greatest common divisor).

2.1 Rational Parking Functions

Definition 14 (a, b - Parking Function). An a, b - parking function is a sequence (a_1, a_2, \dots, a_n) such that :

- $n = a$
- its non-decreasing reordering (b_1, b_2, \dots, b_n) has $b_i \leq \frac{b}{a}(i - 1) + 1$ for all i .

We denote by $\mathcal{PF}_{a,b}$ the set of a, b - parking functions.

Example.

- *Ex. 1* : $a > b$

$$a = 7$$

$$b = 3$$

Limits of the non-decreasing reordering of any $f \in \mathcal{PF}_{7,3}$:

$$[1, 1\frac{3}{7}, 1\frac{6}{7}, 2\frac{2}{7}, 2\frac{5}{7}, 3\frac{1}{7}, 3\frac{4}{7}]$$

$$f_1 = (2, 1, 1, 3, 2, 3, 1) \in \mathcal{PF}_{7,3}$$

$$f_2 = (2, 1, 2, 3, 2, 3, 1) \notin \mathcal{PF}_{7,3}, \text{ even though } f_2 \in \mathcal{PF}_7$$

- *Ex. 2* : $a < b$

$$a = 5$$

$$b = 7$$

Limits of the non-decreasing reordering of any $f \in \mathcal{PF}_{5,7}$:

$$[1, 2\frac{2}{5}, 3\frac{4}{5}, 5\frac{1}{5}, 6\frac{3}{5}]$$

$$f_3 = (6, 3, 5, 1, 2) \in \mathcal{PF}_{5,7}, \text{ even though } f_3 \notin \mathcal{PF}_5$$

$$f_4 = (6, 3, 5, 1, 3) \notin \mathcal{PF}_{5,7}$$

Theorem 6. *Let $pf_{a,b}$ be the cardinal of $\mathcal{PF}_{a,b}$. We have*

$$pf_{a,b} = b^{a-1}$$

Example ($a = 3, b = 5$).

- $pf_{a,b} = 25$
- *Limits* : $[1, 2\frac{2}{3}, 4\frac{1}{3}]$

(1, 1, 1)	(1, 1, 2)	(1, 1, 3)	(1, 1, 4)	(1, 2, 1)	(1, 2, 2)	(1, 2, 3)
(1, 2, 4)	(1, 3, 1)	(1, 3, 2)	(1, 4, 1)	(1, 4, 2)	(2, 1, 1)	(2, 1, 2)
(2, 1, 3)	(2, 1, 4)	(2, 2, 1)	(2, 3, 1)	(2, 4, 1)	(3, 1, 1)	(3, 1, 2)
(3, 2, 1)	(4, 1, 1)	(4, 1, 2)	(4, 2, 1)			

Remark. $\mathcal{PF}_{n,n+1} = \mathcal{PF}_n$. In fact, we do have $b^{a-1} = (n+1)^{n-1}$.

2.1.1 Rational primitive parking functions

Definition 15 (Rational Primitive). *A rational parking function f is said primitive if it is already in non-decreasing order.*

We denote by $\mathcal{PF}'_{a,b}$ the set of primitive a, b - parking functions.

Example ($a = 4, b = 3$). *Limits* : $[1, 1\frac{3}{4}, 2\frac{1}{2}, 3\frac{1}{4}]$

$$f_1 = (1, 1, 2, 2) \in \mathcal{PF}'_{4,3}$$

$$f_2 = (1, 1, 2, 1) \notin \mathcal{PF}'_{4,3}, \text{ even though } f_2 \in \mathcal{PF}_{4,3}.$$

Theorem 7. *Let $pf'_{a,b}$ be the cardinal of $\mathcal{PF}'_{a,b}$. We have*

$$pf'_{a,b} = \frac{1}{a+b} \binom{a+b}{b}$$

which is the rational Catalan number $Cat(a, b)$.

Example ($a = 3, b = 5$).

• $pf'_{a,b} = 7$ • *Limits* : $[1, 2\frac{2}{3}, 4\frac{1}{3}]$

(1, 1, 1) (1, 1, 2) (1, 1, 3) (1, 1, 4) (1, 2, 2) (1, 2, 3) (1, 2, 4)

Remark. $\mathcal{PF}'_{n,n+1} = \mathcal{PF}'_n$. In fact, we do have

$$\begin{aligned} \frac{1}{n+n+1} \binom{n+n+1}{n+1} &= \frac{1}{2n+1} \binom{2n+1}{n+1} = \frac{1}{2n+1} \frac{(2n+1)!}{n!(n+1)!} \\ &= \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1} \frac{(2n)!}{n!n!} = \frac{1}{n+1} \binom{2n}{n} \end{aligned}$$

2.2 Rational Non-crossing Partitions

Definition 16 (a, b - Non-crossing Partition). An a, b - non-crossing partition is *TODO*

Example. An *abncp*

Theorem 8. number of *abncp*

Example. all *abncp* for some a b

Proposition. This means we can create a bijection between $\mathcal{PF}'_{a,b}$ and $\mathcal{NC}_{a,b}$.

Proof.

• $\mathcal{NC}_{a,b} \rightarrow \mathcal{PF}'_{a,b}$:

• $\mathcal{PF}'_{a,b} \rightarrow \mathcal{NC}_{a,b}$:

□

Definition 17. *ncab2*

Example. some *ncab2*

Theorem 9. number of *ncab2*

Example. all *ncab2* for some a b

Proposition. *bijection*

Proof. bijection proof

□

Chapter 3

Trees

3.1 Parking Trees

Definition 18 (Parking Trees). A parking tree is defined from a parking function $f = (a_1, \dots, a_n) \in \mathcal{PF}_n$ as follows :

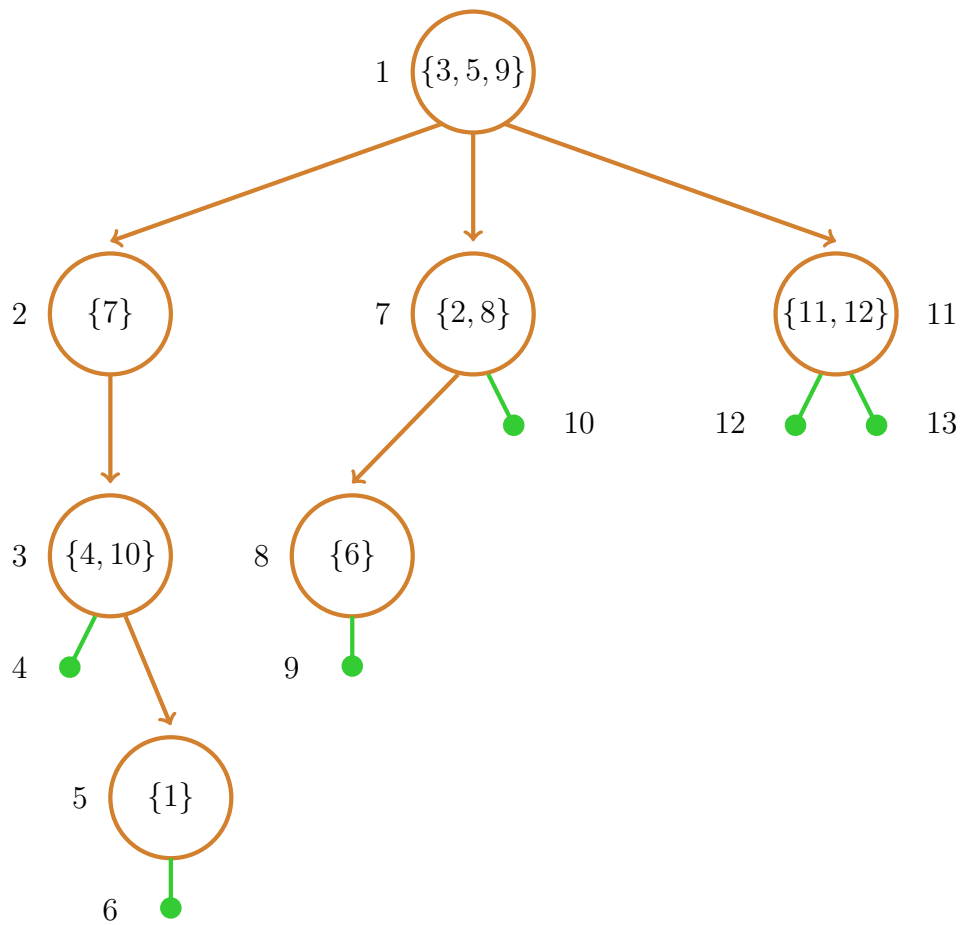
- For $1 \leq i \leq n+1$, we define s_i as $\{j \mid a_j = i\}$
- $[s_1, \dots, s_{n+1}]$ describes the pre-order depth-first traversal of the tree.
- Each node labeled by a set of size k has k children.

Remark. The leaves of the tree are those corresponding to an element i such that $1 \leq i \leq n+1$, and i is not in f .

Furthermore, as we will have a total edges by definition, the presence of a node corresponding to $n+1$ is necessary, even though it will always be empty.

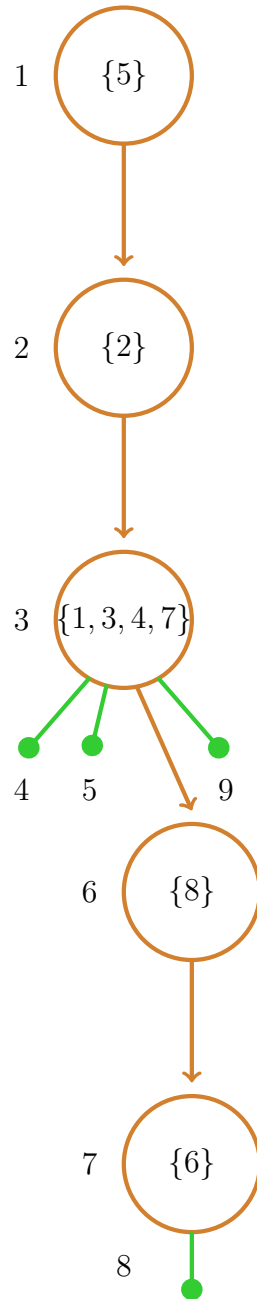
Example ($n = 12$).

- $f = (5, 7, 1, 3, 1, 8, 2, 7, 1, 3, 11, 11)$
- Labels : $[\{3, 5, 9\}, \{7\}, \{4, 10\}, \emptyset, \{1\}, \emptyset, \{2, 8\}, \{6\}, \emptyset, \emptyset, \{11, 12\}, \emptyset, \emptyset]$



Conversely, by reading the labels of a parking tree depth-first in pre-order, we get the list of positions of each number in the corresponding parking function, thus creating a *bijection*.

Example (From parking tree to parking function).



- The labels are $[\{5\}, \{2\}, \{1, 3, 4, 7\}, \emptyset, \emptyset, \{8\}, \{6\}, \emptyset, \emptyset]$.
- Thus the corresponding parking function is $(3, 2, 3, 3, 1, 7, 3, 6) \in \mathcal{PF}_8$.

3.2 Rational Parking Trees

Definition 19 (Rational Parking Trees). *A rational parking tree is defined from a rational parking function $f = (a_1, \dots, a_a) \in \mathcal{PF}_{a,b}$ as follows :*

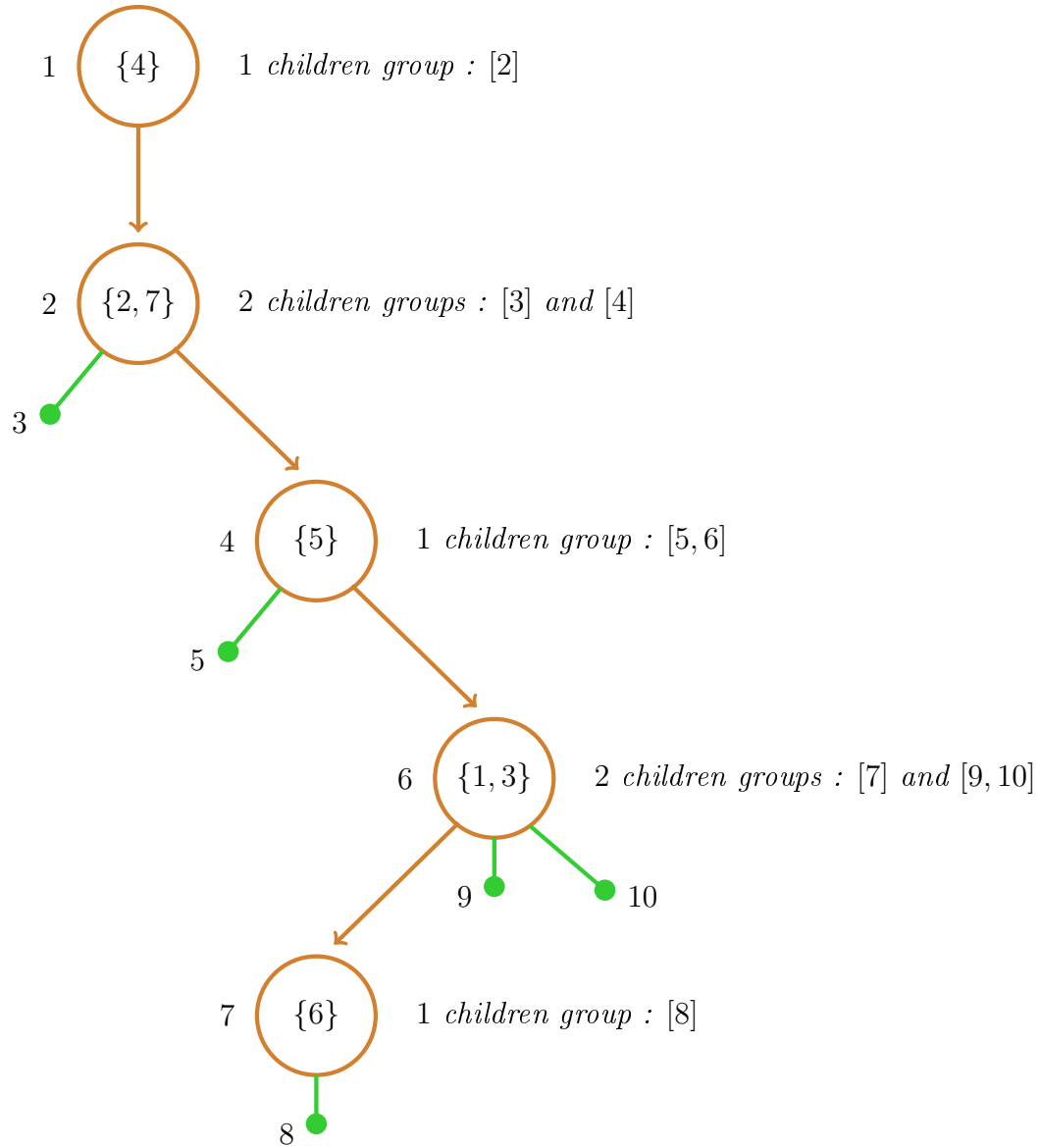
- *For $1 \leq i \leq n + 1$, we define the limit l_i as the integer portion of $\frac{b}{a}(i - 1) + 1$.
Let $l_0 = 0$.*
- *From these limits, we deduce the intervals $itv_i =]l_{i-1}, l_i]$ for $1 \leq i \leq a + 1$.*
- *For $1 \leq i \leq b + 1$, define s_i as $\{j \mid a_j = i\}$.*
- *$[s_1, \dots, s_{b+1}]$ describes the pre-order depth-first traversal of the tree.*
- *Each node labeled by a set of size k has k groups of children, which are defined by the intervals.*

Example ($a < b$).

- $a = 7$
- $b = 9$
- *Limits :* $[1, 2\frac{2}{7}, 3\frac{4}{7}, 4\frac{6}{7}, 6\frac{1}{7}, 7\frac{3}{7}, 8\frac{5}{7}, 10]$
- *Integral limits :* $[0, 1, 2, 3, 4, 6, 7, 8, 10]$
- *Intervals :*

$$]0, 1] \quad]1, 2] \quad]2, 3] \quad]3, 4] \quad]4, 6] \quad]6, 7] \quad]7, 8] \quad]8, 10]$$
- *Children groups :*

$$[1] \quad [2] \quad [3] \quad [4] \quad [5, 6] \quad [7] \quad [8]$$
- $f = (6, 2, 6, 1, 4, 7, 2)$
- *Labels :* $\{\{4\}, \{2, 7\}, \emptyset, \{5\}, \emptyset, \{1, 3\}, \{6\}, \emptyset, \emptyset, \emptyset\}$



Example ($a > b$).

- $a = 9$
- $b = 7$
- *Limits* : $[1, 1\frac{7}{9}, 2\frac{5}{9}, 3\frac{3}{9}, 4\frac{1}{9}, 4\frac{8}{9}, 5\frac{6}{9}, 6\frac{4}{9}, 7\frac{2}{9}, 8]$

- *Integral limits* : $[0, 1, 1, 2, 3, 4, 4, 5, 6, 7, 8]$

- *Intervals* :

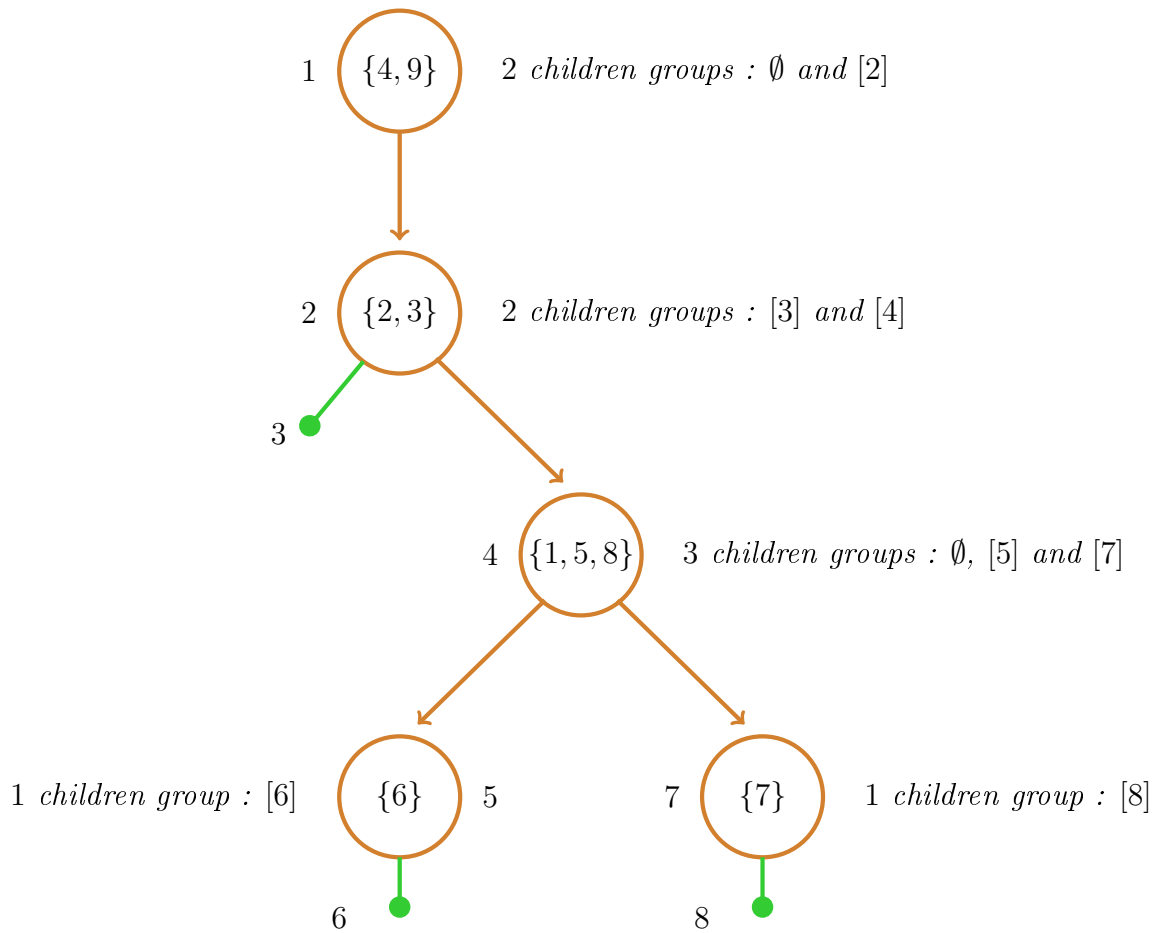
$]0, 1]$	$]1, 1]$	$]1, 2]$	$]2, 3]$	$]3, 4]$
$]4, 4]$	$[4, 5]$	$]5, 6]$	$]6, 7]$	$]7, 8]$

- *Children groups* :

$[1]$	\emptyset	$[2]$	$[3]$	$[4]$	\emptyset	$[5]$	$[6]$	$[7]$	$[8]$
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- $f = (4, 2, 2, 1, 4, 5, 7, 4, 1)$

- *Labels* : $\{\{4, 9\}, \{2, 3\}, \emptyset, \{1, 5, 8\}, \{6\}, \emptyset, \{7\}, \emptyset\}$



In both cases, the converse direction of the *bijection* is obtained with the same method as for classical parking trees.