

# Rational Parking Functions

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## **Abstract**

This is an abstract about Rational Parking Functions



# Chapter 1

## The integer case

### 1.1 Parking Functions

**Definition 1** (Parking Function). A parking function is a sequence  $(a_1, a_2, \dots, a_n)$  such that its non-decreasing reordering  $(b_1, b_2, \dots, b_n)$  has  $b_i < i$  for all  $i$ .

We denote by  $\mathcal{PF}_n$  the set of parking functions of length  $n$ .

$$\mathcal{PF} = \bigcup_{n>0} \mathcal{PF}_n$$

.

**Example.**

$$f_1 = (7, 3, 1, 4, 2, 5, 2) \in \mathcal{PF}_7$$

$$f_2 = (7, 3, 1, 4, 2, 5, 4) \notin \mathcal{PF}_7$$

**Theorem 1.** Let  $pf_n$  be the cardinal of  $\mathcal{PF}_n$ . We have  $pf_n = (n+1)^{n-1}$ .

**Example** ( $n = 1, 2, 3$ ).

- $n = 1$  :  $pf_1 = 1$

(1)

- $n = 2$  :  $pf_2 = 3$

(1, 1)    (1, 2)    (2, 1)

- $n = 3$  :  $pf_3 = 16$

(1, 1, 1)	(1, 1, 2)	(1, 1, 3)	(1, 2, 1)	(1, 2, 2)	(1, 2, 3)	(1, 3, 1)
(1, 3, 2)	(2, 1, 1)	(2, 1, 2)	(2, 1, 3)	(2, 2, 1)	(2, 3, 1)	(3, 1, 1)
(3, 1, 2)	(3, 2, 1)					

### 1.1.1 Primitive parking functions

**Definition 2** (Primitive). A parking function  $(a_1, a_2, \dots, a_n)$  is said primitive if it is already in non-decreasing order.

We denote by  $\mathcal{PF}'_n$  the set of primitive parking functions of length  $n$ .

$$\mathcal{PF}' = \bigcup_{n \geq 0} \mathcal{PF}'_n$$

**Example.**

$$\begin{aligned} f_1 &= (1, 2, 2, 3) \in \mathcal{PF}'_4 \\ f_2 &= (1, 2, 3, 2) \notin \mathcal{PF}'_4, \text{ even though } f_2 \in \mathcal{PF}_4 \end{aligned}$$

**Theorem 2.** Let  $pf'_n$  be the cardinal of  $\mathcal{PF}'_n$ . We have  $pf'_n = \frac{1}{n+1} \binom{2n}{n}$ , which is the  $n^{\text{th}}$  Catalan number.

**Example** ( $n = 1, 2, 3$ ).

- $n = 1$  :  $pf'_1 = 1$   
(1)
- $n = 2$  :  $pf'_2 = 2$   
(1, 1)    (1, 2)
- $n = 3$  :  $pf'_3 = 5$   
(1, 1, 1)    (1, 1, 2)    (1, 1, 3)    (1, 2, 2)    (1, 2, 3)

### 1.1.2 The parking functions poset

**Definition 3** (Rank). Given  $f = (a_1, \dots, a_n) \in \mathcal{PF}_n$ , let

$$b_i = \begin{cases} 1 & \text{if } \exists j \mid a_j = i \\ 0 & \text{otherwise} \end{cases}$$

We define the rank of  $f$ , noted  $rk(f)$ , as

$$\sum_{1 \leq i \leq n} b_i$$

**Example.**

$$\begin{aligned}rk((1, 5, 4, 2, 3, 3, 1)) &= 5 \\rk((4, 7, 1, 1, 3, 2, 2, 8)) &= 6\end{aligned}$$

**Definition 4** ( $\succ_{pf}$ ). We say that  $f = (a_1, \dots, a_n)$  covers  $g = (b_1, \dots, gb_n)$ , written  $f \succ_{pf} g$ , if  $\exists i, j$  such that :

- $i < j$
- There is at least 1 occurrence of  $i$  in  $f$
- There is at least 1 occurrence of  $j$  in  $f$

$$b_k = \begin{cases} i & \text{if } a_k = j \\ a_k & \text{otherwise} \end{cases}$$

**Remark.** If  $f \succ_{pf} g$ , then  $rk(f) = rk(g) + 1$ .

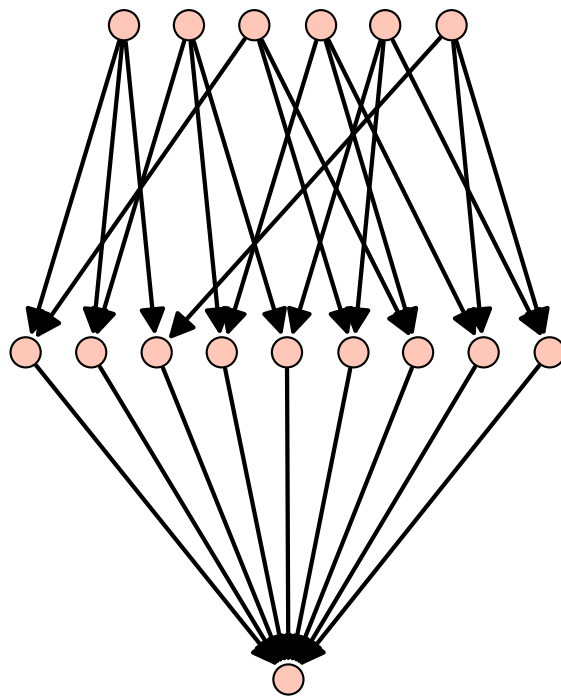
**Example.**

$$\begin{aligned}(1, 4, 2, 6, 1, 1, 3) &\succ_{pf} (1, 4, 2, 1, 1, 1, 3) \\(1, 4, 2, 6, 1, 1, 3) &\not\succ_{pf} (1, 4, 2, 5, 1, 1, 3)\end{aligned}$$

**Proposition.** This covering relation defines the poset of  $\mathcal{PF}_n$ .

**Remark.** The bottom element of this poset is  $(\underbrace{1, \dots, 1}_n)$ , and the top elements are the permutations of  $\{1, \dots, n\}$ .

**Example** (Shape of the poset of  $\mathcal{PF}_3$ ).



*This figure was generated with Sagemath. There are  $4^2 = 16$  elements in this poset.*

## 1.2 Non-crossing Partitions

**Definition 5** (Non-crossing Partition). *A non-crossing partition of a set  $E$  is a set partition  $P = \{E_1, E_2, \dots, E_k\}$  such that if  $a, c \in E_i$ ,  $b, d \in E_j$ , and  $i \neq j$ , then we do not have  $a < b < c < d$ , nor  $a > b > c > d$ .*

*We denote by  $\mathcal{NC}_n$  the set of non-crossing partitions of  $\{1, 2, \dots, n\}$ .*

$$\mathcal{NC} = \bigcup_{n \geq 0} \mathcal{NC}_n$$

From this point, we assume that every partition  $P = \{B_1, \dots, B_l\}$  is sorted such that :

- For each block  $B_i = \{b_1, \dots, b_k\} \in P$ ,  $b_1 < \dots < b_k$

- $\min(B_1) < \dots < \min(B_k)$

**Notation.**  $[n] = \{1, 2, \dots, n\}$

**Example** ( $E = [6]$ ).

$$P_1 = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\} \in \mathcal{NC}_6$$

$$P_2 = \{\{1, 2, 4\}, \{3, 5\}, \{6\}\} \notin \mathcal{NC}_6$$

**Theorem 3.** Let  $nc_n$  be the cardinal of  $\mathcal{NC}_n$ . We have  $nc_n = \frac{1}{n+1} \binom{2n}{n}$ , which is the  $n^{\text{th}}$  Catalan number.

**Example** ( $n = 1, 2, 3$ ).

- $n = 1$  :  $nc_1 = 1$   
 $\{\{1\}\}$
- $n = 2$  :  $nc_2 = 2$   
 $\{\{1, 2\}\} \quad \{\{1\}, \{2\}\}$
- $n = 3$  :  $nc_3 = 5$   
 $\{\{1, 2, 3\}\} \quad \{\{1\}, \{2, 3\}\} \quad \{\{1, 3\}, \{2\}\} \quad \{\{1, 2\}, \{3\}\} \quad \{\{1\}, \{2\}, \{3\}\}$

**Proposition.** This means we can create a bijection between  $\mathcal{PF}'_n$  and  $\mathcal{NC}_n$ .

- $\mathcal{NC}_n \rightarrow \mathcal{PF}'_n$  : For each block  $B$  in the non-crossing partition, take  $i = \min(B)$ , and  $k_i = \text{size}(B)$ .  
 $k_i = 0$  if  $i$  is not the minimum of a block.  
The corresponding parking function is  $(\underbrace{1, \dots, 1}_{k_1}, \underbrace{2, \dots, 2}_{k_2}, \dots, \underbrace{n, \dots, n}_{k_n})$ .

- $\mathcal{PF}'_n \rightarrow \mathcal{NC}_n$  : For each  $i$  in  $[n]$ , if  $i$  appears  $n_i$  times in the parking function,  $B_i$  will be of size  $n_i$  with minimum element  $i$ . There is a unique set partition  $P = \bigcup_i B_i$  of  $[n]$  respecting these conditions that is non-crossing.

**Example** ( $n = 6$ ).

$$P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\} \quad f = (1, 1, 1, 3, 3, 6)$$



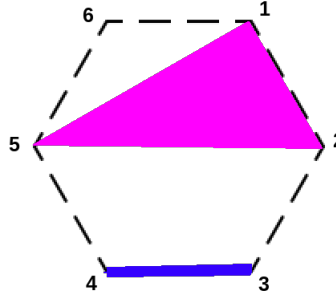
**Corollary.** *A non-crossing partition can be represented by the minimums and sizes of its blocks.*

**Example.**  $\{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$  can be represented by the following dictionary :

- 1 : 3
- 3 : 2
- 6 : 1

A non-crossing partition of  $[n]$  can be represented graphically on a regular  $n$ -vertices polygon, with vertices labeled from 1 to  $n$  clockwise. We then represent each block  $B = \{b_1, \dots, b_k\}$  by the convex hull of  $\{b_1, \dots, b_k\}$ .

**Example** ( $P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$ ).



Thus non-crossing meaning the hulls are *disjoint*.

### 1.2.1 The non-crossing partitions poset

**Definition 6** ( $\succ$ ). *We say that  $P$  covers  $Q$ , written  $P \succ Q$ , if  $\exists B_i, B_j \in P$  such that  $Q = P - \{B_i, B_j\} \cup \{B_i \cup B_j\}$*

**Example.**  $\{\{1, 6\}, \{2, 3\}, \{4, 5\}\} \succ \{\{1, 2, 3, 6\}, \{4, 5\}\}$

- $B_i = \{1, 6\}$
- $B_j = \{2, 3\}$

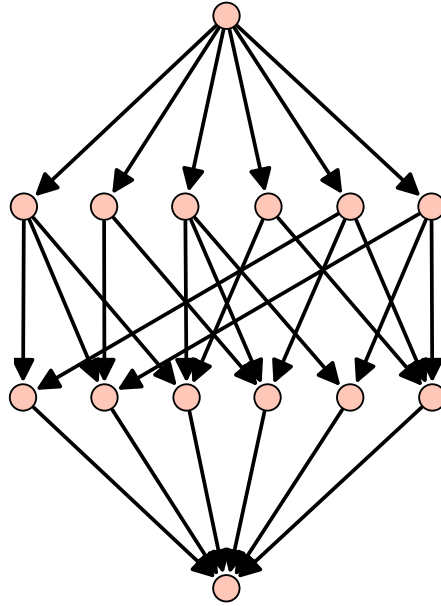
**Proposition.** *This covering relation defines the poset of  $\mathcal{NC}_n$ . We denote by  $\mathcal{NCC}_n$  the set of maximal chains in the poset of  $\mathcal{NC}_n$ .*

$$\mathcal{NCC} = \bigcup_{n \geq 0} \mathcal{NCC}_n$$

**Remark.** *The bottom element of this poset is  $\{\{1, \dots, n\}\}$ , and the top element is  $\{\{1\}, \dots, \{n\}\}$ .*

**Theorem 4.** *Let  $ncc_n$  be the cardinal of  $\mathcal{NCC}_n$ . We have  $ncc_n = n^{n-2}$ .*

**Example** (Shape of the poset of  $\mathcal{NC}_4$ ).



*This figure was generated with Sagemath. There are  $4^2 = 16$  different maximal chains, and  $\frac{1}{5} \binom{8}{4} = \frac{70}{5} = 14$  elements in this poset.*

### 1.2.2 Kreweras complement

**Definition 7** (Associated permutation). *The permutation  $\sigma$  associated to a non-crossing partition has a cycle  $(b_1, \dots, b_k)$  for each block  $B = \{b_1, \dots, b_k\}$  of the partition.*

**Example.** The permutation associated to  $\{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$  is  $(1\ 2\ 5)(3\ 4)(6) = 254316$ .

**Definition 8** (Kreweras complement). The Kreweras complement  $K(P)$  of a non-crossing partition  $P$  is defined as follows :

- Let  $\sigma$  be the permutation associated to  $P$
- Let  $\pi$  be the permutation  $(n\ n-1\ n-2\ \dots\ 3\ 2\ 1) = n123\dots n-1$
- $K(P)$  is the non-crossing partition associated to  $\pi\sigma$ .

**Example** ( $P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$ ).

- $\sigma = (1\ 2\ 5)(3\ 4)(6) = 254316$
- $\pi = (6\ 5\ 4\ 3\ 2\ 1) = 612345$
- $\pi\sigma = 143265 = (1)(2\ 4)(3)(5\ 6)$
- $K(P) = \{\{1\}, \{2, 4\}, \{3\}, \{5, 6\}\}$

**Proposition** (Kreweras minimums). Let  $P = \{B_1, \dots, B_k\}$  be a non-crossing partition. Let  $K(P) = \{B'_1, \dots, B'_l\}$  be its Kreweras complement. Then

$$\bigcup_{1 \leq i \leq l} \min(B'_i) = B_1 \cup \bigcup_{1 \leq j \leq k} B_j - \max(B_i)$$

**Example** ( $P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$ ).

- $K(P) = \{\{1\}, \{2, 4\}, \{3\}, \{5, 6\}\}$
- $\bigcup \min(B'_i) = \{1, 2, 3, 5\}$
- $B_1 \cup \bigcup B_i - \max(B_i) = \{1, 2, 5\} \cup \{3, 4\} - \{4\} \cup \{6\} - \{6\} = \{1, 2, 5\} \cup \{3\} \cup \emptyset = \{1, 2, 3, 5\}$

**Notation.**  $B_{[i]}$  = block containing  $i$ .

**Proposition** (Kreweras block sizes). Let  $P = \{B_1, \dots, B_k\}$  be a non-crossing partition. Let  $K(P) = \{B'_1, \dots, B'_l\}$  be its Kreweras complement. Then the size of the block  $B'_i$  is defined as follows :

- Let  $m_i$  be the  $i^{th}$  minimum of  $K(P)$
- Define a transition  $\phi(e)$  as  
Let  $j = e + 1$  (or 1 if  $e = n$ )  
 $\phi(e) = \max(B_{[j]})$
- The size of  $B'_i$  is  $k_{min}$  such that  $k_{min} = \min\{k > 0 \mid \phi^k(m_i) \in B_{[m_i]}\}$ .

**Example** ( $P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$ ).

- $mins = \{1, 2, 3, 5\}$
- $m_1 = 1$   
 $B_{[1]} = B_1$   
 $\max(B_{[2]} = \max(B_1) = 5$   
The size for  $m_1$  is 1.
- $m_2$   
 $B_{[2]} = B_1$   
 $\max(B_{[3]} = \max(B_2) = 4$   
 $\max(B_{[5]} = \max(B_1) = 5$   
The size for  $m_2$  is 2.
- $m_3 = 3$   
 $B_{[3]} = B_2$   
 $\max(B_{[4]} = \max(B_2) = 4$   
The size for  $m_3$  is 1.
- $m_4 = 5$   
 $B_{[5]} = B_1$   
 $\max(B_{[6]} = \max(B_3) = 6$   
 $\max(B_{[1]} = \max(B_1) = 5$   
The size for  $m_4$  is 2.

### 1.2.3 Action of $\mathfrak{S}_n$ on $\mathcal{NC}_n$

**Definition 9** (Action of  $\mathfrak{S}_n$ ). *The action of  $\mathfrak{S}_n$  on a non-crossing partition  $P = \{B_1, \dots, B_l\} \in \mathcal{NC}_n$  is defined by :*

- For each block  $B_i = \{b_1, \dots, b_k\} : \sigma(B_i) = \{\sigma(b_1), \dots, \sigma(b_k)\}$
- We denote  $\rho = \sigma(P) = \{\sigma(B_1), \dots, \sigma(B_l)\}$

**Example** ( $\sigma = 415362$ ).

$$\sigma(\{\{1, 6\}, \{2, 3, 5\}, \{4\}\}) = \{\{1, 5, 6\}, \{2, 4\}, \{3\}\}$$

**Remark.** Note that  $\sigma(P)$  is not necessarily non-crossing.

**Definition 10** (Rotation). *We define the rotation operator  $rot$  of  $P \in \mathcal{NC}_n$  as  $rot(P) = (1 \ 2 \ 3 \ \dots \ n)(P) = 23 \dots n1(P)$ .*

*Conversely, we define  $rot^{-1}$  of  $P$  as  $rot^{-1}(P) = (n \ n-1 \ \dots \ 3 \ 2 \ 1)(P) = n12 \dots n-1(P)$ .*

**Remark.**  $K(K(P)) = rot^{-1}(P)$ .

**Example** ( $P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$ ).

- $rot(P) = \{\{1, 2\}, \{3, 4, 6\}, \{5\}\}$
- $rot^{-1}(P) = \{\{1, 2, 4\}, \{3\}, \{5, 6\}\}$

## 1.3 Non-crossing 2-partitions

**Definition 11** (Non-crossing 2-partition). *A non-crossing 2-partition of a set  $E$  is a pair  $(P, \sigma)$  where :*

- $P$  is a non-crossing partition of  $E$
- $\sigma$  is a permutation of the elements of  $E$
- For each sorted block  $B_i = \{b_1, \dots, b_k\} \in P$ , we have  $\sigma(b_i) < \dots < \sigma(b_k)$

We denote by  $\mathcal{NC}_n^2$  the set of non-crossing 2-partitions of  $[n]$ .

$$\mathcal{NC}^2 = \bigcup_{n>0} \mathcal{NC}_n^2$$

**Example** ( $\mathcal{NC}_6^2$ ).  $P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\} \quad \sigma = 413265$   
 $\rho = \{\{1, 3, 6\}, \{2\}, \{4, 5\}\}$

**Theorem 5.** Let  $nc_n^2$  be the cardinal of  $\mathcal{NC}_n^2$ . We have  $nc_n^2 = (n+1)^{n-1}$ .

**Example** ( $n = 1, 2, 3$ ).

- $n = 1$  :  $nc_1^2 = 1$   
 $\{\{1\}\} \quad 1 \quad \rho = P$
- $n = 2$  :  $nc_2^2 = 3$   
 $\{\{1\}, \{2\}\} \quad 12 \quad \rho = P$   
 $\{\{1\}, \{2\}\} \quad 21 \quad \rho = P$   
 $\{\{1, 2\}\} \quad 12 \quad \rho = P$
- $n = 3$  :  $nc_3^2 = 16$   
 $\{\{1\}, \{2\}, \{3\}\} \quad 123 \quad \rho = P$   
 $\{\{1\}, \{2\}, \{3\}\} \quad 132 \quad \rho = P$   
 $\{\{1\}, \{2\}, \{3\}\} \quad 213 \quad \rho = P$   
 $\{\{1\}, \{2\}, \{3\}\} \quad 231 \quad \rho = P$   
 $\{\{1\}, \{2\}, \{3\}\} \quad 312 \quad \rho = P$   
 $\{\{1\}, \{2\}, \{3\}\} \quad 321 \quad \rho = P$   
 $\{\{1, 2\}, \{3\}\} \quad 123 \quad \rho = P$   
 $\{\{1, 2\}, \{3\}\} \quad 132 \quad \rho = \{\{1, 3\}, \{2\}\}$   
 $\{\{1, 2\}, \{3\}\} \quad 231 \quad \rho = \{\{1\}, \{2, 3\}\}$   
 $\{\{1\}, \{2, 3\}\} \quad 123 \quad \rho = P$   
 $\{\{1\}, \{2, 3\}\} \quad 213 \quad \rho = \{\{1, 3\}, \{2\}\}$   
 $\{\{1\}, \{2, 3\}\} \quad 312 \quad \rho = \{\{1, 2\}, \{3\}\}$   
 $\{\{1, 3\}, \{2\}\} \quad 123 \quad \rho = P$   
 $\{\{1, 3\}, \{2\}\} \quad 132 \quad \rho = \{\{1, 2\}, \{3\}\}$   
 $\{\{1, 3\}, \{2\}\} \quad 213 \quad \rho = \{\{1\}, \{2, 3\}\}$   
 $\{\{1, 2, 3\}\} \quad 123 \quad \rho = P$

**Proposition.** This means we can create a bijection between  $\mathcal{PF}_n$  and  $\mathcal{NC}_n^2$ .

- $\mathcal{PF}_n \rightarrow \mathcal{NC}_n^2$  : Let  $f = (a_1, \dots, a_n) \in \mathcal{PF}_n$  be our parking function. For  $i \in \{1, \dots, n\}$ , we define :

$l_i$  : the number of occurrences of  $i$  in  $f$ .

$im_i$  :  $\{j \mid a_j = i\}$

The corresponding non-crossing partition will have the following constraints :

For each  $i \in \{1, \dots, n\}$ , if  $l_i > 0$ , then there is a block  $B_{[i]}$  of length  $l_i$  with minimum element  $i$ .

$$\sigma(B_{[i]}) = im_i$$

There is a unique set partition  $P = \bigcup_i B_{[i]}$  of  $[n]$  and a unique permutation  $\sigma$  respecting these conditions such that  $(P, \sigma) \in \mathcal{NC}_n^2$ .

- $\mathcal{NC}_n^2 \rightarrow \mathcal{PF}_n$  : Let  $(P, \sigma)$  with  $P = \{B_1, \dots, B_l\}$  be our non-crossing 2-partition. For each block  $B_i = \{b_1, \dots, b_k\} \in P$  :

$$m_i = \min(B_i) = b_1$$

$$pos_i = \sigma(B_i)$$

For each  $j \in pos_i$ , we define  $a_j = m_i$

The corresponding parking function is  $(a_1, \dots, a_n)$ .

**Example** ( $n = 8$ ).

$$P = \{\{1, 2, 5\}, \{3, 4\}, \{6, 8\}, \{7\}\}$$

$$\sigma = 36187245$$

$$f = (3, 6, 1, 7, 6, 1, 1, 3)$$

### 1.3.1 The non-crossing 2-partitions poset

**Definition 12** ( $\succ^2$ ). We say that  $(P, \sigma)$  covers  $(Q, \tau)$ , written  $(P, \sigma) \succ^2 (Q, \tau)$ , if  $\exists B_i, B_j \in P$  such that

- $Q = P - \{B_i, B_j\} \cup \{B_i \cup B_j\}$
- $l \neq i, j b \in B_l \rightarrow \tau(b) = \sigma(b)$
- Let  $B_i \cup B_j = \{b_1, \dots, b_k\}$  :  
 $\tau(B_i \cup B_j) = \sigma(B_i \cup B_j)$   
 $\tau(b_1) < \dots < \tau(b_k)$

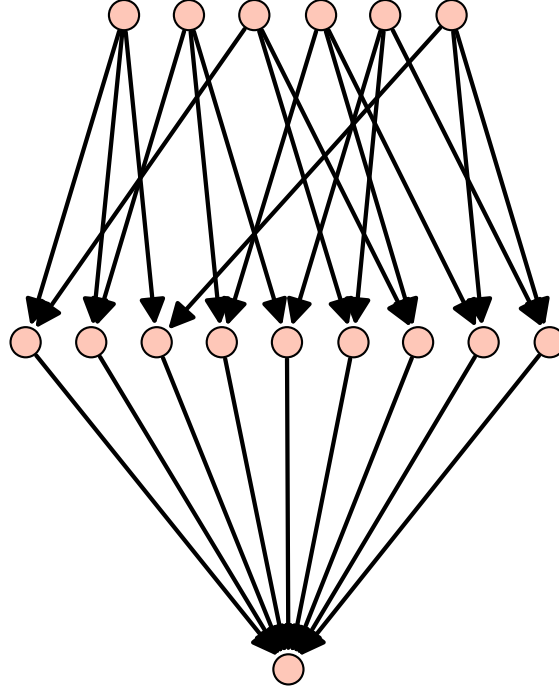
**Example.**

- $P = \{\{1, 6\}, \{2, 3\}, \{4\}, \{5\}\}$
- $\sigma = 236154$
- $Q = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$
- $\tau = 235164$
- $(P, \sigma) \succ^2 (Q, \tau)$
- $(P, \sigma) \not\succ^2 (Q, \sigma)$ , because  $\sigma(\{2, 3, 5\}) = \{3, 6, 5\}$  is not ordered.

**Proposition.** This covering relation defines the poset of  $\mathcal{NC}_n^2$ .

**Remark.** The bottom element of this poset is  $(\{\{1, \dots, n\}\}, 12 \dots n)$ , and the top elements are  $\{(\{\{1\}, \dots, \{n\}\}, \sigma) \mid \sigma \in \mathfrak{S}_n\}$ .

**Example** (Shape of the poset of  $\mathcal{NC}_3^2$ ).



*This figure was generated with Sagemath. There are  $4^2 = 16$  elements in this poset.*



**Remark.** *This poset is in bijection with the poset of  $\mathcal{PF}_n$ , and each can be obtained by applying the aforementioned bijection between  $\mathcal{NC}_n$  and  $\mathcal{PF}_n$  to the elements of the other :  $f \in \mathcal{PF}_n \succ_{pf} g \in \mathcal{PF}_n$  if and only if :*

- $(P, \sigma)$  is the non-crossing 2-partition associated to  $f$
- $(Q, \tau)$  is the non-crossing 2-partition associated to  $g$
- $(P, \sigma) \succ^2 (Q, \tau)$

**Example.**

- $P = \{\{1, 6\}, \{2, 3\}, \{4\}, \{5\}\}$
- $\sigma = 236154$
- $Q = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$
- $\tau = 235164$
- $f = (4, 1, 2, 1, 5, 2) \succ_{pf} g = (4, 1, 2, 1, 2, 2)$

## Chapter 2

### The rational case

# Chapter 3

## Trees