

# Rational Parking Functions

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## **Abstract**

This is an abstract about Rational Parking Functions

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# Chapter 1

## The integer case

### 1.1 Parking Functions

**Definition 1** (Parking Function). *A parking function is a sequence of positive integers  $(a_1, a_2, \dots, a_n)$  such that its non-decreasing reordering  $(b_1, b_2, \dots, b_n)$  has  $b_i \leq i$  for all  $i$ .*

*We denote by  $\mathcal{PF}_n$  the set of parking functions of length  $n$ .*

$$\mathcal{PF} = \bigcup_{n \geq 0} \mathcal{PF}_n$$

.

**Example.**

$$f_1 = (7, 3, 1, 4, 2, 5, 2) \in \mathcal{PF}_7$$

$$f_2 = (7, 3, 1, 4, 2, 5, 4) \notin \mathcal{PF}_7$$

**Theorem 1.** *Let  $pf_n$  be the cardinal of  $\mathcal{PF}_n$ . We have*

$$pf_n = (n + 1)^{n-1}$$

.

**Example** ( $n = 1, 2, 3$ ).

- $n = 1$  :  $pf_1 = 1$   
(1)

- $n = 2$  :  $pf_2 = 3$   
 $(1, 1)$      $(1, 2)$      $(2, 1)$
- $n = 3$  :  $pf_3 = 16$   
 $(1, 1, 1)$      $(1, 1, 2)$      $(1, 1, 3)$      $(1, 2, 1)$      $(1, 2, 2)$      $(1, 2, 3)$      $(1, 3, 1)$   
 $(1, 3, 2)$      $(2, 1, 1)$      $(2, 1, 2)$      $(2, 1, 3)$      $(2, 2, 1)$      $(2, 3, 1)$      $(3, 1, 1)$   
 $(3, 1, 2)$      $(3, 2, 1)$

### 1.1.1 Primitive parking functions

**Definition 2** (Primitive). A parking function  $(a_1, a_2, \dots, a_n)$  is said primitive if it is already in non-decreasing order.

We denote by  $\mathcal{PF}'_n$  the set of primitive parking functions of length  $n$ .

$$\mathcal{PF}' = \bigcup_{n \geq 0} \mathcal{PF}'_n$$

**Example.**

$$\begin{aligned} f_1 &= (1, 2, 2, 3) \in \mathcal{PF}'_4 \\ f_2 &= (1, 2, 3, 2) \notin \mathcal{PF}'_4, \text{ even though } f_2 \in \mathcal{PF}_4 \end{aligned}$$

**Theorem 2.** Let  $pf'_n$  be the cardinal of  $\mathcal{PF}'_n$ . We have

$$pf'_n = \frac{1}{n+1} \binom{2n}{n}$$

which is the  $n^{\text{th}}$  Catalan number  $Cat(n)$ .

**Example** ( $n = 1, 2, 3$ ).

- $n = 1$  :  $pf'_1 = 1$   
 $(1)$
- $n = 2$  :  $pf'_2 = 2$   
 $(1, 1)$      $(1, 2)$
- $n = 3$  :  $pf'_3 = 5$   
 $(1, 1, 1)$      $(1, 1, 2)$      $(1, 1, 3)$      $(1, 2, 2)$      $(1, 2, 3)$

## 1.2 Non-crossing Partitions

**Definition 3** (Non-crossing Partition). A non-crossing partition of a set  $E$  is a set partition  $P = \{E_1, E_2, \dots, E_k\}$  such that if  $a, c \in E_i$ ,  $b, d \in E_j$ , and  $i \neq j$ , then we do not have  $a < b < c < d$ , nor  $a > b > c > d$ .

We denote by  $\mathcal{NC}_n$  the set of non-crossing partitions of  $\{1, 2, \dots, n\}$ .

$$\mathcal{NC} = \bigcup_{n>0} \mathcal{NC}_n$$

From this point, we assume that every partition  $P = \{B_1, \dots, B_l\}$  is sorted such that :

- For each block  $B_i = \{b_1, \dots, b_k\} \in P$ ,  $b_1 < \dots < b_k$
- $\min(B_1) < \dots < \min(B_k)$

**Notation.**  $[n] = \{1, 2, \dots, n\}$

**Example** ( $E = [6]$ ).

$$\begin{aligned} P_1 &= \{\{1, 2, 5\}, \{3, 4\}, \{6\}\} \in \mathcal{NC}_6 \\ P_2 &= \{\{1, 2, 4\}, \{3, 5\}, \{6\}\} \notin \mathcal{NC}_6 \end{aligned}$$

**Theorem 3.** Let  $nc_n$  be the cardinal of  $\mathcal{NC}_n$ . We have

$$nc_n = \frac{1}{n+1} \binom{2n}{n}$$

which is again the  $n^{\text{th}}$  Catalan number  $Cat(n)$ .

**Example** ( $n = 1, 2, 3$ ).

- $n = 1$  :  $nc_1 = 1$   
 $\{\{1\}\}$
- $n = 2$  :  $nc_2 = 2$   
 $\{\{1, 2\}\} \quad \{\{1\}, \{2\}\}$
- $n = 3$  :  $nc_3 = 5$   
 $\{\{1, 2, 3\}\} \quad \{\{1\}, \{2, 3\}\} \quad \{\{1, 3\}, \{2\}\} \quad \{\{1, 2\}, \{3\}\} \quad \{\{1\}, \{2\}, \{3\}\}$

**Proposition.** This means we can create a bijection between  $\mathcal{PF}'_n$  and  $\mathcal{NC}_n$ .

*Proof.*

- $\mathcal{NC}_n \rightarrow \mathcal{PF}'_n$  : For each block  $B$  in the non-crossing partition, take  $i = \min(B)$ , and let  $k_i = \text{size}(B)$ .  
 $k_i = 0$  if  $i$  is not the minimum of a block.  
 The corresponding parking function is  $(\underbrace{1, \dots, 1}_{k_1}, \underbrace{2, \dots, 2}_{k_2}, \dots, \underbrace{n, \dots, n}_{k_n})$ .
- $\mathcal{PF}'_n \rightarrow \mathcal{NC}_n$  : For each  $i$  in  $[n]$ , if  $i$  appears  $n_i$  times in the parking function,  $B_i$  will be of size  $n_i$  with minimum element  $i$ . There is a unique set partition  $P = \bigcup_i B_i$  of  $[n]$  respecting these conditions that is non-crossing : for each minimum  $i$  in *decreasing order*, add the  $n_i$  first free elements of  $[i + 1, i + 2, \dots, n, 1, \dots, i - 1]$  to  $B_i$ .

□

**Example** ( $n = 6$ ).

$$P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\} \quad f = (1, 1, 1, 3, 3, 6)$$

**Corollary.** *A non-crossing partition can be represented by the minimums and sizes of its blocks.*

**Example.**  $\{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$  can be represented by the following dictionary :

- $1 : 3$
- $3 : 2$
- $6 : 1$

A non-crossing partition of  $[n]$  can be represented graphically on a regular  $n$ -vertices polygon, with vertices labeled from 1 to  $n$  clockwise. We then represent each block  $B = \{b_1, \dots, b_k\}$  by the convex hull of  $\{b_1, \dots, b_k\}$ .

**Example** ( $P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$ ).



Thus non-crossing meaning the hulls are disjoint.

**Example** (Counter-example :  $P = \{\{1, 5, 2\}, \{3, 6\}, \{4\}\}$ ).



This partition is not non-crossing, as the convex hulls of  $\{1, 2, 5\}$  and  $\{3, 6\}$  are not disjoint.

### 1.2.1 The non-crossing partitions poset

**Definition 4** ( $\succ$ ). We say that  $P$  covers  $Q$ , written  $P \succ Q$ , if  $\exists B_i, B_j \in P$  such that  $Q = P - \{B_i, B_j\} \cup \{B_i \cup B_j\}$

**Example.**  $\{\{1, 6\}, \{2, 3\}, \{4, 5\}\} \succ \{\{1, 2, 3, 6\}, \{4, 5\}\}$

- $B_i = \{1, 6\}$



- $B_j = \{2, 3\}$

**Proposition.** *This covering relation defines the poset of  $\mathcal{NC}_n$ . We denote by  $\mathcal{NCC}_n$  the set of maximal chains in the poset of  $\mathcal{NC}_n$ .*

$$\mathcal{NCC} = \bigcup_{n>0} \mathcal{NCC}_n$$

**Remark.** *The bottom element of this poset is  $\{\{1, \dots, n\}\}$ , and the top element is  $\{\{1\}, \dots, \{n\}\}$ .*

**Theorem 4.** *Let  $ncc_n$  be the cardinal of  $\mathcal{NCC}_n$ . We have*

$$ncc_n = n^{n-2}$$

.

**Example** (The poset of  $\mathcal{NC}_4$ ).

To shorten labels, we represent  $\{\{1\}, \{2, 3\}, \{4\}\}$  by  $1|23|4$ .



There are  $4^2 = 16$  different maximal chains, and  $\frac{1}{5} \binom{8}{4} = \frac{70}{5} = 14$  elements in this poset.

### 1.2.2 Kreweras complement

**Definition 5** (Associated Permutation). *The permutation  $\sigma$  associated to a non-crossing partition has a cycle  $(b_1, \dots, b_k)$  for each block  $B = \{b_1, \dots, b_k\}$  of the partition.*

**Example.** *The permutation associated to  $\{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$  is  $(1\ 2\ 5)(3\ 4)(6) = 254316$ .*

**Definition 6** (Kreweras Complement). *The Kreweras complement  $K(P)$  of a non-crossing partition  $P$  is defined as follows :*

- *Let  $\sigma$  be the permutation associated to  $P$*
- *Let  $\pi$  be the permutation  $(n\ n-1\ n-2\ \dots\ 3\ 2\ 1) = n123\dots n-1$*
- *$K(P)$  is the non-crossing partition associated to  $\pi\sigma$ .*

**Example** ( $P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$ ).

- $\sigma = (1\ 2\ 5)(3\ 4)(6) = 254316$
- $\pi = (6\ 5\ 4\ 3\ 2\ 1) = 612345$
- $\pi\sigma = 143265 = (1)(2\ 4)(3)(5\ 6)$
- $K(P) = \{\{1\}, \{2, 4\}, \{3\}, \{5, 6\}\}$

**Proposition** (Kreweras minimums). *Let  $P = \{B_1, \dots, B_k\}$  be a non-crossing partition. Let  $K(P) = \{B'_1, \dots, B'_l\}$  be its Kreweras complement. Then*

$$\bigcup_{1 \leq i \leq l} \min(B'_i) = B_1 \cup \bigcup_{1 < j \leq k} B_j - \max(B_i)$$

**Example** ( $P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$ ).

- $K(P) = \{\{1\}, \{2, 4\}, \{3\}, \{5, 6\}\}$
- $\bigcup \min(B'_i) = \{1, 2, 3, 5\}$

- $B_1 \cup \bigcup B_i - \max(B_i) = \{1, 2, 5\} \cup \{3, 4\} - \{4\} \cup \{6\} - \{6\} = \{1, 2, 5\} \cup \{3\} \cup \emptyset = \{1, 2, 3, 5\}$

**Notation.**  $B_{[i]}$  = block containing  $i$ .

**Proposition** (Kreweras block sizes). *Let  $P = \{B_1, \dots, B_k\}$  be a non-crossing partition. Let  $K(P) = \{B'_1, \dots, B'_l\}$  be its Kreweras complement. Then the size of the block  $B'_i$  is defined as follows :*

- Let  $m_i$  be the  $i^{\text{th}}$  minimum of  $K(P)$
- Define a transition  $\phi(e)$  as  

$$\text{Let } j = e + 1 \text{ (or 1 if } e = n)$$

$$\phi(e) = \max(B_{[j]})$$
- The size of  $B'_i$  is  $k_{\min}$  such that  $k_{\min} = \min\{k > 0 \mid \phi^k(m_i) \in B_{[m_i]}\}$ .

**Example** ( $P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$ ).

- $\text{mins} = \{1, 2, 3, 5\}$
- $m_1 = 1$   

$$B_{[1]} = B_1$$

$$\max(B_{[2]}) = \max(B_1) = 5$$

The size for  $m_1$  is 1.
- $m_2$   

$$B_{[2]} = B_1$$

$$\max(B_{[3]}) = \max(B_2) = 4$$

$$\max(B_{[5]}) = \max(B_1) = 5$$

The size for  $m_2$  is 2.
- $m_3 = 3$   

$$B_{[3]} = B_2$$

$$\max(B_{[4]}) = \max(B_2) = 4$$

The size for  $m_3$  is 1.

- $m_4 = 5$

$$B_{[5]} = B_1$$

$$\max(B_{[6]}) = \max(B_3) = 6$$

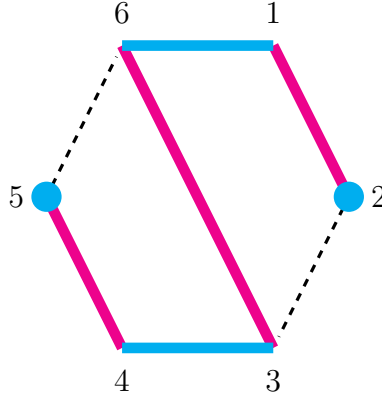
$$\max(B_{[1]}) = \max(B_1) = 5$$

The size for  $m_4$  is 2.

**Definition 7** (Mutually Non-crossing Partitions). 2 partitions  $P$  and  $Q$  are said mutually non-crossing if :

- $P$  is non-crossing
- $Q$  is non-crossing
- For every block  $B_i$  of  $P$  and every block  $B_j$  of  $Q$ , if  $a, c \in B_i$  and  $b, d \in B_j$ , then we can not have  $a < b < c < d$ , nor  $a > b > c > d$ .

**Example** ( $P = \{\{1, 2\}, \{3, 6\}, \{4, 5\}\}, Q = \{\{1, 6\}, \{2\}, \{3, 4\}, \{5\}\}$ ).



**Example** (Counter-example :  $P = \{\{1, 2\}, \{3, 6\}, \{4, 5\}\}, Q = \{\{1, 6\}, \{2, 5\}, \{3, 4\}\}$ ).

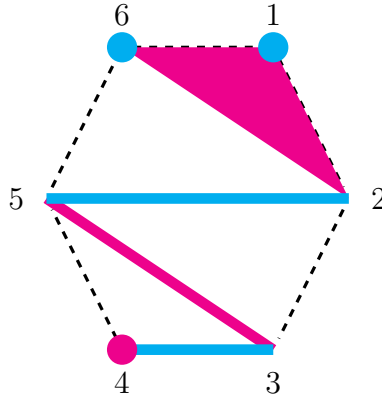


**Remark.** Note that vertices can touch, but the edges of the convex hulls can not cross.

**Proposition.** For any non-crossing partition  $P$ ,  $P$  and  $K(P)$  are mutually non-crossing.

Furthermore,  $K(P)$  is a densest partition that is mutually non-crossing with  $P$ . That is, no partition  $Q$  that is mutually non-crossing with  $P$  has less blocks than  $K(P)$ .

**Example** ( $P = \{1, 2, 6\}, \{3, 5\}, \{4\}\}$ ).  $Q = \{\{1\}, \{2, 5\}, \{3, 4\}, \{6\}\}$

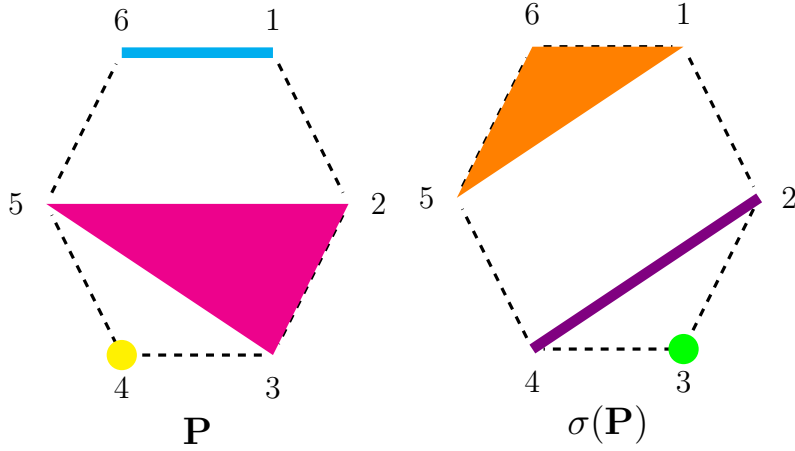


### 1.2.3 Action of $\mathfrak{S}_n$ on partitions of $[n]$

**Definition 8** (Action of  $\mathfrak{S}_n$ ). The action of  $\mathfrak{S}_n$  on a partition  $P = \{B_1, \dots, B_l\}$  of  $[n]$  is defined by :

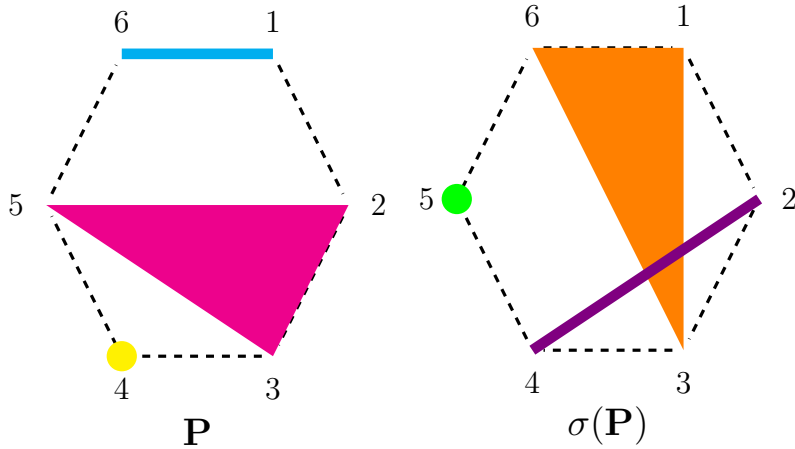
- For each block  $B_i = \{b_1, \dots, b_k\} : \sigma(B_i) = \{\sigma(b_1), \dots, \sigma(b_k)\}$
- When  $P \in \mathcal{NC}_n$ , we denote  $\rho = \sigma(P) = \{\sigma(B_1), \dots, \sigma(B_l)\}$

**Example** ( $\sigma = 415362, P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$ ).  
 $\sigma(P) = \{\{1, 5, 6\}, \{2, 4\}, \{3\}\}$



**Remark.** Note that  $\mathcal{NC}_n$  is not stable under the action of  $\mathfrak{S}_n$ . That is, even if  $P$  is non-crossing,  $\sigma(P)$  is not necessarily non-crossing.

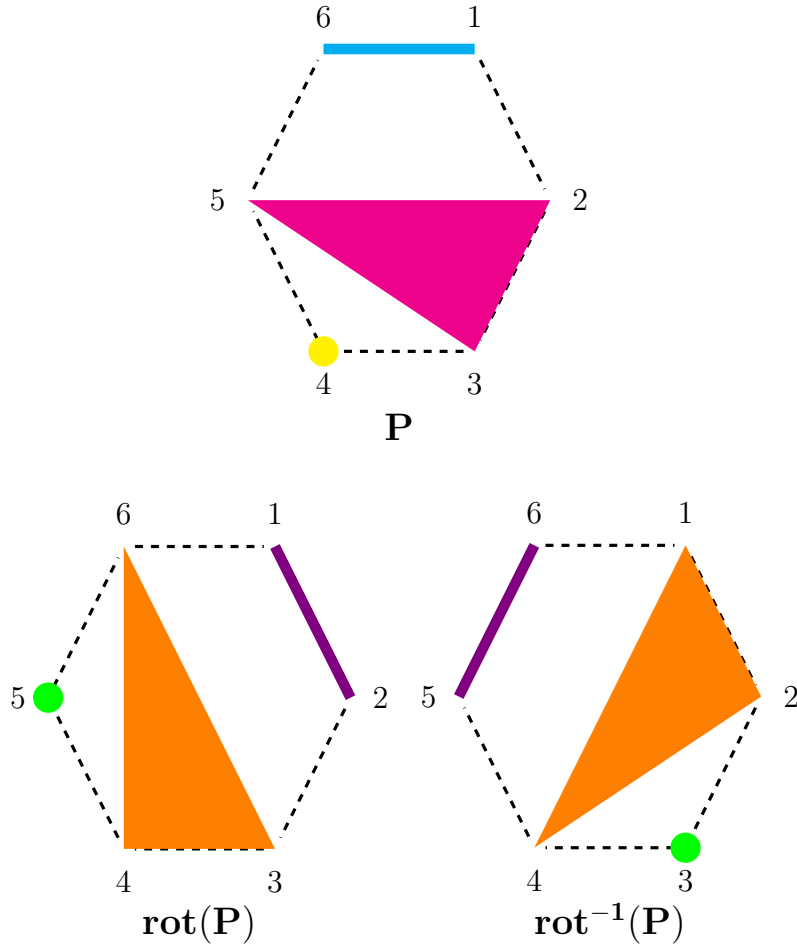
**Example** (Counter-example :  $\sigma = 413562, P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$ ).  
 $\sigma(P) = \{\{1, 3, 6\}, \{2, 4\}, \{5\}\}$



**Definition 9** (Rotation). We define the rotation operator  $rot$  of  $P \in \mathcal{NC}_n$  as  $rot(P) = (1\ 2\ 3\ \dots\ n)(P) = 23\dots n1(P)$ . Conversely, we define  $rot^{-1}$  of  $P$  as  $rot^{-1}(P) = (n\ n-1\ \dots\ 3\ 2\ 1)(P) = n12\dots n-1(P)$ .

**Example** ( $P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$ ).

- $rot(P) = \{\{1, 2\}, \{3, 4, 6\}, \{5\}\}$
- $rot^{-1}(P) = \{\{1, 2, 4\}, \{3\}, \{5, 6\}\}$



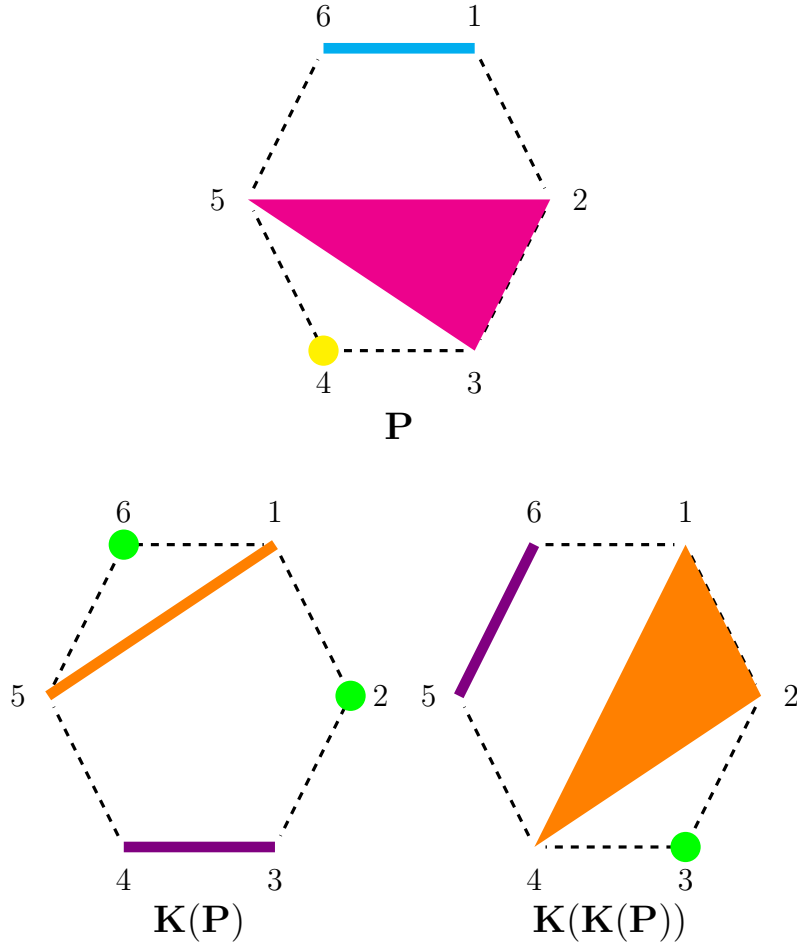
**Remark.**

- $rot(rot^{-1}(P)) = rot^{-1}(rot(P)) = P$

- $rot(P)$  and  $rot^{-1}(P)$  are always non-crossing partitions.
- If  $P \in \mathcal{NC}_n$ , then  $rot^n(P) = rot^{-n}(P) = P$ .

**Proposition.**  $K(K(P)) = rot^{-1}(P)$ .

**Example** ( $P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$ ).



### 1.3 Non-crossing 2-partitions

**Definition 10** (Non-crossing 2-partition). A non-crossing 2-partition of a set  $E$  is a pair  $(P, \sigma)$  where :



- $P$  is a non-crossing partition of  $E$
- $\sigma$  is a permutation of the elements of  $E$
- For each sorted block  $B_i = \{b_1, \dots, b_k\} \in P$ , we have  $\sigma(b_i) < \dots < \sigma(b_k)$

We denote by  $\mathcal{NC}_n^2$  the set of non-crossing 2-partitions of  $[n]$ .

$$\mathcal{NC}^2 = \bigcup_{n>0} \mathcal{NC}_n^2$$

**Example** ( $\mathcal{NC}_6^2$ ).  $P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$   $\sigma = 413265$   
 $\rho = \{\{1, 3, 6\}, \{2\}, \{4, 5\}\}$

**Theorem 5.** Let  $nc_n^2$  be the cardinal of  $\mathcal{NC}_n^2$ . We have

$$nc_n^2 = (n+1)^{n-1}$$

**Example** ( $n = 1, 2, 3$ ).

- $n = 1$  :  $nc_1^2 = 1$   
 $\{\{1\}\} \quad 1 \quad \rho = P$
- $n = 2$  :  $nc_2^2 = 3$   
 $\{\{1\}, \{2\}\} \quad 12 \quad \rho = P$   
 $\{\{1\}, \{2\}\} \quad 21 \quad \rho = P$   
 $\{\{1, 2\}\} \quad 12 \quad \rho = P$
- $n = 3$  :  $nc_3^2 = 16$   
 $\{\{1\}, \{2\}, \{3\}\} \quad 123 \quad \rho = P$   
 $\{\{1\}, \{2\}, \{3\}\} \quad 132 \quad \rho = P$   
 $\{\{1\}, \{2\}, \{3\}\} \quad 213 \quad \rho = P$   
 $\{\{1\}, \{2\}, \{3\}\} \quad 231 \quad \rho = P$   
 $\{\{1\}, \{2\}, \{3\}\} \quad 312 \quad \rho = P$   
 $\{\{1\}, \{2\}, \{3\}\} \quad 321 \quad \rho = P$   
 $\{\{1, 2\}, \{3\}\} \quad 123 \quad \rho = P$

$\{\{1, 2\}, \{3\}\}$	132	$\rho = \{\{1, 3\}, \{2\}\}$
$\{\{1, 2\}, \{3\}\}$	231	$\rho = \{\{1\}, \{2, 3\}\}$
$\{\{1\}, \{2, 3\}\}$	123	$\rho = P$
$\{\{1\}, \{2, 3\}\}$	213	$\rho = \{\{1, 3\}, \{2\}\}$
$\{\{1\}, \{2, 3\}\}$	312	$\rho = \{\{1, 2\}, \{3\}\}$
$\{\{1, 3\}, \{2\}\}$	123	$\rho = P$
$\{\{1, 3\}, \{2\}\}$	132	$\rho = \{\{1, 2\}, \{3\}\}$
$\{\{1, 3\}, \{2\}\}$	213	$\rho = \{\{1\}, \{2, 3\}\}$
$\{\{1, 2, 3\}\}$	123	$\rho = P$

**Proposition.** *This means we can create a bijection between  $\mathcal{PF}_n$  and  $\mathcal{NC}_n^2$ .*

*Proof.*

- $\mathcal{PF}_n \rightarrow \mathcal{NC}_n^2$  : Let  $f = (a_1, \dots, a_n) \in \mathcal{PF}_n$  be our parking function. For  $i \in \{1, \dots, n\}$ , we define :

$l_i$  : the number of occurrences of  $i$  in  $f$ .

$im_i : \{j \mid a_j = i\}$

The corresponding non-crossing partition will have the following constraints :

For each  $i \in \{1, \dots, n\}$ , if  $l_i > 0$ , then there is a block  $B_{[i]}$  of length  $l_i$  with minimum element  $i$ .

$\sigma(B_{[i]}) = im_i$

There is a unique set partition  $P = \bigcup_i B_{[i]}$  of  $[n]$  and a unique per-

mutation  $\sigma$  respecting these conditions such that  $(P, \sigma) \in \mathcal{NC}_n^2$  : for each minimum  $i$  in *decreasing order*, add the  $n_i$  first free elements of  $[i+1, i+2, \dots, n, 1, \dots, i-1]$  to  $B_i$ .  $\sigma$  is then trivially obtained by the second constraint.

- $\mathcal{NC}_n^2 \rightarrow \mathcal{PF}_n$  : Let  $(P, \sigma)$  with  $P = \{B_1, \dots, B_l\}$  be our non-crossing 2-partition. For each block  $B_i = \{b_1, \dots, b_k\} \in P$  :

$$m_i = \min(B_i) = b_1$$

$$pos_i = \sigma(B_i)$$

For each  $j \in pos_i$ , we define  $a_j = m_i$   
The corresponding parking function is  $(a_1, \dots, a_n)$ .

□

**Example** ( $n = 8$ ).

$$P = \{\{1, 2, 5\}, \{3, 4\}, \{6, 8\}, \{7\}\}$$

$$\sigma = 36187245$$

$$f = (3, 6, 1, 7, 6, 1, 1, 3)$$

### 1.3.1 The non-crossing 2-partitions poset

**Definition 11** ( $\succ^2$ ). We say that  $(P, \sigma)$  covers  $(Q, \tau)$ , written  $(P, \sigma) \succ^2 (Q, \tau)$ , if  $\exists B_i, B_j \in P$  such that

- $Q = P - \{B_i, B_j\} \cup \{B_i \cup B_j\}$
- $l \neq i, j, b \in B_l \rightarrow \tau(b) = \sigma(b)$
- Let  $B_i \cup B_j = \{b_1, \dots, b_k\}$  :  
 $\tau(B_i \cup B_j) = \sigma(B_i \cup B_j)$   
 $\tau(b_1) < \dots < \tau(b_k)$

**Example.**

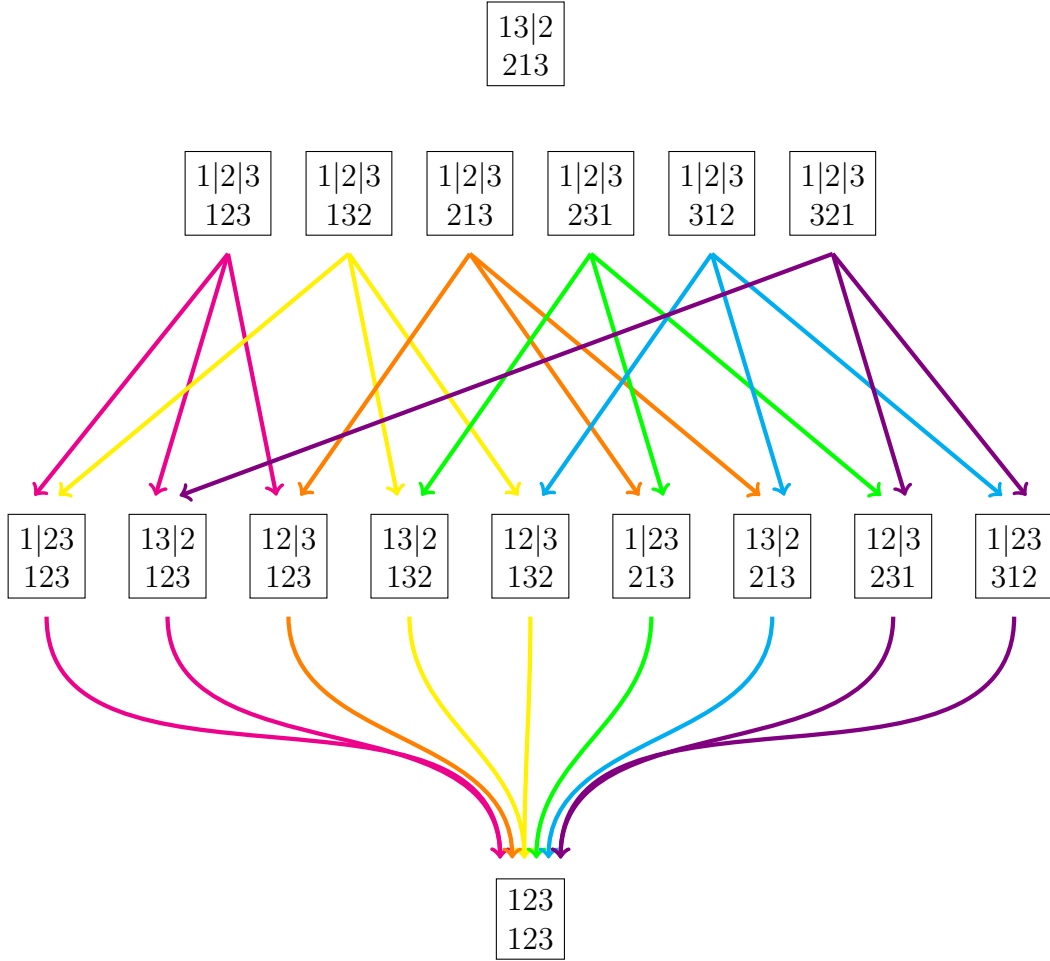
- $P = \{\{1, 6\}, \{2, 3\}, \{4\}, \{5\}\}$
- $\sigma = 236154$
- $Q = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$
- $\tau = 235164$
- $(P, \sigma) \succ^2 (Q, \tau)$
- $(P, \sigma) \not\succ^2 (Q, \sigma)$ , because  $\sigma(\{2, 3, 5\}) = \{3, 6, 5\}$  is not *ordemagenta*.

**Proposition.** This covering relation defines the poset of  $\mathcal{NC}_n^2$ .

**Remark.** The bottom element of this poset is  $(\{\{1, \dots, n\}\}, 12 \dots n)$ , and the top elements are  $\{(\{\{1\}, \dots, \{n\}\}, \sigma) \mid \sigma \in \mathfrak{S}_n\}$ .

**Example** (The poset of  $\mathcal{NC}_3^2$ ).

To shorten labels, we represent  $(\{\{1, 3\}, \{2\}\}, 213)$  by



There are  $4^2 = 16$  elements in this poset.

### 1.3.2 The parking functions poset

**Definition 12** (Rank). Given  $f = (a_1, \dots, a_n) \in \mathcal{PF}_n$ , let

$$b_i = \begin{cases} 1 & \text{if } \exists j \mid a_j = i \\ 0 & \text{otherwise} \end{cases}$$

We define the rank of  $f$ , noted  $rk(f)$ , as

$$\sum_{1 \leq i \leq n} b_i$$

**Example.**

$$\begin{aligned} rk((1, 5, 4, 2, 3, 3, 1)) &= 5 \\ rk((4, 7, 1, 1, 3, 2, 2, 8)) &= 6 \end{aligned}$$

**Definition 13** ( $\succ_{pf}$ ). Since  $\mathcal{PF}_n$  and  $\mathcal{NC}_n^2$  are in bijection, we can define a covering relation  $\succ_{pf}$  for  $\mathcal{PF}_n$  as follows :  
 $f \in \mathcal{PF}_n \succ_{pf} g \in \mathcal{PF}_n$  if and only if :

- $(P, \sigma)$  is the non-crossing 2-partition associated to  $f$
- $(Q, \tau)$  is the non-crossing 2-partition associated to  $g$
- $(P, \sigma) \succ^2 (Q, \tau)$

**Example.**

- $P = \{\{1, 6\}, \{2, 3\}, \{4\}, \{5\}\}$
- $\sigma = 236154$
- $Q = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$
- $\tau = 235164$
- $f = (4, 1, 2, 1, 5, 2) \succ_{pf} g = (4, 1, 2, 1, 2, 2)$

**Remark.** If  $f \succ_{pf} g$ , then  $rk(f) = rk(g) + 1$ , and there exists  $i$  and  $j$  such that :

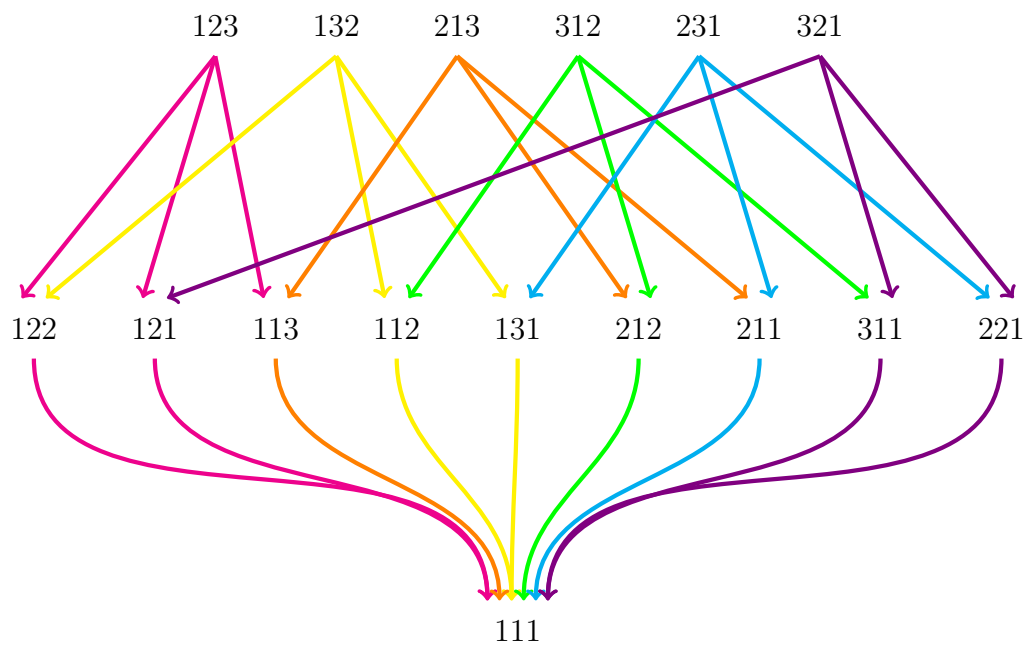
- $i < j$
- There is at least 1 occurrence of  $i$  in  $f$
- There is at least 1 occurrence of  $j$  in  $f$

$$b_k = \begin{cases} i & \text{if } a_k = j \\ a_k & \text{otherwise} \end{cases}$$

**Proposition.** This covering relation defines the poset of  $\mathcal{PF}_n$ .

**Remark.** The bottom element of this poset is  $(\underbrace{1, \dots, 1}_n)$ , and the top elements are the permutations of  $\{1, \dots, n\}$ .

**Example** (The poset of  $\mathcal{PF}_3$ ).



# Chapter 2

## The rational case

For the whole chapter, we will consider 2 *coprime* integers  $a$  and  $b$  (meaning  $a$  and  $b$  have 1 as their greatest common divisor).

### 2.1 Rational Parking Functions

**Definition 14** ( $a, b$  - Parking Function). An  $a, b$  - parking function is a sequence  $(a_1, a_2, \dots, a_n)$  such that :

- $n = a$
- its non-decreasing reordering  $(b_1, b_2, \dots, b_n)$  has  $b_i \leq \frac{b}{a}(i - 1) + 1$  for all  $i$ .

We denote by  $\mathcal{PF}_{a,b}$  the set of  $a, b$  - parking functions.

**Example.**

- *Ex. 1* :  $a > b$

$$a = 7$$

$$b = 3$$

Limits of the non-decreasing reordering of any  $f \in \mathcal{PF}_{7,3}$  :

$$[1, 1\frac{3}{7}, 1\frac{6}{7}, 2\frac{2}{7}, 2\frac{5}{7}, 3\frac{1}{7}, 3\frac{4}{7}]$$

$$f_1 = (2, 1, 1, 3, 2, 3, 1) \in \mathcal{PF}_{7,3}$$

$$f_2 = (2, 1, 2, 3, 2, 3, 1) \notin \mathcal{PF}_{7,3}, \text{ even though } f_2 \in \mathcal{PF}_7$$

- *Ex. 2* :  $a < b$

$$a = 5$$

$$b = 7$$

*Limits of the non-decreasing reordering of any  $f \in \mathcal{PF}_{5,7}$  :*

$$[1, 2\frac{2}{5}, 3\frac{4}{5}, 5\frac{1}{5}, 6\frac{3}{5}]$$

$$f_3 = (6, 3, 5, 1, 2) \in \mathcal{PF}_{5,7}, \text{ even though } f_3 \notin \mathcal{PF}_5$$

$$f_4 = (6, 3, 5, 1, 3) \notin \mathcal{PF}_{5,7}$$

**Theorem 6.** *Let  $pf_{a,b}$  be the cardinal of  $\mathcal{PF}_{a,b}$ . We have*

$$pf_{a,b} = b^{a-1}$$

**Example** ( $a = 3, b = 5$ ).

- $pf_{a,b} = 25$
- *Limits* :  $[1, 2\frac{2}{3}, 4\frac{1}{3}]$

(1, 1, 1)	(1, 1, 2)	(1, 1, 3)	(1, 1, 4)	(1, 2, 1)	(1, 2, 2)	(1, 2, 3)
(1, 2, 4)	(1, 3, 1)	(1, 3, 2)	(1, 4, 1)	(1, 4, 2)	(2, 1, 1)	(2, 1, 2)
(2, 1, 3)	(2, 1, 4)	(2, 2, 1)	(2, 3, 1)	(2, 4, 1)	(3, 1, 1)	(3, 1, 2)
(3, 2, 1)	(4, 1, 1)	(4, 1, 2)	(4, 2, 1)			

**Remark.**  $\mathcal{PF}_{n,n+1} = \mathcal{PF}_n$ . In fact, we do have  $b^{a-1} = (n+1)^{n-1}$ .

### 2.1.1 Rational primitive parking functions

**Definition 15** (Rational Primitive). *A rational parking function  $f$  is said primitive if it is already in non-decreasing order.*

*We denote by  $\mathcal{PF}'_{a,b}$  the set of primitive  $a, b$  - parking functions.*

**Example** ( $a = 4, b = 3$ ). *Limits* :  $[1, 1\frac{3}{4}, 2\frac{1}{2}, 3\frac{1}{4}]$

$$f_1 = (1, 1, 2, 2) \in \mathcal{PF}'_{4,3}$$

$$f_2 = (1, 1, 2, 1) \notin \mathcal{PF}'_{4,3}, \text{ even though } f_2 \in \mathcal{PF}_{4,3}.$$

**Theorem 7.** *Let  $pf'_{a,b}$  be the cardinal of  $\mathcal{PF}'_{a,b}$ . We have*

$$pf'_{a,b} = \frac{1}{a+b} \binom{a+b}{b}$$

*which is the rational Catalan number  $Cat(a, b)$ .*



**Example** ( $a = 3, b = 5$ ).

•  $pf'_{a,b} = 7$  • *Limits* :  $[1, 2\frac{2}{3}, 4\frac{1}{3}]$

(1, 1, 1)    (1, 1, 2)    (1, 1, 3)    (1, 1, 4)    (1, 2, 2)    (1, 2, 3)    (1, 2, 4)

**Remark.**  $\mathcal{PF}'_{n,n+1} = \mathcal{PF}'_n$ . In fact, we do have

$$\begin{aligned} \frac{1}{n+n+1} \binom{n+n+1}{n+1} &= \frac{1}{2n+1} \binom{2n+1}{n+1} = \frac{1}{2n+1} \frac{(2n+1)!}{n!(n+1)!} \\ &= \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1} \frac{(2n)!}{n!n!} = \frac{1}{n+1} \binom{2n}{n} \end{aligned}$$

## 2.2 Rational Non-crossing Partitions

**Definition 16** (a, b - Non-crossing Partition). An a, b - non-crossing partition is *TODO*

**Example.** An *abncp*

**Theorem 8.** number of *abncp*

**Example.** all *abncp* for some a b

**Proposition.** This means we can create a bijection between  $\mathcal{PF}'_{a,b}$  and  $\mathcal{NC}_{a,b}$ .

*Proof.*

•  $\mathcal{NC}_{a,b} \rightarrow \mathcal{PF}'_{a,b}$  :

•  $\mathcal{PF}'_{a,b} \rightarrow \mathcal{NC}_{a,b}$  :

□

**Definition 17.** *ncab2*

**Example.** some *ncab2*

**Theorem 9.** number of *ncab2*

**Example.** all *ncab2* for some a b

**Proposition.** *bijection*

*Proof.* bijection proof

□

# Chapter 3

## Trees

### 3.1 Parking Trees

**Definition 18** (Parking Trees). A parking tree is defined from a parking function  $f = (a_1, \dots, a_n) \in \mathcal{PF}_n$  as follows :

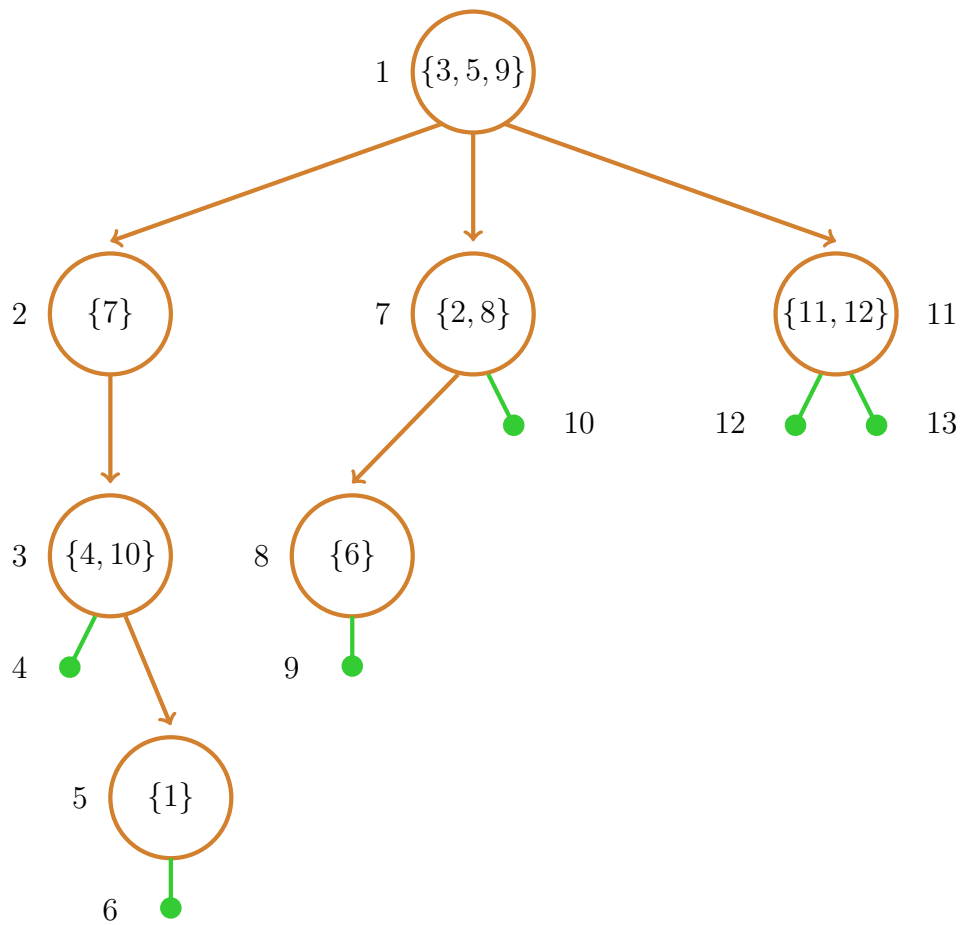
- For  $1 \leq i \leq n+1$ , we define  $s_i$  as  $\{j \mid a_j = i\}$
- $[s_1, \dots, s_{n+1}]$  describes the pre-order depth-first traversal of the tree.
- Each node labeled by a set of size  $k$  has  $k$  children.

**Remark.** The leaves of the tree are those corresponding to an element  $i$  such that  $1 \leq i \leq n+1$ , and  $i$  is not in  $f$ .

Furthermore, as we will have a total edges by definition, the presence of a node corresponding to  $n+1$  is necessary, even though it will always be empty.

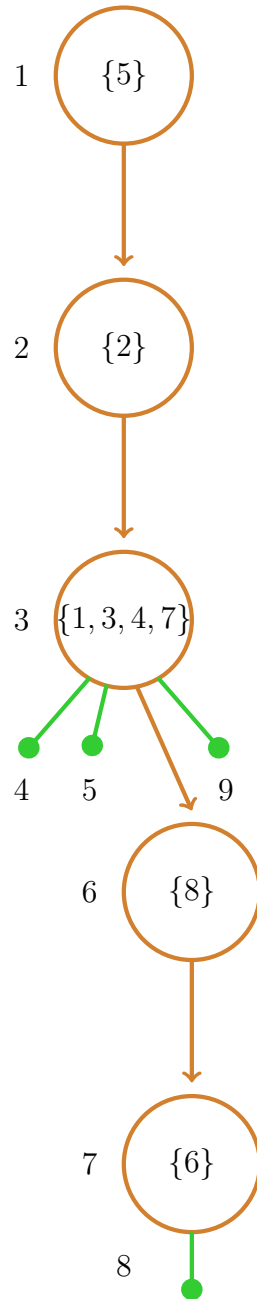
**Example** ( $n = 12$ ).

- $f = (5, 7, 1, 3, 1, 8, 2, 7, 1, 3, 11, 11)$
- Labels :  $[\{3, 5, 9\}, \{7\}, \{4, 10\}, \emptyset, \{1\}, \emptyset, \{2, 8\}, \{6\}, \emptyset, \emptyset, \{11, 12\}, \emptyset, \emptyset]$



Conversely, by reading the labels of a parking tree depth-first in pre-order, we get the list of positions of each number in the corresponding parking function, thus creating a *bijection*.

**Example** (From parking tree to parking function).



- The labels are  $[\{5\}, \{2\}, \{1, 3, 4, 7\}, \emptyset, \emptyset, \{8\}, \{6\}, \emptyset, \emptyset]$ .
- Thus the corresponding parking function is  $(3, 2, 3, 3, 1, 7, 3, 6) \in \mathcal{PF}_8$ .

## 3.2 Rational Parking Trees

**Definition 19** (Rational Parking Trees). *A rational parking tree is defined from a rational parking function  $f = (a_1, \dots, a_a) \in \mathcal{PF}_{a,b}$  as follows :*

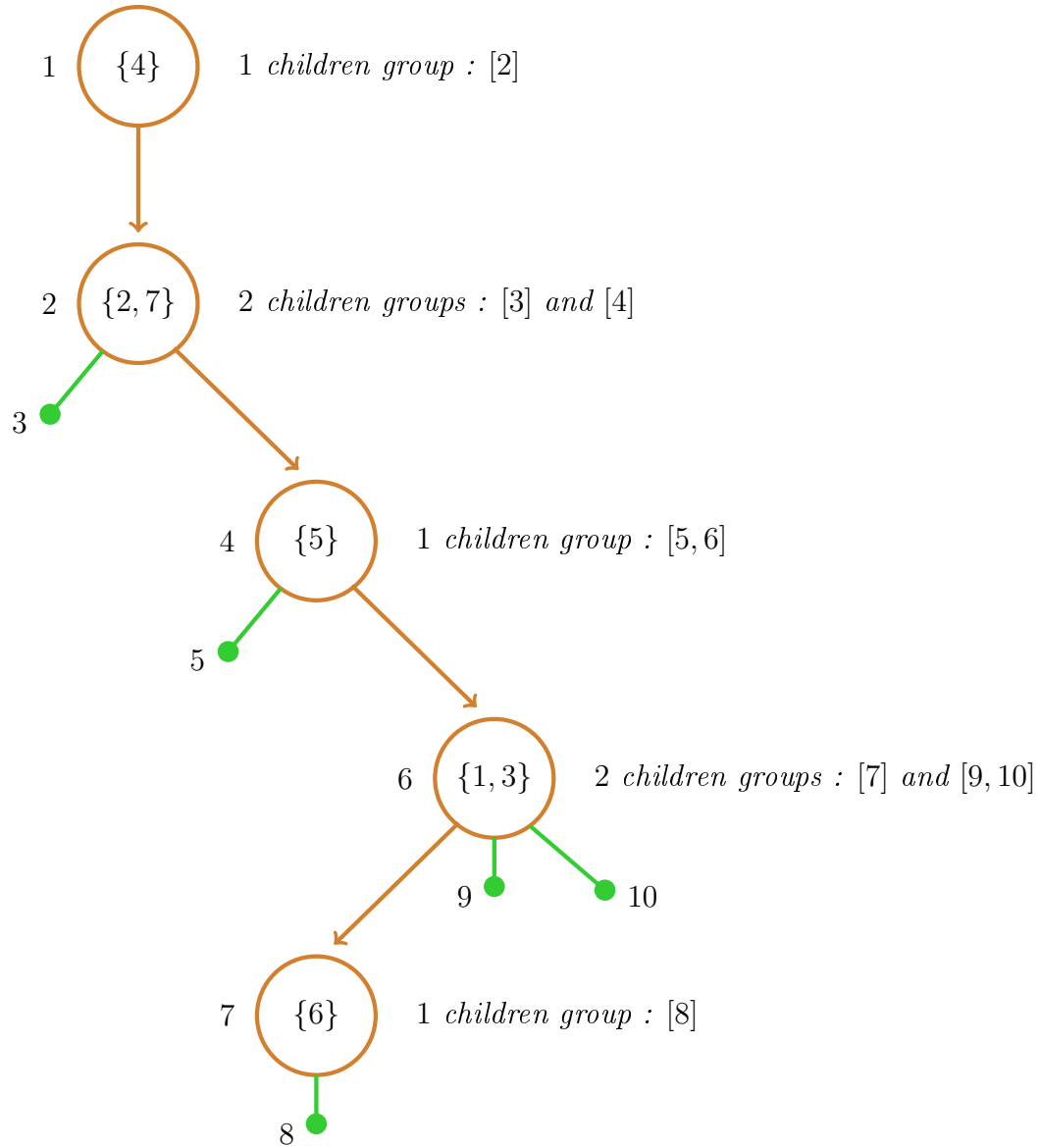
- *For  $1 \leq i \leq n + 1$ , we define the limit  $l_i$  as the integer portion of  $\frac{b}{a}(i - 1) + 1$ .  
Let  $l_0 = 0$ .*
- *From these limits, we deduce the intervals  $itv_i = ]l_{i-1}, l_i]$  for  $1 \leq i \leq a + 1$ .*
- *For  $1 \leq i \leq b + 1$ , define  $s_i$  as  $\{j \mid a_j = i\}$ .*
- *$[s_1, \dots, s_{b+1}]$  describes the pre-order depth-first traversal of the tree.*
- *Each node labeled by a set of size  $k$  has  $k$  groups of children, which are defined by the intervals.*

**Example** ( $a < b$ ).

- $a = 7$
- $b = 9$
- *Limits :*  $[1, 2\frac{2}{7}, 3\frac{4}{7}, 4\frac{6}{7}, 6\frac{1}{7}, 7\frac{3}{7}, 8\frac{5}{7}, 10]$
- *Integral limits :*  $[0, 1, 2, 3, 4, 6, 7, 8, 10]$
- *Intervals :*  

$$]0, 1] \quad ]1, 2] \quad ]2, 3] \quad ]3, 4] \quad ]4, 6] \quad ]6, 7] \quad ]7, 8] \quad ]8, 10]$$
- *Children groups :*  

$$[1] \quad [2] \quad [3] \quad [4] \quad [5, 6] \quad [7] \quad [8]$$
- $f = (6, 2, 6, 1, 4, 7, 2)$
- *Labels :*  $\{\{4\}, \{2, 7\}, \emptyset, \{5\}, \emptyset, \{1, 3\}, \{6\}, \emptyset, \emptyset, \emptyset\}$



**Example** ( $a > b$ ).

- $a = 9$
- $b = 7$
- *Limits* :  $[1, 1\frac{7}{9}, 2\frac{5}{9}, 3\frac{3}{9}, 4\frac{1}{9}, 4\frac{8}{9}, 5\frac{6}{9}, 6\frac{4}{9}, 7\frac{2}{9}, 8]$

- *Integral limits* :  $[0, 1, 1, 2, 3, 4, 4, 5, 6, 7, 8]$

- *Intervals* :

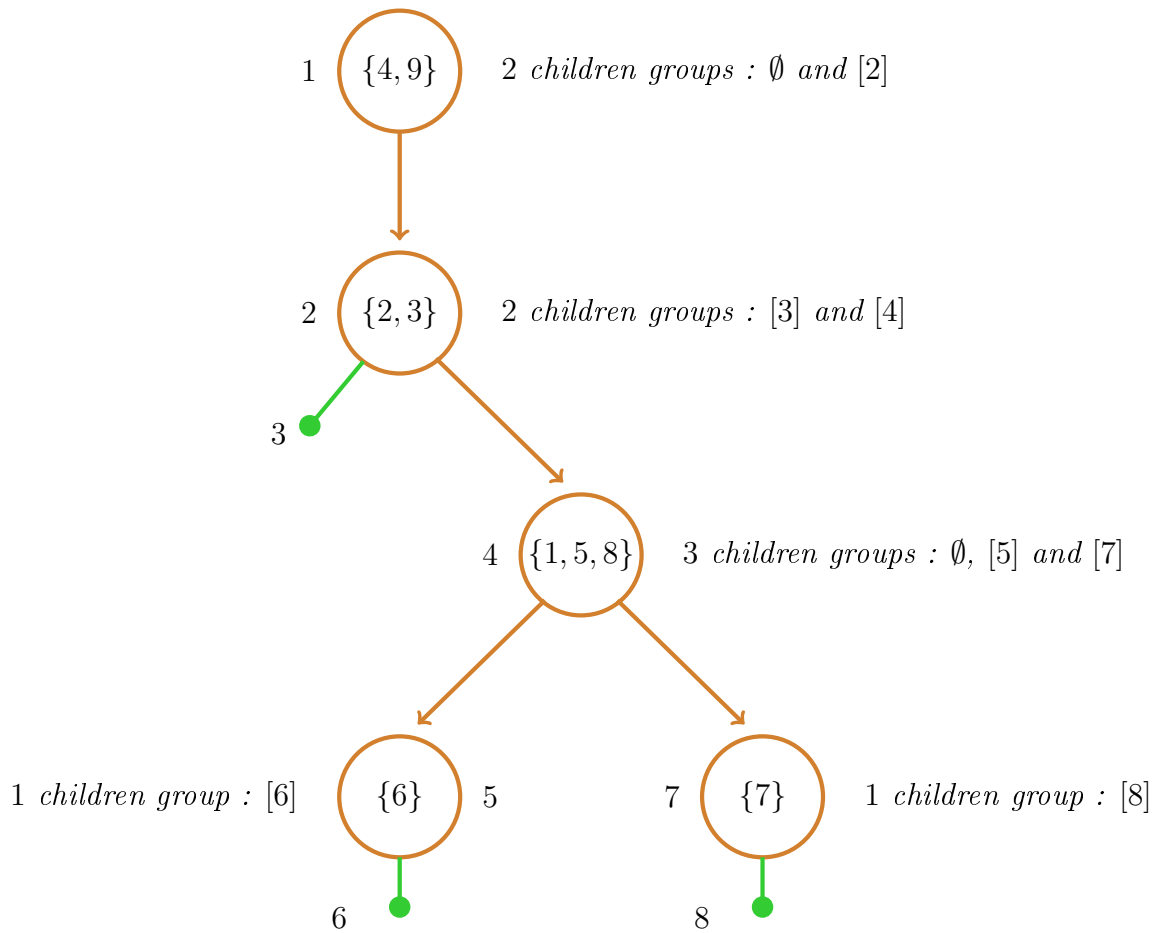
$]0, 1]$	$]1, 1]$	$]1, 2]$	$]2, 3]$	$]3, 4]$
$]4, 4]$	$[4, 5]$	$]5, 6]$	$]6, 7]$	$]7, 8]$

- *Children groups* :

$[1]$	$\emptyset$	$[2]$	$[3]$	$[4]$	$\emptyset$	$[5]$	$[6]$	$[7]$	$[8]$
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- $f = (4, 2, 2, 1, 4, 5, 7, 4, 1)$

- *Labels* :  $\{\{4, 9\}, \{2, 3\}, \emptyset, \{1, 5, 8\}, \{6\}, \emptyset, \{7\}, \emptyset\}$



In both cases, the converse direction of the *bijection* is obtained with the same method as for classical parking trees.