Rational Parking Functions

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Abstract

This is an abstract about Rational Parking Functions

Chapter 1

The integer case

1.1 Parking Functions

Definition 1 (Parking Function)**.** *A* parking function *is a sequence* $(a_1, a_2, \ldots, a_n)$ *such that its non-decreasing reordering* $(b_1, b_2, \ldots, b_n)$ *has* $b_i < i$ *for all* i*.*
We denote by $\mathcal{PF}_n$ *the set of parking functions of length* n*.*

$$\mathcal{PF} = \bigcup_{n>0} \mathcal{PF}_n$$

.

Example.

$$f_1 = (7,3,1,4,2,5,2) \in \mathcal{PF}_7$$
$$f_2 = (7,3,1,4,2,5,4) \notin \mathcal{PF}_7$$

Theorem 1. *Let* pf_n *be the cardinal of* $\mathcal{PF}_n$*. We have* $pf_n = (n+1)^{n-1}$*.*

Example $(n = 1,2,3)$**.**

• $n = 1 \quad : \quad pf_1 = 1$

(1)

• $n = 2 \quad : \quad pf_2 = 3$

$(1,1) \qquad (1,2) \qquad (2,1)$

• $n = 3 \quad : \quad pf_3 = 16$

$(1,1,1)$	$(1,1,2)$	$(1,1,3)$	$(1,2,1)$	$(1,2,2)$	$(1,2,3)$	$(1,3,1)$
$(1,3,2)$	$(2,1,1)$	$(2,1,2)$	$(2,1,3)$	$(2,2,1)$	$(2,3,1)$	$(3,1,1)$

$(3, 1, 2) \qquad (3, 2, 1)$

Definition 2 (Primitive)**.** *A parking function* $(a_1, a_2, \ldots, a_n)$ *is said* primitive *if it is already in non-decreasing order.*
We denote by $\mathcal{PF'}_n$ *the set of primitive parking functions of length* n*.*

$$\mathcal{PF'} = \bigcup_{n>0} \mathcal{PF'}_n$$

Example.

$$f_1 = (1, 2, 2, 3) \in \mathcal{PF'}_4$$
$$f_2 = (1, 2, 3, 2) \notin \mathcal{PF'}_4, \text{ even though } f_2 \in \mathcal{PF}_4$$

Theorem 2. *Let* pf'_n *be the cardinal of* $\mathcal{PF'}_n$*. We have* $pf'_n = \frac{1}{n+1}\binom{2n}{n}$*, which is the* n^{th} *Catalan number.*

Example $(n = 1, 2, 3)$**.**
- $n = 1 \quad : \quad pf'_1 = 1$
  (1)
- $n = 2 \quad : \quad pf'_2 = 2$
  $(1, 1) \qquad (1, 2)$
- $n = 3 \quad : \quad pf'_3 = 5$
  $(1, 1, 1) \qquad (1, 1, 2) \qquad (1, 1, 3) \qquad (1, 2, 2) \qquad (1, 2, 3)$

1.2 Non-crossing Partitions

Definition 3 (Non-crossing Partition)**.** *A* non-crossing partition *of a set* E *is a set partition* $P = \{E_1, E_2, \ldots, E_k\}$ *such that if* $a, c \in E_i$*,* $b, d \in E_j$*, and* $i \neq j$*, then we do* not *have* $a < b < c < d$*, nor* $a > b > c > d$*.*
We denote by $\mathcal{NC}_n$ *the set of non-crossing partitions of* $\{1, 2, \ldots, n\}$*.*

$$\mathcal{NC} = \bigcup_{n>0} \mathcal{NC}_n$$

From this point, we assume that every partition $P = \{B_1, \ldots, B_l\}$ is *sorted* such that :
- For each block $B_i = \{b_1, \ldots, b_k\} \in P$, $b_1 < \ldots < b_k$
- $min(B_1) < \ldots < min(B_k)$

Notation. $[n] = \{1, 2, \ldots, n\}$

Example ($E = [6]$)**.**

$$P_1 = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\} \in \mathcal{NC}_6$$
$$P_2 = \{\{1, 2, 4\}, \{3, 5\}, \{6\}\} \notin \mathcal{NC}_6$$

Theorem 3. *Let nc_n be the cardinal of $\mathcal{NC}_n$. We have $nc_n = \frac{1}{n+1}\binom{2n}{n}$, which is the n^{th} Catalan number.*

Example ($n = 1, 2, 3$)**.**
- $n = 1 \quad : \quad nc_1 = 1$

  $\{\{1\}\}$
- $n = 2 \quad : \quad nc_2 = 2$

  $\{\{1, 2\}\} \qquad \{\{1\}, \{2\}\}$
- $n = 3 \quad : \quad nc_3 = 5$

  $\{\{1, 2, 3\}\} \qquad \{\{1\}, \{2, 3\}\} \qquad \{\{1, 3\}, \{2\}\} \qquad \{\{1, 2\}, \{3\}\} \qquad \{\{1\}, \{2\}, \{3\}\}$

Proposition. *This means we can create a* bijection *between $\mathcal{PF}'_n$ and $\mathcal{NC}_n$.*

- $\mathcal{NC}_n \rightarrow \mathcal{PF}'_n$: For each block B in the non-crossing partition, take $i = min(B)$, and $k_i = size(B)$.
  $k_i = 0$ if i is not the minimum of a block.
  The corresponding parking function is $(\underbrace{1, \ldots, 1}_{k_1}, \underbrace{2, \ldots, 2}_{k_2}, \ldots, \underbrace{n, \ldots, n}_{k_n})$.

- $\mathcal{PF}'_n \rightarrow \mathcal{NC}_n$: For each i in $[n]$, if i appears n_i times in the parking function, B_i will be of size n_i with minimum element i. There is a unique set partition $P = \bigcup_i B_i$ of $[n]$ respecting these conditions that is non-crossing.

Example ($E = [6]$)**.**

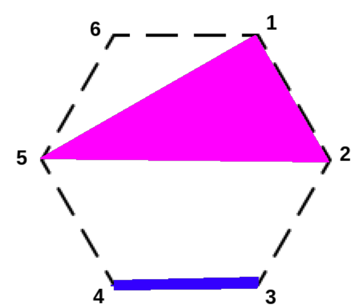
$$P = \{\{1,2,5\},\{3,4\},\{6\}\} \qquad\qquad f = (1,1,1,3,3,6)$$

Corollary. *A non-crossing partition can be represented by the minimums and sizes of its blocks.*

Example. $\{\{1,2,5\},\{3,4\},\{6\}\}$ *can be represented by the following dictionnary :*
• *1 : 3*
• *3 : 2*
• *6 : 1*

A non-crossing partition of $[n]$ can be represented graphically on a regular n-vertices polygon, with vertices labeled from 1 to n clockwise. We then represent each block $B = \{b_1, \ldots, b_k\}$ by the convex hull of $\{b_1, \ldots, b_k\}$.

Example ($P = \{\{1,2,5\},\{3,4\},\{6\}\}$)**.**



Thus non-crossing meaning the hulls are *disjoint*.

1.2.1 The non-crossing partitions poset

Definition 4 ($\succ$)**.** *We say that P covers Q, written $P \succ Q$, if $\exists B_i, B_j \in P$ such that $Q = P - \{B_i, B_j\} \cup \{B_i \cup B_j\}$*

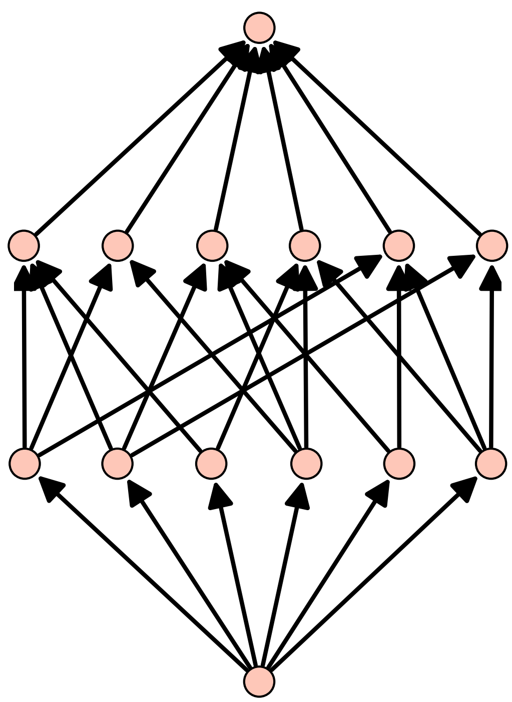
Example. $\{\{1,6\},\{2,3\},\{4,5\}\} \succ \{\{1,2,3,6\},\{4,5\}\}$
- $B_i = \{1,6\}$
- $B_j = \{2,3\}$

Proposition. *This covering relation defines the* poset *of non-crossing partitions of* $[n]$*. We denote by* $\mathcal{NCC}_n$ *the set of* maximal chains *in the poset of* $\mathcal{NC}_n$*.*

$$\mathcal{NCC} = \bigcup_{n>0} \mathcal{NCC}_n$$

Theorem 4. *Let* ncc_n *be the cardinal of* $\mathcal{NC}_n$*. We have* $ncc_n = n^{n-2}$*.*

Example (Shape of the poset of $\mathcal{NC}_4$)**.**



This figure was generated with Sagemath. There are $4^2 = 16$ *different maximal chains, and* $\frac{1}{5}\binom{8}{4} = \frac{70}{5} = 14$ *elements in this poset.*

1.2.2 Kreweras complement

Definition 5 (Associated permutation)**.** *The* permutation σ *associated to a non-crossing partition has a cycle* $(b_1, \ldots, b_k)$ *for each block* $B = \{b_1, \ldots, b_k\}$ *of the partition.*

Example. *The permutation associated to* $\{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$ *is* $(1\ 2\ 5)\ (3\ 4)\ (6) = 254316$*.*

Definition 6 (Kreweras complement)**.** *The* Kreweras complement $K(P)$ *of a non-crossing partition* P *is defined as follows :*
- *Let* σ *be the permutation associated to* P
- *Let* π *be the permutation* $(n\ n-1\ n-2\ \ldots\ 3\ 2\ 1) = n123\ldots n-1$
- $K(P)$ *is the* non-crossing partition *associated to* $\pi\sigma$*.*

Example $(P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\})$**.**
- $\sigma = (1\ 2\ 5)\ (3\ 4)\ (6) = 254316$
- $\pi = (6\ 5\ 4\ 3\ 2\ 1) = 612345$
- $\pi\sigma = 143265 = (1)\ (2\ 4)\ (3)\ (5\ 6)$
- $K(P) = \{\{1\}, \{2, 4\}, \{3\}, \{5, 6\}\}$

Proposition (Kreweras minimums)**.** *Let* $P = \{B_1, \ldots, B_k\}$ *be a non-crossing partition. Let* $K(P) = \{B'_1, \ldots, B'_l\}$ *be its Kreweras complement. Then*

$$\bigcup_{1 \le i \le l} min(B'_i) = B_1 \cup \bigcup_{1 < j \le k} B_i - max(B_i)$$

Example $(P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\})$**.**

- $K(P) = \{\{1\}, \{2, 4\}, \{3\}, \{5, 6\}\}$
- $\bigcup min(B'_i) = \{1, 2, 3, 5\}$
- $B_1 \cup \bigcup B_i - max(B_i) = \{1, 2, 5\} \cup \{3, 4\} - \{4\} \cup \{6\} - \{6\} = \{1, 2, 5\} \cup \{3\} \cup \emptyset = \{1, 2, 3, 5\}$

Notation. $B_{[i]}$ = *block containing* i.

Proposition (Kreweras block sizes)**.** *Let* $P = \{B_1, \dots, B_k\}$ *be a non-crossing partition. Let* $K(P) = \{B'_1, \dots, B'_l\}$ *be its Kreweras complement. Then the size of the block* B'_i *is defined as follows :*

- *Let* m_i *be the the* i^{th} *minimum of* $K(P)$
- *Define a* transition $\phi(e)$ *as*

  Let $j = e + 1$ *(or* 1 *if* $e = n$*)*

  $\phi(e) = max(B_{[j]})$
- *The size of* B'_i *is* k_{min} *such that* $k_{min} = min\{k > 0 \mid \phi^k(m_i) \in B_{[m_i]}\}$.

Example ($P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$)**.**

- $mins = \{1, 2, 3, 5\}$
- $m_1 = 1$

  $B_{[1]} = B_1$

  $max(B_{[2]} = max(B_1) = 5$

  The size for m_1 *is* 1.
- m_2

  $B_{[2]} = B_1$

  $max(B_{[3]}) = max(B_2) = 4$

  $max(B_{[5]}) = max(B_1) = 5$

  The size for m_2 *is* 2.
- $m_3 = 3$

  $B_{[3]} = B_2$

  $max(B_{[4]}) = max(B_2) = 4$

  The size for m_3 *is* 1.

- $m_4 = 5$

  $B_{[5]} = B_1$

  $max(B_{[6]}) = max(B_3) = 6$

  $max(B_{[1]}) = max(B_1) = 5$

  The size for m_4 is 2.

1.3 Non-crossing 2-partitions

Definition 7 (Action of $\mathfrak{S}_n$)**.** *The action of $\mathfrak{S}_n$ on a non-crossing partition $P = \{B_1, \dots, B_l\} \in \mathcal{NC}_n$ is defined by :*
• For each block $B_i = \{b_1, \dots, b_k\}$: $\sigma(Bi) = \{\sigma(b_1), \dots, \sigma(b_k)\}$
• We denote $\rho = \sigma(P) = \{\sigma(B_1), \dots, \sigma(B_l)\}$

Definition 8 (Non-crossing 2-partition)**.** *A* non-crossing 2-partition *of a set E is a pair (P, σ) where :*
• P is a non-crossing partition of E
• σ is a permutation of the elements of E
• For each sorted *block $B_i = \{b_1, \dots, b_k\} \in P$, we have $\sigma(b_i) < \dots < \sigma(b_k)$*

We denote by $\mathcal{NC}^2_n$ the set of non-crossing 2-partitions of $[n]$.

$$\mathcal{NC}^2 = \bigcup_{n>0} \mathcal{NC}^2_n$$

.

Example ($\mathcal{NC}^2_6$)**.** $P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$ $\sigma = 413265$
$\rho = \{\{1, 3, 6\}, \{2\}, \{4, 5\}\}$

Theorem 5. *Let nc^2_n be the cardinal of $\mathcal{NC}^2_n$. We have $nc^2_n = (n + 1)^{n-1}$.*

Example ($n = 1, 2, 3$)**.**

- $n = 1$: $nc^2_1 = 1$

  $\{\{1\}\}$ 1 $\rho = P$

- $n = 2$: $nc^2_2 = 3$

  $\{\{1\}, \{2\}\}$ 12 $\rho = P$

  $\{\{1\}, \{2\}\}$ 21 $\rho = P$

  $\{\{1, 2\}\}$ 12 $\rho = P$

- $n = 3 \quad : \quad nc^2_3 = 16$

$\{\{1\},\{2\},\{3\}\}$	123	$\rho = P$
$\{\{1\},\{2\},\{3\}\}$	132	$\rho = P$
$\{\{1\},\{2\},\{3\}\}$	213	$\rho = P$
$\{\{1\},\{2\},\{3\}\}$	231	$\rho = P$
$\{\{1\},\{2\},\{3\}\}$	312	$\rho = P$
$\{\{1\},\{2\},\{3\}\}$	321	$\rho = P$
$\{\{1,2\},\{3\}\}$	123	$\rho = P$
$\{\{1,2\},\{3\}\}$	132	$\rho = \{\{1,3\},\{2\}\}$
$\{\{1,2\},\{3\}\}$	231	$\rho = \{\{1\},\{2,3\}\}$
$\{\{1\},\{2,3\}\}$	123	$\rho = P$
$\{\{1\},\{2,3\}\}$	213	$\rho = \{\{1,3\},\{2\}\}$
$\{\{1\},\{2,3\}\}$	312	$\rho = \{\{1,2\},\{3\}\}$
$\{\{1,3\},\{2\}\}$	123	$\rho = P$
$\{\{1,3\},\{2\}\}$	132	$\rho = \{\{1,2\},\{3\}\}$
$\{\{1,3\},\{2\}\}$	213	$\rho = \{\{1\},\{2,3\}\}$
$\{\{1,2,3\}\}$	123	$\rho = P$

Proposition. *This means we can create a* bijection *between* $\mathcal{PF}_n$ *and* $\mathcal{NC}^2_n$*.*

- $\mathcal{PF}_n \to \mathcal{NC}^2_n$: Let $f = (a_1, \dots, a_n) \in \mathcal{PF}_n$ be our parking function. For $i \in \{1, \dots, n\}$, we define :

  l_i : the number of occurences of i in f.

  im_i : $\{j \mid a_j = i\}$

  The corresponding non-crossing partition will have the following constraints :

  For each $i \in \{1, \dots, n\}$, if $l_i > 0$, then there is a block $B_{[i]}$ of length l_i with minimum element i.

  $\sigma(B_{[i]}) = im_i$

  There is a unique set partition $P = \bigcup_i B_{[i]}$ of $[n]$ and a unique permutation σ respecting these conditions such that $(P, \sigma) \in \mathcal{NC}^2_n$.

- $\mathcal{NC}_n^2 \to \mathcal{PF}_n$: Let (P, σ) with $P = \{B_1, \ldots, B_l\}$ be our non-crossing 2-partition. For each block $B_i = \{b_1, \ldots, b_k\} \in P$:

  $m_i = min(B_i) = b_1$

  $pos_i = \sigma(B_i)$

  For each $j \in pos_i$, we define $a_j = m_i$

  The corresponding parking function is $(a_1, \ldots, a_n)$.

Example $(n = 8)$**.**

$$P = \{\{1,2,5\},\{3,4\},\{6,8\},\{7\}\}$$
$$\sigma = 36187245$$
$$f = (3,6,1,7,6,1,1,3)$$

Chapter 2

The rational case

Chapter 3

Trees