PHY459: Turbulence

Homework 7
Burger's Equation w. Smagorinsky

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Contents

1 Objective **2**

2 Master **3**
2.1 Time March 5
2.1.1 Derivatives 6
2.1.2 Filter 7
2.1.3 Seed 8
2.1.4 Smagorinsky 8
2.1.4.1 Dealiasing 8
2.2 Figures 9
2.2.1 Results 11

The code featured here is written for `MATLAB`.

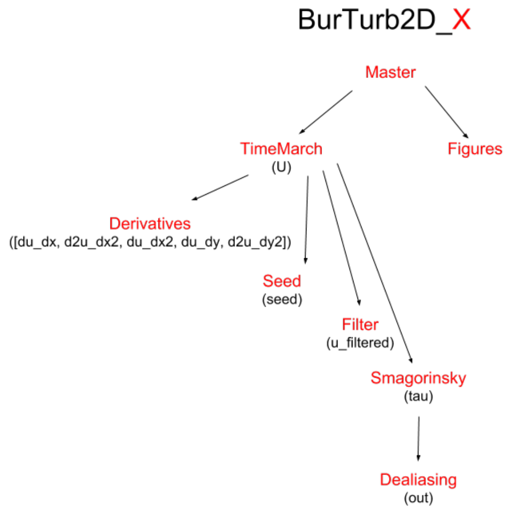


Figure 1: BurTurb2D code flow.

1 Objective

To tackle this problem of implementing the Burger's equation with Smagorinsky's turbulence model[1], I've decided to do some fun trickery and make a 2-dimensional simulation. However if I am to make this a two dimensional model, then it may be neat to observe some asymmetry. Thus, I am choosing a different initial condition than specified for the original simulation of Burger's equation, where we utilized an initial condition with a sine-wave. Our simulation domain $(x \times y)$ will be 0×0 to $2\pi \times 2\pi$, however in the x-direction IC will be $u_x(x, y; t = 0) = 1 - \cos(x(i))$ for all i, and for the y-direction, $u_y(x, y; t = 0) = 1 - \cos(2y(j))$ where $\forall j < n_y/2 + 1$, otherwise the solution is flipped, i.e. $u_y(x, y; t = 0) = -(1 - \cos(2y))$.

In order to solve this equation, we've reformatted it to incorporate two dimensions, and then effectively apply it twice in the code. The equation we are now using is,

$$\partial_t u_i + u_j \partial_j u_i = -\frac{1}{2}\partial_i \tau_{ij} + \text{seed},$$

where the diffusion term does not make way as we have set $\nu = 0$. I have solved the 2D Burger's equation using a pseudo-spectral method. In my code I have broken up the components using functions in `MATLAB`, as illustrated in 1. Each component of the code will be delegated to it's own section in this document for easy finding. We have chosen to populate our space with 100 n_x and n_y. Our time step is 0.51×10^3, as we found this to be the sweet spot before the solution goes unstable, and experiences a lot of numerical instability. Overall we hope to see two shocks colliding, or just about to collide.

[1]To follow the Smagorinksy model, I am referencing https://www.cfd-online.com/Wiki/Smagorinsky-Lilly_model, and using a $C_s = 0.16$.

2 Master

We utilize a **master**, or "main," as referred to in Python nomenclature, where all the the effort done by the sub-function scripts refer back, and then provide results for the figures via the **master**. Here we initialize and the velocities in each direction, and specific the user inputs. We then calculate the initial condition for all n_x and n_y. From there we calculate the initial superposition, or square-root of the two. Then we set things in motion utilizing the **Seed**, detailed in § 2.1.3, via the **Time March** (§ 2.1). Using the **Time March**, we then calculate both `Ux` and `Uy`, then calculate the matrix `Z` so that we can plot the superposition of the two waves.

```
%% Solves 2D Burgers Equation using Pseudo-spectral Method
% ~Marissa B. Adams~
%--------------------------------------------------------------------------
tic;
clc;
clear;
close all;

%% User-defined Inputs
n_x         = 100;          %ROWS
dx          = 2*pi/n_x;
n_y         = 100;          %COLUMNS
dy          = 2*pi/n_y;
t_steps     = 0.51e3;
dt          = 1e-3;
viscosity   = 0;                    %CHANGE IFF NEEDED

%% Calculations
% Initialize
% NOTES :: Starts from sinusoidal u,then take derivatives, introduce seed ...
% ... then introduce smag tau, use derivatives again to calculate the     ...
% ... derivative of tau, then calculate the RHS, then finite different to
    update

ux = zeros(n_x,1);
Ux = zeros(n_x,t_steps);
uy = zeros(n_y,1);
Uy = zeros(n_y,t_steps);

t = (0:dt:(t_steps-1)*dt)';
x = (0:dx:(n_x-1)*dx)';
y = (0:dy:(n_y-1)*dy)';

%%
for i=1:n_x
```

```
    for j=1:n_y
        ux(i,j,1)=1-cos(x(i));%sin(x(i));
        if(j<n_y/2+1)
            uy(i,j,1)=1-cos(2*(y(j)));%sin(y(j));
        else
            uy(i,j,1)=-(1-cos(2*y(j)));
        end
        Z(i,j,1) = sqrt(ux(i,j,1)^2 + uy(i,j,1)^2);
    end
end

%%
% Time-marching along x
seed_diff_x = 1e-6;                    %CHANGE IFF NEEDED
seed_diff_y = 1e-6;
Ux = BurTurb2D_TimeMarch(ux,uy,viscosity,t_steps,dx,dy,dt,n_x,n_y,seed_diff_x)
    ;
ux = ux';
uy = uy';
Uy = BurTurb2D_TimeMarch(uy,ux,viscosity,t_steps,dy,dx,dt,n_y,n_x,seed_diff_y)
    ;

%% Plot figures
Z = zeros(n_x,n_y,t_steps);
for k = 1:t_steps
    for i=1:n_x
        for j=1:n_y
            Z(i,j,k) = sqrt(Ux(i,j,k)^2 + (Uy(j,i,k)')^2);
        end
    end
end


BurTurb2D_Figures(Z,Ux,Uy,ux,uy,x,y,t,dt,t_steps,n_x,n_y,viscosity);


%%
clearvars -except t x y Ux Uy Z viscosity n_x n_y dt dx dy t_steps
toc;
```

2.1 Time March

The **Time March** is effectively where all of the hard work goes into motion. For each time step, the derivatives and model are calculated for each side of the equation, and then updated.

```
function U = BurTurb2D_TimeMarch(u,v,viscosity,t_steps,dx,dy,dt,n_x,n_y,
   seed_diff_x)
 for i = 1:t_steps
    [du_dx,d2u_dx2,du_dx2,du_dy,d2u_dy2] = BurTurb2D_Derivatives(u,dx,v,dy);
    seed_x                               = BurTurb2D_Seed(0.75,n_x,n_y)';
    seed_filtered_x                      = BurTurb2D_Filter(seed_x,2);
    tau_x                                = BurTurb2D_Smagorinsky(u,du_dx,dx);
    dtau_xdx                             = BurTurb2D_Derivatives(tau_x,dx,
   tau_x,dy);

    RHS_x = viscosity*d2u_dx2 + viscosity*d2u_dy2 - 0.5*du_dx2 - v.*du_dy +
    ...
         sqrt(2*seed_diff_x/dt)*seed_filtered_x - 0.5*dtau_xdx;
    % FD for time dervative (velocity verlet-ish)
    if i == 1
        u_mod = u + dt*RHS_x;
    else
        u_mod = u + dt*(1.5*RHS_x - 0.5*RHS_xp);
    end
    u_mod_k          = fft(u_mod);
    u_mod_k(n_x/2+1) = 0;
    u_mod            = real(ifft(u_mod_k));
    u                = u_mod;
    U(:,:,i)         = u_mod;
    RHS_xp           = RHS_x;

    fprintf('%f\n',i*dt);

 end
```

2.1.1 Derivatives

Here I've calculated the derivatives for the equation. Pretty self explanatory.

```
%From notes
function [du_dx,d2u_dx2,du_dx2,du_dy,d2u_dy2] = BurTurb2D_Derivatives(u,dx,v,
    dy)

[n_x,n_y]      = size(u);

pre_x          = 2*pi/n_x/dx;
pre_y          = 2*pi/n_y/dy;

k_x            = [0  1:(n_x/2-1)  0  -(n_x/2-1):1:-1]';
k_y            = [0  1:(n_y/2-1)  0  -(n_y/2-1):1:-1]';

u_k            = fft2(u);
du_dx          = (pre_x)*real(ifft2(sqrt(-1)*k_x.*u_k));
d2u_dx2        = (pre_x^2)*real(ifft2(-k_x.*k_x.*u_k));

u_k_buffer     = [u_k(1:n_x/2+1,:)' zeros(n_y,n_x) u_k(n_x/2+2:n_x,:)']';
u_buffer       = real(ifft2(u_k_buffer));
u2_buffer      = u_buffer.*u_buffer;
u_k2_buffer    = fft2(u2_buffer);
u_k2           = [u_k2_buffer(1:n_x/2+1,:)' u_k2_buffer(n_x+n_x/2+2:n_x+n_x,:)
    ']';
du_dx2         = (2)*(pre_x)*real(ifft2(sqrt(-1)*k_x.*u_k2));

v_k            = fft2(v);
du_dy          = (pre_y)*real(ifft2(sqrt(-1)*k_y.*v_k));
d2u_dy2        = (pre_y^2)*real(ifft2(-k_y.*k_y.*v_k));

end
```

2.1.2 Filter

Here I've apply a Large Eddy Simulation (LES) filter as described by the CFD-online wikipedia. One has the option of choosing a box or Gaussian filter. You can switch between the two in **Time March**. For this run, I used the Gaussian filter.

```
%Choose the box or gaussian, 1 or 2 respectively
%Formulas from CFD wiki: https://www.cfd-online.com/Wiki/LES_filters
function u_filtered = BurTurb2D_Filter(u,option)

n_x         = length(u);
delta       = (2*pi)/n_x;
u_k         = fft(u);
k           = [0 1:(n_x/2-1) n_x/2 -(n_x/2-1):1:-1]';

if option == 1      % Box Filter
    F                   = sin(0.5*k*delta)./(0.5*k*delta);
    F(1)                = 1;
    u_proxy             = F.*u_k;
    u_proxy(n_x/2+1)=  0;

elseif option == 2  % Gaussian Filter
    F                   = exp(-k.^2.*delta^2/24);
    F(1)                = 1;
    u_proxy             = F.*u_k;
    u_proxy(n_x/2+1)=  0;
end
u_filtered = real(ifft(u_proxy));

end
```

2.1.3 Seed

```
%Copied from stack exchange
function seed = BurTurb2D_Seed(alpha,n_x,n_y)

lol                    = sqrt(n_x)*randn(n_x,n_y);
freq                   = [ 1 1:n_x/2 (n_x/2-1):-1:1 ];
lol_k                  = fft2(lol);
lol_k(1,:)               = 0;
lol_k(n_x/2+1,:)         = 0;
lol_mod                = lol_k .* (freq .^ (- alpha/2 ) );
seed                   = real( ifft2(lol_mod) );

end
```

2.1.4 Smagorinsky

```
function tau = BurTurb2D_Smagorinsky(u,du_dx,dx)

[n_x,n_y]        = size(u);
C_s_2            = 0.16^2; %from notes
mod_du_dx_smag   = BurTurb2D_Dealiasing(abs(du_dx),n_x,n_y,1);
du_dx_smag       = BurTurb2D_Dealiasing(du_dx,n_x,n_y,1);
prod_du_dx       = BurTurb2D_Dealiasing(mod_du_dx_smag.*du_dx_smag,n_x,n_y,2);
tau              = -2*C_s_2*((dx)^2)*prod_du_dx;

end
```

2.1.4.1 Dealiasing

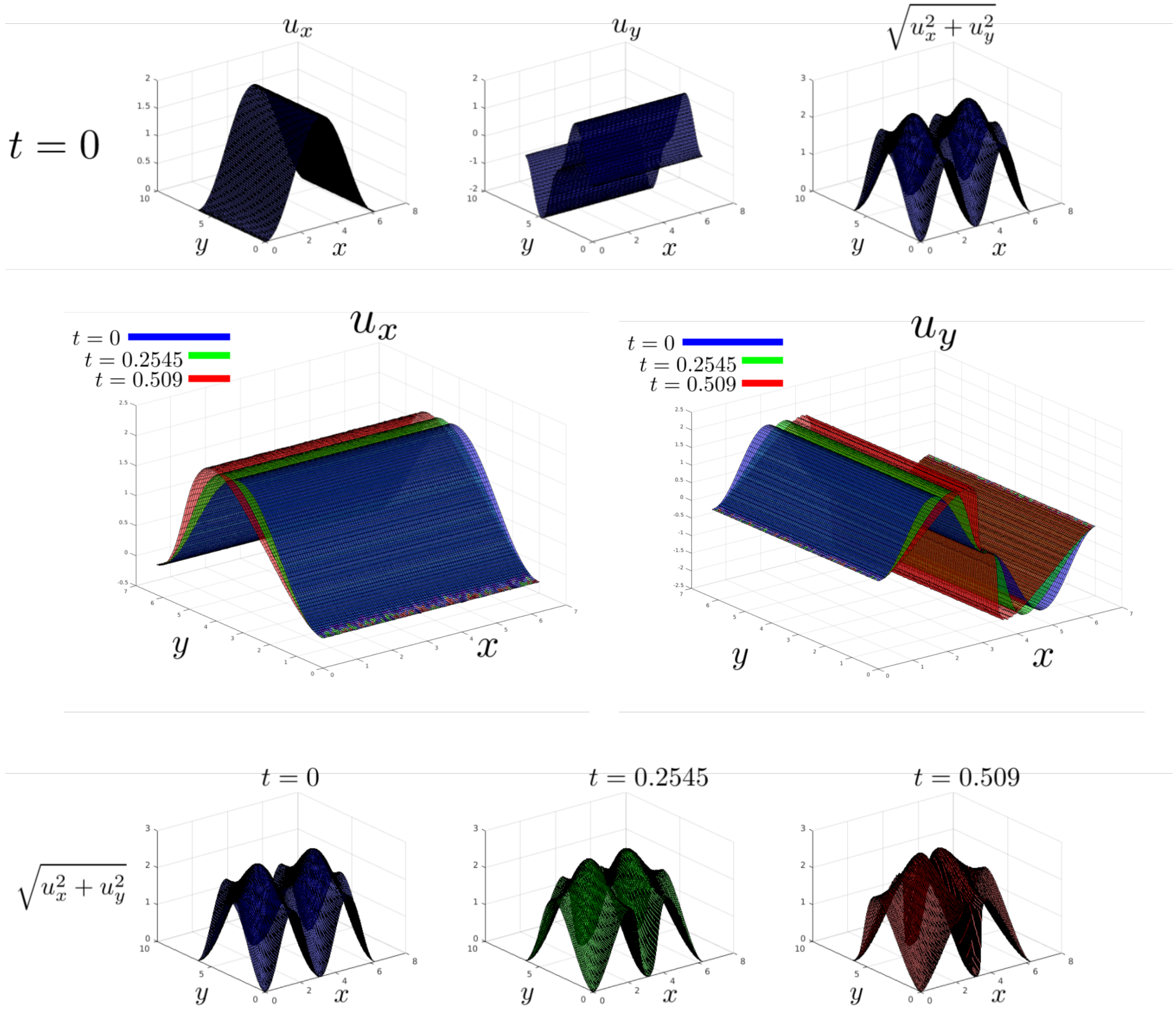
```
function out = BurTurb2D_Dealiasing(term,points1,points2,option)

if option == 1
    num1       = points1/2;
    out_k      = fft2(term);
    out_k_mod  = [out_k(1:points1/2+1,:)' zeros(points2,num1) out_k(points1
   /2+2:points1,:)']';
    out        = real(ifft2(out_k_mod));
elseif option == 2
    num1                         = points1/2;
    out_k                        = fft2(term);
    out_k_mod                    = [out_k(1:points1/2+1,:)' out_k(num1+points1
   /2+2:num1+points1,:)']';
    out_k_mod(points1/2+1,:)     = 0;
    out                          = (3/2)*real(ifft2(out_k_mod));

end
```

2.2 Figures

```
function BurTurb2D_Figures(Z,Ux,Uy,ux,uy,x,y,t,dt,t_steps,n_x,n_y, viscosity)
%%
[X,Y] = meshgrid(x(1:n_x),y(1:n_y));

CRed=zeros(100,100,3);
CRed(:,:,1)=1;

CGreen=zeros(100,100,3);
CGreen(:,:,2)=1;

CBlue=zeros(100,100,3);
CBlue(:,:,3)=1;

figure(1)
subplot(1,3,1)
surf(x,y,ux,CBlue,'FaceAlpha',0.5)%,'EdgeColor','none');
subplot(1,3,2)
surf(x,y,uy,CBlue,'FaceAlpha',0.5)%,'EdgeColor','none');
subplot(1,3,3)
surf(x,y,Z(:,:,1),CBlue,'FaceAlpha',0.5)%,'EdgeColor','none');
print('IC','-dpng');

figure(2)
surf(x,y,Ux(:,:,1),CBlue,'FaceAlpha',0.5)%,'EdgeColor','none')
hold on;
surf(x,y,Ux(:,:,end/2),CGreen,'FaceAlpha',0.5)%,'EdgeColor','none')
hold on;
surf(x,y,Ux(:,:,end),CRed,'FaceAlpha',0.5)%,'EdgeColor','none')
hold off;
print('UxTimeEvol','-dpng');

figure(3)
surf(x,y,Uy(:,:,1)',CBlue,'FaceAlpha',0.5)%,'EdgeColor','none')
hold on;
surf(x,y,Uy(:,:,end/2)',CGreen,'FaceAlpha',0.5)%,'EdgeColor','none')
hold on;
surf(x,y,Uy(:,:,end)',CRed,'FaceAlpha',0.5)%,'EdgeColor','none')
hold off;
print('UyTimeEvol','-dpng');

figure(5)
subplot(1,3,1)
%figure(5)
surf(X,Y,Z(1:n_x,1:n_y,1),CBlue,'FaceAlpha',0.5)%,'EdgeColor','none');
xlabel('x');
ylabel('y');
title(['t = ', num2str(t(1))]);

subplot(1,3,2)
```

```
%figure(6)
surf(X,Y,Z(1:n_x,1:n_y,t_steps/2),CGreen,'FaceAlpha',0.5)%,'EdgeColor','none');
xlabel('x');
ylabel('y');
title(['t = ', num2str(t(end)/2)]);

subplot(1,3,3)
%figure(7)
surf(X,Y,Z(1:n_x,1:n_y,t_steps-1),CRed,'FaceAlpha',0.5)%,'EdgeColor','none');
xlabel('x');
ylabel('y');
title(['t = ', num2str(t(end))]);
print('SuperPosVelEvolution','-dpng');

end
```

2.2.1 Results



Notes:
Viewing the figures about as a table:

- `(row 1)` is the initial condition, from left to right we have the initial condition for u_x, u_y, and their mod-square combination, $U = \sqrt{u_x^2 + u_y^2}$.
- `(row 2)` is the time evolution for both u_x and u_y for $t \approx \{0.0, 0.25, 0.5\}$. Continuing with `(row 1)`, you can see that the initial times are blue, the intermediate, green, and the final, red.
- `(row 3)` is the time evolution for the mod-square of the contents in `(row 2)`, following the same color pattern.