

Rational Parking Functions

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Abstract

This is an abstract about Rational Parking Functions.

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Chapter 1

The integer case

1.1 Parking Functions

Definition 1 (Parking Function). *A parking function is a sequence of positive integers (a_1, a_2, \dots, a_n) such that its non-decreasing reordering (b_1, b_2, \dots, b_n) has $b_i \leq i$ for all i .*

We denote by \mathcal{PF}_n the set of parking functions of length n .

$$\mathcal{PF} = \bigcup_{n \geq 0} \mathcal{PF}_n$$

.

Example.

$$f_1 = (7, 3, 1, 4, 2, 5, 2) \in \mathcal{PF}_7$$

$$f_2 = (7, 3, 1, 4, 2, 5, 4) \notin \mathcal{PF}_7$$

Theorem 1. *Let pf_n be the cardinal of \mathcal{PF}_n . We have*

$$pf_n = (n + 1)^{n-1}$$

.

Example ($n = 1, 2, 3$).

- $n = 1$: $pf_1 = 1$
(1)

- $n = 2$: $pf_2 = 3$
 $(1, 1)$ $(1, 2)$ $(2, 1)$
- $n = 3$: $pf_3 = 16$
 $(1, 1, 1)$ $(1, 1, 2)$ $(1, 1, 3)$ $(1, 2, 1)$ $(1, 2, 2)$ $(1, 2, 3)$ $(1, 3, 1)$
 $(1, 3, 2)$ $(2, 1, 1)$ $(2, 1, 2)$ $(2, 1, 3)$ $(2, 2, 1)$ $(2, 3, 1)$ $(3, 1, 1)$
 $(3, 1, 2)$ $(3, 2, 1)$

1.1.1 Primitive parking functions

Definition 2 (Primitive). A parking function (a_1, a_2, \dots, a_n) is said primitive if it is already in non-decreasing order.

We denote by \mathcal{PF}'_n the set of primitive parking functions of length n .

$$\mathcal{PF}' = \bigcup_{n \geq 0} \mathcal{PF}'_n$$

Example.

$$\begin{aligned} f_1 &= (1, 2, 2, 3) \in \mathcal{PF}'_4 \\ f_2 &= (1, 2, 3, 2) \notin \mathcal{PF}'_4, \text{ even though } f_2 \in \mathcal{PF}_4 \end{aligned}$$

Theorem 2. Let pf'_n be the cardinal of \mathcal{PF}'_n . We have

$$pf'_n = \frac{1}{n+1} \binom{2n}{n}$$

which is the n^{th} Catalan number $Cat(n)$.

Example ($n = 1, 2, 3$).

- $n = 1$: $pf'_1 = 1$
 (1)
- $n = 2$: $pf'_2 = 2$
 $(1, 1)$ $(1, 2)$
- $n = 3$: $pf'_3 = 5$
 $(1, 1, 1)$ $(1, 1, 2)$ $(1, 1, 3)$ $(1, 2, 2)$ $(1, 2, 3)$

1.2 Non-crossing Partitions

Definition 3 (Non-crossing Partition). *A non-crossing partition of a totally ordered set E is a set partition $P = \{E_1, E_2, \dots, E_k\}$ such that if $a, c \in E_i$, $b, d \in E_j$, and $i \neq j$, then we do not have $a < b < c < d$, nor $a > b > c > d$. We denote by \mathcal{NC}_n the set of non-crossing partitions of $\{1, 2, \dots, n\}$.*

$$\mathcal{NC} = \bigcup_{n \geq 0} \mathcal{NC}_n$$

From this point, we assume that every partition $P = \{B_1, \dots, B_l\}$ is *sorted* such that :

- For each block $B_i = \{b_1, \dots, b_k\} \in P$, $b_1 < \dots < b_k$
- $\min(B_1) < \dots < \min(B_k)$

Notation. $[n] = \{1, 2, \dots, n\}$

Example ($E = [6]$).

$$\begin{aligned} P_1 &= \{\{1, 2, 5\}, \{3, 4\}, \{6\}\} \in \mathcal{NC}_6 \\ P_2 &= \{\{1, 2, 4\}, \{3, 5\}, \{6\}\} \notin \mathcal{NC}_6 \end{aligned}$$

Theorem 3. *Let nc_n be the cardinal of \mathcal{NC}_n . We have*

$$nc_n = \frac{1}{n+1} \binom{2n}{n}$$

which is again the n^{th} Catalan number $Cat(n)$.

Example ($n = 1, 2, 3$).

- $n = 1$: $nc_1 = 1$
 $\{\{1\}\}$
- $n = 2$: $nc_2 = 2$
 $\{\{1, 2\}\} \quad \{\{1\}, \{2\}\}$
- $n = 3$: $nc_3 = 5$
 $\{\{1, 2, 3\}\} \quad \{\{1\}, \{2, 3\}\} \quad \{\{1, 3\}, \{2\}\} \quad \{\{1, 2\}, \{3\}\} \quad \{\{1\}, \{2\}, \{3\}\}$

Proposition. *This means we can create a bijection between \mathcal{PF}'_n and \mathcal{NC}_n .*

Proof.

- $\mathcal{NC}_n \rightarrow \mathcal{PF}'_n$: For each block B in the non-crossing partition, take $i = \min(B)$, and let $k_i = \text{size}(B)$.
 $k_i = 0$ if i is not the minimum of a block.
 The corresponding parking function is $(\underbrace{1, \dots, 1}_{k_1}, \underbrace{2, \dots, 2}_{k_2}, \dots, \underbrace{n, \dots, n}_{k_n})$.
- $\mathcal{PF}'_n \rightarrow \mathcal{NC}_n$: For each i in $[n]$, if i appears n_i times in the parking function, B_i will be of size n_i with minimum element i . There is a unique set partition $P = \bigcup_i B_i$ of $[n]$ respecting these conditions that is non-crossing : for each minimum i in *decreasing order*, add the n_i first free elements of $[i + 1, i + 2, \dots, n, 1, \dots, i - 1]$ to B_i .

□

Example ($n = 6$).

$$P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\} \quad f = (1, 1, 1, 3, 3, 6)$$

Corollary. *A non-crossing partition can be represented by the minimums and sizes of its blocks.*

Example. $\{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$ can be represented by the following dictionary :

- $1 : 3$
- $3 : 2$
- $6 : 1$

A non-crossing partition of $[n]$ can be represented graphically on a regular n -vertices polygon, with vertices labeled from 1 to n clockwise. We then represent each block $B = \{b_1, \dots, b_k\}$ by the convex hull of $\{b_1, \dots, b_k\}$.

Example ($P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$).



Thus non-crossing meaning the hulls are disjoint.

Example (Counter-example : $P = \{\{1, 5, 2\}, \{3, 6\}, \{4\}\}$).



This partition is not non-crossing, as the convex hulls of $\{1, 2, 5\}$ and $\{3, 6\}$ are not disjoint.

1.2.1 The non-crossing partitions poset

Definition 4 (\succ). We say that P covers Q , written $P \succ Q$, if $\exists B_i, B_j \in P$ such that $Q = P - \{B_i, B_j\} \cup \{B_i \cup B_j\}$

Example. $\{\{1, 6\}, \{2, 3\}, \{4, 5\}\} \succ \{\{1, 2, 3, 6\}, \{4, 5\}\}$

- $B_i = \{1, 6\}$

- $B_j = \{2, 3\}$

Proposition. *This covering relation defines the poset of \mathcal{NC}_n . We denote by \mathcal{NCC}_n the set of maximal chains in the poset of \mathcal{NC}_n .*

$$\mathcal{NCC} = \bigcup_{n>0} \mathcal{NCC}_n$$

Remark. *The bottom element of this poset is $\{\{1, \dots, n\}\}$, and the top element is $\{\{1\}, \dots, \{n\}\}$.*

Theorem 4. *Let ncc_n be the cardinal of \mathcal{NCC}_n . We have*

$$ncc_n = n^{n-2}$$

.

Example (The poset of \mathcal{NC}_4).

To shorten labels, we represent $\{\{1\}, \{2, 3\}, \{4\}\}$ by $1|23|4$.



There are $4^2 = 16$ different maximal chains, and $\frac{1}{5} \binom{8}{4} = \frac{70}{5} = 14$ elements in this poset.

1.2.2 Kreweras complement

Definition 5 (Associated Permutation). *The permutation σ associated to a non-crossing partition has a cycle (b_1, \dots, b_k) for each block $B = \{b_1, \dots, b_k\}$ of the partition.*

Example. *The permutation associated to $\{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$ is $(1\ 2\ 5)(3\ 4)(6) = 254316$.*

Definition 6 (Kreweras Complement). *The Kreweras complement $K(P)$ of a non-crossing partition P is defined as follows :*

- *Let σ be the permutation associated to P*
- *Let π be the permutation $(n\ n-1\ n-2\ \dots\ 3\ 2\ 1) = n123\dots n-1$*
- *$K(P)$ is the non-crossing partition associated to $\pi\sigma$.*

Example ($P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$).

- $\sigma = (1\ 2\ 5)(3\ 4)(6) = 254316$
- $\pi = (6\ 5\ 4\ 3\ 2\ 1) = 612345$
- $\pi\sigma = 143265 = (1)(2\ 4)(3)(5\ 6)$
- $K(P) = \{\{1\}, \{2, 4\}, \{3\}, \{5, 6\}\}$

Proposition (Kreweras minimums). *Let $P = \{B_1, \dots, B_k\}$ be a non-crossing partition. Let $K(P) = \{B'_1, \dots, B'_l\}$ be its Kreweras complement. Then*

$$\bigcup_{1 \leq i \leq l} \min(B'_i) = B_1 \cup \bigcup_{1 < j \leq k} B_j - \max(B_i)$$

Example ($P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$).

- $K(P) = \{\{1\}, \{2, 4\}, \{3\}, \{5, 6\}\}$
- $\bigcup \min(B'_i) = \{1, 2, 3, 5\}$

- $B_1 \cup \bigcup B_i - \max(B_i) = \{1, 2, 5\} \cup \{3, 4\} - \{4\} \cup \{6\} - \{6\} = \{1, 2, 5\} \cup \{3\} \cup \emptyset = \{1, 2, 3, 5\}$

Notation. $B_{[i]}$ = block containing i .

Proposition (Kreweras block sizes). *Let $P = \{B_1, \dots, B_k\}$ be a non-crossing partition. Let $K(P) = \{B'_1, \dots, B'_l\}$ be its Kreweras complement. Then the size of the block B'_i is defined as follows :*

- Let m_i be the i^{th} minimum of $K(P)$
- Define a transition $\phi(e)$ as

$$\text{Let } j = e + 1 \text{ (or 1 if } e = n)$$

$$\phi(e) = \max(B_{[j]})$$
- The size of B'_i is k_{\min} such that $k_{\min} = \min\{k > 0 \mid \phi^k(m_i) \in B_{[m_i]}\}$.

Example ($P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$).

- $\text{mins} = \{1, 2, 3, 5\}$
- $m_1 = 1$

$$B_{[1]} = B_1$$

$$\max(B_{[2]}) = \max(B_1) = 5$$

The size for m_1 is 1.
- m_2

$$B_{[2]} = B_1$$

$$\max(B_{[3]}) = \max(B_2) = 4$$

$$\max(B_{[5]}) = \max(B_1) = 5$$

The size for m_2 is 2.
- $m_3 = 3$

$$B_{[3]} = B_2$$

$$\max(B_{[4]}) = \max(B_2) = 4$$

The size for m_3 is 1.

- $m_4 = 5$

$$B_{[5]} = B_1$$

$$\max(B_{[6]}) = \max(B_3) = 6$$

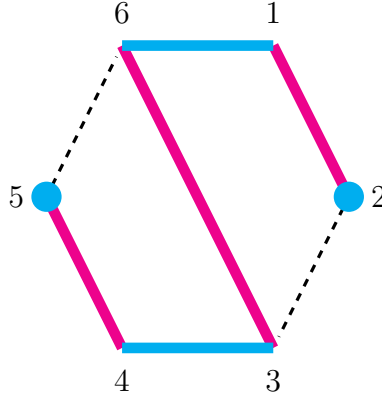
$$\max(B_{[1]}) = \max(B_1) = 5$$

The size for m_4 is 2.

Definition 7 (Mutually Non-crossing Partitions). 2 partitions P and Q are said mutually non-crossing if :

- P is non-crossing
- Q is non-crossing
- For every block B_i of P and every block B_j of Q , if $a, c \in B_i$ and $b, d \in B_j$, then we can not have $a < b < c < d$, nor $a > b > c > d$.

Example ($P = \{\{1, 2\}, \{3, 6\}, \{4, 5\}\}, Q = \{\{1, 6\}, \{2\}, \{3, 4\}, \{5\}\}$).



Example (Counter-example : $P = \{\{1, 2\}, \{3, 6\}, \{4, 5\}\}, Q = \{\{1, 6\}, \{2, 5\}, \{3, 4\}\}$).

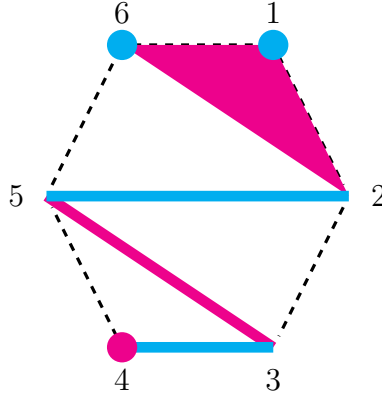


Remark. Note that vertices can touch, but the edges of the convex hulls can not cross.

Proposition. For any non-crossing partition P , P and $K(P)$ are mutually non-crossing.

Furthermore, $K(P)$ is a densest partition that is mutually non-crossing with P . That is, no partition Q that is mutually non-crossing with P has less blocks than $K(P)$.

Example ($P = \{1, 2, 6\}, \{3, 5\}, \{4\}\}$). $Q = \{\{1\}, \{2, 5\}, \{3, 4\}, \{6\}\}$

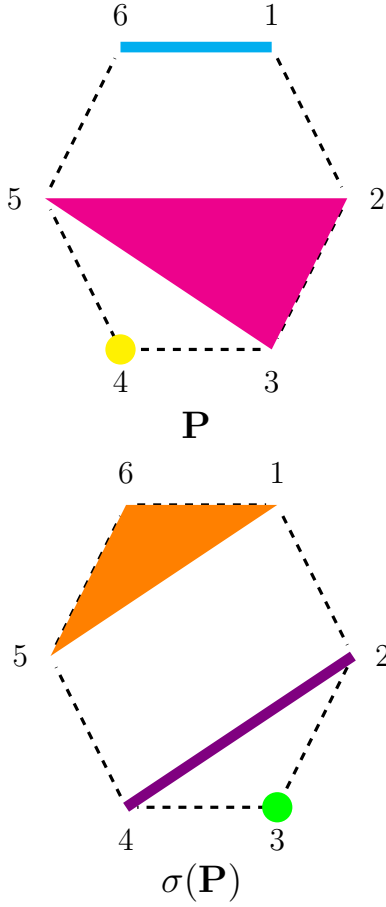


1.2.3 Action of \mathfrak{S}_n on partitions of $[n]$

Definition 8 (Action of \mathfrak{S}_n). The action of \mathfrak{S}_n on a partition $P = \{B_1, \dots, B_l\}$ of $[n]$ is defined by :

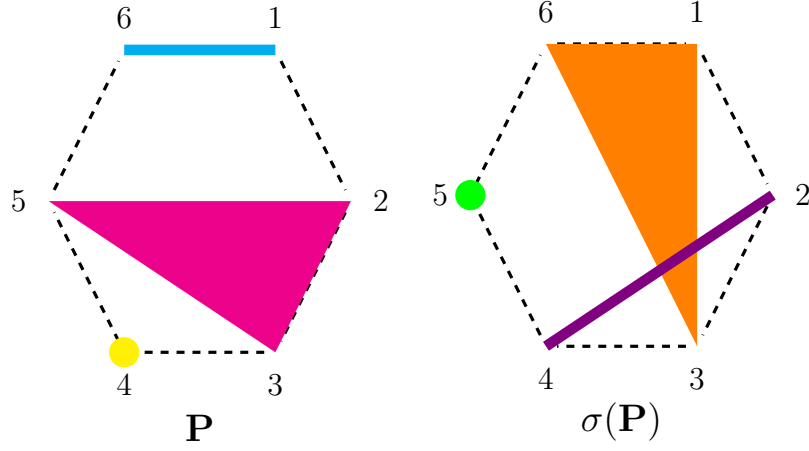
- For each block $B_i = \{b_1, \dots, b_k\} : \sigma(B_i) = \{\sigma(b_1), \dots, \sigma(b_k)\}$
- When $P \in \mathcal{NC}_n$, we denote $\rho = \sigma(P) = \{\sigma(B_1), \dots, \sigma(B_l)\}$

Example ($\sigma = 415362, P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$).
 $\sigma(P) = \{\{1, 5, 6\}, \{2, 4\}, \{3\}\}$



Remark. Note that \mathcal{NC}_n is not stable under the action of \mathfrak{S}_n . That is, even if P is non-crossing, $\sigma(P)$ is not necessarily non-crossing.

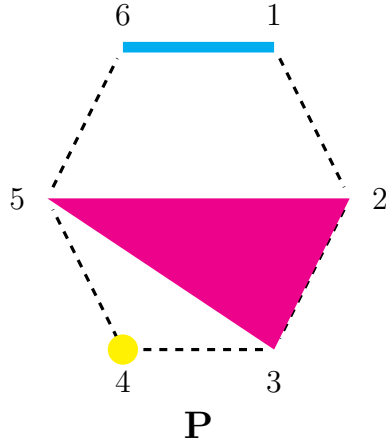
Example (Counter-example : $\sigma = 413562, P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$).
 $\sigma(P) = \{\{1, 3, 6\}, \{2, 4\}, \{5\}\}$

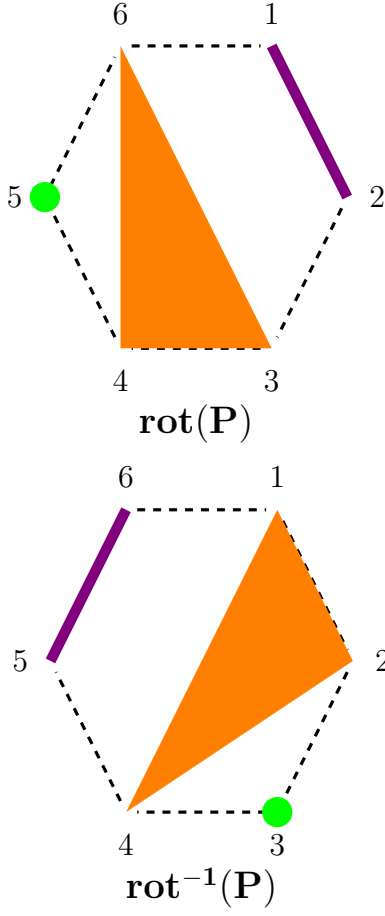


Definition 9 (Rotation). We define the rotation operator rot of $P \in \mathcal{NC}_n$ as $rot(P) = (1\ 2\ 3\ \dots\ n)(P) = 23\dots n1(P)$. Conversely, we define rot^{-1} of P as $rot^{-1}(P) = (n\ n-1\ \dots\ 3\ 2\ 1)(P) = n12\dots n-1(P)$.

Example ($P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$).

- $rot(P) = \{\{1, 2\}, \{3, 4, 6\}, \{5\}\}$
- $rot^{-1}(P) = \{\{1, 2, 4\}, \{3\}, \{5, 6\}\}$



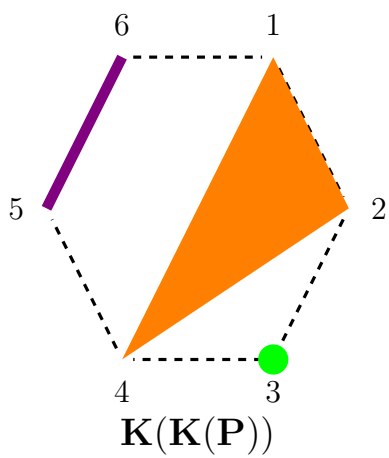
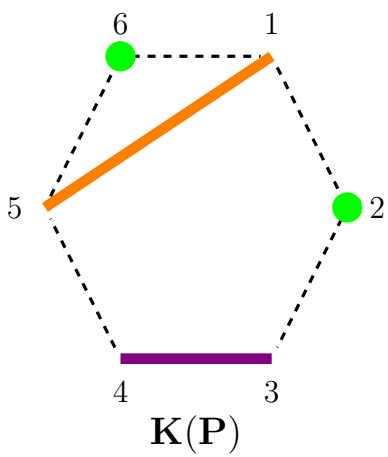
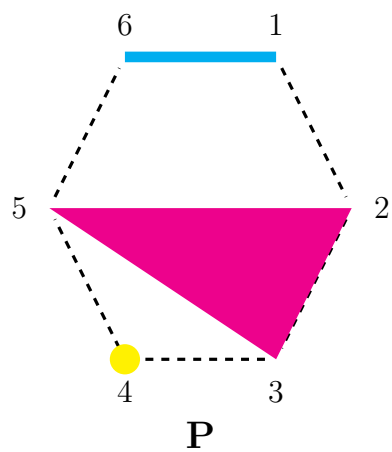


Remark.

- $\text{rot}(\text{rot}^{-1}(P)) = \text{rot}^{-1}(\text{rot}(P)) = P$
- $\text{rot}(P)$ and $\text{rot}^{-1}(P)$ are always non-crossing partitions.
- If $P \in \mathcal{NC}_n$, then $\text{rot}^n(P) = \text{rot}^{-n}(P) = P$.

Proposition. $K(K(P)) = \text{rot}^{-1}(P)$.

Example ($P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$).



1.3 Non-crossing 2-partitions

Definition 10 (Non-crossing 2-partition). A non-crossing 2-partition of a totally ordered set E is a pair (P, σ) where :

- P is a non-crossing partition of E
- σ is a permutation of the elements of E
- For each sorted block $B_i = \{b_1, \dots, b_k\} \in P$, we have $\sigma(b_i) < \dots < \sigma(b_k)$

We denote by \mathcal{NC}_n^2 the set of non-crossing 2-partitions of $[n]$.

$$\mathcal{NC}^2 = \bigcup_{n>0} \mathcal{NC}_n^2$$

Example (\mathcal{NC}_6^2). $P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$ $\sigma = 413265$
 $\rho = \{\{1, 3, 6\}, \{2\}, \{4, 5\}\}$

Theorem 5. Let nc_n^2 be the cardinal of \mathcal{NC}_n^2 . We have

$$nc_n^2 = (n+1)^{n-1}$$

Example ($n = 1, 2, 3$).

- $n = 1$: $nc_1^2 = 1$
 $\{\{1\}\} \quad 1 \quad \rho = P$
- $n = 2$: $nc_2^2 = 3$
 $\{\{1\}, \{2\}\} \quad 12 \quad \rho = P$
 $\{\{1\}, \{2\}\} \quad 21 \quad \rho = P$
 $\{\{1, 2\}\} \quad 12 \quad \rho = P$
- $n = 3$: $nc_3^2 = 16$
 $\{\{1\}, \{2\}, \{3\}\} \quad 123 \quad \rho = P$
 $\{\{1\}, \{2\}, \{3\}\} \quad 132 \quad \rho = P$
 $\{\{1\}, \{2\}, \{3\}\} \quad 213 \quad \rho = P$
 $\{\{1\}, \{2\}, \{3\}\} \quad 231 \quad \rho = P$

$\{\{1\}, \{2\}, \{3\}\}$	312	$\rho = P$
$\{\{1\}, \{2\}, \{3\}\}$	321	$\rho = P$
$\{\{1, 2\}, \{3\}\}$	123	$\rho = P$
$\{\{1, 2\}, \{3\}\}$	132	$\rho = \{\{1, 3\}, \{2\}\}$
$\{\{1, 2\}, \{3\}\}$	231	$\rho = \{\{1\}, \{2, 3\}\}$
$\{\{1\}, \{2, 3\}\}$	123	$\rho = P$
$\{\{1\}, \{2, 3\}\}$	213	$\rho = \{\{1, 3\}, \{2\}\}$
$\{\{1\}, \{2, 3\}\}$	312	$\rho = \{\{1, 2\}, \{3\}\}$
$\{\{1, 3\}, \{2\}\}$	123	$\rho = P$
$\{\{1, 3\}, \{2\}\}$	132	$\rho = \{\{1, 2\}, \{3\}\}$
$\{\{1, 3\}, \{2\}\}$	213	$\rho = \{\{1\}, \{2, 3\}\}$
$\{\{1, 2, 3\}\}$	123	$\rho = P$

Proposition. *This means we can create a bijection between \mathcal{PF}_n and \mathcal{NC}_n^2 .*

Proof.

- $\mathcal{PF}_n \rightarrow \mathcal{NC}_n^2$: Let $f = (a_1, \dots, a_n) \in \mathcal{PF}_n$ be our parking function. For $i \in \{1, \dots, n\}$, we define :

l_i : the number of occurrences of i in f .

$im_i : \{j \mid a_j = i\}$

The corresponding non-crossing partition will have the following constraints :

For each $i \in \{1, \dots, n\}$, if $l_i > 0$, then there is a block $B_{[i]}$ of length l_i with minimum element i .

$\sigma(B_{[i]}) = im_i$

There is a unique set partition $P = \bigcup_i B_{[i]}$ of $[n]$ and a unique per-

mutation σ respecting these conditions such that $(P, \sigma) \in \mathcal{NC}_n^2$: for each minimum i in *decreasing order*, add the n_i first free elements of $[i+1, i+2, \dots, n, 1, \dots, i-1]$ to B_i . σ is then trivially obtained by the second constraint.

- $\mathcal{NC}_n^2 \rightarrow \mathcal{PF}_n$: Let (P, σ) with $P = \{B_1, \dots, B_l\}$ be our non-crossing 2-partition. For each block $B_i = \{b_1, \dots, b_k\} \in P$:

$$m_i = \min(B_i) = b_1$$

$$pos_i = \sigma(B_i)$$

For each $j \in pos_i$, we define $a_j = m_i$

The corresponding parking function is (a_1, \dots, a_n) .

□

Example ($n = 8$).

$$P = \{\{1, 2, 5\}, \{3, 4\}, \{6, 8\}, \{7\}\}$$

$$\sigma = 36187245$$

$$f = (3, 6, 1, 7, 6, 1, 1, 3)$$

1.3.1 The non-crossing 2-partitions poset

Definition 11 (\succ^2). We say that (P, σ) covers (Q, τ) , written $(P, \sigma) \succ^2 (Q, \tau)$, if $\exists B_i, B_j \in P$ such that

- $Q = P - \{B_i, B_j\} \cup \{B_i \cup B_j\}$
- $l \neq i, j, b \in B_l \rightarrow \tau(b) = \sigma(b)$
- Let $B_i \cup B_j = \{b_1, \dots, b_k\}$:
 $\tau(B_i \cup B_j) = \sigma(B_i \cup B_j)$
 $\tau(b_1) < \dots < \tau(b_k)$

Example.

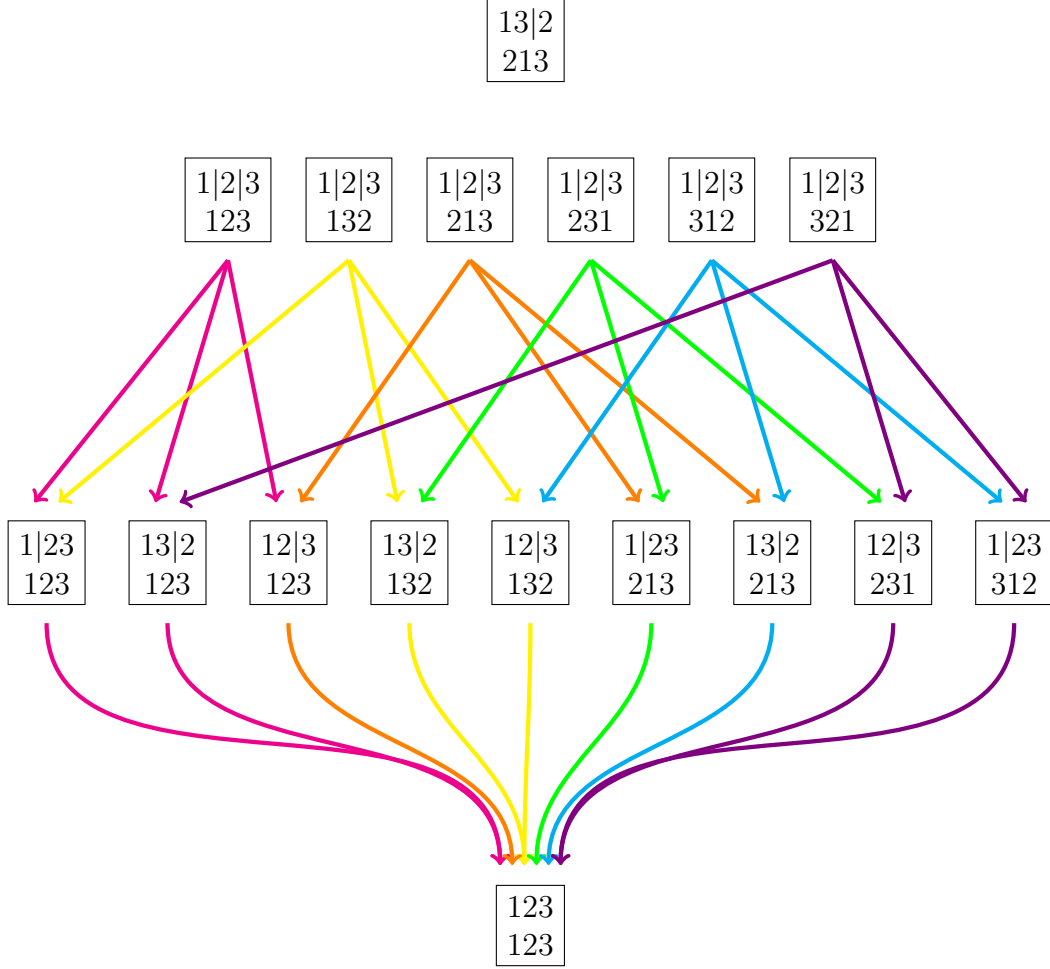
- $P = \{\{1, 6\}, \{2, 3\}, \{4\}, \{5\}\}$
- $\sigma = 236154$
- $Q = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$
- $\tau = 235164$
- $(P, \sigma) \succ^2 (Q, \tau)$
- $(P, \sigma) \not\succ^2 (Q, \sigma)$, because $\sigma(\{2, 3, 5\}) = \{3, 6, 5\}$ is not ordemagenta.

Proposition. This covering relation defines the poset of \mathcal{NC}_n^2 .

Remark. The bottom element of this poset is $(\{\{1, \dots, n\}\}, 12 \dots n)$, and the top elements are $(\{\{1\}, \dots, \{n\}\}, \sigma) \mid \sigma \in \mathfrak{S}_n$.

Example (The poset of \mathcal{NC}_3^2).

To shorten labels, we represent $(\{\{1, 3\}, \{2\}\}, 213)$ by



There are $4^2 = 16$ elements in this poset.

1.3.2 The parking functions poset

Definition 12 (Rank). Given $f = (a_1, \dots, a_n) \in \mathcal{PF}_n$, let

$$b_i = \begin{cases} 1 & \text{if } \exists j \mid a_j = i \\ 0 & \text{otherwise} \end{cases}$$

We define the rank of f , noted $rk(f)$, as

$$\sum_{1 \leq i \leq n} b_i$$

Example.

$$\begin{aligned} rk((1, 5, 4, 2, 3, 3, 1)) &= 5 \\ rk((4, 7, 1, 1, 3, 2, 2, 8)) &= 6 \end{aligned}$$

Definition 13 (\succ_{pf}). Since \mathcal{PF}_n and \mathcal{NC}_n^2 are in bijection, we can define a covering relation \succ_{pf} for \mathcal{PF}_n as follows :
 $f \in \mathcal{PF}_n \succ_{pf} g \in \mathcal{PF}_n$ if and only if :

- (P, σ) is the non-crossing 2-partition associated to f
- (Q, τ) is the non-crossing 2-partition associated to g
- $(P, \sigma) \succ^2 (Q, \tau)$

Example.

- $P = \{\{1, 6\}, \{2, 3\}, \{4\}, \{5\}\}$
- $\sigma = 236154$
- $Q = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$
- $\tau = 235164$
- $f = (4, 1, 2, 1, 5, 2) \succ_{pf} g = (4, 1, 2, 1, 2, 2)$

Remark. If $f \succ_{pf} g$, then $rk(f) = rk(g) + 1$, and there exists i and j such that :

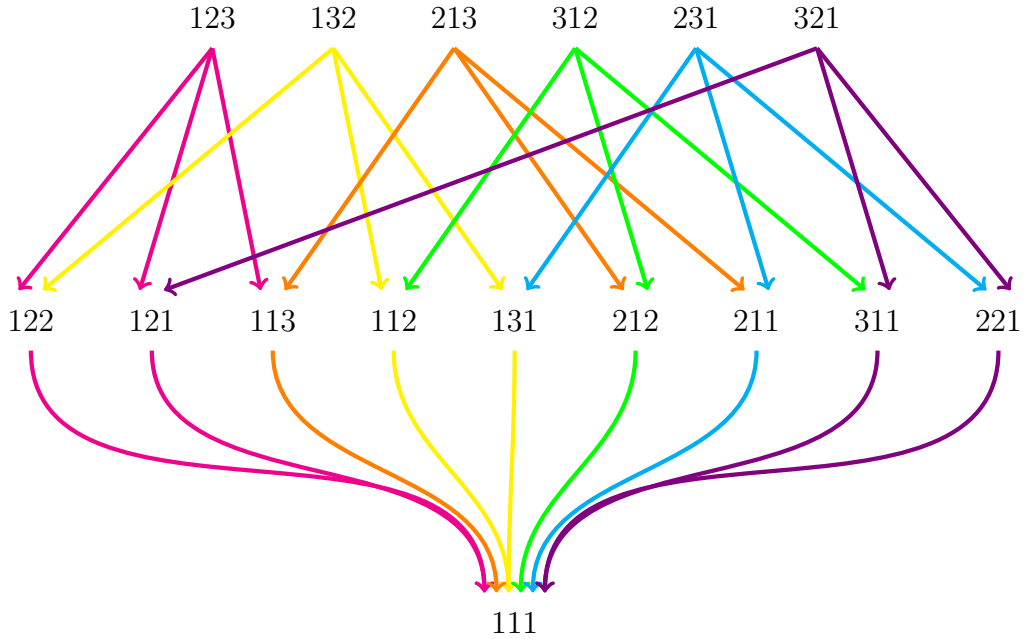
- $i < j$
- There is at least 1 occurrence of i in f
- There is at least 1 occurrence of j in f

$$b_k = \begin{cases} i & \text{if } a_k = j \\ a_k & \text{otherwise} \end{cases}$$

Proposition. *This covering relation defines the poset of \mathcal{PF}_n .*

Remark. *The bottom element of this poset is $(\underbrace{1, \dots, 1}_n)$, and the top elements are the permutations of $\{1, \dots, n\}$.*

Example (The poset of \mathcal{PF}_3).



1.4 A direct poset linked to Dyck paths

1.4.1 Dyck Paths

Notation. *We denote the number of occurrences of a symbol s in a word w by $|w|_s$.*

Definition 14 (Dyck path). *A Dyck word is a word $w \in \{0, 1\}^*$ such that :*

- *for each suffix w' of w , $|w'|_1 \geq |w'|_0$.*
- $|w|_0 = |w|_1$.

A Dyck word of length $2n$ can be represented as a path from $(0,0)$ to (n,n) that stays over $x = y$, called a Dyck path :

- Each 1 corresponds to a North step \uparrow .
- Each 0 corresponds to an East step \rightarrow .

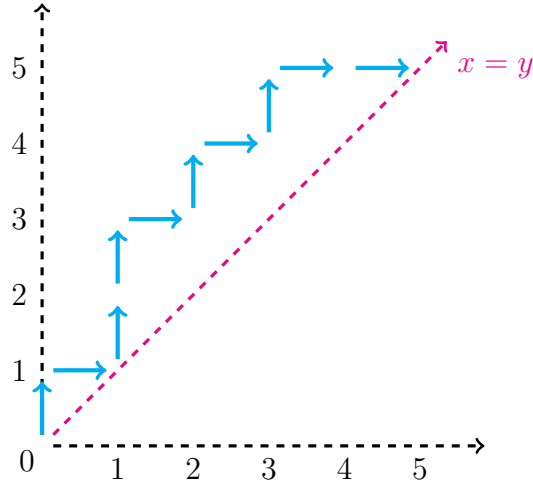
We denote by \mathcal{D}_n the set of Dyck words of length $2n$.

Example ($n = 5$).

$w_1 = 1011000110$ is not a Dyck word, because $|1011000|_0 > |1011000|_1$.

$w_2 = 1011010101$ is not a Dyck word, because $|w_2|_0 \neq |w_2|_1$.

$w_3 = 1011010100$ is a Dyck word :

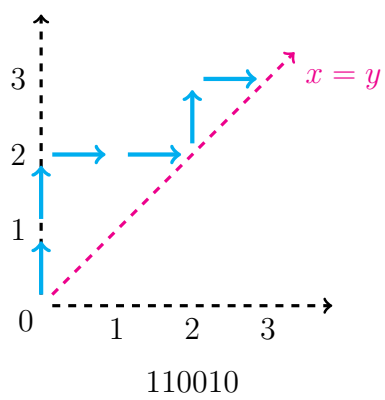
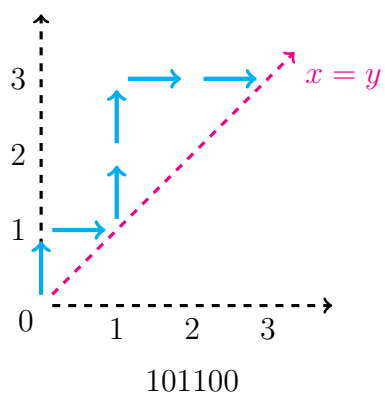
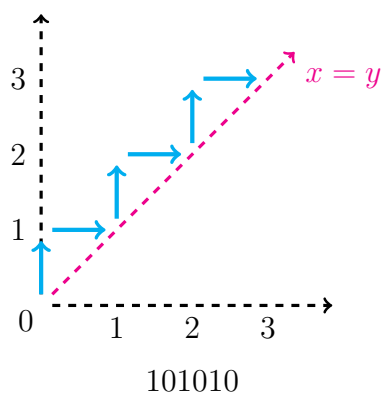


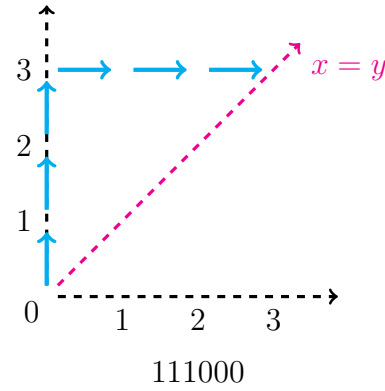
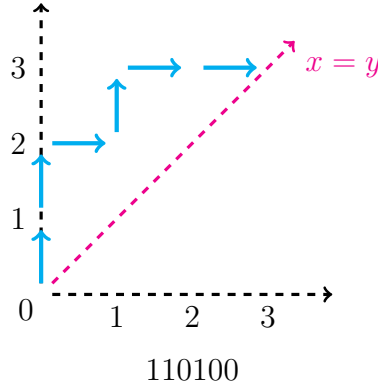
Theorem 6. Let d_n be the cardinal of \mathcal{D}_n . We have

$$d_n = \frac{1}{n+1} \binom{2n}{n}$$

which is the n^{th} Catalan number.

Example ($n = 3$). $d_n = 5$.





Proposition. *This means we can create a bijection between \mathcal{PF}'_n and \mathcal{D}_n .*

Proof.

- $\mathcal{PF}'_n \rightarrow \mathcal{D}_n$: Let $f = (a_1, \dots, a_n) \in \mathcal{PF}'_n$ be our primitive parking function. For $i \in \{1, \dots, n\}$, we define l_i the number of occurrences of i in f .

The corresponding Dyck word will be $\underbrace{1 \dots 1}_l 0 \underbrace{1 \dots 1}_l 0 \dots \underbrace{1 \dots 1}_l 0$.

- $\mathcal{D}_n \rightarrow \mathcal{PF}'_n$: Let w be our Dyck word, and consider its path representation. We define s_i to be the distance between the segment from $(0, i-1)$ to $(0, i)$ and the i^{th} North step. Then, let $a_i = s_i + 1$. The corresponding primitive parking function is (a_1, \dots, a_n) .

□

Example ($n = 6, \mathcal{PF}'_n \rightarrow \mathcal{D}_n$).

- $f = (1, 1, 2, 4, 5, 5)$

$$l_1 = 2$$

$$l_2 = 1$$

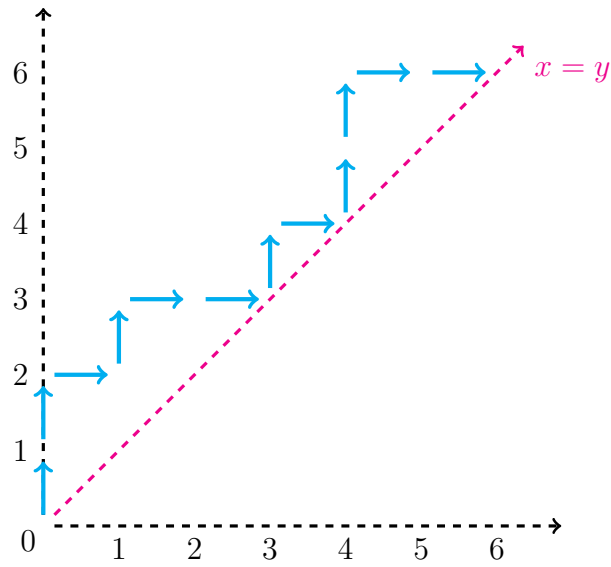
$$l_3 = 0$$

$$l_4 = 1$$

$$l_5 = 2$$

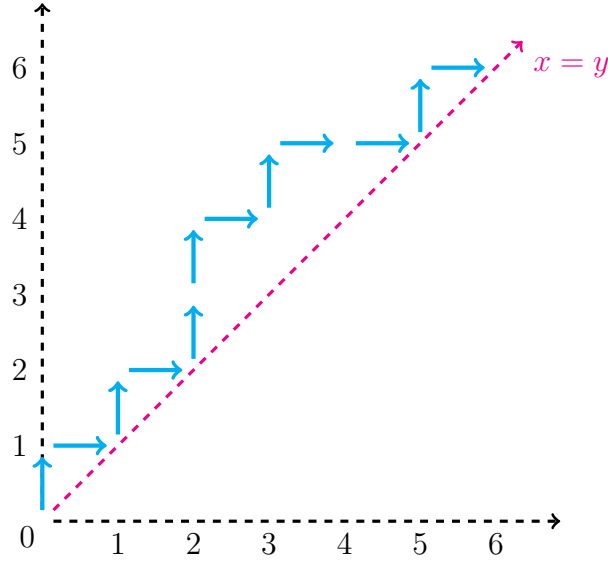
$$l_6 = 0$$

- $w = (110100101100)$



Example $(n = 6, \mathcal{D}_n \rightarrow \mathcal{PF}'_n)$.

- $w = 101011010010$



- *Distances :*

$s_1 = 0$	$a_1 = 1$
$s_2 = 1$	$a_2 = 2$
$s_3 = 2$	$a_3 = 3$
$s_4 = 2$	$a_4 = 3$
$s_5 = 3$	$a_5 = 4$
$s_6 = 5$	$a_6 = 6$

- $f = (1, 2, 3, 3, 4, 6)$

1.4.2 Labeled Dyck Paths

Definition 15 (Labeled Dyck Path). A labeled Dyck word is a word $w \in \{0, \dots, n\}^*$ such that :

- for each suffix w' of w , $|w'|_{\neq 0} \geq |w'|_0$.
- $|w|_0 = |w|_{\neq 0}$.
- for each $i \in \{1, \dots, n\}$, w has exactly one occurrence of i .

- if $w_i \neq 0$ and $w_{i+1} \neq 0$, then $w_i < w_{i+1}$. That is, consecutive North steps have increasing labels.

A labeled Dyck word of length $2n$ can be represented as a path from $(0,0)$ to (n,n) , where each North step is associated to a label :

- Each $i \neq 0$ corresponds to a North step \uparrow labeled i .
- Each 0 corresponds to an East step \rightarrow .

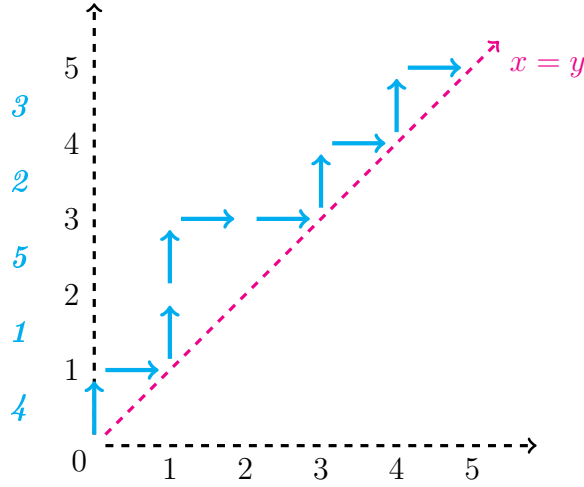
Those paths are called labeled Dyck paths.

We denote by \mathcal{LD}_n the set of labeled Dyck words of length $2n$.

Example ($n = 5$).

$w_1 = 4051002030$ is not a labeled Dyck word, because $5 > 1$.

$w_2 = 4015002030$ is a labeled Dyck word :



Theorem 7. Let ld_n be the cardinal of \mathcal{LD}_n . We have

$$ld_n = (n+1)^{n-1}$$

.

Example ($n = 3$). $ld_n = 4^2 = 16$

- Word of shape $XXX000$:
123000
- Words of shape $XX0X00$:
120300 130200 230100
- Words of shape $XX00X0$:
120030 130020 230010
- Words of shape $X0XX00$:
102300 201300 301200
- Words of shape $X0X0X0$:
102030 103020 201030
203010 301020 302010

Proposition. *This means we can create a bijection between \mathcal{PF}_n and \mathcal{LD}_n .*

Proof.

- $\mathcal{PF}_n \rightarrow \mathcal{LD}_n$: Let $f = (a_1, \dots, a_n) \in \mathcal{PF}_n$ be our parking function. For $i \in \{1, \dots, n\}$, we define $im_i : \{j \mid a_j = i\}$. We then define $im_{i,1}, \dots, im_{i,k_i}$ to be the elements of im_i in increasing order. The corresponding labeled Dyck word will be

$$\underbrace{im_{1,1} \cdots im_{1,k_1}}_{im_1} 0 \underbrace{im_{2,1} \cdots im_{2,k_2}}_{im_2} 0 \cdots \underbrace{im_{n,1} \cdots im_{n,k_n}}_{im_n} 0.$$
- $\mathcal{LD}_n \rightarrow \mathcal{PF}_n$: Let w be our labeled Dyck word, and consider its path representation. We define s_i to be the distance between the segment from $(0, i-1)$ to $(0, i)$ and the i^{th} North step. Then, let $label(i)$ be the label of the i^{th} North step, and $dist_i = \{label(j) \mid s_j = i\}$ be the set of the labels of all North steps at distance i . Then, if $j \in dist_i$, let $a_j = i+1$. The corresponding parking function is (a_1, \dots, a_n) .

□

Example $(n = 6, \mathcal{PF}_n \rightarrow \mathcal{LD}_n)$.

- $f = (5, 2, 1, 4, 5, 1)$

$$im_1 = \{3, 6\}$$

$$im_2 = \{2\}$$

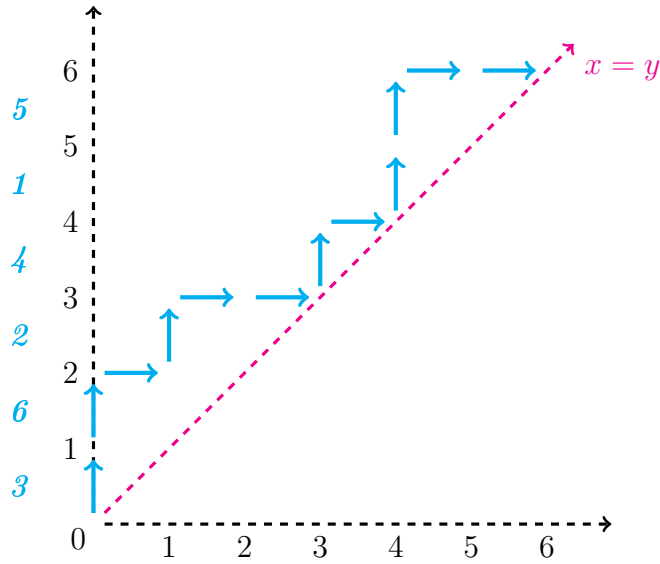
$$im_3 = \emptyset$$

$$im_4 = \{4\}$$

$$im_5 = \{1, 5\}$$

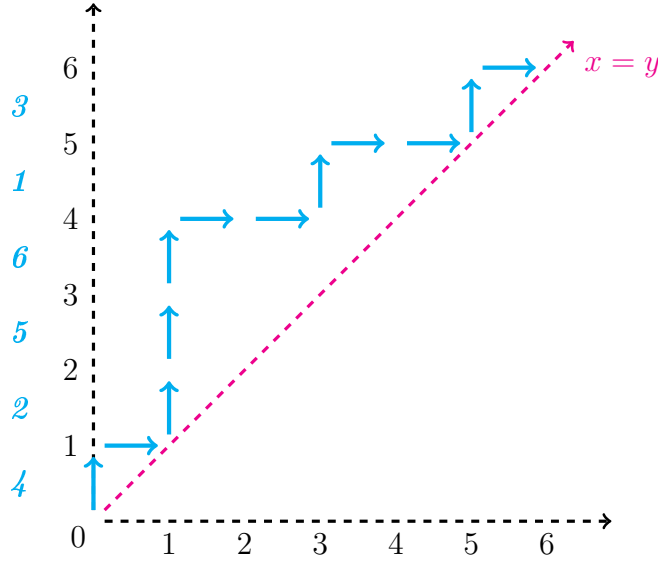
$$im_6 = \emptyset$$

- $w = 360200401500$



Example $(n = 6, \mathcal{LD}_n \rightarrow \mathcal{PF}_n)$.

- $w = 402560010030$



- *Distances :*

$$\begin{array}{lll} s_1 = 0 & s_2 = 1 & s_3 = 1 \\ s_4 = 1 & s_5 = 3 & s_6 = 5 \end{array}$$

- *Labels :*

$$\begin{array}{lll} dist_0 = \{4\} & dist_1 = \{2, 5, 6\} & dist_2 = \emptyset \\ dist_3 = \{1\} & dist_4 = \emptyset & dist_5 = \{3\} \end{array}$$

- $f = (4, 2, 6, 1, 2, 2)$

Remark. The primitive parking functions are exactly the parking functions corresponding to labeled Dyck paths where the i^{th} North step is labeled i .

1.4.3 Dyck - Parking Posets

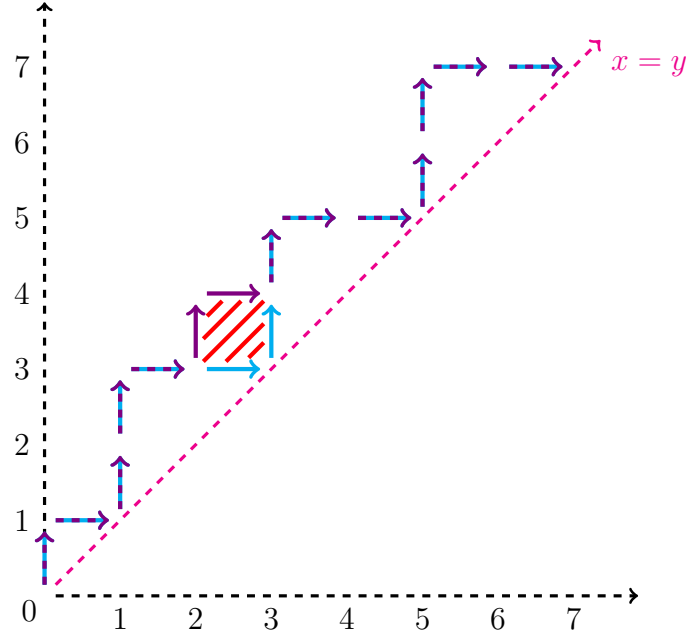
Primitive Dyck - Parking Posets

Definition 16 (\succ_d). For w and w' two Dyck words, we say that w covers w' , written $w \succ_d w'$, if $\exists w_1, w_2$ such that :

- $w = w_1 01 w_2$
- $w' = w_1 10 w_2$

Example ($n = 7$). $10110011001100 \succ_d 10110101001100$

- $w_1 = 10110$
- $w_2 = 1001100$

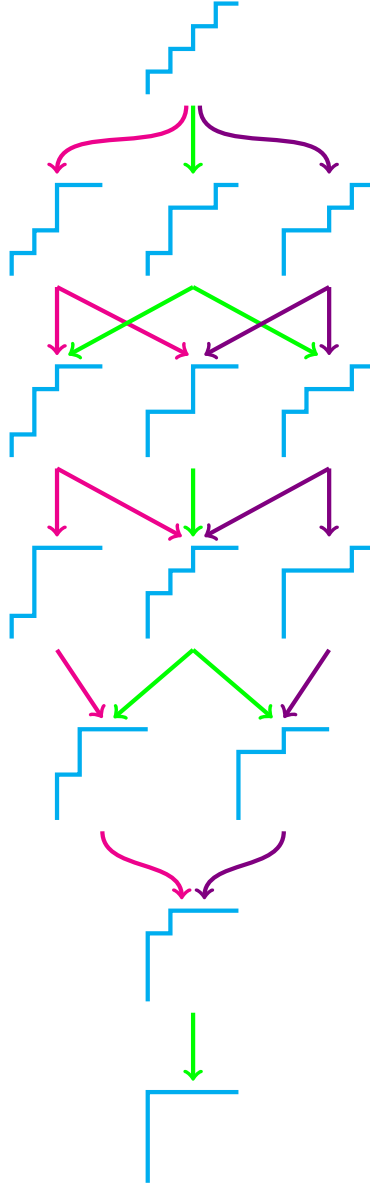


■ 10110011001100
■ 10110101001100
■ *difference*

Remark. If $w_1 \succ_d w_2$, then the path corresponding to w_2 is over the path corresponding to w_1 , and the difference between the two paths is a square of size 1 by 1.

Proposition. This covering relation defines a poset for \mathcal{D}_n .

Example (The poset of \mathcal{D}_4).



There are $\frac{1}{5} \binom{8}{4} = \frac{70}{5} = 14$ elements in this poset.

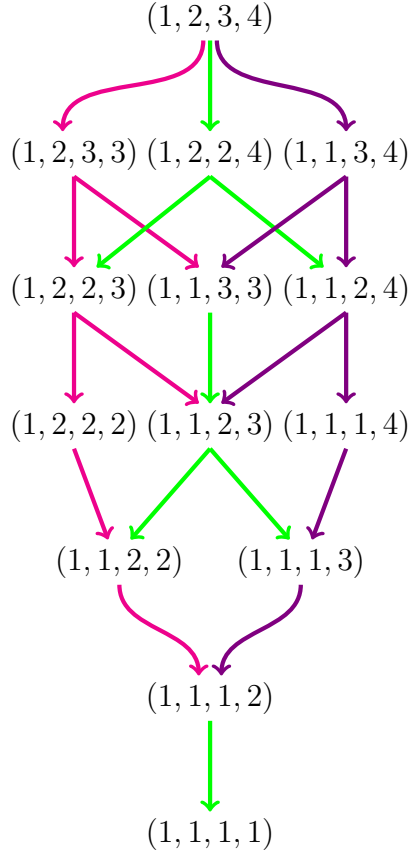
Definition 17 (\succ'). For f and g two primitive parking functions, we say that f covers g , written $f \succ' g$, if $\exists i$ such that :

- $f = (a_1, \dots, a_{i-1}, a_i, \quad a_{i+1}, \dots, a_n)$
- $g = (a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_n)$

Example ($n = 6$). $(1, 1, 2, 3, 4, 5) \succ' (1, 1, 2, 3, 3, 5)$

Proposition. *This covering relation defines a poset for \mathcal{PF}'_n .*

Example (The poset of \mathcal{PF}'_4).



There are $\frac{1}{5} \binom{8}{4} = \frac{70}{5} = 14$ elements in this poset.

Remark. *The two posets are isomorphic, and one can be obtained by applying the aforementioned bijection to the other.*

Theorem 8. *The number of intervals in those posets is equal to the $n + 1^{\text{th}}$ term of the integer sequence defined by <https://oeis.org/A005700>. The first terms of this sequence are 1, 1, 3, 14, 84, 594, 4719, 40898, 379236, 3711916, ...*

Alec Mihailovs proved this sequence to be equal to

$$\frac{6(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!}$$

Classical Dyck - Parking Posets

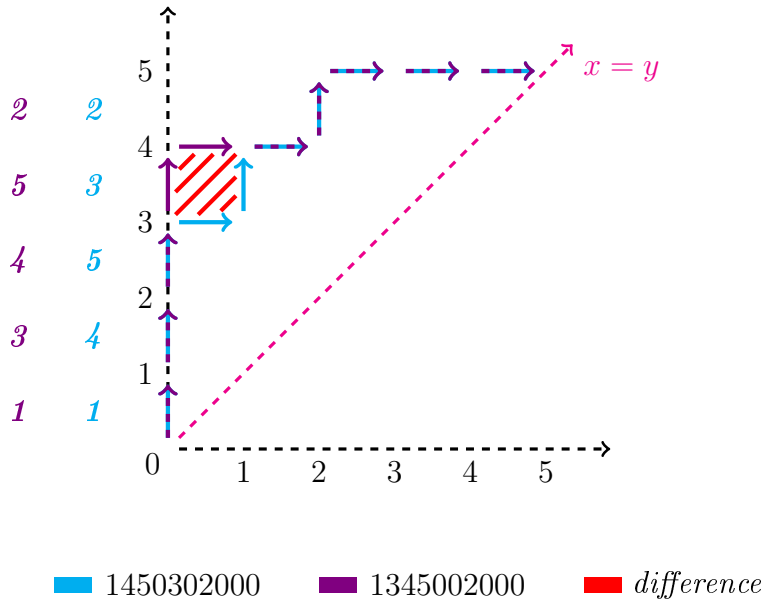
Definition 18 (\succ_{ld}). For w and w' two labeled Dyck words, we say that w covers w' , written $w \succ_{ld} w'$, if either :

- $\exists w_2, x, x', y$ such that :
 - $x = x_1x_2 \cdots x_n$ has all its digits > 0
 - $x' = x$ where y is correctly inserted regarding the order condition
 - $y > 0$
 - $w = x0yw_2$
 - $w' = x'0w_2$
- $\exists w_1, w_2, x, x', y$ such that :
 - $x = x_1x_2 \cdots x_n$ has all its digits > 0
 - $y > 0$
 - $x' = x$ where y is correctly inserted regarding the order condition
 - $w = w_10x0yw_2$
 - $w' = w_10x'0w_2$
- or $\exists w_1, w_2, y$ such that :
 - $y > 0$
 - $w = w_100yw_2$
 - $w' = w_10y0w_2$

Example ($n = 5$, first case). $1450302000 \succ_{ld} 1345002000$

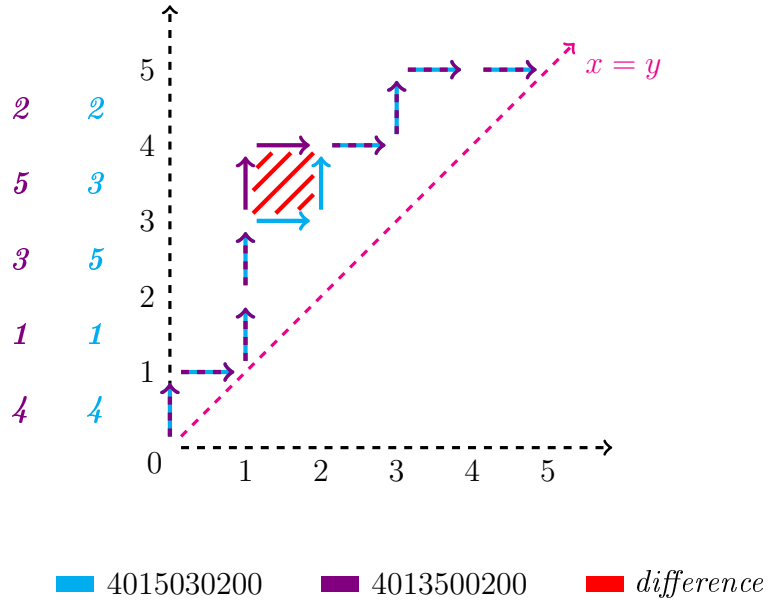
- $w_2 = 02000$
- $x = 145$

- $x' = 1345$
- $y = 3$



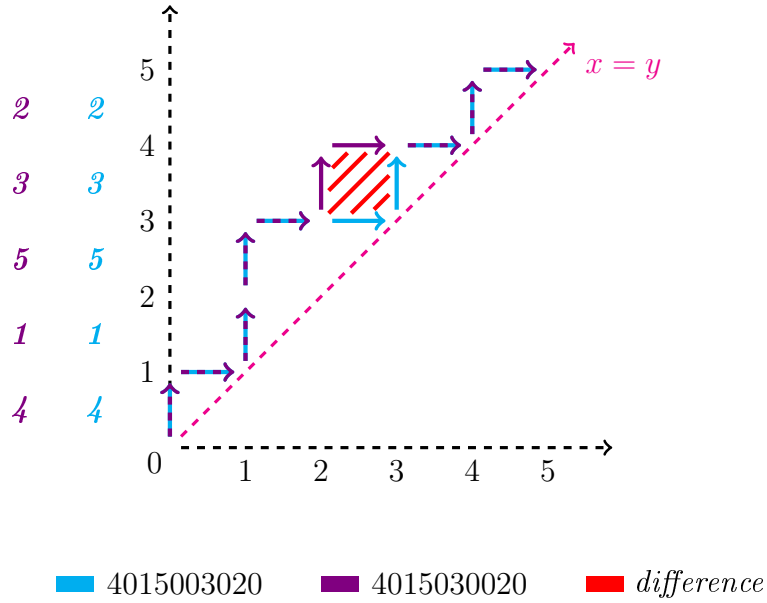
Example ($n = 5$, second case). $4015030200 \succ_{ld} 4013500200$

- $w_1 = 4$
- $w_2 = 0200$
- $x = 15$
- $x' = 135$
- $y = 3$



Example ($n = 5$, third case). $4015003020 \succ_{ld} 4015030020$

- $w_1 = 4015$
- $w_2 = 020$
- $y = 3$

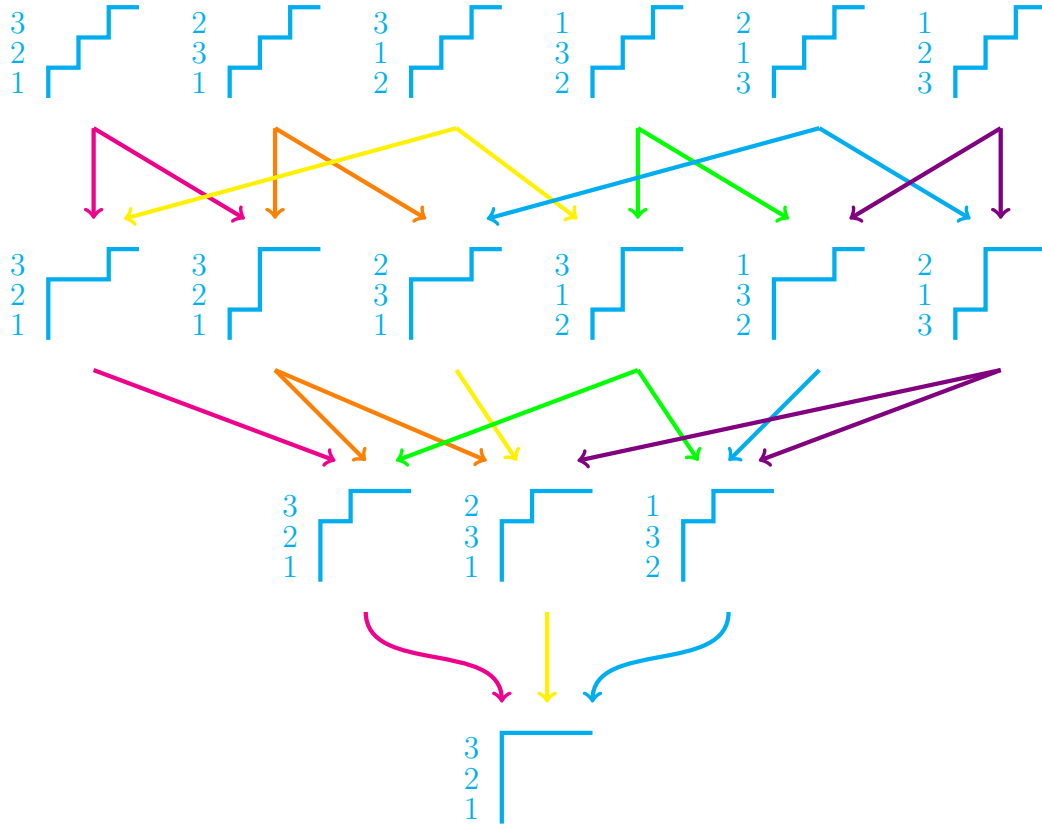


Remark. If $w_1 \succ_{ld} w_2$, then the path corresponding to w_2 is over the path corresponding to w_1 , and the difference between the two paths is a square of size 1 by 1.

Furthermore, the labeling can be seen as follows : if one has to merge two sequences of North steps of w_1 to make w_2 , then the merging will be made by ordering the corresponding labels.

Proposition. This covering relation defines a poset for \mathcal{LD}_n .

Example (The poset of \mathcal{LD}_3).



There are $4^2 = 16$ elements in this poset.

Definition 19 (\succ). For f and g two parking functions, we say that f covers g , written $f \succ g$, if $\exists i$ such that :

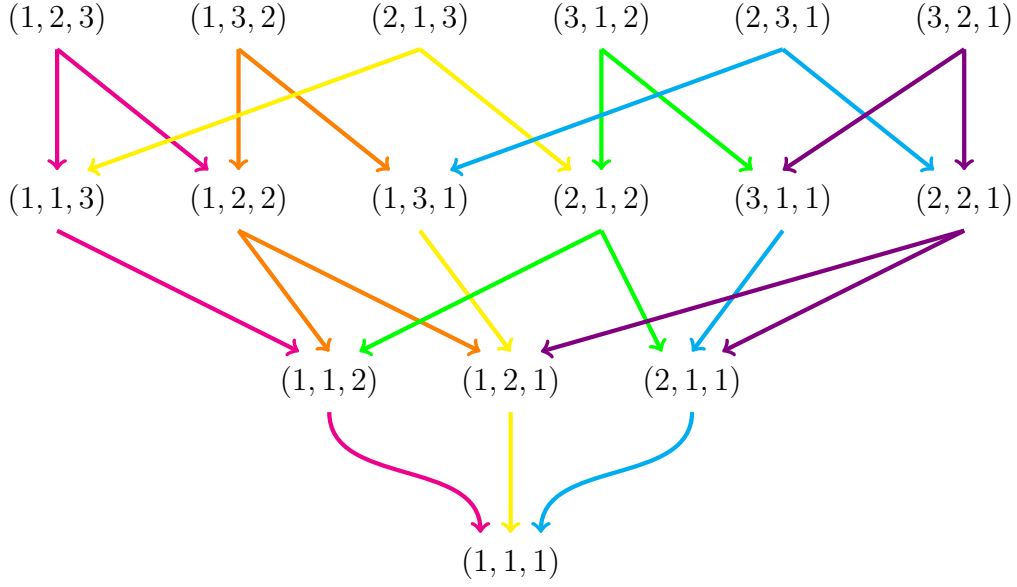
- $f = (a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n)$
- $g = (a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_n)$

That is, the same relation as for primitive parking functions.

Example ($n = 6$). $(2, 1, 5, 3, 1, 4) \succ (2, 1, 5, 2, 1, 4)$

Proposition. This covering relation defines a poset for \mathcal{PF}_n .

Example (The poset of \mathcal{PF}_3).



There are $4^2 = 16$ elements in this poset.

Remark. *The two posets are isomorphic, and one can be obtained by applying the aforementioned bijection to the other.*

Theorem 9. *The number of intervals in those posets is equal to the $n + 1^{\text{th}}$ term of the integer sequence defined by <https://oeis.org/A196304>.*

The first terms of this sequence are 1, 1, 5, 64, 1587, 65421, 4071178, 357962760, 4237910716,

Chapter 2

The rational case

For the whole chapter, we will consider 2 *coprime* integers a and b (meaning a and b have 1 as their greatest common divisor).

2.1 Rational Parking Functions

Definition 20 (a, b - Parking Function). An a, b - parking function is a sequence (a_1, a_2, \dots, a_n) such that :

- $n = a$
- its non-decreasing reordering (b_1, b_2, \dots, b_n) has $b_i \leq \frac{b}{a}(i - 1) + 1$ for all i .

We denote by $\mathcal{PF}_{a,b}$ the set of a, b - parking functions.

Example.

- *Ex. 1* : $a > b$

$$a = 7$$

$$b = 3$$

Limits of the non-decreasing reordering of any $f \in \mathcal{PF}_{7,3}$:

$$[1, 1\frac{3}{7}, 1\frac{6}{7}, 2\frac{2}{7}, 2\frac{5}{7}, 3\frac{1}{7}, 3\frac{4}{7}]$$

$$f_1 = (2, 1, 1, 3, 2, 3, 1) \in \mathcal{PF}_{7,3}$$

$$f_2 = (2, 1, 2, 3, 2, 3, 1) \notin \mathcal{PF}_{7,3}, \text{ even though } f_2 \in \mathcal{PF}_7$$

- *Ex. 2* : $a < b$

$$a = 5$$

$$b = 7$$

Limits of the non-decreasing reordering of any $f \in \mathcal{PF}_{5,7}$:

$$[1, 2\frac{2}{5}, 3\frac{4}{5}, 5\frac{1}{5}, 6\frac{3}{5}]$$

$$f_3 = (6, 3, 5, 1, 2) \in \mathcal{PF}_{5,7}, \text{ even though } f_3 \notin \mathcal{PF}_5$$

$$f_4 = (6, 3, 5, 1, 3) \notin \mathcal{PF}_{5,7}$$

Theorem 10. *Let $pf_{a,b}$ be the cardinal of $\mathcal{PF}_{a,b}$. We have*

$$pf_{a,b} = b^{a-1}$$

Example ($a = 3, b = 5$).

- $pf_{a,b} = 25$
- *Limits* : $[1, 2\frac{2}{3}, 4\frac{1}{3}]$

(1, 1, 1)	(1, 1, 2)	(1, 1, 3)	(1, 1, 4)	(1, 2, 1)	(1, 2, 2)	(1, 2, 3)
(1, 2, 4)	(1, 3, 1)	(1, 3, 2)	(1, 4, 1)	(1, 4, 2)	(2, 1, 1)	(2, 1, 2)
(2, 1, 3)	(2, 1, 4)	(2, 2, 1)	(2, 3, 1)	(2, 4, 1)	(3, 1, 1)	(3, 1, 2)
(3, 2, 1)	(4, 1, 1)	(4, 1, 2)	(4, 2, 1)			

Remark. $\mathcal{PF}_{n,n+1} = \mathcal{PF}_n$. In fact, we do have $b^{a-1} = (n+1)^{n-1}$.

2.1.1 Rational primitive parking functions

Definition 21 (Rational Primitive). *A rational parking function f is said primitive if it is already in non-decreasing order.*

We denote by $\mathcal{PF}'_{a,b}$ the set of primitive a, b - parking functions.

Example ($a = 4, b = 3$). *Limits* : $[1, 1\frac{3}{4}, 2\frac{1}{2}, 3\frac{1}{4}]$

$$f_1 = (1, 1, 2, 2) \in \mathcal{PF}'_{4,3}$$

$$f_2 = (1, 1, 2, 1) \notin \mathcal{PF}'_{4,3}, \text{ even though } f_2 \in \mathcal{PF}_{4,3}.$$

Theorem 11. *Let $pf'_{a,b}$ be the cardinal of $\mathcal{PF}'_{a,b}$. We have*

$$pf'_{a,b} = \frac{1}{a+b} \binom{a+b}{b}$$

which is the rational Catalan number $Cat(a, b)$.

Example ($a = 3, b = 5$).

• $pf'_{a,b} = 7$ • *Limits* : $[1, 2\frac{2}{3}, 4\frac{1}{3}]$

(1, 1, 1) (1, 1, 2) (1, 1, 3) (1, 1, 4) (1, 2, 2) (1, 2, 3) (1, 2, 4)

Remark. $\mathcal{PF}'_{n,n+1} = \mathcal{PF}'_n$. In fact, we do have

$$\begin{aligned} \frac{1}{n+n+1} \binom{n+n+1}{n+1} &= \frac{1}{2n+1} \binom{2n+1}{n+1} = \frac{1}{2n+1} \frac{(2n+1)!}{n!(n+1)!} \\ &= \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1} \frac{(2n)!}{n!n!} = \frac{1}{n+1} \binom{2n}{n} \end{aligned}$$

2.2 Rational Non-crossing Partitions

Definition 22 (a, b - Non-crossing Partition). An a, b - non-crossing partition is *TODO*

We denote by $\mathcal{NC}_{a,b}$ the set of a, b - non-crossing partitions.

Example. An *abncp*

Theorem 12. Let $nc_{a,b}$ be the cardinal of $\mathcal{NC}_{a,b}$. We have

$$nc_{a,b} = \frac{1}{a+b} \binom{a+b}{a} = \frac{(a+b-1)!}{a!b!}$$

, which is the rational Catalan number.

Example. all *abncp* for some a, b

Proposition. This means we can create a bijection between $\mathcal{PF}'_{a,b}$ and $\mathcal{NC}_{a,b}$.

Proof.

- $\mathcal{NC}_{a,b} \rightarrow \mathcal{PF}'_{a,b}$:
- $\mathcal{PF}'_{a,b} \rightarrow \mathcal{NC}_{a,b}$:

□

Definition 23 (a, b - Non-crossing 2-Partition). An a, b - Non-crossing 2-partition is *TODO*

We denote by $\mathcal{NC}_{a,b}^2$ the set of a, b - non-crossing 2-partitions.

Example. some *ncab2*

Theorem 13. Let $nc_{a,b}^2$ be the cardinal of $\mathcal{NC}_{a,b}^2$. We have

$$nc_{a,b}^2 = b^{a-1}$$

.

Example. all *ncab2* for some a b

Proposition. *bijection*

Proof. bijection proof

□

2.3 A direct poset linked to Rational Dyck paths

2.3.1 Rational Dyck Paths

Definition 24 (a, b - Dyck path). An a, b - Dyck word is a word $w \in \{0, 1\}^*$ such that :

- for each suffix w' of w ,

$$|w'|_1 \geq \frac{a}{b} |w'|_0$$

.

- $|w|_0 = b$.

- $|w|_1 = a$.

An a, b - Dyck word can be represented as a path from $(0, 0)$ to (b, a) that stays over $y = \frac{a}{b}x$, called an a, b - Dyck path :

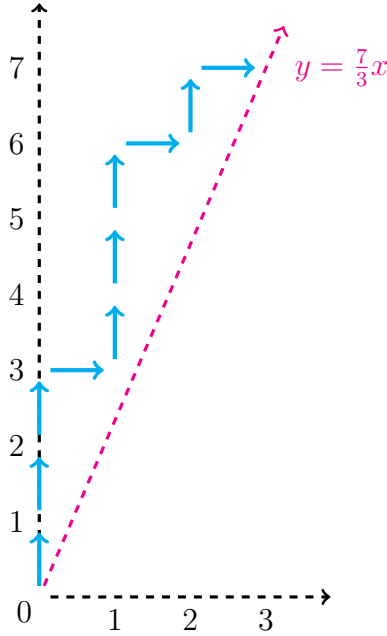
- Each 1 corresponds to a North step \uparrow .
- Each 0 corresponds to an East step \rightarrow .

We denote by $\mathcal{R}_{a,b}$ the set of a, b - Dyck words.

Example ($a > b : a = 7, b = 3$).

$w_1 = 1110011110$ is not a 7, 3 - Dyck word, because $|11100|_1 = 3$
 $< \frac{7}{3}|11100|_0 = \frac{14}{3} = 4\frac{1}{3}$.

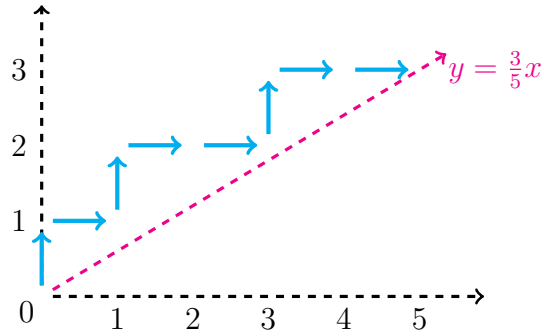
$w_2 = 1110111010$ is a 7, 3 - Dyck word :



Example ($a < b : a = 3, b = 5$).

$w_1 = 10100010$ is not a 3, 5 - Dyck word, because $|101000|_1 = 2$
 $< \frac{3}{5}|101000|_0 = \frac{12}{5} = 2\frac{2}{5}$.

$w_2 = 10100100$ is a 3, 5 - Dyck word :

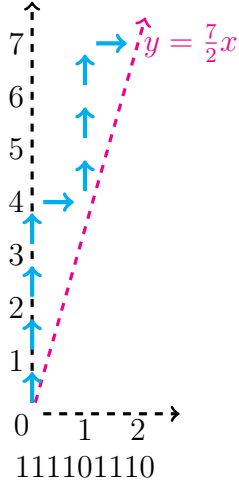


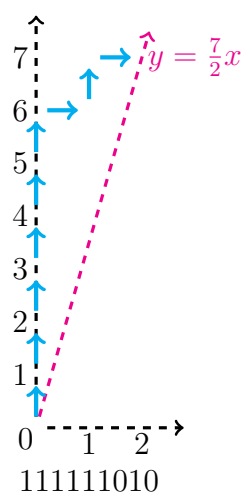
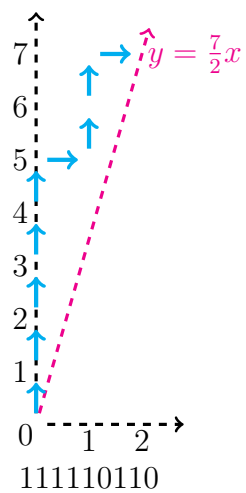
Theorem 14. Let $r_{a,b}$ be the cardinal of $\mathcal{R}_{a,b}$. We have

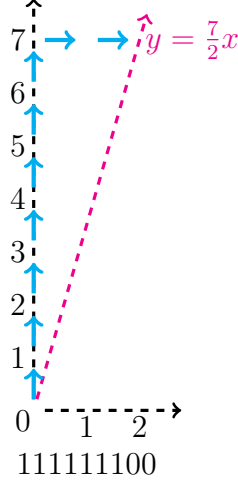
$$r_{a,b} = \frac{1}{a+b} \binom{a+b}{a} = \frac{(a+b-1)!}{a!b!}$$

, which is again the rational Catalan number.

Example ($a = 7, b = 2$). $r_n = 4$.







Proposition. *This means we can create a bijection between $\mathcal{PF}'_{a,b}$ and $\mathcal{R}_{a,b}$.*

Proof.

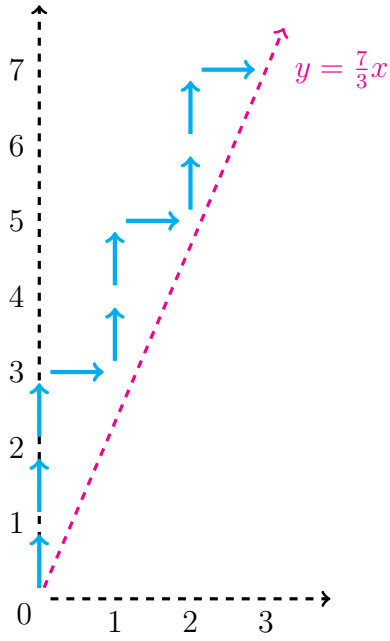
- $\mathcal{PF}'_{a,b} \rightarrow \mathcal{R}_{a,b}$: Let $f = (a_1, \dots, a_n) \in \mathcal{PF}'_{a,b}$ be our rational primitive parking function. For $i \in \{1, \dots, b\}$, we define l_i the number of occurrences of i in f .
The corresponding rational Dyck word will be $\underbrace{1 \dots 1}_l 0 \underbrace{1 \dots 1}_l 0 \dots \underbrace{1 \dots 1}_l 0$.
- $\mathcal{R}_{a,b} \rightarrow \mathcal{PF}'_{a,b}$: Let w be our rational Dyck word, and consider its path representation. We define s_i to be the distance between the segment from $(0, i-1)$ to $(0, i)$ and the i^{th} North step. Then, let $a_i = s_i + 1$.
The corresponding rational primitive parking function is (a_1, \dots, a_a) .

□

Remark. *This bijection is exactly the same as the one between classical primitive parking functions and Dyck paths.*

Example ($a > b : a = 7, b = 3, \mathcal{PF}'_{a,b} \rightarrow \mathcal{R}_{a,b}$).

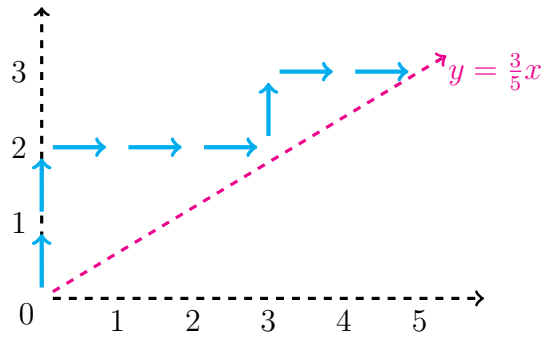
- $f = (1, 1, 1, 2, 2, 3, 3)$
 $l_1 = 3 \qquad l_2 = 2 \qquad l_3 = 2$
- $w = (1110110110)$



Example ($a < b : a = 3, b = 5, \mathcal{PF}'_{a,b} \rightarrow \mathcal{R}_{a,b}$).

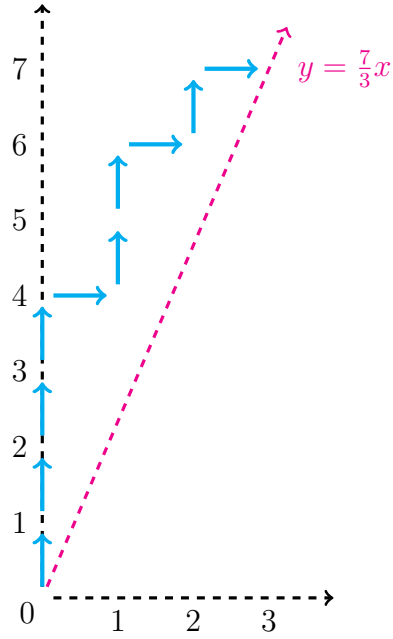
- $f = (1, 1, 4)$

$l_1 = 2$	l_2	$l_3 = 0$
$l_4 = 1$	$l_5 = 0$	
- $w = 11000100$



Example ($a > b : a = 7, b = 3, \mathcal{R}_{a,b} \rightarrow \mathcal{PF}'_{a,b}$).

- $w = 1111011010$



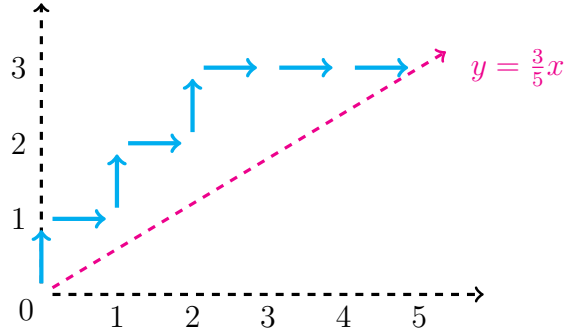
- *Distances :*

$s_1 = 0$	$a_1 = 1$
$s_2 = 0$	$a_2 = 1$
$s_3 = 0$	$a_3 = 1$
$s_4 = 0$	$a_4 = 1$
$s_5 = 1$	$a_5 = 2$
$s_6 = 1$	$a_6 = 2$
$s_7 = 2$	$a_7 = 3$

- $f = (1, 1, 1, 1, 2, 2, 3)$

Example ($a < b : a = 3, b = 5, \mathcal{R}_{a,b} \rightarrow \mathcal{PF}'_{a,b}$).

- $w = 10101000$



- *Distances :*

$$\begin{array}{ll} s_1 = 0 & a_1 = 1 \\ s_2 = 1 & a_2 = 2 \\ s_3 = 2 & a_3 = 3 \end{array}$$

- $f = (1, 2, 3)$

2.3.2 Rational Labeled Dyck Paths

Definition 25 (Labeled a, b - Dyck Path). A labeled a, b - Dyck word is a word $w \in \{0, \dots, n\}^*$ such that :

- for each suffix w' of w ,

$$|w'|_{\neq 0} \geq \frac{a}{b} |w'|_0$$

.

- $|w|_0 = b$.
- $|w|_{\neq 0} = a$.
- for each $i \in \{1, \dots, a\}$, w has exactly one occurrence of i .
- if $w_i \neq 0$ and $w_{i+1} \neq 0$, then $w_i < w_{i+1}$. That is, consecutive North steps have increasing labels.

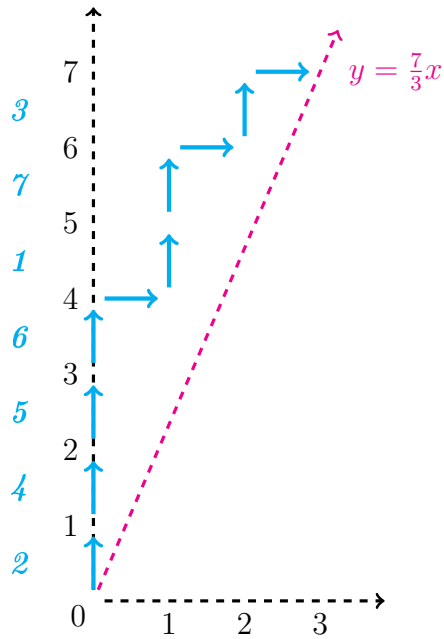
A labeled a, b - Dyck word can be represented as a path from $(0, 0)$ to (b, a) , where each North step is associated to a label :

- Each $i \neq 0$ corresponds to a North step \uparrow labeled i .
- Each 0 corresponds to an East step \rightarrow .

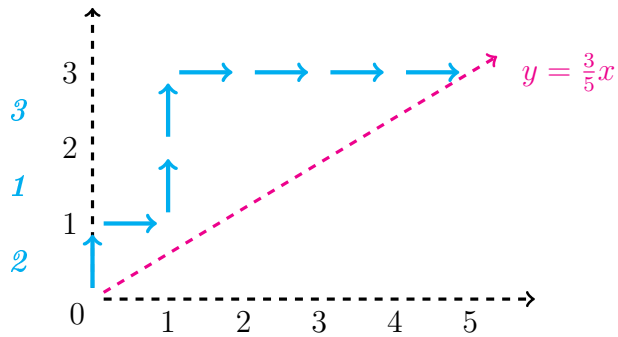
Those paths are called labeled a, b - Dyck paths.

We denote by $\mathcal{LR}_{a,b}$ the set of labeled a, b - Dyck words.

Example ($a > b : a = 7, b = 3$). $w_2 = 2456017030$:



Example ($a < b : a = 3, b = 5$). $w = 20130000$:



Theorem 15. *Let $lr_{a,b}$ be the cardinal of $\mathcal{LR}_{a,b}$. We have*

$$lr_{a,b} = b^{a-1}$$

Example ($a > b : a = 4, b = 3$). $lr_{a,b} = 3^3 = 27$

- *Word of shape $XXXX000$:*

1234000

- *Words of shape $XXX0X00$:*

1230400

1240300

1340200

2340100

- *Words of shape $XX0XX00$:*

1203400

1302400

1402300

2301400

2401300

3401200

- *Words of shape $XXX00X0$:*

1230040

1240030

1340020

2340010

- *Words of shape $XX0X0X0$:*

1203040

1204030

1302040

1304020

1402030

1403020

2301040

2304010

2401030

2403010

3401020

3402010

Proposition. *This means we can create a bijection between $\mathcal{PF}_{a,b}$ and $\mathcal{LR}_{a,b}$.*

Proof.

- $\mathcal{PF}_{a,b} \rightarrow \mathcal{LR}_{a,b}$: Let $f = (a_1, \dots, a_n) \in \mathcal{PF}_{a,b}$ be our a, b - parking function. For $i \in \{1, \dots, b\}$, we define $im_i : \{j \mid a_j = i\}$. We then define $im_{i,1}, \dots, im_{i,k_i}$ to be the elements of im_i in increasing order.

The corresponding labeled a, b - Dyck word will be

$$\underbrace{im_{1,1} \cdots im_{1,k_1}}_{im_1} 0 \underbrace{im_{2,1} \cdots im_{2,k_2}}_{im_2} 0 \cdots \underbrace{im_{n,1} \cdots im_{n,k_n}}_{im_b} 0.$$

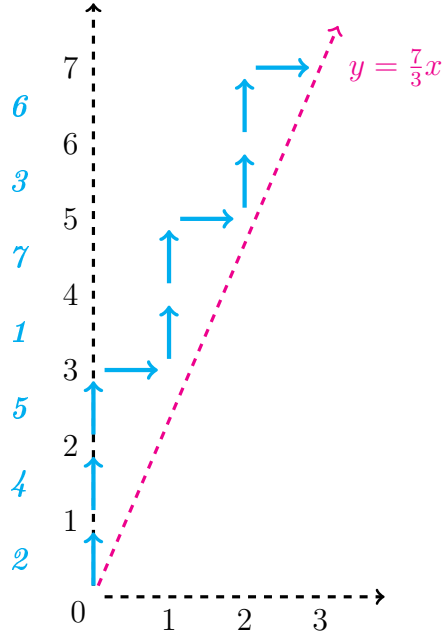
- $\mathcal{LR}_{a,b} \rightarrow \mathcal{PF}_n$: Let w be our labeled a, b - Dyck word, and consider its path representation. We define s_i to be the distance between the segment from $(0, i-1)$ to $(0, i)$ and the i^{th} North step. Then, let $label(i)$ be the label of the i^{th} North step, and $dist_i = \{label(j) \mid s_j = i\}$ be the set of the labels of all North steps at distance i . Then, if $j \in dist_i$, let $a_j = i+1$. The corresponding parking function is (a_1, \dots, a_a) .

□

Remark. *This bijection is exactly the same as the one between classical parking functions and labeled Dyck paths.*

Example ($a > b : a = 7, b = 3, \mathcal{PF}_{a,b} \rightarrow \mathcal{LR}_{a,b}$).

- $f = (2, 1, 3, 1, 1, 3, 2)$
 $im_1 = \{2, 4, 5\} \qquad im_2 = \{1, 7\} \qquad im_3 = \{3, 6\}$
- $w = 2450170360$



Example ($a < b : a = 3, b = 5, \mathcal{PF}_{a,b} \rightarrow \mathcal{LR}_{a,b}$).

- $f = (4, 1, 2)$

$$im_1 = \{2\}$$

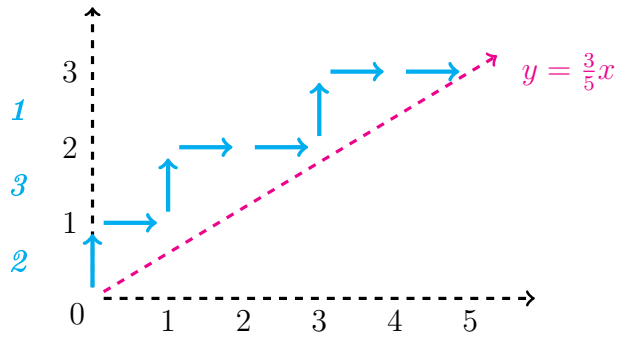
$$im_2 = \{3\}$$

$$im_3 = \emptyset$$

$$im_4 = \{1\}$$

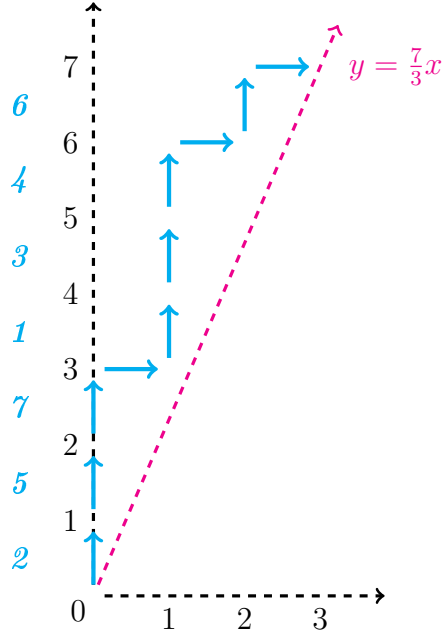
$$im_5 = \emptyset$$

- $w = 20300100$



Example ($a > b : a = 7, b = 3, \mathcal{LR}_{a,b} \rightarrow \mathcal{PF}_{a,b}$).

- $w = 2570134060$



- *Distances :*

$$\begin{array}{lll} s_1 = 0 & s_2 = 0 & s_3 = 0 \\ s_4 = 1 & s_5 = 1 & s_6 = 1 \\ s_7 = 2 \end{array}$$

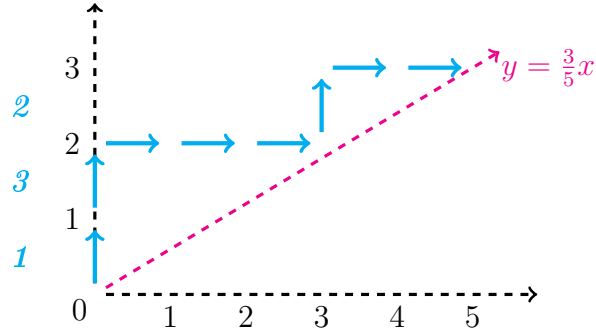
- *Labels :*

$$\begin{array}{lll} dist_0 = \{2, 5, 7\} & dist_1 = \{1, 3, 4\} & dist_2 = \\ \{6\} & & \end{array}$$

- $f = (2, 1, 2, 2, 1, 3, 1)$

Example ($a < b : a = 3, b = 5, \mathcal{LR}_{a,b} \rightarrow \mathcal{PF}_{a,b}$).

- $w = 13000200$



- Distances :

$$s_1 = 0$$

$$s_2 = 0$$

$$s_3 = 3$$

- Labels :

$$dist_0 = \{1, 3\}$$

$$dist_1 = \emptyset$$

$$dist_2 = \emptyset$$

$$dist_3 = \{2\}$$

$$dist_4 = \emptyset$$

- $f = (1, 4, 1)$

Remark. The rational primitive parking functions are exactly the rational parking functions corresponding to rational labeled Dyck paths where the i^{th} North step is labeled i .

2.3.3 Rational Dyck - Parking Posets

Rational Primitive Dyck - Parking Posets

Definition 26 (\succ_r). For w and w' two a, b - Dyck words, we say that w covers w' , written $w \succ_r w'$, if $\exists w_1, w_2$ such that :

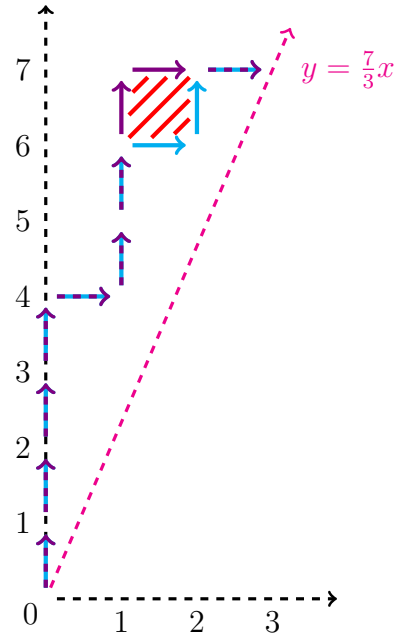
- $w = w_1 0 1 w_2$

- $w' = w_1 1 0 w_2$

Example ($a = 7, b = 3$). $1111011010 \succ_r 1111011100$

- $w_1 = 1111011$

- $w_2 = 0$

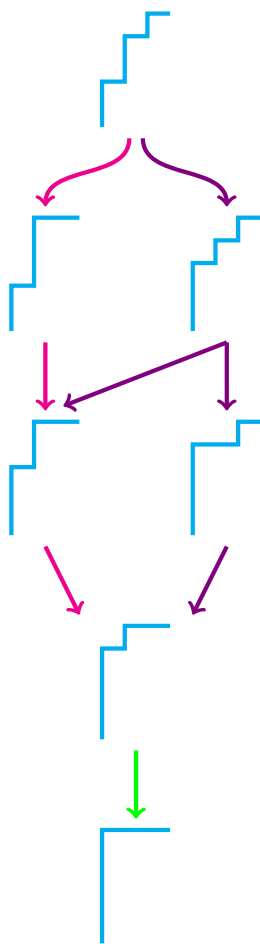


■ 1111011010 ■ 1111011100 ■ difference

Remark. If $w_1 \succ_r w_2$, then the path corresponding to w_2 is over the path corresponding to w_1 , and the difference between the two paths is a square of size 1 by 1.

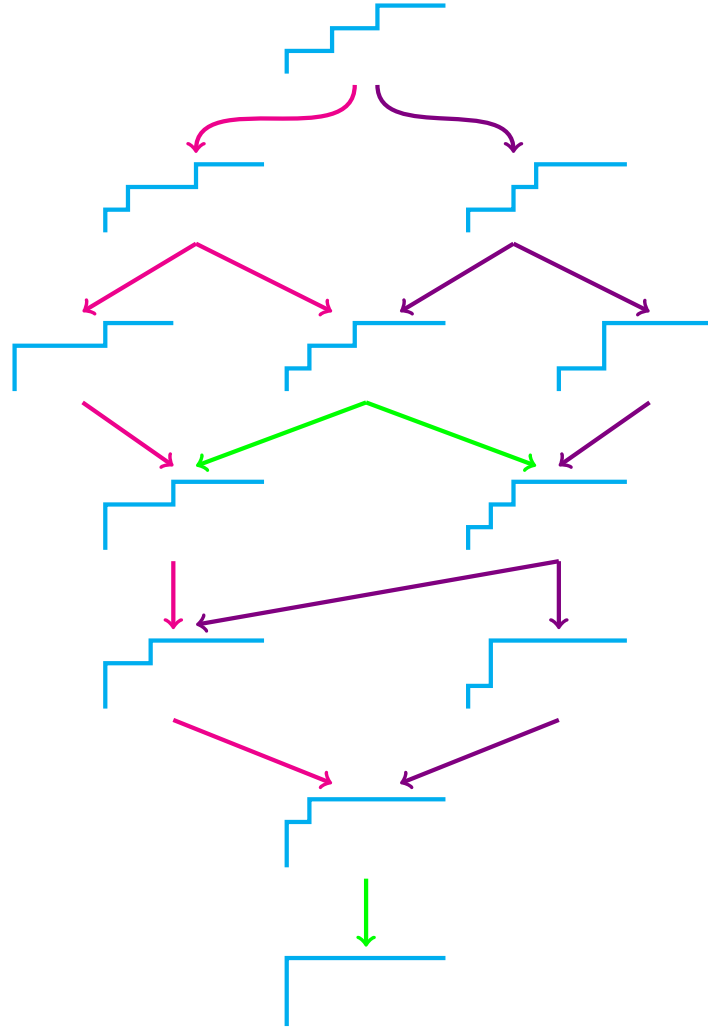
Proposition. This covering relation defines a poset for $\mathcal{R}_{a,b}$.

Example ($a > b$: The poset of $\mathcal{R}_{5,3}$).



There are $\frac{1}{8} \binom{8}{5} = \frac{42}{6} = 7$ elements in this poset.

Example ($a < b$: The poset of $\mathcal{R}_{3,7}$).



There are $\frac{1}{10} \binom{10}{3} = \frac{72}{6} = 12$ elements in this poset.

Definition 27 (\succ'). For f and g two rational primitive a, b - parking functions, we say that f covers g , written $f \succ' g$, if $\exists i$ such that :

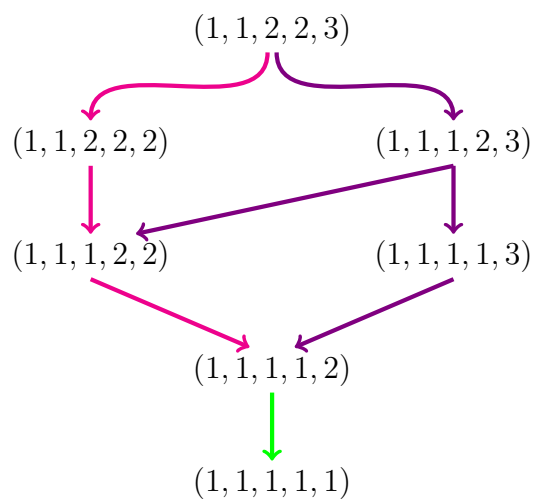
- $f = (a_1, \dots, a_{i-1}, a_i, \quad a_{i+1}, \dots, a_n)$
- $g = (a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_n)$

Example ($a > b : a = 7, b = 3$). $(1, 1, 1, 2, 2, 2, 3) \succ' (1, 1, 1, 1, 2, 2, 3)$

Example ($a < b : a = 3, b = 5$). $(1, 2, 4) \succ' (1, 1, 4)$

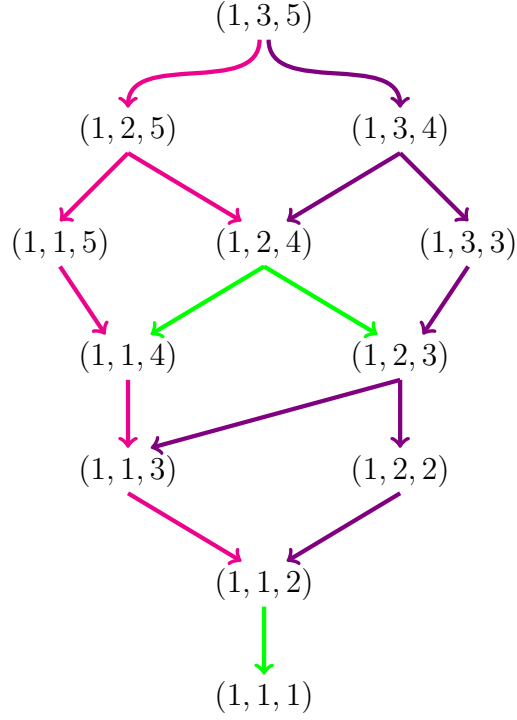
Proposition. *This covering relation defines a poset for $\mathcal{PF}'_{a,b}$.*

Example ($a > b$: The poset of $\mathcal{PF}'_{5,3}$).



There are $\frac{1}{8} \binom{8}{5} = \frac{42}{6} = 7$ elements in this poset.

Example ($a < b$: The poset of $\mathcal{PF}'_{3,7}$).



There are $\frac{1}{10} \binom{10}{3} = \frac{72}{6} = 12$ elements in this poset.

Remark. The posets of $\mathcal{PF}'_{a,b}$ and $\mathcal{R}_{a,b}$ are isomorphic, and one can be obtained by applying the aforementioned bijection to the other.

Rational Dyck - Parking Posets

Definition 28 (\succ_{lr}). For w and w' two labeled a, b - Dyck words, we say that w covers w' , written $w \succ_{lr} w'$, if either :

- $\exists w_2, x, x', y$ such that :

$x = x_1 x_2 \cdots x_n$ has all its digits > 0

$x' = x$ where y is correctly inserted regarding the order condition

$y > 0$

$w = x0yw_2$

$w' = x'0w_2$

- $\exists w_1, w_2, x, x', y$ such that :

$x = x_1x_2 \cdots x_n$ has all its digits > 0

$y > 0$

$x' = x$ where y is correctly inserted regarding the order condition

$w = w_10x0yw_2$

$w' = w_10x'0w_2$

- or $\exists w_1, w_2, y$ such that :

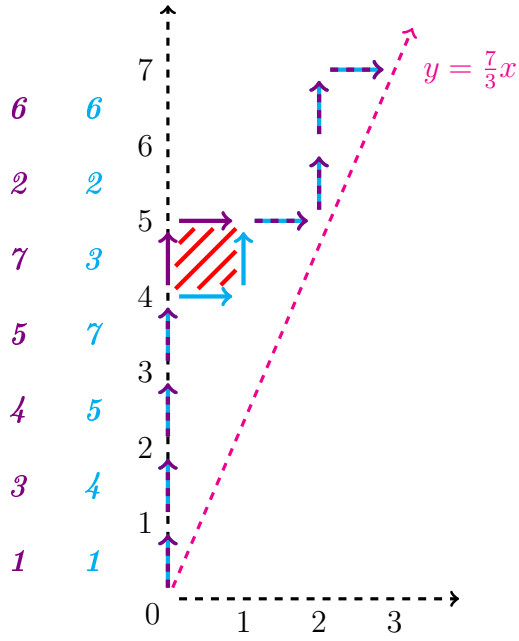
$y > 0$

$w = w_100yw_2$

$w' = w_10y0w_2$

Example ($a > b : a = 7, b = 3$, first case). $1457030260 \succ_{lr} 1345700260$

- $w_2 = 0260$
- $x = 1457$
- $x' = 13457$
- $y = 3$



■ 145703026
 ■ 1345700260
 ■ *difference*

Example ($a < b : a = 3, b = 5$, first case). $20130000 \succ_{lr} 12030000$

- $w_2 = 30000$
- $x = 2$
- $x' = 12$
- $y = 1$

Example ($a > b : a = 7, b = 3$, second case). $2460150370 \succ_{lr} 2460135070$

- $w_1 = 246$
- $w_2 = 70$
- $x = 15$
- $x' = 135$

- $y = 3$

Example ($a < b : a = 3, b = 5$, second case). $20301000 \succ_{lr} 20130000$

- $w_1 = 2$
- $w_2 = 000$
- $x = 3$
- $x' = 13$
- $y = 1$

Example ($a > b : a = 7, b = 3$, third case). $235670014 \succ_{lr} 235670104$

- $w_1 = 23567$
- $w_2 = 4$
- $y = 1$

Example ($a < b : a = 3, b = 5$, third case). $23000100 \succ_{lr} 23001000$

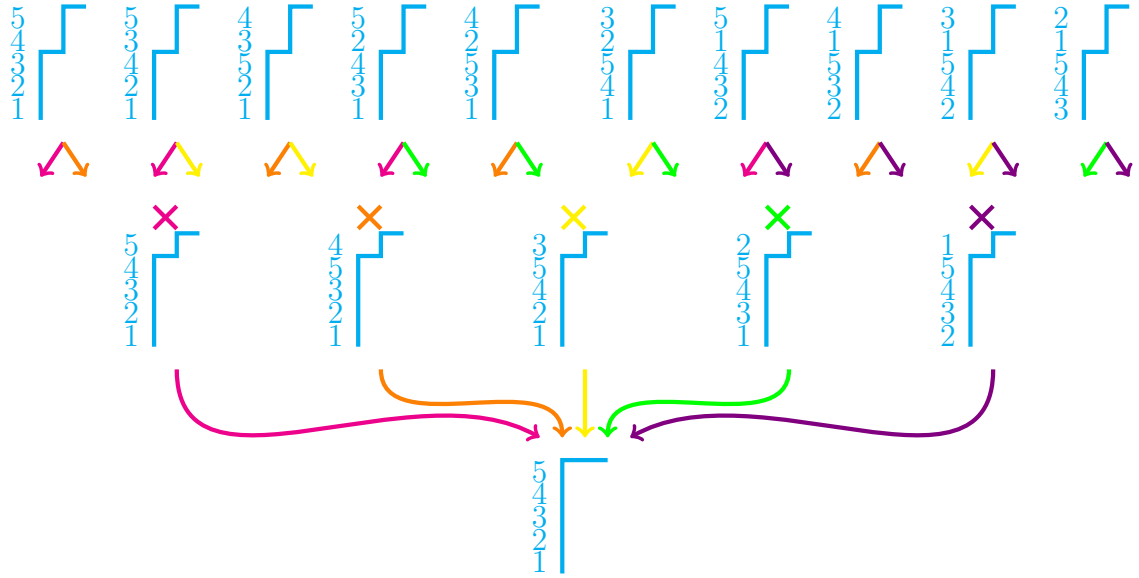
- $w_1 = 230$
- $w_2 = 00$
- $y = 1$

Remark. If $w_1 \succ_{lr} w_2$, then the path corresponding to w_2 is over the path corresponding to w_1 , and the difference between the two paths is a square of size 1 by 1.

Furthermore, the labeling can be seen as follows : if one has to merge two sequences of North steps of w_1 to make w_2 , then the merging will be made by ordering the corresponding labels.

Proposition. This covering relation defines a poset for $\mathcal{LR}_{a,b}$.

Example ($a > b$: The poset of $\mathcal{LR}_{5,2}$).

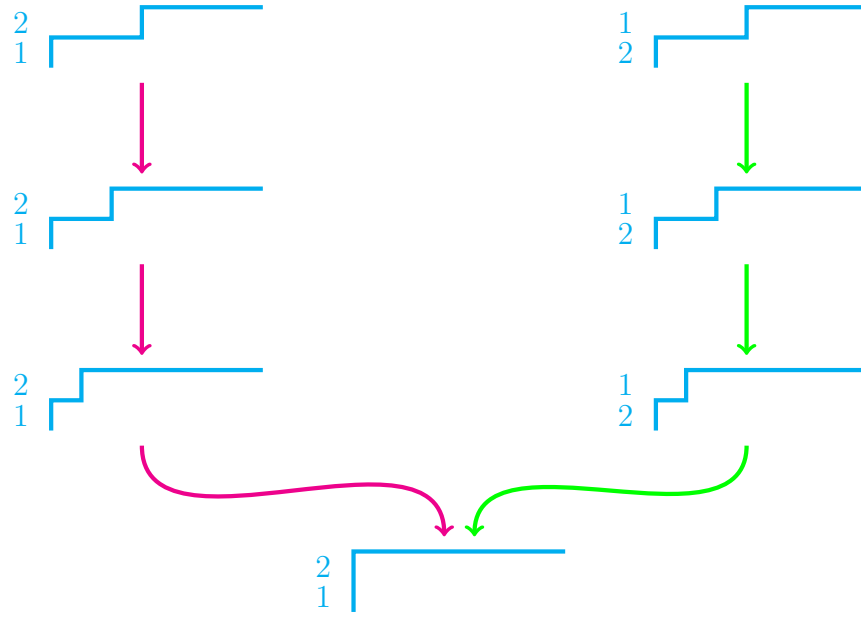


Arrows have been simplified for readability.

Arrows between the top 2 levels are to be read as ending at the cross of the corresponding color.

There are $2^4 = 16$ elements in this poset.

Example ($a < b$: The poset of $\mathcal{LR}_{2,7}$).



There are $7^1 = 7$ elements in this poset.

Definition 29 (\succ). For f and g two rational parking functions, we say that f covers g , written $f \succ g$, if $\exists i$ such that :

- $f = (a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n)$
- $g = (a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_n)$

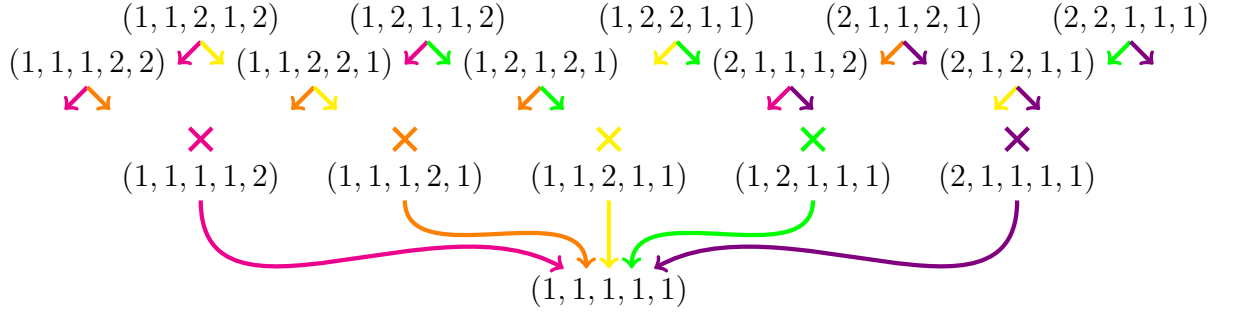
That is, the same relation as for rational primitive parking functions.

Example ($a > b : a = 7, b = 3$). $(2, 3, 1, 1, 2, 1, 3) \succ (2, 3, 1, 1, 1, 1, 3)$

Example ($a < b : a = 3, b = 5$). $(4, 1, 2) \succ (3, 1, 2)$

Proposition. This covering relation defines a poset for $\mathcal{PF}_{a,b}$.

Example ($a > b$: The poset of $\mathcal{PF}_{5,2}$).

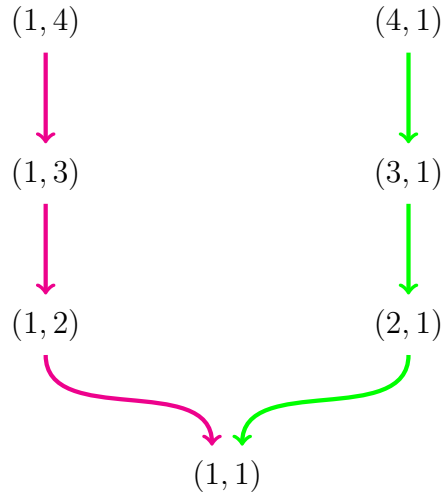


Arrows have been simplified for readability.

Arrows between the top 2 levels are to be read as ending at the cross of the corresponding color.

There are $2^4 = 16$ elements in this poset.

Example ($a < b$: The poset of $\mathcal{PF}_{2,7}$).



There are $7^1 = 7$ elements in this poset.

Remark. *The posets of $\mathcal{PF}_{a,b}$ and $\mathcal{LR}_{a,b}$ are isomorphic, and one can be obtained by applying the aforementioned bijection to the other.*

Chapter 3

Trees

3.1 Parking Trees

Definition 30 (Parking Trees). A parking tree is defined from a parking function $f = (a_1, \dots, a_n) \in \mathcal{PF}_n$ as follows :

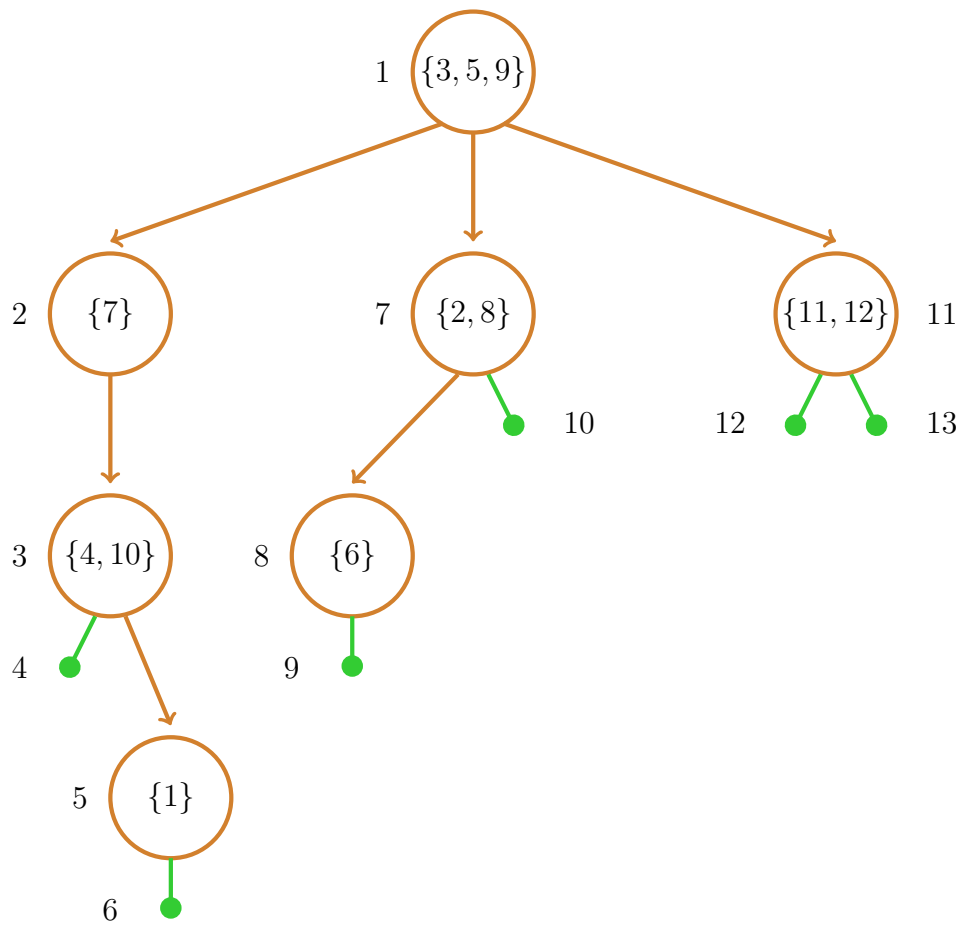
- For $1 \leq i \leq n+1$, we define s_i as $\{j \mid a_j = i\}$
- $[s_1, \dots, s_{n+1}]$ describes the pre-order depth-first traversal of the tree.
- Each node labeled by a set of size k has k children.

Remark. The leaves of the tree are those corresponding to an element i such that $1 \leq i \leq n+1$, and i is not in f .

Furthermore, as we will have a total edges by definition, the presence of a node corresponding to $n+1$ is necessary, even though it will always be empty.

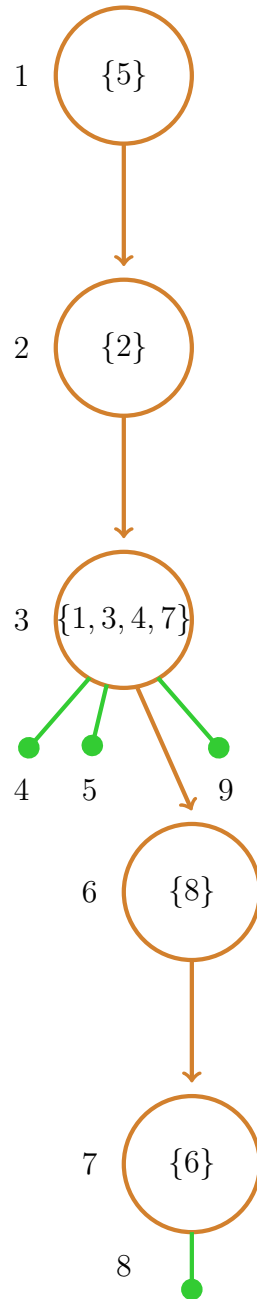
Example ($n = 12$).

- $f = (5, 7, 1, 3, 1, 8, 2, 7, 1, 3, 11, 11)$
- Labels : $[\{3, 5, 9\}, \{7\}, \{4, 10\}, \emptyset, \{1\}, \emptyset, \{2, 8\}, \{6\}, \emptyset, \emptyset, \{11, 12\}, \emptyset, \emptyset]$



Conversely, by reading the labels of a parking tree depth-first in pre-order, we get the list of positions of each number in the corresponding parking function, thus creating a *bijection*.

Example (From parking tree to parking function).



- The labels are $[\{5\}, \{2\}, \{1, 3, 4, 7\}, \emptyset, \emptyset, \{8\}, \{6\}, \emptyset, \emptyset]$.
- Thus the corresponding parking function is $(3, 2, 3, 3, 1, 7, 3, 6) \in \mathcal{PF}_8$.

3.2 Rational Parking Trees

Definition 31 (Rational Parking Trees). *A rational parking tree is defined from a rational parking function $f = (a_1, \dots, a_a) \in \mathcal{PF}_{a,b}$ as follows :*

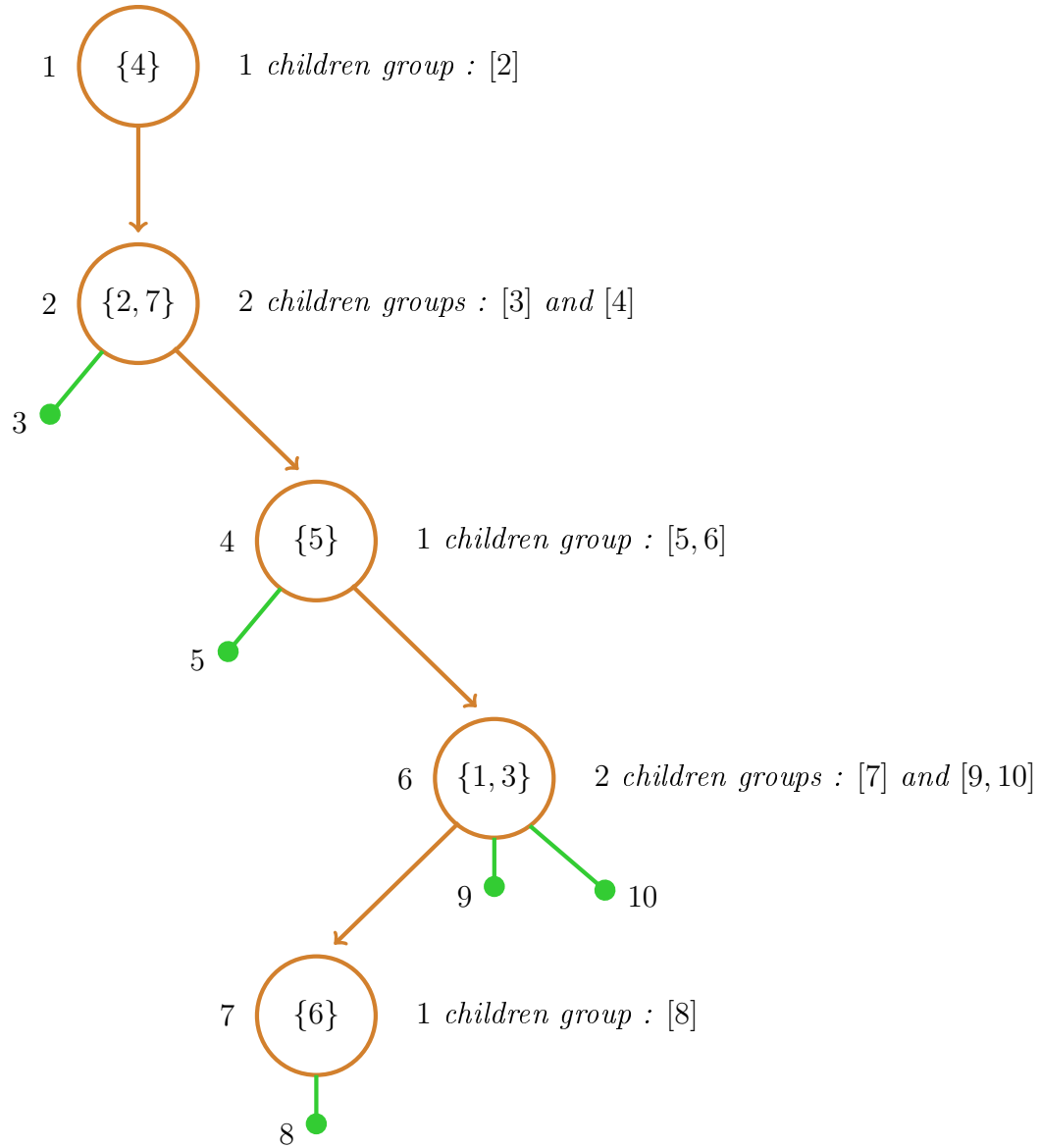
- *For $1 \leq i \leq n + 1$, we define the limit l_i as the integer portion of $\frac{b}{a}(i - 1) + 1$.
Let $l_0 = 0$.*
- *From these limits, we deduce the intervals $itv_i =]l_{i-1}, l_i]$ for $1 \leq i \leq a + 1$.*
- *For $1 \leq i \leq b + 1$, define s_i as $\{j \mid a_j = i\}$.*
- *$[s_1, \dots, s_{b+1}]$ describes the pre-order depth-first traversal of the tree.*
- *Each node labeled by a set of size k has k groups of children, which are defined by the intervals.*

Example ($a < b$).

- $a = 7$
- $b = 9$
- *Limits :* $[1, 2\frac{2}{7}, 3\frac{4}{7}, 4\frac{6}{7}, 6\frac{1}{7}, 7\frac{3}{7}, 8\frac{5}{7}, 10]$
- *Integral limits :* $[0, 1, 2, 3, 4, 6, 7, 8, 10]$
- *Intervals :*

$$]0, 1] \quad]1, 2] \quad]2, 3] \quad]3, 4] \quad]4, 6] \quad]6, 7] \quad]7, 8] \quad]8, 10]$$
- *Children groups :*

$$[1] \quad [2] \quad [3] \quad [4] \quad [5, 6] \quad [7] \quad [8]$$
- $f = (6, 2, 6, 1, 4, 7, 2)$
- *Labels :* $\{\{4\}, \{2, 7\}, \emptyset, \{5\}, \emptyset, \{1, 3\}, \{6\}, \emptyset, \emptyset, \emptyset\}$



Example ($a > b$).

- $a = 9$
- $b = 7$
- *Limits* : $[1, 1\frac{7}{9}, 2\frac{5}{9}, 3\frac{3}{9}, 4\frac{1}{9}, 4\frac{8}{9}, 5\frac{6}{9}, 6\frac{4}{9}, 7\frac{2}{9}, 8]$

- *Integral limits* : $[0, 1, 1, 2, 3, 4, 4, 5, 6, 7, 8]$

- *Intervals* :

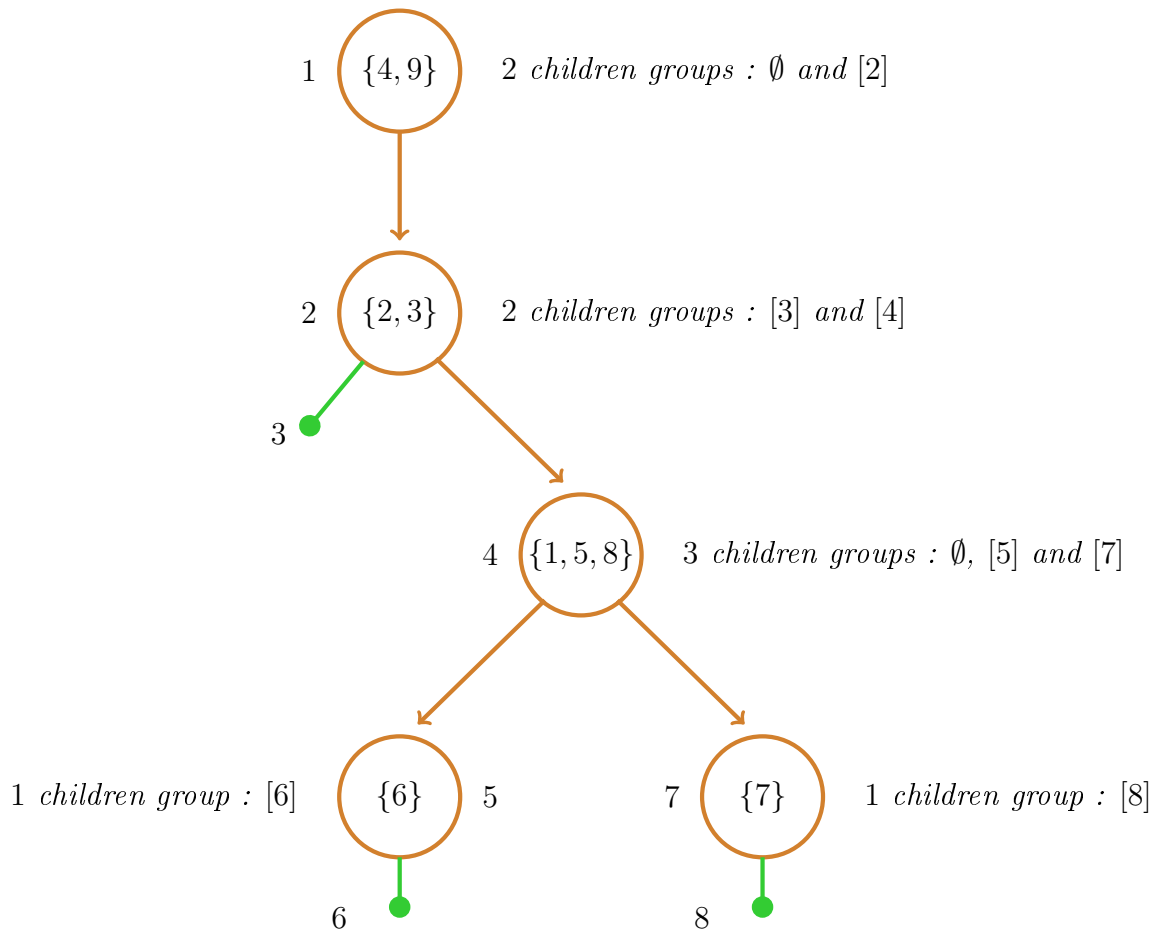
$]0, 1]$	$]1, 1]$	$]1, 2]$	$]2, 3]$	$]3, 4]$
$]4, 4]$	$[4, 5]$	$]5, 6]$	$]6, 7]$	$]7, 8]$

- *Children groups* :

$[1]$	\emptyset	$[2]$	$[3]$	$[4]$	\emptyset	$[5]$	$[6]$	$[7]$	$[8]$
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- $f = (4, 2, 2, 1, 4, 5, 7, 4, 1)$

- *Labels* : $\{\{4, 9\}, \{2, 3\}, \emptyset, \{1, 5, 8\}, \{6\}, \emptyset, \{7\}, \emptyset\}$



In both cases, the converse direction of the *bijection* is obtained with the same method as for classical parking trees.