

# Rational Parking Functions

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August 27, 2020

## **Abstract**

This is an abstract about Rational Parking Functions.

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# Chapter 1

## The integer case

### 1.1 Parking Functions

**Definition 1** (Parking Function). *A parking function is a sequence of positive integers  $(a_1, a_2, \dots, a_n)$  such that its non-decreasing reordering  $(b_1, b_2, \dots, b_n)$  has  $b_i \leq i$  for all  $i$ .*

*We denote by  $\mathcal{PF}_n$  the set of parking functions of length  $n$ .*

$$\mathcal{PF} = \bigcup_{n \geq 0} \mathcal{PF}_n$$

.

**Example.**

$$f_1 = (7, 3, 1, 4, 2, 5, 2) \in \mathcal{PF}_7$$

$$f_2 = (7, 3, 1, 4, 2, 5, 4) \notin \mathcal{PF}_7$$

**Theorem 1.** *Let  $pf_n$  be the cardinal of  $\mathcal{PF}_n$ . We have*

$$pf_n = (n + 1)^{n-1}$$

.

**Example** ( $n = 1, 2, 3$ ).

- $n = 1$  :  $pf_1 = 1$   
(1)

- $n = 2$  :  $pf_2 = 3$   
 $(1, 1)$      $(1, 2)$      $(2, 1)$
- $n = 3$  :  $pf_3 = 16$   
 $(1, 1, 1)$      $(1, 1, 2)$      $(1, 1, 3)$      $(1, 2, 1)$      $(1, 2, 2)$      $(1, 2, 3)$      $(1, 3, 1)$   
 $(1, 3, 2)$      $(2, 1, 1)$      $(2, 1, 2)$      $(2, 1, 3)$      $(2, 2, 1)$      $(2, 3, 1)$      $(3, 1, 1)$   
 $(3, 1, 2)$      $(3, 2, 1)$

### 1.1.1 Primitive parking functions

**Definition 2** (Primitive). A parking function  $(a_1, a_2, \dots, a_n)$  is said primitive if it is already in non-decreasing order.

We denote by  $\mathcal{PF}'_n$  the set of primitive parking functions of length  $n$ .

$$\mathcal{PF}' = \bigcup_{n \geq 0} \mathcal{PF}'_n$$

**Example.**

$$\begin{aligned} f_1 &= (1, 2, 2, 3) \in \mathcal{PF}'_4 \\ f_2 &= (1, 2, 3, 2) \notin \mathcal{PF}'_4, \text{ even though } f_2 \in \mathcal{PF}_4 \end{aligned}$$

**Theorem 2.** Let  $pf'_n$  be the cardinal of  $\mathcal{PF}'_n$ . We have

$$pf'_n = \frac{1}{n+1} \binom{2n}{n}$$

which is the  $n^{\text{th}}$  Catalan number  $Cat(n)$ .

**Example** ( $n = 1, 2, 3$ ).

- $n = 1$  :  $pf'_1 = 1$   
 $(1)$
- $n = 2$  :  $pf'_2 = 2$   
 $(1, 1)$      $(1, 2)$
- $n = 3$  :  $pf'_3 = 5$   
 $(1, 1, 1)$      $(1, 1, 2)$      $(1, 1, 3)$      $(1, 2, 2)$      $(1, 2, 3)$

## 1.2 Non-crossing Partitions

**Definition 3** (Non-crossing Partition). *A non-crossing partition of a totally ordered set  $E$  is a set partition  $P = \{E_1, E_2, \dots, E_k\}$  such that if  $a, c \in E_i$ ,  $b, d \in E_j$ , and  $i \neq j$ , then we do not have  $a < b < c < d$ , nor  $a > b > c > d$ . We denote by  $\mathcal{NC}_n$  the set of non-crossing partitions of  $\{1, 2, \dots, n\}$ .*

$$\mathcal{NC} = \bigcup_{n \geq 0} \mathcal{NC}_n$$

From this point, we assume that every partition  $P = \{B_1, \dots, B_l\}$  is *sorted* such that :

- For each block  $B_i = \{b_1, \dots, b_k\} \in P$ ,  $b_1 < \dots < b_k$
- $\min(B_1) < \dots < \min(B_k)$

**Notation.**  $[n] = \{1, 2, \dots, n\}$

**Example** ( $E = [6]$ ).

$$\begin{aligned} P_1 &= \{\{1, 2, 5\}, \{3, 4\}, \{6\}\} \in \mathcal{NC}_6 \\ P_2 &= \{\{1, 2, 4\}, \{3, 5\}, \{6\}\} \notin \mathcal{NC}_6 \end{aligned}$$

**Theorem 3.** *Let  $nc_n$  be the cardinal of  $\mathcal{NC}_n$ . We have*

$$nc_n = \frac{1}{n+1} \binom{2n}{n}$$

*which is again the  $n^{\text{th}}$  Catalan number  $Cat(n)$ .*

**Example** ( $n = 1, 2, 3$ ).

- $n = 1$  :  $nc_1 = 1$   
 $\{\{1\}\}$
- $n = 2$  :  $nc_2 = 2$   
 $\{\{1, 2\}\} \quad \{\{1\}, \{2\}\}$
- $n = 3$  :  $nc_3 = 5$   
 $\{\{1, 2, 3\}\} \quad \{\{1\}, \{2, 3\}\} \quad \{\{1, 3\}, \{2\}\} \quad \{\{1, 2\}, \{3\}\} \quad \{\{1\}, \{2\}, \{3\}\}$

**Proposition.** *This means we can create a bijection between  $\mathcal{PF}'_n$  and  $\mathcal{NC}_n$ .*

*Proof.*

- $\mathcal{NC}_n \rightarrow \mathcal{PF}'_n$  : For each block  $B$  in the non-crossing partition, take  $i = \min(B)$ , and let  $k_i = \text{size}(B)$ .  
 $k_i = 0$  if  $i$  is not the minimum of a block.  
 The corresponding parking function is  $(\underbrace{1, \dots, 1}_{k_1}, \underbrace{2, \dots, 2}_{k_2}, \dots, \underbrace{n, \dots, n}_{k_n})$ .
- $\mathcal{PF}'_n \rightarrow \mathcal{NC}_n$  : For each  $i$  in  $[n]$ , if  $i$  appears  $n_i$  times in the parking function,  $B_i$  will be of size  $n_i$  with minimum element  $i$ . There is a unique set partition  $P = \bigcup_i B_i$  of  $[n]$  respecting these conditions that is non-crossing : for each minimum  $i$  in *decreasing order*, add the  $n_i$  first free elements of  $[i + 1, i + 2, \dots, n, 1, \dots, i - 1]$  to  $B_i$ .

□

**Example** ( $n = 6$ ).

$$P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\} \quad f = (1, 1, 1, 3, 3, 6)$$

**Corollary.** *A non-crossing partition can be represented by the minimums and sizes of its blocks.*

**Example.**  $\{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$  can be represented by the following dictionary :

- $1 : 3$
- $3 : 2$
- $6 : 1$

A non-crossing partition of  $[n]$  can be represented graphically on a regular  $n$ -vertices polygon, with vertices labeled from 1 to  $n$  clockwise. We then represent each block  $B = \{b_1, \dots, b_k\}$  by the convex hull of  $\{b_1, \dots, b_k\}$ .

**Example** ( $P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$ ).



Thus non-crossing meaning the hulls are disjoint.

**Example** (Counter-example :  $P = \{\{1, 5, 2\}, \{3, 6\}, \{4\}\}$ ).



This partition is not non-crossing, as the convex hulls of  $\{1, 2, 5\}$  and  $\{3, 6\}$  are not disjoint.

### 1.2.1 The non-crossing partitions poset

**Definition 4** ( $\succ$ ). We say that  $P$  covers  $Q$ , written  $P \succ Q$ , if  $\exists B_i, B_j \in P$  such that  $Q = P - \{B_i, B_j\} \cup \{B_i \cup B_j\}$

**Example.**  $\{\{1, 6\}, \{2, 3\}, \{4, 5\}\} \succ \{\{1, 2, 3, 6\}, \{4, 5\}\}$

- $B_i = \{1, 6\}$



- $B_j = \{2, 3\}$

**Proposition.** *This covering relation defines the poset of  $\mathcal{NC}_n$ . We denote by  $\mathcal{NCC}_n$  the set of maximal chains in the poset of  $\mathcal{NC}_n$ .*

$$\mathcal{NCC} = \bigcup_{n>0} \mathcal{NCC}_n$$

**Remark.** *The bottom element of this poset is  $\{\{1, \dots, n\}\}$ , and the top element is  $\{\{1\}, \dots, \{n\}\}$ .*

**Theorem 4.** *Let  $ncc_n$  be the cardinal of  $\mathcal{NCC}_n$ . We have*

$$ncc_n = n^{n-2}$$

.

**Example** (The poset of  $\mathcal{NC}_4$ ).

*To shorten labels, we represent  $\{\{1\}, \{2, 3\}, \{4\}\}$  by  $1|23|4$ .*



There are  $4^2 = 16$  different maximal chains, and  $\frac{1}{5} \binom{8}{4} = \frac{70}{5} = 14$  elements in this poset.

### 1.2.2 Kreweras complement

**Definition 5** (Associated Permutation). *The permutation  $\sigma$  associated to a non-crossing partition has a cycle  $(b_1, \dots, b_k)$  for each block  $B = \{b_1, \dots, b_k\}$  of the partition.*

**Example.** *The permutation associated to  $\{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$  is  $(1\ 2\ 5)(3\ 4)(6) = 254316$ .*

**Definition 6** (Kreweras Complement). *The Kreweras complement  $K(P)$  of a non-crossing partition  $P$  is defined as follows :*

- *Let  $\sigma$  be the permutation associated to  $P$*
- *Let  $\pi$  be the permutation  $(n\ n-1\ n-2\ \dots\ 3\ 2\ 1) = n123\dots n-1$*
- *$K(P)$  is the non-crossing partition associated to  $\pi\sigma$ .*

**Example** ( $P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$ ).

- $\sigma = (1\ 2\ 5)(3\ 4)(6) = 254316$
- $\pi = (6\ 5\ 4\ 3\ 2\ 1) = 612345$
- $\pi\sigma = 143265 = (1)(2\ 4)(3)(5\ 6)$
- $K(P) = \{\{1\}, \{2, 4\}, \{3\}, \{5, 6\}\}$

**Proposition** (Kreweras minimums). *Let  $P = \{B_1, \dots, B_k\}$  be a non-crossing partition. Let  $K(P) = \{B'_1, \dots, B'_l\}$  be its Kreweras complement. Then*

$$\bigcup_{1 \leq i \leq l} \min(B'_i) = B_1 \cup \bigcup_{1 < j \leq k} B_j - \max(B_i)$$

**Example** ( $P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$ ).

- $K(P) = \{\{1\}, \{2, 4\}, \{3\}, \{5, 6\}\}$
- $\bigcup \min(B'_i) = \{1, 2, 3, 5\}$

- $B_1 \cup \bigcup B_i - \max(B_i) = \{1, 2, 5\} \cup \{3, 4\} - \{4\} \cup \{6\} - \{6\} = \{1, 2, 5\} \cup \{3\} \cup \emptyset = \{1, 2, 3, 5\}$

**Notation.**  $B_{[i]}$  = block containing  $i$ .

**Proposition** (Kreweras block sizes). *Let  $P = \{B_1, \dots, B_k\}$  be a non-crossing partition. Let  $K(P) = \{B'_1, \dots, B'_l\}$  be its Kreweras complement. Then the size of the block  $B'_i$  is defined as follows :*

- Let  $m_i$  be the  $i^{\text{th}}$  minimum of  $K(P)$
- Define a transition  $\phi(e)$  as  

$$\text{Let } j = e + 1 \text{ (or 1 if } e = n)$$

$$\phi(e) = \max(B_{[j]})$$
- The size of  $B'_i$  is  $k_{\min}$  such that  $k_{\min} = \min\{k > 0 \mid \phi^k(m_i) \in B_{[m_i]}\}$ .

**Example** ( $P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$ ).

- $\text{mins} = \{1, 2, 3, 5\}$
- $m_1 = 1$   

$$B_{[1]} = B_1$$

$$\max(B_{[2]}) = \max(B_1) = 5$$

The size for  $m_1$  is 1.
- $m_2$   

$$B_{[2]} = B_1$$

$$\max(B_{[3]}) = \max(B_2) = 4$$

$$\max(B_{[5]}) = \max(B_1) = 5$$

The size for  $m_2$  is 2.
- $m_3 = 3$   

$$B_{[3]} = B_2$$

$$\max(B_{[4]}) = \max(B_2) = 4$$

The size for  $m_3$  is 1.

- $m_4 = 5$

$$B_{[5]} = B_1$$

$$\max(B_{[6]}) = \max(B_3) = 6$$

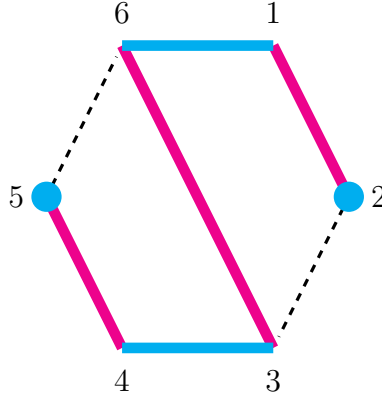
$$\max(B_{[1]}) = \max(B_1) = 5$$

The size for  $m_4$  is 2.

**Definition 7** (Mutually Non-crossing Partitions). 2 partitions  $P$  and  $Q$  are said mutually non-crossing if :

- $P$  is non-crossing
- $Q$  is non-crossing
- For every block  $B_i$  of  $P$  and every block  $B_j$  of  $Q$ , if  $a, c \in B_i$  and  $b, d \in B_j$ , then we can not have  $a < b < c < d$ , nor  $a > b > c > d$ .

**Example** ( $P = \{\{1, 2\}, \{3, 6\}, \{4, 5\}\}, Q = \{\{1, 6\}, \{2\}, \{3, 4\}, \{5\}\}$ ).



**Example** (Counter-example :  $P = \{\{1, 2\}, \{3, 6\}, \{4, 5\}\}, Q = \{\{1, 6\}, \{2, 5\}, \{3, 4\}\}$ ).

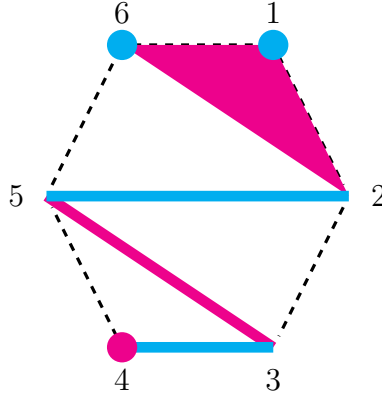


**Remark.** Note that vertices can touch, but the edges of the convex hulls can not cross.

**Proposition.** For any non-crossing partition  $P$ ,  $P$  and  $K(P)$  are mutually non-crossing.

Furthermore,  $K(P)$  is a densest partition that is mutually non-crossing with  $P$ . That is, no partition  $Q$  that is mutually non-crossing with  $P$  has less blocks than  $K(P)$ .

**Example** ( $P = \{1, 2, 6\}, \{3, 5\}, \{4\}\}$ ).  $Q = \{\{1\}, \{2, 5\}, \{3, 4\}, \{6\}\}$

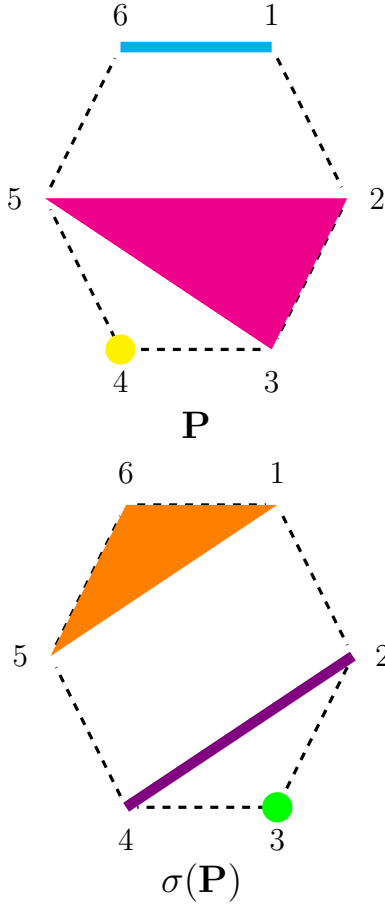


### 1.2.3 Action of $\mathfrak{S}_n$ on partitions of $[n]$

**Definition 8** (Action of  $\mathfrak{S}_n$ ). The action of  $\mathfrak{S}_n$  on a partition  $P = \{B_1, \dots, B_l\}$  of  $[n]$  is defined by :

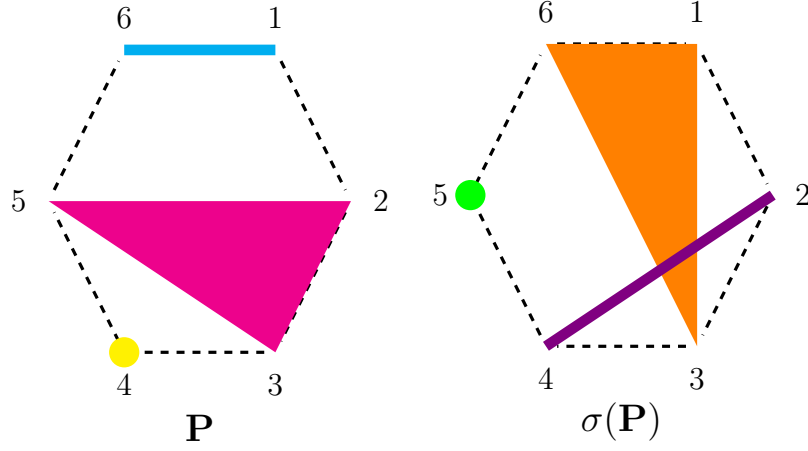
- For each block  $B_i = \{b_1, \dots, b_k\} : \sigma(B_i) = \{\sigma(b_1), \dots, \sigma(b_k)\}$
- When  $P \in \mathcal{NC}_n$ , we denote  $\rho = \sigma(P) = \{\sigma(B_1), \dots, \sigma(B_l)\}$

**Example** ( $\sigma = 415362, P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$ ).  
 $\sigma(P) = \{\{1, 5, 6\}, \{2, 4\}, \{3\}\}$



**Remark.** Note that  $\mathcal{NC}_n$  is not stable under the action of  $\mathfrak{S}_n$ . That is, even if  $P$  is non-crossing,  $\sigma(P)$  is not necessarily non-crossing.

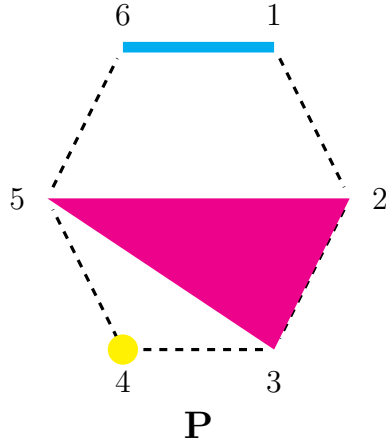
**Example** (Counter-example :  $\sigma = 413562, P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$ ).  
 $\sigma(P) = \{\{1, 3, 6\}, \{2, 4\}, \{5\}\}$

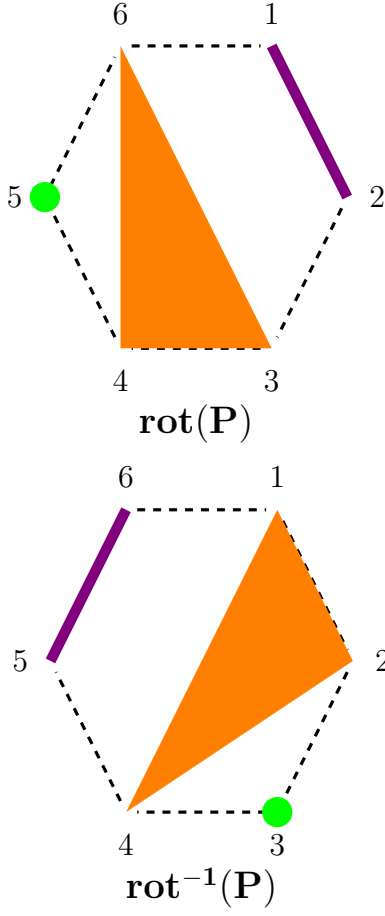


**Definition 9** (Rotation). We define the rotation operator  $rot$  of  $P \in \mathcal{NC}_n$  as  $rot(P) = (1\ 2\ 3\ \dots\ n)(P) = 23\dots n1(P)$ . Conversely, we define  $rot^{-1}$  of  $P$  as  $rot^{-1}(P) = (n\ n-1\ \dots\ 3\ 2\ 1)(P) = n12\dots n-1(P)$ .

**Example** ( $P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$ ).

- $rot(P) = \{\{1, 2\}, \{3, 4, 6\}, \{5\}\}$
- $rot^{-1}(P) = \{\{1, 2, 4\}, \{3\}, \{5, 6\}\}$





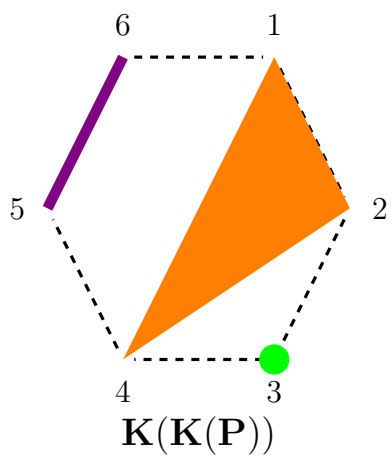
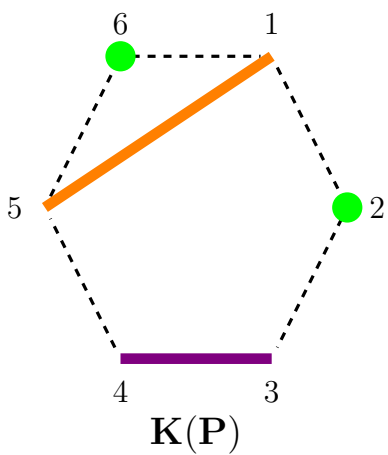
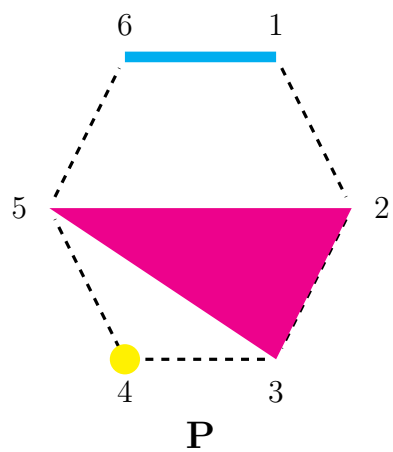
**Remark.**

- $\text{rot}(\text{rot}^{-1}(P)) = \text{rot}^{-1}(\text{rot}(P)) = P$
- $\text{rot}(P)$  and  $\text{rot}^{-1}(P)$  are always non-crossing partitions.
- If  $P \in \mathcal{NC}_n$ , then  $\text{rot}^n(P) = \text{rot}^{-n}(P) = P$ .

**Proposition.**  $K(K(P)) = \text{rot}^{-1}(P)$ .

**Example** ( $P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$ ).





### 1.3 Non-crossing 2-partitions

**Definition 10** (Non-crossing 2-partition). A non-crossing 2-partition of a totally ordered set  $E$  is a pair  $(P, \sigma)$  where :

- $P$  is a non-crossing partition of  $E$
- $\sigma$  is a permutation of the elements of  $E$
- For each sorted block  $B_i = \{b_1, \dots, b_k\} \in P$ , we have  $\sigma(b_i) < \dots < \sigma(b_k)$

We denote by  $\mathcal{NC}_n^2$  the set of non-crossing 2-partitions of  $[n]$ .

$$\mathcal{NC}^2 = \bigcup_{n>0} \mathcal{NC}_n^2$$

**Example** ( $\mathcal{NC}_6^2$ ).  $P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$   $\sigma = 413265$   
 $\rho = \{\{1, 3, 6\}, \{2\}, \{4, 5\}\}$

**Theorem 5.** Let  $nc_n^2$  be the cardinal of  $\mathcal{NC}_n^2$ . We have

$$nc_n^2 = (n+1)^{n-1}$$

**Example** ( $n = 1, 2, 3$ ).

- $n = 1$  :  $nc_1^2 = 1$   
 $\{\{1\}\} \quad 1 \quad \rho = P$
- $n = 2$  :  $nc_2^2 = 3$   
 $\{\{1\}, \{2\}\} \quad 12 \quad \rho = P$   
 $\{\{1\}, \{2\}\} \quad 21 \quad \rho = P$   
 $\{\{1, 2\}\} \quad 12 \quad \rho = P$
- $n = 3$  :  $nc_3^2 = 16$   
 $\{\{1\}, \{2\}, \{3\}\} \quad 123 \quad \rho = P$   
 $\{\{1\}, \{2\}, \{3\}\} \quad 132 \quad \rho = P$   
 $\{\{1\}, \{2\}, \{3\}\} \quad 213 \quad \rho = P$   
 $\{\{1\}, \{2\}, \{3\}\} \quad 231 \quad \rho = P$

$\{\{1\}, \{2\}, \{3\}\}$	312	$\rho = P$
$\{\{1\}, \{2\}, \{3\}\}$	321	$\rho = P$
$\{\{1, 2\}, \{3\}\}$	123	$\rho = P$
$\{\{1, 2\}, \{3\}\}$	132	$\rho = \{\{1, 3\}, \{2\}\}$
$\{\{1, 2\}, \{3\}\}$	231	$\rho = \{\{1\}, \{2, 3\}\}$
$\{\{1\}, \{2, 3\}\}$	123	$\rho = P$
$\{\{1\}, \{2, 3\}\}$	213	$\rho = \{\{1, 3\}, \{2\}\}$
$\{\{1\}, \{2, 3\}\}$	312	$\rho = \{\{1, 2\}, \{3\}\}$
$\{\{1, 3\}, \{2\}\}$	123	$\rho = P$
$\{\{1, 3\}, \{2\}\}$	132	$\rho = \{\{1, 2\}, \{3\}\}$
$\{\{1, 3\}, \{2\}\}$	213	$\rho = \{\{1\}, \{2, 3\}\}$
$\{\{1, 2, 3\}\}$	123	$\rho = P$

**Proposition.** *This means we can create a bijection between  $\mathcal{PF}_n$  and  $\mathcal{NC}_n^2$ .*

*Proof.*

- $\mathcal{PF}_n \rightarrow \mathcal{NC}_n^2$  : Let  $f = (a_1, \dots, a_n) \in \mathcal{PF}_n$  be our parking function. For  $i \in \{1, \dots, n\}$ , we define :

$l_i$  : the number of occurrences of  $i$  in  $f$ .

$im_i : \{j \mid a_j = i\}$

The corresponding non-crossing partition will have the following constraints :

For each  $i \in \{1, \dots, n\}$ , if  $l_i > 0$ , then there is a block  $B_{[i]}$  of length  $l_i$  with minimum element  $i$ .

$\sigma(B_{[i]}) = im_i$

There is a unique set partition  $P = \bigcup_i B_{[i]}$  of  $[n]$  and a unique per-

mutation  $\sigma$  respecting these conditions such that  $(P, \sigma) \in \mathcal{NC}_n^2$  : for each minimum  $i$  in *decreasing order*, add the  $n_i$  first free elements of  $[i+1, i+2, \dots, n, 1, \dots, i-1]$  to  $B_i$ .  $\sigma$  is then trivially obtained by the second constraint.

- $\mathcal{NC}_n^2 \rightarrow \mathcal{PF}_n$  : Let  $(P, \sigma)$  with  $P = \{B_1, \dots, B_l\}$  be our non-crossing 2-partition. For each block  $B_i = \{b_1, \dots, b_k\} \in P$  :

$$m_i = \min(B_i) = b_1$$

$$pos_i = \sigma(B_i)$$

For each  $j \in pos_i$ , we define  $a_j = m_i$

The corresponding parking function is  $(a_1, \dots, a_n)$ .

□

**Example** ( $n = 8$ ).

$$P = \{\{1, 2, 5\}, \{3, 4\}, \{6, 8\}, \{7\}\}$$

$$\sigma = 36187245$$

$$f = (3, 6, 1, 7, 6, 1, 1, 3)$$

### 1.3.1 The non-crossing 2-partitions poset

**Definition 11** ( $\succ^2$ ). We say that  $(P, \sigma)$  covers  $(Q, \tau)$ , written  $(P, \sigma) \succ^2 (Q, \tau)$ , if  $\exists B_i, B_j \in P$  such that

- $Q = P - \{B_i, B_j\} \cup \{B_i \cup B_j\}$
- $l \neq i, j, b \in B_l \rightarrow \tau(b) = \sigma(b)$
- Let  $B_i \cup B_j = \{b_1, \dots, b_k\}$  :  
 $\tau(B_i \cup B_j) = \sigma(B_i \cup B_j)$   
 $\tau(b_1) < \dots < \tau(b_k)$

**Example.**

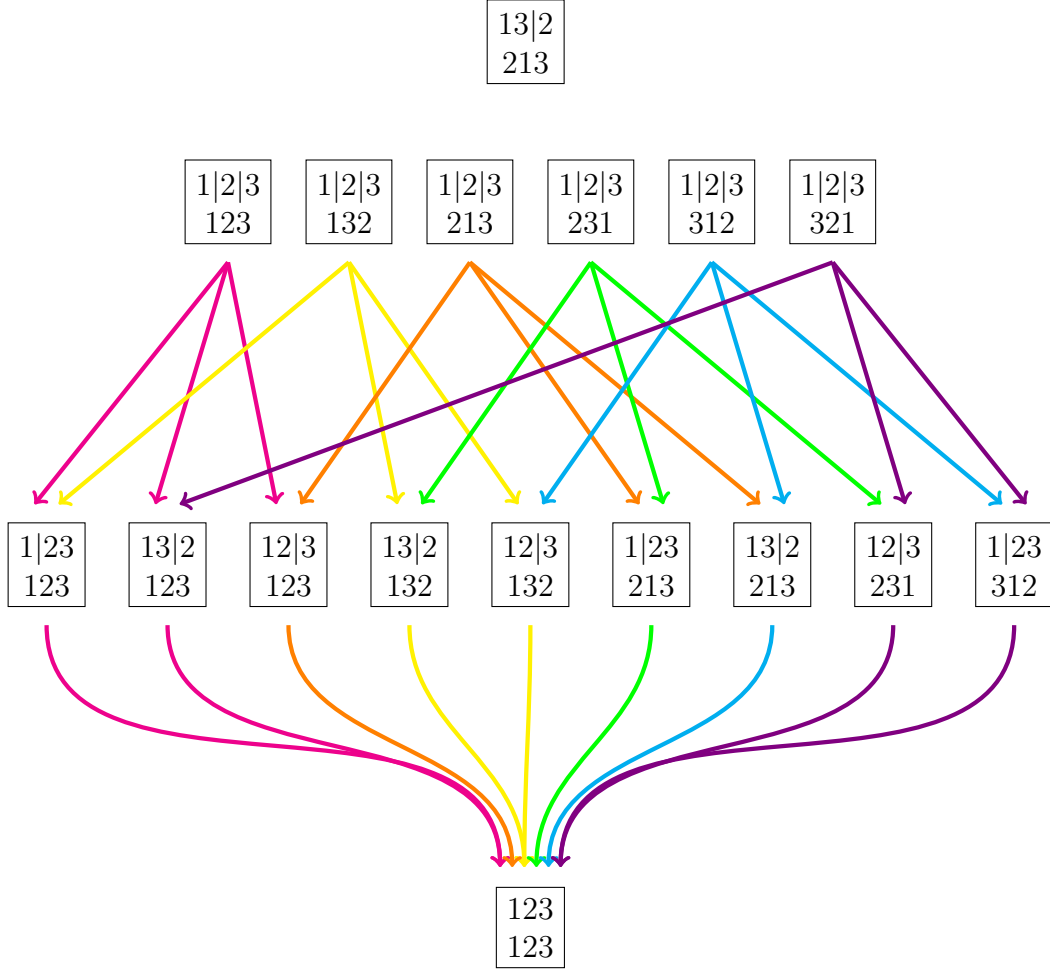
- $P = \{\{1, 6\}, \{2, 3\}, \{4\}, \{5\}\}$
- $\sigma = 236154$
- $Q = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$
- $\tau = 235164$
- $(P, \sigma) \succ^2 (Q, \tau)$
- $(P, \sigma) \not\succ^2 (Q, \sigma)$ , because  $\sigma(\{2, 3, 5\}) = \{3, 6, 5\}$  is not ordemagenta.

**Proposition.** This covering relation defines the poset of  $\mathcal{NC}_n^2$ .

**Remark.** The bottom element of this poset is  $(\{\{1, \dots, n\}\}, 12 \dots n)$ , and the top elements are  $(\{\{1\}, \dots, \{n\}\}, \sigma) \mid \sigma \in \mathfrak{S}_n$ .

**Example** (The poset of  $\mathcal{NC}_3^2$ ).

To shorten labels, we represent  $(\{\{1, 3\}, \{2\}\}, 213)$  by



There are  $4^2 = 16$  elements in this poset.

### 1.3.2 The parking functions poset

**Definition 12** (Rank). Given  $f = (a_1, \dots, a_n) \in \mathcal{PF}_n$ , let

$$b_i = \begin{cases} 1 & \text{if } \exists j \mid a_j = i \\ 0 & \text{otherwise} \end{cases}$$

We define the rank of  $f$ , noted  $rk(f)$ , as

$$\sum_{1 \leq i \leq n} b_i$$

**Example.**

$$\begin{aligned} rk((1, 5, 4, 2, 3, 3, 1)) &= 5 \\ rk((4, 7, 1, 1, 3, 2, 2, 8)) &= 6 \end{aligned}$$

**Definition 13** ( $\succ_{pf}$ ). Since  $\mathcal{PF}_n$  and  $\mathcal{NC}_n^2$  are in bijection, we can define a covering relation  $\succ_{pf}$  for  $\mathcal{PF}_n$  as follows :  
 $f \in \mathcal{PF}_n \succ_{pf} g \in \mathcal{PF}_n$  if and only if :

- $(P, \sigma)$  is the non-crossing 2-partition associated to  $f$
- $(Q, \tau)$  is the non-crossing 2-partition associated to  $g$
- $(P, \sigma) \succ^2 (Q, \tau)$

**Example.**

- $P = \{\{1, 6\}, \{2, 3\}, \{4\}, \{5\}\}$
- $\sigma = 236154$
- $Q = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$
- $\tau = 235164$
- $f = (4, 1, 2, 1, 5, 2) \succ_{pf} g = (4, 1, 2, 1, 2, 2)$

**Remark.** If  $f \succ_{pf} g$ , then  $rk(f) = rk(g) + 1$ , and there exists  $i$  and  $j$  such that :

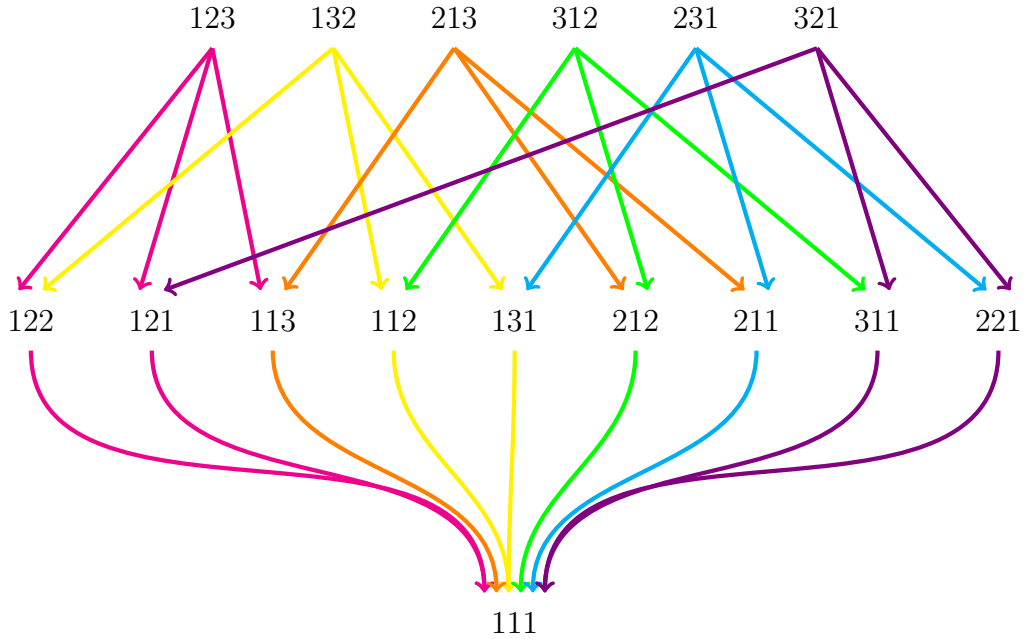
- $i < j$
- There is at least 1 occurrence of  $i$  in  $f$
- There is at least 1 occurrence of  $j$  in  $f$

$$b_k = \begin{cases} i & \text{if } a_k = j \\ a_k & \text{otherwise} \end{cases}$$

**Proposition.** *This covering relation defines the poset of  $\mathcal{PF}_n$ .*

**Remark.** *The bottom element of this poset is  $(\underbrace{1, \dots, 1}_n)$ , and the top elements are the permutations of  $\{1, \dots, n\}$ .*

**Example** (The poset of  $\mathcal{PF}_3$ ).



## 1.4 A direct poset linked to Dyck paths

### 1.4.1 Dyck Paths

**Notation.** *We denote the number of occurrences of a symbol  $s$  in a word  $w$  by  $|w|_s$ .*

**Definition 14** (Dyck path). *A Dyck word is a word  $w \in \{0, 1\}^*$  such that :*

- *for each suffix  $w'$  of  $w$ ,  $|w'|_1 \geq |w'|_0$ .*
- $|w|_0 = |w|_1$ .

A Dyck word of length  $2n$  can be represented as a path from  $(0,0)$  to  $(n,n)$  that stays over  $x = y$ , called a Dyck path :

- Each 1 corresponds to a North step  $\uparrow$ .
- Each 0 corresponds to an East step  $\rightarrow$ .

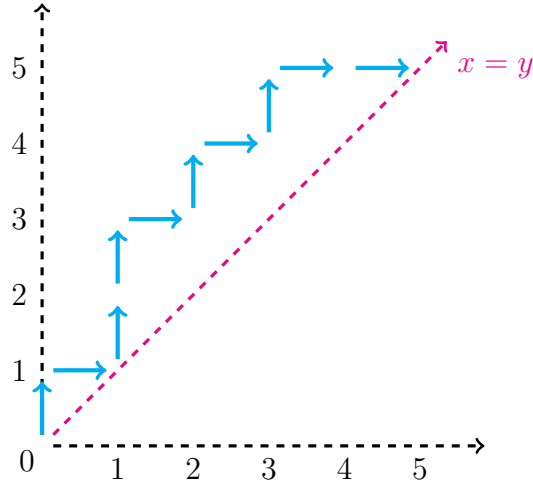
We denote by  $\mathcal{D}_n$  the set of Dyck words of length  $2n$ .

**Example** ( $n = 5$ ).

$w_1 = 1011000110$  is not a Dyck word, because  $|1011000|_0 > |1011000|_1$ .

$w_2 = 1011010101$  is not a Dyck word, because  $|w_2|_0 \neq |w_2|_1$ .

$w_3 = 1011010100$  is a Dyck word :



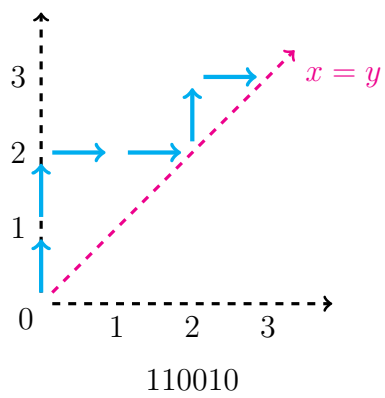
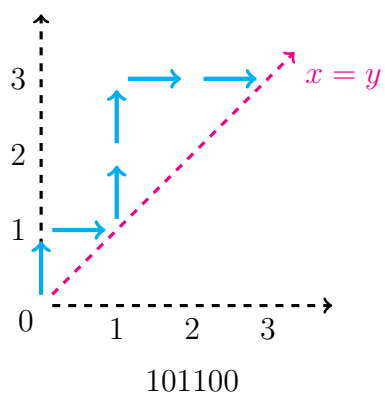
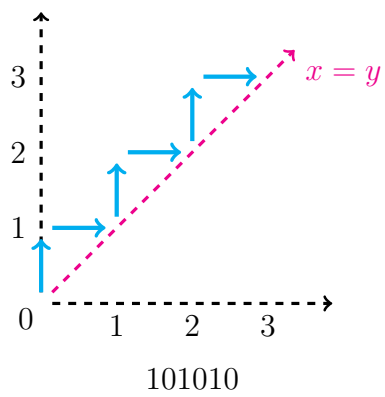
**Theorem 6.** Let  $d_n$  be the cardinal of  $\mathcal{D}_n$ . We have

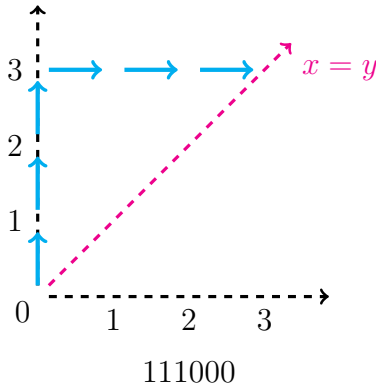
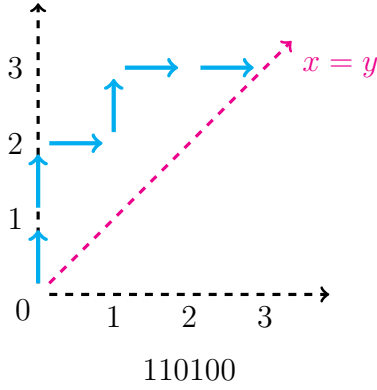
$$d_n = \frac{1}{n+1} \binom{2n}{n}$$

which is the  $n^{\text{th}}$  Catalan number.

**Example** ( $n = 3$ ).  $d_n = 5$ .







**Proposition.** *This means we can create a bijection between  $\mathcal{PF}'_n$  and  $\mathcal{D}_n$ .*

*Proof.*

- $\mathcal{PF}'_n \rightarrow \mathcal{D}_n$  : Let  $f = (a_1, \dots, a_n) \in \mathcal{PF}'_n$  be our primitive parking function. For  $i \in \{1, \dots, n\}$ , we define  $l_i$  the number of occurrences of  $i$  in  $f$ .

The corresponding Dyck word will be  $\underbrace{1 \dots 1}_l 0 \underbrace{1 \dots 1}_l 0 \dots \underbrace{1 \dots 1}_l 0$ .

- $\mathcal{D}_n \rightarrow \mathcal{PF}'_n$  : Let  $w$  be our Dyck word, and consider its path representation. We define  $s_i$  to be the distance between the segment from  $(0, i-1)$  to  $(0, i)$  and the  $i^{th}$  North step. Then, let  $a_i = s_i + 1$ . The corresponding primitive parking function is  $(a_1, \dots, a_n)$ .

□

**Example** ( $n = 6, \mathcal{PF}'_n \rightarrow \mathcal{D}_n$ ).

- $f = (1, 1, 2, 4, 5, 5)$

$$l_1 = 2$$

$$l_2 = 1$$

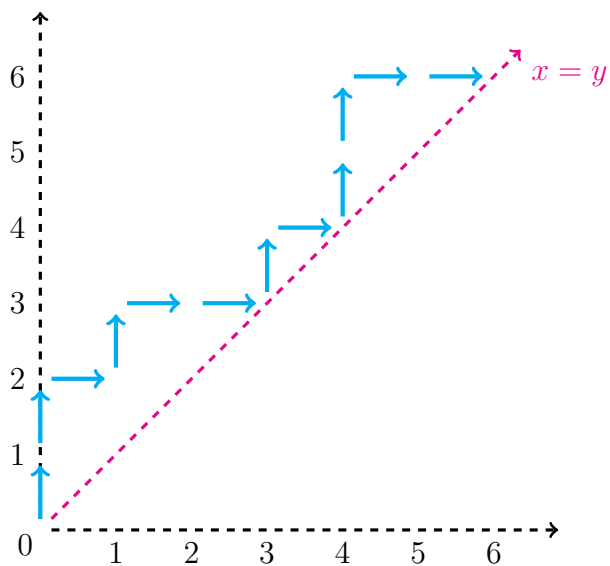
$$l_3 = 0$$

$$l_4 = 1$$

$$l_5 = 2$$

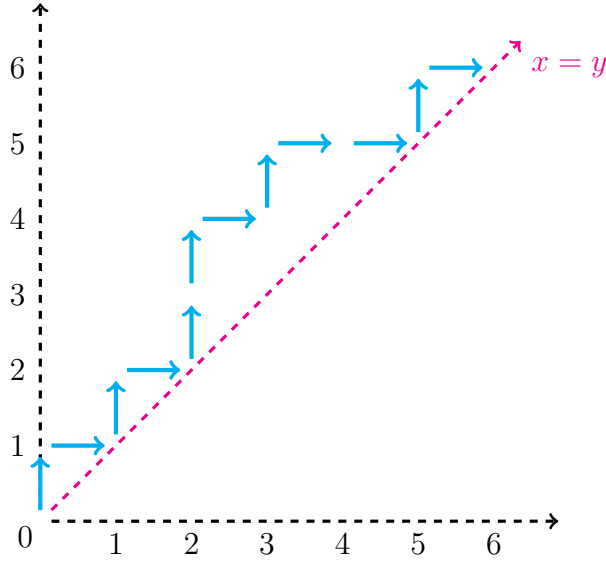
$$l_6 = 0$$

- $w = (110100101100)$



**Example**  $(n = 6, \mathcal{D}_n \rightarrow \mathcal{PF}'_n)$ .

- $w = 101011010010$



- *Distances :*

$s_1 = 0$	$a_1 = 1$
$s_2 = 1$	$a_2 = 2$
$s_3 = 2$	$a_3 = 3$
$s_4 = 2$	$a_4 = 3$
$s_5 = 3$	$a_5 = 4$
$s_6 = 5$	$a_6 = 6$

- $f = (1, 2, 3, 3, 4, 6)$

### 1.4.2 Labeled Dyck Paths

**Definition 15** (Labeled Dyck Path). A labeled Dyck word is a word  $w \in \{0, \dots, n\}^*$  such that :

- for each suffix  $w'$  of  $w$ ,  $|w'|_{\neq 0} \geq |w'|_0$ .
- $|w|_0 = |w|_{\neq 0}$ .
- for each  $i \in \{1, \dots, n\}$ ,  $w$  has exactly one occurrence of  $i$ .

- if  $w_i \neq 0$  and  $w_{i+1} \neq 0$ , then  $w_i < w_{i+1}$ . That is, consecutive North steps have increasing labels.

A labeled Dyck word of length  $2n$  can be represented as a path from  $(0,0)$  to  $(n,n)$ , where each North step is associated to a label :

- Each  $i \neq 0$  corresponds to a North step  $\uparrow$  labeled  $i$ .
- Each  $0$  corresponds to an East step  $\rightarrow$ .

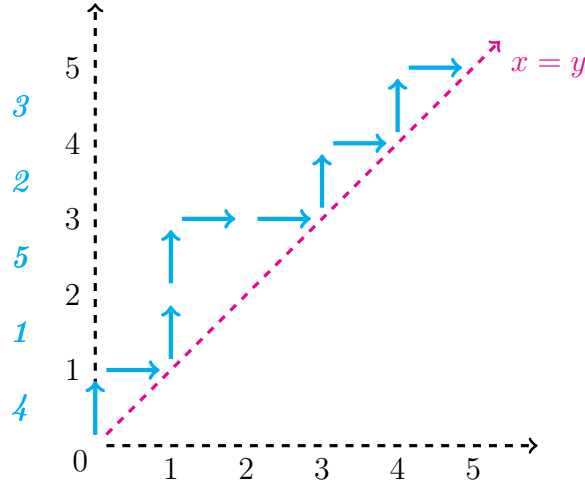
Those paths are called labeled Dyck paths.

We denote by  $\mathcal{LD}_n$  the set of labeled Dyck words of length  $2n$ .

**Example** ( $n = 5$ ).

$w_1 = 4051002030$  is not a labeled Dyck word, because  $5 > 1$ .

$w_2 = 4015002030$  is a labeled Dyck word :



**Theorem 7.** Let  $ld_n$  be the cardinal of  $\mathcal{LD}_n$ . We have

$$ld_n = (n+1)^{n-1}$$

.

**Example** ( $n = 3$ ).  $ld_n = 4^2 = 16$

- Word of shape  $XXX000$  :  
123000
- Words of shape  $XX0X00$  :  
120300                      130200                      230100
- Words of shape  $XX00X0$  :  
120030                      130020                      230010
- Words of shape  $X0XX00$  :  
102300                      201300                      301200
- Words of shape  $X0X0X0$  :  
102030                      103020                      201030  
203010                      301020                      302010

**Proposition.** *This means we can create a bijection between  $\mathcal{PF}_n$  and  $\mathcal{LD}_n$ .*

*Proof.*

- $\mathcal{PF}_n \rightarrow \mathcal{LD}_n$  : Let  $f = (a_1, \dots, a_n) \in \mathcal{PF}_n$  be our parking function. For  $i \in \{1, \dots, n\}$ , we define  $im_i : \{j \mid a_j = i\}$ . We then define  $im_{i,1}, \dots, im_{i,k_i}$  to be the elements of  $im_i$  in increasing order. The corresponding labeled Dyck word will be  

$$\underbrace{im_{1,1} \cdots im_{1,k_1}}_{im_1} 0 \underbrace{im_{2,1} \cdots im_{2,k_2}}_{im_2} 0 \cdots \underbrace{im_{n,1} \cdots im_{n,k_n}}_{im_n} 0.$$
- $\mathcal{LD}_n \rightarrow \mathcal{PF}_n$  : Let  $w$  be our labeled Dyck word, and consider its path representation. We define  $s_i$  to be the distance between the segment from  $(0, i-1)$  to  $(0, i)$  and the  $i^{th}$  North step. Then, let  $label(i)$  be the label of the  $i^{th}$  North step, and  $dist_i = \{label(j) \mid s_j = i\}$  be the set of the labels of all North steps at distance  $i$ . Then, if  $j \in dist_i$ , let  $a_j = i+1$ . The corresponding parking function is  $(a_1, \dots, a_n)$ .

□

**Example**  $(n = 6, \mathcal{PF}_n \rightarrow \mathcal{LD}_n)$ .

- $f = (5, 2, 1, 4, 5, 1)$

$$im_1 = \{3, 6\}$$

$$im_2 = \{2\}$$

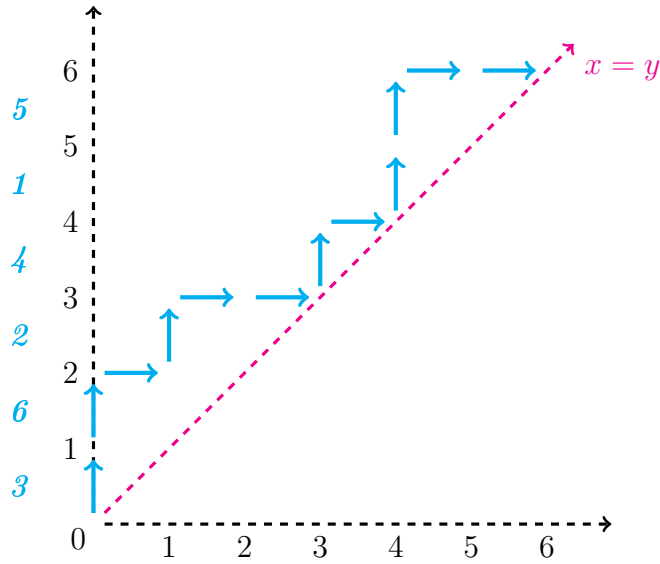
$$im_3 = \emptyset$$

$$im_4 = \{4\}$$

$$im_5 = \{1, 5\}$$

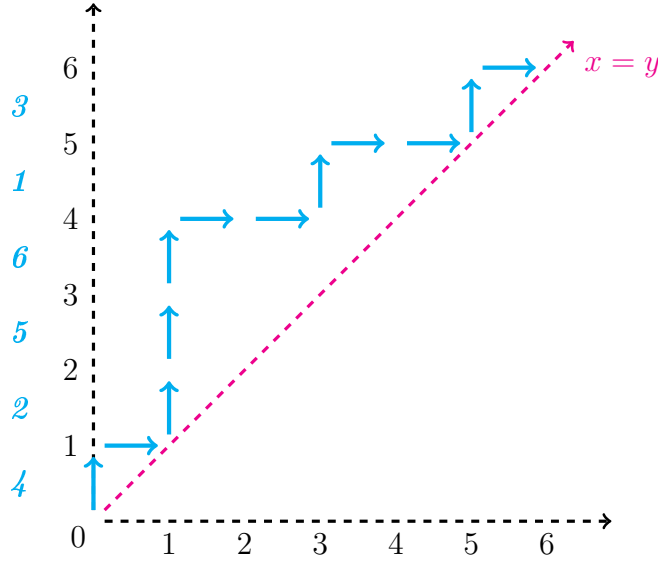
$$im_6 = \emptyset$$

- $w = 360200401500$



**Example**  $(n = 6, \mathcal{LD}_n \rightarrow \mathcal{PF}_n)$ .

- $w = 402560010030$



- *Distances :*

$$\begin{array}{lll} s_1 = 0 & s_2 = 1 & s_3 = 1 \\ s_4 = 1 & s_5 = 3 & s_6 = 5 \end{array}$$

- *Labels :*

$$\begin{array}{lll} dist_0 = \{4\} & dist_1 = \{2, 5, 6\} & dist_2 = \emptyset \\ dist_3 = \{1\} & dist_4 = \emptyset & dist_5 = \{3\} \end{array}$$

- $f = (4, 2, 6, 1, 2, 2)$

**Remark.** The primitive parking functions are exactly the parking functions corresponding to labeled Dyck paths where the  $i^{\text{th}}$  North step is labeled  $i$ .

### 1.4.3 Dyck - Parking Posets

#### Primitive Dyck - Parking Posets

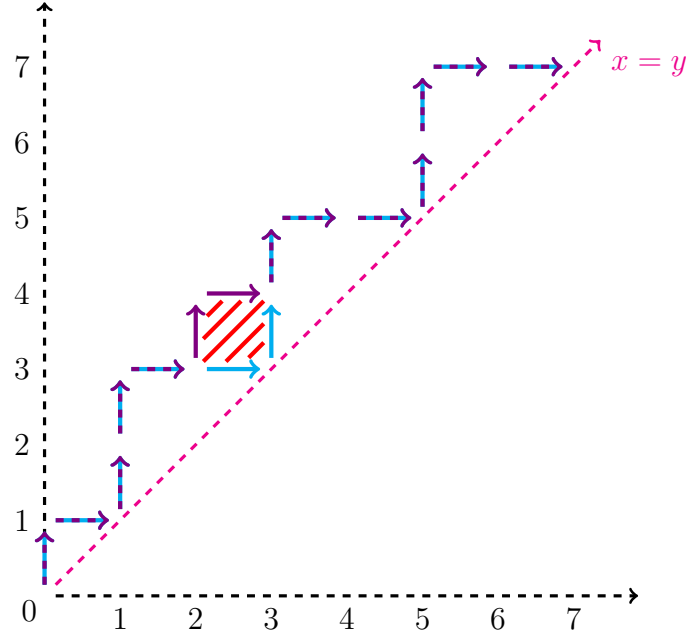
**Definition 16** ( $\succ_d$ ). For  $w$  and  $w'$  two Dyck words, we say that  $w$  covers  $w'$ , written  $w \succ_d w'$ , if  $\exists w_1, w_2$  such that :

- $w = w_1 01 w_2$
- $w' = w_1 10 w_2$



**Example** ( $n = 7$ ).  $10110011001100 \succ_d 10110101001100$

- $w_1 = 10110$
- $w_2 = 1001100$

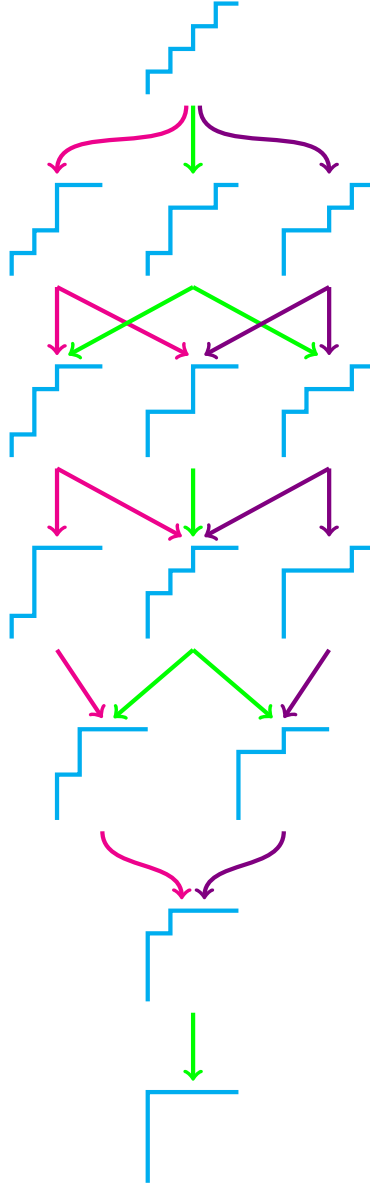


■  $10110011001100$ 
■  $10110101001100$ 
■ *difference*

**Remark.** If  $w_1 \succ_d w_2$ , then the path corresponding to  $w_2$  is over the path corresponding to  $w_1$ , and the difference between the two paths is a square of size 1 by 1.

**Proposition.** This covering relation defines a poset for  $\mathcal{D}_n$ .

**Example** (The poset of  $\mathcal{D}_4$ ).



There are  $\frac{1}{5} \binom{8}{4} = \frac{70}{5} = 14$  elements in this poset.

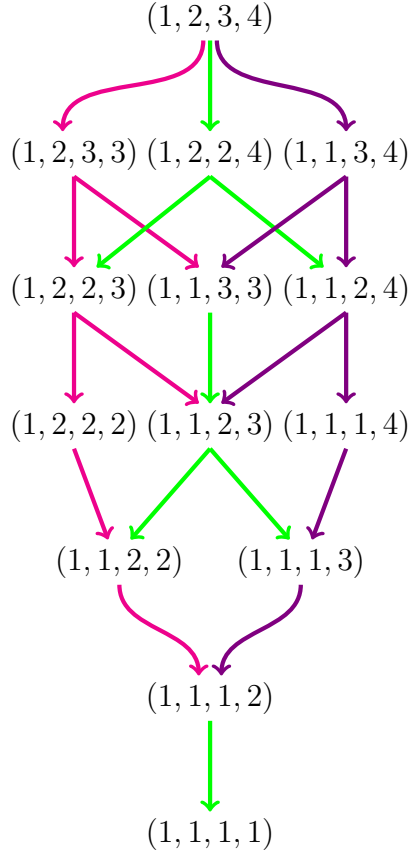
**Definition 17** ( $\succ'$ ). For  $f$  and  $g$  two primitive parking functions, we say that  $f$  covers  $g$ , written  $f \succ' g$ , if  $\exists i$  such that :

- $f = (a_1, \dots, a_{i-1}, a_i, \quad a_{i+1}, \dots, a_n)$
- $g = (a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_n)$

**Example** ( $n = 6$ ).  $(1, 1, 2, 3, 4, 5) \succ' (1, 1, 2, 3, 3, 5)$

**Proposition.** *This covering relation defines a poset for  $\mathcal{PF}'_n$ .*

**Example** (The poset of  $\mathcal{PF}'_4$ ).



There are  $\frac{1}{5} \binom{8}{4} = \frac{70}{5} = 14$  elements in this poset.

**Remark.** *The two posets are isomorphic, and one can be obtained by applying the aforementioned bijection to the other.*

**Theorem 8.** *The number of intervals in those posets is equal to the  $n + 1^{\text{th}}$  term of the integer sequence defined by <https://oeis.org/A005700>. The first terms of this sequence are 1, 1, 3, 14, 84, 594, 4719, 40898, 379236, 3711916, ...*

Alec Mihailovs proved this sequence to be equal to

$$\frac{6(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!}$$

.

## Classical Dyck - Parking Posets

**Definition 18** ( $\succ_{ld}$ ). For  $w$  and  $w'$  two labeled Dyck words, we say that  $w$  covers  $w'$ , written  $w \succ_{ld} w'$ , if either :

- $\exists w_2, x, x', y, z, z'$  such that :

$x = x_1x_2 \cdots x_n$  has all its digits  $> 0$

$z = z_1z_2 \cdots z_m$  has all its digits  $> 0$

$y$  is in  $z$

$x' = x$  where  $y$  is correctly inserted regarding the order condition

$z' = z$  where  $y$  is removed

$w = x0zw_2$

$w' = x'0z'w_2$

- $\exists w_1, w_2, x, x', y, z, z'$  such that :

$x = x_1x_2 \cdots x_n$  has all its digits  $> 0$

$z = z_1z_2 \cdots z_m$  has all its digits  $> 0$

$y$  is in  $z$

$x' = x$  where  $y$  is correctly inserted regarding the order condition

$z' = z$  where  $y$  is removed

$w = w_10x0zw_2$

$w' = w_10x'0z'w_2$

- or  $\exists w_1, w_2, y, z$  such that :

$z = z_1z_2 \cdots z_m$  has all its digits  $> 0$

$y$  is in  $z$

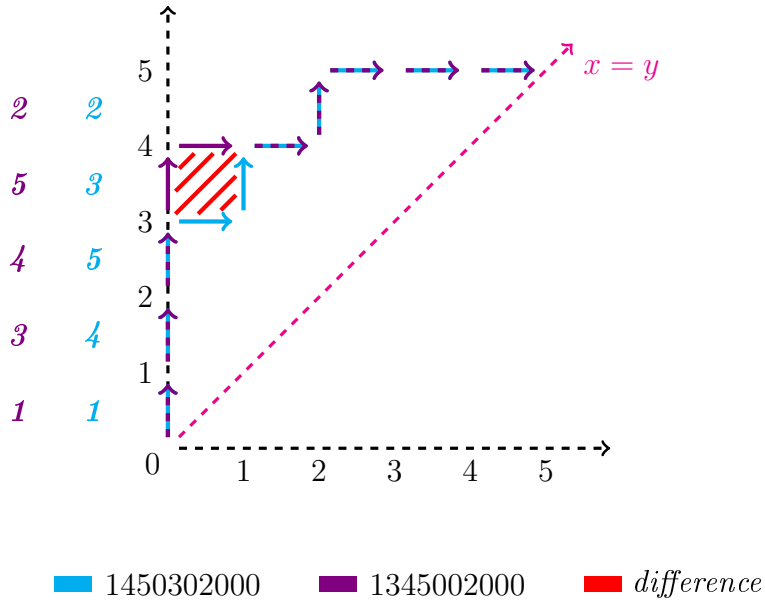
$z' = z$  where  $y$  is removed

$w = w_100zw_2$

$w' = w_10y0z'w_2$

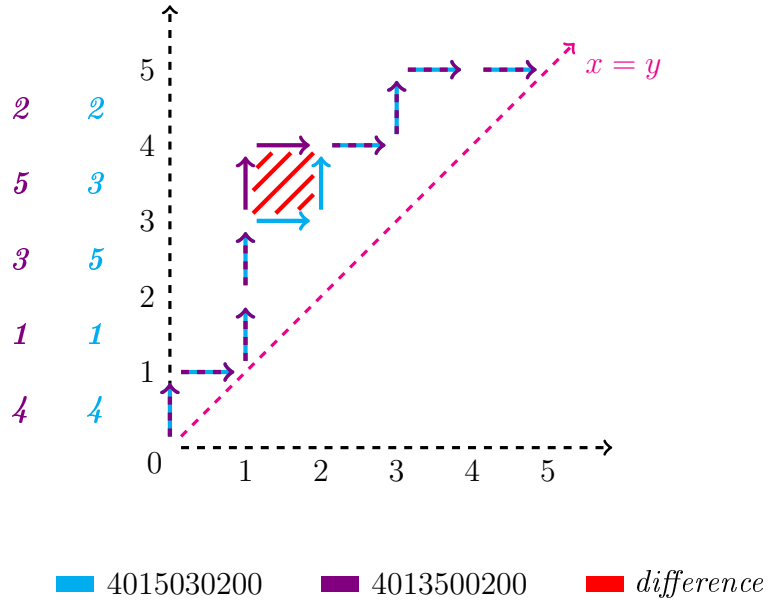
**Example** ( $n = 5$ , first case).  $1450302000 \succ_{ld} 1345002000$

- $w_2 = 02000$
- $x = 145$
- $x' = 1345$
- $y = 3$



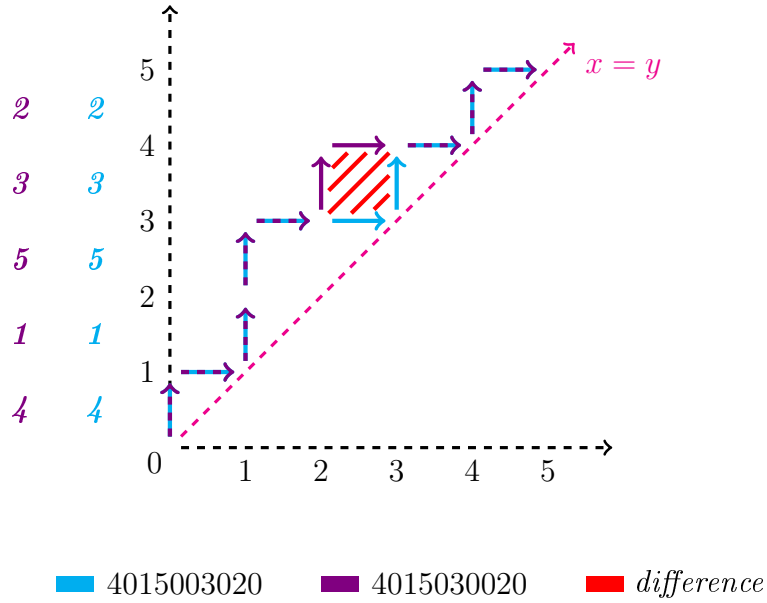
**Example** ( $n = 5$ , second case).  $4015030200 \succ_{ld} 4013500200$

- $w_1 = 4$
- $w_2 = 0200$
- $x = 15$
- $x' = 135$
- $y = 3$



**Example** ( $n = 5$ , third case).  $4015003020 \succ_{ld} 4015030020$

- $w_1 = 4015$
- $w_2 = 020$
- $y = 3$

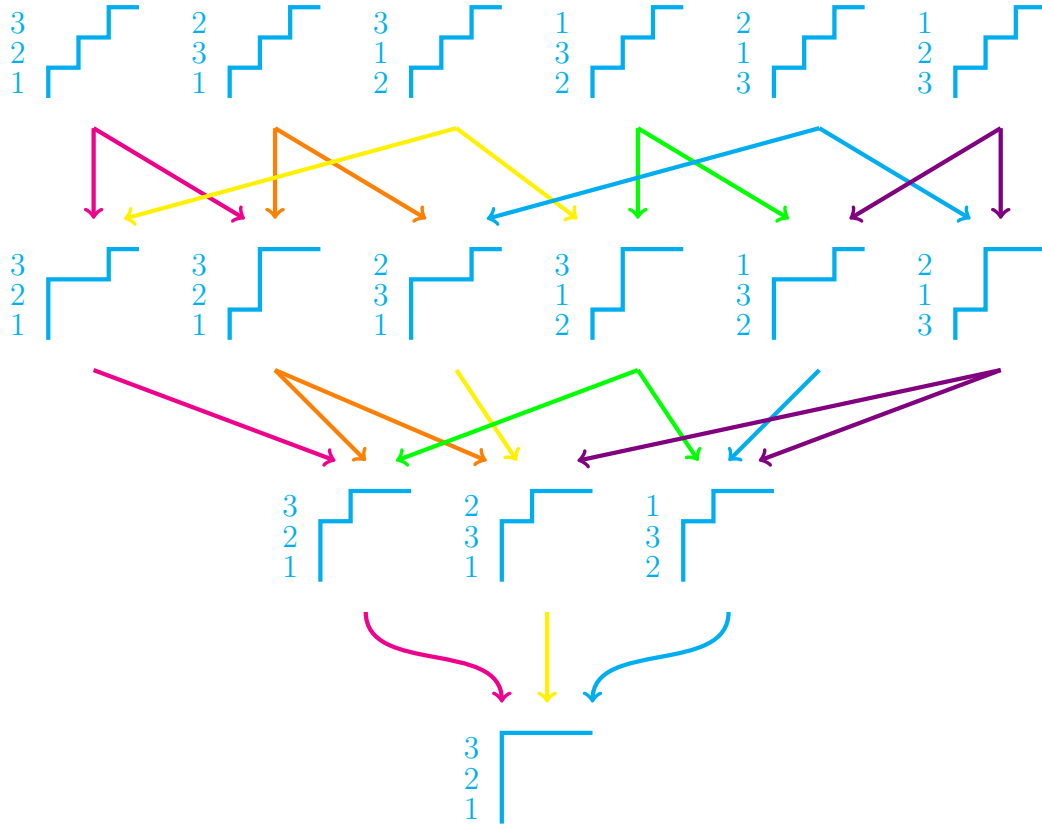


**Remark.** If  $w_1 \succ_{ld} w_2$ , then the path corresponding to  $w_2$  is over the path corresponding to  $w_1$ , and the difference between the two paths is a square of size 1 by 1.

Furthermore, the labeling can be seen as follows : if one has to merge two sequences of North steps of  $w_1$  to make  $w_2$ , then the merging will be made by ordering the corresponding labels.

**Proposition.** This covering relation defines a poset for  $\mathcal{LD}_n$ .

**Example** (The poset of  $\mathcal{LD}_3$ ).



There are  $4^2 = 16$  elements in this poset.

**Definition 19** ( $\succ$ ). For  $f$  and  $g$  two parking functions, we say that  $f$  covers  $g$ , written  $f \succ g$ , if  $\exists i$  such that :

- $f = (a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n)$
- $g = (a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_n)$

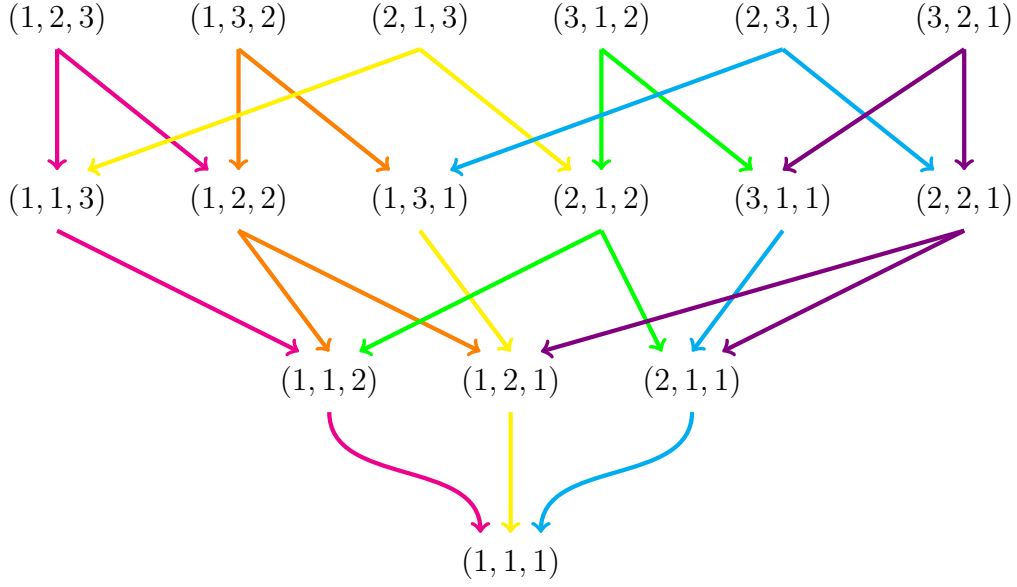
That is, the same relation as for primitive parking functions.

**Example** ( $n = 6$ ).  $(2, 1, 5, 3, 1, 4) \succ (2, 1, 5, 2, 1, 4)$

**Proposition.** This covering relation defines a poset for  $\mathcal{PF}_n$ .

**Example** (The poset of  $\mathcal{PF}_3$ ).





*There are  $4^2 = 16$  elements in this poset.*

**Remark.** *The two posets are isomorphic, and one can be obtained by applying the aforementioned bijection to the other.*

**Theorem 9.** *The number of intervals in those posets is equal to the  $n + 1^{\text{th}}$  term of the integer sequence defined by <https://oeis.org/A196304>.*

*The first terms of this sequence are 1, 1, 5, 64, 1587, 65421, 4071178, 357962760, 4237910716, ....*

# Chapter 2

## The rational case

For the whole chapter, we will consider 2 *coprime* integers  $a$  and  $b$  (meaning  $a$  and  $b$  have 1 as their greatest common divisor).

### 2.1 Rational Parking Functions

**Definition 20** ( $a, b$  - Parking Function). An  $a, b$  - parking function is a sequence  $(a_1, a_2, \dots, a_n)$  such that :

- $n = a$
- its non-decreasing reordering  $(b_1, b_2, \dots, b_n)$  has  $b_i \leq \frac{b}{a}(i - 1) + 1$  for all  $i$ .

We denote by  $\mathcal{PF}_{a,b}$  the set of  $a, b$  - parking functions.

**Example.**

- *Ex. 1* :  $a > b$

$$a = 7$$

$$b = 3$$

Limits of the non-decreasing reordering of any  $f \in \mathcal{PF}_{7,3}$  :

$$[1, 1\frac{3}{7}, 1\frac{6}{7}, 2\frac{2}{7}, 2\frac{5}{7}, 3\frac{1}{7}, 3\frac{4}{7}]$$

$$f_1 = (2, 1, 1, 3, 2, 3, 1) \in \mathcal{PF}_{7,3}$$

$$f_2 = (2, 1, 2, 3, 2, 3, 1) \notin \mathcal{PF}_{7,3}, \text{ even though } f_2 \in \mathcal{PF}_7$$

- *Ex. 2* :  $a < b$

$$a = 5$$

$$b = 7$$

*Limits of the non-decreasing reordering of any  $f \in \mathcal{PF}_{5,7}$  :*

$$[1, 2\frac{2}{5}, 3\frac{4}{5}, 5\frac{1}{5}, 6\frac{3}{5}]$$

$$f_3 = (6, 3, 5, 1, 2) \in \mathcal{PF}_{5,7}, \text{ even though } f_3 \notin \mathcal{PF}_5$$

$$f_4 = (6, 3, 5, 1, 3) \notin \mathcal{PF}_{5,7}$$

**Theorem 10.** *Let  $pf_{a,b}$  be the cardinal of  $\mathcal{PF}_{a,b}$ . We have*

$$pf_{a,b} = b^{a-1}$$

**Example** ( $a = 3, b = 5$ ).

- $pf_{a,b} = 25$
- *Limits* :  $[1, 2\frac{2}{3}, 4\frac{1}{3}]$

(1, 1, 1)	(1, 1, 2)	(1, 1, 3)	(1, 1, 4)	(1, 2, 1)	(1, 2, 2)	(1, 2, 3)
(1, 2, 4)	(1, 3, 1)	(1, 3, 2)	(1, 4, 1)	(1, 4, 2)	(2, 1, 1)	(2, 1, 2)
(2, 1, 3)	(2, 1, 4)	(2, 2, 1)	(2, 3, 1)	(2, 4, 1)	(3, 1, 1)	(3, 1, 2)
(3, 2, 1)	(4, 1, 1)	(4, 1, 2)	(4, 2, 1)			

**Remark.**  $\mathcal{PF}_{n,n+1} = \mathcal{PF}_n$ . In fact, we do have  $b^{a-1} = (n+1)^{n-1}$ .

### 2.1.1 Rational primitive parking functions

**Definition 21** (Rational Primitive). *A rational parking function  $f$  is said primitive if it is already in non-decreasing order.*

*We denote by  $\mathcal{PF}'_{a,b}$  the set of primitive  $a, b$  - parking functions.*

**Example** ( $a = 4, b = 3$ ). *Limits* :  $[1, 1\frac{3}{4}, 2\frac{1}{2}, 3\frac{1}{4}]$

$$f_1 = (1, 1, 2, 2) \in \mathcal{PF}'_{4,3}$$

$$f_2 = (1, 1, 2, 1) \notin \mathcal{PF}'_{4,3}, \text{ even though } f_2 \in \mathcal{PF}_{4,3}.$$

**Theorem 11.** *Let  $pf'_{a,b}$  be the cardinal of  $\mathcal{PF}'_{a,b}$ . We have*

$$pf'_{a,b} = \frac{1}{a+b} \binom{a+b}{b}$$

*which is the rational Catalan number  $Cat(a, b)$ .*

**Example** ( $a = 3, b = 5$ ).

•  $pf'_{a,b} = 7$  • *Limits* :  $[1, 2\frac{2}{3}, 4\frac{1}{3}]$

(1, 1, 1)    (1, 1, 2)    (1, 1, 3)    (1, 1, 4)    (1, 2, 2)    (1, 2, 3)    (1, 2, 4)

**Remark.**  $\mathcal{PF}'_{n,n+1} = \mathcal{PF}'_n$ . In fact, we do have

$$\begin{aligned} \frac{1}{n+n+1} \binom{n+n+1}{n+1} &= \frac{1}{2n+1} \binom{2n+1}{n+1} = \frac{1}{2n+1} \frac{(2n+1)!}{n!(n+1)!} \\ &= \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1} \frac{(2n)!}{n!n!} = \frac{1}{n+1} \binom{2n}{n} \end{aligned}$$

## 2.2 Rational Non-crossing Partitions

**Definition 22** ( $a, b$  - Non-crossing Partition). An  $a, b$  - non-crossing partition is *TODO*

We denote by  $\mathcal{NC}_{a,b}$  the set of  $a, b$  - non-crossing partitions.

**Example.** An *abncp*

**Theorem 12.** Let  $nc_{a,b}$  be the cardinal of  $\mathcal{NC}_{a,b}$ . We have

$$nc_{a,b} = \frac{1}{a+b} \binom{a+b}{a} = \frac{(a+b-1)!}{a!b!}$$

, which is the rational Catalan number.

**Example.** all *abncp* for some  $a, b$

**Proposition.** This means we can create a bijection between  $\mathcal{PF}'_{a,b}$  and  $\mathcal{NC}_{a,b}$ .

*Proof.*

- $\mathcal{NC}_{a,b} \rightarrow \mathcal{PF}'_{a,b}$  :
- $\mathcal{PF}'_{a,b} \rightarrow \mathcal{NC}_{a,b}$  :

□

**Definition 23** (a, b - Non-crossing 2-Partition). An a, b - Non-crossing 2-partition is *TODO*

We denote by  $\mathcal{NC}_{a,b}^2$  the set of a, b - non-crossing 2-partitions.

**Example.** some *ncab2*

**Theorem 13.** Let  $nc_{a,b}^2$  be the cardinal of  $\mathcal{NC}_{a,b}^2$ . We have

$$nc_{a,b}^2 = b^{a-1}$$

.

**Example.** all *ncab2* for some a b

**Proposition.** *bijection*

*Proof.* bijection proof

□

## 2.3 A direct poset linked to Rational Dyck paths

### 2.3.1 Rational Dyck Paths

**Definition 24** (a, b - Dyck path). An a, b - Dyck word is a word  $w \in \{0, 1\}^*$  such that :

- for each suffix  $w'$  of  $w$ ,

$$|w'|_1 \geq \frac{a}{b} |w'|_0$$

.

- $|w|_0 = b$ .

- $|w|_1 = a$ .

An a, b - Dyck word can be represented as a path from  $(0, 0)$  to  $(b, a)$  that stays over  $y = \frac{a}{b}x$ , called an a, b - Dyck path :

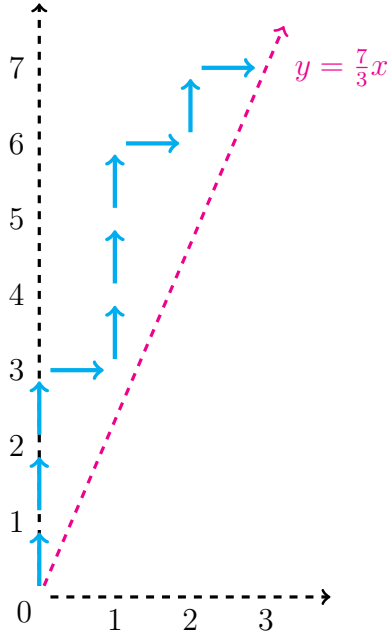
- Each 1 corresponds to a North step  $\uparrow$ .
- Each 0 corresponds to an East step  $\rightarrow$ .

We denote by  $\mathcal{R}_{a,b}$  the set of a, b - Dyck words.

**Example** ( $a > b : a = 7, b = 3$ ).

$w_1 = 1110011110$  is not a 7, 3 - Dyck word, because  $|11100|_1 = 3$   
 $< \frac{7}{3}|11100|_0 = \frac{14}{3} = 4\frac{1}{3}$ .

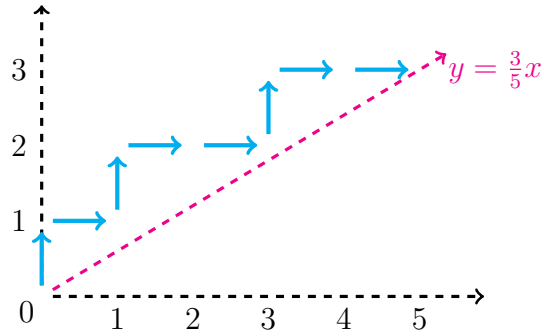
$w_2 = 1110111010$  is a 7, 3 - Dyck word :



**Example** ( $a < b : a = 3, b = 5$ ).

$w_1 = 10100010$  is not a 3, 5 - Dyck word, because  $|101000|_1 = 2$   
 $< \frac{3}{5}|101000|_0 = \frac{12}{5} = 2\frac{2}{5}$ .

$w_2 = 10100100$  is a 3, 5 - Dyck word :

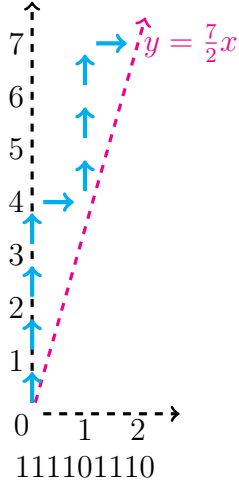


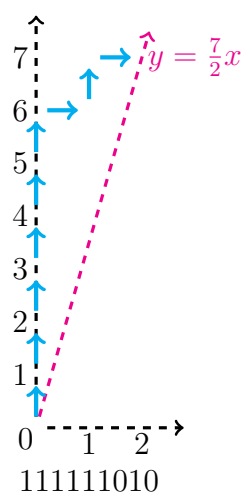
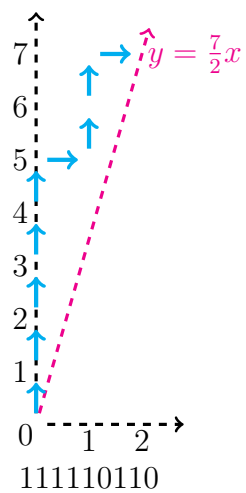
**Theorem 14.** Let  $r_{a,b}$  be the cardinal of  $\mathcal{R}_{a,b}$ . We have

$$r_{a,b} = \frac{1}{a+b} \binom{a+b}{a} = \frac{(a+b-1)!}{a!b!}$$

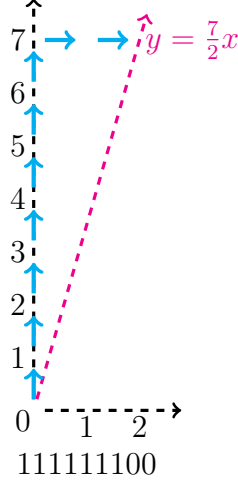
, which is again the rational Catalan number.

**Example** ( $a = 7, b = 2$ ).  $r_n = 4$ .









**Proposition.** *This means we can create a bijection between  $\mathcal{PF}'_{a,b}$  and  $\mathcal{R}_{a,b}$ .*

*Proof.*

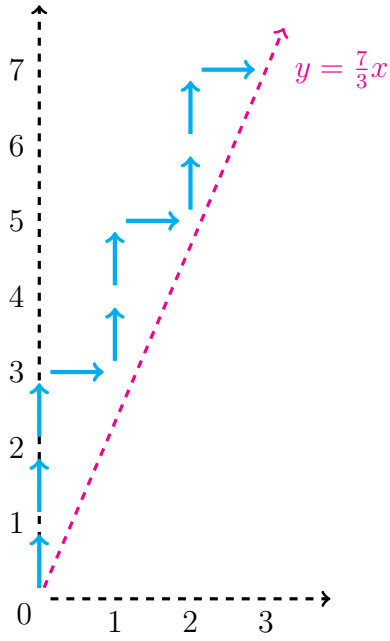
- $\mathcal{PF}'_{a,b} \rightarrow \mathcal{R}_{a,b}$  : Let  $f = (a_1, \dots, a_n) \in \mathcal{PF}'_{a,b}$  be our rational primitive parking function. For  $i \in \{1, \dots, b\}$ , we define  $l_i$  the number of occurrences of  $i$  in  $f$ .  
The corresponding rational Dyck word will be  $\underbrace{1 \dots 1}_l 0 \underbrace{1 \dots 1}_l 0 \dots \underbrace{1 \dots 1}_l 0$ .
- $\mathcal{R}_{a,b} \rightarrow \mathcal{PF}'_{a,b}$  : Let  $w$  be our rational Dyck word, and consider its path representation. We define  $s_i$  to be the distance between the segment from  $(0, i-1)$  to  $(0, i)$  and the  $i^{\text{th}}$  North step. Then, let  $a_i = s_i + 1$ .  
The corresponding rational primitive parking function is  $(a_1, \dots, a_a)$ .

□

**Remark.** *This bijection is exactly the same as the one between classical primitive parking functions and Dyck paths.*

**Example** ( $a > b : a = 7, b = 3, \mathcal{PF}'_{a,b} \rightarrow \mathcal{R}_{a,b}$ ).

- $f = (1, 1, 1, 2, 2, 3, 3)$   
 $l_1 = 3 \qquad l_2 = 2 \qquad l_3 = 2$
- $w = (1110110110)$



**Example** ( $a < b : a = 3, b = 5, \mathcal{PF}'_{a,b} \rightarrow \mathcal{R}_{a,b}$ ).

- $f = (1, 1, 4)$

$$l_1 = 2$$

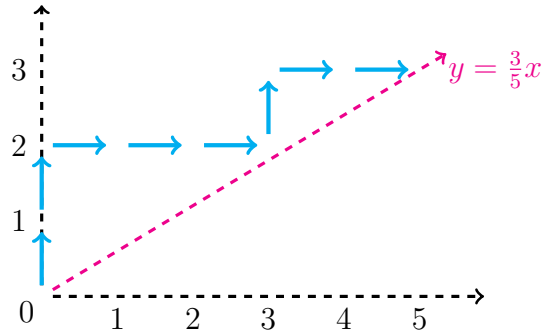
$$l_2$$

$$l_3 = 0$$

$$l_4 = 1$$

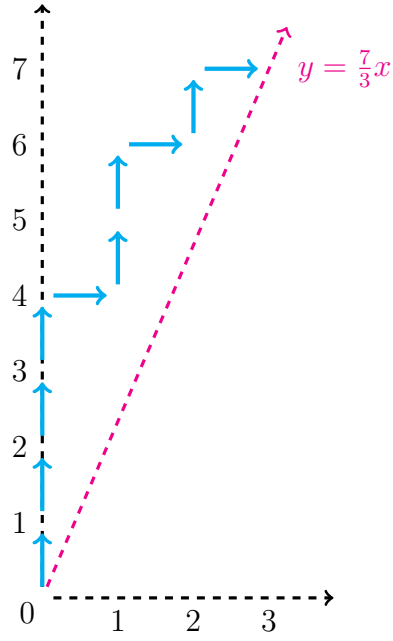
$$l_5 = 0$$

- $w = 11000100$



**Example** ( $a > b : a = 7, b = 3, \mathcal{R}_{a,b} \rightarrow \mathcal{PF}'_{a,b}$ ).

- $w = 1111011010$



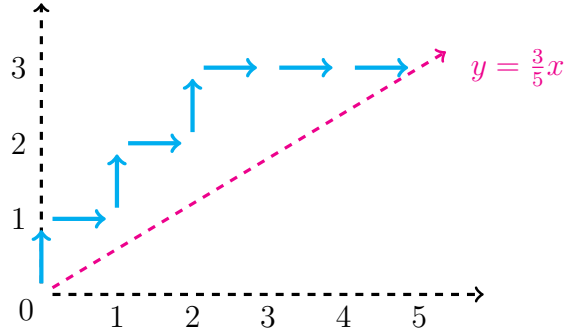
- *Distances :*

$s_1 = 0$	$a_1 = 1$
$s_2 = 0$	$a_2 = 1$
$s_3 = 0$	$a_3 = 1$
$s_4 = 0$	$a_4 = 1$
$s_5 = 1$	$a_5 = 2$
$s_6 = 1$	$a_6 = 2$
$s_7 = 2$	$a_7 = 3$

- $f = (1, 1, 1, 1, 2, 2, 3)$

**Example**  $(a < b : a = 3, b = 5, \mathcal{R}_{a,b} \rightarrow \mathcal{PF}'_{a,b})$ .

- $w = 10101000$



- *Distances :*

$$\begin{array}{ll} s_1 = 0 & a_1 = 1 \\ s_2 = 1 & a_2 = 2 \\ s_3 = 2 & a_3 = 3 \end{array}$$

- $f = (1, 2, 3)$

### 2.3.2 Rational Labeled Dyck Paths

**Definition 25** (Labeled a, b - Dyck Path). A labeled a, b - Dyck word is a word  $w \in \{0, \dots, n\}^*$  such that :

- for each suffix  $w'$  of  $w$ ,

$$|w'|_{\neq 0} \geq \frac{a}{b} |w'|_0$$

.

- $|w|_0 = b$ .
- $|w|_{\neq 0} = a$ .
- for each  $i \in \{1, \dots, a\}$ ,  $w$  has exactly one occurrence of  $i$ .
- if  $w_i \neq 0$  and  $w_{i+1} \neq 0$ , then  $w_i < w_{i+1}$ . That is, consecutive North steps have increasing labels.

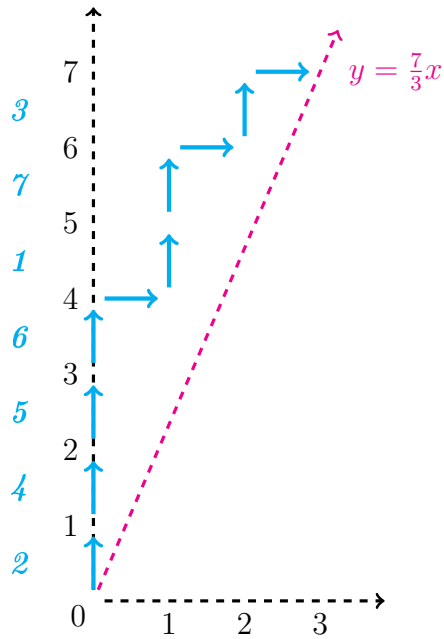
A labeled a, b - Dyck word can be represented as a path from  $(0, 0)$  to  $(b, a)$ , where each North step is associated to a label :

- Each  $i \neq 0$  corresponds to a North step  $\uparrow$  labeled  $i$ .
- Each 0 corresponds to an East step  $\rightarrow$ .

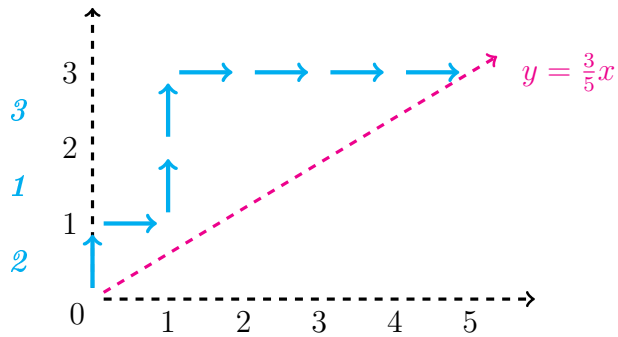
Those paths are called labeled a, b - Dyck paths.

We denote by  $\mathcal{LR}_{a,b}$  the set of labeled a, b - Dyck words.

**Example** ( $a > b : a = 7, b = 3$ ).  $w_2 = 2456017030$  :



**Example** ( $a < b : a = 3, b = 5$ ).  $w = 20130000$  :



**Theorem 15.** *Let  $lr_{a,b}$  be the cardinal of  $\mathcal{LR}_{a,b}$ . We have*

$$lr_{a,b} = b^{a-1}$$

**Example** ( $a > b : a = 4, b = 3$ ).  $lr_{a,b} = 3^3 = 27$

- *Word of shape XXXX000 :*

1234000

- *Words of shape XXX0X00 :*

1230400

1240300

1340200

2340100

- *Words of shape XX0XX00 :*

1203400

1302400

1402300

2301400

2401300

3401200

- *Words of shape XXX00X0 :*

1230040

1240030

1340020

2340010

- *Words of shape XX0X0X0 :*

1203040

1204030

1302040

1304020

1402030

1403020

2301040

2304010

2401030

2403010

3401020

3402010

**Proposition.** *This means we can create a bijection between  $\mathcal{PF}_{a,b}$  and  $\mathcal{LR}_{a,b}$ .*

*Proof.*

- $\mathcal{PF}_{a,b} \rightarrow \mathcal{LR}_{a,b}$  : Let  $f = (a_1, \dots, a_n) \in \mathcal{PF}_{a,b}$  be our a, b - parking function. For  $i \in \{1, \dots, b\}$ , we define  $im_i : \{j \mid a_j = i\}$ . We then define  $im_{i,1}, \dots, im_{i,k_i}$  to be the elements of  $im_i$  in increasing order.

The corresponding labeled a, b - Dyck word will be

$$\underbrace{im_{1,1} \cdots im_{1,k_1}}_{im_1} 0 \underbrace{im_{2,1} \cdots im_{2,k_2}}_{im_2} 0 \cdots \underbrace{im_{n,1} \cdots im_{n,k_n}}_{im_b} 0.$$

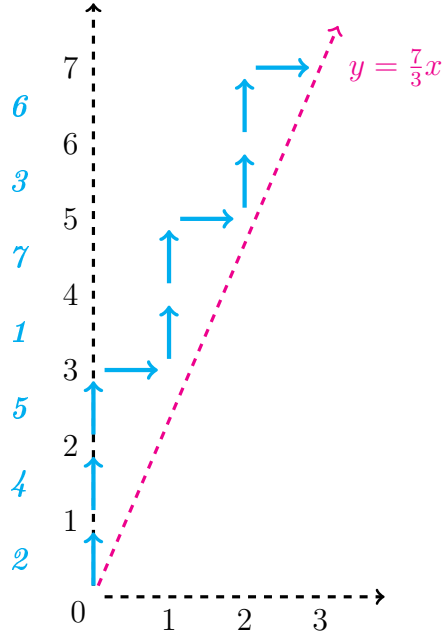
- $\mathcal{LR}_{a,b} \rightarrow \mathcal{PF}_n$  : Let  $w$  be our labeled a, b - Dyck word, and consider its path representation. We define  $s_i$  to be the distance between the segment from  $(0, i-1)$  to  $(0, i)$  and the  $i^{th}$  North step. Then, let  $label(i)$  be the label of the  $i^{th}$  North step, and  $dist_i = \{label(j) \mid s_j = i\}$  be the set of the labels of all North steps at distance  $i$ . Then, if  $j \in dist_i$ , let  $a_j = i+1$ . The corresponding parking function is  $(a_1, \dots, a_a)$ .

□

**Remark.** *This bijection is exactly the same as the one between classical parking functions and labeled Dyck paths.*

**Example** ( $a > b : a = 7, b = 3, \mathcal{PF}_{a,b} \rightarrow \mathcal{LR}_{a,b}$ ).

- $f = (2, 1, 3, 1, 1, 3, 2)$   
 $im_1 = \{2, 4, 5\} \qquad im_2 = \{1, 7\} \qquad im_3 = \{3, 6\}$
- $w = 2450170360$



**Example** ( $a < b : a = 3, b = 5, \mathcal{PF}_{a,b} \rightarrow \mathcal{LR}_{a,b}$ ).

- $f = (4, 1, 2)$

$$im_1 = \{2\}$$

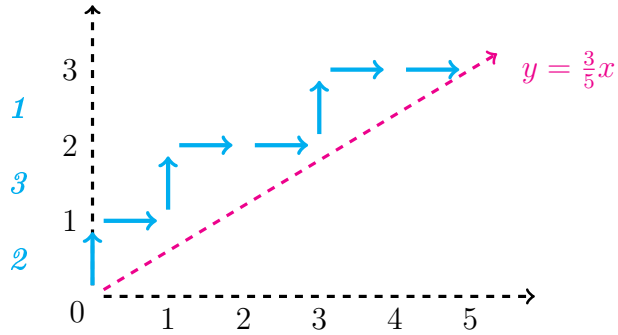
$$im_2 = \{3\}$$

$$im_3 = \emptyset$$

$$im_4 = \{1\}$$

$$im_5 = \emptyset$$

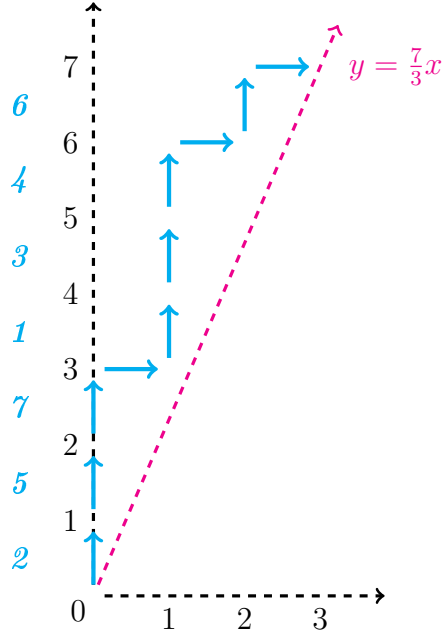
- $w = 20300100$



**Example** ( $a > b : a = 7, b = 3, \mathcal{LR}_{a,b} \rightarrow \mathcal{PF}_{a,b}$ ).



- $w = 2570134060$



- *Distances :*

$$\begin{array}{lll} s_1 = 0 & s_2 = 0 & s_3 = 0 \\ s_4 = 1 & s_5 = 1 & s_6 = 1 \\ s_7 = 2 \end{array}$$

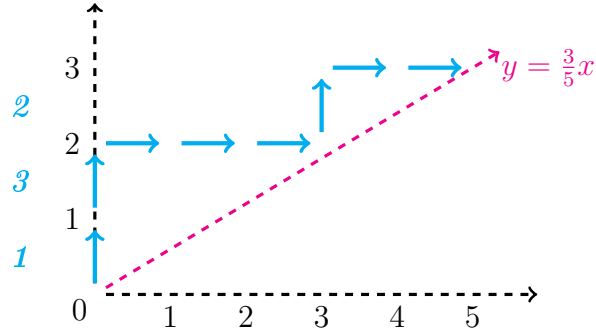
- *Labels :*

$$\begin{array}{lll} dist_0 = \{2, 5, 7\} & dist_1 = \{1, 3, 4\} & dist_2 = \\ \{6\} & & \end{array}$$

- $f = (2, 1, 2, 2, 1, 3, 1)$

**Example** ( $a < b : a = 3, b = 5, \mathcal{LR}_{a,b} \rightarrow \mathcal{PF}_{a,b}$ ).

- $w = 13000200$



- Distances :

$$s_1 = 0$$

$$s_2 = 0$$

$$s_3 = 3$$

- Labels :

$$dist_0 = \{1, 3\}$$

$$dist_1 = \emptyset$$

$$dist_2 = \emptyset$$

$$dist_3 = \{2\}$$

$$dist_4 = \emptyset$$

- $f = (1, 4, 1)$

**Remark.** The rational primitive parking functions are exactly the rational parking functions corresponding to rational labeled Dyck paths where the  $i^{th}$  North step is labeled  $i$ .

### 2.3.3 Rational Dyck - Parking Posets

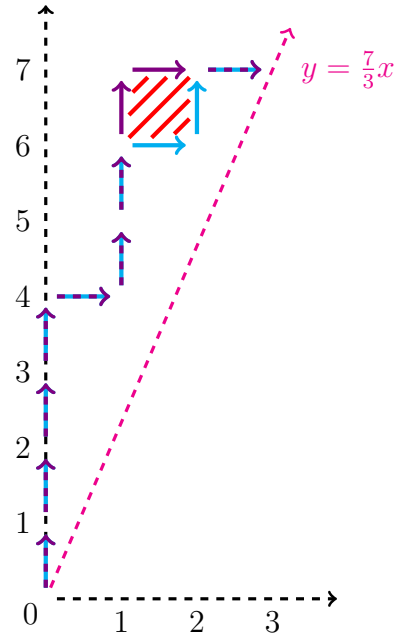
#### Rational Primitive Dyck - Parking Posets

**Definition 26** ( $\succ_r$ ). For  $w$  and  $w'$  two  $a, b$  - Dyck words, we say that  $w$  covers  $w'$ , written  $w \succ_r w'$ , if  $\exists w_1, w_2$  such that :

- $w = w_1 0 1 w_2$
- $w' = w_1 1 0 w_2$

**Example** ( $a = 7, b = 3$ ).  $1111011010 \succ_r 1111011100$

- $w_1 = 1111011$
- $w_2 = 0$

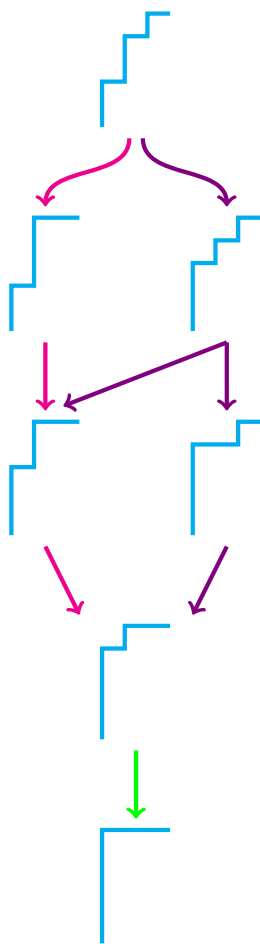


■ 1111011010      
 ■ 1111011100      
 ■ difference

**Remark.** If  $w_1 \succ_r w_2$ , then the path corresponding to  $w_2$  is over the path corresponding to  $w_1$ , and the difference between the two paths is a square of size 1 by 1.

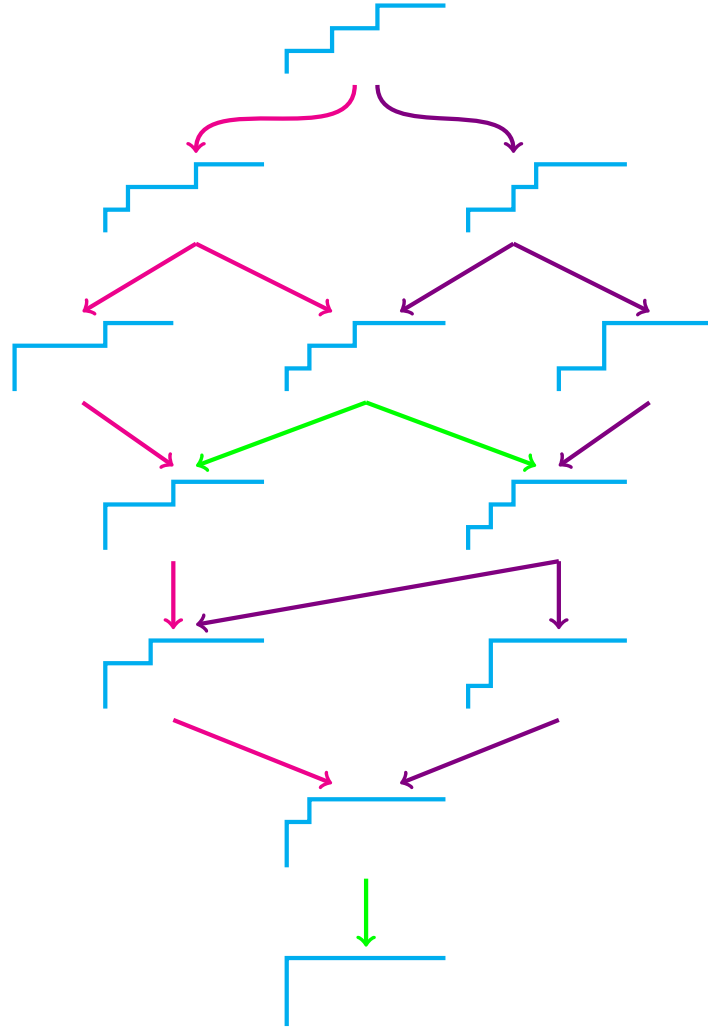
**Proposition.** This covering relation defines a poset for  $\mathcal{R}_{a,b}$ .

**Example** ( $a > b$  : The poset of  $\mathcal{R}_{5,3}$ ).



There are  $\frac{1}{8} \binom{8}{5} = \frac{42}{6} = 7$  elements in this poset.

**Example** ( $a < b$  : The poset of  $\mathcal{R}_{3,7}$ ).



There are  $\frac{1}{10} \binom{10}{3} = \frac{72}{6} = 12$  elements in this poset.

**Definition 27** ( $\succ'$ ). For  $f$  and  $g$  two rational primitive  $a, b$  - parking functions, we say that  $f$  covers  $g$ , written  $f \succ' g$ , if  $\exists i$  such that :

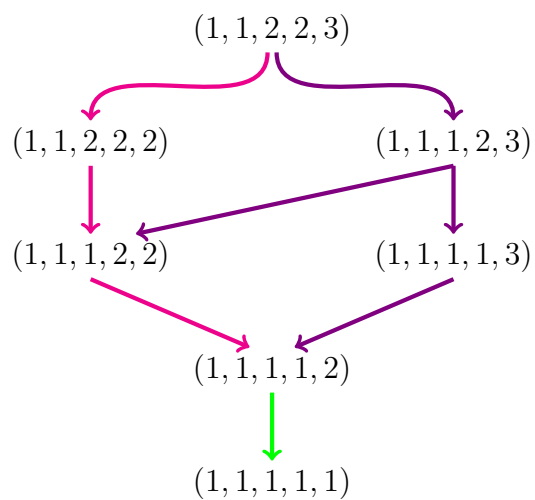
- $f = (a_1, \dots, a_{i-1}, a_i, \quad a_{i+1}, \dots, a_n)$
- $g = (a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_n)$

**Example** ( $a > b : a = 7, b = 3$ ).  $(1, 1, 1, 2, 2, 2, 3) \succ' (1, 1, 1, 1, 2, 2, 3)$

**Example** ( $a < b : a = 3, b = 5$ ).  $(1, 2, 4) \succ' (1, 1, 4)$

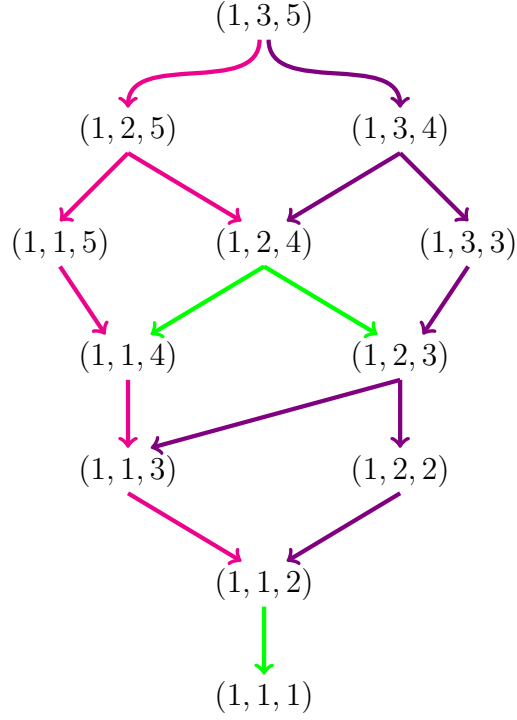
**Proposition.** *This covering relation defines a poset for  $\mathcal{PF}'_{a,b}$ .*

**Example** ( $a > b$  : The poset of  $\mathcal{PF}'_{5,3}$ ).



*There are  $\frac{1}{8} \binom{8}{5} = \frac{42}{6} = 7$  elements in this poset.*

**Example** ( $a < b$  : The poset of  $\mathcal{PF}'_{3,7}$ ).



There are  $\frac{1}{10} \binom{10}{3} = \frac{72}{6} = 12$  elements in this poset.

**Remark.** The posets of  $\mathcal{PF}'_{a,b}$  and  $\mathcal{R}_{a,b}$  are isomorphic, and one can be obtained by applying the aforementioned bijection to the other.

### Rational Dyck - Parking Posets

**Definition 28** ( $\succ_{lr}$ ). For  $w$  and  $w'$  two labeled  $a, b$  - Dyck words, we say that  $w$  covers  $w'$ , written  $w \succ_{lr} w'$ , if either :

- $\exists w_2, x, x', y$  such that :

$x = x_1 x_2 \cdots x_n$  has all its digits  $> 0$

$x' = x$  where  $y$  is correctly inserted regarding the order condition

$y > 0$

$w = x0yw_2$

$w' = x'0w_2$

- $\exists w_1, w_2, x, x', y$  such that :

$x = x_1x_2 \cdots x_n$  has all its digits  $> 0$

$y > 0$

$x' = x$  where  $y$  is correctly inserted regarding the order condition

$w = w_10x0yw_2$

$w' = w_10x'0w_2$

- or  $\exists w_1, w_2, y$  such that :

$y > 0$

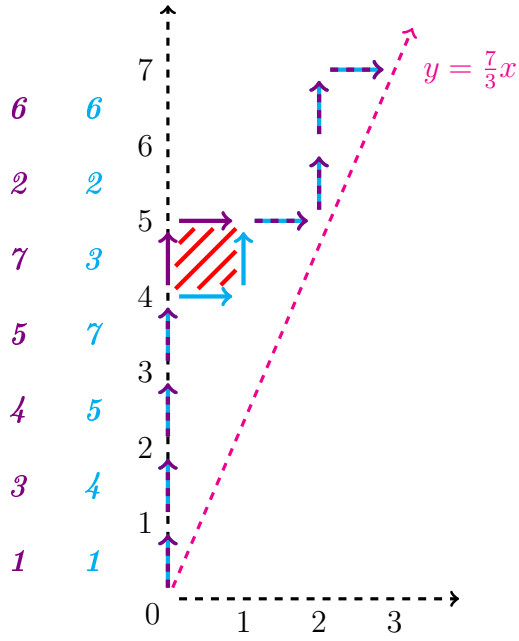
$w = w_100yw_2$

$w' = w_10y0w_2$

**Example** ( $a > b : a = 7, b = 3$ , first case).  $1457030260 \succ_{lr} 1345700260$

- $w_2 = 0260$
- $x = 1457$
- $x' = 13457$
- $y = 3$





■ 145703026     
 ■ 1345700260     
 ■ *difference*

**Example** ( $a < b : a = 3, b = 5$ , first case).  $20130000 \succ_{lr} 12030000$

- $w_2 = 30000$
- $x = 2$
- $x' = 12$
- $y = 1$

**Example** ( $a > b : a = 7, b = 3$ , second case).  $2460150370 \succ_{lr} 2460135070$

- $w_1 = 246$
- $w_2 = 70$
- $x = 15$
- $x' = 135$

- $y = 3$

**Example** ( $a < b : a = 3, b = 5$ , second case).  $20301000 \succ_{lr} 20130000$

- $w_1 = 2$
- $w_2 = 000$
- $x = 3$
- $x' = 13$
- $y = 1$

**Example** ( $a > b : a = 7, b = 3$ , third case).  $235670014 \succ_{lr} 235670104$

- $w_1 = 23567$
- $w_2 = 4$
- $y = 1$

**Example** ( $a < b : a = 3, b = 5$ , third case).  $23000100 \succ_{lr} 23001000$

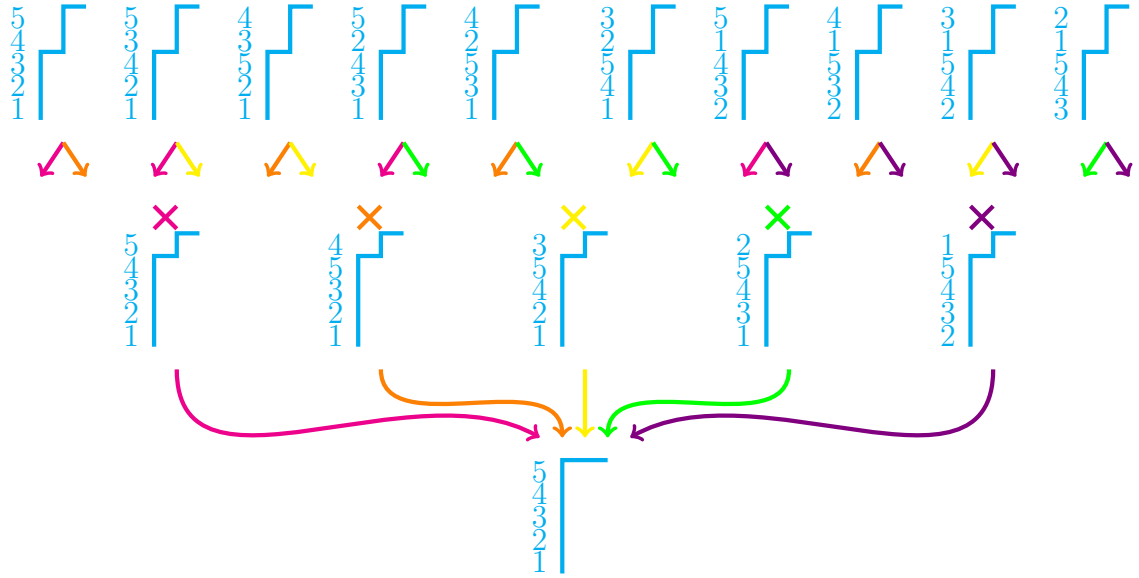
- $w_1 = 230$
- $w_2 = 00$
- $y = 1$

**Remark.** If  $w_1 \succ_{lr} w_2$ , then the path corresponding to  $w_2$  is over the path corresponding to  $w_1$ , and the difference between the two paths is a square of size 1 by 1.

Furthermore, the labeling can be seen as follows : if one has to merge two sequences of North steps of  $w_1$  to make  $w_2$ , then the merging will be made by ordering the corresponding labels.

**Proposition.** This covering relation defines a poset for  $\mathcal{LR}_{a,b}$ .

**Example** ( $a > b$  : The poset of  $\mathcal{LR}_{5,2}$ ).

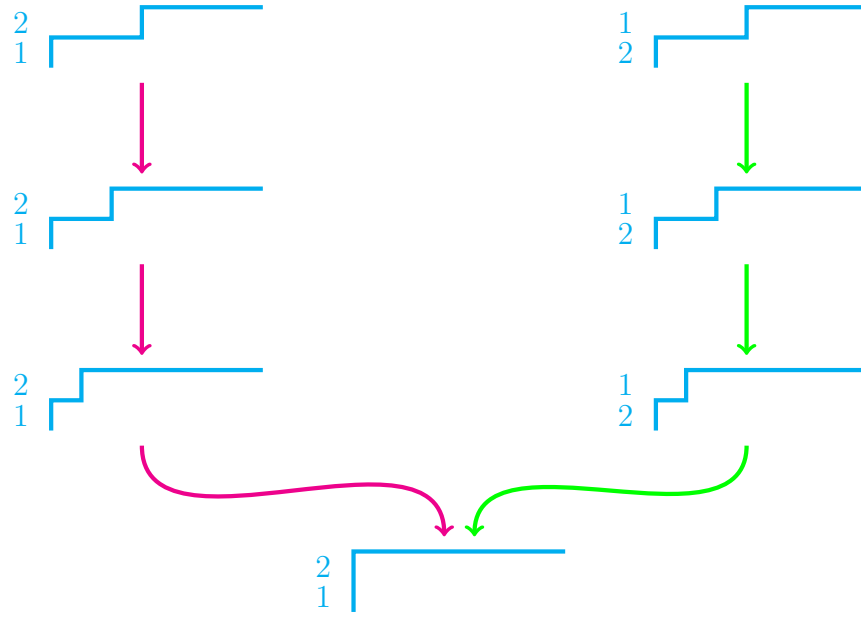


*Arrows have been simplified for readability.*

*Arrows between the top 2 levels are to be read as ending at the cross of the corresponding color.*

*There are  $2^4 = 16$  elements in this poset.*

**Example** ( $a < b$  : The poset of  $\mathcal{LR}_{2,7}$ ).



There are  $7^1 = 7$  elements in this poset.

**Definition 29 ( $\succ$ ).** For  $f$  and  $g$  two rational parking functions, we say that  $f$  covers  $g$ , written  $f \succ g$ , if  $\exists i$  such that :

- $f = (a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n)$
- $g = (a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_n)$

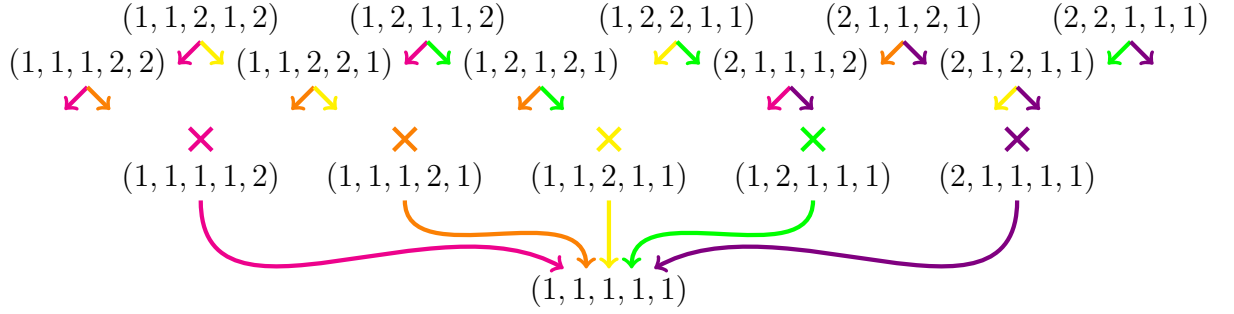
That is, the same relation as for rational primitive parking functions.

**Example** ( $a > b : a = 7, b = 3$ ).  $(2, 3, 1, 1, 2, 1, 3) \succ (2, 3, 1, 1, 1, 1, 3)$

**Example** ( $a < b : a = 3, b = 5$ ).  $(4, 1, 2) \succ (3, 1, 2)$

**Proposition.** This covering relation defines a poset for  $\mathcal{PF}_{a,b}$ .

**Example** ( $a > b$  : The poset of  $\mathcal{PF}_{5,2}$ ).

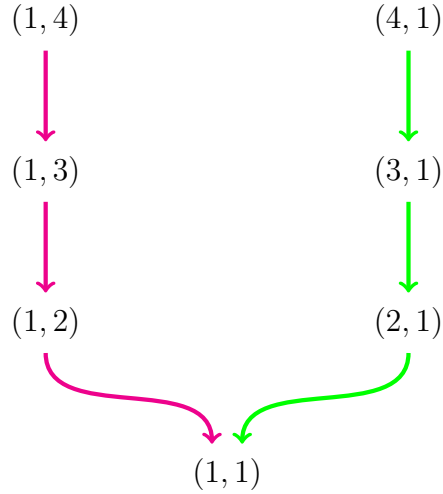


*Arrows have been simplified for readability.*

*Arrows between the top 2 levels are to be read as ending at the cross of the corresponding color.*

*There are  $2^4 = 16$  elements in this poset.*

**Example** ( $a < b$  : The poset of  $\mathcal{PF}_{2,7}$ ).



*There are  $7^1 = 7$  elements in this poset.*

**Remark.** *The posets of  $\mathcal{PF}_{a,b}$  and  $\mathcal{LR}_{a,b}$  are isomorphic, and one can be obtained by applying the aforementioned bijection to the other.*

# Chapter 3

## Trees

### 3.1 Parking Trees

**Definition 30** (Parking Trees). A parking tree is defined from a parking function  $f = (a_1, \dots, a_n) \in \mathcal{PF}_n$  as follows :

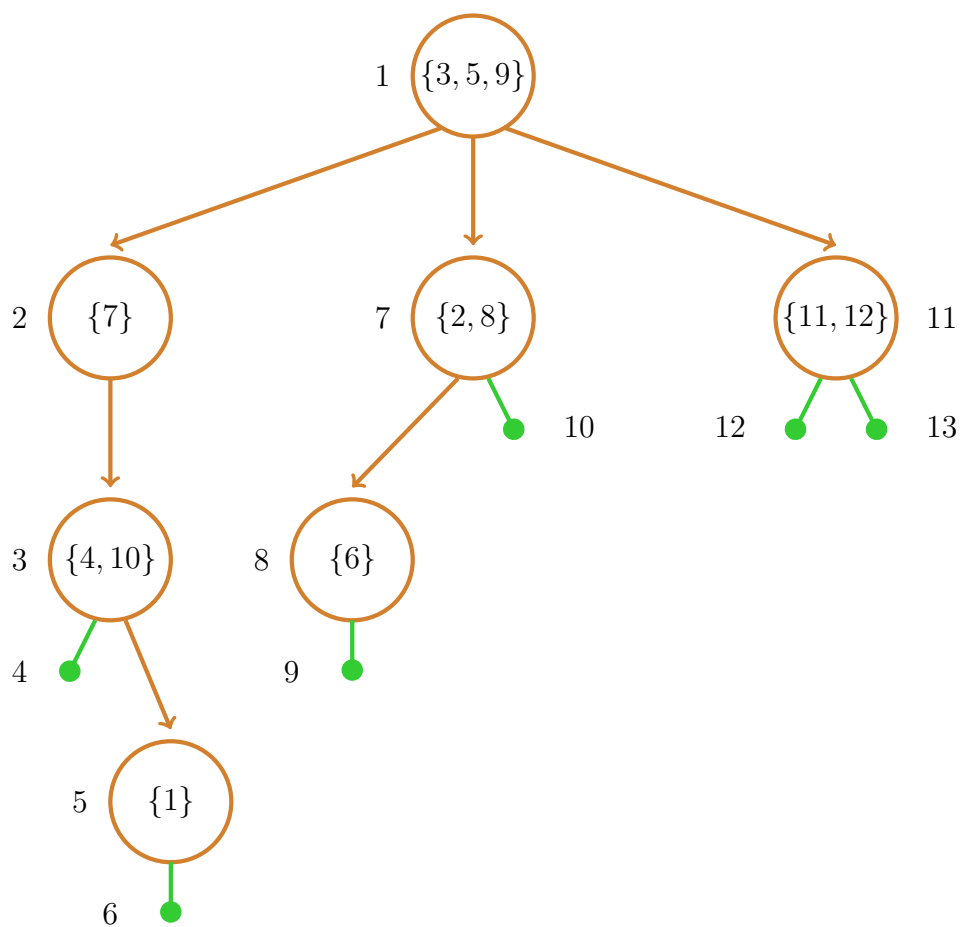
- For  $1 \leq i \leq n+1$ , we define  $s_i$  as  $\{j \mid a_j = i\}$
- $[s_1, \dots, s_{n+1}]$  describes the pre-order depth-first traversal of the tree.
- Each node labeled by a set of size  $k$  has  $k$  children.

**Remark.** The leaves of the tree are those corresponding to an element  $i$  such that  $1 \leq i \leq n+1$ , and  $i$  is not in  $f$ .

Furthermore, as we will have a total edges by definition, the presence of a node corresponding to  $n+1$  is necessary, even though it will always be empty.

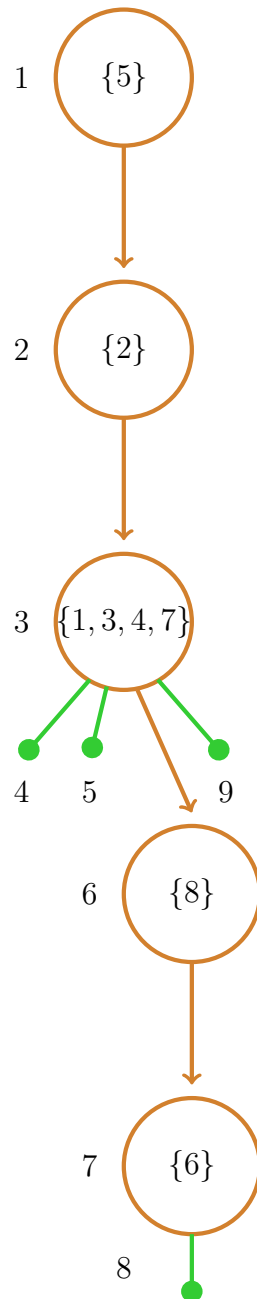
**Example** ( $n = 12$ ).

- $f = (5, 7, 1, 3, 1, 8, 2, 7, 1, 3, 11, 11)$
- Labels :  $[\{3, 5, 9\}, \{7\}, \{4, 10\}, \emptyset, \{1\}, \emptyset, \{2, 8\}, \{6\}, \emptyset, \emptyset, \{11, 12\}, \emptyset, \emptyset]$



Conversely, by reading the labels of a parking tree depth-first in pre-order, we get the list of positions of each number in the corresponding parking function, thus creating a *bijection*.

**Example** (From parking tree to parking function).



- The labels are  $[\{5\}, \{2\}, \{1, 3, 4, 7\}, \emptyset, \emptyset, \{8\}, \{6\}, \emptyset, \emptyset]$ .
- Thus the corresponding parking function is  $(3, 2, 3, 3, 1, 7, 3, 6) \in \mathcal{PF}_8$ .



## 3.2 Rational Parking Trees

**Definition 31** (Rational Parking Trees). *A rational parking tree is defined from a rational parking function  $f = (a_1, \dots, a_a) \in \mathcal{PF}_{a,b}$  as follows :*

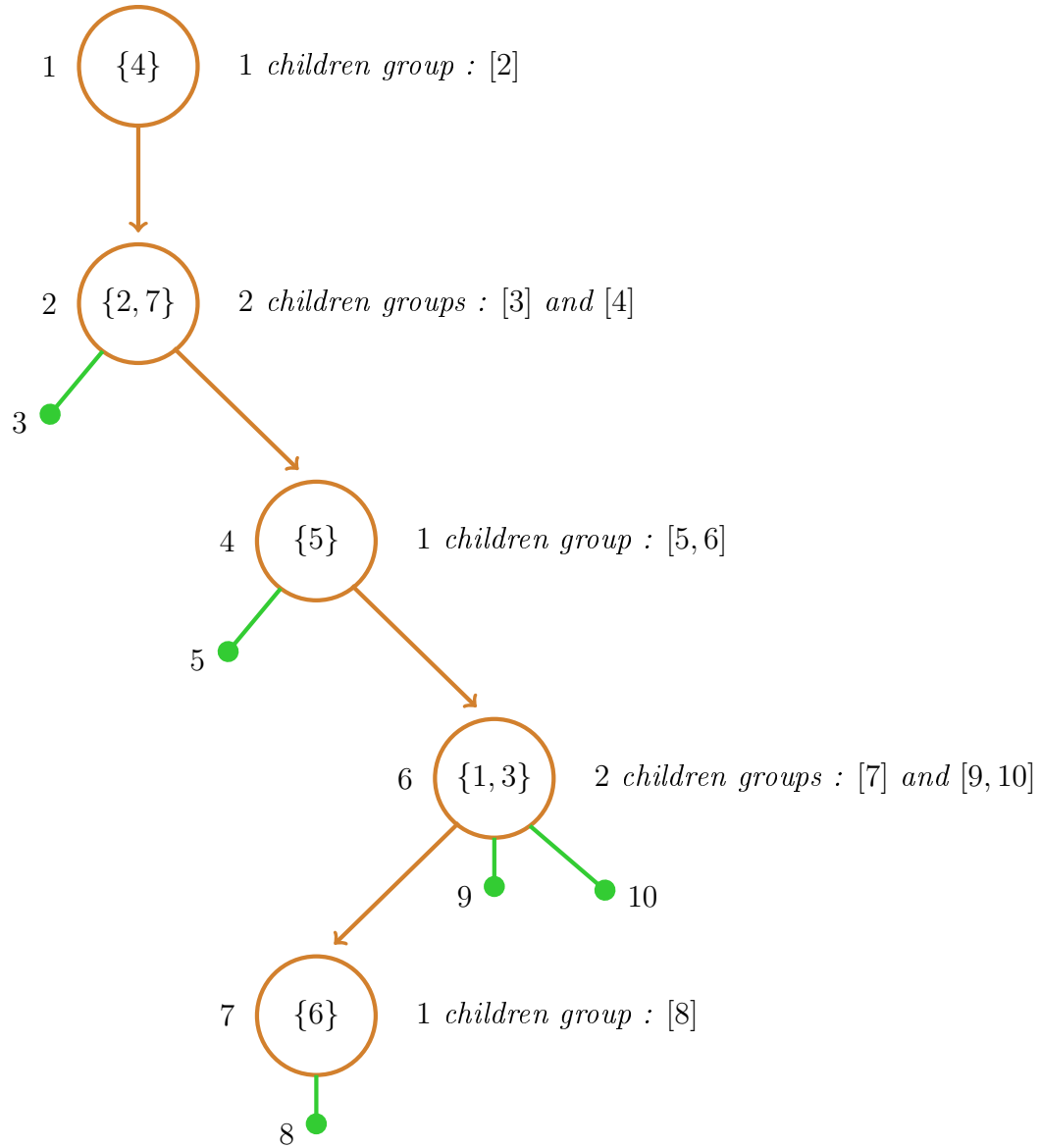
- *For  $1 \leq i \leq n + 1$ , we define the limit  $l_i$  as the integer portion of  $\frac{b}{a}(i - 1) + 1$ .  
Let  $l_0 = 0$ .*
- *From these limits, we deduce the intervals  $itv_i = ]l_{i-1}, l_i]$  for  $1 \leq i \leq a + 1$ .*
- *For  $1 \leq i \leq b + 1$ , define  $s_i$  as  $\{j \mid a_j = i\}$ .*
- *$[s_1, \dots, s_{b+1}]$  describes the pre-order depth-first traversal of the tree.*
- *Each node labeled by a set of size  $k$  has  $k$  groups of children, which are defined by the intervals.*

**Example** ( $a < b$ ).

- $a = 7$
- $b = 9$
- *Limits :*  $[1, 2\frac{2}{7}, 3\frac{4}{7}, 4\frac{6}{7}, 6\frac{1}{7}, 7\frac{3}{7}, 8\frac{5}{7}, 10]$
- *Integral limits :*  $[0, 1, 2, 3, 4, 6, 7, 8, 10]$
- *Intervals :*  

$$]0, 1] \quad ]1, 2] \quad ]2, 3] \quad ]3, 4] \quad ]4, 6] \quad ]6, 7] \quad ]7, 8] \quad ]8, 10]$$
- *Children groups :*  

$$[1] \quad [2] \quad [3] \quad [4] \quad [5, 6] \quad [7] \quad [8]$$
- $f = (6, 2, 6, 1, 4, 7, 2)$
- *Labels :*  $\{\{4\}, \{2, 7\}, \emptyset, \{5\}, \emptyset, \{1, 3\}, \{6\}, \emptyset, \emptyset, \emptyset\}$



**Example** ( $a > b$ ).

- $a = 9$
- $b = 7$
- *Limits* :  $[1, 1\frac{7}{9}, 2\frac{5}{9}, 3\frac{3}{9}, 4\frac{1}{9}, 4\frac{8}{9}, 5\frac{6}{9}, 6\frac{4}{9}, 7\frac{2}{9}, 8]$

- *Integral limits* :  $[0, 1, 1, 2, 3, 4, 4, 5, 6, 7, 8]$

- *Intervals* :

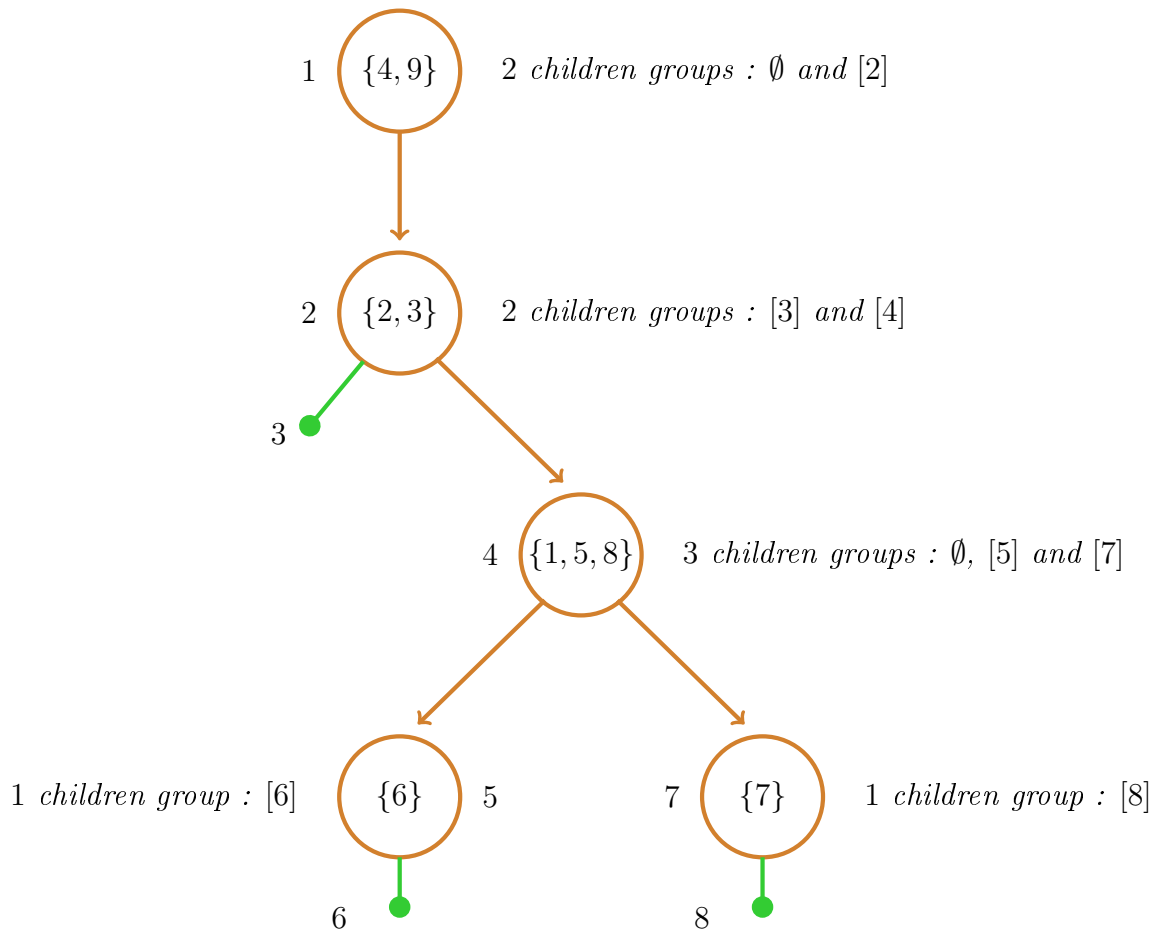
$]0, 1]$	$]1, 1]$	$]1, 2]$	$]2, 3]$	$]3, 4]$
$]4, 4]$	$[4, 5]$	$]5, 6]$	$]6, 7]$	$]7, 8]$

- *Children groups* :

$[1]$	$\emptyset$	$[2]$	$[3]$	$[4]$	$\emptyset$	$[5]$	$[6]$	$[7]$	$[8]$
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- $f = (4, 2, 2, 1, 4, 5, 7, 4, 1)$

- *Labels* :  $\{\{4, 9\}, \{2, 3\}, \emptyset, \{1, 5, 8\}, \{6\}, \emptyset, \{7\}, \emptyset\}$



In both cases, the converse direction of the *bijection* is obtained with the same method as for classical parking trees.