

(Rational) Parking Functions

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Abstract

Introduced in 1966 by Konheim and Weiss [1], a (classical) Parking Function is a combinatorial structure represented by an integer sequence (a_1, \dots, a_n) such that $\#\{i \mid a_i \leq k\} \geq k \ \forall k$.

In this article, we will address bijections and posets on four types of parking functions : classical (also called *integer*), primitive classical, rational, and primitive rational. Those four types are defined throughout the article.

Each of these types has been proved to be counted by a well-known number – respectively $(n+1)^{n-1}$, Cat_n , b^{a-1} , and $Cat_{a,b}$. Nevertheless, to the best of our knowledge, the posets we will present here have not been exploited on parking functions specifically.

Furthermore, Theorem 8 and its equivalent Conjecture for classical parking functions reveal interesting results on the number of intervals in those posets.

This report goes hand in hand with a Sagemath implementation. Notions found in [2], [3], [4], [5], [6], [7], [8], and [9] are encoded alongside the aforementioned posets. All of the source code and premade tests can be found here ¹.

Finally, The notion of Parking Trees defined in [9] is extended to the rational case, hence creating a bijection between rational parking functions and Rational Parking Trees.

¹github.com/tessalsifi/ParkingFunctions

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Chapter 1

The integer case

1.1 Parking Functions

Definition 1 (Parking Function). *A parking function is a sequence of positive integers (a_1, a_2, \dots, a_n) such that its non-decreasing reordering (b_1, b_2, \dots, b_n) has $b_i \leq i$ for all i . In other words, $\#\{i \mid a_i \leq k\} \geq k \ \forall k$.*

We denote by \mathcal{PF}_n the set of parking functions of length n .

$$\mathcal{PF} = \bigcup_{n \geq 0} \mathcal{PF}_n$$

.

Example.

$$f_1 = (7, 3, 1, 4, 2, 5, 2) \in \mathcal{PF}_7$$

$$f_2 = (7, 3, 1, 4, 2, 5, 4) \notin \mathcal{PF}_7$$

Theorem 1 (Konheim and Weiss, 1966). *Let pf_n be the cardinal of \mathcal{PF}_n . We have*

$$pf_n = (n + 1)^{n-1}$$

.

Example ($n = 1, 2, 3$).

- $n = 1 \quad : \quad pf_1 = 1$

- (1)
- $n = 2$: $pf_2 = 3$
 - (1, 1) (1, 2) (2, 1)
- $n = 3$: $pf_3 = 16$
 - (1, 1, 1) (1, 1, 2) (1, 1, 3) (1, 2, 1) (1, 2, 2) (1, 2, 3) (1, 3, 1)
 - (1, 3, 2) (2, 1, 1) (2, 1, 2) (2, 1, 3) (2, 2, 1) (2, 3, 1) (3, 1, 1)
 - (3, 1, 2) (3, 2, 1)

1.1.1 Primitive parking functions

Definition 2 (Primitive). *A parking function (a_1, a_2, \dots, a_n) is said primitive if it is already in non-decreasing order.*

We denote by \mathcal{PF}'_n the set of primitive parking functions of length n .

$$\mathcal{PF}' = \bigcup_{n \geq 0} \mathcal{PF}'_n$$

Example.

$$\begin{aligned} f_1 &= (1, 2, 2, 3) \in \mathcal{PF}'_4 \\ f_2 &= (1, 2, 3, 2) \notin \mathcal{PF}'_4, \text{ even though } f_2 \in \mathcal{PF}_4 \end{aligned}$$

Theorem 2 (Stanley, 1999). *Let pf'_n be the cardinal of \mathcal{PF}'_n . We have*

$$pf'_n = \frac{1}{n+1} \binom{2n}{n}$$

which is the n^{th} Catalan number $Cat(n)$.

Example ($n = 1, 2, 3$).

- $n = 1$: $pf'_1 = 1$
 - (1)
- $n = 2$: $pf'_2 = 2$
 - (1, 1) (1, 2)
- $n = 3$: $pf'_3 = 5$
 - (1, 1, 1) (1, 1, 2) (1, 1, 3) (1, 2, 2) (1, 2, 3)

As comparing two parking functions can be tricky, one could wonder whether there exists an other combinatorial structure with a well-defined order that is in bijection with \mathcal{PF}_n or \mathcal{PF}'_n . The next section will present such a candidate.

1.2 Non-crossing Partitions

Definition 3 (Non-crossing Partition). *A non-crossing partition of a totally ordered set E is a set partition $P = \{E_1, E_2, \dots, E_k\}$ such that if $a, c \in E_i$, $b, d \in E_j$, and $i \neq j$, then we do not have $a < b < c < d$, nor $a > b > c > d$.*

We denote by \mathcal{NC}_n the set of non-crossing partitions of $\{1, 2, \dots, n\}$.

$$\mathcal{NC} = \bigcup_{n \geq 0} \mathcal{NC}_n$$

From this point, we assume that every partition $P = \{B_1, \dots, B_l\}$ is *sorted* such that :

- For each block $B_i = \{b_1, \dots, b_k\} \in P$, $b_1 < \dots < b_k$
- $\min(B_1) < \dots < \min(B_k)$

Notation. $[n] = \{1, 2, \dots, n\}$

Example ($E = [6]$).

$$\begin{aligned} P_1 &= \{\{1, 2, 5\}, \{3, 4\}, \{6\}\} \in \mathcal{NC}_6 \\ P_2 &= \{\{1, 2, 4\}, \{3, 5\}, \{6\}\} \notin \mathcal{NC}_6 \end{aligned}$$

Theorem 3 (Kreweras, 1972). *Let nc_n be the cardinal of \mathcal{NC}_n . We have*

$$nc_n = \frac{1}{n+1} \binom{2n}{n}$$

which is, again, the n^{th} Catalan number $Cat(n)$.

Example ($n = 1, 2, 3$).

- $n = 1$: $nc_1 = 1$
 $\{\{1\}\}$
- $n = 2$: $nc_2 = 2$

$$\begin{array}{l}
\{\{1, 2\}\} \quad \{\{1\}, \{2\}\} \\
\bullet \ n = 3 \ : \ nc_3 = 5 \\
\{\{1, 2, 3\}\} \quad \{\{1\}, \{2, 3\}\} \quad \{\{1, 3\}, \{2\}\} \quad \{\{1, 2\}, \{3\}\} \quad \{\{1\}, \{2\}, \{3\}\}
\end{array}$$

Proposition. *This means we can create a bijection between \mathcal{PF}'_n and \mathcal{NC}_n .*

Proof.

- $\mathcal{NC}_n \rightarrow \mathcal{PF}'_n$: For each block B in the non-crossing partition, take $i = \min(B)$, and let $k_i = \text{size}(B)$.
 $k_i = 0$ if i is not the minimum of a block.
The corresponding parking function is $(\underbrace{1, \dots, 1}_{k_1}, \underbrace{2, \dots, 2}_{k_2}, \dots, \underbrace{n, \dots, n}_{k_n})$.
- $\mathcal{PF}'_n \rightarrow \mathcal{NC}_n$: For each i in $[n]$, if i appears n_i times in the parking function, B_i will be of size n_i with minimum element i . There is a unique set partition $P = \bigcup_i B_i$ of $[n]$ respecting these conditions that is non-crossing : for each minimum i in *decreasing order*, add the n_i first free elements of $[i + 1, i + 2, \dots, n, 1, \dots, i - 1]$ to B_i .

□

Example ($n = 6$).

$$P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\} \quad f = (1, 1, 1, 3, 3, 6)$$

From the construction of the proof, we can deduce the following corollary:

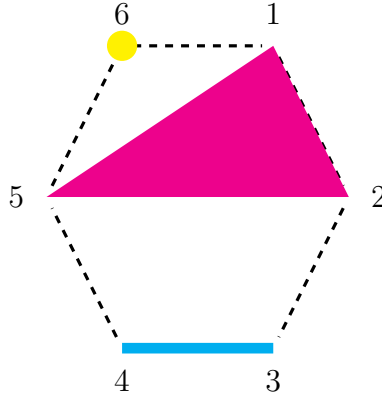
Corollary. *A non-crossing partition can be represented by the minimums and sizes of its blocks.*

Example. $\{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$ can be represented by the following dictionary :

- $1 : 3$
- $3 : 2$
- $6 : 1$

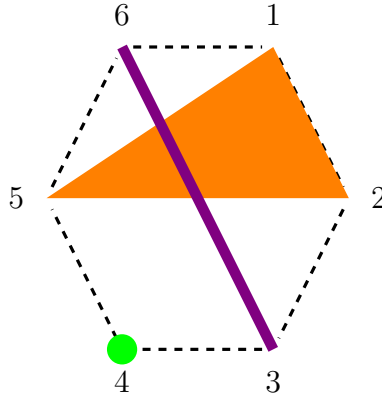
A non-crossing partition of $[n]$ can be represented graphically on a regular n -vertices polygon, with vertices labeled from 1 to n clockwise. We then represent each block $B = \{b_1, \dots, b_k\}$ by the convex hull of $\{b_1, \dots, b_k\}$.

Example ($P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$).



Thus non-crossing meaning the hulls are disjoint.

Example (Counter-example : $P = \{\{1, 5, 2\}, \{3, 6\}, \{4\}\}$).



This partition is not non-crossing, as the convex hulls of $\{1, 2, 5\}$ and $\{3, 6\}$ are not disjoint.

Now that we have a bijection between the two combinatorial structures, we wish to use the well-known order on \mathcal{NC}_n and apply it to \mathcal{PF}_n .

1.2.1 The non-crossing partitions poset

Definition 4 (\succ). We say that P covers Q , written $P \succ Q$, if $\exists B_i, B_j \in P$ such that $Q = P - \{B_i, B_j\} \cup \{B_i \cup B_j\}$

Example. $\{\{1, 6\}, \{2, 3\}, \{4, 5\}\} \succ \{\{1, 2, 3, 6\}, \{4, 5\}\}$

- $B_i = \{1, 6\}$
- $B_j = \{2, 3\}$

Proposition. This covering relation defines the poset of \mathcal{NC}_n . We denote by \mathcal{NCC}_n the set of maximal chains in the poset of \mathcal{NC}_n .

$$\mathcal{NCC} = \bigcup_{n \geq 0} \mathcal{NCC}_n$$

Remark. The bottom element of this poset is $\{\{1, \dots, n\}\}$, and the top element is $\{\{1\}, \dots, \{n\}\}$.

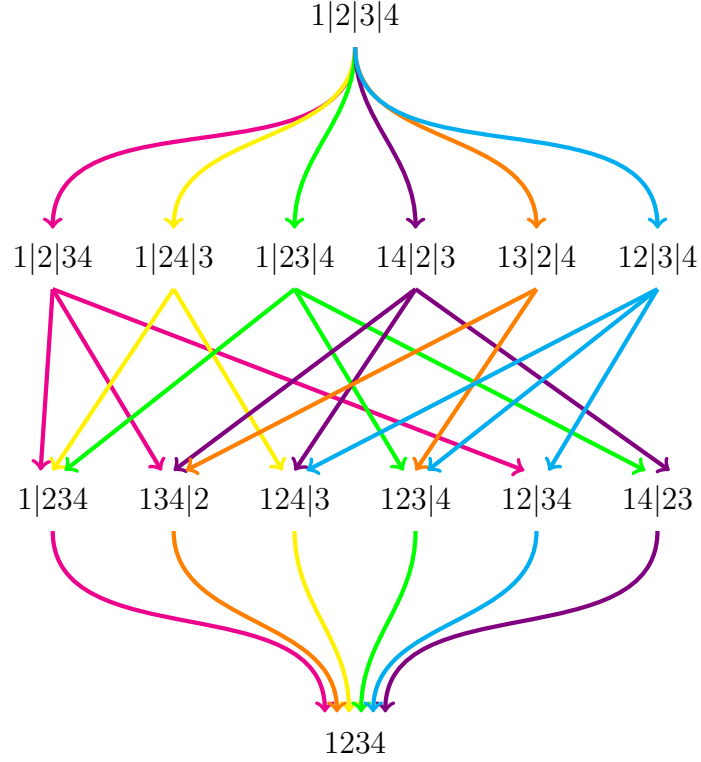
Theorem 4 (Kreweras, 1972). Let ncc_n be the cardinal of \mathcal{NCC}_n . We have

$$ncc_n = n^{n-2}$$

.

Example (The poset of \mathcal{NC}_4).

To shorten labels, we represent $\{\{1\}, \{2, 3\}, \{4\}\}$ by $1|23|4$.



There are $4^2 = 16$ different maximal chains, and $\frac{1}{5} \binom{8}{4} = \frac{70}{5} = 14$ elements in this poset.

One of the main constructions on non-crossing partitions is the *complement*, defined by Kreweras. Not only does it possess multiple interesting properties that will be presented in the next part, but will also be helpful once we attempt to generalize the aforementioned bijection to the rational case, in the second chapter.

1.2.2 Kreweras complement

Definition 5 (Associated Permutation). *The permutation σ associated to a non-crossing partition has a cycle (b_1, \dots, b_k) for each block $B = \{b_1, \dots, b_k\}$ of the partition.*

Example. *The permutation associated to $\{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$ is $(1\ 2\ 5)(3\ 4)(6) = 254316$.*

Definition 6 (Kreweras Complement). *The Kreweras complement $K(P)$ of a non-crossing partition P is defined as follows :*

- *Let σ be the permutation associated to P*
- *Let π be the permutation $(n \ n-1 \ n-2 \ \dots \ 3 \ 2 \ 1) = n123 \dots n-1$*
- *$K(P)$ is the non-crossing partition associated to $\pi\sigma$.*

Example ($P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$).

- $\sigma = (1 \ 2 \ 5) (3 \ 4) (6) = 254316$
- $\pi = (6 \ 5 \ 4 \ 3 \ 2 \ 1) = 612345$
- $\pi\sigma = 143265 = (1) (2 \ 4) (3) (5 \ 6)$
- $K(P) = \{\{1\}, \{2, 4\}, \{3\}, \{5, 6\}\}$

Proposition (Kreweras minimums). *Let $P = \{B_1, \dots, B_k\}$ be a non-crossing partition. Let $K(P) = \{B'_1, \dots, B'_l\}$ be its Kreweras complement. Then*

$$\bigcup_{1 \leq i \leq l} \min(B'_i) = B_1 \cup \bigcup_{1 < j \leq k} B_j - \max(B_k)$$

Example ($P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$).

- $K(P) = \{\{1\}, \{2, 4\}, \{3\}, \{5, 6\}\}$
- $\bigcup \min(B'_i) = \{1, 2, 3, 5\}$
- $B_1 \cup \bigcup B_j - \max(B_k) = \{1, 2, 5\} \cup \{3, 4\} - \{4\} \cup \{6\} - \{6\} = \{1, 2, 5\} \cup \{3\} \cup \emptyset = \{1, 2, 3, 5\}$

Notation. $B_{[i]}$ = block containing i .

Proposition (Kreweras block sizes). *Let $P = \{B_1, \dots, B_k\}$ be a non-crossing partition. Let $K(P) = \{B'_1, \dots, B'_l\}$ be its Kreweras complement. Then the size of the block B'_i is defined as follows :*

- *Let m_i be the i^{th} minimum of $K(P)$*

- Define a transition $\phi(e)$ as

$$\text{Let } j = e + 1 \text{ (or 1 if } e = n)$$

$$\phi(e) = \max(B_{[j]})$$
- The size of B'_i is k_{\min} such that $k_{\min} = \min\{k > 0 \mid \phi^k(m_i) \in B_{[m_i]}\}$.

Example ($P = \{\{1, 2, 5\}, \{3, 4\}, \{6\}\}$).

- $\text{mins} = \{1, 2, 3, 5\}$
- $m_1 = 1$

$$B_{[1]} = B_1$$

$$\max(B_{[2]} = \max(B_1) = 5$$

The size for m_1 is 1.
- m_2

$$B_{[2]} = B_1$$

$$\max(B_{[3]}) = \max(B_2) = 4$$

$$\max(B_{[5]}) = \max(B_1) = 5$$

The size for m_2 is 2.
- $m_3 = 3$

$$B_{[3]} = B_2$$

$$\max(B_{[4]}) = \max(B_2) = 4$$

The size for m_3 is 1.
- $m_4 = 5$

$$B_{[5]} = B_1$$

$$\max(B_{[6]}) = \max(B_3) = 6$$

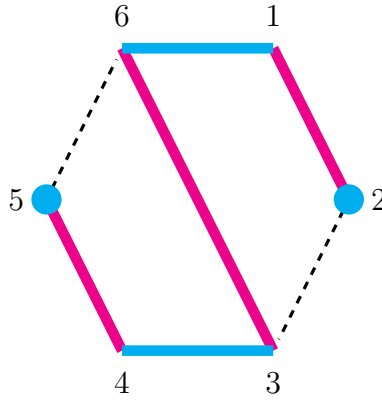
$$\max(B_{[1]}) = \max(B_1) = 5$$

The size for m_4 is 2.

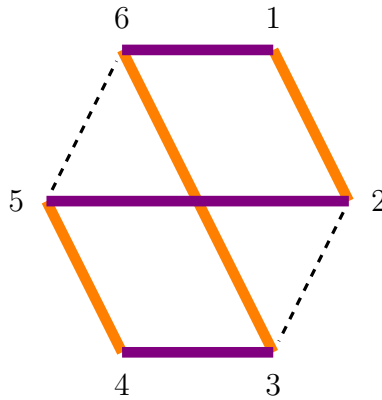
Definition 7 (Mutually Non-crossing Partitions). *2 partitions P and Q are said mutually non-crossing if :*

- *P is non-crossing*
- *Q is non-crossing*
- *For every block B_i of P and every block B_j of Q , if $a, c \in B_i$ and $b, d \in B_j$, then we can not have $a < b < c < d$, nor $a > b > c > d$.*

Example ($P = \{\{1, 2\}, \{3, 6\}, \{4, 5\}\}, Q = \{\{1, 6\}, \{2\}, \{3, 4\}, \{5\}\}$).



Example (Counter-example : $P = \{\{1, 2\}, \{3, 6\}, \{4, 5\}\}, Q = \{\{1, 6\}, \{2, 5\}, \{3, 4\}\}$).

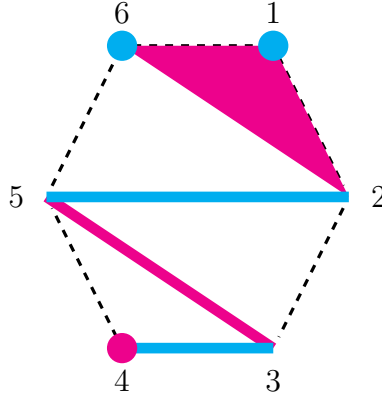


Remark. *Note that vertices can touch, but the edges of the convex hulls can not cross.*

Proposition (Bodnar, 2017). *For any non-crossing partition P , P and $K(P)$ are mutually non-crossing.*

Furthermore, $K(P)$ is a *densest* partition that is mutually non-crossing with P . That is, *no* partition Q that is mutually non-crossing with P has less blocks than $K(P)$.

Example ($P = \{1, 2, 6\}, \{3, 5\}, \{4\}\}$). $Q = \{\{1\}, \{2, 5\}, \{3, 4\}, \{6\}\}$



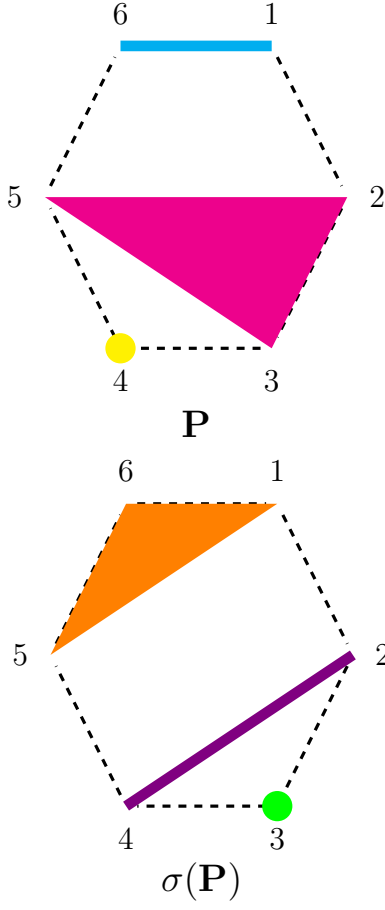
By construction, the symmetric group of order n carries a natural action on \mathcal{NC}_n . The following section presents some of the constructions and properties that emerge from this action.

1.2.3 Action of \mathfrak{S}_n on partitions of $[n]$

Definition 8 (Action of \mathfrak{S}_n). *The action of \mathfrak{S}_n on a partition $P = \{B_1, \dots, B_l\}$ of $[n]$ is defined by :*

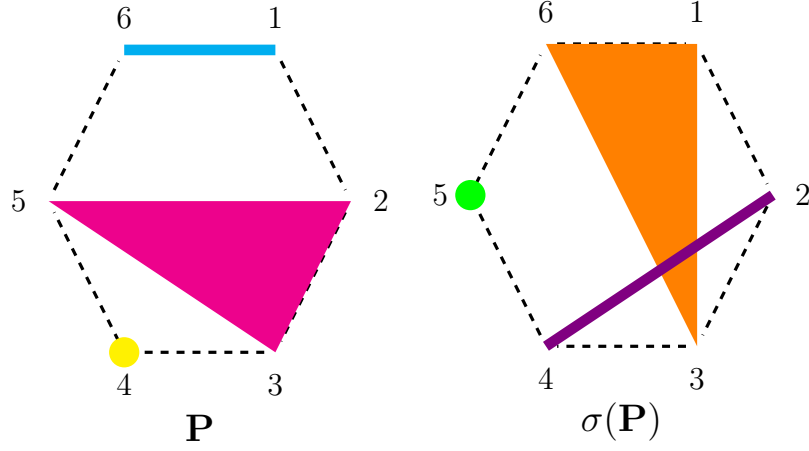
- For each block $B_i = \{b_1, \dots, b_k\} : \sigma(B_i) = \{\sigma(b_1), \dots, \sigma(b_k)\}$
- When $P \in \mathcal{NC}_n$, we denote $\rho = \sigma(P) = \{\sigma(B_1), \dots, \sigma(B_l)\}$

Example ($\sigma = 415362, P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$).
 $\sigma(P) = \{\{1, 5, 6\}, \{2, 4\}, \{3\}\}$



Remark. Note that \mathcal{NC}_n is not stable under the action of \mathfrak{S}_n . That is, even if P is non-crossing, $\sigma(P)$ is not necessarily non-crossing.

Example (Counter-example : $\sigma = 413562, P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$).
 $\sigma(P) = \{\{1, 3, 6\}, \{2, 4\}, \{5\}\}$

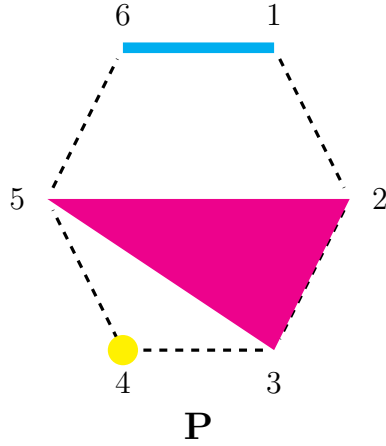


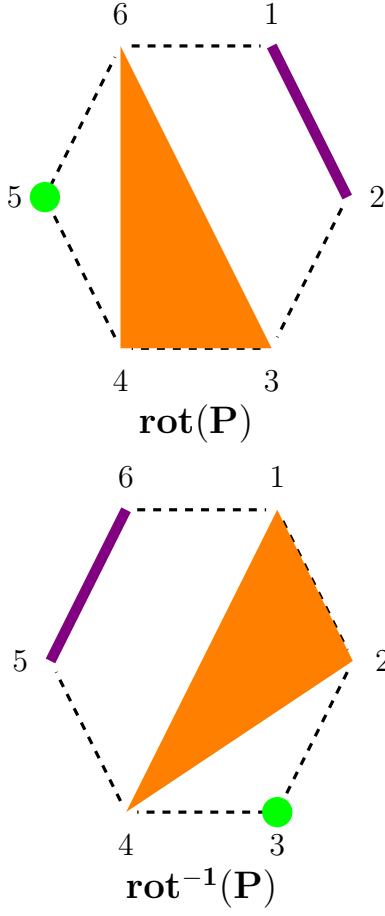
Definition 9 (Rotation). We define the rotation operator rot of $P \in \mathcal{NC}_n$ as $rot(P) = (1\ 2\ 3\ \dots\ n)(P) = 23\dots n1(P)$.

Conversely, we define rot^{-1} of P as $rot^{-1}(P) = (n\ n-1\ \dots\ 3\ 2\ 1)(P) = n12\dots n-1(P)$.

Example ($P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$).

- $rot(P) = \{\{1, 2\}, \{3, 4, 6\}, \{5\}\}$
- $rot^{-1}(P) = \{\{1, 2, 4\}, \{3\}, \{5, 6\}\}$



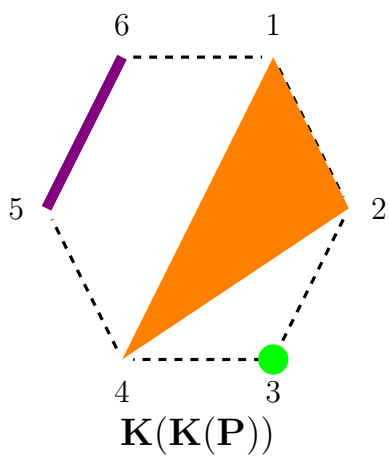
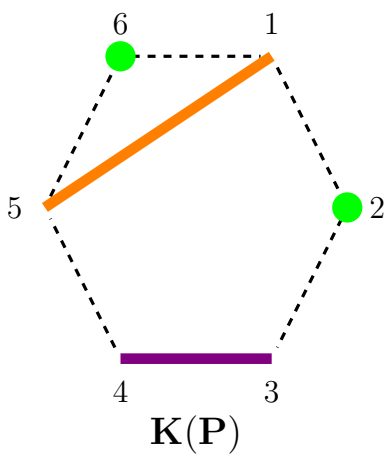
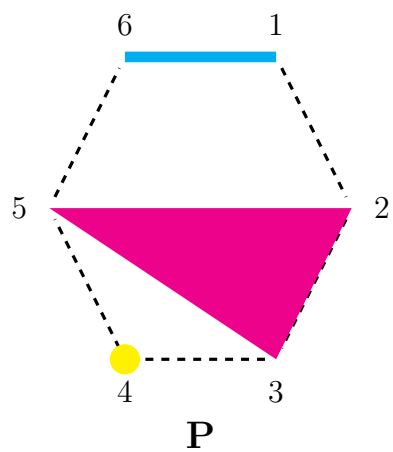


Remark.

- $\text{rot}(\text{rot}^{-1}(P)) = \text{rot}^{-1}(\text{rot}(P)) = P$
- $\text{rot}(P)$ and $\text{rot}^{-1}(P)$ are always non-crossing partitions.
- If $P \in \mathcal{NC}_n$, then $\text{rot}^n(P) = \text{rot}^{-n}(P) = P$.

Proposition (Bodnar, 2017). $K(K(P)) = \text{rot}^{-1}(P)$.

Example ($P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$).



Now that we assessed the link between primitive parking functions and non-crossing partitions, we wish to find a similar structure for classical parking functions. Hence, we introduce another combinatorial object : the *non-crossing 2-partition*.

1.3 Non-crossing 2-partitions

Definition 10 (Non-crossing 2-partition). *A non-crossing 2-partition of a totally ordered set E is a pair (P, σ) where :*

- *P is a non-crossing partition of E*
- *σ is a permutation of the elements of E*
- *For each sorted block $B_i = \{b_1, \dots, b_k\} \in P$, we have $\sigma(b_i) < \dots < \sigma(b_k)$*

We denote by \mathcal{NC}_n^2 the set of non-crossing 2-partitions of $[n]$.

$$\mathcal{NC}^2 = \bigcup_{n \geq 0} \mathcal{NC}_n^2$$

Example (\mathcal{NC}_6^2). $P = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$ $\sigma = 413265$
 $\rho = \{\{1, 3, 6\}, \{2\}, \{4, 5\}\}$

Theorem 5 (Edelman, 1979). *Let nc_n^2 be the cardinal of \mathcal{NC}_n^2 . We have*

$$nc_n^2 = (n+1)^{n-1}$$

Example ($n = 1, 2, 3$).

- $n = 1$: $nc_1^2 = 1$
 $\{\{1\}\} \quad 1 \quad \rho = P$
- $n = 2$: $nc_2^2 = 3$
 $\{\{1\}, \{2\}\} \quad 12 \quad \rho = P$
 $\{\{1\}, \{2\}\} \quad 21 \quad \rho = P$
 $\{\{1, 2\}\} \quad 12 \quad \rho = P$

- $n = 3$: $nc_3^2 = 16$

$\{\{1\}, \{2\}, \{3\}\}$	123	$\rho = P$
$\{\{1\}, \{2\}, \{3\}\}$	132	$\rho = P$
$\{\{1\}, \{2\}, \{3\}\}$	213	$\rho = P$
$\{\{1\}, \{2\}, \{3\}\}$	231	$\rho = P$
$\{\{1\}, \{2\}, \{3\}\}$	312	$\rho = P$
$\{\{1\}, \{2\}, \{3\}\}$	321	$\rho = P$
$\{\{1, 2\}, \{3\}\}$	123	$\rho = P$
$\{\{1, 2\}, \{3\}\}$	132	$\rho = \{\{1, 3\}, \{2\}\}$
$\{\{1, 2\}, \{3\}\}$	231	$\rho = \{\{1\}, \{2, 3\}\}$
$\{\{1\}, \{2, 3\}\}$	123	$\rho = P$
$\{\{1\}, \{2, 3\}\}$	213	$\rho = \{\{1, 3\}, \{2\}\}$
$\{\{1\}, \{2, 3\}\}$	312	$\rho = \{\{1, 2\}, \{3\}\}$
$\{\{1, 3\}, \{2\}\}$	123	$\rho = P$
$\{\{1, 3\}, \{2\}\}$	132	$\rho = \{\{1, 2\}, \{3\}\}$
$\{\{1, 3\}, \{2\}\}$	213	$\rho = \{\{1\}, \{2, 3\}\}$
$\{\{1, 2, 3\}\}$	123	$\rho = P$

Proposition. *This means we can create a bijection between \mathcal{PF}_n and \mathcal{NC}_n^2 .*

Proof.

- $\mathcal{PF}_n \rightarrow \mathcal{NC}_n^2$: Let $f = (a_1, \dots, a_n) \in \mathcal{PF}_n$ be our parking function. For $i \in \{1, \dots, n\}$, we define :

l_i : the number of occurrences of i in f .

im_i : $\{j \mid a_j = i\}$

The corresponding non-crossing partition will have the following constraints :

For each $i \in \{1, \dots, n\}$, if $l_i > 0$, then there is a block $B_{[i]}$ of length l_i with minimum element i .

$$\sigma(B_{[i]}) = im_i$$

There is a unique set partition $P = \bigcup_i B_{[i]}$ of $[n]$ and a unique permutation σ respecting these conditions such that $(P, \sigma) \in \mathcal{NC}_n^2$: for each minimum i in *decreasing order*, add the n_i first free elements of $[i+1, i+2, \dots, n, 1, \dots, i-1]$ to B_i . σ is then trivially obtained by the second constraint.

- $\mathcal{NC}_n^2 \rightarrow \mathcal{PF}_n$: Let (P, σ) with $P = \{B_1, \dots, B_l\}$ be our non-crossing 2-partition. For each block $B_i = \{b_1, \dots, b_k\} \in P$:

$$m_i = \min(B_i) = b_1$$

$$pos_i = \sigma(B_i)$$

For each $j \in pos_i$, we define $a_j = m_i$

The corresponding parking function is (a_1, \dots, a_n) .

□

Example ($n = 8$).

$$P = \{\{1, 2, 5\}, \{3, 4\}, \{6, 8\}, \{7\}\}$$

$$\sigma = 36187245$$

$$f = (3, 6, 1, 7, 6, 1, 1, 3)$$

Following the path of the classical primitive case, we recall the cover relation defined in [9] in order to deduce a poset for \mathcal{PF}_n .

1.3.1 The non-crossing 2-partitions poset

Definition 11 (\succ^2). We say that (P, σ) covers (Q, τ) , written $(P, \sigma) \succ^2 (Q, \tau)$, if $\exists B_i, B_j \in P$ such that

- $Q = P - \{B_i, B_j\} \cup \{B_i \cup B_j\}$
- $l \neq i, j, b \in B_l \rightarrow \tau(b) = \sigma(b)$
- Let $B_i \cup B_j = \{b_1, \dots, b_k\}$:
 $\tau(B_i \cup B_j) = \sigma(B_i \cup B_j)$
 $\tau(b_1) < \dots < \tau(b_k)$

Example.

- $P = \{\{1, 6\}, \{2, 3\}, \{4\}, \{5\}\}$
- $\sigma = 236154$
- $Q = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$
- $\tau = 235164$
- $(P, \sigma) \succ^2 (Q, \tau)$
- $(P, \sigma) \not\succ^2 (Q, \sigma)$, because $\sigma(\{2, 3, 5\}) = \{3, 6, 5\}$ is not ordered.

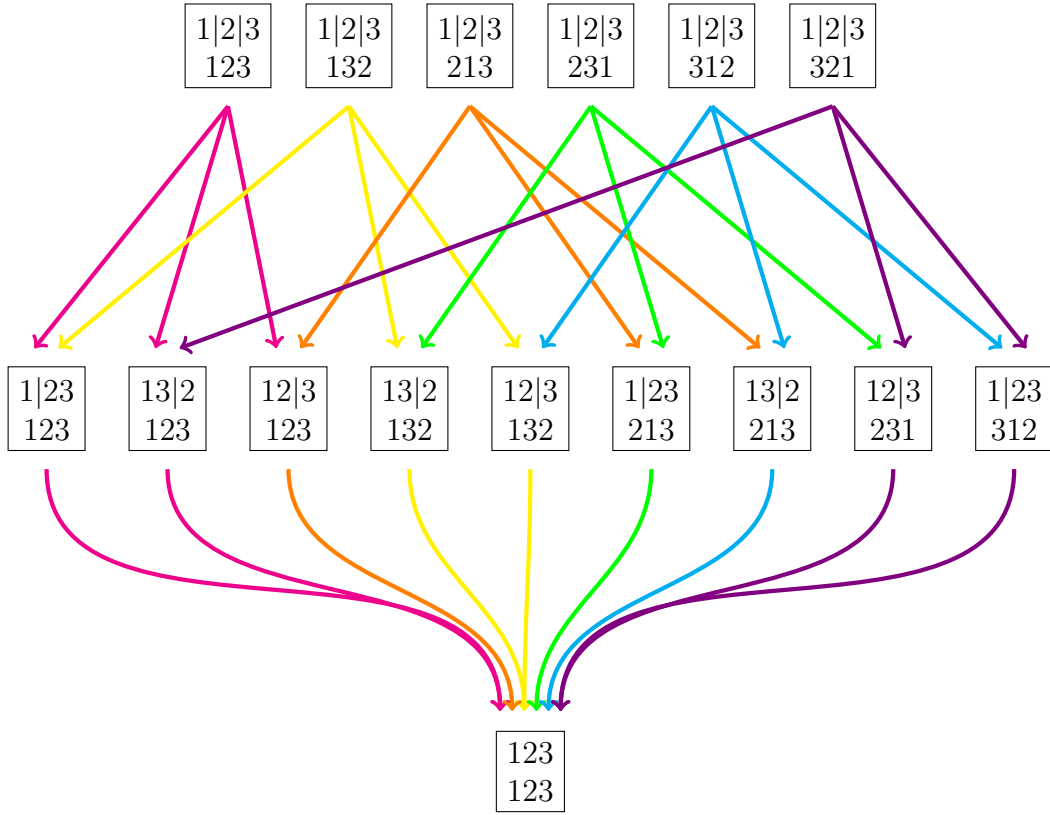
Proposition. *This covering relation defines the poset of \mathcal{NC}_n^2 .*

Remark. *The bottom element of this poset is $(\{\{1, \dots, n\}\}, 12 \dots n)$, and the top elements are $(\{\{1\}, \dots, \{n\}\}, \sigma) \mid \sigma \in \mathfrak{S}_n$.*

Example (The poset of \mathcal{NC}_3^2).

To shorten labels, we represent $(\{\{1, 3\}, \{2\}\}, 213)$ by

13 2
213



There are $4^2 = 16$ elements in this poset.

We can now define a poset for \mathcal{PF}_n , in which the *height* of any parking function will be defined by its *rank*.

1.3.2 The parking functions poset

Definition 12 (Rank). Given $f = (a_1, \dots, a_n) \in \mathcal{PF}_n$, let

$$b_i = \begin{cases} 1 & \text{if } \exists j \mid a_j = i \\ 0 & \text{otherwise} \end{cases}$$

We define the rank of f , noted $rk(f)$, as

$$\sum_{1 \leq i \leq n} b_i$$

Example.

$$\begin{aligned}rk((1, 5, 4, 2, 3, 3, 1)) &= 5 \\rk((4, 7, 1, 1, 3, 2, 2, 8)) &= 6\end{aligned}$$

Definition 13 (\succ_{pf}). Since \mathcal{PF}_n and \mathcal{NC}_n^2 are in bijection, we can define a covering relation \succ_{pf} for \mathcal{PF}_n as follows :
 $f \in \mathcal{PF}_n \succ_{pf} g \in \mathcal{PF}_n$ if and only if :

- (P, σ) is the non-crossing 2-partition associated to f
- (Q, τ) is the non-crossing 2-partition associated to g
- $(P, \sigma) \succ^2 (Q, \tau)$

Example.

- $P = \{\{1, 6\}, \{2, 3\}, \{4\}, \{5\}\}$
- $\sigma = 236154$
- $Q = \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$
- $\tau = 235164$
- $f = (4, 1, 2, 1, 5, 2) \succ_{pf} g = (4, 1, 2, 1, 2, 2)$

Remark. If $f \succ_{pf} g$, then $rk(f) = rk(g) + 1$, and there exists i and j such that :

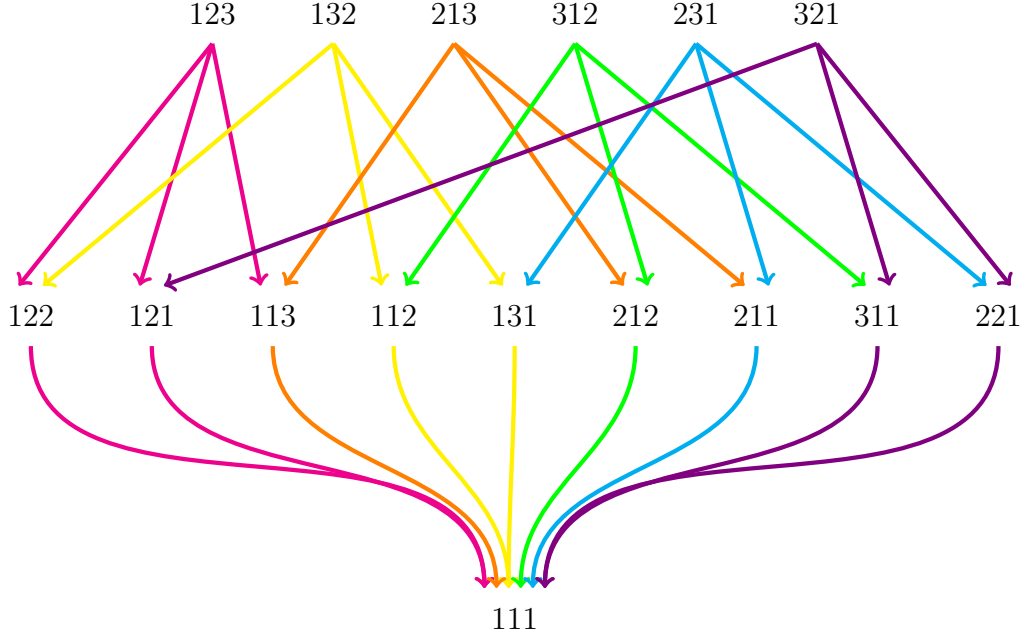
- $i < j$
- There is at least 1 occurrence of i in f
- There is at least 1 occurrence of j in f

$$b_k = \begin{cases} i & \text{if } a_k = j \\ a_k & \text{otherwise} \end{cases}$$

Proposition. This covering relation defines the poset of \mathcal{PF}_n .

Remark. The bottom element of this poset is $(\underbrace{1, \dots, 1}_n)$, and the top elements are the permutations of $\{1, \dots, n\}$.

Example (The poset of \mathcal{PF}_3).



While non-crossing (2-)partitions are frequently used to define a poset for (primitive) parking functions, it is rather unpractical that using a bijection to define the cover relation is necessary.

Thereby, the following section will present our tentative at finding a more direct definition for the cover relation of \mathcal{PF}_n and \mathcal{PF}'_n , this time using a bijection with (decorated) Dyck Paths. A main benefit of this solution is that we define a cover relation for both structures, and the poset of one can be obtained by applying the given bijection to the poset of the other.

Furthermore, the two main results of this article – Theorem 8 and the corresponding Conjecture for the classical case – rise from the number of intervals in those posets.

1.4 A direct poset linked to Dyck paths

1.4.1 Dyck Paths

Notation. We denote the number of occurrences of a symbol s in a word w by $|w|_s$.

Definition 14 (Dyck path). A Dyck word is a word $w \in \{0, 1\}^*$ such that :

- for each suffix w' of w , $|w'|_1 \geq |w'|_0$.
- $|w|_0 = |w|_1$.

A Dyck word of length $2n$ can be represented as a path from $(0, 0)$ to (n, n) that stays over $x = y$, called a Dyck path :

- Each 1 corresponds to a North step \uparrow .
- Each 0 corresponds to an East step \rightarrow .

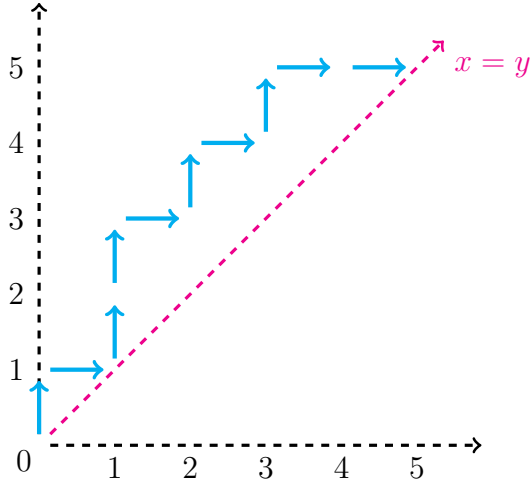
We denote by \mathcal{D}_n the set of Dyck words of length $2n$.

Example ($n = 5$).

$w_1 = 1011000110$ is not a Dyck word, because $|1011000|_0 > |1011000|_1$.

$w_2 = 1011010101$ is not a Dyck word, because $|w_2|_0 \neq |w_2|_1$.

$w_3 = 1011010100$ is a Dyck word :

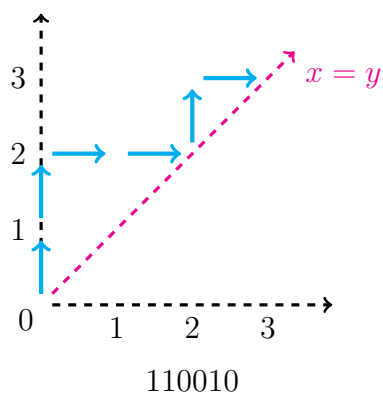
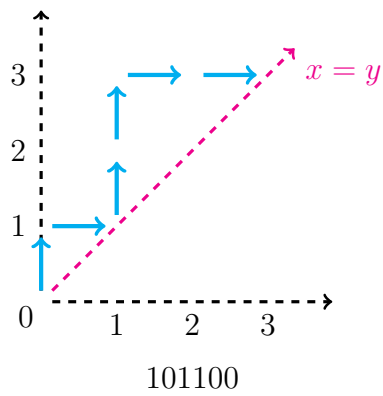
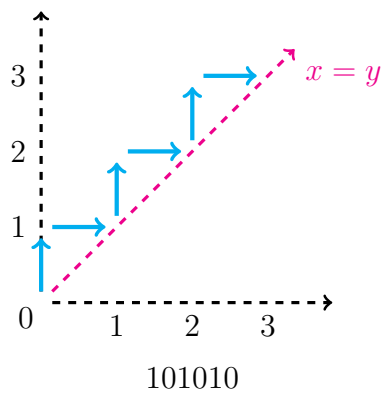


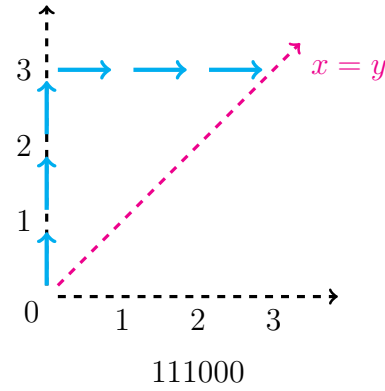
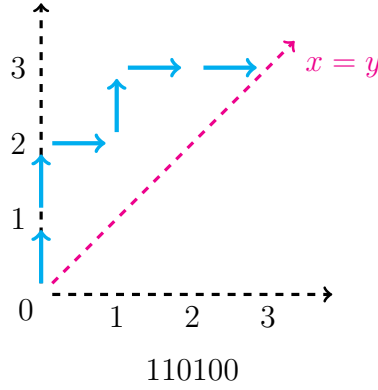
Theorem 6 (André, 1887). Let d_n be the cardinal of \mathcal{D}_n . We have

$$d_n = \frac{1}{n+1} \binom{2n}{n}$$

which is the n^{th} Catalan number.

Example ($n = 3$). $d_n = 5$.





Proposition. *This means we can create a bijection between \mathcal{PF}'_n and \mathcal{D}_n .*

Proof.

- $\mathcal{PF}'_n \rightarrow \mathcal{D}_n$: Let $f = (a_1, \dots, a_n) \in \mathcal{PF}'_n$ be our primitive parking function. For $i \in \{1, \dots, n\}$, we define l_i the number of occurrences of i in f .

The corresponding Dyck word will be $\underbrace{1 \dots 1}_l 0 \underbrace{1 \dots 1}_l 0 \dots \underbrace{1 \dots 1}_l 0$.

- $\mathcal{D}_n \rightarrow \mathcal{PF}'_n$: Let w be our Dyck word, and consider its path representation. We define s_i to be the distance between the segment from $(0, i-1)$ to $(0, i)$ and the i^{th} North step. Then, let $a_i = s_i + 1$. The corresponding primitive parking function is (a_1, \dots, a_n) .

□

Example ($n = 6, \mathcal{PF}'_n \rightarrow \mathcal{D}_n$).

- $f = (1, 1, 2, 4, 5, 5)$

$$l_1 = 2$$

$$l_2 = 1$$

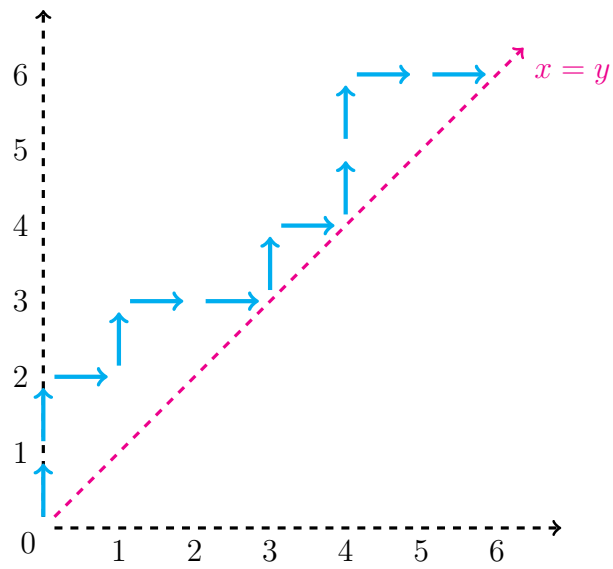
$$l_3 = 0$$

$$l_4 = 1$$

$$l_5 = 2$$

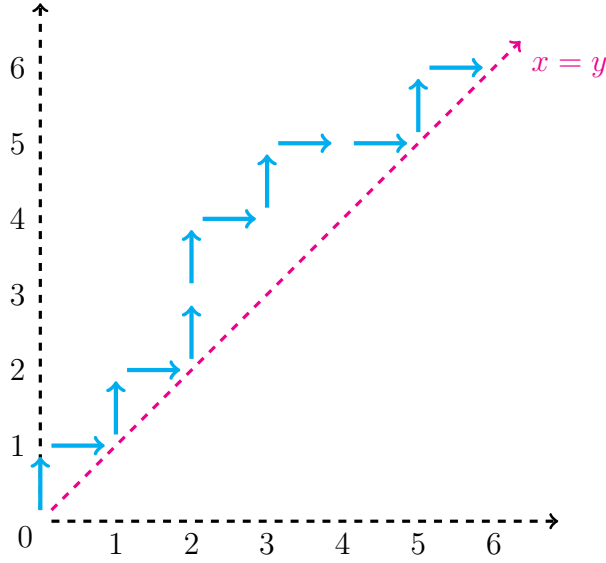
$$l_6 = 0$$

- $w = (110100101100)$



Example $(n = 6, \mathcal{D}_n \rightarrow \mathcal{PF}'_n)$.

- $w = 101011010010$



- *Distances :*

$s_1 = 0$	$a_1 = 1$
$s_2 = 1$	$a_2 = 2$
$s_3 = 2$	$a_3 = 3$
$s_4 = 2$	$a_4 = 3$
$s_5 = 3$	$a_5 = 4$
$s_6 = 5$	$a_6 = 6$

- $f = (1, 2, 3, 3, 4, 6)$

To apply a similar bijection to non-primitive parking functions, we will need an upgraded version of Dyck paths, that is, Labeled Dyck Paths.

1.4.2 Labeled Dyck Paths

Definition 15 (Labeled Dyck Word). *A labeled Dyck word is a word $w \in \{0, \dots, n\}^*$ such that :*

- *for each suffix w' of w , $|w'|_{\neq 0} \geq |w'|_0$.*
- $|w|_0 = |w|_{\neq 0}$.

- for each $i \in \{1, \dots, n\}$, w has exactly one occurrence of i .
- if $w_i \neq 0$ and $w_{i+1} \neq 0$, then $w_i < w_{i+1}$. That is, consecutive North steps have increasing labels.

A labeled Dyck word of length $2n$ can be represented as a *path* from $(0, 0)$ to (n, n) , where each North step is associated to a label :

- Each $i \neq 0$ corresponds to a *North step* \uparrow labeled i .
- Each 0 corresponds to an *East step* \rightarrow .

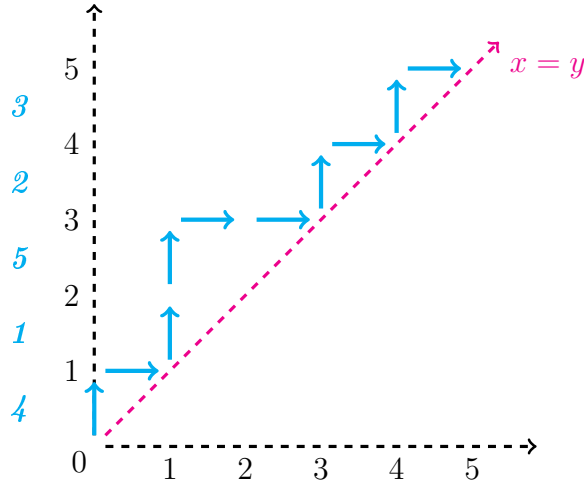
Those paths are called *labeled Dyck paths*.

We denote by \mathcal{LD}_n the set of labeled Dyck words of length $2n$.

Example ($n = 5$).

$w_1 = 4051002030$ is not a labeled Dyck word, because $5 > 1$.

$w_2 = 4015002030$ is a labeled Dyck word :



Theorem 7. Let ld_n be the cardinal of \mathcal{LD}_n . We have

$$ld_n = (n + 1)^{n-1}$$

.

Example ($n = 3$). $ld_n = 4^2 = 16$

- Word of shape $XXX000$:
123000
- Words of shape $XX0X00$:
120300 130200 230100
- Words of shape $XX00X0$:
120030 130020 230010
- Words of shape $X0XX00$:
102300 201300 301200
- Words of shape $X0X0X0$:
102030 103020 201030
203010 301020 302010

Proposition. This means we can create a bijection between \mathcal{PF}_n and \mathcal{LD}_n .

Proof.

- $\mathcal{PF}_n \rightarrow \mathcal{LD}_n$: Let $f = (a_1, \dots, a_n) \in \mathcal{PF}_n$ be our parking function. For $i \in \{1, \dots, n\}$, we define $im_i : \{j \mid a_j = i\}$. We then define $im_{i,1}, \dots, im_{i,k_i}$ to be the elements of im_i in increasing order. The corresponding labeled Dyck word will be

$$\underbrace{im_{1,1} \dots im_{1,k_1}}_{im_1} 0 \underbrace{im_{2,1} \dots im_{2,k_2}}_{im_2} 0 \dots \underbrace{im_{n,1} \dots im_{n,k_n}}_{im_n} 0.$$
- $\mathcal{LD}_n \rightarrow \mathcal{PF}_n$: Let w be our labeled Dyck word, and consider its path representation. We define s_i to be the distance between the segment from $(0, i-1)$ to $(0, i)$ and the i^{th} North step. Then, let $label(i)$ be the label of the i^{th} North step, and $dist_i = \{label(j) \mid s_j = i\}$ be the set of the labels of all North steps at distance i . Then, if $j \in dist_i$, let $a_j = i + 1$. The corresponding parking function is (a_1, \dots, a_n) .

□

Example $(n = 6, \mathcal{PF}_n \rightarrow \mathcal{LD}_n)$.

- $f = (5, 2, 1, 4, 5, 1)$

$$im_1 = \{3, 6\}$$

$$im_2 = \{2\}$$

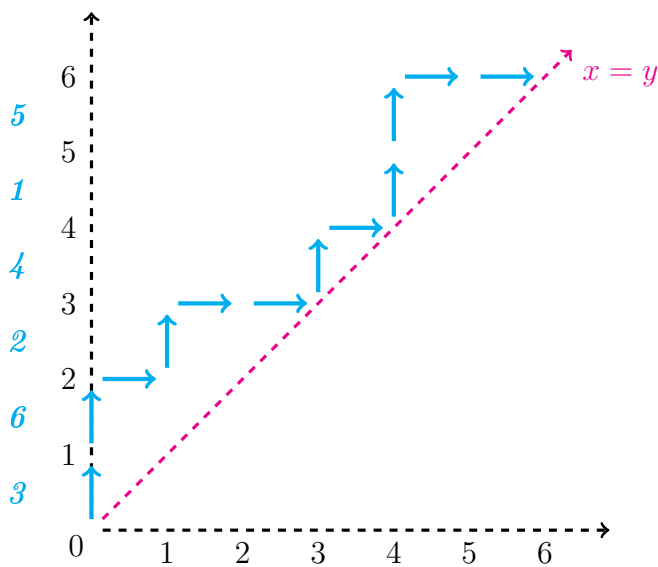
$$im_3 = \emptyset$$

$$im_4 = \{4\}$$

$$im_5 = \{1, 5\}$$

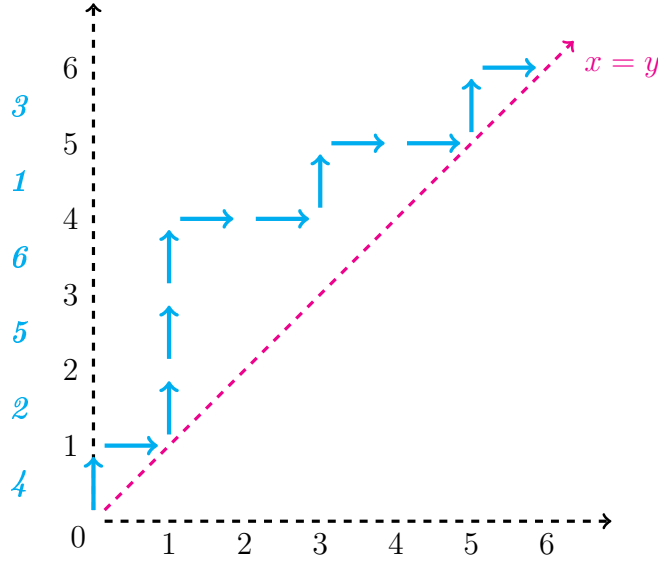
$$im_6 = \emptyset$$

- $w = 360200401500$



Example $(n = 6, \mathcal{LD}_n \rightarrow \mathcal{PF}_n)$.

- $w = 402560010030$



- *Distances :*

$$\begin{array}{lll} s_1 = 0 & s_2 = 1 & s_3 = 1 \\ s_4 = 1 & s_5 = 3 & s_6 = 5 \end{array}$$

- *Labels :*

$$\begin{array}{lll} dist_0 = \{4\} & dist_1 = \{2, 5, 6\} & dist_2 = \emptyset \\ dist_3 = \{1\} & dist_4 = \emptyset & dist_5 = \{3\} \end{array}$$

- $f = (4, 2, 6, 1, 2, 2)$

Remark. *The primitive parking functions are exactly the parking functions corresponding to labeled Dyck paths where the i^{th} North step is labeled i .*

With those bijections in mind, we can now define cover relations that will issue in the expected bijective posets.

1.4.3 Dyck - Parking Posets

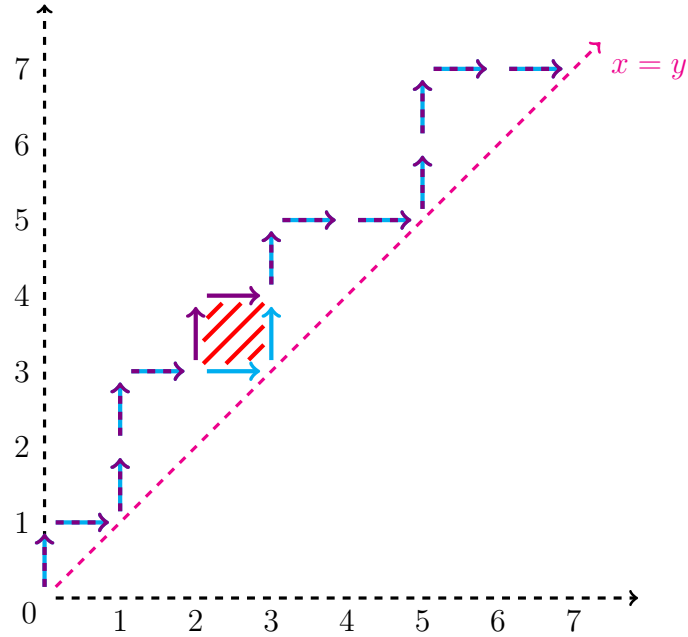
Primitive Dyck - Parking Posets

Definition 16 (\succ_d). *For w and w' two Dyck words, we say that w covers w' , written $w \succ_d w'$, if $\exists w_1, w_2$ such that :*

- $w = w_1 01 w_2$
- $w' = w_1 10 w_2$

Example ($n = 7$). $10110011001100 \succ_d 10110101001100$

- $w_1 = 10110$
- $w_2 = 1001100$

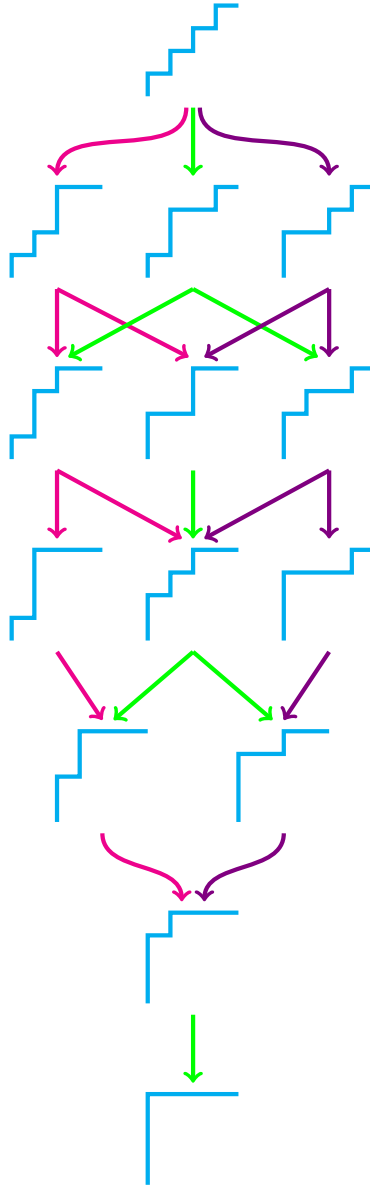


■ 10110011001100 ■ 10110101001100 ■ difference

Remark. If $w_1 \succ_d w_2$, then the path corresponding to w_2 is over the path corresponding to w_1 , and the difference between the two paths is a square of size 1 by 1.

Proposition. This covering relation defines a poset for \mathcal{D}_n .

Example (The poset of \mathcal{D}_4).



There are $\frac{1}{5} \binom{8}{4} = \frac{70}{5} = 14$ elements in this poset.

Definition 17 (Nested Dyck paths). *Two Dyck Paths w_1 and w_2 are said nested if w_1 is equal to w_2 or over w_2 .*

We can thus easily deduce the following proposition from the preceding remark.

Proposition. *If there exists a sequence $w_1 \succ_d w_2 \succ_d w_3 \succ_d \cdots \succ_d w_k$ with $k \geq 0$, then w_1 and w_k are nested.*

Now that we have defined the cover relation for Dyck paths, we have to define the corresponding relation for primitive parking functions, that will allow us to create the wanted bijective posets.

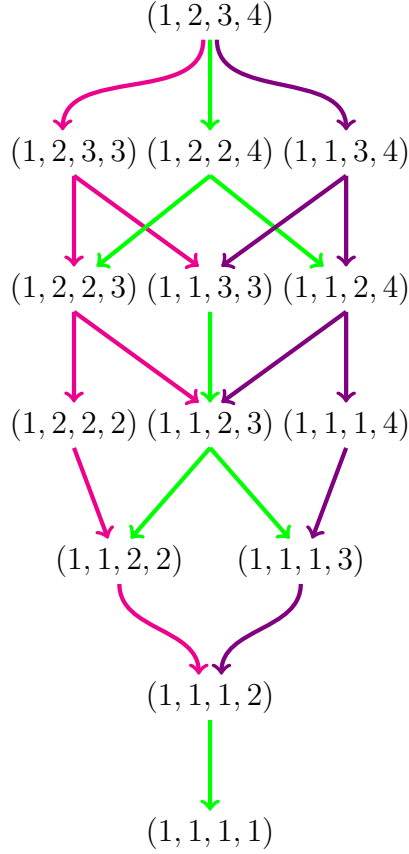
Definition 18 (\succ'). *For f and g two primitive parking functions, we say that f covers g , written $f \succ' g$, if $\exists i$ such that :*

- $f = (a_1, \dots, a_{i-1}, a_i, \quad a_{i+1}, \dots, a_n)$
- $g = (a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_n)$

Example ($n = 6$). $(1, 1, 2, 3, 4, 5) \succ' (1, 1, 2, 3, 3, 5)$

Proposition. *This covering relation defines a poset for \mathcal{PF}'_n .*

Example (The poset of \mathcal{PF}'_4).



There are $\frac{1}{5} \binom{8}{4} = \frac{70}{5} = 14$ elements in this poset.

Remark. The two posets are isomorphic, and one can be obtained by applying the aforementioned bijection to the other.

Theorem 8 (Main Theorem). The number of intervals in those posets is equal to the $n+1^{th}$ term of the integer sequence defined by <https://oeis.org/A005700>. The first terms of this sequence are 1, 1, 3, 14, 84, 594, 4719, 40898, 379236, 3711916, ... Alec Mihailovs proved this sequence to be equal to

$$\frac{6(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!}$$

Proof. As the number of intervals in the poset for Dyck words can be seen as the number of pairs (w_1, w_k) such that $w_1 \succ_d w_2 \succ_d \dots \succ_d w_k$, we can define

it as the number of *nested pairs of Dyck paths*. This has been proved to be equal to this integer sequence by Bruce Westbury in 2013. \square

We now extend this construction to the non-primitive case.

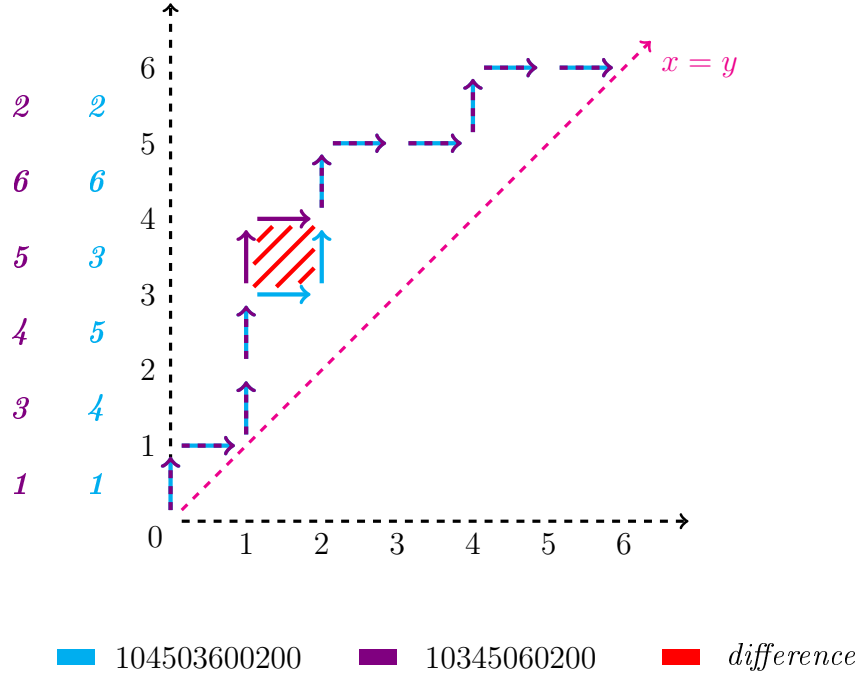
Classical Dyck - Parking Posets

Definition 19 (\succ_{ld}). For w and w' two labeled Dyck words, we say that w covers w' , written $w \succ_{ld} w'$, if $\exists l, r, x, x', y, z, z'$ such that :

- l is either empty or ends with 0
- r is either empty or starts with 0
- $x = x_1x_2\cdots$ has all its digits > 0
- $z = z_1z_2\cdots$ has all its digits > 0
- $x' = x$ where y is correctly inserted regarding the order condition
- y is in z , and $z' = z$ where y is removed
- $w = lx0zr$
- $w' = lx'0z'r$

Example ($n = 5$). $104503600200 \succ_{ld} 10345060200$

- $l = 10$
- $r = 0200$
- $x = 45$
- $x' = 345$
- $y = 3$
- $z = 36$
- $z' = 6$



Definition 20 (Rise). A rise of a decorated Dyck word is a maximal sequence of non-zero digits preceding a zero.

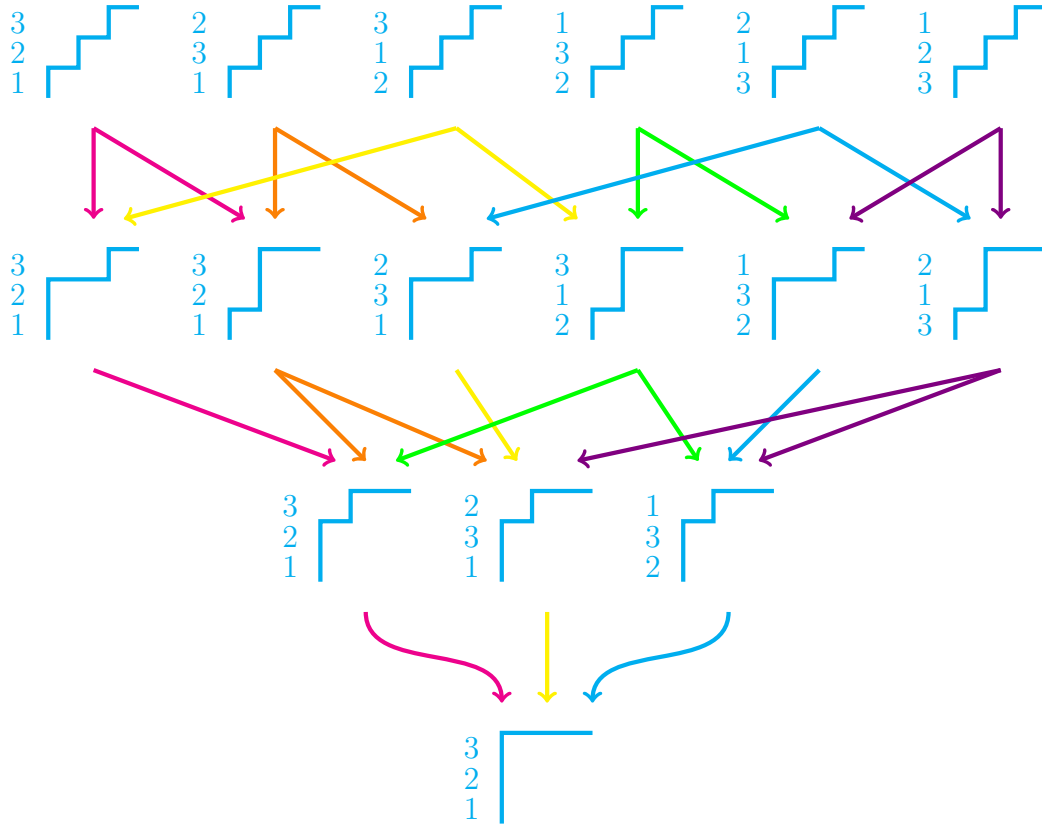
Example ($n = 5$). In order, the rises of 104503600200 are :
 $\bullet 1 \bullet 45 \bullet 36 \bullet \emptyset \bullet 2 \bullet \emptyset$

Remark. If $w_1 \succ_{ld} w_2$, then the path corresponding to w_2 is over the path corresponding to w_1 , and the difference between the two paths is a square of size 1 by 1.

Furthermore, the covering relation can be seen as follows : w_1 covers w_2 if we can obtain w_2 by taking a digit from the $i + 1^{th}$ rise of w_1 , and inserting it into the i^{th} rise of w_1 in increasing order.

Proposition. This covering relation defines a poset for \mathcal{LD}_n .

Example (The poset of \mathcal{LD}_3).



There are $4^2 = 16$ elements in this poset.

Definition 21 (\succ). For f and g two parking functions, we say that f covers g , written $f \succ g$, if $\exists i$ such that :

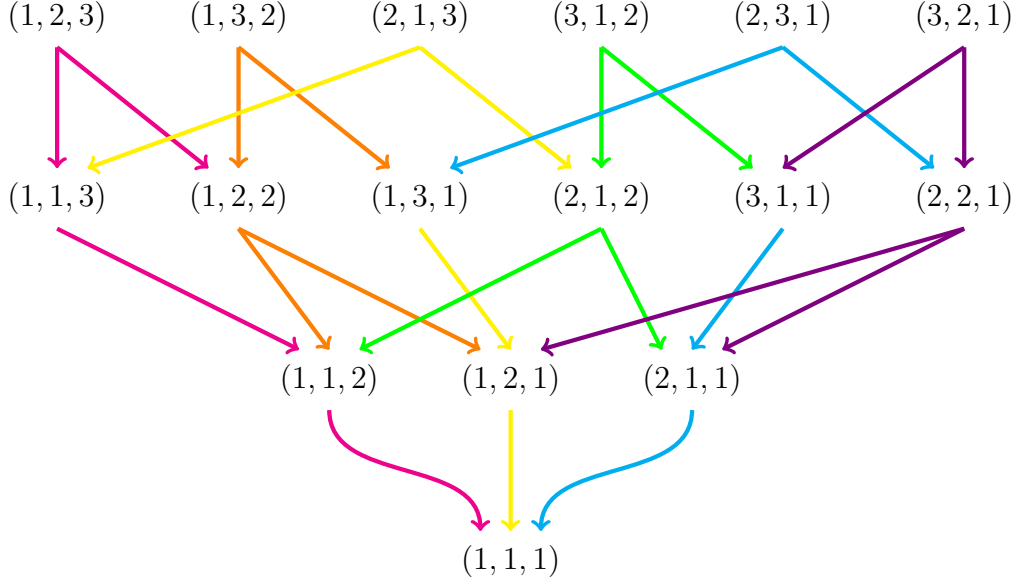
- $f = (a_1, \dots, a_{i-1}, a_i, \quad a_{i+1}, \dots, a_n)$
- $g = (a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_n)$

That is, the same relation as for primitive parking functions.

Example ($n = 6$). $(2, 1, 5, 3, 1, 4) \succ (2, 1, 5, 2, 1, 4)$

Proposition. This covering relation defines a poset for \mathcal{PF}_n .

Example (The poset of \mathcal{PF}_3).



There are $4^2 = 16$ elements in this poset.

Remark. The two posets are isomorphic, and one can be obtained by applying the aforementioned bijection to the other.

Conjecture (Main Conjecture). The number of intervals in those posets is equal to the $n+1^{\text{th}}$ term of the integer sequence defined by <https://oeis.org/A196304>. The first terms of this sequence are 1, 1, 5, 64, 1587, 65421, 4071178, 357962760, 4237910716,

While, to the best of our knowledge, there is no combinatorial structure proved to follow this integer sequence, tests on $n = 1, 2, \dots, 8$ suggest that the number of intervals in those posets might be one.

To go further, the next chapter will tackle a generalization of classical parking functions called *Rational Parking Functions*. The upgrade can be seen as such : Let (a_1, \dots, a_n) be a sequence of positive integers, and (b_1, \dots, b_n) its non-decreasing rearrangement. In the classical case, the bounds for (b_1, \dots, b_n) were $(1, \dots, n)$, thus simply depending on the integer n . In the rational case, the bounds will depend on *two coprime integers* a and b , namely $(1, 1 + \frac{b}{a}, 1 + \frac{2b}{a}, 1 + \frac{3b}{a}, \dots)$, with $a = n$.

Chapter 2

The rational case

For the whole chapter, we will consider 2 *coprime* integers a and b (meaning a and b have 1 as their greatest common divisor).

2.1 Rational Parking Functions

Definition 22 (a, b - Parking Function). An a, b - parking function is a sequence (a_1, a_2, \dots, a_n) such that :

- $n = a$
- its non-decreasing reordering (b_1, b_2, \dots, b_n) has $b_i \leq \frac{b}{a}(i - 1) + 1$ for all i .

We denote by $\mathcal{PF}_{a,b}$ the set of a, b - parking functions.

Example.

- *Ex. 1* : $a > b$

$$a = 7$$

$$b = 3$$

Limits of the non-decreasing reordering of any $f \in \mathcal{PF}_{7,3}$:

$$[1, 1\frac{3}{7}, 1\frac{6}{7}, 2\frac{2}{7}, 2\frac{5}{7}, 3\frac{1}{7}, 3\frac{4}{7}]$$

$$f_1 = (2, 1, 1, 3, 2, 3, 1) \in \mathcal{PF}_{7,3}$$

$$f_2 = (2, 1, 2, 3, 2, 3, 1) \notin \mathcal{PF}_{7,3}, \text{ even though } f_2 \in \mathcal{PF}_7$$

- *Ex. 2* : $a < b$

$$a = 5$$

$$b = 7$$

Limits of the non-decreasing reordering of any $f \in \mathcal{PF}_{5,7}$:

$$[1, 2\frac{2}{5}, 3\frac{4}{5}, 5\frac{1}{5}, 6\frac{3}{5}]$$

$$f_3 = (6, 3, 5, 1, 2) \in \mathcal{PF}_{5,7}, \text{ even though } f_3 \notin \mathcal{PF}_5$$

$$f_4 = (6, 3, 5, 1, 3) \notin \mathcal{PF}_{5,7}$$

Theorem 9. *Let $pf_{a,b}$ be the cardinal of $\mathcal{PF}_{a,b}$. We have*

$$pf_{a,b} = b^{a-1}$$

Example ($a = 3, b = 5$).

- $pf_{a,b} = 25$
- *Limits* : $[1, 2\frac{2}{3}, 4\frac{1}{3}]$

(1, 1, 1)	(1, 1, 2)	(1, 1, 3)	(1, 1, 4)	(1, 2, 1)	(1, 2, 2)	(1, 2, 3)
(1, 2, 4)	(1, 3, 1)	(1, 3, 2)	(1, 4, 1)	(1, 4, 2)	(2, 1, 1)	(2, 1, 2)
(2, 1, 3)	(2, 1, 4)	(2, 2, 1)	(2, 3, 1)	(2, 4, 1)	(3, 1, 1)	(3, 1, 2)
(3, 2, 1)	(4, 1, 1)	(4, 1, 2)	(4, 2, 1)			

Remark. $\mathcal{PF}_{n,n+1} = \mathcal{PF}_n$. In fact, we do have $b^{a-1} = (n+1)^{n-1}$.

2.1.1 Rational primitive parking functions

Definition 23 (Rational Primitive). *A rational parking function f is said primitive if it is already in non-decreasing order.*

We denote by $\mathcal{PF}'_{a,b}$ the set of primitive a, b - parking functions.

Example ($a = 4, b = 3$). *Limits* : $[1, 1\frac{3}{4}, 2\frac{1}{2}, 3\frac{1}{4}]$

$$f_1 = (1, 1, 2, 2) \in \mathcal{PF}'_{4,3}$$

$$f_2 = (1, 1, 2, 1) \notin \mathcal{PF}'_{4,3}, \text{ even though } f_2 \in \mathcal{PF}_{4,3}.$$

Theorem 10. *Let $pf'_{a,b}$ be the cardinal of $\mathcal{PF}'_{a,b}$. We have*

$$pf'_{a,b} = \frac{1}{a+b} \binom{a+b}{b}$$

which is the rational Catalan number $Cat(a, b)$.

Example ($a = 3, b = 5$).

• $pf'_{a,b} = 7$ • *Limits* : $[1, 2\frac{2}{3}, 4\frac{1}{3}]$

(1, 1, 1) (1, 1, 2) (1, 1, 3) (1, 1, 4) (1, 2, 2) (1, 2, 3) (1, 2, 4)

Remark. $\mathcal{PF}'_{n,n+1} = \mathcal{PF}'_n$. In fact, we do have

$$\begin{aligned} \frac{1}{n+n+1} \binom{n+n+1}{n+1} &= \frac{1}{2n+1} \binom{2n+1}{n+1} = \frac{1}{2n+1} \frac{(2n+1)!}{n!(n+1)!} \\ &= \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1} \frac{(2n)!}{n!n!} = \frac{1}{n+1} \binom{2n}{n} \end{aligned}$$

2.2 Rational Non-crossing Partitions

Definition 24 (a, b - Non-crossing Partition). An a, b - non-crossing partition is *TODO*

We denote by $\mathcal{NC}_{a,b}$ the set of a, b - non-crossing partitions.

Example. An *abncp*

Theorem 11. Let $nc_{a,b}$ be the cardinal of $\mathcal{NC}_{a,b}$. We have

$$nc_{a,b} = \frac{1}{a+b} \binom{a+b}{a} = \frac{(a+b-1)!}{a!b!}$$

, which is the rational Catalan number.

Example. all *abncp* for some a, b

Proposition. This means we can create a bijection between $\mathcal{PF}'_{a,b}$ and $\mathcal{NC}_{a,b}$.

Proof.

- $\mathcal{NC}_{a,b} \rightarrow \mathcal{PF}'_{a,b}$:
- $\mathcal{PF}'_{a,b} \rightarrow \mathcal{NC}_{a,b}$:

□

Definition 25 (a, b - Non-crossing 2-Partition). An a, b - Non-crossing 2-partition is *TODO*

We denote by $\mathcal{NC}_{a,b}^2$ the set of a, b - non-crossing 2-partitions.

Example. some *ncab2*

Theorem 12. Let $nc_{a,b}^2$ be the cardinal of $\mathcal{NC}_{a,b}^2$. We have

$$nc_{a,b}^2 = b^{a-1}$$

.

Example. all *ncab2* for some a b

Proposition. *bijection*

Proof. bijection proof

□

2.3 A direct poset linked to Rational Dyck paths

2.3.1 Rational Dyck Paths

Definition 26 (a, b - Dyck path). An a, b - Dyck word is a word $w \in \{0, 1\}^*$ such that :

- for each suffix w' of w ,

$$|w'|_1 \geq \frac{a}{b} |w'|_0$$

.

- $|w|_0 = b$.

- $|w|_1 = a$.

An a, b - Dyck word can be represented as a path from $(0, 0)$ to (b, a) that stays over $y = \frac{a}{b}x$, called an a, b - Dyck path :

- Each 1 corresponds to a North step \uparrow .
- Each 0 corresponds to an East step \rightarrow .

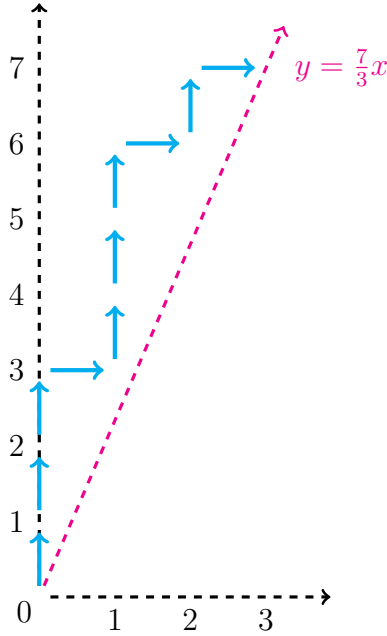
We denote by $\mathcal{R}_{a,b}$ the set of a, b - Dyck words.

Example ($a > b : a = 7, b = 3$).

$w_1 = 1110011110$ is not a 7, 3 - Dyck word, because $|11100|_1 = 3$

$$< \frac{7}{3}|11100|_0 = \frac{14}{3} = 4\frac{1}{3}.$$

$w_2 = 1110111010$ is a 7, 3 - Dyck word :

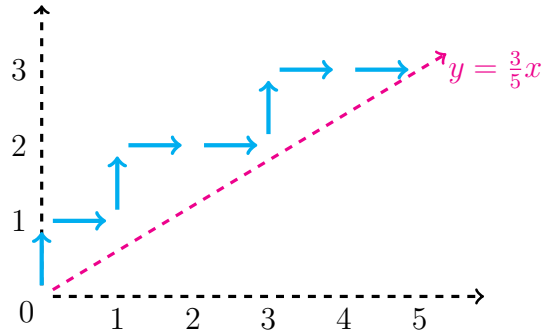


Example ($a < b : a = 3, b = 5$).

$w_1 = 10100010$ is not a 3, 5 - Dyck word, because $|101000|_1 = 2$

$$< \frac{3}{5}|101000|_0 = \frac{12}{5} = 2\frac{2}{5}.$$

$w_2 = 10100100$ is a 3, 5 - Dyck word :

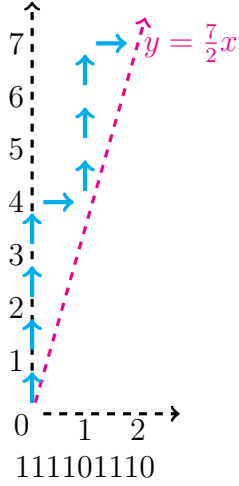


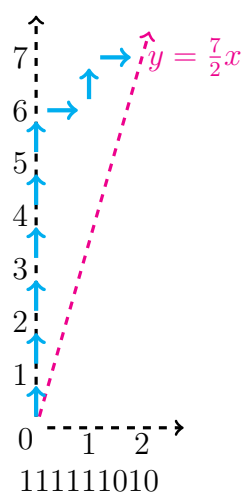
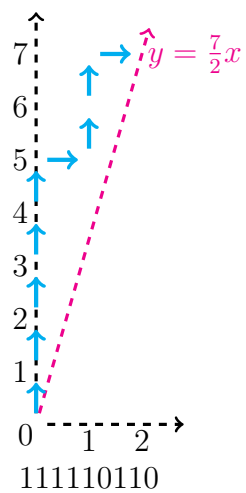
Theorem 13. Let $r_{a,b}$ be the cardinal of $\mathcal{R}_{a,b}$. We have

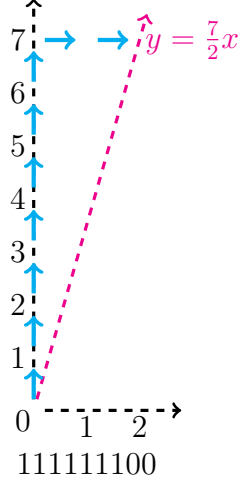
$$r_{a,b} = \frac{1}{a+b} \binom{a+b}{a} = \frac{(a+b-1)!}{a!b!}$$

, which is again the rational Catalan number.

Example ($a = 7, b = 2$). $r_n = 4$.







Proposition. *This means we can create a bijection between $\mathcal{PF}'_{a,b}$ and $\mathcal{R}_{a,b}$.*

Proof.

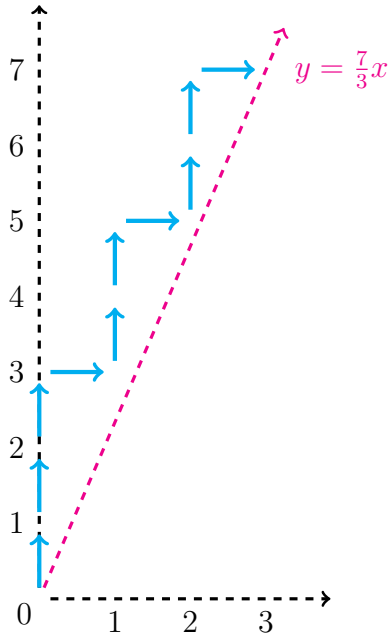
- $\mathcal{PF}'_{a,b} \rightarrow \mathcal{R}_{a,b}$: Let $f = (a_1, \dots, a_n) \in \mathcal{PF}'_{a,b}$ be our rational primitive parking function. For $i \in \{1, \dots, b\}$, we define l_i the number of occurrences of i in f .
The corresponding rational Dyck word will be $\underbrace{1 \dots 1}_l 0 \underbrace{1 \dots 1}_l 0 \dots \underbrace{1 \dots 1}_l 0$.
- $\mathcal{R}_{a,b} \rightarrow \mathcal{PF}'_{a,b}$: Let w be our rational Dyck word, and consider its path representation. We define s_i to be the distance between the segment from $(0, i-1)$ to $(0, i)$ and the i^{th} North step. Then, let $a_i = s_i + 1$.
The corresponding rational primitive parking function is (a_1, \dots, a_a) .

□

Remark. *This bijection is exactly the same as the one between classical primitive parking functions and Dyck paths.*

Example ($a > b : a = 7, b = 3, \mathcal{PF}'_{a,b} \rightarrow \mathcal{R}_{a,b}$).

- $f = (1, 1, 1, 2, 2, 3, 3)$
 $l_1 = 3 \qquad l_2 = 2 \qquad l_3 = 2$
- $w = (1110110110)$



Example ($a < b : a = 3, b = 5, \mathcal{PF}'_{a,b} \rightarrow \mathcal{R}_{a,b}$).

- $f = (1, 1, 4)$

$$l_1 = 2$$

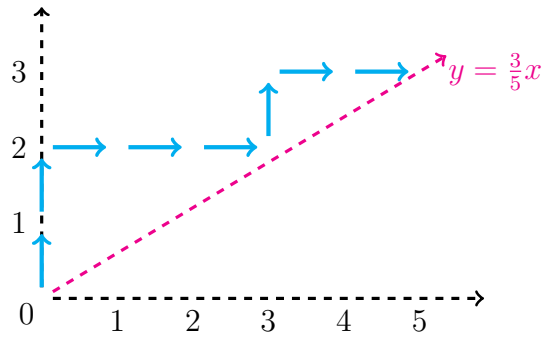
$$l_2$$

$$l_3 = 0$$

$$l_4 = 1$$

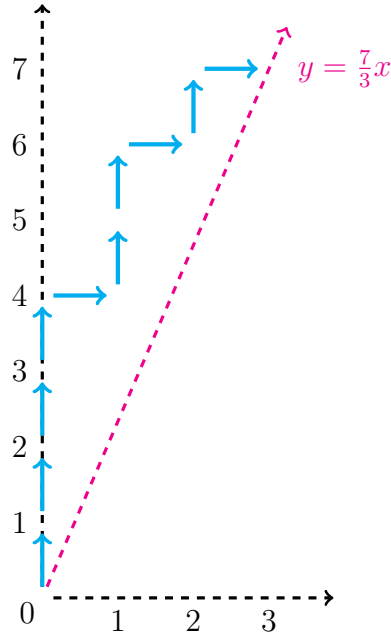
$$l_5 = 0$$

- $w = 11000100$



Example ($a > b : a = 7, b = 3, \mathcal{R}_{a,b} \rightarrow \mathcal{PF}'_{a,b}$).

- $w = 1111011010$



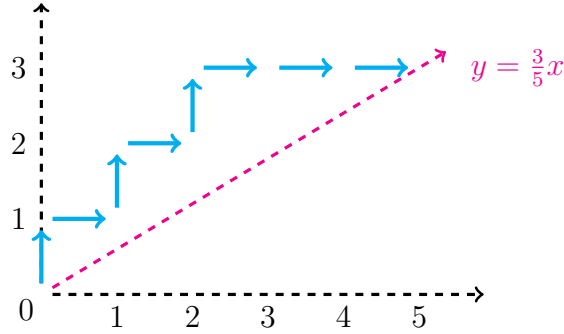
- *Distances :*

$s_1 = 0$	$a_1 = 1$
$s_2 = 0$	$a_2 = 1$
$s_3 = 0$	$a_3 = 1$
$s_4 = 0$	$a_4 = 1$
$s_5 = 1$	$a_5 = 2$
$s_6 = 1$	$a_6 = 2$
$s_7 = 2$	$a_7 = 3$

- $f = (1, 1, 1, 1, 2, 2, 3)$

Example ($a < b : a = 3, b = 5, \mathcal{R}_{a,b} \rightarrow \mathcal{PF}'_{a,b}$).

- $w = 10101000$



- *Distances :*

$$\begin{array}{ll} s_1 = 0 & a_1 = 1 \\ s_2 = 1 & a_2 = 2 \\ s_3 = 2 & a_3 = 3 \end{array}$$

- $f = (1, 2, 3)$

2.3.2 Rational Labeled Dyck Paths

Definition 27 (Labeled a, b - Dyck Path). A labeled a, b - Dyck word is a word $w \in \{0, \dots, n\}^*$ such that :

- for each suffix w' of w ,

$$|w'|_{\neq 0} \geq \frac{a}{b} |w'|_0$$

.

- $|w|_0 = b$.
- $|w|_{\neq 0} = a$.
- for each $i \in \{1, \dots, a\}$, w has exactly one occurrence of i .
- if $w_i \neq 0$ and $w_{i+1} \neq 0$, then $w_i < w_{i+1}$. That is, consecutive North steps have increasing labels.

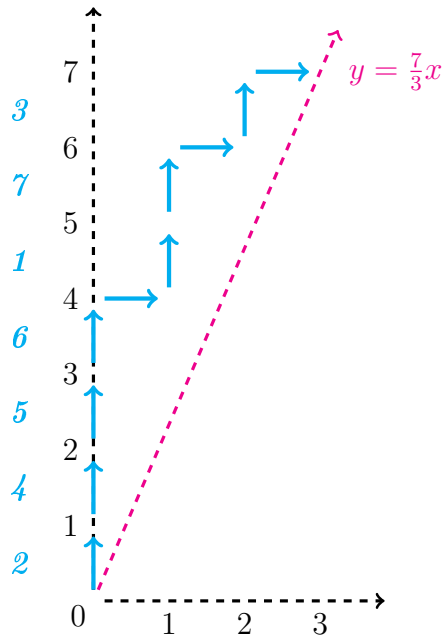
A labeled a, b - Dyck word can be represented as a path from $(0, 0)$ to (b, a) , where each North step is associated to a label :

- Each $i \neq 0$ corresponds to a North step \uparrow labeled i .
- Each 0 corresponds to an East step \rightarrow .

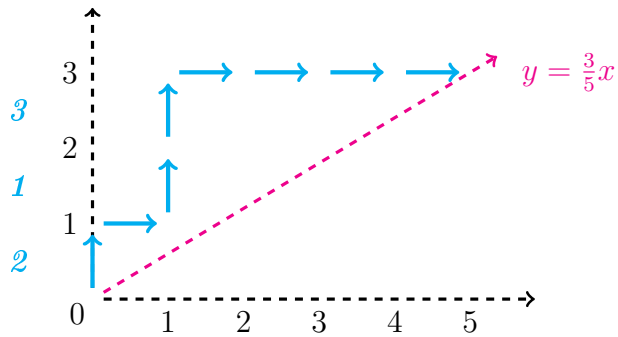
Those paths are called labeled a, b - Dyck paths.

We denote by $\mathcal{LR}_{a,b}$ the set of labeled a, b - Dyck words.

Example ($a > b : a = 7, b = 3$). $w_2 = 2456017030$:



Example ($a < b : a = 3, b = 5$). $w = 20130000$:



Theorem 14. *Let $lr_{a,b}$ be the cardinal of $\mathcal{LR}_{a,b}$. We have*

$$lr_{a,b} = b^{a-1}$$

Example ($a > b : a = 4, b = 3$). $lr_{a,b} = 3^3 = 27$

- *Word of shape XXXX000 :*

1234000

- *Words of shape XXX0X00 :*

1230400

1240300

1340200

2340100

- *Words of shape XX0XX00 :*

1203400

1302400

1402300

2301400

2401300

3401200

- *Words of shape XXX00X0 :*

1230040

1240030

1340020

2340010

- *Words of shape XX0X0X0 :*

1203040

1204030

1302040

1304020

1402030

1403020

2301040

2304010

2401030

2403010

3401020

3402010

Proposition. *This means we can create a bijection between $\mathcal{PF}_{a,b}$ and $\mathcal{LR}_{a,b}$.*

Proof.

- $\mathcal{PF}_{a,b} \rightarrow \mathcal{LR}_{a,b}$: Let $f = (a_1, \dots, a_n) \in \mathcal{PF}_{a,b}$ be our a, b - parking function. For $i \in \{1, \dots, b\}$, we define $im_i : \{j \mid a_j = i\}$. We then define $im_{i,1}, \dots, im_{i,k_i}$ to be the elements of im_i in increasing order.

The corresponding labeled a, b - Dyck word will be

$$\underbrace{im_{1,1} \cdots im_{1,k_1}}_{im_1} 0 \underbrace{im_{2,1} \cdots im_{2,k_2}}_{im_2} 0 \cdots \underbrace{im_{n,1} \cdots im_{n,k_n}}_{im_b} 0.$$

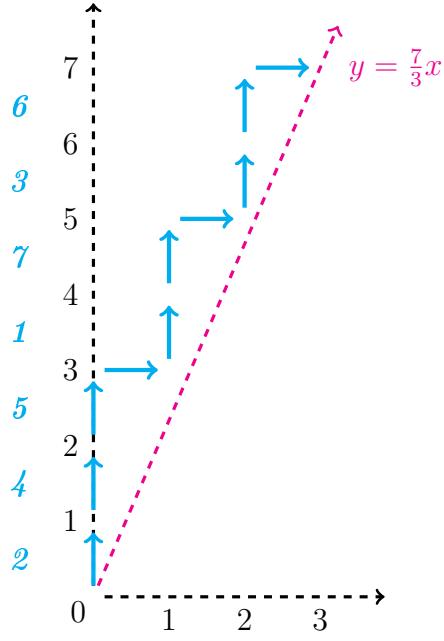
- $\mathcal{LR}_{a,b} \rightarrow \mathcal{PF}_n$: Let w be our labeled a, b - Dyck word, and consider its path representation. We define s_i to be the distance between the segment from $(0, i-1)$ to $(0, i)$ and the i^{th} North step. Then, let $label(i)$ be the label of the i^{th} North step, and $dist_i = \{label(j) \mid s_j = i\}$ be the set of the labels of all North steps at distance i . Then, if $j \in dist_i$, let $a_j = i+1$. The corresponding parking function is (a_1, \dots, a_a) .

□

Remark. *This bijection is exactly the same as the one between classical parking functions and labeled Dyck paths.*

Example ($a > b : a = 7, b = 3, \mathcal{PF}_{a,b} \rightarrow \mathcal{LR}_{a,b}$).

- $f = (2, 1, 3, 1, 1, 3, 2)$
 $im_1 = \{2, 4, 5\} \qquad im_2 = \{1, 7\} \qquad im_3 = \{3, 6\}$
- $w = 2450170360$



Example ($a < b : a = 3, b = 5, \mathcal{PF}_{a,b} \rightarrow \mathcal{LR}_{a,b}$).

- $f = (4, 1, 2)$

$$im_1 = \{2\}$$

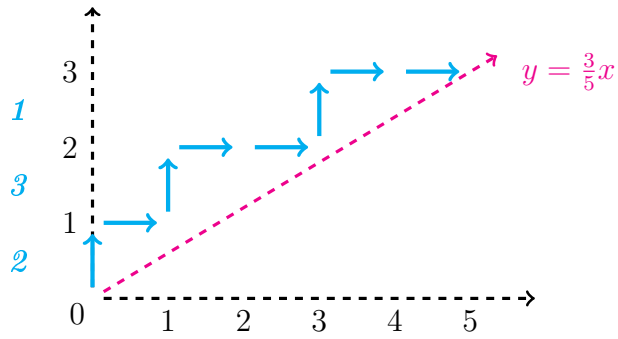
$$im_2 = \{3\}$$

$$im_3 = \emptyset$$

$$im_4 = \{1\}$$

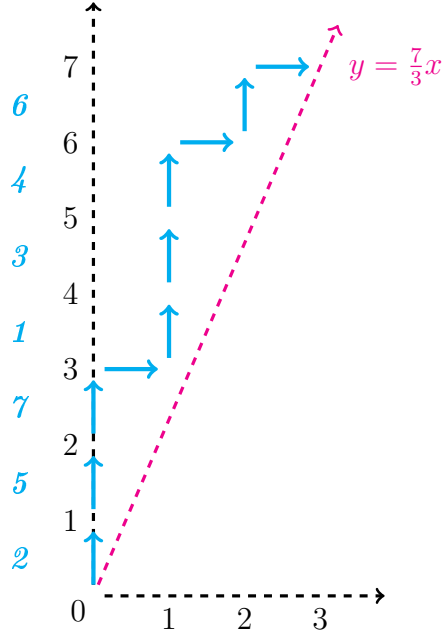
$$im_5 = \emptyset$$

- $w = 20300100$



Example ($a > b : a = 7, b = 3, \mathcal{LR}_{a,b} \rightarrow \mathcal{PF}_{a,b}$).

- $w = 2570134060$



- *Distances :*

$$\begin{array}{lll} s_1 = 0 & s_2 = 0 & s_3 = 0 \\ s_4 = 1 & s_5 = 1 & s_6 = 1 \\ s_7 = 2 \end{array}$$

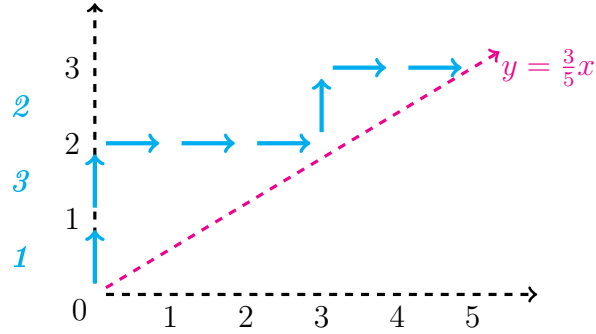
- *Labels :*

$$\begin{array}{lll} dist_0 = \{2, 5, 7\} & dist_1 = \{1, 3, 4\} & dist_2 = \\ \{6\} & & \end{array}$$

- $f = (2, 1, 2, 2, 1, 3, 1)$

Example ($a < b : a = 3, b = 5, \mathcal{LR}_{a,b} \rightarrow \mathcal{PF}_{a,b}$).

- $w = 13000200$



- Distances :

$$s_1 = 0 \qquad s_2 = 0 \qquad s_3 = 3$$

- Labels :

$$\begin{array}{lll} dist_0 = \{1, 3\} & dist_1 = \emptyset & dist_2 = \emptyset \\ dist_3 = \{2\} & dist_4 = \emptyset & \end{array}$$

- $f = (1, 4, 1)$

Remark. The rational primitive parking functions are exactly the rational parking functions corresponding to rational labeled Dyck paths where the i^{th} North step is labeled i .

2.3.3 Rational Dyck - Parking Posets

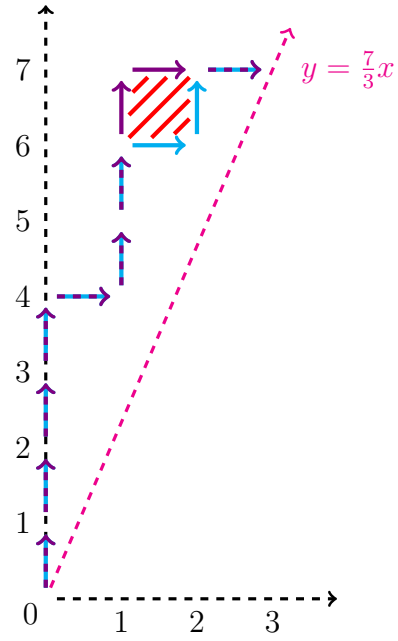
Rational Primitive Dyck - Parking Posets

Definition 28 (\succ_r). For w and w' two a, b - Dyck words, we say that w covers w' , written $w \succ_r w'$, if $\exists w_1, w_2$ such that :

- $w = w_1 0 1 w_2$
- $w' = w_1 1 0 w_2$

Example ($a = 7, b = 3$). $1111011010 \succ_r 1111011100$

- $w_1 = 1111011$
- $w_2 = 0$

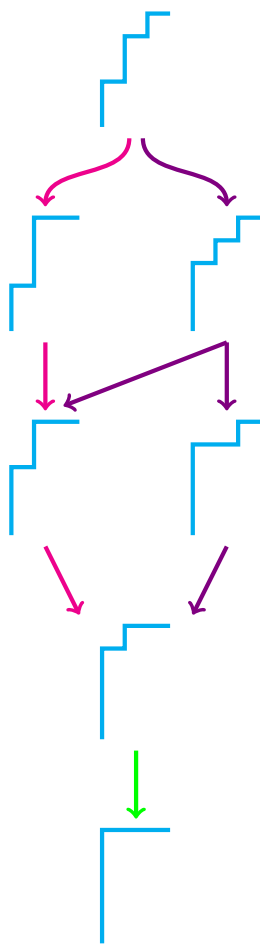


■ 1111011010 ■ 1111011100 ■ difference

Remark. If $w_1 \succ_r w_2$, then the path corresponding to w_2 is over the path corresponding to w_1 , and the difference between the two paths is a square of size 1 by 1.

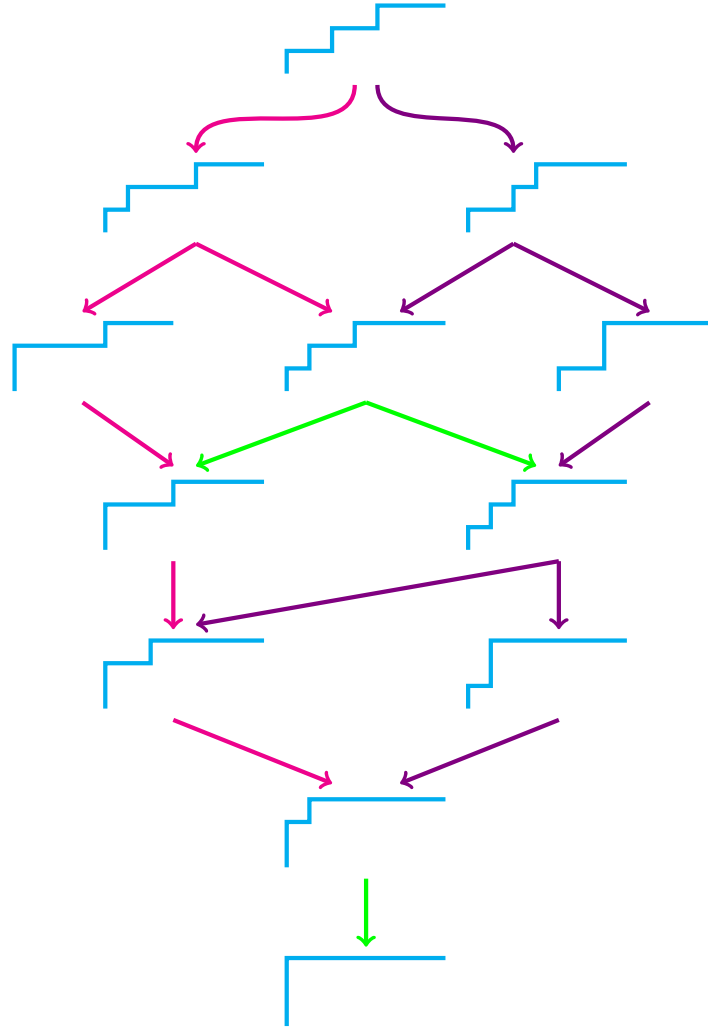
Proposition. This covering relation defines a poset for $\mathcal{R}_{a,b}$.

Example ($a > b$: The poset of $\mathcal{R}_{5,3}$).



There are $\frac{1}{8} \binom{8}{5} = \frac{42}{6} = 7$ elements in this poset.

Example ($a < b$: The poset of $\mathcal{R}_{3,7}$).



There are $\frac{1}{10} \binom{10}{3} = \frac{72}{6} = 12$ elements in this poset.

Definition 29 (\succ'). For f and g two rational primitive a, b - parking functions, we say that f covers g , written $f \succ' g$, if $\exists i$ such that :

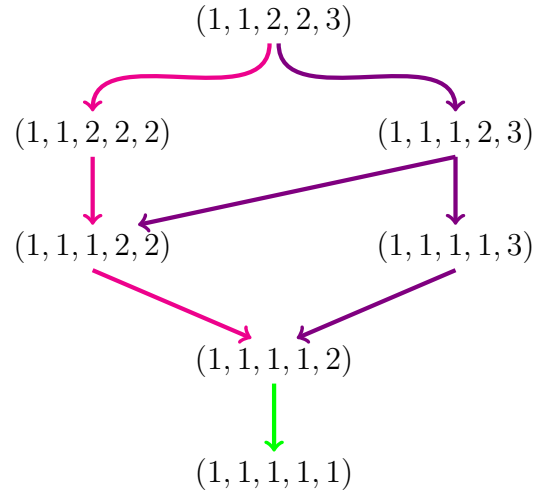
- $f = (a_1, \dots, a_{i-1}, a_i, \quad a_{i+1}, \dots, a_n)$
- $g = (a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_n)$

Example ($a > b : a = 7, b = 3$). $(1, 1, 1, 2, 2, 2, 3) \succ' (1, 1, 1, 1, 2, 2, 3)$

Example ($a < b : a = 3, b = 5$). $(1, 2, 4) \succ' (1, 1, 4)$

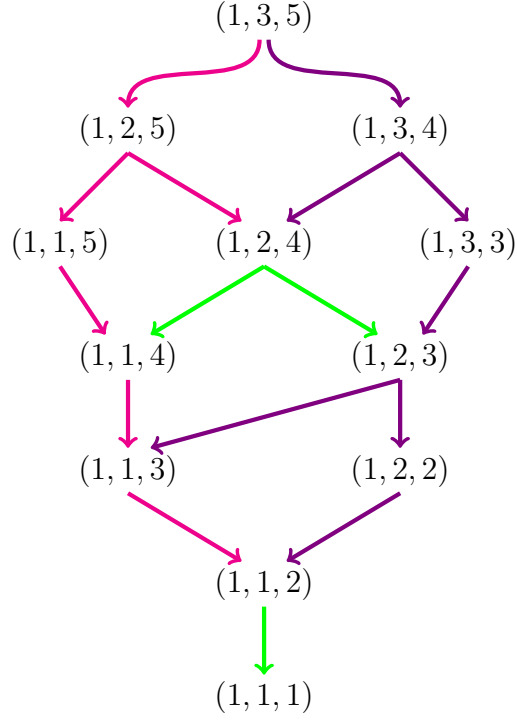
Proposition. *This covering relation defines a poset for $\mathcal{PF}'_{a,b}$.*

Example ($a > b$: The poset of $\mathcal{PF}'_{5,3}$).



There are $\frac{1}{8} \binom{8}{5} = \frac{42}{6} = 7$ elements in this poset.

Example ($a < b$: The poset of $\mathcal{PF}'_{3,7}$).



There are $\frac{1}{10} \binom{10}{3} = \frac{72}{6} = 12$ elements in this poset.

Remark. The posets of $\mathcal{PF}'_{a,b}$ and $\mathcal{R}_{a,b}$ are isomorphic, and one can be obtained by applying the aforementioned bijection to the other.

Rational Dyck - Parking Posets

Definition 30 (\succ_{lr}). For w and w' two labeled a, b - Dyck words, we say that w covers w' , written $w \succ_{lr} w'$, if $\exists l, r, x, x', y, z, z'$ such that :

- l is either empty or ends with 0
- r is either empty or starts with 0
- $x = x_1 x_2 \cdots$ has all its digits > 0
- $z = z_1 z_2 \cdots$ has all its digits > 0
- $x' = x$ where y is correctly inserted regarding the order condition

- y is in z , and $z' = z$ where y is removed
- $w = lx0zr$
- $w' = lx'0z'r$

Example ($a > b : a = 7, b = 3$). $2460150370 \succ_{lr} 2460135070$

- $l = 2460$
- $r = 0$
- $x = 15$
- $x' = 135$
- $y = 3$
- $z = 37$
- $z' = 7$

Example ($a < b : a = 3, b = 5$). $20301000 \succ_{lr} 20130000$

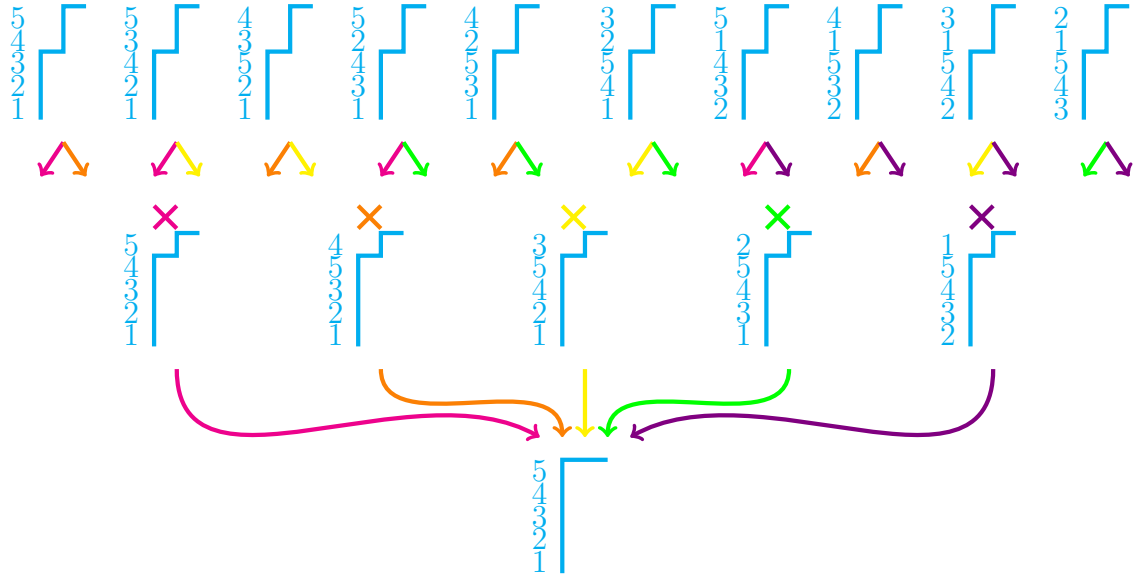
- $l = 20$
- $r = 000$
- $x = 3$
- $x' = 13$
- $y = 1$
- $z = 1$
- $z' = \emptyset$

Remark. If $w_1 \succ_{lr} w_2$, then the path corresponding to w_2 is over the path corresponding to w_1 , and the difference between the two paths is a square of size 1 by 1.

Furthermore, the covering relation can be seen as follows : w_1 covers w_2 if we can obtain w_2 by taking a digit from the $i + 1^{th}$ rise of w_1 , and inserting it into the i^{th} rise of w_1 in increasing order.

Proposition. *This covering relation defines a poset for $\mathcal{LR}_{a,b}$.*

Example ($a > b$: The poset of $\mathcal{LR}_{5,2}$).

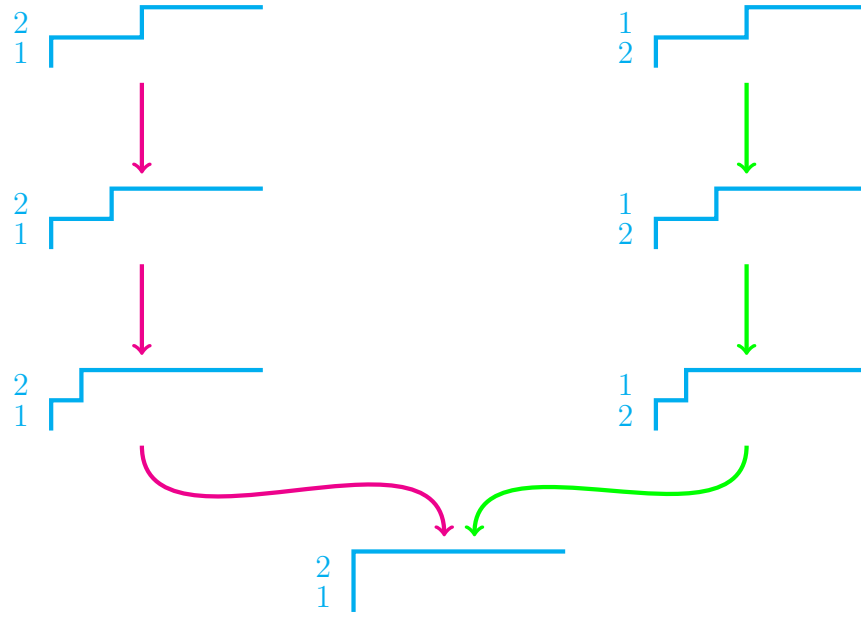


Arrows have been simplified for readability.

Arrows between the top 2 levels are to be read as ending at the cross of the corresponding color.

There are $2^4 = 16$ elements in this poset.

Example ($a < b$: The poset of $\mathcal{LR}_{2,7}$).



There are $7^1 = 7$ elements in this poset.

Definition 31 (\succ). For f and g two rational parking functions, we say that f covers g , written $f \succ g$, if $\exists i$ such that :

- $f = (a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n)$
- $g = (a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_n)$

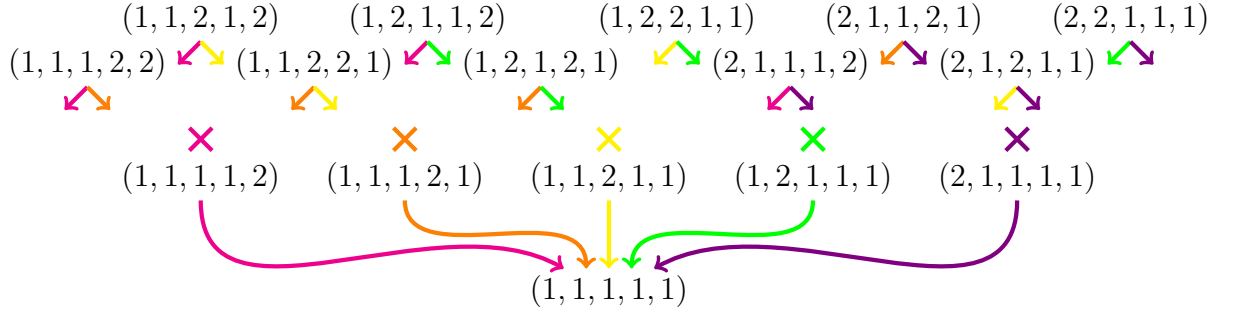
That is, the same relation as for rational primitive parking functions.

Example ($a > b : a = 7, b = 3$). $(2, 3, 1, 1, 2, 1, 3) \succ (2, 3, 1, 1, 1, 1, 3)$

Example ($a < b : a = 3, b = 5$). $(4, 1, 2) \succ (3, 1, 2)$

Proposition. This covering relation defines a poset for $\mathcal{PF}_{a,b}$.

Example ($a > b$: The poset of $\mathcal{PF}_{5,2}$).

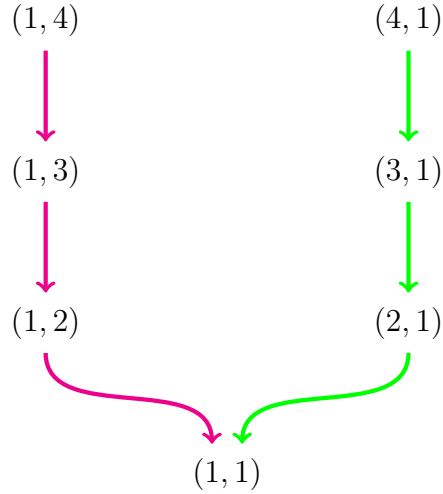


Arrows have been simplified for readability.

Arrows between the top 2 levels are to be read as ending at the cross of the corresponding color.

There are $2^4 = 16$ elements in this poset.

Example ($a < b$: The poset of $\mathcal{PF}_{2,7}$).



There are $7^1 = 7$ elements in this poset.

Remark. *The posets of $\mathcal{PF}_{a,b}$ and $\mathcal{LR}_{a,b}$ are isomorphic, and one can be obtained by applying the aforementioned bijection to the other.*

Chapter 3

Trees

3.1 Parking Trees

Definition 32 (Parking Trees). A parking tree is defined from a parking function $f = (a_1, \dots, a_n) \in \mathcal{PF}_n$ as follows :

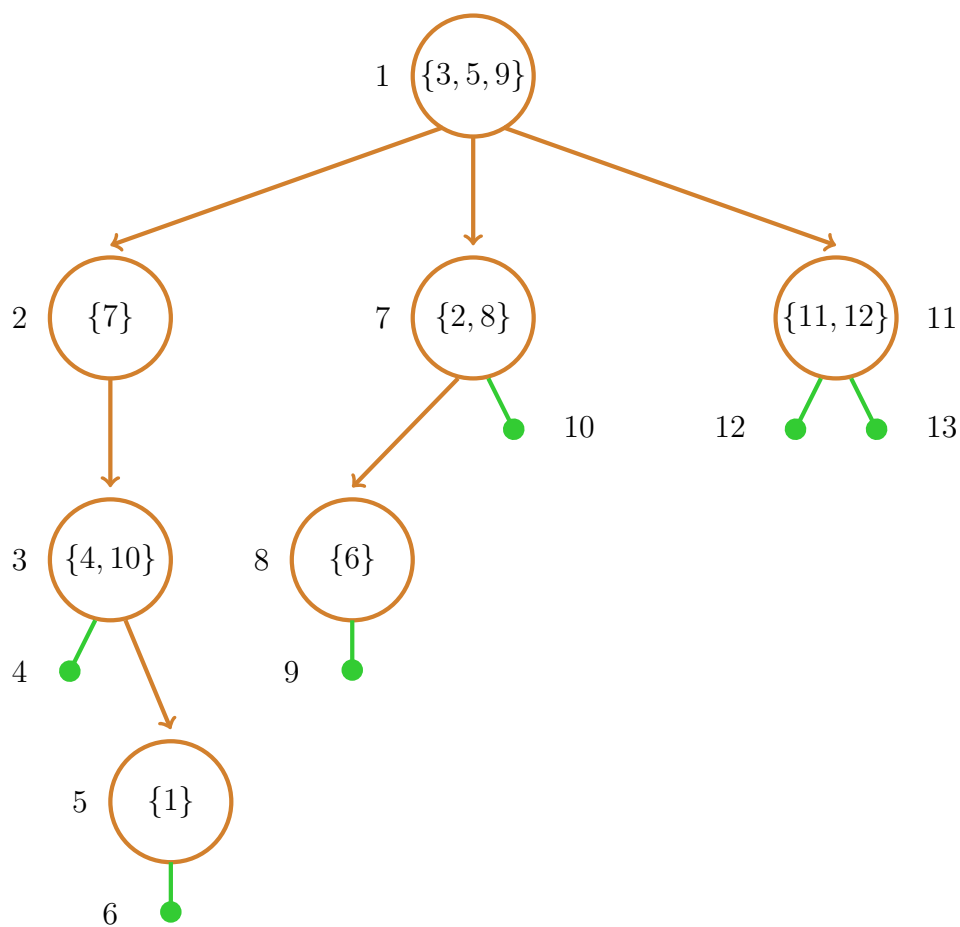
- For $1 \leq i \leq n+1$, we define s_i as $\{j \mid a_j = i\}$
- $[s_1, \dots, s_{n+1}]$ describes the pre-order depth-first traversal of the tree.
- Each node labeled by a set of size k has k children.

Remark. The leaves of the tree are those corresponding to an element i such that $1 \leq i \leq n+1$, and i is not in f .

Furthermore, as we will have a total edges by definition, the presence of a node corresponding to $n+1$ is necessary, even though it will always be empty.

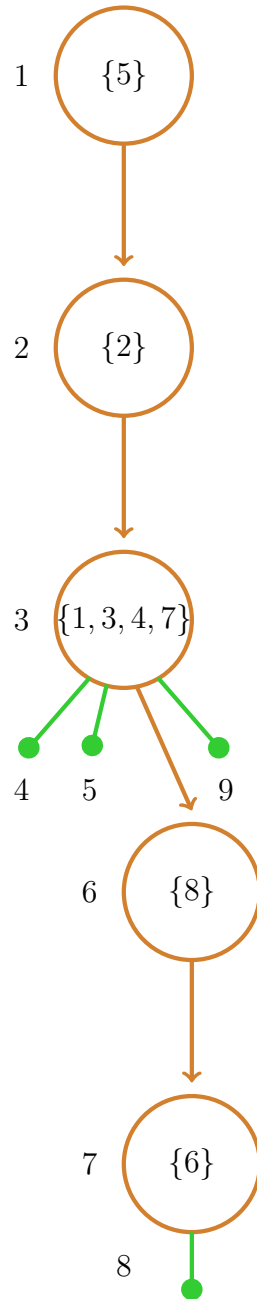
Example ($n = 12$).

- $f = (5, 7, 1, 3, 1, 8, 2, 7, 1, 3, 11, 11)$
- Labels : $[\{3, 5, 9\}, \{7\}, \{4, 10\}, \emptyset, \{1\}, \emptyset, \{2, 8\}, \{6\}, \emptyset, \emptyset, \{11, 12\}, \emptyset, \emptyset]$



Conversely, by reading the labels of a parking tree depth-first in pre-order, we get the list of positions of each number in the corresponding parking function, thus creating a *bijection*.

Example (From parking tree to parking function).



- The labels are $[\{5\}, \{2\}, \{1, 3, 4, 7\}, \emptyset, \emptyset, \{8\}, \{6\}, \emptyset, \emptyset]$.
- Thus the corresponding parking function is $(3, 2, 3, 3, 1, 7, 3, 6) \in \mathcal{PF}_8$.

3.2 Rational Parking Trees

Definition 33 (Rational Parking Trees). *A rational parking tree is defined from a rational parking function $f = (a_1, \dots, a_a) \in \mathcal{PF}_{a,b}$ as follows :*

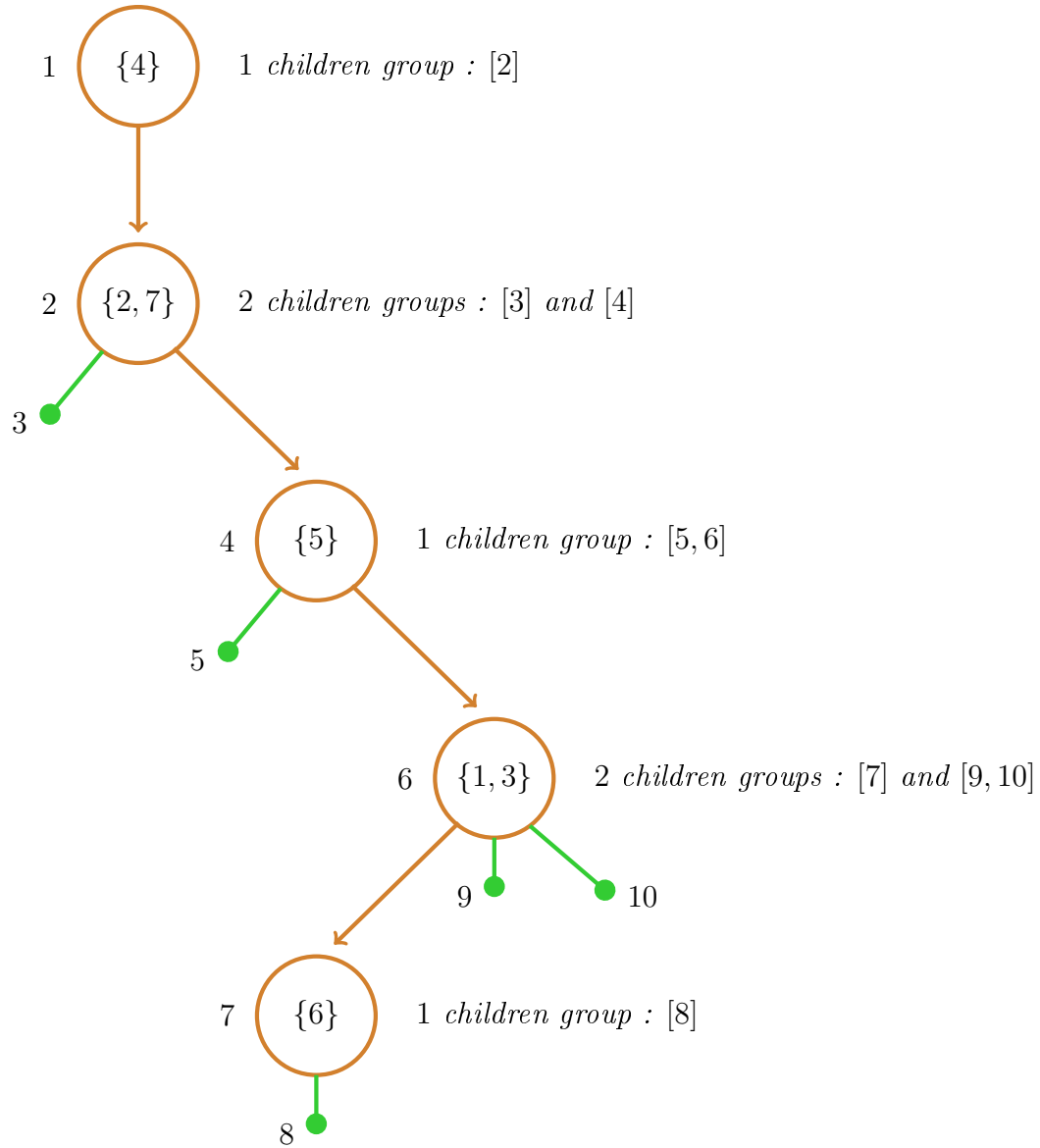
- *For $1 \leq i \leq n + 1$, we define the limit l_i as the integer portion of $\frac{b}{a}(i - 1) + 1$.
Let $l_0 = 0$.*
- *From these limits, we deduce the intervals $itv_i =]l_{i-1}, l_i]$ for $1 \leq i \leq a + 1$.*
- *For $1 \leq i \leq b + 1$, define s_i as $\{j \mid a_j = i\}$.*
- *$[s_1, \dots, s_{b+1}]$ describes the pre-order depth-first traversal of the tree.*
- *Each node labeled by a set of size k has k groups of children, which are defined by the intervals.*

Example ($a < b$).

- $a = 7$
- $b = 9$
- *Limits :* $[1, 2\frac{2}{7}, 3\frac{4}{7}, 4\frac{6}{7}, 6\frac{1}{7}, 7\frac{3}{7}, 8\frac{5}{7}, 10]$
- *Integral limits :* $[0, 1, 2, 3, 4, 6, 7, 8, 10]$
- *Intervals :*

$$]0, 1] \quad]1, 2] \quad]2, 3] \quad]3, 4] \quad]4, 6] \quad]6, 7] \quad]7, 8] \quad]8, 10]$$
- *Children groups :*

$$[1] \quad [2] \quad [3] \quad [4] \quad [5, 6] \quad [7] \quad [8]$$
- $f = (6, 2, 6, 1, 4, 7, 2)$
- *Labels :* $\{\{4\}, \{2, 7\}, \emptyset, \{5\}, \emptyset, \{1, 3\}, \{6\}, \emptyset, \emptyset, \emptyset\}$



Example ($a > b$).

- $a = 9$
- $b = 7$
- *Limits* : $[1, 1\frac{7}{9}, 2\frac{5}{9}, 3\frac{3}{9}, 4\frac{1}{9}, 4\frac{8}{9}, 5\frac{6}{9}, 6\frac{4}{9}, 7\frac{2}{9}, 8]$

- *Integral limits* : $[0, 1, 1, 2, 3, 4, 4, 5, 6, 7, 8]$

- *Intervals* :

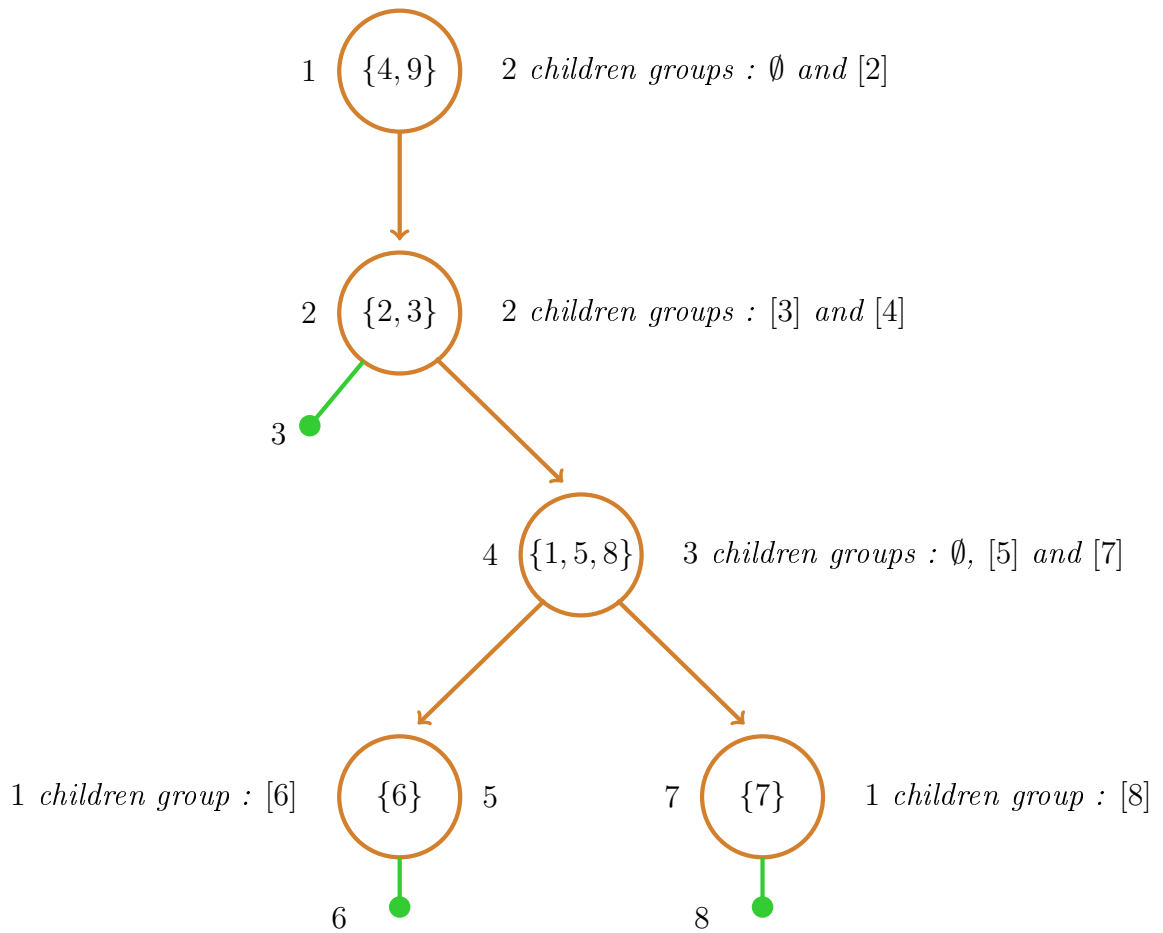
$]0, 1]$	$]1, 1]$	$]1, 2]$	$]2, 3]$	$]3, 4]$
$]4, 4]$	$[4, 5]$	$]5, 6]$	$]6, 7]$	$]7, 8]$

- *Children groups* :

$[1]$	\emptyset	$[2]$	$[3]$	$[4]$	\emptyset	$[5]$	$[6]$	$[7]$	$[8]$
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- $f = (4, 2, 2, 1, 4, 5, 7, 4, 1)$

- *Labels* : $\{\{4, 9\}, \{2, 3\}, \emptyset, \{1, 5, 8\}, \{6\}, \emptyset, \{7\}, \emptyset\}$



In both cases, the converse direction of the *bijection* is obtained with the same method as for classical parking trees.

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