Rational Parking Functions

Matthieu Josuat-Vergès Tessa Lelièvre-Osswald

August 27, 2020

Abstract

This is an abstract about Rational Parking Functions

1 Parking Functions

Definition 1 (Parking Function)**.** *A* parking function *is a sequence* $(a_1, a_2, \ldots, a_n)$ *such that its non-decreasing reordering* $(b_1, b_2, \ldots, b_n)$ *has* $b_i < i$ *for all* i*.*
We denote by $\mathcal{PF}_n$ *the set of parking functions of length* n*.*

$$\mathcal{PF} = \bigcup_{n>0} \mathcal{PF}_n$$

.

Example.

$$f_1 = (7, 3, 1, 4, 2, 5, 2) \in \mathcal{PF}_7$$
$$f_2 = (7, 3, 1, 4, 2, 5, 4) \notin \mathcal{PF}_7$$

Theorem 1. *Let* pf_n *be the cardinal of* $\mathcal{PF}_n$*. We have* $pf_n = (n+1)^{n-1}$*.*

Example $(n = 1, 2, 3)$**.**
• $n = 1 \quad : \quad pf_1 = 1$
(1)

• $n = 2 \quad : \quad pf_2 = 3$
$(1,1) \qquad (1,2) \qquad (2,1)$
• $n = 3 \quad : \quad pf_3 = 16$
$(1,1,1) \qquad (1,1,2) \qquad (1,1,3) \qquad (1,2,1) \qquad (1,2,2) \qquad (1,2,3) \qquad (1,3,1)$
$(1,3,2) \qquad (2,1,1) \qquad (2,1,2) \qquad (2,1,3) \qquad (2,2,1) \qquad (2,3,1) \qquad (3,1,1)$
$(3,1,2) \qquad (3,2,1)$

Definition 2 (Primitive)**.** *A parking function* $(a_1, a_2, \ldots, a_n)$ *is said* primitive *if it is already in non-decreasing order.*
We denote by $\mathcal{PF}'_n$ *the set of primitive parking functions of length* n*.*

$$\mathcal{PF}' = \bigcup_{n>0} \mathcal{PF}'_n$$

Example.

$$f_1 = (1,2,2,3) \in \mathcal{PF}'_4$$
$$f_2 = (1,2,3,2) \notin \mathcal{PF}'_4, \text{ even though } f_2 \in \mathcal{PF}_4$$

Theorem 2. *Let* pf'_n *be the cardinal of* $\mathcal{PF}'_n$*. We have* $pf'_n = \frac{1}{n+1}\binom{2n}{n}$*, which is the* n^{th} *Catalan number.*

Example $(n = 1, 2, 3)$**.**
• $n = 1 \quad : \quad pf'_1 = 1$
(1)
• $n = 2 \quad : \quad pf'_2 = 2$
$(1,1) \qquad (1,2)$
• $n = 3 \quad : \quad pf'_3 = 5$
$(1,1,1) \qquad (1,1,2) \qquad (1,1,3) \qquad (1,2,2) \qquad (1,2,3)$

2 Non-crossing Partitions

Definition 3 (Non-crossing Partition)**.** *A* non-crossing partition *of a set* E *is a set partition* $P = \{E_1, E_2, \ldots, E_k\}$ *such that if* $a, c \in E_i$*,* $b, d \in E_j$*, and*

$i \neq j$, then we do not *have $a < b < c < d$, nor $a > b > c > d$.*
We denote by $\mathcal{NC}_n$ the set of non-crossing partitions of $\{1, 2, \ldots, n\}$.

$$\mathcal{NC} = \bigcup_{n>0} \mathcal{NC}_n$$

Notation. $[n] = \{1, 2, \ldots, n\}$

Example ($E = [6]$)**.**

$$P_1 = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\} \in \mathcal{NC}_6$$
$$P_2 = \{\{1, 2, 4\}, \{3, 5\}, \{6\}\} \notin \mathcal{NC}_6$$

Theorem 3. *Let nc_n be the cardinal of $\mathcal{NC}_n$. We have $nc_n = \frac{1}{n+1}\binom{2n}{n}$, which is the n^{th} Catalan number.*

Example ($n = 1, 2, 3$)**.**
- $n = 1 \quad : \quad nc_1 = 1$
  $\{\{1\}\}$
- $n = 2 \quad : \quad nc_2 = 2$
  $\{\{1, 2\}\} \qquad \{\{1\}, \{2\}\}$
- $n = 3 \quad : \quad nc_3 = 5$
  $\{\{1, 2, 3\}\} \qquad \{\{1\}, \{2, 3\}\} \qquad \{\{1, 3\}, \{2\}\} \qquad \{\{1, 2\}, \{3\}\} \qquad \{\{1\}, \{2\}, \{3\}\}$

Proposition. *This means we can create a* bijection *between $\mathcal{PF}'_n$ and $\mathcal{NC}_n$.*

- $\mathcal{NC}_n \to \mathcal{PF}'_n$: For each block B in the non-crossing partition, take $i = min(B)$, and $k_i = size(B)$.
  $k_i = 0$ if i is not the minimum of a block.
  The corresponding parking function is $(\underbrace{1, \ldots, 1}_{k_1}, \underbrace{2, \ldots, 2}_{k_2}, \ldots, \underbrace{n, \ldots, n}_{k_n})$.

- $\mathcal{PF}'_n \to \mathcal{NC}_n$: For each i in $[n]$, if i appears n_i times in the parking function, B_i will be of size n_i with minimum element i. There is a unique set partition $P = \bigcup_i B_i$ of $[n]$ respecting these conditions that is non-crossing.

Example ($E = [6]$)**.**

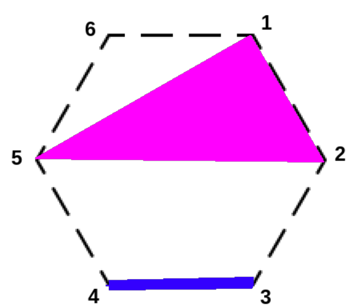
$$P = \{\{1,2,5\},\{3,4\},\{6\}\} \qquad\qquad f = (1,1,1,3,3,6)$$

Corollary. *A non-crossing partition can be represented by the minimums and sizes of its blocks.*

Example. $\{\{1,2,5\},\{3,4\},\{6\}\}$ *can be represented by the following dictionnary :*
• *1 : 3*
• *3 : 2*
• *6 : 1*

A non-crossing partition of $[n]$ can be represented graphically on a regular n-vertices polygon, with vertices labeled from 1 to n clockwise. We then represent each block $B = \{b_1, \ldots, b_k\}$ by the convex hull of $\{b_1, \ldots, b_k\}$.

Example ($P = \{\{1,2,5\},\{3,4\},\{6\}\}$)**.**



Thus non-crossing meaning the hulls are *disjoint*.

2.1 The non-crossing partitions poset

Definition 4 ($\succ$)**.** *We say that P covers Q, written $P \succ Q$, if $\exists B_i, B_j \in P$ such that $Q = P - \{B_i, B_j\} \cup \{B_i \cup B_j\}$*

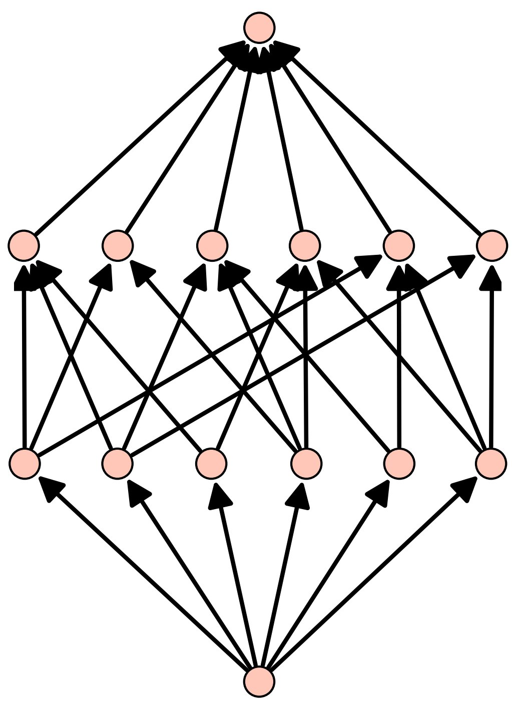
Example. $\{\{1,6\},\{2,3\},\{4,5\}\} \succ \{\{1,2,3,6\},\{4,5\}\}$
• $B_i = \{1,6\}$
• $B_j = \{2,3\}$

Proposition. *This covering relation defines the* poset *of non-crossing partitions of* $[n]$*. We denote by* $\mathcal{NCC}_n$ *the set of* maximal chains *in the poset of* $\mathcal{NC}_n$*.*

$$\mathcal{NCC} = \bigcup_{n>0} \mathcal{NCC}_n$$

Theorem 4. *Let* ncc_n *be the cardinal of* $\mathcal{NC}_n$*. We have* $ncc_n = n^{n-2}$*.*

Example (Shape of the poset of $\mathcal{NC}_4$)**.**



This figure was generated with Sagemath. There are $4^2 = 16$ *different maximal chains, and* $\frac{1}{5}\binom{8}{4} = \frac{70}{5} = 14$ *elements in this poset.*

2.2 Kreweras complement

Definition 5 (Associated permutation)**.** *The* permutation σ *associated to a non-crossing partition has a cycle* $(b_1, \dots, b_k)$ *for each block* $B = \{b_1, \dots, b_k\}$ *of the partition.*

Example. *The permutation associated to* $\{\{1,2,5\},\{3,4\},\{6\}\}$ *is* $(1\ 2\ 5)\ (3\ 4)\ (6) = 254316$.

Definition 6 (Kreweras complement)**.** *The* Kreweras complement $K(P)$ *of a non-crossing partition* P *is defined as follows :*
- *Let* σ *be the permutation associated to* P
- *Let* π *be the permutation* $(n\ n-1\ n-2\ \dots\ 3\ 2\ 1) = n123\dots n-1$
- $K(P)$ *is the* non-crossing partition *associated to* $\pi\sigma$.

Example $(P = \{\{1,2,5\},\{3,4\},\{6\}\})$**.**
- $\sigma = (1\ 2\ 5)\ (3\ 4)\ (6) = 254316$
- $\pi = (6\ 5\ 4\ 3\ 2\ 1) = 612345$
- $\pi\sigma = 143265 = (1)\ (2\ 4)\ (3)\ (5\ 6)$
- $K(P) = \{\{1\},\{2,4\},\{3\},\{5,6\}\}$

Proposition (Kreweras minimums)**.** *Let* $P = \{B_1, \dots, B_k\}$ *be a non-crossing partition. Let* $K(P) = \{B'_1, \dots, B'_l\}$ *be its Kreweras complement. Then*

$$\bigcup_{1 \le i \le l} min(B'_i) = B_1 \cup \bigcup_{1 < j \le k} B_i - max(B_i)$$

Example $(P = \{\{1,2,5\},\{3,4\},\{6\}\})$**.**

- $K(P) = \{\{1\},\{2,4\},\{3\},\{5,6\}\}$
- $\bigcup min(B'_i) = \{1,2,3,5\}$
- $B_1 \cup \bigcup B_i - max(B_i) = \{1,2,5\} \cup \{3,4\} - \{4\} \cup \{6\} - \{6\} = \{1,2,5\} \cup \{3\} \cup \emptyset = \{1,2,3,5\}$

Notation. $B_{[i]}$ = *block containing* i.

Proposition (Kreweras block sizes)**.** *Let* $P = \{B_1, \ldots, B_k\}$ *be a non-crossing partition. Let* $K(P) = \{B'_1, \ldots, B'_l\}$ *be its Kreweras complement. Then the size of the block* B'_i *is defined as follows :*

- *Let* m_i *be the the* i^{th} *minimum of* $K(P)$
- *Define a* transition $\phi(e)$ *as*

  Let $j = e + 1$ *(or* 1 *if* $e = n$*)*

  $\phi(e) = max(B_{[j]})$
- *The size of* B'_i *is* k_{min} *such that* $k_{min} = min\{k > 0 \mid \phi^k(m_i) \in B_{[m_i]}\}$.

Example $(P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\})$**.**

- $mins = \{1, 2, 3, 5\}$
- $m_1 = 1$

  $B_{[1]} = B_1$

  $max(B_{[2]} = max(B_1) = 5$

  The size for m_1 *is* 1.
- m_2

  $B_{[2]} = B_1$

  $max(B_{[3]}) = max(B_2) = 4$

  $max(B_{[5]}) = max(B_1) = 5$

  The size for m_2 *is* 2.
- $m_3 = 3$

  $B_{[3]} = B_2$

  $max(B_{[4]}) = max(B_2) = 4$

  The size for m_3 *is* 1.

- $m_4 = 5$

  $B_{[5]} = B_1$

  $max(B_{[6]}) = max(B_3) = 6$

  $max(B_{[1]}) = max(B_1) = 5$

  The size for m_4 is 2.

3 Non-crossing 2-partitions

Definition 7 (Non-crossing 2-partition)**.** *A* non-crossing 2-partition *of a set E is a pair (P, σ) where :*
• P is a non-crossing partition of E
• σ is a permutation of the elements of E
We denote by $\mathcal{NC}^2_n$ the set of non-crossing 2-partitions of $[n]$.

$$\mathcal{NC}^2 = \bigcup_{n>0} \mathcal{NC}^2_n$$

.

Example ($\mathcal{NC}^2_6$)**.** $P = \{\{1,6\},\{2,3,5\},\{4\}\}$ $\sigma = 416235$

Theorem 5. *Let nc^2_n be the cardinal of $\mathcal{NC}^2_n$. We have $nc^2_n = (n+1)^{n-1}$.*

Example ($n = 1, 2, 3$)**.**

- $n = 1$: $nc^2_1 = 1$

  $\{\{1\}\}$ 1

- $n = 2$: $nc^2_2 = 3$

  $\{\{1\},\{2\}\}$ 12

  $\{\{1\},\{2\}\}$ 21

  $\{\{1,2\}\}$ 12

- $n = 3$: $nc^2_3 = 16$

  $\{\{1\},\{2\},\{3\}\}$ 123

  $\{\{1\},\{2\},\{3\}\}$ 132

  $\{\{1\},\{2\},\{3\}\}$ 213

$\{\{1\},\{2\},\{3\}\}$	231
$\{\{1\},\{2\},\{3\}\}$	312
$\{\{1\},\{2\},\{3\}\}$	321
$\{\{1,2\},\{3\}\}$	123
$\{\{1,2\},\{3\}\}$	132
$\{\{1,2\},\{3\}\}$	231
$\{\{1\},\{2,3\}\}$	123
$\{\{1\},\{2,3\}\}$	213
$\{\{1\},\{2,3\}\}$	312
$\{\{1,3\},\{2\}\}$	123
$\{\{1,3\},\{2\}\}$	132
$\{\{1,3\},\{2\}\}$	231
$\{\{1,2,3\}\}$	123