Rational Parking Functions

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Abstract

This is an abstract about Rational Parking Functions

1 Parking Functions

Definition 1 (Parking Function). *A* parking function *is a sequence* $(a_1, a_2, \ldots, a_n)$ *such that its non-decreasing reordering* $(b_1, b_2, \ldots, b_n)$ *has* $b_i < i$ *for all* i.
We denote by $\mathcal{PF}_n$ *the set of parking functions of length* n.

$$\mathcal{PF} = \bigcup_{n>0} \mathcal{PF}_n$$

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Example.

$$f_1 = (7, 3, 1, 4, 2, 5, 2) \in \mathcal{PF}_7$$
$$f_2 = (7, 3, 1, 4, 2, 5, 4) \notin \mathcal{PF}_7$$

Proposition. *Let* pf_n *be the cardinal of* $\mathcal{PF}_n$*. We have* $pf_n = (n+1)^{n-1}$.

Example $(n = 1, 2, 3)$**.**
• $n = 1 \quad : \quad pf_1 = 1$
(1)

• $n = 2 \quad : \quad pf_2 = 3$
$(1,1) \qquad (1,2) \qquad (2,1)$
• $n = 3 \quad : \quad pf_3 = 16$
$(1,1,1) \qquad (1,1,2) \qquad (1,1,3) \qquad (1,2,1) \qquad (1,2,2) \qquad (1,2,3) \qquad (1,3,1)$
$(1,3,2) \qquad (2,1,1) \qquad (2,1,2) \qquad (2,1,3) \qquad (2,2,1) \qquad (2,3,1) \qquad (3,1,1)$
$(3,1,2) \qquad (3,2,1)$

Definition 2 (Primitive)**.** *A parking function* $(a_1, a_2, \ldots, a_n)$ *is said* primitive *if it is already in non-decreasing order.*
We denote by $\mathcal{PPF}_n$ *the set of primitive parking functions of length* n*.*

$$\mathcal{PPF} = \bigcup_{n>0} \mathcal{PPF}_n$$

Example.

$$f_1 = (1,2,2,3) \in \mathcal{PPF}_4$$
$$f_2 = (1,2,3,2) \notin \mathcal{PPF}_4, \textit{ even though } f_2 \in \mathcal{PF}_4$$

Proposition. *Let* ppf_n *be the cardinal of* $\mathcal{PPF}_n$*. We have* $ppf_n = \frac{1}{n+1}\binom{2n}{n}$*, which is the* n^{th} *Catalan number.*

Example $(n = 1, 2, 3)$**.**
• $n = 1 \quad : \quad ppf_1 = 1$
(1)
• $n = 2 \quad : \quad ppf_2 = 2$
$(1,1) \qquad (1,2)$
• $n = 3 \quad : \quad ppf_3 = 5$
$(1,1,1) \qquad (1,1,2) \qquad (1,1,3) \qquad (1,2,2) \qquad (1,2,3)$

2 Non-crossing Partitions

Definition 3 (Non-crossing Partition)**.** *A* non-crossing partition *of a set* E *is a set partition* $P = \{E_1, E_2, \ldots, E_k\}$ *such that if* $a, c \in E_i$*,* $b, d \in E_j$*, and*

$i \neq j$, then we do not *have $a < b < c < d$, nor $a > b > c > d$.*
We denote by $\mathcal{NC}_n$ the set of non-crossing partitions of $\{1, 2, \ldots, n\}$.

$$\mathcal{NC} = \bigcup_{n>0} \mathcal{NC}_n$$

Notation. $[n] = \{1, 2, \ldots, n\}$

Example $(E = [6])$**.**

$$P_1 = \{\{1,2,5\},\{3,4\},\{6\}\} \in \mathcal{NC}_6$$
$$P_2 = \{\{1,2,4\},\{3,5\},\{6\}\} \notin \mathcal{NC}_6$$

Proposition. *Let nc_n be the cardinal of $\mathcal{NC}_n$. We have $nc_n = \frac{1}{n+1}\binom{2n}{n}$, which is the n^{th} Catalan number.*

Example $(n = 1, 2, 3)$**.**
• $n = 1 \quad : \quad nc_1 = 1$
$\{\{1\}\}$
• $n = 2 \quad : \quad nc_2 = 2$
$\{\{1,2\}\} \qquad \{\{1\},\{2\}\}$
• $n = 3 \quad : \quad nc_3 = 5$
$\{\{1,2,3\}\} \qquad \{\{1\},\{2,3\}\} \qquad \{\{1,3\},\{2\}\} \qquad \{\{1,2\},\{3\}\} \qquad \{\{1\},\{2\},\{3\}\}$

Proposition. *This means we can create a* bijection *between $\mathcal{PPF}_n$ and $\mathcal{NC}_n$.*

- $\mathcal{NC}_n \to \mathcal{PPF}_n$: For each block B in the non-crossing partition, take $i = min(B)$, and $k_i = size(B)$.
  $k_i = 0$ if i is not the minimum of a block.
  The corresponding parking function is $(\underbrace{1,\ldots,1}_{k_1},\underbrace{2,\ldots,2}_{k_2},\ldots,\underbrace{n,\ldots,n}_{k_n})$.

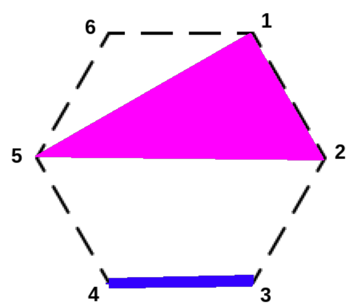
- $\mathcal{PPF}_n \to \mathcal{NC}_n$: For each i in $[n]$, if i appears n_i times in the parking function, B_i will be of size n_i with minimum element i. There is a unique set partition $P = \bigcup_i B_i$ of $[n]$ respecting these conditions that is non-crossing.

Example ($E = [6]$)**.**

$$P = \{\{1,2,5\},\{3,4\},\{6\}\} \qquad\qquad f = (1,1,1,3,3,6)$$

A non-crossing partition of $[n]$ can be represented graphically on a regular n-vertices polygon, with vertices labeled from 1 to n clockwise. We then represent each block $B = \{b_1, \ldots, b_k\}$ by the convex hull of $\{b_1, \ldots, b_k\}$.

Example ($P = \{\{1,2,5\},\{3,4\},\{6\}\}$)**.**



Thus non-crossing meaning the hulls are *disjoint*.

3 Non-crossing 2-partitions

Definition 4 (Non-crossing 2-partition)**.** *A* non-crossing 2-partition *of a set E is a pair (P, σ) where :*
- *P is a non-crossing partition of E*
- *σ is a permutation of the elements of E*