Rational Parking Functions

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August 27, 2020

Abstract

This is an abstract about Rational Parking Functions

Chapter 1

The integer case

1.1 Parking Functions

Definition 1 (Parking Function)**.** *A* parking function *is a sequence* $(a_1, a_2, \ldots, a_n)$ *such that its non-decreasing reordering* $(b_1, b_2, \ldots, b_n)$ *has* $b_i < i$ *for all* i*.*
We denote by $\mathcal{PF}_n$ *the set of parking functions of length* n*.*

$$\mathcal{PF} = \bigcup_{n>0} \mathcal{PF}_n$$

.

Example.

$$f_1 = (7, 3, 1, 4, 2, 5, 2) \in \mathcal{PF}_7$$
$$f_2 = (7, 3, 1, 4, 2, 5, 4) \notin \mathcal{PF}_7$$

Theorem 1. *Let* pf_n *be the cardinal of* $\mathcal{PF}_n$*. We have*

$$pf_n = (n+1)^{n-1}$$

.

Example $(n = 1, 2, 3)$**.**
• $n = 1$: $pf_1 = 1$
(1)
• $n = 2$: $pf_2 = 3$

$(1,1) \quad (1,2) \quad (2,1)$

• $n = 3 \; : \; pf_3 = 16$

$(1,1,1) \quad (1,1,2) \quad (1,1,3) \quad (1,2,1) \quad (1,2,2) \quad (1,2,3) \quad (1,3,1)$
$(1,3,2) \quad (2,1,1) \quad (2,1,2) \quad (2,1,3) \quad (2,2,1) \quad (2,3,1) \quad (3,1,1)$
$(3,1,2) \quad (3,2,1)$

1.1.1 Primitive parking functions

Definition 2 (Primitive). *A parking function $(a_1, a_2, \ldots, a_n)$ is said* primitive *if it is already in non-decreasing order.*
We denote by $\mathcal{PF}'_n$ the set of primitive parking functions of length n.

$$\mathcal{PF}' = \bigcup_{n>0} \mathcal{PF}'_n$$

Example.

$$f_1 = (1,2,2,3) \in \mathcal{PF}'_4$$
$$f_2 = (1,2,3,2) \notin \mathcal{PF}'_4, \text{ even though } f_2 \in \mathcal{PF}_4$$

Theorem 2. *Let pf'_n be the cardinal of $\mathcal{PF}'_n$. We have*

$$pf'_n = \frac{1}{n+1}\binom{2n}{n}$$

which is the n^{th} Catalan number $Cat(n)$.

Example $(n = 1,2,3)$**.**

• $n = 1 \; : \; pf'_1 = 1$

(1)

• $n = 2 \; : \; pf'_2 = 2$

$(1,1) \quad (1,2)$

• $n = 3 \; : \; pf'_3 = 5$

$(1,1,1) \quad (1,1,2) \quad (1,1,3) \quad (1,2,2) \quad (1,2,3)$

1.2 Non-crossing Partitions

Definition 3 (Non-crossing Partition). *A* non-crossing partition *of a set E is a set partition $P = \{E_1, E_2, \ldots, E_k\}$ such that if $a, c \in E_i$, $b, d \in E_j$, and $i \neq j$, then we do* not *have $a < b < c < d$, nor $a > b > c > d$.*
We denote by $\mathcal{NC}_n$ the set of non-crossing partitions of $\{1, 2, \ldots, n\}$.

$$\mathcal{NC} = \bigcup_{n>0} \mathcal{NC}_n$$

From this point, we assume that every partition $P = \{B_1, \ldots, B_l\}$ is *sorted* such that :
- For each block $B_i = \{b_1, \ldots, b_k\} \in P$, $b_1 < \ldots < b_k$
- $min(B_1) < \ldots < min(B_k)$

Notation. $[n] = \{1, 2, \ldots, n\}$

Example ($E = [6]$).

$$P_1 = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\} \in \mathcal{NC}_6$$
$$P_2 = \{\{1, 2, 4\}, \{3, 5\}, \{6\}\} \notin \mathcal{NC}_6$$

Theorem 3. *Let nc_n be the cardinal of $\mathcal{NC}_n$. We have*

$$nc_n = \frac{1}{n+1}\binom{2n}{n}$$

which is again the n^{th} Catalan number $Cat(n)$.

Example ($n = 1, 2, 3$).
- $n = 1 \ : \ nc_1 = 1$
  $\{\{1\}\}$
- $n = 2 \ : \ nc_2 = 2$
  $\{\{1, 2\}\}$ $\{\{1\}, \{2\}\}$
- $n = 3 \ : \ nc_3 = 5$
  $\{\{1, 2, 3\}\}$ $\{\{1\}, \{2, 3\}\}$ $\{\{1, 3\}, \{2\}\}$ $\{\{1, 2\}, \{3\}\}$ $\{\{1\}, \{2\}, \{3\}\}$

Proposition. *This means we can create a* bijection *between $\mathcal{PF}'_n$ and $\mathcal{NC}_n$.*

Proof.

- $\mathcal{NC}_n \to \mathcal{PF'}_n$: For each block B in the non-crossing partition, take $i = min(B)$, and $k_i = size(B)$.
  $k_i = 0$ if i is not the minimum of a block.
  The corresponding parking function is $(\underbrace{1, \ldots, 1}_{k_1}, \underbrace{2, \ldots, 2}_{k_2}, \ldots, \underbrace{n, \ldots, n}_{k_n})$.

- $\mathcal{PF'}_n \to \mathcal{NC}_n$: For each i in $[n]$, if i appears n_i times in the parking function, B_i will be of size n_i with minimum element i. There is a unique set partition $P = \bigcup_i B_i$ of $[n]$ respecting these conditions that is non-crossing.

$\square$

Example $(n = 6)$**.**

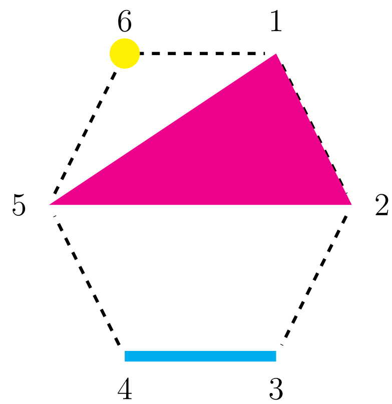
$$P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\} \qquad\qquad f = (1, 1, 1, 3, 3, 6)$$

Corollary. *A non-crossing partition can be represented by the minimums and sizes of its blocks.*

Example. $\{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$ *can be represented by the following dictionnary :*
- *1 : 3*
- *3 : 2*
- *6 : 1*

A non-crossing partition of $[n]$ can be represented graphically on a regular n-vertices polygon, with vertices labeled from 1 to n clockwise. We then represent each block $B = \{b_1, \ldots, b_k\}$ by the convex hull of $\{b_1, \ldots, b_k\}$.

Example $(P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\})$**.**



Thus non-crossing meaning the hulls are *disjoint*.

1.2.1 The non-crossing partitions poset

Definition 4 ($\succ$)**.** *We say that P covers Q, written $P \succ Q$, if $\exists B_i, B_j \in P$ such that $Q = P - \{B_i, B_j\} \cup \{B_i \cup B_j\}$*

Example. $\{\{1,6\},\{2,3\},\{4,5\}\} \succ \{\{1,2,3,6\},\{4,5\}\}$

- $B_i = \{1,6\}$
- $B_j = \{2,3\}$

Proposition. *This covering relation defines the* poset *of $\mathcal{NC}_n$. We denote by $\mathcal{NCC}_n$ the set of* maximal chains *in the poset of $\mathcal{NC}_n$.*

$$\mathcal{NCC} = \bigcup_{n>0} \mathcal{NCC}_n$$

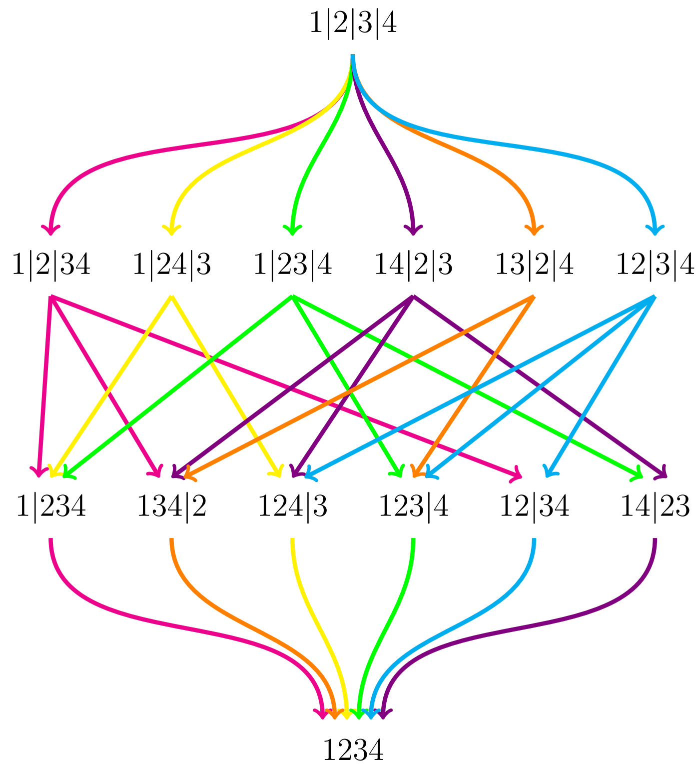
Remark. *The bottom element of this poset is $\{\{1,\ldots,n\}\}$, and the top element is $\{\{1\},\ldots,\{n\}\}$.*

Theorem 4. *Let ncc_n be the cardinal of $\mathcal{NCC}_n$. We have*

$$ncc_n = n^{n-2}$$

.

Example (The poset of $\mathcal{NC}_4$)**.**
To shorten labels, we represent $\{\{1\},\{2,3\},\{4\}\}$ *by* $1|23|4$.



There are $4^2 = 16$ *different maximal chains, and* $\frac{1}{5}\binom{8}{4} = \frac{70}{5} = 14$ *elements in this poset.*

1.2.2 Kreweras complement

Definition 5 (Associated Permutation)**.** *The* permutation σ *associated to a non-crossing partition has a cycle* $(b_1, \dots, b_k)$ *for each block* $B = \{b_1, \dots, b_k\}$ *of the partition.*

Example. *The permutation associated to* $\{\{1,2,5\},\{3,4\},\{6\}\}$ *is* $(1\ 2\ 5)\ (3\ 4)\ (6) = 254316$.

Definition 6 (Kreweras Complement)**.** *The* Kreweras complement $K(P)$ *of a non-crossing partition* P *is defined as follows :*

- *Let σ be the permutation associated to P*
- *Let π be the permutation $(n\ n-1\ n-2\ \dots\ 3\ 2\ 1) = n123\dots n-1$*
- *$K(P)$ is the* non-crossing partition *associated to $\pi\sigma$.*

Example ($P = \{\{1,2,5\},\{3,4\},\{6\}\}$)**.**
- $\sigma = (1\ 2\ 5)\ (3\ 4)\ (6) = 254316$
- $\pi = (6\ 5\ 4\ 3\ 2\ 1) = 612345$
- $\pi\sigma = 143265 = (1)\ (2\ 4)\ (3)\ (5\ 6)$
- $K(P) = \{\{1\},\{2,4\},\{3\},\{5,6\}\}$

Proposition (Kreweras minimums)**.** *Let $P = \{B_1,\dots,B_k\}$ be a non-crossing partition. Let $K(P) = \{B'_1,\dots,B'_l\}$ be its Kreweras complement. Then*

$$\bigcup_{1\leq i\leq l} min(B'_i) = B_1 \cup \bigcup_{1<j\leq k} B_i - max(B_i)$$

Example ($P = \{\{1,2,5\},\{3,4\},\{6\}\}$)**.**

- $K(P) = \{\{1\},\{2,4\},\{3\},\{5,6\}\}$
- $\bigcup min(B'_i) = \{1,2,3,5\}$
- $B_1 \cup \bigcup B_i - max(B_i) = \{1,2,5\} \cup \{3,4\} - \{4\} \cup \{6\} - \{6\} = \{1,2,5\} \cup \{3\} \cup \emptyset = \{1,2,3,5\}$

Notation. $B_{[i]}$ *= block containing i.*

Proposition (Kreweras block sizes)**.** *Let $P = \{B_1,\dots,B_k\}$ be a non-crossing partition. Let $K(P) = \{B'_1,\dots,B'_l\}$ be its Kreweras complement. Then the size of the block B'_i is defined as follows :*

- *Let m_i be the the i^{th} minimum of $K(P)$*
- *Define a* transition *$\phi(e)$ as*

  Let $j = e+1$ (or 1 if $e = n$)

  $\phi(e) = max(B_{[j]})$

- *The size of* B'_i *is* k_{min} *such that* $k_{min} = min\{k > 0 \mid \phi^k(m_i) \in B_{[m_i]}\}$.

Example ($P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$)**.**

- $mins = \{1, 2, 3, 5\}$
- $m_1 = 1$

  $B_{[1]} = B_1$

  $max(B_{[2]} = max(B_1) = 5$

  The size for m_1 *is* 1.
- m_2

  $B_{[2]} = B_1$

  $max(B_{[3]}) = max(B_2) = 4$

  $max(B_{[5]}) = max(B_1) = 5$

  The size for m_2 *is* 2.
- $m_3 = 3$

  $B_{[3]} = B_2$

  $max(B_{[4]}) = max(B_2) = 4$

  The size for m_3 *is* 1.
- $m_4 = 5$

  $B_{[5]} = B_1$

  $max(B_{[6]}) = max(B_3) = 6$

  $max(B_{[1]}) = max(B_1) = 5$

  The size for m_4 *is* 2.

1.2.3 Action of $\mathfrak{S}_n$ on $\mathcal{NC}_n$

Definition 7 (Action of $\mathfrak{S}_n$). *The action of $\mathfrak{S}_n$ on a non-crossing partition $P = \{B_1, \dots, B_l\} \in \mathcal{NC}_n$ is defined by :*
• *For each block $B_i = \{b_1, \dots, b_k\}$: $\sigma(Bi) = \{\sigma(b_1), \dots, \sigma(b_k)\}$*
• *We denote $\rho = \sigma(P) = \{\sigma(B_1), \dots, \sigma(B_l)\}$*

Example ($\sigma = 415362$).
$\sigma(\{\{1,6\},\{2,3,5\},\{4\}\}) = \{\{1,5,6\},\{2,4\},\{3\}\}$

Remark. *Note that $\sigma(P)$ is* not *necessarily non-crossing.*

Definition 8 (Rotation). *We define the* rotation operator *rot of $P \in \mathcal{NC}_n$ as $rot(P) = (1\ 2\ 3\ \ \dots\ n)(P) = 23\dots n1(P)$.*
Conversely, we define rot^{-1} of P as $rot^{-1}(P) = (n\ n-1\ \ \dots 3\ 2\ 1)(P) = n12\dots n-1(P)$.

Remark. $K(K(P)) = rot^{-1}(P)$.

Example ($P = \{\{1,6\},\{2,3,5\},\{4\}\}$).

- $rot(P) = \{\{1,2\},\{3,4,6\},\{5\}\}$
- $rot^{-1}(P) = \{\{1,2,4\},\{3\},\{5,6\}\}$

1.3 Non-crossing 2-partitions

Definition 9 (Non-crossing 2-partition). *A* non-crossing 2-partition *of a set E is a pair (P, σ) where :*
• *P is a non-crossing partition of E*
• *σ is a permutation of the elements of E*
• *For each* sorted *block $B_i = \{b_1, \dots, b_k\} \in P$, we have $\sigma(b_i) < \dots < \sigma(b_k)$*

We denote by $\mathcal{NC}_n^2$ the set of non-crossing 2-partitions of $[n]$.

$$\mathcal{NC}^2 = \bigcup_{n>0} \mathcal{NC}_n^2$$

.

Example ($\mathcal{NC}_6^2$). $P = \{\{1,6\},\{2,3,5\},\{4\}\}$ $\sigma = 413265$
$\rho = \{\{1,3,6\},\{2\},\{4,5\}\}$

Theorem 5. *Let nc_n^2 be the cardinal of $\mathcal{NC}_n^2$. We have*

$$nc_n^2 = (n+1)^{n-1}$$

Example ($n = 1, 2, 3$).

- $n = 1$: $nc_1^2 = 1$

  $\{\{1\}\}$ 1 $\rho = P$

- $n = 2$: $nc_2^2 = 3$

$\{\{1\},\{2\}\}$	12	$\rho = P$
$\{\{1\},\{2\}\}$	21	$\rho = P$
$\{\{1,2\}\}$	12	$\rho = P$

- $n = 3$: $nc_3^2 = 16$

$\{\{1\},\{2\},\{3\}\}$	123	$\rho = P$
$\{\{1\},\{2\},\{3\}\}$	132	$\rho = P$
$\{\{1\},\{2\},\{3\}\}$	213	$\rho = P$
$\{\{1\},\{2\},\{3\}\}$	231	$\rho = P$
$\{\{1\},\{2\},\{3\}\}$	312	$\rho = P$
$\{\{1\},\{2\},\{3\}\}$	321	$\rho = P$
$\{\{1,2\},\{3\}\}$	123	$\rho = P$
$\{\{1,2\},\{3\}\}$	132	$\rho = \{\{1,3\},\{2\}\}$
$\{\{1,2\},\{3\}\}$	231	$\rho = \{\{1\},\{2,3\}\}$
$\{\{1\},\{2,3\}\}$	123	$\rho = P$
$\{\{1\},\{2,3\}\}$	213	$\rho = \{\{1,3\},\{2\}\}$
$\{\{1\},\{2,3\}\}$	312	$\rho = \{\{1,2\},\{3\}\}$
$\{\{1,3\},\{2\}\}$	123	$\rho = P$
$\{\{1,3\},\{2\}\}$	132	$\rho = \{\{1,2\},\{3\}\}$

$\{\{1,3\},\{2\}\}$	213	$\rho = \{\{1\},\{2,3\}\}$
$\{\{1,2,3\}\}$	123	$\rho = P$

Proposition. *This means we can create a* bijection *between* $\mathcal{PF}_n$ *and* $\mathcal{NC}_n^2$.

Proof.

- $\mathcal{PF}_n \to \mathcal{NC}_n^2$: Let $f = (a_1, \ldots, a_n) \in \mathcal{PF}_n$ be our parking function. For $i \in \{1, \ldots, n\}$, we define :

  l_i : the number of occurences of i in f.

  im_i : $\{j \mid a_j = i\}$

  The corresponding non-crossing partition will have the following constraints :

  For each $i \in \{1, \ldots, n\}$, if $l_i > 0$, then there is a block $B_{[i]}$ of length l_i with minimum element i.

  $\sigma(B_{[i]}) = im_i$

  There is a unique set partition $P = \bigcup\limits_i B_{[i]}$ of $[n]$ and a unique permutation σ respecting these conditions such that $(P, \sigma) \in \mathcal{NC}_n^2$.

- $\mathcal{NC}_n^2 \to \mathcal{PF}_n$: Let (P, σ) with $P = \{B_1, \ldots, B_l\}$ be our non-crossing 2-partition. For each block $B_i = \{b_1, \ldots, b_k\} \in P$:

  $m_i = min(B_i) = b_1$

  $pos_i = \sigma(B_i)$

  For each $j \in pos_i$, we define $a_j = m_i$

  The corresponding parking function is $(a_1, \ldots, a_n)$.

□

Example $(n = 8)$**.**

$$P = \{\{1,2,5\},\{3,4\},\{6,8\},\{7\}\}$$
$$\sigma = 36187245$$
$$f = (3,6,1,7,6,1,1,3)$$

1.3.1 The non-crossing 2-partitions poset

Definition 10 ($\succ^2$)**.** *We say that* (P,σ) *covers* (Q,τ)*, written* $(P,\sigma) \succ^2 (Q,\tau)$*, if* $\exists B_i, B_j \in P$ *such that*
- $Q = P - \{B_i, B_j\} \cup \{B_i \cup B_j\}$
- $l \neq i, jb \in B_l \to \tau(b) = \sigma(b)$
- *Let* $B_i \cup B_j = \{b_1, \ldots, b_k\}$ *:*
  $\tau(B_i \cup B_j) = \sigma(B_i \cup B_j)$
  $\tau(b_1) < \ldots < \tau(b_k)$

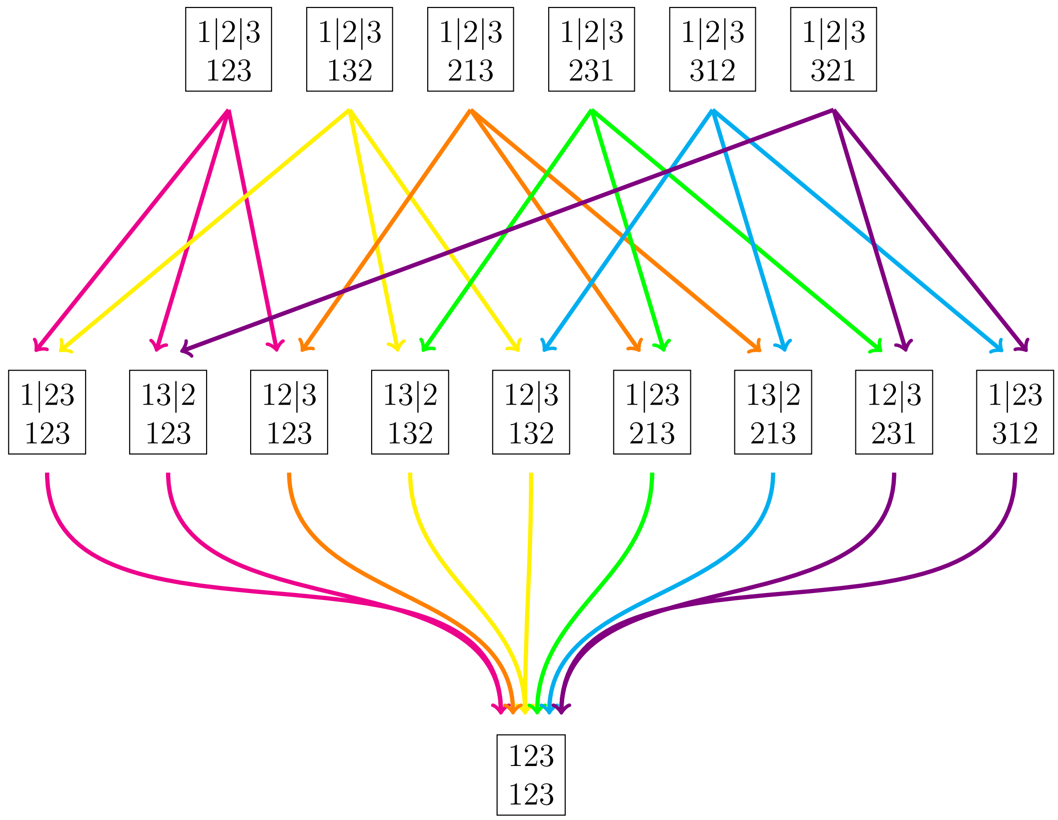
Example.
- $P = \{\{1,6\},\{2,3\},\{4\},\{5\}\}$
- $\sigma = 236154$
- $Q = \{\{1,6\},\{2,3,5\},\{4\}\}$
- $\tau = 235164$
- $(P,\sigma) \succ^2 (Q,\tau)$
- $(P,\sigma) \not\succ^2 (Q,\sigma)$*, because* $\sigma(\{2,3,5\}) = \{3,6,5\}$ *is* not *ordemagenta.*

Proposition. *This covering relation defines the* poset *of* $\mathcal{NC}_n^2$*.*

Remark. *The bottom element of this poset is* $(\{\{1,\ldots,n\}\}, 12\ldots n)$*, and the top elements are* $\{(\{\{1\},\ldots,\{n\}\},\sigma) \mid \sigma \in \mathfrak{S}_n\}$*.*

Example (The poset of $\mathcal{NC}_3^2$)**.**
To shorten labels, we represent $(\{\{1,3\},\{2\}\}, 213)$ *by*

$$\boxed{\begin{array}{c} 13|2 \\ 213 \end{array}}$$



There are $4^2 = 16$ *elements in this poset.*

1.3.2 The parking functions poset

Definition 11 (Rank)**.** *Given* $f = (a_1, \ldots, a_n) \in \mathcal{PF}_n$*, let*

$$b_i = \begin{cases} 1 \text{ if } \exists j \mid a_j = i \\ 0 \text{ otherwise} \end{cases}$$

We define the rank *of* f*, noted* $rk(f)$*, as*

$$\sum_{1 \leq i \leq n} b_i$$

Example.

$$rk((1,5,4,2,3,3,1)) = 5$$
$$rk((4,7,1,1,3,2,2,8)) = 6$$

Definition 12 ($\succ_{pf}$)**.** *Since $\mathcal{PF}_n$ and $\mathcal{NC}_n^2$ are in bijection, we can define a* covering relation *$\succ_{pf}$ for $\mathcal{PF}_n$ as follows :*
$f \in \mathcal{PF}_n \succ_{pf} g \in \mathcal{PF}_n$ if and only if :

- *(P,σ) is the non-crossing 2-partition associated to f*
- *(Q,τ) is the non-crossing 2-partition associated to g*
- *$(P,\sigma) \succ^2 (Q,\tau)$*

Example.
- $P = \{\{1,6\},\{2,3\},\{4\},\{5\}\}$
- $\sigma = 236154$
- $Q = \{\{1,6\},\{2,3,5\},\{4\}\}$
- $\tau = 235164$
- $f = (4,1,2,1,5,2) \succ_{pf} g = (4,1,2,1,2,2)$

Remark. *If $f \succ_{pf} g$, then $rk(f) = rk(g)+1$, and there exists i and j such that :*

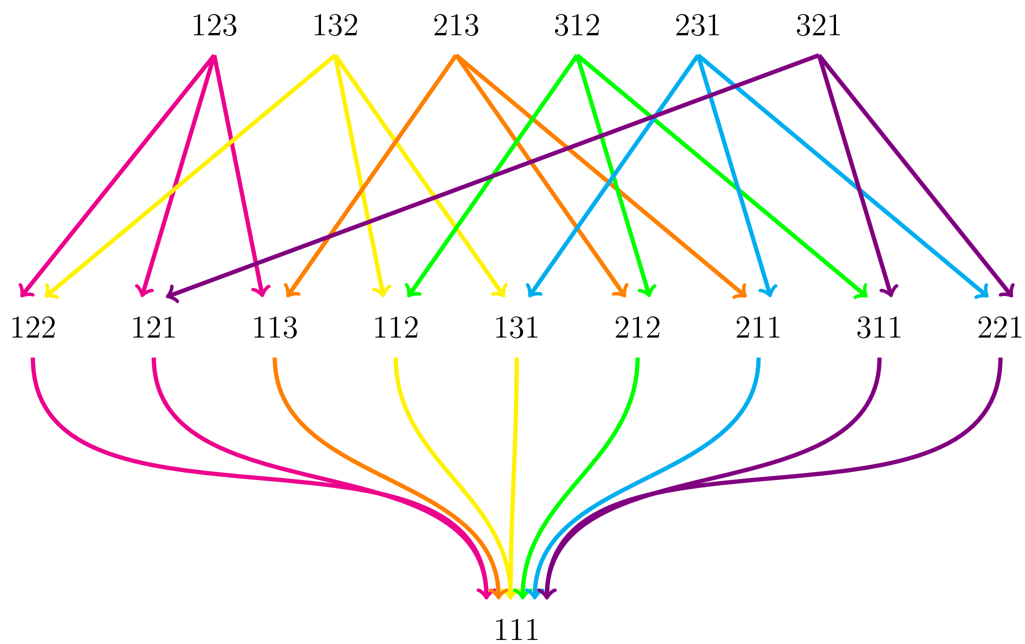
- *$i < j$*
- *There is at least 1 occurence of i in f*
- *There is at least 1 occurence of j in f*

$$b_k = \begin{cases} i \text{ if } a_k = j \\ a_k \text{ otherwise} \end{cases}$$

Proposition. *This covering relation defines the* poset *of $\mathcal{PF}_n$.*

Remark. *The bottom element of this poset is $(\underbrace{1,\ldots,1}_{n})$, and the top elements are the* permutations *of $\{1,\ldots,n\}$.*

Example (The poset of $\mathcal{PF}_3$).



Chapter 2

The rational case

For the whole chapter, we will consider 2 *coprime* integers a and b (meaning a and b have 1 as their greatest common divisor).

2.1 Rational Parking Functions

Definition 13 (a, b - Parking Function)**.** *An* a, b - parking function *is a sequence* $(a_1, a_2, \ldots, a_n)$ *such that :*
• $n = a$
• *its non-decreasing reordering* $(b_1, b_2, \ldots, b_n)$ *has* $b_i \leqslant \frac{b}{a}(i-1)+1$ *for all* i.

We denote by $\mathcal{PF}_{a,b}$ *the set of a, b - parking functions.*

Example.

- *Ex. 1 :* $a > b$

  $a = 7$

  $b = 3$

  Limits of the non-decreasing reordering of any $f \in \mathcal{PF}_{7,3}$ *:*

  $[1,\ 1\frac{3}{7},\ 1\frac{6}{7},\ 2\frac{2}{7},\ 2\frac{5}{7},\ 3\frac{1}{7},\ 3\frac{4}{7}]$

  $f_1 = (2,1,1,3,2,3,1) \in \mathcal{PF}_{7,3}$

  $f_2 = (2,1,2,3,2,3,1) \notin \mathcal{PF}_{7,3},$ *even though* $f_2 \in \mathcal{PF}_7$

- *Ex. 2 :* $a < b$

  $a = 5$

  $b = 7$

  Limits of the non-decreasing reordering of any $f \in \mathcal{PF}_{5,7}$ *:*

  $[1,\ 2\frac{2}{5},\ 3\frac{4}{5},\ 5\frac{1}{5},\ 6\frac{3}{5}]$

  $f_3 = (6, 3, 5, 1, 2) \in \mathcal{PF}_{5,7}$, *even though* $f_3 \notin \mathcal{PF}_5$

  $f_4 = (6, 3, 5, 1, 3) \notin \mathcal{PF}_{5,7}$

Theorem 6. *Let* $pf_{a,b}$ *be the cardinal of* $\mathcal{PF}_{a,b}$*. We have*

$$pf_{a,b} = b^{a-1}$$

Example ($a = 3, b = 5$)**.**
• $pf_{a,b} = 25$ • *Limits :* $[1,\ 2\frac{2}{3},\ 4\frac{1}{3}]$

$(1,1,1)$	$(1,1,2)$	$(1,1,3)$	$(1,1,4)$	$(1,2,1)$	$(1,2,2)$	$(1,2,3)$
$(1,2,4)$	$(1,3,1)$	$(1,3,2)$	$(1,4,1)$	$(1,4,2)$	$(2,1,1)$	$(2,1,2)$
$(2,1,3)$	$(2,1,4)$	$(2,2,1)$	$(2,3,1)$	$(2,4,1)$	$(3,1,1)$	$(3,1,2)$
$(3,2,1)$	$(4,1,1)$	$(4,1,2)$	$(4,2,1)$			

Remark. $\mathcal{PF}_{n,n+1} = \mathcal{PF}_n$*. In fact, we do have* $b^{a-1} = (n+1)^{n-1}$.

2.1.1 Rational primitives parking functions

Definition 14 (Rational Primitive)**.** *A rational parking function* f *is said* primitive *if it is already in non-decreasing order.*
We denote by $\mathcal{PF}'_{a,b}$ *the set of primitive* a, b *- parking functions.*

Example ($a = 4, b = 3$)**.** *Limits :* $[1,\ 1\frac{3}{4},\ 2\frac{1}{2},\ 3\frac{1}{4}]$

$$f_1 = (1, 1, 2, 2) \in \mathcal{PF}'_{4,3}$$
$$f_2 = (1, 1, 2, 1) \notin \mathcal{PF}'_{4,3},\ \textit{even though } f_2 \in \mathcal{PF}_{4,3}.$$

Theorem 7. *Let* $pf'_{a,b}$ *be the cardinal of* $\mathcal{PF}'_{a,b}$*. We have*

$$pf'_{a,b} = \frac{1}{a+b}\binom{a+b}{b}$$

which is the rational Catalan number $Cat(a, b)$.

Example $(a = 3, b = 5)$.
• $pf'_{a,b} = 7$ • *Limits :* $[1,\ 2\frac{2}{3},\ 4\frac{1}{3}]$

$$(1,1,1) \qquad (1,1,2) \qquad (1,1,3) \qquad (1,1,4) \qquad (1,2,2) \qquad (1,2,3) \qquad (1,2,4)$$

Remark. $\mathcal{PF'}_{n,n+1} = \mathcal{PF'}_n$. *In fact, we do have*

$$\frac{1}{n+n+1}\binom{n+n+1}{n+1} = \frac{1}{2n+1}\binom{2n+1}{n+1} = \frac{1}{2n+1}\frac{(2n+1)!}{n!(n+1)!}$$

$$= \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1}\frac{(2n)!}{n!n!} = \frac{1}{n+1}\binom{2n}{n}$$

2.2 Rational Non-crossing Partitions

Definition 15 (Mutually Non-crossing Partitions). *2 partitions P and Q are said* mutually non-crossing *if :*
• *P is non-crossing*
• *Q is non-crossing*
• *For every block B_i of P and every block B_j of Q, if $a, c \in B_i$ and $b, d \in B_j$, then we can* not *have $a < b < c < d$, nor $a > b > c > d$.*

Definition 16 (a, b - Non-crossing Partition). *An* a, b - non-crossing partition *is a pair (P, Q) such that :*
• $P, Q \in \mathcal{NC}_{b-1}$
• *P and Q are mutually non-crossing.*
• *TODO : ADD THE CONDITION*

Proposition. *This means we can create a* bijection *between $\mathcal{PF'}_{a,b}$ and $\mathcal{NC}_{a,b}$.*

Proof.

- $\mathcal{NC}_{a,b} \to \mathcal{PF'}_{a,b}$:
- $\mathcal{PF'}_{a,b} \to \mathcal{NC}_{a,b}$:

□

Chapter 3

Trees