

Rational Parking Functions

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Abstract

This is an abstract about Rational Parking Functions

Chapter 1

The integer case

1.1 Parking Functions

Definition 1 (Parking Function). A parking function is a sequence (a_1, a_2, \dots, a_n) such that its non-decreasing reordering (b_1, b_2, \dots, b_n) has $b_i < i$ for all i .

We denote by \mathcal{PF}_n the set of parking functions of length n .

$$\mathcal{PF} = \bigcup_{n>0} \mathcal{PF}_n$$

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Example.

$$f_1 = (7, 3, 1, 4, 2, 5, 2) \in \mathcal{PF}_7$$

$$f_2 = (7, 3, 1, 4, 2, 5, 4) \notin \mathcal{PF}_7$$

Theorem 1. Let pf_n be the cardinal of \mathcal{PF}_n . We have $pf_n = (n+1)^{n-1}$.

Example ($n = 1, 2, 3$).

- $n = 1$: $pf_1 = 1$

(1)

- $n = 2$: $pf_2 = 3$

(1, 1) (1, 2) (2, 1)

- $n = 3$: $pf_3 = 16$

(1, 1, 1)	(1, 1, 2)	(1, 1, 3)	(1, 2, 1)	(1, 2, 2)	(1, 2, 3)	(1, 3, 1)
(1, 3, 2)	(2, 1, 1)	(2, 1, 2)	(2, 1, 3)	(2, 2, 1)	(2, 3, 1)	(3, 1, 1)
(3, 1, 2)	(3, 2, 1)					

1.1.1 Primitive parking functions

Definition 2 (Primitive). A parking function (a_1, a_2, \dots, a_n) is said primitive if it is already in non-decreasing order.

We denote by \mathcal{PF}'_n the set of primitive parking functions of length n .

$$\mathcal{PF}' = \bigcup_{n>0} \mathcal{PF}'_n$$

Example.

$$\begin{aligned} f_1 &= (1, 2, 2, 3) \in \mathcal{PF}'_4 \\ f_2 &= (1, 2, 3, 2) \notin \mathcal{PF}'_4, \text{ even though } f_2 \in \mathcal{PF}_4 \end{aligned}$$

Theorem 2. Let pf'_n be the cardinal of \mathcal{PF}'_n . We have $pf'_n = \frac{1}{n+1} \binom{2n}{n}$, which is the n^{th} Catalan number.

Example ($n = 1, 2, 3$).

- $n = 1$: $pf'_1 = 1$
(1)
- $n = 2$: $pf'_2 = 2$
(1, 1) (1, 2)
- $n = 3$: $pf'_3 = 5$
(1, 1, 1) (1, 1, 2) (1, 1, 3) (1, 2, 2) (1, 2, 3)

1.2 Non-crossing Partitions

Definition 3 (Non-crossing Partition). A non-crossing partition of a set E is a set partition $P = \{E_1, E_2, \dots, E_k\}$ such that if $a, c \in E_i$, $b, d \in E_j$, and $i \neq j$, then we do not have $a < b < c < d$, nor $a > b > c > d$.

We denote by \mathcal{NC}_n the set of non-crossing partitions of $\{1, 2, \dots, n\}$.

$$\mathcal{NC} = \bigcup_{n>0} \mathcal{NC}_n$$

From this point, we assume that every partition $P = \{B_1, \dots, B_l\}$ is sorted such that :

- For each block $B_i = \{b_1, \dots, b_k\} \in P$, $b_1 < \dots < b_k$
- $\min(B_1) < \dots < \min(B_k)$

Notation. $[n] = \{1, 2, \dots, n\}$

Example ($E = [6]$).

$$P_1 = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\} \in \mathcal{NC}_6$$

$$P_2 = \{\{1, 2, 4\}, \{3, 5\}, \{6\}\} \notin \mathcal{NC}_6$$

Theorem 3. Let nc_n be the cardinal of \mathcal{NC}_n . We have $nc_n = \frac{1}{n+1} \binom{2n}{n}$, which is the n^{th} Catalan number.

Example ($n = 1, 2, 3$).

- $n = 1$: $nc_1 = 1$
 $\{\{1\}\}$
- $n = 2$: $nc_2 = 2$
 $\{\{1, 2\}\} \quad \{\{1\}, \{2\}\}$
- $n = 3$: $nc_3 = 5$
 $\{\{1, 2, 3\}\} \quad \{\{1\}, \{2, 3\}\} \quad \{\{1, 3\}, \{2\}\} \quad \{\{1, 2\}, \{3\}\} \quad \{\{1\}, \{2\}, \{3\}\}$

Proposition. This means we can create a bijection between \mathcal{PF}'_n and \mathcal{NC}_n .

- $\mathcal{NC}_n \rightarrow \mathcal{PF}'_n$: For each block B in the non-crossing partition, take $i = \min(B)$, and $k_i = \text{size}(B)$.
 $k_i = 0$ if i is not the minimum of a block.
The corresponding parking function is $(\underbrace{1, \dots, 1}_{k_1}, \underbrace{2, \dots, 2}_{k_2}, \dots, \underbrace{n, \dots, n}_{k_n})$.

- $\mathcal{PF}'_n \rightarrow \mathcal{NC}_n$: For each i in $[n]$, if i appears n_i times in the parking function, B_i will be of size n_i with minimum element i . There is a unique set partition $P = \bigcup_i B_i$ of $[n]$ respecting these conditions that is non-crossing.

Example ($n = 6$).

$$P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\} \quad f = (1, 1, 1, 3, 3, 6)$$

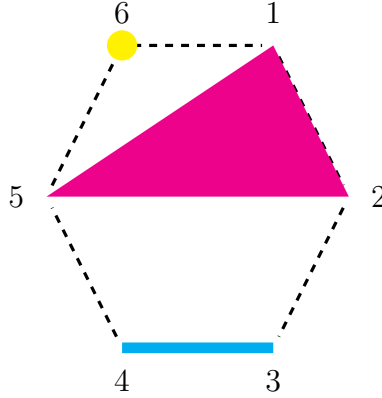
Corollary. *A non-crossing partition can be represented by the minimums and sizes of its blocks.*

Example. $\{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$ can be represented by the following dictionary :

- $1 : 3$
- $3 : 2$
- $6 : 1$

A non-crossing partition of $[n]$ can be represented graphically on a regular n -vertices polygon, with vertices labeled from 1 to n clockwise. We then represent each block $B = \{b_1, \dots, b_k\}$ by the convex hull of $\{b_1, \dots, b_k\}$.

Example ($P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$).



Thus non-crossing meaning the hulls are *disjoint*.

1.2.1 The non-crossing partitions poset

Definition 4 (\succ). *We say that P covers Q , written $P \succ Q$, if $\exists B_i, B_j \in P$ such that $Q = P - \{B_i, B_j\} \cup \{B_i \cup B_j\}$*

Example. $\{\{1, 6\}, \{2, 3\}, \{4, 5\}\} \succ \{\{1, 2, 3, 6\}, \{4, 5\}\}$

- $B_i = \{1, 6\}$
- $B_j = \{2, 3\}$

Proposition. *This covering relation defines the poset of \mathcal{NC}_n . We denote by \mathcal{NCC}_n the set of maximal chains in the poset of \mathcal{NC}_n .*

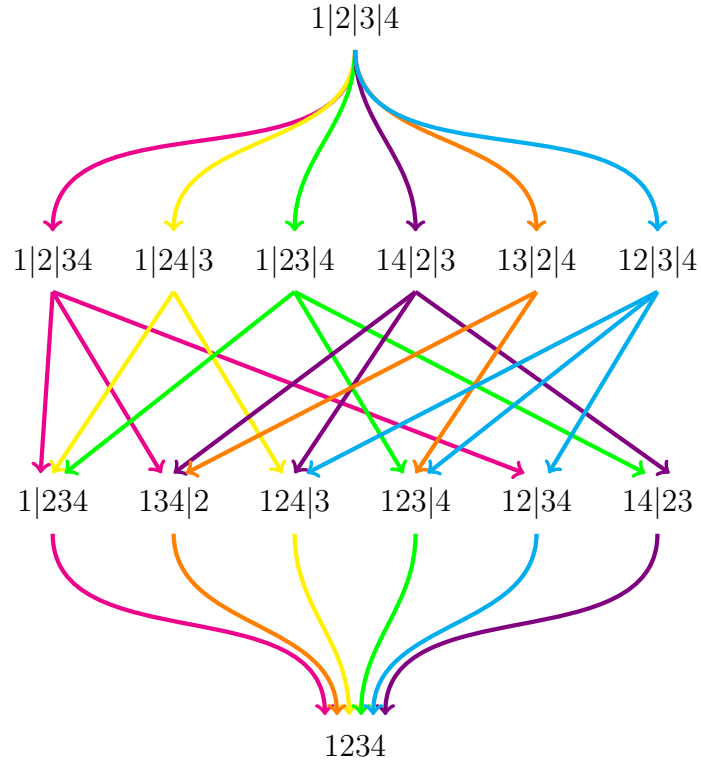
$$\mathcal{NCC} = \bigcup_{n>0} \mathcal{NCC}_n$$

Remark. *The bottom element of this poset is $\{\{1, \dots, n\}\}$, and the top element is $\{\{1\}, \dots, \{n\}\}$.*

Theorem 4. *Let ncc_n be the cardinal of \mathcal{NCC}_n . We have $ncc_n = n^{n-2}$.*

Example (The poset of \mathcal{NC}_4).

To shorten labels, we represent $\{\{1\}, \{2, 3\}, \{4\}\}$ by $1|23|4$.



There are $4^2 = 16$ different maximal chains, and $\frac{1}{5} \binom{8}{4} = \frac{70}{5} = 14$ elements in this poset.

1.2.2 Kreweras complement

Definition 5 (Associated Permutation). *The permutation σ associated to a non-crossing partition has a cycle (b_1, \dots, b_k) for each block $B = \{b_1, \dots, b_k\}$ of the partition.*

Example. *The permutation associated to $\{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$ is $(1\ 2\ 5)(3\ 4)(6) = 254316$.*

Definition 6 (Kreweras Complement). *The Kreweras complement $K(P)$ of a non-crossing partition P is defined as follows :*

- *Let σ be the permutation associated to P*
- *Let π be the permutation $(n\ n-1\ n-2\ \dots\ 3\ 2\ 1) = n123\dots n-1$*
- *$K(P)$ is the non-crossing partition associated to $\pi\sigma$.*

Example ($P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$).

- $\sigma = (1\ 2\ 5)(3\ 4)(6) = 254316$
- $\pi = (6\ 5\ 4\ 3\ 2\ 1) = 612345$
- $\pi\sigma = 143265 = (1)(2\ 4)(3)(5\ 6)$
- $K(P) = \{\{1\}, \{2, 4\}, \{3\}, \{5, 6\}\}$

Proposition (Kreweras minimums). *Let $P = \{B_1, \dots, B_k\}$ be a non-crossing partition. Let $K(P) = \{B'_1, \dots, B'_l\}$ be its Kreweras complement. Then*

$$\bigcup_{1 \leq i \leq l} \min(B'_i) = B_1 \cup \bigcup_{1 < j \leq k} B_j - \max(B_i)$$

Example ($P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$).

- $K(P) = \{\{1\}, \{2, 4\}, \{3\}, \{5, 6\}\}$
- $\bigcup \min(B'_i) = \{1, 2, 3, 5\}$
- $B_1 \cup \bigcup B_i - \max(B_i) = \{1, 2, 5\} \cup \{3, 4\} - \{4\} \cup \{6\} - \{6\} = \{1, 2, 5\} \cup \{3\} \cup \emptyset = \{1, 2, 3, 5\}$

Notation. $B_{[i]}$ = block containing i .

Proposition (Kreweras block sizes). *Let $P = \{B_1, \dots, B_k\}$ be a non-crossing partition. Let $K(P) = \{B'_1, \dots, B'_l\}$ be its Kreweras complement. Then the size of the block B'_i is defined as follows :*

- *Let m_i be the i^{th} minimum of $K(P)$*
- *Define a transition $\phi(e)$ as*

$$\text{Let } j = e + 1 \text{ (or 1 if } e = n)$$

$$\phi(e) = \max(B_{[j]})$$
- *The size of B'_i is k_{min} such that $k_{min} = \min\{k > 0 \mid \phi^k(m_i) \in B_{[m_i]}\}$.*

Example ($P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$).

- $mins = \{1, 2, 3, 5\}$
- $m_1 = 1$

$$B_{[1]} = B_1$$

$$\max(B_{[2]} = \max(B_1) = 5$$
The size for m_1 is 1.
- m_2

$$B_{[2]} = B_1$$

$$\max(B_{[3]}) = \max(B_2) = 4$$

$$\max(B_{[5]}) = \max(B_1) = 5$$
The size for m_2 is 2.
- $m_3 = 3$

$$B_{[3]} = B_2$$

$$\max(B_{[4]}) = \max(B_2) = 4$$
The size for m_3 is 1.

- $m_4 = 5$

$$B_{[5]} = B_1$$

$$\max(B_{[6]}) = \max(B_3) = 6$$

$$\max(B_{[1]}) = \max(B_1) = 5$$

The size for m_4 is 2.

1.2.3 Action of \mathfrak{S}_n on \mathcal{NC}_n

Definition 7 (Action of \mathfrak{S}_n). The action of \mathfrak{S}_n on a non-crossing partition $P = \{B_1, \dots, B_l\} \in \mathcal{NC}_n$ is defined by :

- For each block $B_i = \{b_1, \dots, b_k\} : \sigma(B_i) = \{\sigma(b_1), \dots, \sigma(b_k)\}$
- We denote $\rho = \sigma(P) = \{\sigma(B_1), \dots, \sigma(B_l)\}$

Example ($\sigma = 415362$).

$$\sigma(\{\{1, 6\}, \{2, 3, 5\}, \{4\}\}) = \{\{1, 5, 6\}, \{2, 4\}, \{3\}\}$$

Remark. Note that $\sigma(P)$ is not necessarily non-crossing.

Definition 8 (Rotation). We define the rotation operator rot of $P \in \mathcal{NC}_n$ as $\text{rot}(P) = (1 \ 2 \ 3 \ \dots \ n)(P) = 23 \dots n1(P)$.

Conversely, we define rot^{-1} of P as $\text{rot}^{-1}(P) = (n \ n-1 \ \dots \ 3 \ 2 \ 1)(P) = n12 \dots n-1(P)$.

Remark. $K(K(P)) = \text{rot}^{-1}(P)$.

Example ($P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$).

- $\text{rot}(P) = \{\{1, 2\}, \{3, 4, 6\}, \{5\}\}$
- $\text{rot}^{-1}(P) = \{\{1, 2, 4\}, \{3\}, \{5, 6\}\}$

1.3 Non-crossing 2-partitions

Definition 9 (Non-crossing 2-partition). A non-crossing 2-partition of a set E is a pair (P, σ) where :

- P is a non-crossing partition of E
- σ is a permutation of the elements of E

- For each sorted block $B_i = \{b_1, \dots, b_k\} \in P$, we have $\sigma(b_i) < \dots < \sigma(b_k)$

We denote by \mathcal{NC}_n^2 the set of non-crossing 2-partitions of $[n]$.

$$\mathcal{NC}^2 = \bigcup_{n \geq 0} \mathcal{NC}_n^2$$

Example (\mathcal{NC}_6^2). $P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$ $\sigma = 413265$
 $\rho = \{\{1, 3, 6\}, \{2\}, \{4, 5\}\}$

Theorem 5. Let nc_n^2 be the cardinal of \mathcal{NC}_n^2 . We have $nc_n^2 = (n+1)^{n-1}$.

Example ($n = 1, 2, 3$).

- $n = 1$: $nc_1^2 = 1$
 $\{\{1\}\} \quad 1 \quad \rho = P$
- $n = 2$: $nc_2^2 = 3$
 $\{\{1\}, \{2\}\} \quad 12 \quad \rho = P$
 $\{\{1\}, \{2\}\} \quad 21 \quad \rho = P$
 $\{\{1, 2\}\} \quad 12 \quad \rho = P$
- $n = 3$: $nc_3^2 = 16$
 $\{\{1\}, \{2\}, \{3\}\} \quad 123 \quad \rho = P$
 $\{\{1\}, \{2\}, \{3\}\} \quad 132 \quad \rho = P$
 $\{\{1\}, \{2\}, \{3\}\} \quad 213 \quad \rho = P$
 $\{\{1\}, \{2\}, \{3\}\} \quad 231 \quad \rho = P$
 $\{\{1\}, \{2\}, \{3\}\} \quad 312 \quad \rho = P$
 $\{\{1\}, \{2\}, \{3\}\} \quad 321 \quad \rho = P$
 $\{\{1, 2\}, \{3\}\} \quad 123 \quad \rho = P$
 $\{\{1, 2\}, \{3\}\} \quad 132 \quad \rho = \{\{1, 3\}, \{2\}\}$
 $\{\{1, 2\}, \{3\}\} \quad 231 \quad \rho = \{\{1\}, \{2, 3\}\}$
 $\{\{1\}, \{2, 3\}\} \quad 123 \quad \rho = P$

$\{\{1\}, \{2, 3\}\}$	213	$\rho = \{\{1, 3\}, \{2\}\}$
$\{\{1\}, \{2, 3\}\}$	312	$\rho = \{\{1, 2\}, \{3\}\}$
$\{\{1, 3\}, \{2\}\}$	123	$\rho = P$
$\{\{1, 3\}, \{2\}\}$	132	$\rho = \{\{1, 2\}, \{3\}\}$
$\{\{1, 3\}, \{2\}\}$	213	$\rho = \{\{1\}, \{2, 3\}\}$
$\{\{1, 2, 3\}\}$	123	$\rho = P$

Proposition. *This means we can create a bijection between \mathcal{PF}_n and \mathcal{NC}_n^2 .*

- $\mathcal{PF}_n \rightarrow \mathcal{NC}_n^2$: Let $f = (a_1, \dots, a_n) \in \mathcal{PF}_n$ be our parking function. For $i \in \{1, \dots, n\}$, we define :

l_i : the number of occurrences of i in f .

$im_i : \{j \mid a_j = i\}$

The corresponding non-crossing partition will have the following constraints :

For each $i \in \{1, \dots, n\}$, if $l_i > 0$, then there is a block $B_{[i]}$ of length l_i with minimum element i .

$\sigma(B_{[i]}) = im_i$

There is a unique set partition $P = \bigcup_i B_{[i]}$ of $[n]$ and a unique permutation σ respecting these conditions such that $(P, \sigma) \in \mathcal{NC}_n^2$.

- $\mathcal{NC}_n^2 \rightarrow \mathcal{PF}_n$: Let (P, σ) with $P = \{B_1, \dots, B_l\}$ be our non-crossing 2-partition. For each block $B_i = \{b_1, \dots, b_k\} \in P$:

$m_i = \min(B_i) = b_1$

$pos_i = \sigma(B_i)$

For each $j \in pos_i$, we define $a_j = m_i$

The corresponding parking function is (a_1, \dots, a_n) .

Example ($n = 8$).

$$P = \{\{1, 2, 5\}, \{3, 4\}, \{6, 8\}, \{7\}\}$$

$$\sigma = 36187245$$

$$f = (3, 6, 1, 7, 6, 1, 1, 3)$$

1.3.1 The non-crossing 2-partitions poset

Definition 10 (\succ^2). We say that (P, σ) covers (Q, τ) , written $(P, \sigma) \succ^2 (Q, \tau)$, if $\exists B_i, B_j \in P$ such that

- $Q = P - \{B_i, B_j\} \cup \{B_i \cup B_j\}$
- $l \neq i, j, b \in B_l \rightarrow \tau(b) = \sigma(b)$
- Let $B_i \cup B_j = \{b_1, \dots, b_k\} :$
 $\tau(B_i \cup B_j) = \sigma(B_i \cup B_j)$
 $\tau(b_1) < \dots < \tau(b_k)$

Example.

- $P = \{\{1, 6\}, \{2, 3\}, \{4\}, \{5\}\}$
- $\sigma = 236154$
- $Q = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$
- $\tau = 235164$
- $(P, \sigma) \succ^2 (Q, \tau)$
- $(P, \sigma) \not\succ^2 (Q, \sigma)$, because $\sigma(\{2, 3, 5\}) = \{3, 6, 5\}$ is not *ordemagenta*.

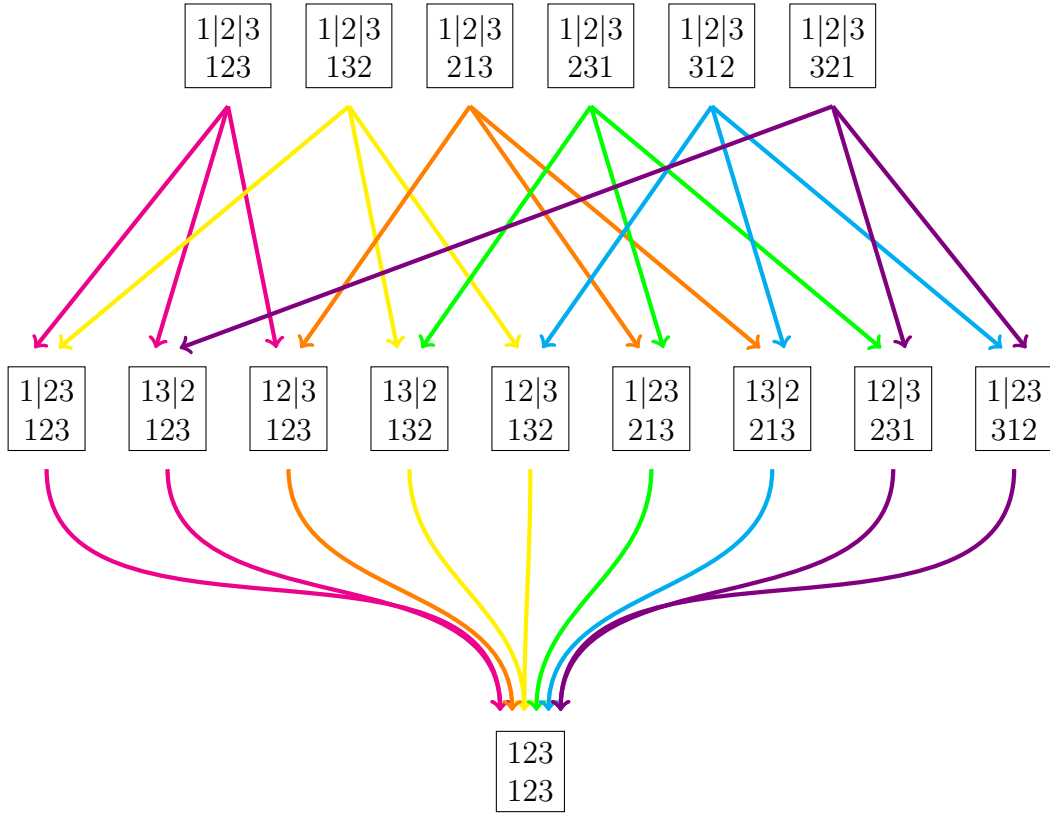
Proposition. This covering relation defines the poset of \mathcal{NC}_n^2 .

Remark. The bottom element of this poset is $(\{\{1, \dots, n\}\}, 12 \dots n)$, and the top elements are $(\{\{1\}, \dots, \{n\}\}, \sigma) \mid \sigma \in \mathfrak{S}_n$.

Example (The poset of \mathcal{NC}_3^2).

To shorten labels, we represent $(\{\{1, 3\}, \{2\}\}, 213)$ by

13 2
213



There are $4^2 = 16$ elements in this poset.

1.3.2 The parking functions poset

Definition 11 (Rank). Given $f = (a_1, \dots, a_n) \in \mathcal{PF}_n$, let

$$b_i = \begin{cases} 1 & \text{if } \exists j \mid a_j = i \\ 0 & \text{otherwise} \end{cases}$$

We define the rank of f , noted $rk(f)$, as

$$\sum_{1 \leq i \leq n} b_i$$

Example.

$$\begin{aligned}rk((1, 5, 4, 2, 3, 3, 1)) &= 5 \\rk((4, 7, 1, 1, 3, 2, 2, 8)) &= 6\end{aligned}$$

Definition 12 (\succ_{pf}). Since \mathcal{PF}_n and \mathcal{NC}_n^2 are in bijection, we can define a covering relation \succ_{pf} for \mathcal{PF}_n as follows :
 $f \in \mathcal{PF}_n \succ_{pf} g \in \mathcal{PF}_n$ if and only if :

- (P, σ) is the non-crossing 2-partition associated to f
- (Q, τ) is the non-crossing 2-partition associated to g
- $(P, \sigma) \succ^2 (Q, \tau)$

Example.

- $P = \{\{1, 6\}, \{2, 3\}, \{4\}, \{5\}\}$
- $\sigma = 236154$
- $Q = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$
- $\tau = 235164$
- $f = (4, 1, 2, 1, 5, 2) \succ_{pf} g = (4, 1, 2, 1, 2, 2)$

Remark. If $f \succ_{pf} g$, then $rk(f) = rk(g) + 1$, and there exists i and j such that :

- $i < j$
- There is at least 1 occurrence of i in f
- There is at least 1 occurrence of j in f

$$b_k = \begin{cases} i & \text{if } a_k = j \\ a_k & \text{otherwise} \end{cases}$$

Proposition. This covering relation defines the poset of \mathcal{PF}_n .

Remark. The bottom element of this poset is $(\underbrace{1, \dots, 1}_n)$, and the top elements are the permutations of $\{1, \dots, n\}$.

Example (The poset of \mathcal{PF}_3).



Chapter 2

The rational case

For the whole chapter, we will consider 2 *coprime* integers a and b (meaning a and b have 1 as their greatest common divisor).

2.1 Rational Parking Functions

Definition 13 (a, b - Parking Function). An a, b - parking function is a sequence (a_1, a_2, \dots, a_n) such that :

- $n = a$
- its non-decreasing reordering (b_1, b_2, \dots, b_n) has $b_i < \frac{b}{a}(i - 1) + 1$ for all i .

We denote by \mathcal{PF}_a^b the set of a, b - parking functions.

Example.

- *Ex. 1* : $a > b$

$$a = 7$$

$$b = 3$$

Limits of the non-decreasing reordering of any $f \in \mathcal{PF}_7^3$:

$$[1, 1\frac{3}{7}, 1\frac{6}{7}, 2\frac{2}{7}, 2\frac{5}{7}, 3\frac{1}{7}, 3\frac{4}{7}]$$

$$f_1 = (2, 1, 1, 3, 2, 3, 1) \in \mathcal{PF}_7^3$$

$$f_2 = (2, 1, 2, 3, 2, 3, 1) \notin \mathcal{PF}_7^3, \text{ even though } f_2 \in \mathcal{PF}_7$$

- *Ex. 2* : $a < b$

$$a = 5$$

$$b = 7$$

Limits of the non-decreasing reordering of any $f \in \mathcal{PF}_5^7$:

$$[1, 2\frac{2}{5}, 3\frac{4}{5}, 5\frac{1}{5}, 6\frac{3}{5}]$$

$$f_3 = (6, 3, 5, 1, 2) \in \mathcal{PF}_5^7, \text{ even though } f_3 \notin \mathcal{PF}_5$$

$$f_4 = (6, 3, 5, 1, 3) \notin \mathcal{PF}_5^7$$

Theorem 6. *Let pf_a^b be the cardinal of \mathcal{PF}_a^b . We have $pf_a^b = b^{a-1}$.*

Example ($a = 3, b = 5$).

- $pf_a^b = 25$
- *Limits* : $[1, 2\frac{2}{3}, 4\frac{1}{3}]$

(1, 1, 1)	(1, 1, 2)	(1, 1, 3)	(1, 1, 4)	(1, 2, 1)	(1, 2, 2)	(1, 2, 3)
(1, 2, 4)	(1, 3, 1)	(1, 3, 2)	(1, 4, 1)	(1, 4, 2)	(2, 1, 1)	(2, 1, 2)
(2, 1, 3)	(2, 1, 4)	(2, 2, 1)	(2, 3, 1)	(2, 4, 1)	(3, 1, 1)	(3, 1, 2)
(3, 2, 1)	(4, 1, 1)	(4, 1, 2)	(4, 2, 1)			

Chapter 3

Trees