Rational Parking Functions

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Abstract

This is an abstract about Rational Parking Functions.

Contents

Chapter 1

The integer case

1.1 Parking Functions

Definition 1 (Parking Function)**.** *A parking function is a sequence of positive integers $(a_1, a_2, \ldots, a_n)$ such that its non-decreasing reordering $(b_1, b_2, \ldots, b_n)$ has $b_i \leqslant i$ for all i.*
We denote by $\mathcal{PF}_n$ the set of parking functions of length n.

$$\mathcal{PF} = \bigcup_{n>0} \mathcal{PF}_n$$

.

Example.

$$f_1 = (7, 3, 1, 4, 2, 5, 2) \in \mathcal{PF}_7$$
$$f_2 = (7, 3, 1, 4, 2, 5, 4) \notin \mathcal{PF}_7$$

Theorem 1. *Let pf_n be the cardinal of $\mathcal{PF}_n$. We have*

$$pf_n = (n+1)^{n-1}$$

.

Example $(n = 1, 2, 3)$**.**
- $n = 1$: $pf_1 = 1$
    (1)

- $n = 2$: $pf_2 = 3$
  $(1,1)$ $(1,2)$ $(2,1)$
- $n = 3$: $pf_3 = 16$
  $(1,1,1)$ $(1,1,2)$ $(1,1,3)$ $(1,2,1)$ $(1,2,2)$ $(1,2,3)$ $(1,3,1)$
  $(1,3,2)$ $(2,1,1)$ $(2,1,2)$ $(2,1,3)$ $(2,2,1)$ $(2,3,1)$ $(3,1,1)$
  $(3,1,2)$ $(3,2,1)$

1.1.1   Primitive parking functions

Definition 2 (Primitive). *A parking function $(a_1, a_2, \ldots, a_n)$ is said* primitive *if it is already in non-decreasing order.*
We denote by $\mathcal{PF}'_n$ the set of primitive parking functions of length n.

$$\mathcal{PF}' = \bigcup_{n>0} \mathcal{PF}'_n$$

Example.

$$f_1 = (1,2,2,3) \in \mathcal{PF}'_4$$
$$f_2 = (1,2,3,2) \notin \mathcal{PF}'_4, \ \textit{even though } f_2 \in \mathcal{PF}_4$$

Theorem 2. *Let pf'_n be the cardinal of $\mathcal{PF}'_n$. We have*

$$pf'_n = \frac{1}{n+1}\binom{2n}{n}$$

which is the nth Catalan number $Cat(n)$.

Example $(n = 1, 2, 3)$**.**
- $n = 1$: $pf'_1 = 1$
  (1)
- $n = 2$: $pf'_2 = 2$
  $(1,1)$ $(1,2)$
- $n = 3$: $pf'_3 = 5$
  $(1,1,1)$ $(1,1,2)$ $(1,1,3)$ $(1,2,2)$ $(1,2,3)$

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1.2   Non-crossing Partitions

Definition 3 (Non-crossing Partition). *A non-crossing partition of a totally ordered set E is a set partition $P = \{E_1, E_2, \ldots, E_k\}$ such that if $a, c \in E_i$, $b, d \in E_j$, and $i \neq j$, then we do not have $a < b < c < d$, nor $a > b > c > d$. We denote by $\mathcal{NC}_n$ the set of non-crossing partitions of $\{1, 2, \ldots, n\}$.*

$$\mathcal{NC} = \bigcup_{n>0} \mathcal{NC}_n$$

From this point, we assume that every partition $P = \{B_1, \ldots, B_l\}$ is *sorted* such that :
- For each block $B_i = \{b_1, \ldots, b_k\} \in P$, $b_1 < \ldots < b_k$
- $min(B_1) < \ldots < min(B_k)$

Notation. $[n] = \{1, 2, \ldots, n\}$

Example $(E = [6])$.

$$P_1 = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\} \in \mathcal{NC}_6$$
$$P_2 = \{\{1, 2, 4\}, \{3, 5\}, \{6\}\} \notin \mathcal{NC}_6$$

Theorem 3. *Let nc_n be the cardinal of $\mathcal{NC}_n$. We have*

$$nc_n = \frac{1}{n+1}\binom{2n}{n}$$

which is again the nth Catalan number $Cat(n)$.

Example $(n = 1, 2, 3)$.
- $n = 1$: $nc_1 = 1$
    $\{\{1\}\}$
- $n = 2$: $nc_2 = 2$
    $\{\{1, 2\}\}$ $\{\{1\}, \{2\}\}$
- $n = 3$: $nc_3 = 5$
    $\{\{1, 2, 3\}\}$ $\{\{1\}, \{2, 3\}\}$ $\{\{1, 3\}, \{2\}\}$ $\{\{1, 2\}, \{3\}\}$ $\{\{1\}, \{2\}, \{3\}\}$

Proposition. *This means we can create a bijection between $\mathcal{PF'}_n$ and $\mathcal{NC}_n$.*

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Proof.

- $\mathcal{NC}_n \to \mathcal{PF}'_n$: For each block B in the non-crossing partition, take $i = min(B)$, and let $k_i = size(B)$.
  $k_i = 0$ if i is not the minimum of a block.
  The corresponding parking function is $(\underbrace{1,\ldots,1}_{k_1}, \underbrace{2,\ldots,2}_{k_2}, \ldots, \underbrace{n,\ldots,n}_{k_n})$.

- $\mathcal{PF}'_n \to \mathcal{NC}_n$: For each i in $[n]$, if i appears n_i times in the parking function, B_i will be of size n_i with minimum element i. There is a unique set partition $P = \bigcup_i B_i$ of $[n]$ respecting these conditions that is non-crossing : for each minimum i in *decreasing order*, add the n_i first free elements of $[i+1, i+2, \ldots, n, 1, \ldots, i-1]$ to B_i.

$\square$

Example $(n = 6)$.

$$P = \{\{1,2,5\},\{3,4\},\{6\}\} \qquad\qquad f = (1,1,1,3,3,6)$$

Corollary. *A non-crossing partition can be represented by the minimums and sizes of its blocks.*

Example. $\{\{1,2,5\},\{3,4\},\{6\}\}$ *can be represented by the following dictionnary :*
- *1 : 3*
- *3 : 2*
- *6 : 1*

A non-crossing partition of $[n]$ can be represented graphically on a regular n-vertices polygon, with vertices labeled from 1 to n clockwise. We then represent each block $B = \{b_1, \ldots, b_k\}$ by the convex hull of $\{b_1, \ldots, b_k\}$.

Example $(P = \{\{1,2,5\},\{3,4\},\{6\}\})$.

Thus non-crossing meaning the hulls are disjoint.

Example (Counter-example : $P = \{\{1,5,2\},\{3,6\},\{4\}\}$).

This partition is not *non-crossing, as the convex hulls of* $\{1,2,5\}$ *and* $\{3,6\}$ *are* not *disjoint.*

1.2.1   The non-crossing partitions poset

Definition 4 ($\succ$). *We say that P covers Q, written $P \succ Q$, if $\exists B_i, B_j \in P$ such that $Q = P - \{B_i, B_j\} \cup \{B_i \cup B_j\}$*

Example. $\{\{1,6\},\{2,3\},\{4,5\}\} \succ \{\{1,2,3,6\},\{4,5\}\}$
• $B_i = \{1,6\}$

- $B_j = \{2, 3\}$

Proposition. *This covering relation defines the* poset *of* $\mathcal{NC}_n$. *We denote by* $\mathcal{NCC}_n$ *the set of* maximal chains *in the poset of* $\mathcal{NC}_n$.

$$\mathcal{NCC} = \bigcup_{n>0} \mathcal{NCC}_n$$

Remark. *The bottom element of this poset is* $\{\{1, \ldots, n\}\}$, *and the top element is* $\{\{1\}, \ldots, \{n\}\}$.

Theorem 4. *Let* ncc_n *be the cardinal of* $\mathcal{NCC}_n$. *We have*

$$ncc_n = n^{n-2}$$

.

Example (The poset of $\mathcal{NC}_4$).
To shorten labels, we represent $\{\{1\}, \{2, 3\}, \{4\}\}$ *by* $1|23|4$.

There are $4^2 = 16$ different maximal chains, and $\frac{1}{5}\binom{8}{4} = \frac{70}{5} = 14$ elements in this poset.

1.2.2   Kreweras complement

Definition 5 (Associated Permutation). *The permutation σ associated to a non-crossing partition has a cycle $(b_1, \ldots, b_k)$ for each block $B = \{b_1, \ldots, b_k\}$ of the partition.*

Example. *The permutation associated to $\{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$ is $(1\ 2\ 5)\ (3\ 4)\ (6) = 254316$.*

Definition 6 (Kreweras Complement). *The Kreweras complement $K(P)$ of a non-crossing partition P is defined as follows :*
- *Let σ be the permutation associated to P*
- *Let π be the permutation $(n\ n-1\ n-2\ \ldots\ 3\ 2\ 1) = n123\ldots n-1$*
- *$K(P)$ is the non-crossing partition associated to $\pi\sigma$.*

Example ($P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$).
- $\sigma = (1\ 2\ 5)\ (3\ 4)\ (6) = 254316$
- $\pi = (6\ 5\ 4\ 3\ 2\ 1) = 612345$
- $\pi\sigma = 143265 = (1)\ (2\ 4)\ (3)\ (5\ 6)$
- $K(P) = \{\{1\}, \{2, 4\}, \{3\}, \{5, 6\}\}$

Proposition (Kreweras minimums). *Let $P = \{B_1, \ldots, B_k\}$ be a non-crossing partition. Let $K(P) = \{B_1', \ldots, B_l'\}$ be its Kreweras complement. Then*

$$\bigcup_{1 \leq i \leq l} min(B_i') = B_1 \cup \bigcup_{1 < j \leq k} B_i - max(B_i)$$

Example ($P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$).

- $K(P) = \{\{1\}, \{2, 4\}, \{3\}, \{5, 6\}\}$

- $\bigcup min(B_i') = \{1, 2, 3, 5\}$

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- $B_1 \cup \bigcup B_i - max(B_i) = \{1,2,5\} \cup \{3,4\} - \{4\} \cup \{6\} - \{6\} = \{1,2,5\} \cup \{3\} \cup \emptyset = \{1,2,3,5\}$

Notation. $B_{[i]} = block\ containing\ i.$

Proposition (Kreweras block sizes). *Let $P = \{B_1, \ldots, B_k\}$ be a non-crossing partition. Let $K(P) = \{B'_1, \ldots, B'_l\}$ be its Kreweras complement. Then the size of the block B'_i is defined as follows :*

- *Let m_i be the the ith minimum of $K(P)$*

- *Define a* transition *$\phi(e)$ as*

  Let $j = e + 1$ *(or 1 if $e = n$)*

  $\phi(e) = max(B_{[j]})$

- *The size of B'_i is k_{min} such that $k_{min} = min\{k > 0 \mid \phi^k(m_i) \in B_{[m_i]}\}$.*

Example $(P = \{\{1,2,5\}, \{3,4\}, \{6\}\})$.

- $mins = \{1,2,3,5\}$

- $m_1 = 1$

  $B_{[1]} = B_1$

  $max(B_{[2]} = max(B_1) = 5$

  The size for m_1 is 1.

- m_2

  $B_{[2]} = B_1$

  $max(B_{[3]}) = max(B_2) = 4$

  $max(B_{[5]}) = max(B_1) = 5$

  The size for m_2 is 2.

- $m_3 = 3$

  $B_{[3]} = B_2$

  $max(B_{[4]}) = max(B_2) = 4$

  The size for m_3 is 1.

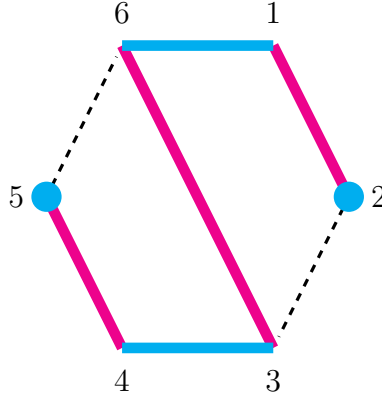- $m_4 = 5$

$$B_{[5]} = B_1$$
$$max(B_{[6]}) = max(B_3) = 6$$
$$max(B_{[1]}) = max(B_1) = 5$$

The size for m_4 is 2.

Definition 7 (Mutually Non-crossing Partitions). *2 partitions P and Q are* *said* mutually non-crossing *if :*
- *P is non-crossing*
- *Q is non-crossing*
- *For every block B_i of P and every block B_j of Q, if $a, c \in B_i$ and $b, d \in B_j$,* *then we can* not *have $a < b < c < d$, nor $a > b > c > d$.*

Example $(P = \{\{1, 2\}, \{3, 6\},\ \{4, 5\}\}, Q = \{\{1, 6\}, \{2\}, \{3, 4\}, \{5\}\})$.

Example (Counter-example : $P = \{\{1, 2\}, \{3, 6\},\ \{4, 5\}\}, Q = \{\{1, 6\}, \{2, 5\}, \{3, 4\}\}$).
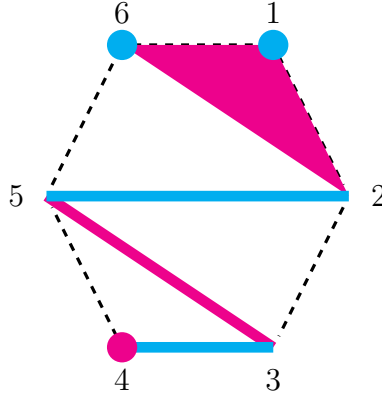
Remark. *Note that vertices can touch, but the edges of the convex hulls can not cross.*

Proposition. *For any non-crossing partition P, P and $K(P)$ are mutually non-crossing.*
Furthermore, $K(P)$ is a densest *partition that is mutually non-crossing with P. That is,* no *partition Q that is mutually non-crossing with P has less blocks than $K(P)$.*

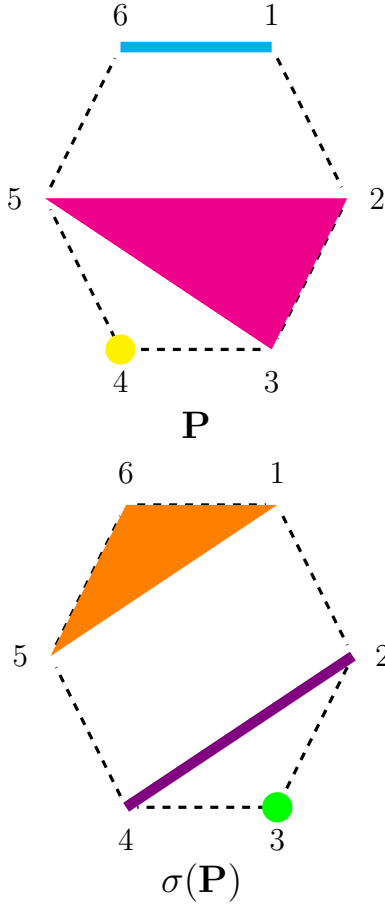**Example** ($P = \{1, 2, 6\}, \{3, 5\}, \{4\}\}$). $Q = \{\{1\}, \{2, 5\}, \{3, 4\}, \{6\}\}$

1.2.3   Action of $\mathfrak{S}_n$ on partitions of $[n]$

Definition 8 (Action of $\mathfrak{S}_n$). *The action of $\mathfrak{S}_n$ on a partition $P = \{B_1, \ldots, B_l\}$ of $[n]$ is defined by :*
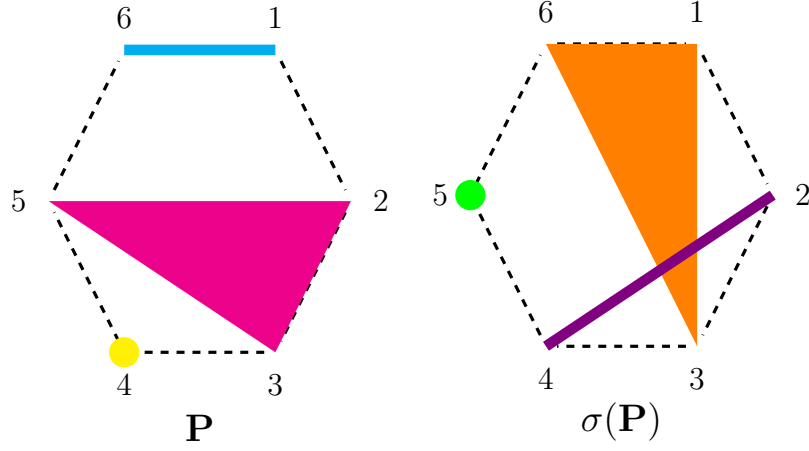• *For each block $B_i = \{b_1, \ldots, b_k\}$: $\sigma(Bi) = \{\sigma(b_1), \ldots, \sigma(b_k)\}$*
• *When $P \in \mathcal{NC}_n$, we denote $\rho = \sigma(P) = \{\sigma(B_1), \ldots, \sigma(B_l)\}$*

Example ($\sigma = 415362, P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$).
$\sigma(P) = \{\{1, 5, 6\}, \{2, 4\}, \{3\}\}$

P

$\sigma(\mathbf{P})$

Remark. *Note that $\mathcal{NC}_n$ is not stable under the action of $\mathfrak{S}_n$. That is, even if P is non-crossing, $\sigma(P)$ is not necessarily non-crossing.*
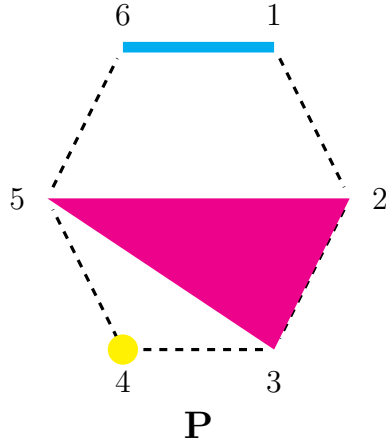
Example (Counter-example : $\sigma = 413562, P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$).
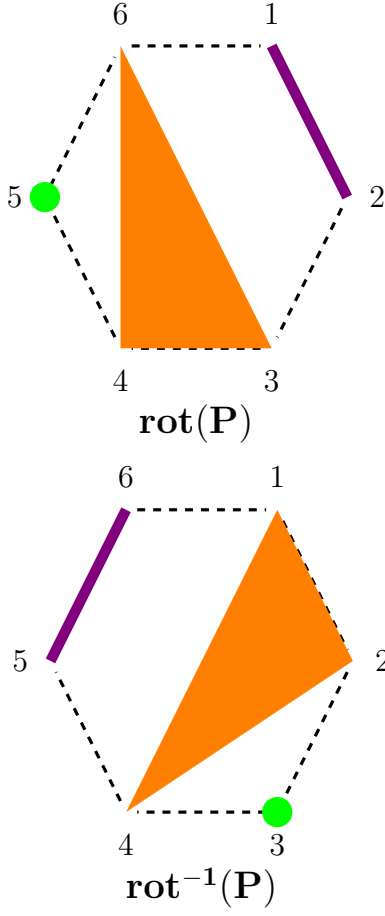$\sigma(P) = \{\{1, 3, 6\}, \{2, 4\}, \{5\}\}$

Definition 9 (Rotation). *We define the* rotation operator *rot of* $P \in \mathcal{NC}_n$ *as* $rot(P) = (1 \ 2 \ 3 \ \ldots \ n)(P) = 23\ldots n1(P)$.
Conversely, we define rot^{-1} *of* P *as* $rot^{-1}(P) = (n \ n-1 \ \ldots 3 \ 2 \ 1)(P) = n12\ldots n-1(P)$.

Example $(P = \{\{1,6\}, \{2,3,5\}, \{4\}\})$.

- $rot(P) = \{\{1,2\}, \{3,4,6\}, \{5\}\}$

- $rot^{-1}(P) = \{\{1,2,4\}, \{3\}, \{5,6\}\}$
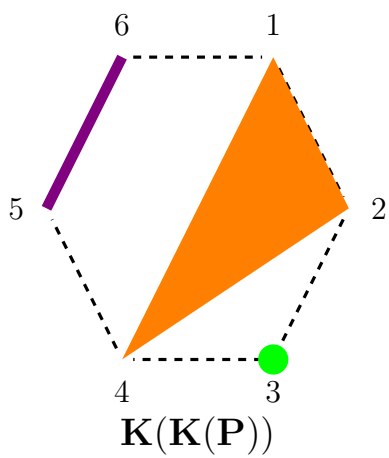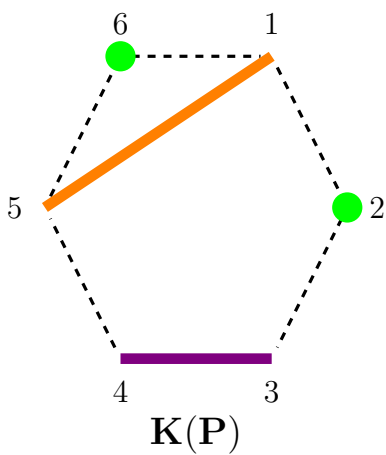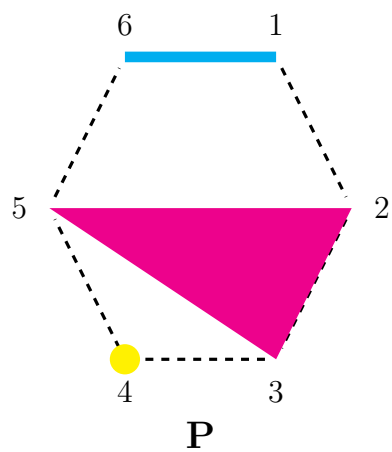
rot(P)

rot⁻¹(P)

Remark.

- $rot(rot^{-1}(P)) = rot^{-1}(rot(P)) = P$

- $rot(P)$ and $rot^{-1}(P)$ are always non-crossing partitions.

- If $P \in \mathcal{NC}_n$, then $rot^n(P) = rot^{-n}(P) = P$.

Proposition. $K(K(P)) = rot^{-1}(P)$.

Example $(P = \{\{1,6\}, \{2,3,5\}, \{4\}\})$.

P

K(P)

K(K(P))

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1.3   Non-crossing 2-partitions

Definition 10 (Non-crossing 2-partition). *A* non-crossing 2-partition *of a totally ordered* set E *is a pair* (P, σ) *where :*
- *P is a non-crossing partition of E*
- *σ is a permutation of the elements of E*
- *For each* sorted *block* $B_i = \{b_1, \ldots, b_k\} \in P$, *we have* $\sigma(b_i) < \ldots < \sigma(b_k)$

We denote by $\mathcal{NC}_n^2$ the set of non-crossing 2-partitions of $[n]$.

$$\mathcal{NC}^2 = \bigcup_{n>0} \mathcal{NC}_n^2$$

.

Example $(\mathcal{NC}_6^2)$. $\begin{aligned} P &= \{\{1,6\}, \{2,3,5\}, \{4\}\} \\ \rho &= \{\{1,3,6\}, \{2\}, \{4,5\}\} \end{aligned}$ $\sigma = 413265$

Theorem 5. *Let nc_n^2 be the cardinal of $\mathcal{NC}_n^2$. We have*

$$nc_n^2 = (n+1)^{n-1}$$

Example $(n = 1, 2, 3)$.

- $n = 1$: $nc_1^2 = 1$

  $\{\{1\}\}$ 1 $\rho = P$

- $n = 2$: $nc_2^2 = 3$

  $\{\{1\}, \{2\}\}$ 12 $\rho = P$

  $\{\{1\}, \{2\}\}$ 21 $\rho = P$

  $\{\{1, 2\}\}$ 12 $\rho = P$

- $n = 3$: $nc_3^2 = 16$

  $\{\{1\}, \{2\}, \{3\}\}$ 123 $\rho = P$

  $\{\{1\}, \{2\}, \{3\}\}$ 132 $\rho = P$

  $\{\{1\}, \{2\}, \{3\}\}$ 213 $\rho = P$

  $\{\{1\}, \{2\}, \{3\}\}$ 231 $\rho = P$

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$$\begin{array}{lll}
\{\{1\},\{2\},\{3\}\} & 312 & \rho = P \\
\{\{1\},\{2\},\{3\}\} & 321 & \rho = P \\
\{\{1,2\},\{3\}\} & 123 & \rho = P \\
\{\{1,2\},\{3\}\} & 132 & \rho = \{\{1,3\},\{2\}\} \\
\{\{1,2\},\{3\}\} & 231 & \rho = \{\{1\},\{2,3\}\} \\
\{\{1\},\{2,3\}\} & 123 & \rho = P \\
\{\{1\},\{2,3\}\} & 213 & \rho = \{\{1,3\},\{2\}\} \\
\{\{1\},\{2,3\}\} & 312 & \rho = \{\{1,2\},\{3\}\} \\
\{\{1,3\},\{2\}\} & 123 & \rho = P \\
\{\{1,3\},\{2\}\} & 132 & \rho = \{\{1,2\},\{3\}\} \\
\{\{1,3\},\{2\}\} & 213 & \rho = \{\{1\},\{2,3\}\} \\
\{\{1,2,3\}\} & 123 & \rho = P \\
\end{array}$$

Proposition. *This means we can create a* bijection *between* $\mathcal{PF}_n$ *and* $\mathcal{NC}_n^2$.

Proof.

- $\mathcal{PF}_n \to \mathcal{NC}_n^2$: Let $f = (a_1, \ldots, a_n) \in \mathcal{PF}_n$ be our parking function. For $i \in \{1, \ldots, n\}$, we define :

    l_i : the number of occurences of i in f.

    im_i : $\{j \mid a_j = i\}$

    The corresponding non-crossing partition will have the following constraints :

    For each $i \in \{1, \ldots, n\}$, if $l_i > 0$, then there is a block $B_{[i]}$ of length l_i with minimum element i.

    $\sigma(B_{[i]}) = im_i$

    There is a unique set partition $P = \bigcup_i B_{[i]}$ of $[n]$ and a unique permutation σ respecting these conditions such that $(P, \sigma) \in \mathcal{NC}_n^2$: for each minimum i in *decreasing order*, add the n_i first free elements of $[i+1, i+2, \ldots, n, 1, \ldots, i-1]$ to B_i. σ is then trivially obtained by the second constraint.

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- $\mathcal{NC}_n^2 \to \mathcal{PF}_n$: Let (P, σ) with $P = \{B_1, \ldots, B_l\}$ be our non-crossing 2-partition. For each block $B_i = \{b_1, \ldots, b_k\} \in P$:

$$m_i = min(B_i) = b_1$$
$$pos_i = \sigma(B_i)$$

For each $j \in pos_i$, we define $a_j = m_i$
The corresponding parking function is $(a_1, \ldots, a_n)$.

$\square$

Example $(n = 8)$.

$$P = \{\{1, 2, 5\}, \{3, 4\}, \{6, 8\}, \{7\}\}$$
$$\sigma = 36187245$$
$$f = (3, 6, 1, 7, 6, 1, 1, 3)$$

1.3.1 The non-crossing 2-partitions poset

Definition 11 $(\succ^2)$. *We say that* (P, σ) *covers* (Q, τ), *written* $(P, \sigma) \succ^2$
(Q, τ), *if* $\exists B_i, B_j \in P$ *such that*
- $Q = P - \{B_i, B_j\} \cup \{B_i \cup B_j\}$
- $l \neq i, j \, b \in B_l \to \tau(b) = \sigma(b)$
- *Let* $B_i \cup B_j = \{b_1, \ldots, b_k\}$:
    $\tau(B_i \cup B_j) = \sigma(B_i \cup B_j)$
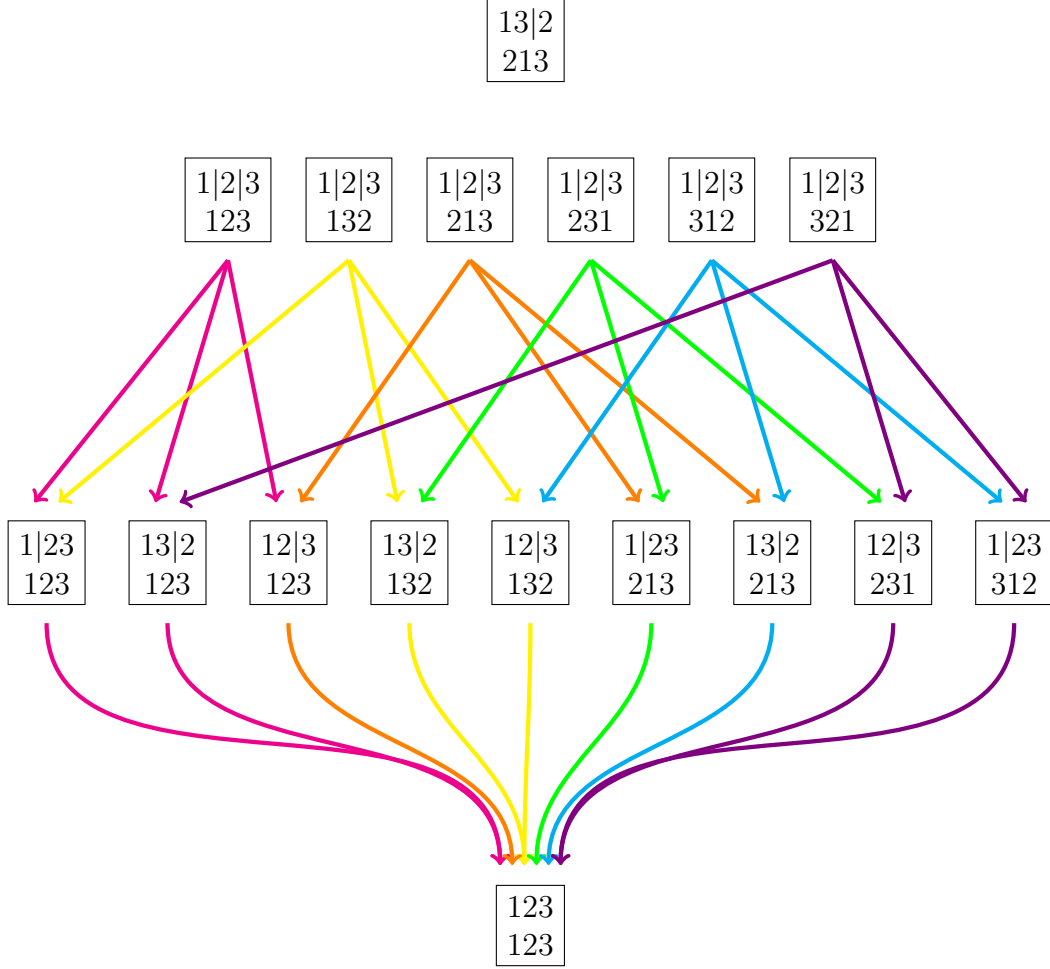 $\tau(b_1) < \ldots < \tau(b_k)$

Example.
- $P = \{\{1, 6\}, \{2, 3\}, \{4\}, \{5\}\}$
- $\sigma = 236154$
- $Q = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$
- $\tau = 235164$
- $(P, \sigma) \succ^2 (Q, \tau)$
- $(P, \sigma) \not\succ^2 (Q, \sigma)$, *because* $\sigma(\{2, 3, 5\}) = \{3, 6, 5\}$ *is* not *ordemagenta.*

Proposition. *This covering relation defines the* poset *of* $\mathcal{NC}_n^2$.

Remark. *The bottom element of this poset is* $(\{\{1, \ldots, n\}\}, 12 \ldots n)$, *and the top elements are* $\{(\{\{1\}, \ldots, \{n\}\}, \sigma) \mid \sigma \in \mathfrak{S}_n\}$.

Example (The poset of $\mathcal{NC}_3^2$).
To shorten labels, we represent $(\{\{1,3\},\{2\}\}, 213)$ *by*

There are $4^2 = 16$ elements in this poset.

1.3.2   The parking functions poset

Definition 12 (Rank). *Given $f = (a_1, \ldots, a_n) \in \mathcal{PF}_n$, let*

$$b_i = \begin{cases} 1 \; \textit{if } \exists j \mid a_j = i \\ 0 \; \textit{otherwise} \end{cases}$$

We define the rank *of f, noted* $rk(f)$, *as*

$$\sum_{1 \leq i \leq n} b_i$$

Example.

$$rk((1, 5, 4, 2, 3, 3, 1)) = 5$$
$$rk((4, 7, 1, 1, 3, 2, 2, 8)) = 6$$

Definition 13 ($\succ_{pf}$). *Since* $\mathcal{PF}_n$ *and* $\mathcal{NC}_n^2$ *are in bijection, we can define a covering relation* $\succ_{pf}$ *for* $\mathcal{PF}_n$ *as follows :*
$f \in \mathcal{PF}_n \succ_{pf} g \in \mathcal{PF}_n$ *if and only if :*

- (P, σ) *is the non-crossing 2-partition associated to f*

- (Q, τ) *is the non-crossing 2-partition associated to g*

- $(P, \sigma) \succ^2 (Q, \tau)$

Example.
- $P = \{\{1, 6\}, \{2, 3\}, \{4\}, \{5\}\}$
- $\sigma = 236154$
- $Q = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$
- $\tau = 235164$
- $f = (4, 1, 2, 1, 5, 2) \succ_{pf} g = (4, 1, 2, 1, 2, 2)$

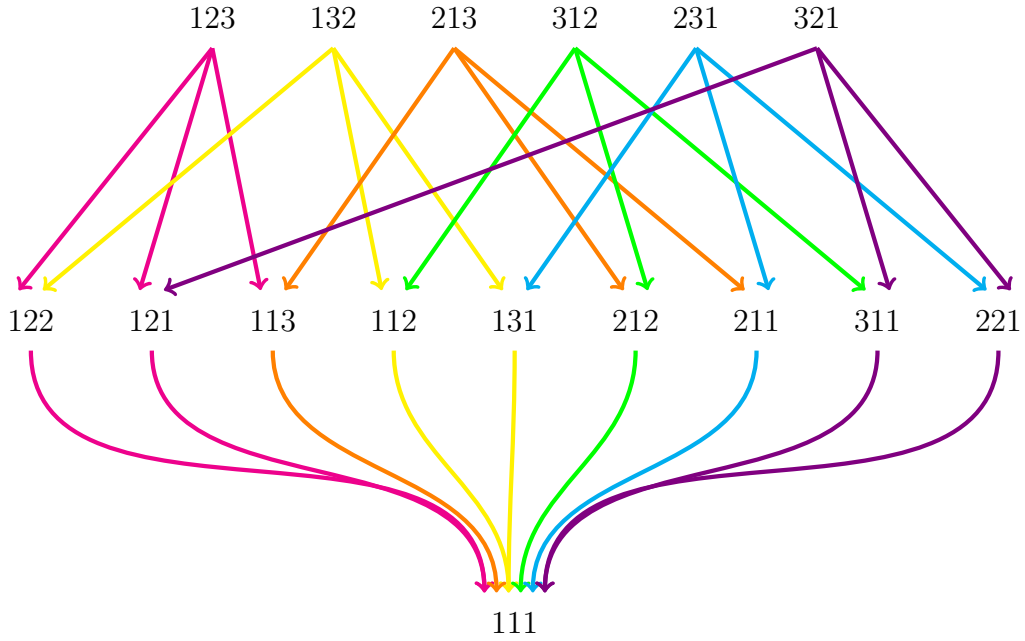**Remark.** *If* $f \succ_{pf} g$, *then* $rk(f) = rk(g) + 1$, *and there exists i and j such that :*

- $i < j$

- *There is at least 1 occurence of i in f*

- *There is at least 1 occurence of j in f*

$$b_k = \begin{cases} i \ if \ a_k = j \\ a_k \ otherwise \end{cases}$$

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Proposition. *This covering relation defines the* poset *of* $\mathcal{PF}_n$.

Remark. *The bottom element of this poset is* $(\underbrace{1, \ldots, 1}_{n})$*, and the top elements are the* permutations *of* $\{1, \ldots, n\}$.

Example (The poset of $\mathcal{PF}_3$).

1.4   A direct poset linked to Dyck paths

1.4.1   Dyck Paths

Notation. *We denote the number of occurences of a symbol s in a word w by $|w|_s$.*

Definition 14 (Dyck path). *A Dyck word is a word $w \in \{0,1\}^*$ such that :*

- *for each* suffix *w' of w, $|w'|_1 \geqslant |w'|_0$.*

- *$|w|_0 = |w|_1$.*

A Dyck word of length $2n$ *can be represented as a* path *from* $(0,0)$ *to* (n,n) *that stays over* $x = y$, *called a* Dyck path :

- *Each* 1 *corresponds to a* North step $\uparrow$.

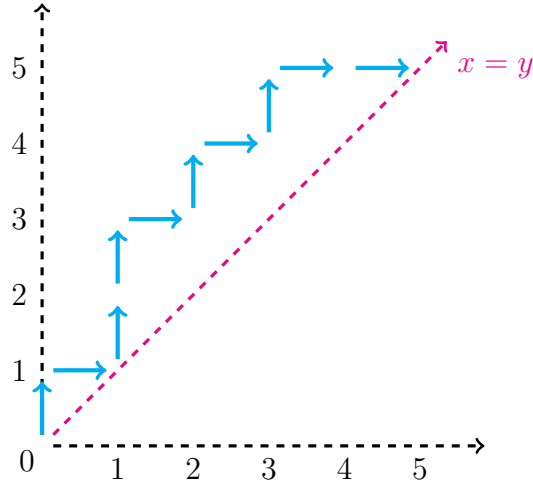- *Each* 0 *corresponds to an* East step $\rightarrow$.

We denote by $\mathcal{D}_n$ *the set of Dyck words of length* $2n$.

Example $(n = 5)$.

$w_1 = 1011000110$ *is* not *a Dyck word, because* $|1011000|_0 > |1011000|_1$.
$w_2 = 1011010101$ *is* not *a Dyck word, because* $|w_2|_0 \neq |w_2|_1$.
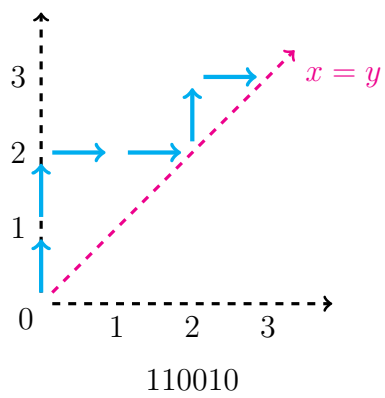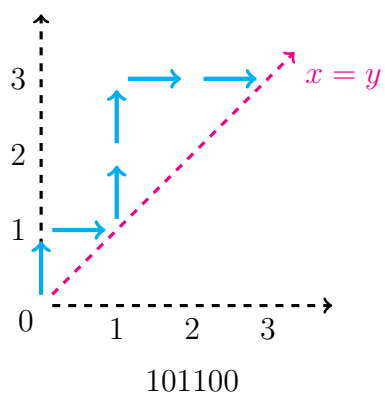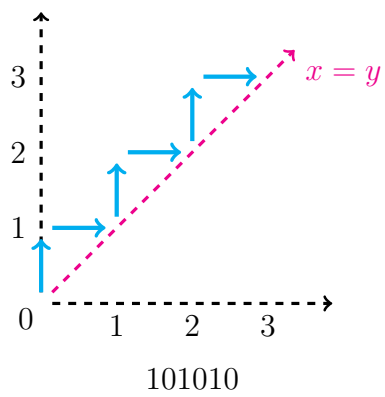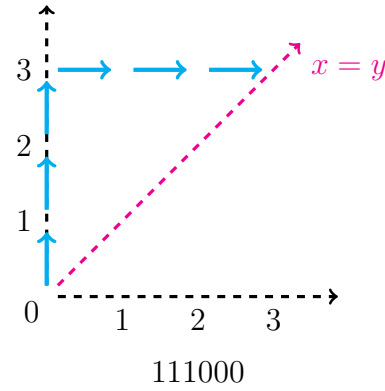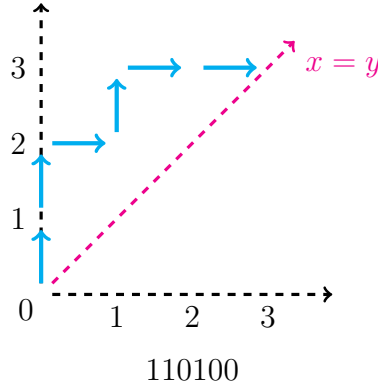$w_3 = 1011010100$ is *a Dyck word* :

Theorem 6. *Let* d_n *be the cardinal of* $\mathcal{D}_n$. *We have*

$$d_n = \frac{1}{n+1}\binom{2n}{n}$$

which is the nth *Catalan number.*

Example $(n = 3)$. $d_n = 5$.

101010

101100

110010

110100

111000

Proposition. *This means we can create a* bijection *between* $\mathcal{PF'}_n$ *and* $\mathcal{D}_n$.

Proof.

- $\mathcal{PF'}_n \to \mathcal{D}_n$: Let $f = (a_1, \dots, a_n) \in \mathcal{PF'}_n$ be our primitive parking function. For $i \in \{1, \dots, n\}$, we define l_i the number of occurences of i in f.
  The corresponding Dyck word will be $\underbrace{1 \cdots 1}_{l_1} 0 \underbrace{1 \cdots 1}_{l_2} 0 \cdots \underbrace{1 \cdots 1}_{l_n} 0$.

- $\mathcal{D}_n \to \mathcal{PF'}_n$: Let w be our Dyck word, and consider its path representation. We define s_i to be the distance between the segment from $(0, i-1)$ to $(0, i)$ and the ith North step. Then, let $a_i = s_i + 1$.
  The corresponding primitive parking function is $(a_1, \dots, a_n)$.

  $\square$

Example $(n = 6, \mathcal{PF'}_n \to \mathcal{D}_n)$**.**

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- $f = (1, 1, 2, 4, 5, 5)$

$$l_1 = 2 \qquad l_2 = 1 \qquad l_3 = 0$$
$$l_4 = 1 \qquad l_5 = 2 \qquad l_6 = 0$$
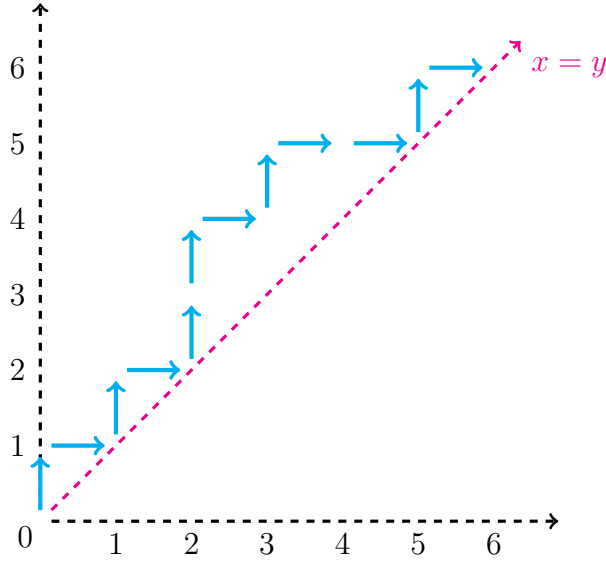
- $w = (110100101100)$
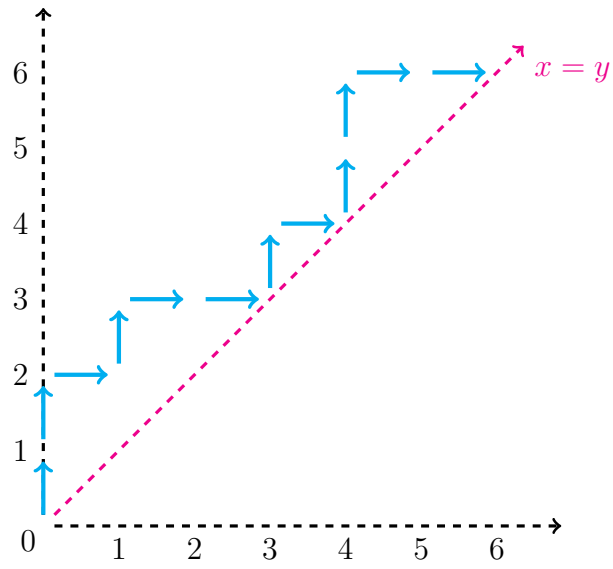
Example $\left(n = 6, \mathcal{D}_n \to \mathcal{PF'}_n\right)$.

- $w = 101011010010$

- *Distances :*

$$s_1 = 0 \qquad a_1 = 1$$
$$s_2 = 1 \qquad a_2 = 2$$
$$s_3 = 2 \qquad a_3 = 3$$
$$s_4 = 2 \qquad a_4 = 3$$
$$s_5 = 3 \qquad a_5 = 4$$
$$s_6 = 5 \qquad a_6 = 6$$

- $f = (1, 2, 3, 3, 4, 6)$

1.4.2   Labeled Dyck Paths

Definition 15 (Labeled Dyck Path). *A labeled Dyck word is a word $w \in \{0, \ldots, n\}^*$ such that :*

- *for each suffix w' of w, $|w'|_{\neq 0} \geqslant |w'|_0$.*

- $|w|_0 = |w|_{\neq 0}.$

- *for each $i \in \{1, \ldots, n\}$, w has exactly one occurence of i.*

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- if $w_i \neq 0$ and $w_{i+1} \neq 0$, then $w_i < w_{i+1}$. That is, consecutive North steps have increasing labels.

A labeled Dyck word of length $2n$ can be represented as a path from $(0,0)$ to (n,n), where each North step is associated to a label :

- Each $i \neq 0$ corresponds to a North step $\uparrow$ labeled i.

- Each 0 corresponds to an East step $\rightarrow$.
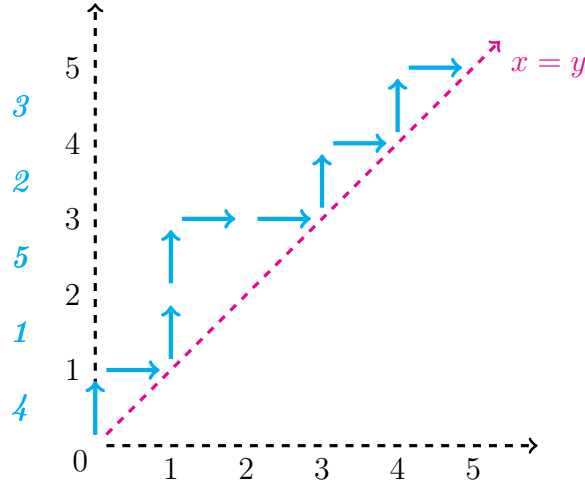
Those paths are called labeled Dyck paths.
We denote by $\mathcal{LD}_n$ the set of labeled Dyck words of length $2n$.

Example $(n = 5)$.

$w_1 = 4051002030$ is not a labeled Dyck word, because $5 > 1$.

$w_2 = 4015002030$ is a labeled Dyck word :

Theorem 7. Let ld_n be the cardinal of $\mathcal{LD}_n$. We have

$$ld_n = (n+1)^{n-1}$$

.

Example $(n = 3)$. $ld_n = 4^2 = 16$

- *Word of shape $XXX000$:*

    123000

- *Words of shape $XX0X00$:*

    120300   130200   230100

- *Words of shape $XX00X0$:*

    120030   130020   230010

- *Words of shape $X0XX00$:*

    102300   201300   301200

- *Words of shape $X0X0X0$:*

    102030   103020   201030

    203010   301020   302010

Proposition. *This means we can create a* bijection *between $\mathcal{PF}_n$ and $\mathcal{LD}_n$.*

Proof.

- $\mathcal{PF}_n \to \mathcal{LD}_n$: Let $f = (a_1, \ldots, a_n) \in \mathcal{PF}_n$ be our parking function. For $i \in \{1, \ldots, n\}$, we define $im_i : \{j \mid a_j = i\}$.
  We then define $im_{i,1}, \ldots, im_{i,k_i}$ to be the elements of im_i in increasing order.
  The corresponding labeled Dyck word will be
  $\underbrace{im_{1,1} \cdots im_{1,k_1}}_{im_1} 0 \underbrace{im_{2,1} \cdots im_{2,k_2}}_{im_2} 0 \cdots \underbrace{im_{n,1} \cdots im_{n,k_n}}_{im_n} 0.$

- $\mathcal{LD}_n \to \mathcal{PF}_n$: Let w be our labeled Dyck word, and consider its path representation. We define s_i to be the distance between the segment from $(0, i-1)$ to $(0, i)$ and the ith North step.
  Then, let $label(i)$ be the label of the ith North step, and $dist_i = \{label(j)|s_j = i\}$ be the set of the labels of all North steps at distance i.
  Then, if $j \in dist_i$, let $a_j = i + 1$.
  The corresponding parking function is $(a_1, \ldots, a_n)$.

$\square$

Example $(n = 6, \mathcal{PF}_n \to \mathcal{LD}_n)$.

- $f = (5, 2, 1, 4, 5, 1)$
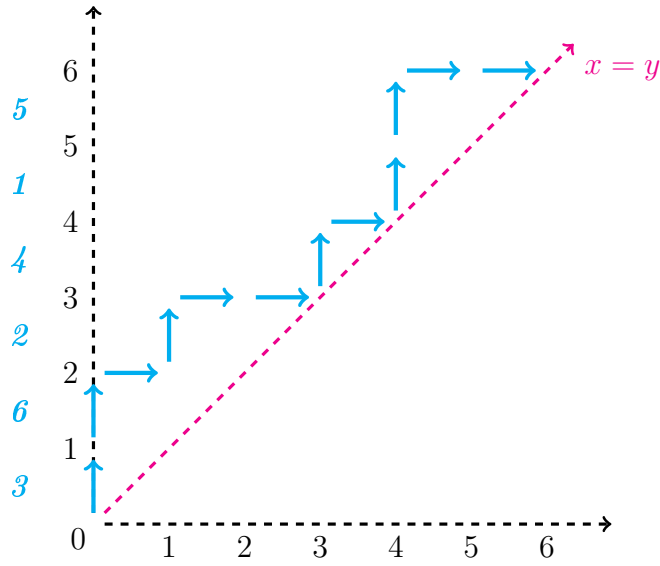
$$im_1 = \{3, 6\} \qquad im_2 = \{2\} \qquad im_3 = \emptyset$$
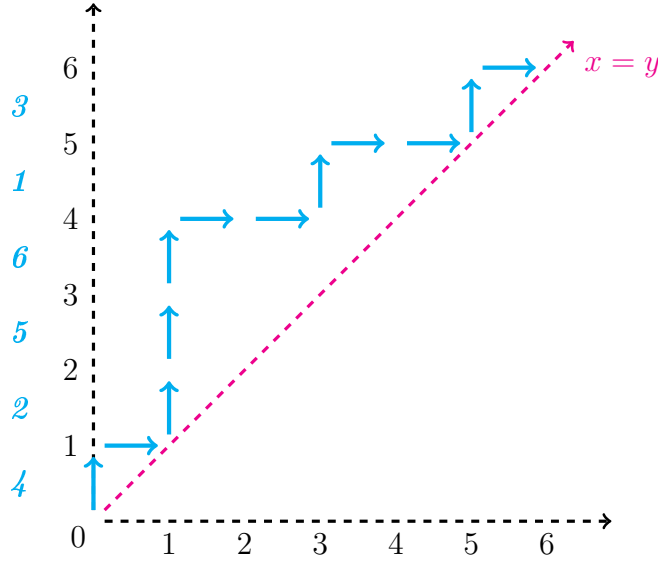$$im_4 = \{4\} \qquad im_5 = \{1, 5\} \qquad im_6 = \emptyset$$

- $w = 360200401500$

Example $(n = 6, \mathcal{LD}_n \to \mathcal{PF}_n)$.

- $w = 402560010030$

- *Distances :*

$$s_1 = 0 \qquad\qquad s_2 = 1 \qquad\qquad s_3 = 1$$
$$s_4 = 1 \qquad\qquad s_5 = 3 \qquad\qquad s_6 = 5$$

- *Labels :*

$$dist_0 = \{4\} \qquad\qquad dist_1 = \{2,5,6\} \qquad\qquad dist_2 = \emptyset$$
$$dist_3 = \{1\} \qquad\qquad dist_4 = \emptyset \qquad\qquad dist_5 = \{3\}$$

- $f = (4,2,6,1,2,2)$

Remark. *The primitive parking functions are exactly the parking functions corresponding to labeled Dyck paths where the ith North step is labeled i.*

1.4.3 Dyck - Parking Posets
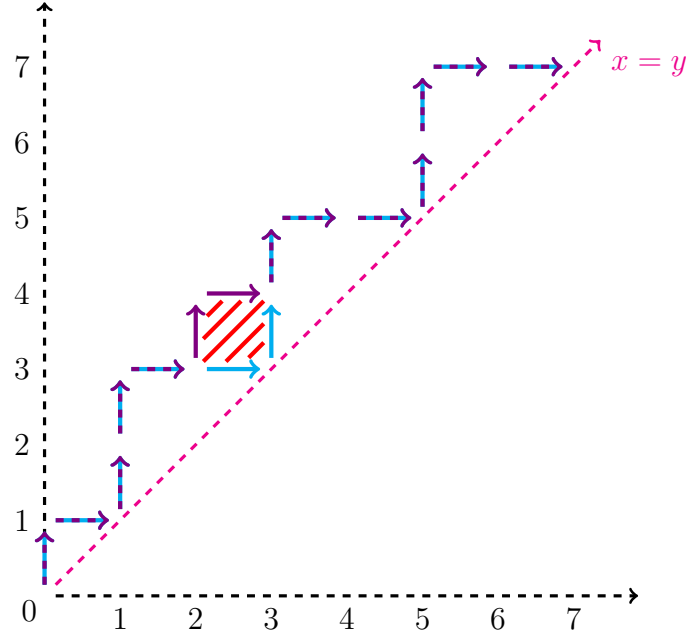
Primitive Dyck - Parking Posets

Definition 16 ($\gtrdot_d$)**.** *For w and w' two Dyck words, we say that w covers w', written $w \gtrdot_d w'$, if $\exists w_1, w_2$ such that :*

- $w = w_1 01 w_2$

- $w' = w_1 10 w_2$

Example $(n = 7)$. $10110011001100 >_d 10110101001100$

- $w_1 = 10110$

- $w_2 = 1001100$
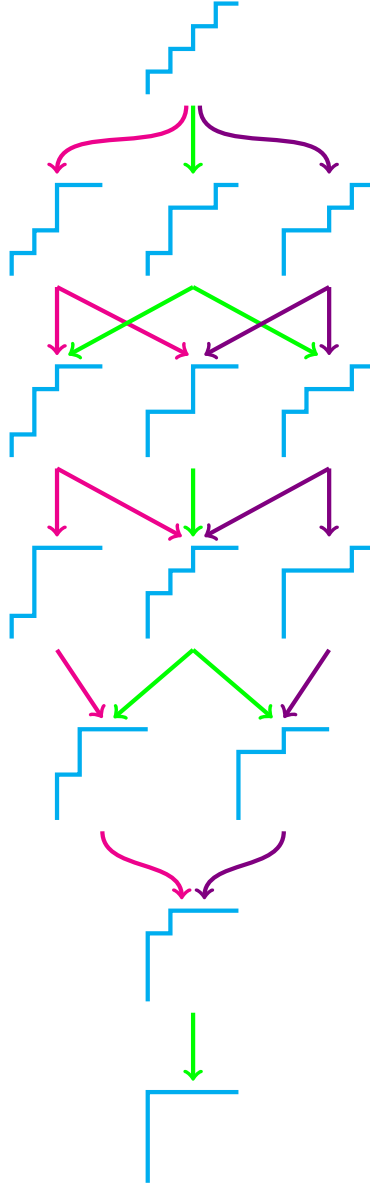

■ 10110011001100	■ 10110101001100	■ *difference*

Remark. *If $w_1 >_d w_2$, then the path corresponding to w_2 is* over *the path corresponding to w_1, and the* difference *between the two paths is a square of size 1 by 1.*

Proposition. *This covering relation defines a* poset *for $\mathcal{D}_n$.*

Example (The poset of $\mathcal{D}_4$).

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There are $\frac{1}{5}\binom{8}{4} = \frac{70}{5} = 14$ elements in this poset.
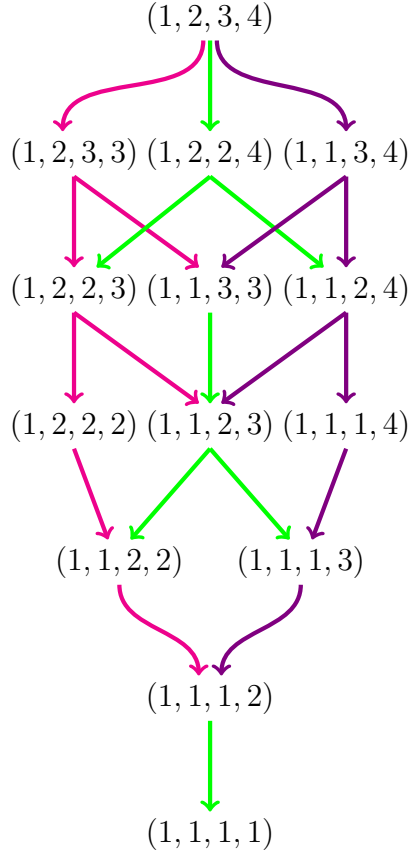
Definition 17 ($>'$). *For f and g two primitive parking functions, we say that f covers g, written $f >' g$, if $\exists i$ such that :*

- $f = (a_1, \ldots, a_{i-1}, a_i, \quad a_{i+1}, \ldots, a_n)$

- $g = (a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_n)$

Example ($n = 6$). $(1, 1, 2, 3, 4, 5) >' (1, 1, 2, 3, 3, 5)$

Proposition. *This covering relation defines a* poset *for $\mathcal{PF}'_n$.*

Example (The poset of $\mathcal{PF}'_4$)**.**

$$(1, 2, 3, 4)$$

$$(1, 2, 3, 3) \quad (1, 2, 2, 4) \quad (1, 1, 3, 4)$$

$$(1, 2, 2, 3) \quad (1, 1, 3, 3) \quad (1, 1, 2, 4)$$

$$(1, 2, 2, 2) \quad (1, 1, 2, 3) \quad (1, 1, 1, 4)$$

$$(1, 1, 2, 2) \qquad (1, 1, 1, 3)$$

$$(1, 1, 1, 2)$$

$$(1, 1, 1, 1)$$

There are $\frac{1}{5}\binom{8}{4} = \frac{70}{5} = 14$ elements in this poset.

Remark. *The two posets are isomorphic, and one can be obtained by applying the aforementioned bijection to the other.*

Theorem 8. *The number of intervals in those posets is equal to the $n + 1^{th}$ term of the integer sequence defined by https://oeis.org/A005700.*
The first terms of this sequence are $1, 1, 3, 14, 84, 594, 4719, 40898, 379236, 3711916, ...$

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Alec Mihailovs proved this sequence to be equal to
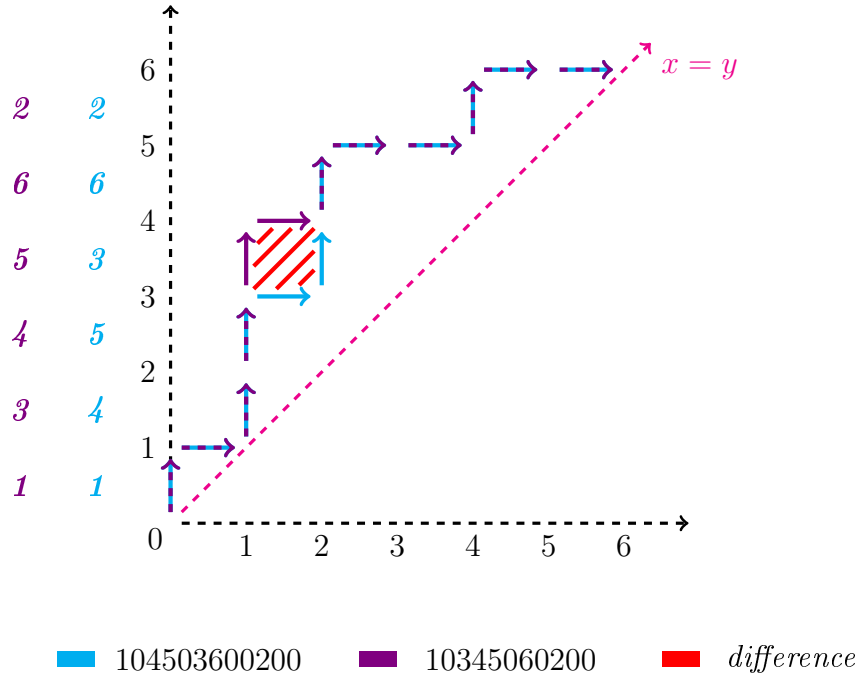
$$\frac{6(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!}$$

.

Classical Dyck - Parking Posets

Definition 18 ($>_{ld}$). *For w and w' two labeled Dyck words, we say that w covers w', written $w >_{ld} w'$, if $\exists l, r, x, x', y, z, z'$ such that :*

- *l is either empty or ends with 0*

- *r is either empty or starts with 0*

- *$x = x_1 x_2 \cdots$ has all its digits > 0*

- *$z = z_1 z_2 \cdots$ has all its digits > 0*

- *$x' = x$ where y is correctly inserted regarding the order condition*

- *y is in z, and $z' = z$ where y is removed*

- *$w = lx0zr$*

- *$w' = lx'0z'r$*

Example $(n = 5)$. $104503600200 >_{ld} 10345060200$

- $l = 10$

- $r = 0200$

- $x = 45$

- $x' = 345$

- $y = 3$

- $z = 36$

- $z' = 6$

2   2

6   6

5   3

4   5

3   4

1   1

$x = y$

██ 104503600200   ██ 10345060200   ██ *difference*

Definition 19 (Rise). *A* rise *of a decorated Dyck word is a maximal sequence of non-zero digits preceding a zero.*

Example ($n = 5$)**.** *In order, the rises of* 104503600200 *are :*
• 1 • 45 • 36 • $\emptyset$ • 2 • $\emptyset$

Remark. *If $w_1 \gtrdot_{ld} w_2$, then the path corresponding to w_2 is* over *the path corresponding to w_1, and the* difference *between the two paths is a square of size 1 by 1.*
Furthermore, the covering relation can be seen as follows : w_1 covers w_2 if we can obtain w_2 by taking a digit from the $i + 1^{th}$ rise of w_1, and inserting it into the ith rise of w_1 in increasing order.

Proposition. *This covering relation defines a* poset *for $\mathcal{LD}_n$.*

Example (The poset of $\mathcal{LD}_3$)**.**
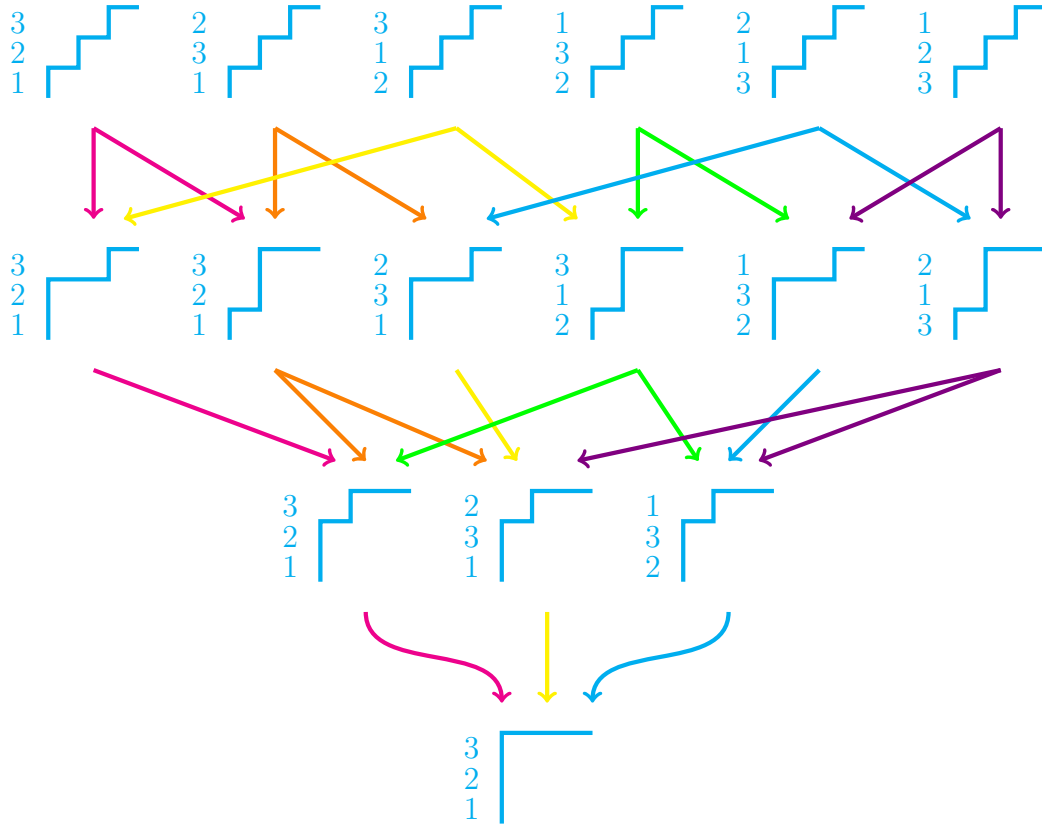
There are $4^2 = 16$ elements in this poset.

Definition 20 ($\gtrdot$)**.** *For f and g two parking functions, we say that f covers g, written $f \gtrdot g$, if $\exists i$ such that :*
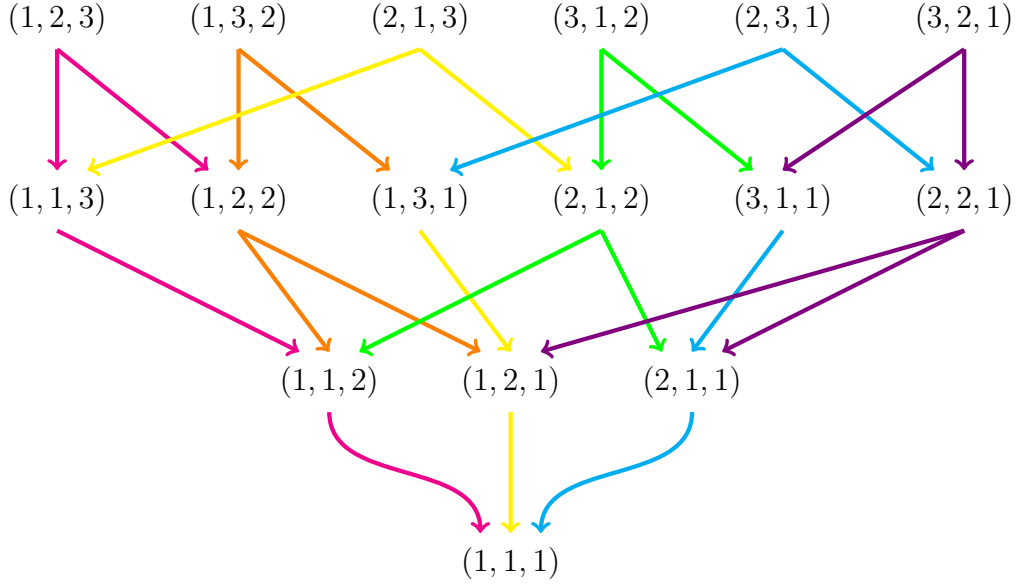
- $f = (a_1, \ldots, a_{i-1}, a_i, \quad a_{i+1}, \ldots, a_n)$

- $g = (a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_n)$

That is, the same relation as for primitive parking functions.

Example $(n = 6)$**.** $(2, 1, 5, 3, 1, 4) \gtrdot (2, 1, 5, 2, 1, 4)$

Proposition. *This covering relation defines a* poset *for $\mathcal{PF}_n$.*

Example (The poset of $\mathcal{PF}_3$)**.**

There are $4^2 = 16$ elements in this poset.

Remark. *The two posets are isomorphic, and one can be obtained by applying the aforementioned bijection to the other.*

Theorem 9. *The number of intervals in those posets is equal to the $n + 1^{th}$ term of the integer sequence defined by https://oeis.org/A196304.*
The first terms of this sequence are $1, 1, 5, 64, 1587, 65421, 4071178, 357962760, 4237910716, \dots$

Chapter 2

The rational case

For the whole chapter, we will consider 2 *coprime* integers a and b (meaning a and b have 1 as their greatest common divisor).

2.1 Rational Parking Functions

Definition 21 (a, b - Parking Function). *An* a, b - parking function *is a sequence* $(a_1, a_2, \ldots, a_n)$ *such that :*
- $n = a$
- *its non-decreasing reordering* $(b_1, b_2, \ldots, b_n)$ *has* $b_i \leqslant \frac{b}{a}(i-1) + 1$ *for all* i.

We denote by $\mathcal{PF}_{a,b}$ *the set of a, b - parking functions.*

Example.

- *Ex. 1 :* $a > b$

    $a = 7$

    $b = 3$

    Limits of the non-decreasing reordering of any $f \in \mathcal{PF}_{7,3}$:

    $[1, \; 1\frac{3}{7}, \; 1\frac{6}{7}, \; 2\frac{2}{7}, \; 2\frac{5}{7}, \; 3\frac{1}{7}, \; 3\frac{4}{7}]$

    $f_1 = (2, 1, 1, 3, 2, 3, 1) \in \mathcal{PF}_{7,3}$

    $f_2 = (2, 1, 2, 3, 2, 3, 1) \notin \mathcal{PF}_{7,3}$, *even though* $f_2 \in \mathcal{PF}_7$

- *Ex. 2 : $a < b$*

  $a = 5$

  $b = 7$

  Limits of the non-decreasing reordering of any $f \in \mathcal{PF}_{5,7}$:

  $[1, \ 2\frac{2}{5}, \ 3\frac{4}{5}, \ 5\frac{1}{5}, \ 6\frac{3}{5}]$

  $f_3 = (6, 3, 5, 1, 2) \in \mathcal{PF}_{5,7}$, *even though* $f_3 \notin \mathcal{PF}_5$

  $f_4 = (6, 3, 5, 1, 3) \notin \mathcal{PF}_{5,7}$

Theorem 10. *Let $pf_{a,b}$ be the cardinal of $\mathcal{PF}_{a,b}$. We have*

$$pf_{a,b} = b^{a-1}$$

Example $(a = 3, b = 5)$.
- $pf_{a,b} = 25$ • *Limits* : $[1, \ 2\frac{2}{3}, \ 4\frac{1}{3}]$

$(1,1,1)$	$(1,1,2)$	$(1,1,3)$	$(1,1,4)$	$(1,2,1)$	$(1,2,2)$	$(1,2,3)$
$(1,2,4)$	$(1,3,1)$	$(1,3,2)$	$(1,4,1)$	$(1,4,2)$	$(2,1,1)$	$(2,1,2)$
$(2,1,3)$	$(2,1,4)$	$(2,2,1)$	$(2,3,1)$	$(2,4,1)$	$(3,1,1)$	$(3,1,2)$
$(3,2,1)$	$(4,1,1)$	$(4,1,2)$	$(4,2,1)$			

Remark. $\mathcal{PF}_{n,n+1} = \mathcal{PF}_n$. *In fact, we do have* $b^{a-1} = (n+1)^{n-1}$.

2.1.1 Rational primitive parking functions

Definition 22 (Rational Primitive). *A rational parking function f is said primitive if it is already in non-decreasing order.*
We denote by $\mathcal{PF}'_{a,b}$ the set of primitive a, b - parking functions.

Example $(a = 4, b = 3)$. *Limits* : $[1, \ 1\frac{3}{4}, \ 2\frac{1}{2}, \ 3\frac{1}{4}]$

$$f_1 = (1, 1, 2, 2) \in \mathcal{PF}'_{4,3}$$
$$f_2 = (1, 1, 2, 1) \notin \mathcal{PF}'_{4,3}, \ \text{even though } f_2 \in \mathcal{PF}_{4,3}.$$

Theorem 11. *Let $pf'_{a,b}$ be the cardinal of $\mathcal{PF}'_{a,b}$. We have*

$$pf'_{a,b} = \frac{1}{a+b} \binom{a+b}{b}$$

which is the rational Catalan number $Cat(a, b)$.

Example ($a = 3, b = 5$).
• $pf'_{a,b} = 7$ • *Limits* : $[1,\ 2\frac{2}{3},\ 4\frac{1}{3}]$

$$(1,1,1) \qquad (1,1,2) \qquad (1,1,3) \qquad (1,1,4) \qquad (1,2,2) \qquad (1,2,3) \qquad (1,2,4)$$

Remark. $\mathcal{PF}'_{n,n+1} = \mathcal{PF}'_n$. *In fact, we do have*

$$\frac{1}{n+n+1}\binom{n+n+1}{n+1} = \frac{1}{2n+1}\binom{2n+1}{n+1} = \frac{1}{2n+1}\frac{(2n+1)!}{n!(n+1)!}$$

$$= \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1}\frac{(2n)!}{n!n!} = \frac{1}{n+1}\binom{2n}{n}$$

2.2 Rational Non-crossing Partitions

Definition 23 (a, b - Non-crossing Partition). *An* a, b - non-crossing partition *is TODO*
We denote by $\mathcal{NC}_{a,b}$ *the set of a, b - non-crossing partitions.*

Example. *An abncp*

Theorem 12. *Let* $nc_{a,b}$ *be the cardinal of* $\mathcal{NC}_{a,b}$. *We have*

$$nc_{a,b} = \frac{1}{a+b}\binom{a+b}{a} = \frac{(a+b-1)!}{a!b!}$$

, which is the rational Catalan number.

Example. *all abncp for some a b*

Proposition. *This means we can create a* bijection *between* $\mathcal{PF}'_{a,b}$ *and* $\mathcal{NC}_{a,b}$.

Proof.

- $\mathcal{NC}_{a,b} \to \mathcal{PF}'_{a,b}$:

- $\mathcal{PF}'_{a,b} \to \mathcal{NC}_{a,b}$:

$\square$

Definition 24 (a, b - Non-crossing 2-Partition)**.** *An* a, b - Non-crossing 2-partition *is TODO*
We denote by $\mathcal{NC}^2_{a,b}$ *the set of a, b - non-crossing 2-partitions.*

Example. *some ncab2*

Theorem 13. *Let* $nc^2_{a,b}$ *be the cardinal of* $\mathcal{NC}^2_{a,b}$*. We have*

$$nc^2_{a,b} = b^{a-1}$$

.

Example. *all ncab2 for some a b*

Proposition. *bijection*

Proof. bijection proof $\qquad\square$

2.3 A direct poset linked to Rational Dyck paths

2.3.1 Rational Dyck Paths

Definition 25 (a, b - Dyck path)**.** *An* a, b - Dyck word *is a word* $w \in \{0,1\}^*$ *such that :*

- *for each* suffix w' *of* w,
$$|w'|_1 \geqslant \frac{a}{b}|w'|_0$$

  .

- $|w|_0 = b.$

- $|w|_1 = a.$

An a, b - Dyck word can be represented as a path *from* $(0,0)$ *to* (b,a) *that stays over* $y = \frac{a}{b}x$, *called an* a, b - Dyck path *:*
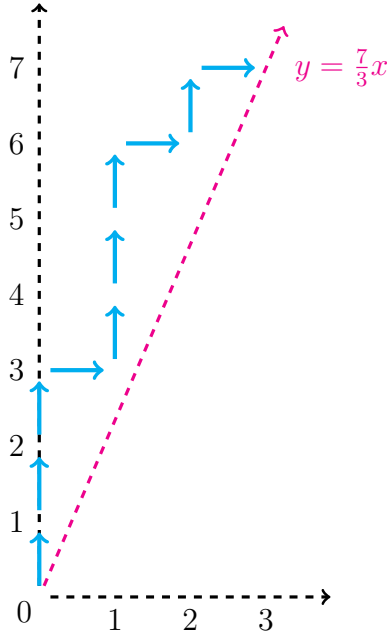
- *Each 1 corresponds to a* North step $\uparrow$.

- *Each 0 corresponds to an* East step $\rightarrow$.

We denote by $\mathcal{R}_{a,b}$ *the set of a, b - Dyck words.*

Example $(a > b : a = 7, b = 3)$.

$w_1 = 1110011110$ *is* not *a 7, 3 - Dyck word, because* $|11100|_1 = 3$
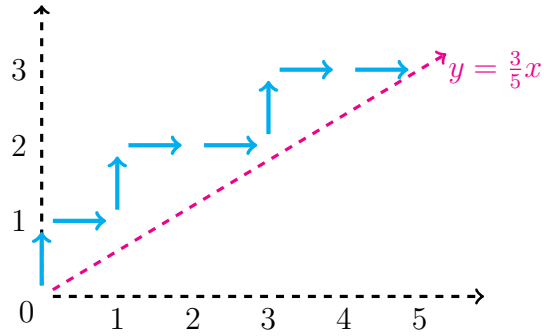$$< \frac{7}{3}|11100|_0 = \frac{14}{3} = 4\frac{1}{3}.$$
$w_2 = 1110111010$ is *a 7, 3 - Dyck word :*

Example $(a < b : a = 3, b = 5)$.

$w_1 = 10100010$ *is* not *a 3, 5 - Dyck word, because* $|101000|_1 = 2$
$$< \frac{3}{5}|101000|_0 = \frac{12}{5} = 2\frac{2}{5}.$$
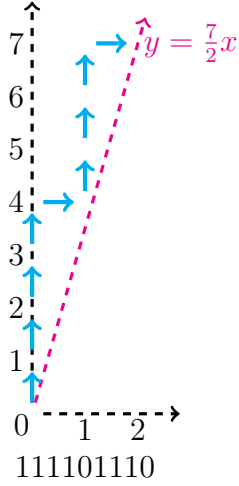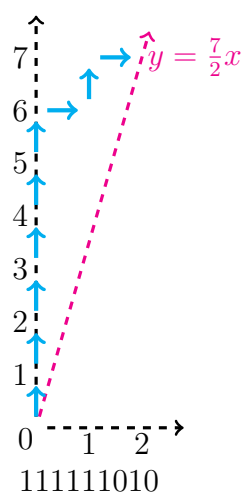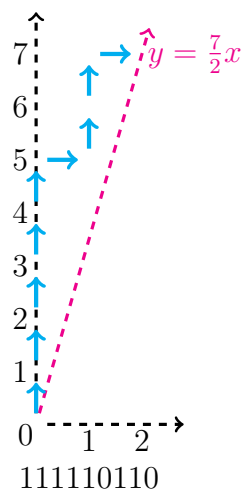$w_2 = 10100100$ is *a 3, 5 - Dyck word :*

Theorem 14. *Let $r_{a,b}$ be the cardinal of $\mathcal{R}_{a,b}$. We have*

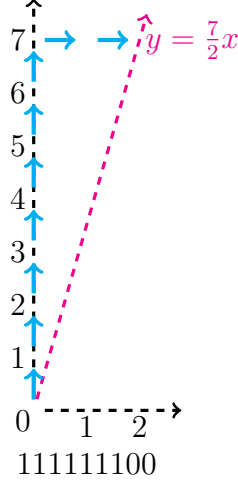$$r_{a,b} = \frac{1}{a+b}\binom{a+b}{a} = \frac{(a+b-1)!}{a!b!}$$

, which is again the rational Catalan number.

Example $(a = 7, b = 2)$**.** $r_n = 4$.

111101110

$y = \frac{7}{2}x$

111110110

$y = \frac{7}{2}x$

111111010

44

7 → → $y = \frac{7}{2}x$
6
5
4
3
2
1
0   1   2
111111100

Proposition. *This means we can create a* bijection *between* $\mathcal{PF}'_{a,b}$ *and* $\mathcal{R}_{a,b}$.

Proof.

- $\mathcal{PF}'_{a,b} \to \mathcal{R}_{a,b}$: Let $f = (a_1, \ldots, a_n) \in \mathcal{PF}'_{a,b}$ be our rational prim-
  itive parking function. For $i \in \{1, \ldots, b\}$, we define l_i the number of
  occurences of i in f.
  The corresponding rational Dyck word will be $\underbrace{1 \cdots 1}_{l_1} 0 \underbrace{1 \cdots 1}_{l_2} 0 \cdots \underbrace{1 \cdots 1}_{l_b} 0$.

- $\mathcal{R}_{a,b} \to \mathcal{PF}'_{a,b}$: Let w be our rational Dyck word, and consider its path
  representation. We define s_i to be the distance between the segment
  from $(0, i-1)$ to $(0, i)$ and the ith North step. Then, let $a_i = s_i + 1$.
  The corresponding rational primitive parking function is $(a_1, \ldots, a_a)$.

$\square$

Remark. *This bijection is exactly the same as the one between classical
primitive parking functions and Dyck paths.*

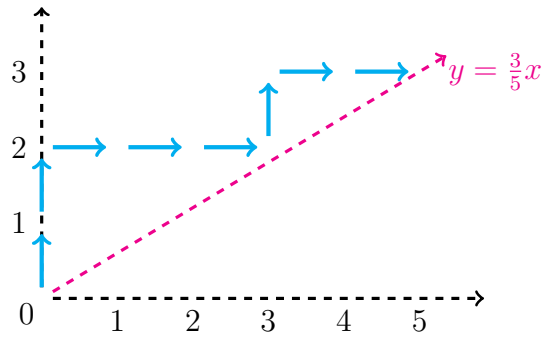**Example** $(a > b : a = 7, b = 3, \mathcal{PF}'_{a,b} \to \mathcal{R}_{a,b})$.

- $f = (1, 1, 1, 2, 2, 3, 3)$
      $l_1 = 3$ $l_2 = 2$ $l_3 = 2$

- $w = (1110110110)$

45
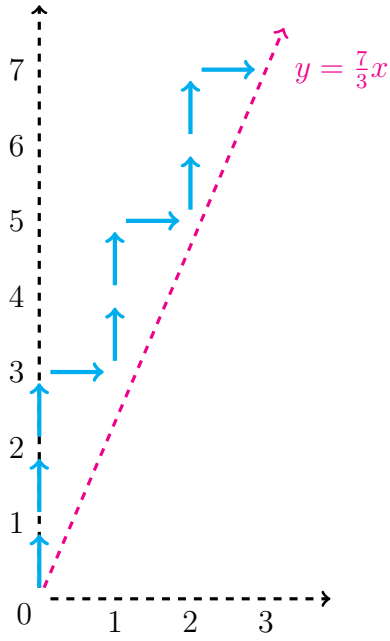
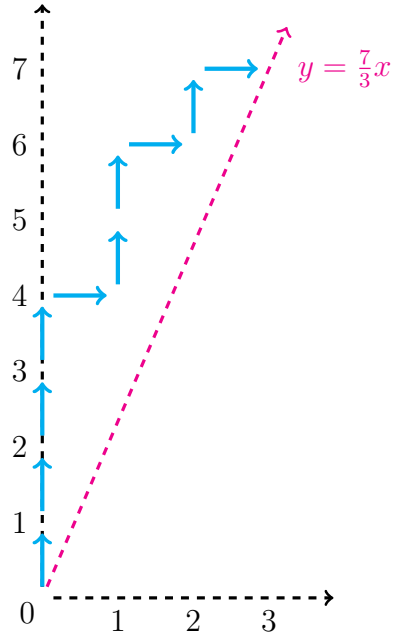**Example** $(a < b : a = 3, b = 5, \mathcal{PF'}_{a,b} \to \mathcal{R}_{a,b})$.

- $f = (1, 1, 4)$

$l_1 = 2$	l_2	$l_3 = 0$
$l_4 = 1$	$l_5 = 0$	

- $w = 11000100$

Example $(a > b : a = 7, b = 3, \mathcal{R}_{a,b} \to \mathcal{PF'}_{a,b})$.
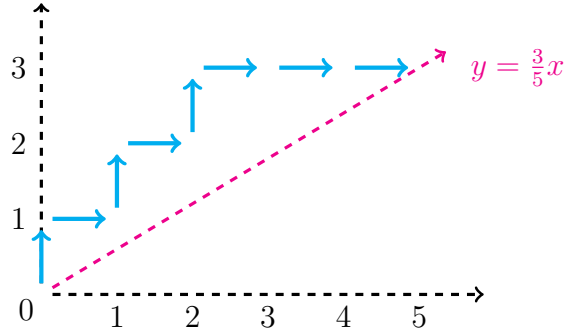
- $w = 1111011010$

- *Distances :*

$$s_1 = 0 \qquad\qquad a_1 = 1$$
$$s_2 = 0 \qquad\qquad a_2 = 1$$
$$s_3 = 0 \qquad\qquad a_3 = 1$$
$$s_4 = 0 \qquad\qquad a_4 = 1$$
$$s_5 = 1 \qquad\qquad a_5 = 2$$
$$s_6 = 1 \qquad\qquad a_6 = 2$$
$$s_7 = 2 \qquad\qquad a_7 = 3$$

- $f = (1, 1, 1, 1, 2, 2, 3)$

Example $(a < b : a = 3, b = 5, \mathcal{R}_{a,b} \to \mathcal{PF'}_{a,b})$.

- $w = 10101000$

- *Distances :*

$$s_1 = 0 \qquad\qquad a_1 = 1$$
$$s_2 = 1 \qquad\qquad a_2 = 2$$
$$s_3 = 2 \qquad\qquad a_3 = 3$$

- $f = (1, 2, 3)$

2.3.2   Rational Labeled Dyck Paths

Definition 26 (Labeled a, b - Dyck Path). *A labeled a, b - Dyck word is a word $w \in \{0, \dots, n\}^*$ such that :*

- *for each suffix w' of w,*

$$|w'|_{\neq 0} \geqslant \frac{a}{b} |w'|_0$$

.

- $|w|_0 = b.$

- $|w|_{\neq 0} = a.$

- *for each $i \in \{1, \dots, a\}$, w has exactly one occurence of i.*

- *if $w_i \neq 0$ and $w_{i+1} \neq 0$, then $w_i < w_{i+1}$. That is, consecutive North steps have increasing labels.*
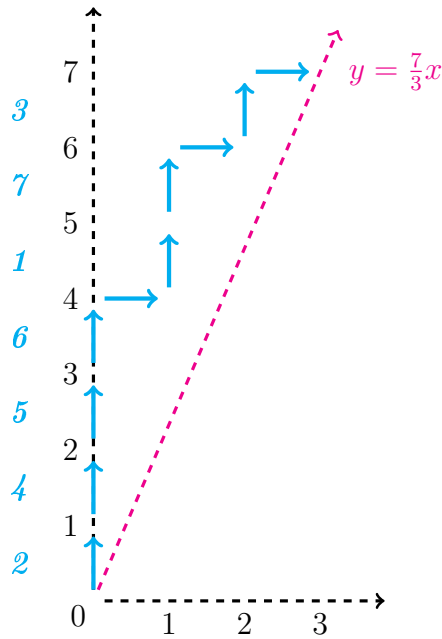
A labeled a, b - Dyck word can be represented as a path from $(0,0)$ to (b,a), where each North step is associated to a label :

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- *Each $i \neq 0$ corresponds to a* North step $\uparrow$ *labeled i.*

- *Each 0 corresponds to an* East step $\rightarrow$.
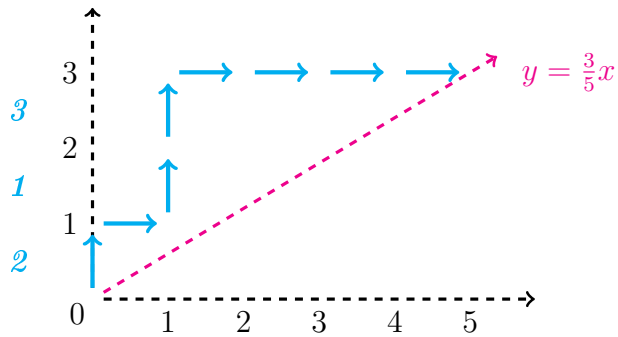
Those paths are called labeled a, b - Dyck paths.
We denote by $\mathcal{LR}_{a,b}$ the set of labeled a, b - Dyck words.

Example $(a > b : a = 7, b = 3)$. $w_2 = 2456017030$:

Example $(a < b : a = 3, b = 5)$. $w = 20130000$:

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Theorem 15. *Let $lr_{a,b}$ be the cardinal of $\mathcal{LR}_{a,b}$. We have*

$$lr_{a,b} = b^{a-1}$$

.

Example $(a > b : a = 4, b = 3)$. $lr_{a,b} = 3^3 = 27$

- *Word of shape $XXXX000$:*

    1234000

- *Words of shape $XXX0X00$:*

    1230400   1240300   1340200

    2340100

- *Words of shape $XX0XX00$:*

    1203400   1302400   1402300

    2301400   2401300   3401200

- *Words of shape $XXX00X0$:*

    1230040   1240030   1340020

    2340010

- *Words of shape $XX0X0X0$:*

    1203040   1204030   1302040

    1304020   1402030   1403020

    2301040   2304010   2401030

    2403010   3401020   3402010

Proposition. *This means we can create a bijection between $\mathcal{PF}_{a,b}$ and $\mathcal{LR}_{a,b}$.*

Proof.

- $\mathcal{PF}_{a,b} \to \mathcal{LR}_{a,b}$: Let $f = (a_1, \ldots, a_n) \in \mathcal{PF}_{a,b}$ be our a, b - parking function. For $i \in \{1, \ldots, b\}$, we define $im_i : \{j \mid a_j = i\}$.
  We then define $im_{i,1}, \ldots, im_{i,k_i}$ to be the elements of im_i in increasing order.
  The corresponding labeled a, b - Dyck word will be
  $$\underbrace{im_{1,1} \cdots im_{1,k_1}}_{im_1} 0 \, \underbrace{im_{2,1} \cdots im_{2,k_2}}_{im_2} 0 \cdots \underbrace{im_{n,1} \cdots im_{b,k_b}}_{im_b} 0.$$

- $\mathcal{LR}_{a,b} \to \mathcal{PF}_n$: Let w be our labeled a, b - Dyck word, and consider its path representation. We define s_i to be the distance between the segment from $(0, i-1)$ to $(0, i)$ and the ith North step.
  Then, let $label(i)$ be the label of the ith North step, and $dist_i = \{label(j) \mid s_j = i\}$ be the set of the labels of all North steps at distance i.
  Then, if $j \in dist_i$, let $a_j = i + 1$.
  The corresponding parking function is $(a_1, \ldots, a_a)$.

  $\square$

Remark. *This bijection is exactly the same as the one between classical parking functions and labeled Dyck paths.*

Example $(a > b : a = 7, b = 3, \mathcal{PF}_{a,b} \to \mathcal{LR}_{a,b})$.

- $f = (2, 1, 3, 1, 1, 3, 2)$

  $im_1 = \{2, 4, 5\}$ $\qquad\qquad$ $im_2 = \{1, 7\}$ $\qquad\qquad$ $im_3 = \{3, 6\}$

- $w = 2450170360$
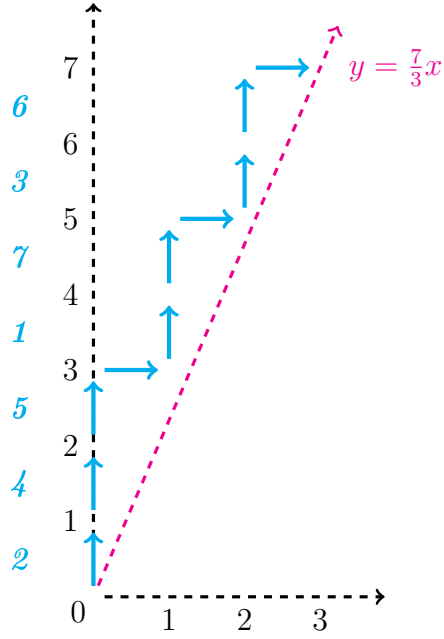
Example $(a < b : a = 3, b = 5, \mathcal{PF}_{a,b} \to \mathcal{LR}_{a,b})$.
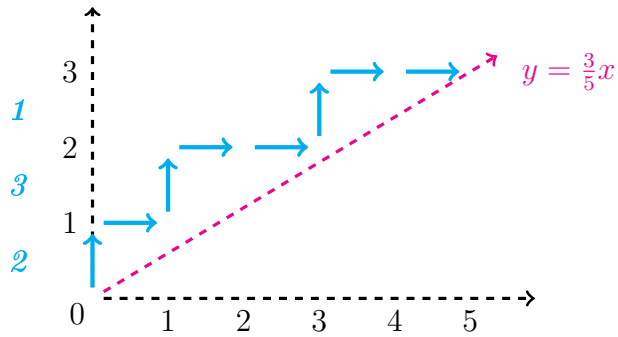
- $f = (4, 1, 2)$

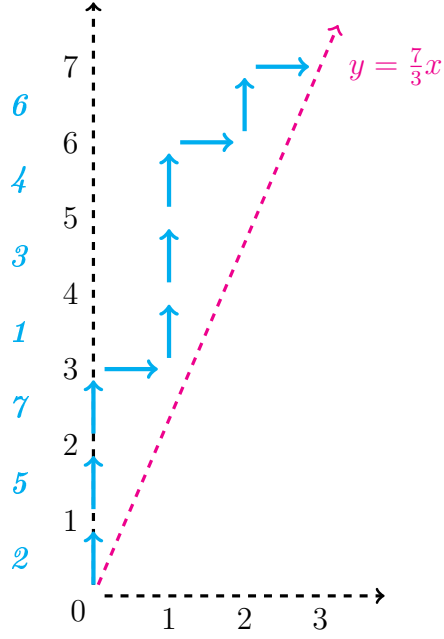$$im_1 = \{2\} \qquad im_2 = \{3\} \qquad im_3 = \emptyset$$
$$im_4 = \{1\} \qquad im_5 = \emptyset$$

- $w = 20300100$

Example $(a > b : a = 7, b = 3, \mathcal{LR}_{a,b} \to \mathcal{PF}_{a,b})$.

- $w = 2570134060$

- *Distances :*

$$s_1 = 0 \qquad s_2 = 0 \qquad s_3 = 0$$
$$s_4 = 1 \qquad s_5 = 1 \qquad s_6 = 1$$
$$s_7 = 2$$

- *Labels :*

$$dist_0 = \{2,5,7\} \qquad dist_1 = \{1,3,4\} \qquad dist_2 = \{6\}$$

- $f = (2,1,2,2,1,3,1)$

Example $(a < b : a = 3, b = 5, \mathcal{LR}_{a,b} \to \mathcal{PF}_{a,b})$.

- $w = 13000200$
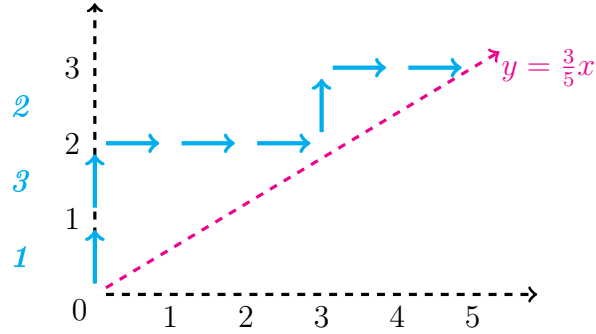
- *Distances :*

$$s_1 = 0 \qquad\qquad s_2 = 0 \qquad\qquad s_3 = 3$$

- *Labels :*

$$dist_0 = \{1, 3\} \qquad\qquad dist_1 = \emptyset \qquad\qquad dist_2 = \emptyset$$
$$dist_3 = \{2\} \qquad\qquad dist_4 = \emptyset$$

- $f = (1, 4, 1)$

Remark. *The rational primitive parking functions are exactly the rational parking functions corresponding to rational labeled Dyck paths where the ith North step is labeled i.*

2.3.3   Rational Dyck - Parking Posets

Rational Primitive Dyck - Parking Posets

Definition 27 $(\gtrdot_r)$**.** *For w and w' two a, b - Dyck words, we say that w covers w', written $w \gtrdot_r w'$, if $\exists w_1, w_2$ such that :*

- $w = w_1 01 w_2$

- $w' = w_1 10 w_2$

Example $(a = 7, b = 3)$**.** $1111011010 \gtrdot_r 1111011100$

- $w_1 = 1111011$

- $w_2 = 0$

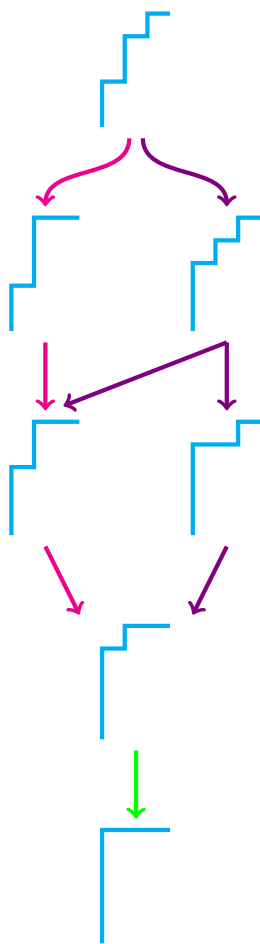$$y = \tfrac{7}{3}x$$
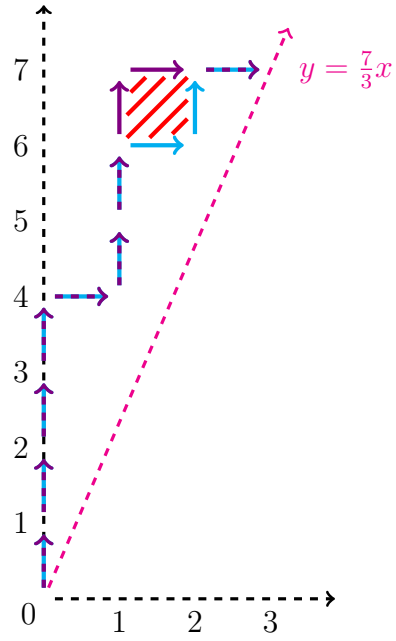
| ▬ 1111011010 | ▬ 1111011100 | ▬ *difference* |

Remark. *If $w_1 \gg_r w_2$, then the path corresponding to w_2 is* over *the path corresponding to w_1, and the* difference *between the two paths is a square of size 1 by 1.*

Proposition. *This covering relation defines a* poset *for $\mathcal{R}_{a,b}$.*

Example $(a > b :$ The poset of $\mathcal{R}_{5,3})$**.**

There are $\frac{1}{8}\binom{8}{5} = \frac{42}{6} = 7$ elements in this poset.

Example $(a < b :$ The poset of $\mathcal{R}_{3,7})$**.**

There are $\frac{1}{10}\binom{10}{3} = \frac{72}{6} = 12$ elements in this poset.

Definition 28 ($\gtrdot'$). *For f and g two rational primitive a, b - parking functions, we say that f covers g, written $f \gtrdot' g$, if $\exists i$ such that :*

- $f = (a_1, \ldots, a_{i-1}, a_i, \quad a_{i+1}, \ldots, a_n)$

- $g = (a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_n)$

Example ($a > b : a = 7, b = 3$). $(1, 1, 1, 2, 2, 2, 3) \gtrdot' (1, 1, 1, 1, 2, 2, 3)$

Example ($a < b : a = 3, b = 5$). $(1, 2, 4) \gtrdot' (1, 1, 4)$
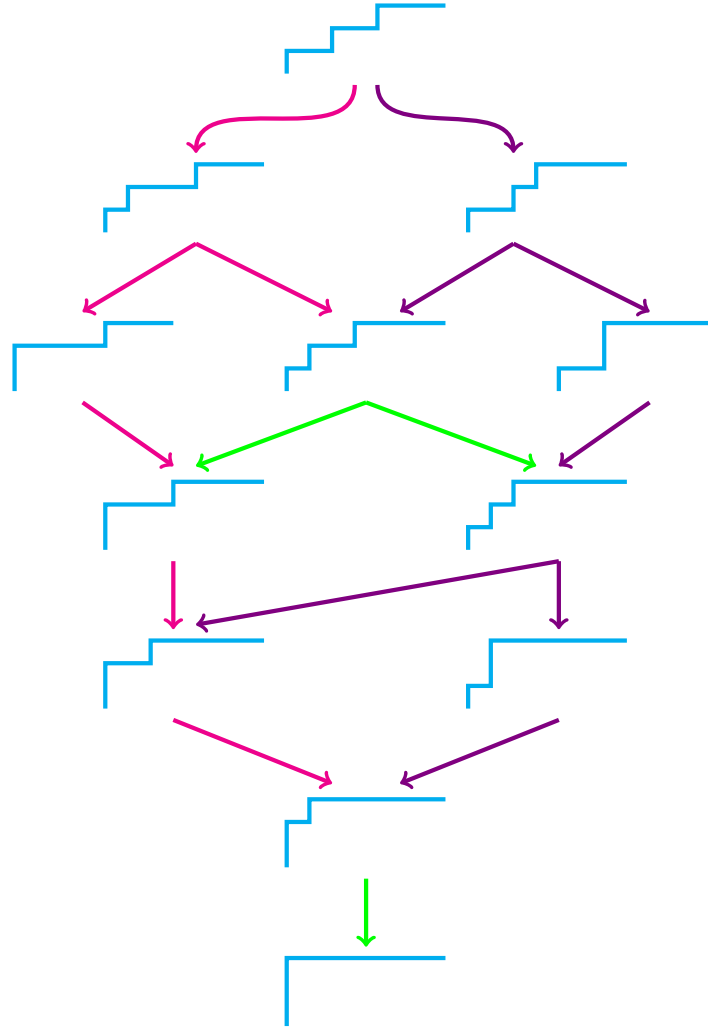
Proposition. *This covering relation defines a* poset *for* $\mathcal{PF}'_{a,b}$.

Example ($a > b$: The poset of $\mathcal{PF}'_{5,3}$).
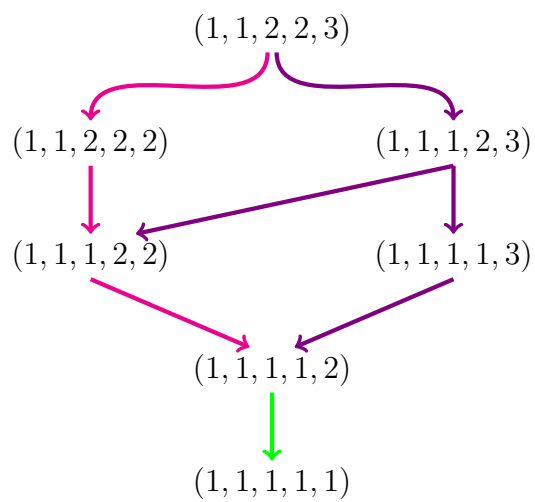
There are $\frac{1}{8}\binom{8}{5} = \frac{42}{6} = 7$ elements in this poset.

Example ($a < b$: The poset of $\mathcal{PF}'_{3,7}$).

There are $\frac{1}{10}\binom{10}{3} = \frac{72}{6} = 12$ elements in this poset.

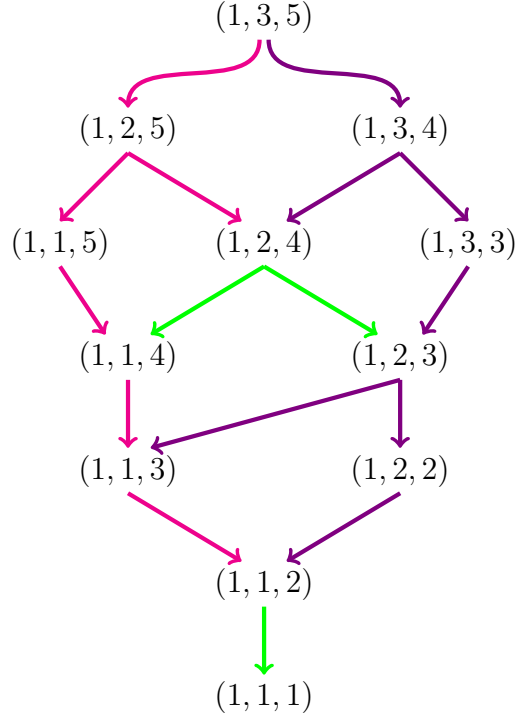**Remark.** *The posets of $\mathcal{PF}'_{a,b}$ and $\mathcal{R}_{a,b}$ are isomorphic, and one can be obtained by applying the aforementioned bijection to the other.*

Rational Dyck - Parking Posets

Definition 29 $(\gtrdot_{lr})$. *For w and w' two labeled a, b - Dyck words, we say that w covers w', written $w \gtrdot_{lr} w'$, if $\exists l, r, x, x', y, z, z'$ such that :*

- *l is either empty or ends with 0*

- *r is either empty or starts with 0*

- *$x = x_1 x_2 \cdots$ has all its digits > 0*

- *$z = z_1 z_2 \cdots$ has all its digits > 0*

- *$x' = x$ where y is correctly inserted regarding the order condition*

- y *is in* z, *and* $z' = z$ *where* y *is removed*

- $w = lx0zr$

- $w' = lx'0z'r$

Example $(a > b : a = 7, b = 3)$. $2460150370 >_{lr} 2460135070$

- $l = 2460$

- $r = 0$

- $x = 15$

- $x' = 135$

- $y = 3$

- $z = 37$

- $z' = 7$

Example $(a < b : a = 3, b = 5)$. $20301000 >_{lr} 20130000$

- $l = 20$

- $r = 000$

- $x = 3$

- $x' = 13$

- $y = 1$

- $z = 1$

- $z' = \emptyset$

Remark. *If* $w_1 >_{lr} w_2$, *then the path corresponding to* w_2 *is over the path corresponding to* w_1, *and the difference between the two paths is a square of size 1 by 1.*
Furthermore, the covering relation can be seen as follows : w_1 *covers* w_2 *if we can obtain* w_2 *by taking a digit from the* $i + 1^{th}$ *rise of* w_1, *and inserting it into the* i^{th} *rise of* w_1 *in increasing order.*

Proposition. *This covering relation defines a* poset *for* $\mathcal{LR}_{a,b}$.

Example ($a > b$: The poset of $\mathcal{LR}_{5,2}$).
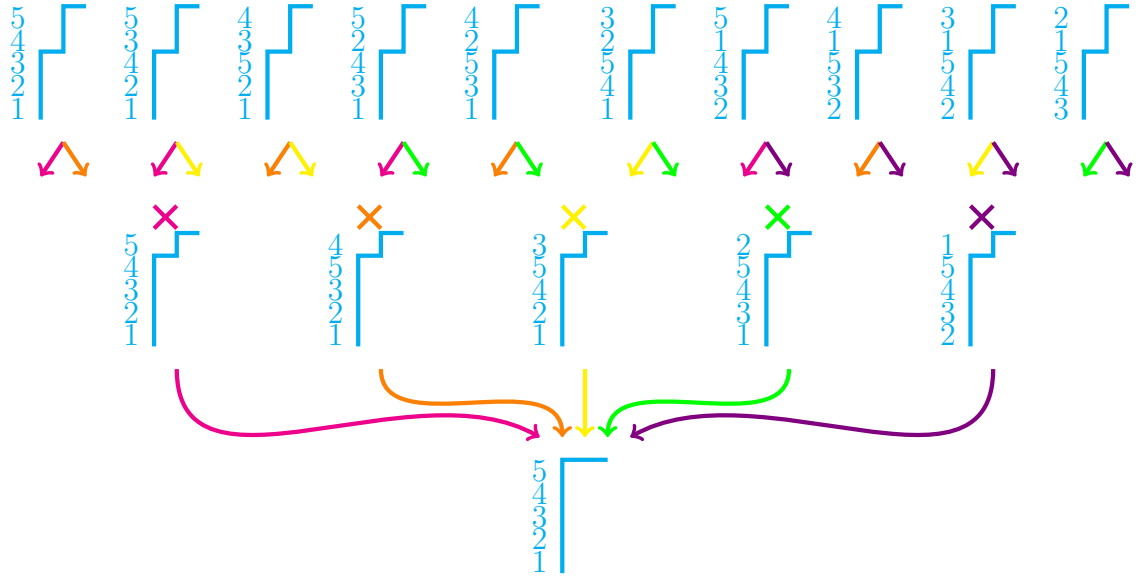
Arrows have been simplified for readability.
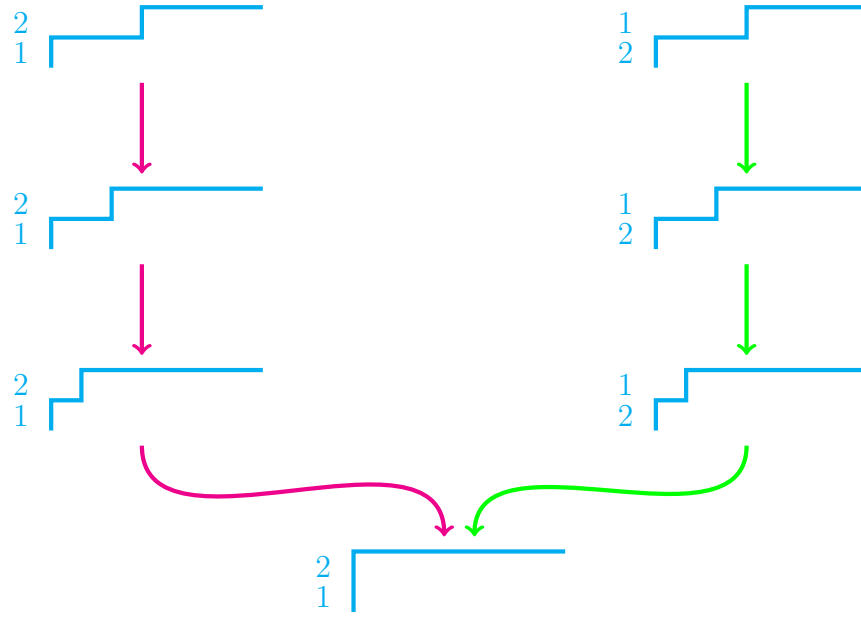*Arrows between the top 2 levels are to be read as ending at the cross of the corresponding color.*
There are $2^4 = 16$ elements in this poset.

Example ($a < b$: The poset of $\mathcal{LR}_{2,7}$).

There are $7^1 = 7$ elements in this poset.

Definition 30 ($\gtrdot$). *For f and g two rational parking functions, we say that f covers g, written $f \gtrdot g$, if $\exists i$ such that :*

- $f = (a_1, \ldots, a_{i-1}, a_i, \quad a_{i+1}, \ldots, a_n)$

- $g = (a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_n)$
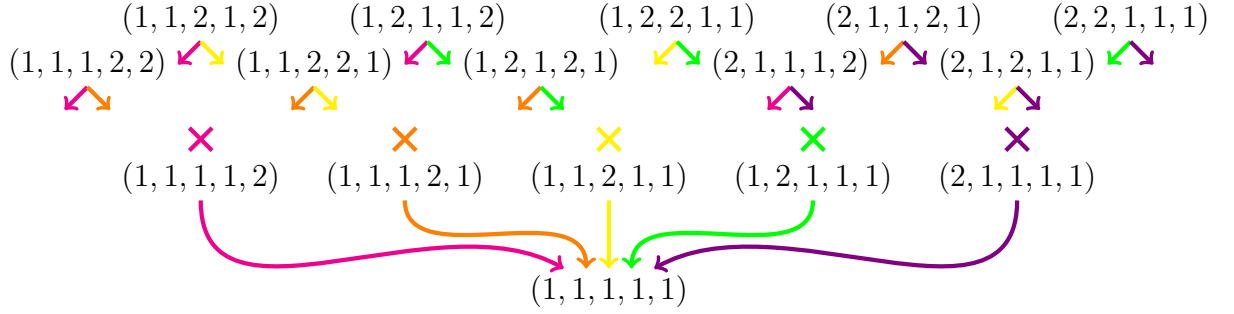
That is, the same relation as for rational primitive parking functions.

Example ($a > b : a = 7, b = 3$). $(2, 3, 1, 1, 2, 1, 3) \gtrdot (2, 3, 1, 1, 1, 1, 3)$

Example ($a < b : a = 3, b = 5$). $(4, 1, 2) \gtrdot (3, 1, 2)$

Proposition. *This covering relation defines a* poset *for $\mathcal{PF}_{a,b}$.*

Example ($a > b :$ The poset of $\mathcal{PF}_{5,2}$).

$$(1,1,2,1,2) \qquad (1,2,1,1,2) \qquad (1,2,2,1,1) \qquad (2,1,1,2,1) \qquad (2,2,1,1,1)$$
$$(1,1,1,2,2) \searrow (1,1,2,2,1) \searrow (1,2,1,2,1) \searrow (2,1,1,1,2) \searrow (2,1,2,1,1) \searrow$$

$$\times \qquad \times \qquad \times \qquad \times \qquad \times$$

$$(1,1,1,1,2) \qquad (1,1,1,2,1) \qquad (1,1,2,1,1) \qquad (1,2,1,1,1) \qquad (2,1,1,1,1)$$
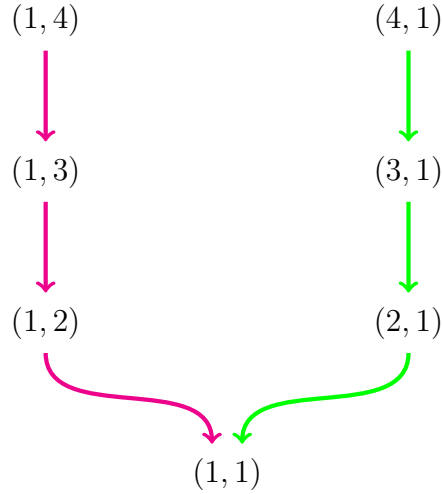
$$(1,1,1,1,1)$$

Arrows have been simplified for readability.
Arrows between the top 2 levels are to be read as ending at the cross of the
corresponding color.
There are $2^4 = 16$ elements in this poset.

Example ($a < b$: The poset of $\mathcal{PF}_{2,7}$).

$$(1,4) \qquad\qquad\qquad (4,1)$$

$$(1,3) \qquad\qquad\qquad (3,1)$$

$$(1,2) \qquad\qquad\qquad (2,1)$$

$$(1,1)$$

There are $7^1 = 7$ elements in this poset.

Remark. *The posets of $\mathcal{PF}_{a,b}$ and $\mathcal{LR}_{a,b}$ are isomorphic, and one can be obtained by applying the aforementioned bijection to the other.*

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Chapter 3

Trees

3.1 Parking Trees

Definition 31 (Parking Trees). *A parking tree is defined from a parking function $f = (a_1, \ldots, a_n) \in \mathcal{PF}_n$ as follows :*

- *For $1 \leqslant i \leqslant n+1$, we define s_i as $\{j \mid a_j = i\}$*

- *$[s_1, \ldots, s_{n+1}]$ describes the pre-order depth-first traversal of the tree.*

- *Each node labeled by a set of size k has k children.*

Remark. *The leaves of the tree are those corresponding to an element i such that $1 \leqslant i \leqslant n+1$, and i is* not *in f.*
Furthermore, as we will have a total edges by definition, the presence of a node corresponding to $n+1$ is necessary, even though it will always be empty.

Example ($n = 12$).
- $f = (5, 7, 1, 3, 1, 8, 2, 7, 1, 3, 11, 11)$
- *Labels :* $[\{3, 5, 9\}, \{7\}, \{4, 10\}, \emptyset, \{1\}, \emptyset, \{2, 8\}, \{6\}, \emptyset, \emptyset, \{11, 12\}, \emptyset, \emptyset]$
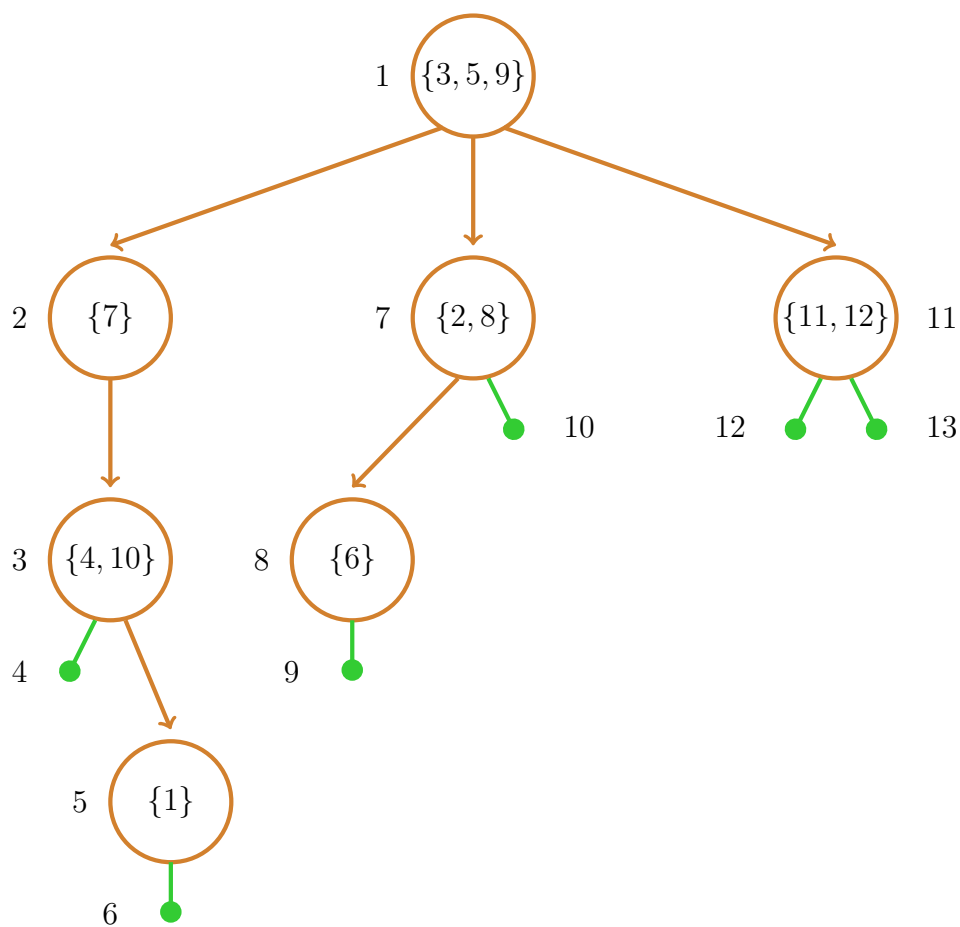
Conversely, by reading the labels of a parking tree depth-first in pre-order, we get the list of positions of each number in the corresponding parking function, thus creating a *bijection*.

Example (From parking tree to parking function)**.**
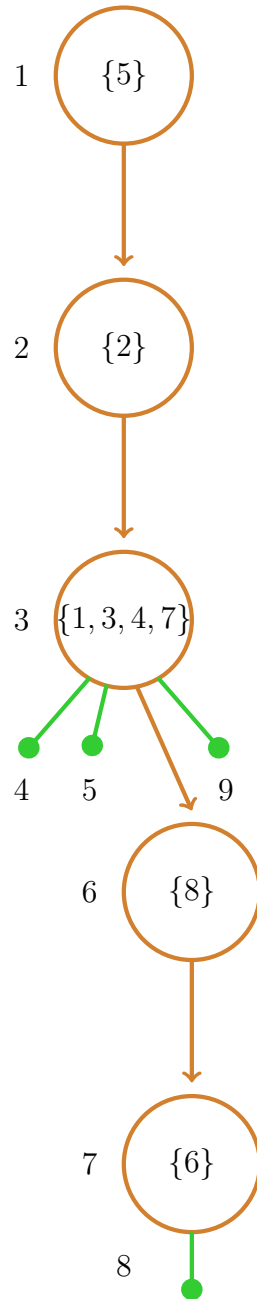
- *The labels are* $[\{5\}, \{2\}, \{1, 3, 4, 7\}, \emptyset, \emptyset, \{8\}, \{6\}, \emptyset, \emptyset]$.

- *Thus the corresponding parking function is* $(3, 2, 3, 3, 1, 7, 3, 6) \in \mathcal{PF}_8$.

3.2 Rational Parking Trees

Definition 32 (Rational Parking Trees)**.** *A rational parking tree is defined from a rational parking function* $f = (a_1, \ldots, a_a) \in \mathcal{PF}_{a,b}$ *as follows :*
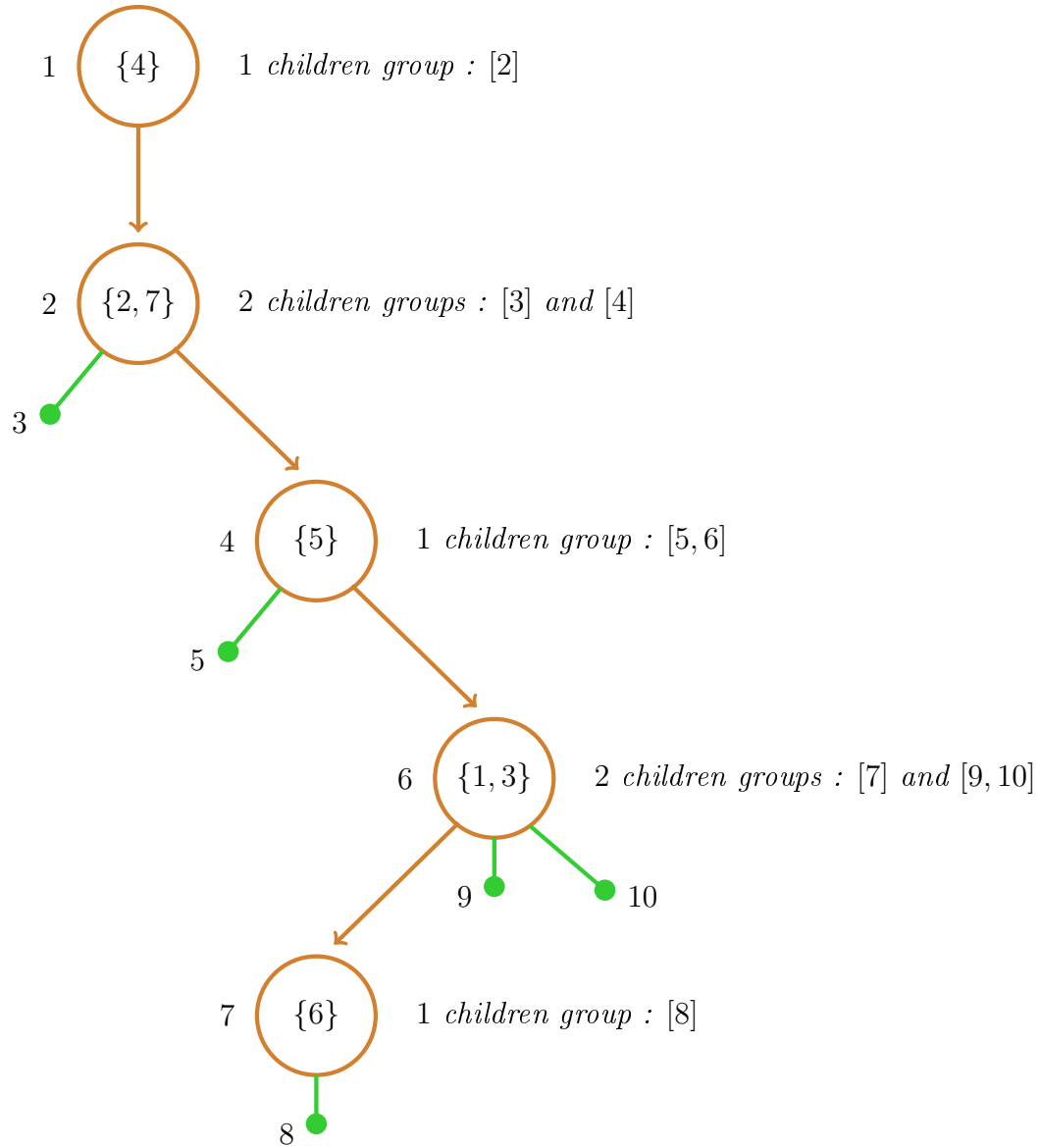
- *For* $1 \leqslant i \leqslant n+1$*, we define the limit* l_i *as the* integer portion *of* $\dfrac{b}{a}(i-1) + 1$.

  Let $l_0 = 0$.

- *From these limits, we deduce the intervals* $itv_i = \,]l_{i-1}, l_i]$ *for* $1 \leqslant i \leqslant a+1$.

- *For* $1 \leqslant i \leqslant b+1$*, define* s_i *as* $\{j \mid a_j = i\}$.

- $[s_1, \ldots, s_{b+1}]$ *describes the pre-order depth-first traversal of the tree.*

- *Each node labeled by a set of size* k *has* k groups *of children, which are defined by the intervals.*

Example $(a < b)$**.**

- $a = 7$

- $b = 9$

- *Limits :* $[1, \; 2\frac{2}{7}, \; 3\frac{4}{7}, \; 4\frac{6}{7}, \; 6\frac{1}{7}, \; 7\frac{3}{7}, \; 8\frac{5}{7}, \; 10]$

- *Integral limits :* $[0, 1, 2, 3, 4, 6, 7, 8, 10]$

- *Intervals :*

  $]0,1] \qquad]1,2] \qquad]2,3] \qquad]3,4] \qquad]4,6] \qquad]6,7] \qquad]7,8] \qquad]8,10]$

- *Children groups :*

  $[1] \qquad [2] \qquad [3] \qquad [4] \qquad [5,6] \qquad [7] \qquad [8]$

- $f = (6, 2, 6, 1, 4, 7, 2)$

- *Labels :* $\{\{4\}, \; \{2,7\}, \; \emptyset, \; \{5\}, \; \emptyset, \; \{1,3\}, \; \{6\}, \; \emptyset, \; \emptyset, \; \emptyset\}$

1   {4}   *1 children group* : $[2]$

2   {2, 7}   *2 children groups* : $[3]$ *and* $[4]$

3

4   {5}   *1 children group* : $[5, 6]$

5

6   {1, 3}   *2 children groups* : $[7]$ *and* $[9, 10]$

9   10

7   {6}   *1 children group* : $[8]$

8

Example $(a > b)$.

- $a = 9$
- $b = 7$
- *Limits* : $[1,\ 1\frac{7}{9},\ 2\frac{5}{9},\ 3\frac{3}{9},\ 4\frac{1}{9},\ 4\frac{8}{9},\ 5\frac{6}{9},\ 6\frac{4}{9},\ 7\frac{2}{9},\ 8]$

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- *Integral limits :* $[0, 1, 1, 2, 3, 4, 4, 5, 6, 7, 8]$

- *Intervals :*

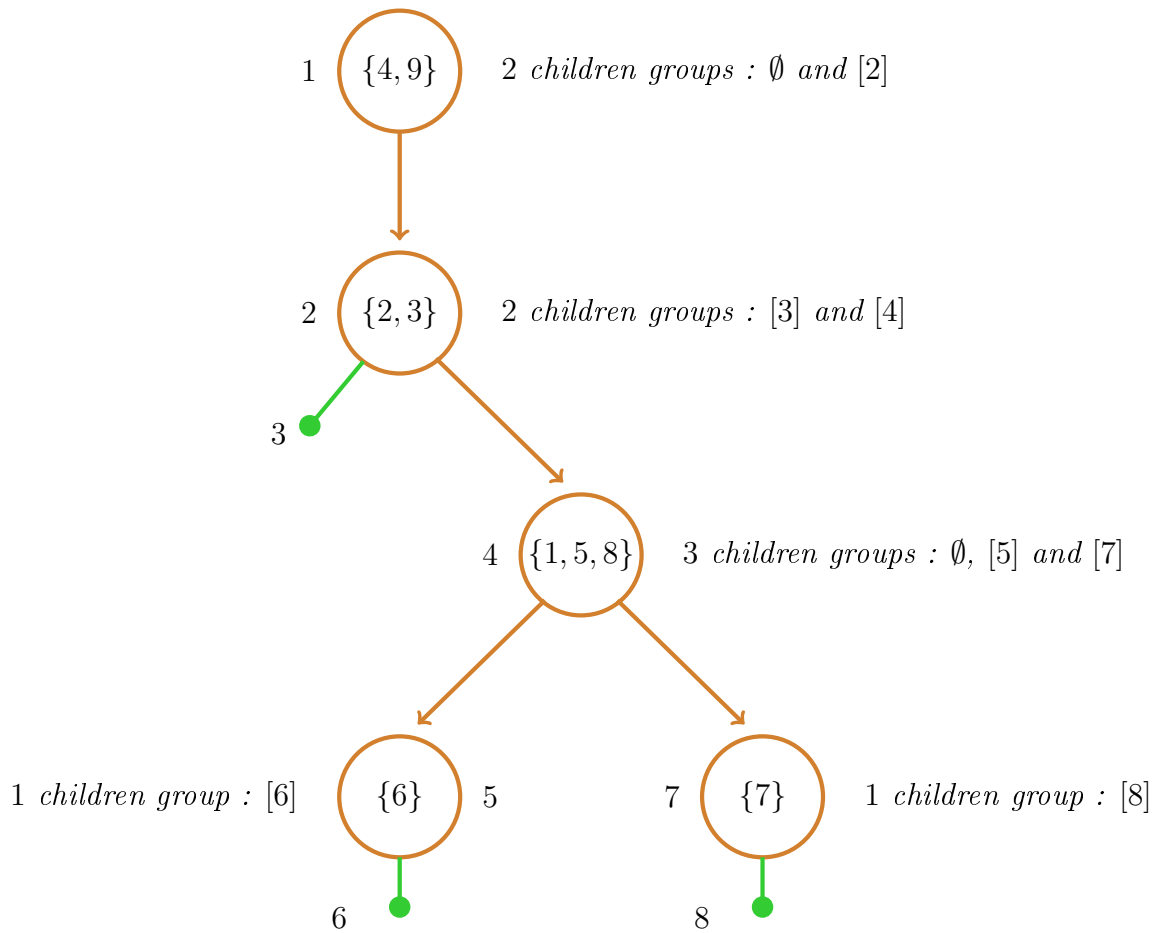 $]0, 1]$ $]1, 1]$ $]1, 2]$ $]2, 3]$ $]3, 4]$

  $]4, 4]$ $[4, 5]$ $]5, 6]$ $]6, 7]$ $]7, 8]$

- *Children groups :*

  $[1]$ $\emptyset$ $[2]$ $[3]$ $[4]$ $\emptyset$ $[5]$ $[6]$ $[7]$ $[8]$

- $f = (4, 2, 2, 1, 4, 5, 7, 4, 1)$

- *Labels :* $\{\{4, 9\}, \{2, 3\}, \emptyset, \{1, 5, 8\}, \{6\}, \emptyset, \{7\}, \emptyset\}$

1   $\{4, 9\}$ 2 *children groups :* $\emptyset$ *and* $[2]$

2   $\{2, 3\}$ 2 *children groups :* $[3]$ *and* $[4]$

3

4   $\{1, 5, 8\}$ 3 *children groups :* $\emptyset$, $[5]$ *and* $[7]$

1 *children group :* $[6]$ $\{6\}$ 5   7 $\{7\}$ 1 *children group :* $[8]$

6   8

In both cases, the converse direction of the *bijection* is obtained with the same method as for classical parking trees.