# Rational Parking Functions

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## Abstract

This is an abstract about Rational Parking Functions

# Chapter 1

# The integer case

## 1.1 Parking Functions

**Definition 1** (Parking Function)**.** *A* parking function *is a sequence* $(a_1, a_2, \ldots, a_n)$ *such that its non-decreasing reordering* $(b_1, b_2, \ldots, b_n)$ *has* $b_i < i$ *for all* $i$.
*We denote by* $\mathcal{PF}_n$ *the set of parking functions of length* $n$.

$$\mathcal{PF} = \bigcup_{n>0} \mathcal{PF}_n$$

.

**Example.**

$$f_1 = (7, 3, 1, 4, 2, 5, 2) \in \mathcal{PF}_7$$
$$f_2 = (7, 3, 1, 4, 2, 5, 4) \notin \mathcal{PF}_7$$

**Theorem 1.** *Let* $pf_n$ *be the cardinal of* $\mathcal{PF}_n$*. We have* $pf_n = (n+1)^{n-1}$.

**Example** $(n = 1, 2, 3)$**.**

• $n = 1 \quad : \quad pf_1 = 1$
  (1)
• $n = 2 \quad : \quad pf_2 = 3$
  (1, 1) (1, 2) (2, 1)
• $n = 3 \quad : \quad pf_3 = 16$
  (1, 1, 1) (1, 1, 2) (1, 1, 3) (1, 2, 1) (1, 2, 2) (1, 2, 3) (1, 3, 1)
  (1, 3, 2) (2, 1, 1) (2, 1, 2) (2, 1, 3) (2, 2, 1) (2, 3, 1) (3, 1, 1)
  (3, 1, 2) (3, 2, 1)

### 1.1.1 Primitive parking functions

**Definition 2** (Primitive)**.** *A parking function* $(a_1, a_2, \ldots, a_n)$ *is said* primitive *if it is already in non-decreasing order.*
*We denote by* $\mathcal{PF}'_n$ *the set of primitive parking functions of length* $n$.

$$\mathcal{PF}' = \bigcup_{n>0} \mathcal{PF}'_n$$

**Example.**

$$f_1 = (1, 2, 2, 3) \in \mathcal{PF}'_4$$
$$f_2 = (1, 2, 3, 2) \notin \mathcal{PF}'_4, \text{ even though } f_2 \in \mathcal{PF}_4$$

**Theorem 2.** *Let* $pf'_n$ *be the cardinal of* $\mathcal{PF}'_n$. *We have* $pf'_n = \frac{1}{n+1}\binom{2n}{n}$, *which is the* $n^{th}$ *Catalan number.*

**Example** $(n = 1, 2, 3)$**.**

- $n = 1 \quad : \quad pf'_1 = 1$
  (1)
- $n = 2 \quad : \quad pf'_2 = 2$
  (1, 1) (1, 2)
- $n = 3 \quad : \quad pf'_3 = 5$
  (1, 1, 1) (1, 1, 2) (1, 1, 3) (1, 2, 2) (1, 2, 3)

## 1.2 Non-crossing Partitions

**Definition 3** (Non-crossing Partition)**.** *A* non-crossing partition *of a set* $E$ *is a set partition* $P = \{E_1, E_2, \ldots, E_k\}$ *such that if* $a, c \in E_i$, $b, d \in E_j$, *and* $i \neq j$, *then we do* not *have* $a < b < c < d$, *nor* $a > b > c > d$.
*We denote by* $\mathcal{NC}_n$ *the set of non-crossing partitions of* $\{1, 2, \ldots, n\}$.

$$\mathcal{NC} = \bigcup_{n>0} \mathcal{NC}_n$$

From this point, we assume that every partition $P = \{B_1, \ldots, B_l\}$ is *sorted* such that :

- For each block $B_i = \{b_1, \ldots, b_k\} \in P$, $b_1 < \ldots < b_k$
- $min(B_1) < \ldots < min(B_k)$

**Notation.** $[n] = \{1, 2, \ldots, n\}$

**Example** $(E = [6])$**.**

$$P_1 = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\} \in \mathcal{NC}_6$$
$$P_2 = \{\{1, 2, 4\}, \{3, 5\}, \{6\}\} \notin \mathcal{NC}_6$$

**Theorem 3.** *Let $nc_n$ be the cardinal of $\mathcal{NC}_n$. We have $nc_n = \frac{1}{n+1}\binom{2n}{n}$, which is the $n^{th}$ Catalan number.*

**Example** $(n = 1, 2, 3)$**.**
- $n = 1 \quad : \quad nc_1 = 1$
  $\{\{1\}\}$
- $n = 2 \quad : \quad nc_2 = 2$
  $\{\{1, 2\}\} \qquad \{\{1\}, \{2\}\}$
- $n = 3 \quad : \quad nc_3 = 5$
  $\{\{1, 2, 3\}\} \qquad \{\{1\}, \{2, 3\}\} \qquad \{\{1, 3\}, \{2\}\} \qquad \{\{1, 2\}, \{3\}\} \qquad \{\{1\}, \{2\}, \{3\}\}$

**Proposition.** *This means we can create a* bijection *between $\mathcal{PF'}_n$ and $\mathcal{NC}_n$.*

- $\mathcal{NC}_n \to \mathcal{PF'}_n$ : For each block $B$ in the non-crossing partition, take $i = min(B)$, and $k_i = size(B)$.
  $k_i = 0$ if $i$ is not the minimum of a block.
  The corresponding parking function is $(\underbrace{1, \ldots, 1}_{k_1}, \underbrace{2, \ldots, 2}_{k_2}, \ldots, \underbrace{n, \ldots, n}_{k_n})$.

- $\mathcal{PF'}_n \to \mathcal{NC}_n$ : For each $i$ in $[n]$, if $i$ appears $n_i$ times in the parking function, $B_i$ will be of size $n_i$ with minimum element $i$. There is a unique set partition $P = \bigcup_i B_i$ of $[n]$ respecting these conditions that is non-crossing.

**Example** $(n = 6)$**.**

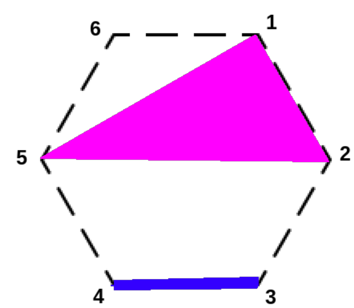
$$P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\} \qquad\qquad f = (1, 1, 1, 3, 3, 6)$$

**Corollary.** *A non-crossing partition can be represented by the minimums and sizes of its blocks.*

**Example.** $\{\{1,2,5\},\{3,4\},\{6\}\}$ *can be represented by the following dictionnary :*
- *1 : 3*
- *3 : 2*
- *6 : 1*

A non-crossing partition of $[n]$ can be represented graphically on a regular $n$-vertices polygon, with vertices labeled from 1 to $n$ clockwise. We then represent each block $B = \{b_1, \ldots, b_k\}$ by the convex hull of $\{b_1, \ldots, b_k\}$.

**Example** ($P = \{\{1,2,5\},\{3,4\},\{6\}\}$).




Thus non-crossing meaning the hulls are *disjoint*.

### 1.2.1 The non-crossing partitions poset

**Definition 4** ($\succ$)**.** *We say that $P$ covers $Q$, written $P \succ Q$, if $\exists B_i, B_j \in P$ such that $Q = P - \{B_i, B_j\} \cup \{B_i \cup B_j\}$*

**Example.** $\{\{1,6\},\{2,3\},\{4,5\}\} \succ \{\{1,2,3,6\},\{4,5\}\}$
- $B_i = \{1,6\}$
- $B_j = \{2,3\}$

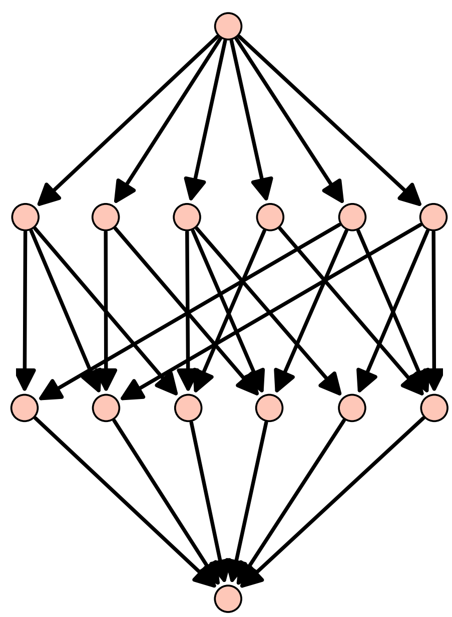
**Proposition.** *This covering relation defines the* poset *of $\mathcal{NC}_n$. We denote by $\mathcal{NCC}_n$ the set of* maximal chains *in the poset of $\mathcal{NC}_n$.*

$$\mathcal{NCC} = \bigcup_{n>0} \mathcal{NCC}_n$$

**Remark.** *The bottom element of this poset is $\{\{1,\ldots,n\}\}$, and the top element is $\{\{1\},\ldots,\{n\}\}$.*

**Theorem 4.** *Let $ncc_n$ be the cardinal of $\mathcal{NCC}_n$. We have $ncc_n = n^{n-2}$.*

**Example** (Shape of the poset of $\mathcal{NC}_4$)**.**



*This figure was generated with Sagemath. There are $4^2 = 16$ different maximal chains, and $\frac{1}{5}\binom{8}{4} = \frac{70}{5} = 14$ elements in this poset.*

### 1.2.2 Kreweras complement

**Definition 5** (Associated Permutation)**.** *The* permutation $\sigma$ *associated to a non-crossing partition has a cycle $(b_1,\ldots,b_k)$ for each block $B = \{b_1,\ldots,b_k\}$ of the partition.*

**Example.** *The permutation associated to* $\{\{1,2,5\},\{3,4\},\{6\}\}$ *is* $(1\ 2\ 5)\ (3\ 4)\ (6) = 254316$.

**Definition 6** (Kreweras Complement)**.** *The* Kreweras complement $K(P)$ *of a non-crossing partition* $P$ *is defined as follows :*
- *Let* $\sigma$ *be the permutation associated to* $P$
- *Let* $\pi$ *be the permutation* $(n\ n-1\ n-2\ \dots\ 3\ 2\ 1) = n123\dots n-1$
- $K(P)$ *is the* non-crossing partition *associated to* $\pi\sigma$*.*

**Example** $(P = \{\{1,2,5\},\{3,4\},\{6\}\})$**.**
- $\sigma = (1\ 2\ 5)\ (3\ 4)\ (6) = 254316$
- $\pi = (6\ 5\ 4\ 3\ 2\ 1) = 612345$
- $\pi\sigma = 143265 = (1)\ (2\ 4)\ (3)\ (5\ 6)$
- $K(P) = \{\{1\},\{2,4\},\{3\},\{5,6\}\}$

**Proposition** (Kreweras minimums)**.** *Let* $P = \{B_1,\dots,B_k\}$ *be a non-crossing partition. Let* $K(P) = \{B'_1,\dots,B'_l\}$ *be its Kreweras complement. Then*

$$\bigcup_{1\le i\le l} min(B'_i) = B_1 \cup \bigcup_{1<j\le k} B_i - max(B_i)$$

**Example** $(P = \{\{1,2,5\},\{3,4\},\{6\}\})$**.**

- $K(P) = \{\{1\},\{2,4\},\{3\},\{5,6\}\}$
- $\bigcup min(B'_i) = \{1,2,3,5\}$
- $B_1 \cup \bigcup B_i - max(B_i) = \{1,2,5\}\cup\{3,4\}-\{4\}\cup\{6\}-\{6\} = \{1,2,5\}\cup\{3\}\cup\emptyset = \{1,2,3,5\}$

**Notation.** $B_{[i]}$ = *block containing* $i$*.*

**Proposition** (Kreweras block sizes)**.** *Let* $P = \{B_1,\dots,B_k\}$ *be a non-crossing partition. Let* $K(P) = \{B'_1,\dots,B'_l\}$ *be its Kreweras complement. Then the size of the block* $B'_i$ *is defined as follows :*

- *Let* $m_i$ *be the the* $i^{th}$ *minimum of* $K(P)$
- *Define a* transition $\phi(e)$ *as*

  *Let* $j = e + 1$ *(or* $1$ *if* $e = n$*)*

  $\phi(e) = max(B_{[j]})$
- *The size of* $B'_i$ *is* $k_{min}$ *such that* $k_{min} = min\{k > 0 \mid \phi^k(m_i) \in B_{[m_i]}\}$*.*

**Example** ($P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$)**.**

- $mins = \{1, 2, 3, 5\}$
- $m_1 = 1$

  $B_{[1]} = B_1$

  $max(B_{[2]} = max(B_1) = 5$

  *The size for* $m_1$ *is* $1$*.*
- $m_2$

  $B_{[2]} = B_1$

  $max(B_{[3]}) = max(B_2) = 4$

  $max(B_{[5]}) = max(B_1) = 5$

  *The size for* $m_2$ *is* $2$*.*
- $m_3 = 3$

  $B_{[3]} = B_2$

  $max(B_{[4]}) = max(B_2) = 4$

  *The size for* $m_3$ *is* $1$*.*
- $m_4 = 5$

  $B_{[5]} = B_1$

  $max(B_{[6]}) = max(B_3) = 6$

  $max(B_{[1]}) = max(B_1) = 5$

  *The size for* $m_4$ *is* $2$*.*

### 1.2.3 Action of $\mathfrak{S}_n$ on $\mathcal{NC}_n$

**Definition 7** (Action of $\mathfrak{S}_n$)**.** *The action of $\mathfrak{S}_n$ on a non-crossing partition $P = \{B_1, \ldots, B_l\} \in \mathcal{NC}_n$ is defined by :*
• *For each block $B_i = \{b_1, \ldots, b_k\}$ : $\sigma(Bi) = \{\sigma(b_1), \ldots, \sigma(b_k)\}$*
• *We denote $\rho = \sigma(P) = \{\sigma(B_1), \ldots, \sigma(B_l)\}$*

**Example** ($\sigma = 415362$)**.**
$\sigma(\{\{1,6\},\{2,3,5\},\{4\}\}) = \{\{1,5,6\},\{2,4\},\{3\}\}$

**Remark.** *Note that $\sigma(P)$ is* not *necessarily non-crossing.*

**Definition 8** (Rotation)**.** *We define the* rotation operator *rot of $P \in \mathcal{NC}_n$ as $rot(P) = (1\ 2\ 3\ \ \ldots\ n)(P) = 23 \ldots n1(P)$.*
*Conversely, we define $rot^{-1}$ of $P$ as $rot^{-1}(P) = (n\ n-1\ \ \ldots 3\ 2\ 1)(P) = n12 \ldots n-1(P)$.*

**Remark.** $K(K(P)) = rot^{-1}(P)$.

**Example** ($P = \{\{1,6\},\{2,3,5\},\{4\}\}$)**.**

- $rot(P) = \{\{1,2\},\{3,4,6\},\{5\}\}$
- $rot^{-1}(P) = \{\{1,2,4\},\{3\},\{5,6\}\}$

## 1.3 Non-crossing 2-partitions

**Definition 9** (Non-crossing 2-partition)**.** *A* non-crossing 2-partition *of a set $E$ is a pair $(P, \sigma)$ where :*
• *$P$ is a non-crossing partition of $E$*
• *$\sigma$ is a permutation of the elements of $E$*
• *For each* sorted *block $B_i = \{b_1, \ldots, b_k\} \in P$, we have $\sigma(b_i) < \ldots < \sigma(b_k)$*

*We denote by $\mathcal{NC}_n^2$ the set of non-crossing 2-partitions of $[n]$.*

$$\mathcal{NC}^2 = \bigcup_{n>0} \mathcal{NC}_n^2$$

.

**Example** ($\mathcal{NC}_6^2$)**.** $P = \{\{1,6\},\{2,3,5\},\{4\}\}$ $\sigma = 413265$
$\rho = \{\{1,3,6\},\{2\},\{4,5\}\}$

**Theorem 5.** *Let $nc_n^2$ be the cardinal of $\mathcal{NC}_n^2$. We have $nc_n^2 = (n+1)^{n-1}$.*

**Example** $(n = 1, 2, 3)$**.**

- $n = 1 \quad : \quad nc_1^2 = 1$

| | | |
|---|---|---|
| $\{\{1\}\}$ | $1$ | $\rho = P$ |

- $n = 2 \quad : \quad nc_2^2 = 3$

| | | |
|---|---|---|
| $\{\{1\},\{2\}\}$ | $12$ | $\rho = P$ |
| $\{\{1\},\{2\}\}$ | $21$ | $\rho = P$ |
| $\{\{1,2\}\}$ | $12$ | $\rho = P$ |

- $n = 3 \quad : \quad nc_3^2 = 16$

| | | |
|---|---|---|
| $\{\{1\},\{2\},\{3\}\}$ | $123$ | $\rho = P$ |
| $\{\{1\},\{2\},\{3\}\}$ | $132$ | $\rho = P$ |
| $\{\{1\},\{2\},\{3\}\}$ | $213$ | $\rho = P$ |
| $\{\{1\},\{2\},\{3\}\}$ | $231$ | $\rho = P$ |
| $\{\{1\},\{2\},\{3\}\}$ | $312$ | $\rho = P$ |
| $\{\{1\},\{2\},\{3\}\}$ | $321$ | $\rho = P$ |
| $\{\{1,2\},\{3\}\}$ | $123$ | $\rho = P$ |
| $\{\{1,2\},\{3\}\}$ | $132$ | $\rho = \{\{1,3\},\{2\}\}$ |
| $\{\{1,2\},\{3\}\}$ | $231$ | $\rho = \{\{1\},\{2,3\}\}$ |
| $\{\{1\},\{2,3\}\}$ | $123$ | $\rho = P$ |
| $\{\{1\},\{2,3\}\}$ | $213$ | $\rho = \{\{1,3\},\{2\}\}$ |
| $\{\{1\},\{2,3\}\}$ | $312$ | $\rho = \{\{1,2\},\{3\}\}$ |
| $\{\{1,3\},\{2\}\}$ | $123$ | $\rho = P$ |
| $\{\{1,3\},\{2\}\}$ | $132$ | $\rho = \{\{1,2\},\{3\}\}$ |
| $\{\{1,3\},\{2\}\}$ | $213$ | $\rho = \{\{1\},\{2,3\}\}$ |
| $\{\{1,2,3\}\}$ | $123$ | $\rho = P$ |

**Proposition.** *This means we can create a* bijection *between $\mathcal{PF}_n$ and $\mathcal{NC}_n^2$.*

- $\mathcal{PF}_n \to \mathcal{NC}_n^2$ : Let $f = (a_1, \ldots, a_n) \in \mathcal{PF}_n$ be our parking function. For $i \in \{1, \ldots, n\}$, we define :

  $l_i$ : the number of occurences of $i$ in $f$.

  $im_i$ : $\{j \mid a_j = i\}$

  The corresponding non-crossing partition will have the following constraints :

  For each $i \in \{1, \ldots, n\}$, if $l_i > 0$, then there is a block $B_{[i]}$ of length $l_i$ with minimum element $i$.

  $\sigma(B_{[i]}) = im_i$

  There is a unique set partition $P = \bigcup_i B_{[i]}$ of $[n]$ and a unique permutation $\sigma$ respecting these conditions such that $(P, \sigma) \in \mathcal{NC}_n^2$.

- $\mathcal{NC}_n^2 \to \mathcal{PF}_n$ : Let $(P, \sigma)$ with $P = \{B_1, \ldots, B_l\}$ be our non-crossing 2-partition. For each block $B_i = \{b_1, \ldots, b_k\} \in P$ :

  $m_i = min(B_i) = b_1$

  $pos_i = \sigma(B_i)$

  For each $j \in pos_i$, we define $a_j = m_i$

  The corresponding parking function is $(a_1, \ldots, a_n)$.

**Example** $(n = 8)$**.**

$$P = \{\{1, 2, 5\}, \{3, 4\}, \{6, 8\}, \{7\}\}$$
$$\sigma = 36187245$$
$$f = (3, 6, 1, 7, 6, 1, 1, 3)$$

### 1.3.1 The non-crossing 2-partitions poset

**Definition 10** $(\succ^2)$**.** *We say that $(P, \sigma)$ covers $(Q, \tau)$, written $(P, \sigma) \succ^2 (Q, \tau)$, if $\exists B_i, B_j \in P$ such that*

- $Q = P - \{B_i, B_j\} \cup \{B_i \cup B_j\}$
- $l \neq i, jb \in B_l \to \tau(b) = \sigma(b)$
- *Let* $B_i \cup B_j = \{b_1, \ldots, b_k\}$ *:*

  $\tau(B_i \cup B_j) = \sigma(B_i \cup B_j)$

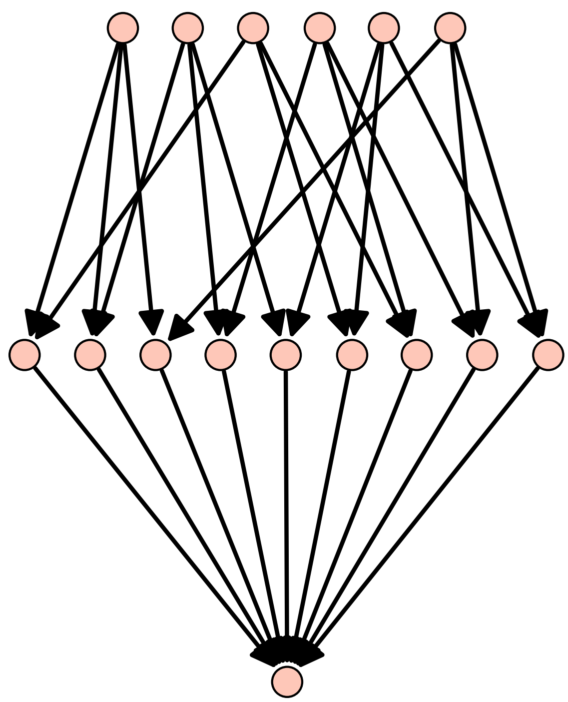
  $\tau(b_1) < \ldots < \tau(b_k)$

**Example.**
- $P = \{\{1,6\},\{2,3\},\{4\},\{5\}\}$
- $\sigma = 236154$
- $Q = \{\{1,6\},\{2,3,5\},\{4\}\}$
- $\tau = 235164$
- $(P,\sigma) \succ^2 (Q,\tau)$
- $(P,\sigma) \not\succ^2 (Q,\sigma)$, *because* $\sigma(\{2,3,5\}) = \{3,6,5\}$ *is* not *ordered.*

**Proposition.** *This covering relation defines the* poset *of* $\mathcal{NC}_n^2$.

**Remark.** *The bottom element of this poset is* $(\{\{1,\dots,n\}\}, 12\dots n)$*, and the top elements are* $\{(\{\{1\},\dots,\{n\}\},\sigma) \mid \sigma \in \mathfrak{S}_n\}$.

**Example** (Shape of the poset of $\mathcal{NC}_3^2$)**.**



*This figure was generated with Sagemath. There are* $4^2 = 16$ *elements in this poset.*

### 1.3.2 The parking functions poset

**Definition 11** (Rank)**.** *Given* $f = (a_1, \ldots, a_n) \in \mathcal{PF}_n$*, let*

$$b_i = \begin{cases} 1 \text{ if } \exists j \mid a_j = i \\ 0 \text{ otherwise} \end{cases}$$

*We define the* rank *of* $f$*, noted* $rk(f)$*, as*

$$\sum_{1 \leq i \leq n} b_i$$

**Example.**

$$rk((1,5,4,2,3,3,1)) = 5$$
$$rk((4,7,1,1,3,2,2,8)) = 6$$

**Definition 12** ($\succ_{pf}$)**.** *Since* $\mathcal{PF}_n$ *and* $\mathcal{NC}_n^2$ *are in bijection, we can define a* covering relation $\succ_{pf}$ *for* $\mathcal{PF}_n$ *as follows :*
$f \in \mathcal{PF}_n \succ_{pf} g \in \mathcal{PF}_n$ *if and only if :*

- $(P, \sigma)$ *is the non-crossing 2-partition associated to* $f$
- $(Q, \tau)$ *is the non-crossing 2-partition associated to* $g$
- $(P, \sigma) \succ^2 (Q, \tau)$

**Example.**
• $P = \{\{1,6\},\{2,3\},\{4\},\{5\}\}$
• $\sigma = 236154$
• $Q = \{\{1,6\},\{2,3,5\},\{4\}\}$
• $\tau = 235164$
• $f = (4,1,2,1,5,2) \succ_{pf} g = (4,1,2,1,2,2)$

**Remark.** *If* $f \succ_{pf} g$*, then* $rk(f) = rk(g) + 1$*, and there exists* $i$ *and* $j$ *such that :*

- $i < j$

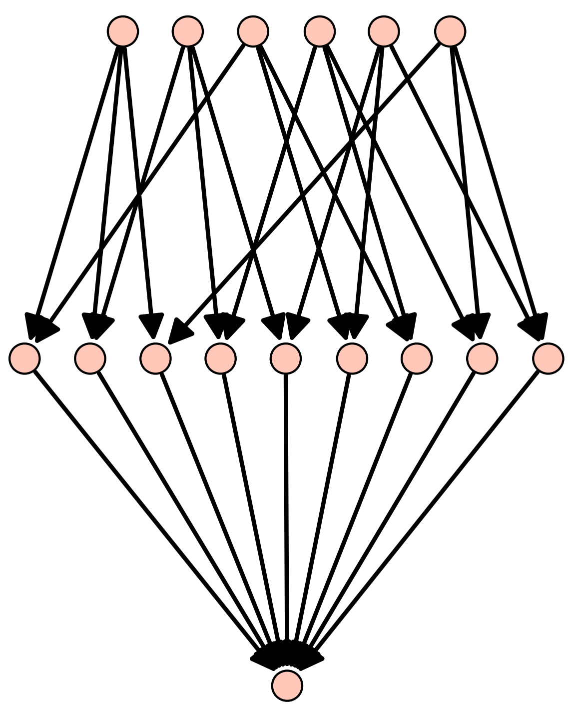
- *There is at least* 1 *occurence of* $i$ *in* $f$
- *There is at least* 1 *occurence of* $j$ *in* $f$

$$b_k = \begin{cases} i \text{ if } a_k = j \\ a_k \text{ otherwise} \end{cases}$$

**Proposition.** *This covering relation defines the* poset *of* $\mathcal{PF}_n$.

**Remark.** *The bottom element of this poset is* $(\underbrace{1, \ldots, 1}_{n})$*, and the top elements are the* permutations *of* $\{1, \ldots, n\}$.

**Example** (Shape of the poset of $\mathcal{PF}_3$)**.**



*This figure was generated with Sagemath. There are* $4^2 = 16$ *elements in this poset.*

# Chapter 2

# The rational case

For the whole chapter, we will consider 2 *coprime* integers $a$ and $b$ (meaning $a$ and $b$ have 1 as their greatest common divisor).

## 2.1 Rational Parking Functions

**Definition 13** (a, b - Parking Function)**.** *An* a, b - parking function *is a sequence* $(a_1, a_2, \dots, a_n)$ *such that :*
• $n = a$
• *its non-decreasing reordering* $(b_1, b_2, \dots, b_n)$ *has* $b_i < \frac{b}{a}(i-1)+1$ *for all* $i$*.*

*We denote by* $\mathcal{PF}_a^b$ *the set of a, b - parking functions.*

**Example.**

- *Ex. 1 :* $a > b$

  $a = 7$

  $b = 3$

  *Limits of the non-decreasing reordering of any* $f \in \mathcal{PF}_7^3$ *:*

  $[1,\ 1\frac{3}{7},\ 1\frac{6}{7},\ 2\frac{2}{7},\ 2\frac{5}{7},\ 3\frac{1}{7},\ 3\frac{4}{7}]$

  $f_1 = (2,1,1,3,2,3,1) \in \mathcal{PF}_7^3$

  $f_2 = (2,1,2,3,2,3,1) \notin \mathcal{PF}_7^3$*, even though* $f_2 \in \mathcal{PF}_7$

- *Ex. 2 :* $a < b$

$a = 5$

$b = 7$

*Limits of the non-decreasing reordering of any* $f \in \mathcal{PF}_5^7$ *:*

$[1,\ 2\frac{2}{5},\ 3\frac{4}{5},\ 5\frac{1}{5},\ 6\frac{3}{5}]$

$f_3 = (6, 3, 5, 1, 2) \in \mathcal{PF}_5^7$, *even though* $f_3 \notin \mathcal{PF}_5$

$f_4 = (6, 3, 5, 1, 3) \notin \mathcal{PF}_5^7$

**Theorem 6.** *Let* $pf_a^b$ *be the cardinal of* $\mathcal{PF}_a^b$*. We have* $pf_a^b = b^{a-1}$.

**Example** $(a = 3, b = 5)$**.** $\bullet$ $pf_a^b = 25$ $\bullet$ *Limits :* $[1,\ 2\frac{2}{3},\ 4\frac{1}{3}]$

| | | | | | | |
|---|---|---|---|---|---|---|
| $(1,1,1)$ | $(1,1,2)$ | $(1,1,3)$ | $(1,1,4)$ | $(1,2,1)$ | $(1,2,2)$ | $(1,2,3)$ |
| $(1,2,4)$ | $(1,3,1)$ | $(1,3,2)$ | $(1,4,1)$ | $(1,4,2)$ | $(2,1,1)$ | $(2,1,2)$ |
| $(2,1,3)$ | $(2,1,4)$ | $(2,2,1)$ | $(2,3,1)$ | $(2,4,1)$ | $(3,1,1)$ | $(3,1,2)$ |
| $(3,2,1)$ | $(4,1,1)$ | $(4,1,2)$ | $(4,2,1)$ | | | |

# Chapter 3

# Trees