Basic Concepts :→

Determine how many zeros end the number 100!

$$100! = 100 \cdot 99 \cdot 98 \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

Solution :→

Adding a zero to the end of a product occurs precisely when we multiply by 10. Multiplication by any number ending in 1,3,7,9 cannot possibly add a zero to a product (∵ non of them divide 10).

Prime factorization of 10 is $10 = 5 \cdot 2$. We solve this problem by counting the factors of 5 in 100!

⇒ In numbers 1-10, only 5 and 10 have factors of 5. The number 5 may be paired with 2 to yield 10 and the number 10 does not need to be paired. The two resulting factors of 10 contribute two zeros to the full product that forms the factorial.

⇒ In numbers 11-20, only the numbers 15 and 20 have factors of 5. This contributes <u>two</u> additional zeros.

In numbers 21-30, only 25 and 30 have factors of 5 but 25 has two factors of 5.

Thus, $22 \times 24 \times 25 = 11 \times 12 \times (5 \times 2) \times (5 \times 2)$ and this will contribute $10 \times 10 = \underline{100}$ or two zeroes. Thus, range 21-30 contributes a total of <u>three zeroes.</u>

⇒ 31-40, only 35 and 40 have factors of 5, thus contributes additional <u>two zeroes</u>.

⇒ 41–50, only 45 and 50 have factors of 5. But 50 has two factors of 5. (5×5×2) So, additional <u>three zeros</u> are contributed.

⇒ 51–60, only 55 and 60 have factors of 5. So, contibution of <u>two zeros</u>.

⇒ 61–70, only 65 and 70 have factors of 5. So, additional <u>two zeros</u>.

⇒ 71–80, only 75 and 80. But 75 has three factors of 5. (5×5×3). So, additional ~~[illegible]~~ <u>three zeros</u>

⇒ 81–90, only 85 and 90. So, additional <u>two zeroes</u>.

⇒ 91–100, only 95 and 100. But 100 has (5×2×5×2) two factors of 5. So, additional <u>three zeros</u>.

Total number of zeros in 100! = 2+2+3+2+3+2 +2+ ~~[illegible]~~ 3+2+3 = 24

Features of Problem

⇒ A trailing zero comes from multiplication by 10. (Essential feature).

⇒ Analysing special case (Product 10·9·8...3·2·1)

⇒ Determined how to pass from the special case to the full problem.

⇒ The numbers from 1 to 100 contain 100÷5 = 20 multiples of 5. Four of these multiples of 5 are in fact multiples of 25 (25, 50, 75, 100), hence contribute two 5's. That gives a total of 24 factors of 5 in 100! Pairing each of these with an even number gives a factor of ten, and hence a zero. Hence there are 24 zeroes at the end of 100!

(2) A math class has 12 students. At the beginning of each class hour, each student shakes hands with each of the other students. How many handshakes take place?

⇒ If 1 student, handshake = 0

⇒ If 2 students, handshake = 1

⇒ When another (third) student comes in, he shake hands with 2 persons. So, in case of 3 students,
Total handshakes = 1 + 2

⇒ When fourth student comes, he shake hands with 3 persons. So, in case of 4 students, Total handshakes = 1+2+3

⇒ When fifth student comes, he shake hands with 4 persons. So, in case of 5 students, Total handshakes = 1+2+3+4

OBSERVE PATTERN

So, for 12 students, number of handshakes
= 1+2+3+4+5+6+7+8+9+10+11
= 66 handshakes.

(3) Assume that K is a positive integer. What is the sum of the integers.

$S = 1 + 2 + 3 + \ldots + (K-1) + K$?

Preliminary discussion

Think S as a function:

$S(K) = 1 + 2 + 3 + \ldots + K$

If a function $f(K)$ increases by a fixed amount, say 3, each time that K is increased by 1, then f must be a linear function and have the form of $f(K) = 3K + b$. Likewise, if the function g increases by a linear function of K each time that K is increased by 1, then g is quadratic. (Concept of derivative: the derivative of a quadratic function is linear). For eg: → if $g(K) = K^2$ then $g(K+1) - g(K) = 2K+1$ and that difference is linear.

Solution: → A useful method for analyzing a sum is to rewrite each term so that cancellations are introduced.

$$(K+1)^2 = K^2 + 2K + 1$$
$$(K+1)^2 - K^2 = 2K + 1$$

$$2^2 - 1^2 = 3 = 2\cdot 1 + 1$$
$$3^2 - 2^2 = 5 = 2\cdot 2 + 1$$
$$4^2 - 3^2 = 7 = 2\cdot 3 + 1$$
$$\cdots\cdots$$
$$K^2 - (K-1)^2 = 2K - 1 = 2\cdot(K-1) + 1$$
$$(K+1)^2 - K^2 = 2K + 1 = 2\cdot K + 1$$

Add both sides

$$[2^2 - 1^2] + [3^2 - 2^2] + [3^2 - 2^2] + [4^2 - 3^2] + \dots + [(K+1)^2 - K^2]$$
$$= 2[1 + 2 + 3 + \dots + K] + [\underbrace{1 + 1 + 1 \dots + 1}_{K \text{ terms}}]$$

LHS "telescopes" (i.e., all but first and last terms cancel)

$$(K+1)^2 - 1^2 = 2\cdot S + K$$
$$K^2 + 2K + 1 - 1 = 2S + K$$
$$S = \frac{K^2 + K}{2}$$

④ Imitate the method used in the last problem to find formula for the sum $1^2 + 2^2 + 3^2 + \dots + K^2$ when K is a positive integer.

We have,

$$(n+1)^3 = n^3+3n^2+3n+1$$

$$(n+1)^3 - n^3 = 3n^2+3n+1$$

Substitute $n=1$ to $n=K$

$$2^3-1^3 = 3\cdot 1^2+3\cdot 1+1$$
$$3^3-2^3 = 3\cdot 2^2+3\cdot 2+1$$
$$4^3-3^3 = 3\cdot 3^2+3\cdot 3+1$$
$$5^3-4^3 = 3\cdot 4^2+3\cdot 4+1$$
$$\cdots\cdots$$
$$K^3-(K-1)^3 = 3\cdot(K-1)^2+3\cdot(K-1)+1$$
$$(K+1)^3-K^3 = 3\cdot K^2+3\cdot K+1$$

Adding both sides,

LHS "telescopes"

$$(K+1)^3-1^3 = 3\cdot[1^2+2^2+3^3+4^2+\dots+K^2] + 3\cdot[1+2+3+4+\dots+K] + \underbrace{[1+1+1+1+\dots+1]}_{K \text{ terms}}$$

$$(K+1)^3-1^3 = 3\cdot S(K^2)+3\cdot S(K)+K$$

$$(K+1)^3-1^3 = 3\cdot S(K^2)+3\left(\frac{K^2+K}{2}\right)+K$$

$$S(K^2) = \frac{1}{3}\left[(K+1)^3-1-3\left(\frac{K^2+K}{2}\right)-K\right]$$

$$= \frac{1}{3}\left[\frac{2K^3+6K^2+6K+\cancel{2}-\cancel{2}-3K^2-3K\overset{-2K}{\cancel{-2K}}}{2}\right]$$

$$= \frac{1}{6}\left[2K^3+3K^2+2K\right] = \frac{1}{6}\left[2K^3+3K^2+K\right]$$

$$= \frac{K}{6}\left[2K^2+3K+\overset{1}{\cancel{2}}\right] = \frac{K}{6}\left[2K^2+2K+K+\overset{1}{\cancel{2}}\right]$$

$$= \frac{K}{6}\left[2K(K+1)+1(K+\overset{1}{\cancel{2}})\right]$$

$$= \frac{K}{6}\left[(2K+1)(K+1)\right]$$

NOTE :→ If there are K students in the class and the class period begins with everyone shaking everyone else's hand, then total number of handshakes ~~is~~ are

$$1+2+3+4+\dots+(K-1)$$

$$=\left[\frac{(K-1)^2+(K-1)}{2}\right]=\left[\frac{K^2-2K+1+K-1}{2}\right]=\left[\frac{K^2-K}{2}\right]$$

5. Suppose there are K students in the class. If K is even then will the number of handshakes that takes place be even or odd? If K is odd then will the number of handshakes that takes place be even or odd?

#students	#handshakes	parity of handshakes
0	0	even
1	0	even
2	1	odd
3	3	odd
4	6	~~odd~~ even
5	10	even
6	15	odd
7	21	odd
8	28	even
9	36	even
10	45	odd
11	55	odd
12	66	even
13	78	even

⇒ The first two number for handshakes are even, then there are two odd, then there are two even, and so forth

The pattern repeats in increments of four.
Number in "# handshakes" column in row for (K+1) students is obtained by adding the two numbers in the row of K students.

(i) In rows $4l, 4l+1$, the number of handshakes is even.

(ii) In rows $4l+2, 4l+3$, the number of handshakes is odd.

→ For K students, "# handshakes" $= \frac{K^2-K}{2}$

For K+1 students, "# handshakes"

$$= K + \left(\frac{K^2-K}{2}\right)$$

$$= \frac{2K + K^2 - K}{2}$$

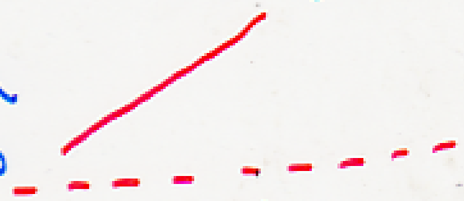
$$= \frac{K^2+K}{2}$$

⑥ What is the greatest number of regions into which ~~one line can divide the~~ ~~plane~~ three straight lines (of infinite extent) can divide the plane?

Simpler Question :→ "What is the greatest number of regions into which one line can divide the plane?" (Two regions)

Look at two lines.

⇒ If two lines coincide, then the plane is divided into two regions.



⇒ If two lines are distinct but parallel, then the plane is divided into three separate

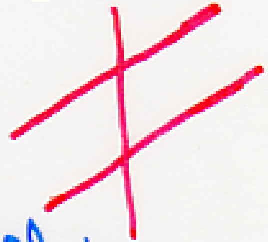
If two lines are skew (non-parallel) [Most Probable Case] (General Position).
The plane is separated into 4 regions.

Three lines

⇒ If all the three lines coincide, then we are in the situation of one line. If two of the lines coincide, then we are in the situation for two lines.

⇒ If three lines are distinct and are parallel, then the plane is separated into four regions.

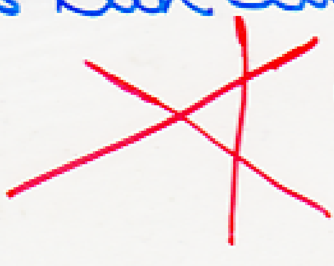
⇒ If two are parallel and the third is skew to them, then the plane is separated into six regions.



⇒ No two of three lines are parallel :→

⇒ If three lines ~~pass through a~~ ~~...~~ single point, the plane is separated into ~~...~~ ~~regions~~ six regions. (~~Most~~ Probable case)

⇒ If three lines do not pass through a single point and no two of them are parallel, then the plane is separated into seven regions. Thus, seven is the maximal regions into which three lines can divide the plane.



(Most Probable Case)

Count :→ We are given k objects $\{a_1, \ldots a_k\}$. How many different ordered pairs may be made up from those k objects?

⇒ There are k possible choices (namely $a_1, a_2, \ldots a_k$) for the first element of the ordered pair. After choosing first element, there are $(k-1)$ choices for the second element. The total number of possible ordered pairs, chosen from among $\{a_1, a_2 \ldots, a_k\}$ is then $k \cdot (k-1)$.

Given k objects $\{a_1, a_2, \ldots, a_k\}$. In how many different orders can we arrange these objects?

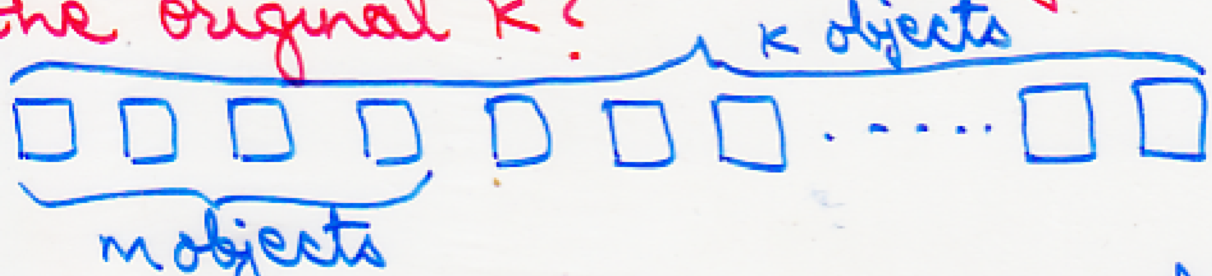
$$\underbrace{\square\ \square\ \square\ \square\ \square \cdots \square}_{k\text{-positions}}$$

suppose we have k positions into which to put the objects. There are k different objects (namely $a_1, a_2 \ldots, a_k$) that we may put in first position, $(k-1)$ different objects to put into second position, $(k-2)$ different objects to put into third and so forth. Hence there are

$$k \cdot (k-1) \cdot (k-2) \cdots 3 \cdot 2 \cdot 1 = k!$$

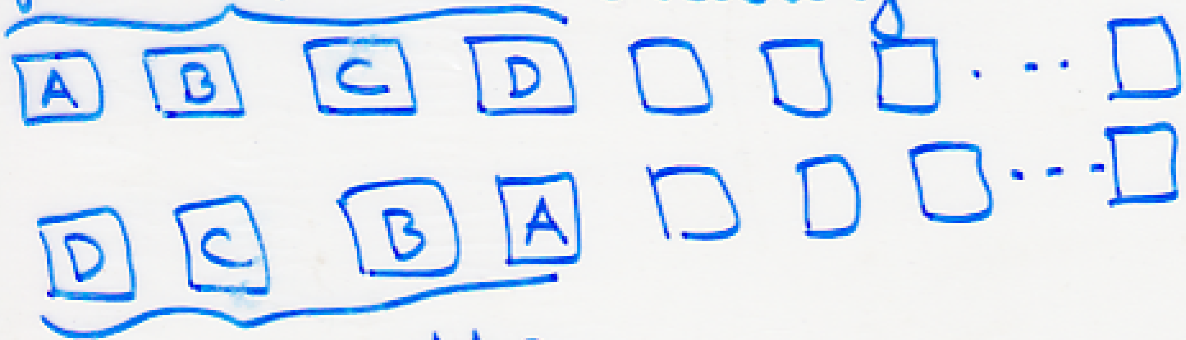
possible different orderings of the k objects $\{a_1, a_2, \ldots, a_k\}$

Given k objects $\{a_1, a_2, \ldots, a_k\}$. Suppose that m is a positive integer that is less than or equal to k. In how many ways can we choose m objects among the original k?



⇒ select an ordering for the entire set of k objects

⇒ select subcollection of m objects, the first m in the ordering.



These are the same subset of $m = 5$ elements, occurring in different order.

⇒ Counting different orderings of those first m objects as different we get $m!$. But in our case the ordering doesn't matter. So, we divide by $m!$

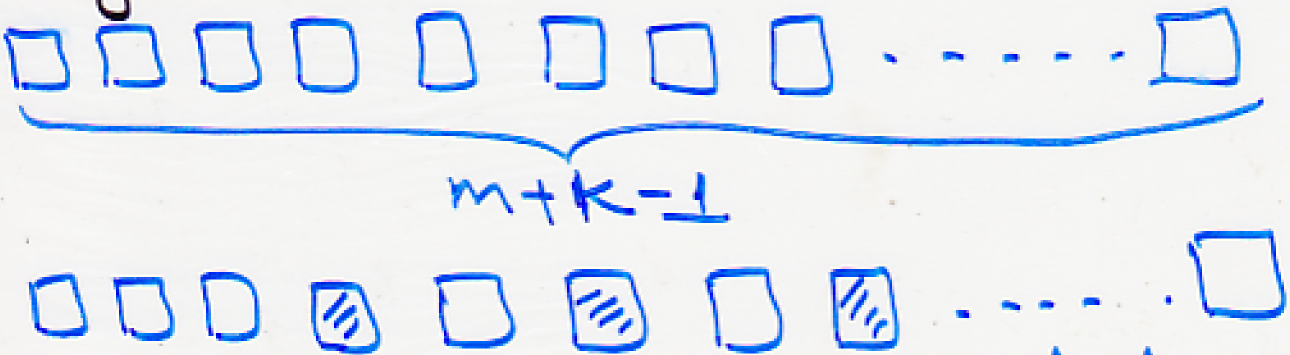
⇒ Also we divide by the different orderings of the remaining $k-m$ objects $(k-m)!$

The total number of pairs of hands for two team members is

$$c_1 \cdot c_2 = \binom{52}{13} \cdot \binom{39}{13}$$

Assume that k and m are positive integers. How many different monomials of degree m are there in R^k?

The space R^k consists of elements of the form $(x_1, x_2, \dots, x_k)$ where $x_1, x_2, \dots, x_k$ are real numbers. A monomial is an expression like $(x_1)^2 \cdot (x_3)^3 \cdot (x_5)^6$. It consists of some powers of some of the variables multiplied together. This particular monomial has degree 11. The problem asks for the number of all possible monomials of a given degree m in R^k.



(k−1) boxes shaded

m boxes unshaded

⇒ B/w the leftmost edge of all boxes and the first shaded box is a group of unshaded boxes say, there are m_1 of them. Note $0 \le m_1 \le m$

Next to the right of the first shaded box and to the left of the second shaded box there is a group of unshaded boxes, say m_2 of them. Continue.

⇒ we see that shading of $k-1$ boxes gives rise to non-negative integers $m_1, m_2, \ldots, m_k$ such that $m_1+m_2+\ldots m_k = m$

In turn, this k-tuple ~~in turn~~ of numbers corresponds to the monomial $(x_1)^{m_1} \cdot (x_2)^{m_2} \ldots (x_k)^{m_k}$

⇒ This process runs in reverse also. Any monomial $(x_1)^{m_1} \cdot (x_2)^{m_2} (x_k)^{m_k}$ has a corresponding k-tuple $(m_1, m_2 \ldots m_k)$ and this k-tuple in turn corresponds to a shading of $k-1$ boxes from among ~~[illegible]~~ $m+k-1$ boxes.

⇒ Thus, the number of monomials that we wish to count is ~~k-1 boxes chosen from~~

$$\binom{m+k-1}{k-1} = \binom{m+k-1}{k-1}$$

Hence, different subcollections of m objects chosen from among a total of K objects is $\frac{K!}{m!(K-m)!}$

$$\binom{K}{m} = \frac{K!}{m!(K-m)!}$$

How many different 5 card poker hands may be had from a deck of 52 cards?

52 choose 5 $\binom{52}{5} = \frac{52!}{5!(52-5)!} = 2598,960$

(2,598,960).

How many pairs of bridge hands may be dealt from a deck of 52 cards?

The bridge is played by two teams of two people. Each person is dealt 13 cards. Consider one of the teams.

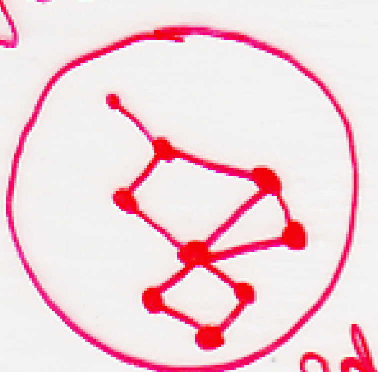
⇒ The first team member is dealt 13 cards from among total of 52

$$C_1 = \binom{52}{13}$$

⇒ The second team member is also dealt 13 cards, chosen from among the remaining 52-13 = 39 cards

$$C_2 = \binom{39}{13}$$

Suppose that we have an admissible graph on the unit sphere in 3D space. Here by "admissible graph" we mean a connected configuration of arcs. Two arcs may be joined only at their endpoints. The endpoints of the arcs in the graph are called vertices. The arcs are called edges. An edge is that portion of an arc that lies between two vertices. A face is any 2D region, without holes, that is bordered by edges and vertices.



admissible

non-admissible

This problem asks you to verify Euler's formula for an admissible graph.

⇒ Let V be the number of vertices, E the number of edges, and F the number of faces. Then Euler's formula is

$$V - E + F = 2$$

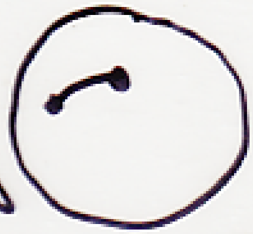
⇒ The simplest graph that is admissible consists of a single vertex and nothing else.

The complement of the single vertex in a sphere is a valid face. Thus, $V=1$, $E=0$ and $F=1$

$$V-E+F = 1-0+1=2$$

⇒ The next most complex graph has one edge, with a vertex on each end and nothing else.

The complement (in the sphere) of this edge with its endpoints is a single valid face. Thus,



$$V=2,\ E=1 \text{ and } F=1$$

$$V-E+F=2-1+1=2$$

⇒ Let $P(K)$ be the statement "Euler's formula is valid for any admissible graph with K edges."

Induction :→

① $P(1)$ is already true.

② Assume Euler's formula is valid for any admissible graph having K edges. Now let G be a graph having $(K+1)$ edges. There is some edge that can be removed from G so that the remaining graph G' is still admissible (for eg:→ an edge that separates two different faces will do.

Say V', E' and F' denote the numbers of vertices, edges and faces for G'.

⇒ The graph G is obtained from G' by adding an edge. If the edge is added by attaching one end leaving the other free, the number of edges and vertices increased by one. Thus

$$V = V'+1\ ,\ E = E'+1 \text{ and } F = F'$$

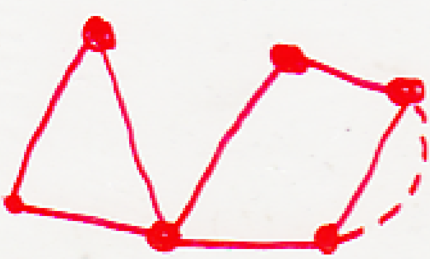
$$E = E'+1$$

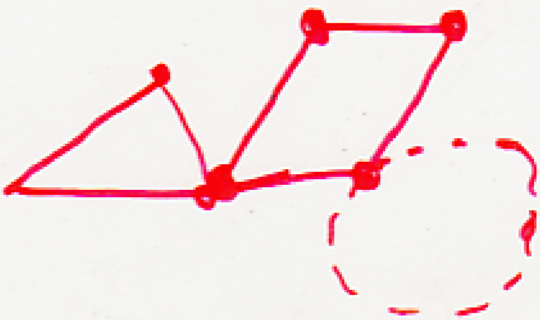
Since by hypothesis

$V' - E' + F = 2$, it follows that

$V - E + F = 2$ as desired

⇒ If instead the edge is added by attaching both ends then number of faces is increased by one, the number of edges by 1 and the number of vertices does not change. Since by hypothesis $V' - E' + F' = 2$, it follows that $V - E + F = 2$





the inductive hypothesis applies to the first j mail-boxes. So, one of them will contain at least two letters.

③ If the mailboxes contains two or more letters, then we are done because some mailbox contains at least two letters

Suppose that six people are in a room. Explain why either three of these people all know each other or else ~~three~~ there are three of the people none of whom knows each other.

⇒ If A knows B, then B knows A and vice versa. Call one of them Ram. Of other 5 people either Ram knows 3 of them or he does not know three of them.

⇒ Let Ram knows {Hari, Krishna, Bishnu}. Now if two of these people know each other (say Hari and Krishna), then the triple {Ram, Hari, Krishna} is a mutually aquianted triple. If, instead, it is not the case that two of these three people know each other, then {Hari, Krishna, Bishnu} is a triple of people no two of them know each other.

Assume that K is a positive integer. If $(K+1)$ letters are delivered to K mailboxes, then show that one mailbox must contain at least two letters.

Let $P(K)$ be "If $(K+1)$ letters are delivered to K mailboxes then some mailbox must receive at least two letters".

For $K=1$, $K+1=2$ letters are delivered to $K=1$ mailboxes then only mailbox will receive two letters.

⇒ Assume that $P(K)$ has been proved. Assume that $(K+1)+1$ letters have been delivered to $K+1$ mailboxes.

⇒ If last mailbox is empty, then all the letters have been delivered to the first j mailboxes. Inductive hypothesis applies and one of these first K mailboxes contains at least two letters.

⇒ If last mailbox contains precisely one letter, then the remaining $(j+1)$ letters have been delivered to the first j mailboxes. Once again

Problems of Logic

You are on an island that is populated by two types of people: truth tellers and liars. When asked a Yes-No question, a truth tellers ~~and liars~~ always tells a truth and a liar always lies. There is no visual method for telling a truth teller from a liar. What single question could you ask anyone that you meet on the island to determine whether that person is a truth teller or a liar?

⇒ Direct question "Are you a truth teller".

Answer :→ Truth Teller "Yes"
Liar "Yes"

Are you a lier :→ Truth Teller "No"
Lier "No"

Solution :→ If you are a truth teller then how would you answer the question "Are you a liar"?

Truth Teller :→ NO

LIAR :→ YES (Liar knows that a truth teller if asked will say "No") But liar must lie. So he/she will say "YES"

Monty Hall Problem :→

D1	D2	D3
G_1	G_2	C

⇒ The issue is whether the contestant will switch from the currently selected door to the remaining door (the one that the contestant has not chosen and Monty Hall did not open)

⇒ Equal probability for there to be a goat behind the remaining door and behind the door that the contestant has already selected. To think this is a naive approach as it doesnot take into account the fact that there are two distinct goats.

Careful Analysis :→ Assume a contestant always choose Door 3 and Monty Hall choose Door 1. There are 6 = 3! possible permutations of three objects.

	Door 1	Door 2	Door 3
1	G_1	G_2	C
2	G_2	G_1	C
3	G_1	C	G_2
4	G_2	C	G_1
5	C	G_1	G_2
6	C	G_2	G_1

① Monty Hall will reveal a goat behind Door 1 or Door 2. It is not to the contestant's advantage to switch, so we record 'N'.

② Similar record 'N'

③ ~~The~~ Monty Hall will reveal a goat behind Door 1, and it is to the contestant's advantage to switch. Record 'Y'.

④ Like ③ so Record 'Y'

⑤ In the fifth case, Monty Hall will reveal a goat behind Door 2. It is advantage to switch. Record 'Y'

⑥ Same as ⑤ so Record 'Y'.

⇒ 4 'Y' and 2 'N'. Advantage to switch after Monty Hall reveals the goat.

There are more adults than boys, more boys than girls, more girls than families. If no family has fewer than 3 children, then what is the least number of families that there could be?

⇒ If there were just one family then there would be at least two girls, at least three boys, and at least four adults. But 4 adults make two families and that is a contradiction

⇒ If two family, at least 3 girls, at least 4 boys, and at least five adults means that there cannot be just two families, there are at least three. Hence contradiction

⇒ If 3 families, at least 4 girls, 5 boys (at least) and at least 6 adults. That is not a contradiction Thus, 3 families might satisfy the conditions.

Suppose there are 3 married couples.
1st family :→ 2 girls and 1 boy
2nd family :→ 2 girls and 1 boy
3rd family :→ 3 boys.
Then there are 6 adults, 5 boys, 4 girls and 3 families.

⇒ 3 families is the smallest number.
(exhaustion)

Explain why there are infinitely many prime numbers

⇒ A prime number is a positive integer, not 1, that has no divisors other than 1 and itself.

Prime numbers :→ 2, 3, 5, 7, 11, 13, 17, 19, 23 . . . . . .

Fundamental Theorem of Arithmetic

⇒ Every +ve integer is divisible by a prime number - indeed can be a factored in a unique manner into prime factors.

⇒ Suppose there ~~are~~ were in fact only finitely many primes in the world. Let $p_1, p_2 \cdots p_1$ be such primes. Consider the number $N = (p_1 \cdot p_2 \cdot p_3 \cdots\cdots p_K) + 1$. Now N must be divisible by some prime number. However, it is not divisible by p_1, since division by p_1 results in a remainder of 1. Likewise, N is not divisible by p_2, as 1 is remainder. In fact, N is not divisible by any of $p_1, p_2 \cdots\cdots p_K$. But these were all the primes in the universe. Yet N must be divisible by some prime! That is a contradiction. Hence there must be infinitely many primes.

<u>Issues of Parity</u> :→

An 8'×8' bathroom is to have its floor tiled. Each tile is 2'×1'. In one corner of the bathroom is a sink, and its plumbing

Since the number of black squares covered will always equal the number of white squares covered, this bathroom floor can never be tiled.

⇒ We have a container that contains 6 quarts and another that contains 4 quarts. We fill these containers by immersing them in the river. How can we use them to fill one of the containers with 3 quarts of water?

Only moves we are allowed are

(i) to fill a container

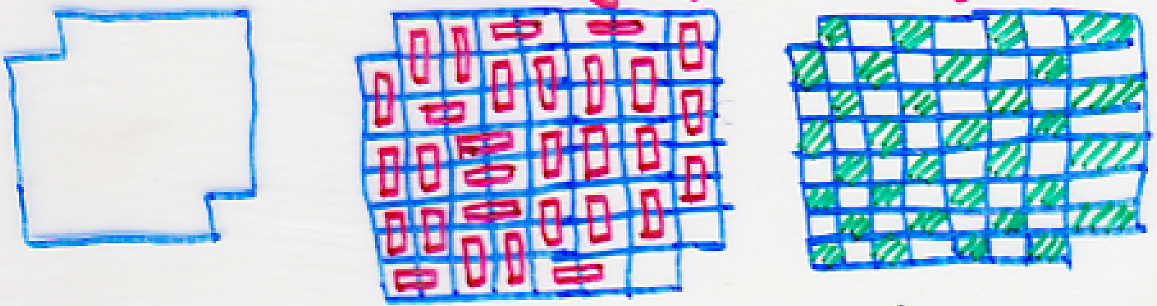
(ii) to empty a container

(iii) to pour one container into the other.

⇒ Our various pouring operations correspond to adding and subtracting multiples of 4 and 6. Now, adding and subtracting of even numbers always results in an even number answer. Thus, there is no way to obtain the number 3.

⇒ The problem cannot be solved.

occupies a 1'×1' square in the floor. In the opposite corner is a toilet, and its plumbing occupies a 1'×1' square in the floor. The situation is shown below. How is it possible to achieve the required tiling of the floor?



⇒ The area to be tiled is 8'×8' less two square feet, i.e., we must tile an area of 62 square feet. Thus, we will use 31 tiles. In second figure, there are two squares remaining and they cannot be covered by a single tile.

⇒ Color the bare floor of the bathroom like a checkerboard.

⇒ When we place a 2'×1' tile on the floor then it will cover two adjacent squares (one black (green) and one white). Thus, if we place two ~~tiles~~ (2'×1') tiles on the floor, they will cover a total of two black squares and two white squares.

⇒ In general, K tiles placed on the floor will cover K black squares and K white squares. The bathroom floor we are dealing has 32 black and 30 white squares.

Imagine a polyhedron with 1981 vertices. [Just place 1981 points on the unit sphere in 3D space. Now connect them with line segments in an obvious way to obtain a polyhedron.]

Imagine that each edge is assigned an electric charge of +1 or -1. Explain why there must be a vertex such that the product of the charges of all the edges that meet at that vertex must be +1.

⇒ Suppose that we multiply together all the products corresponding to all the vertices. Then every edge is counted twice (since each edge has two vertices ~~at~~ on its ends) so every +1 is counted twice and every -1 is counted twice. Thus, the product is +1

⇒ But there are an odd number of vertices. Thus, it cannot be the case that the product coming from each vertex is -1 (since the product of an odd number of -1's is equal to -1). Therefore, at least one vertex has product equalling +1.

A herd of cattle invades a barn dance. Suddenly, the barn is running amok with both cattle ~~plus plus~~ and people. A quick count reveals 120 heads and 300 feet. How many cattle are there? How many people.

Let c denotes cattle and p denotes people. Then total number of cattle plus people is $c+p$ while the total number of feet is $4c+2p$ (since a head of cattle has four feet while a person has two)

Thus, $c+p=120$

$4c+2p=200$

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Solving, we get $c=30$ and $p=90$

A sheep can clear a field, eating the grass, in one day. A cow can clear the same field in half a day. How long does it take the sheep and cow, working together, to clear the field.

A cow is like two sheep. So, the cow and sheep together is like 3 sheep. Thus, they will clear the field in a third of a day

What is the last digit of 3^{4798}?

Notice

$3^1 = 3 \quad 3^2 = 9 \quad 3^3 = 27 \quad 3^4 = 81 \quad 3^5 = 243$ etc.

Only possible last digits are 3, 9, 7 and 1 the the pattern repeats. In this ~~case~~ list, 1 is special because $1 \cdot 1 = 1$. And the digit 1 occurs when we raise 3 to the fourth power.

$$3^{4796} = [3^4]^{1199} \cdot 3^2$$

$$\begin{array}{r} 1199 \\ 4\overline{)4796} \\ \underline{\quad\quad} \\ 2 \end{array}$$

⇒ The expression in bracket will terminate with a 1. If we raise it to power 1199, it will still terminate with a 1. On the other hand 3^2 is 9. Hence 3^{4796} terminates with 9.

A party is held at the house of the Sameer. There were four other couples present (besides Mr. and Mrs. Sameer), and many, ~~not~~ but not all, pairs of people shook hands. Nobody shook hands with anyone twice, and nobody shook hands with his/her spouse. Both the host and hostess shook some hands.

At the end of the party, Mr. Sameer polls each person present to see how many hands each person (other than himself) shook. Each person gives a different answer. Determine how many hands Mrs. Sameer must have shaken.

⇒ We write S for the two Sameer and we denote each of the four couples by A, B, C and D.

⇒ Nobody shook 9 hands, since nobody shook the hand of his/her spouse. Therefore, 0 through 8 are used in describing the different number of handshakes performed by each of the nine people (other than Sameer)

⇒ Let Mr. A shook 8 hands. Everyone in couples B, C, D, S must have shaken Mr. A's hand in order to ~~account~~ account for 8 shakes. So, each of the peoples in B, C, D, S shook hands at least once. But somebody shook hands zero times. It must be Mrs. A.

⇒ Eliminate Mr. and Mrs A from our consideration. Mrs. B shook the hand of Mr. A but she didnot shake the hand of Mrs. A, since nobody did. To obtain a total of 7 shakes, she must have shaken the hands of all the peoples in couples C, D, S. But someone had to shake only one hand. (the people in C, D, S have now each shaken at least two hands, since they each shook Mr. A's hand, as well as Mrs. B's hand. Mr. B shook only one hand.

⇒ Eliminate Mr. and Mrs. B. ~~Mr. C~~ Mr. C shook with all peoples in couples D and S (4) and (one with Mr. A & Mrs. B). Mrs. C shook only 2 hands.

⇒ Eliminate Mr. & Mrs. C
Mr. D shook with all peoples in couples S (2) and (3) (one with Mr. A; Mrs. B and Mr. C). Mrs. D shook only 3 hands.

⇒ Eliminate Mr & Mrs. D. So, ~~in terms~~ in terms of person who have shaken 4 hands, Mrs. Sameer is only remaining. Hence Mrs. Sameer shook 4 hands.

Probabilistic approach to solving counting Problems

① Eight slips of paper with the letters A, B, C, D, E, F, G, H written on them are placed into a bin. The eight slips are drawn one by one from the bin. What is the probability that the first four to come out are A, C, E, H (in some order)?

⇒ We see that we are randomly selecting four objects from among eight. We want to know whether a particular four, in any order, will be the ones that we select.

⇒ The number of different ways to choose four objects from among eight is

$$\binom{8}{4} = \frac{8!}{4!(8-4)!} = 70$$

⇒ Of these different subsets of four, only one will be the set $\{A, C, E, H\}$. ~~The~~ Thus, the probability of the first four slips being the ones that we want will be $\frac{1}{70}$.

② Suppose that you write 37 letters and then you address 37 envelopes to go with them. Closing your eyes, you randomly stuff one letter into each envelope what is the probability that just one envelope contains the wrong letter?

⇒ Assume that the envelopes are numbered 1-37 and the letters are numbered 1-37. If letters 1 through 36 go into envelopes 1-36 then what remains are letter 37 and envelope 37. So that last letter is forced to go into the correct envelope.

⇒ It is impossible to have just one letter in the wrong envelope. If one letter is in the wrong envelope then at least two letters are in the wrong envelope.

The probability is zero.

③ Suppose that you have 37 envelopes and you address 37 letters to go with them. Closing your eyes, you randomly stuff one letter into each envelope. What is the probability that precisely two letters are in the wrong envelopes and all others in the correct envelope?

⇒ If just two letters are to be in the wrong envelope then they will have to be switched. Eg:→ Letter 5 could go into envelope 19 and letter 19 into envelope 5.

⇒ Thus the number of different ways that we can get just two letters in the wrong envelopes is just the same as the number of different ways that we can choose two letters from among 37.

$$N = \binom{37}{2} = \frac{37!}{2!(37-2)!} = 666$$

⇒ Number of different possible ways to distribute 37 letters among 37 envelopes is 37!

⇒ Probability that all letters but two will be in the correct envelopes is

$$P = \frac{666}{37!} \approx 4.86 \cdot 10^{-41}$$

④ A woman goes to visit the house of some friends whom she has not seen in many years. She knows that, besides the two married adults in the household, there are two children of different ages. But she does not know their genders.

On entering the house, she sees a football helmet. What is the probability that both children are boys?

Incorrect Analysis :→ At least one child is a boy. There is equal probability that the other is a boy or a girl. So, the odds of a boy are .5.

⇒ The sample space consists of all possible pairs of children.

If we consider all possible pairs of children, in the order (oldest, youngest), then the possibilities are

(B,B), (B,G), (G,B), (G,G)

⇒ We do not know whether the child who owns the helmet is the youngest or oldest. Thus, any of the first three ordered pairs could be the one describing the children in this family.

⇒ Of these three ordered pairs, two reveal the second child to be a girl and one reveals the second child to be a boy. Thus, there is 1/3 probability in our problem that the second child is a boy.

Skeptical

Boy = Head (H)
Girl = Tail (T)

TH	TH	HT	HH	TH
HH	TH	HT	TH	TT
TH	HT	HH	HH	TT
HT	TH	HT	HH	HH
HH	HT	HH	TT	HH
HT	TH	HT	TH	TT
HH	HH	HT	HH	HT
HT	TH	HT	TT	HT
HT	TH	TH	TT	HH
HT	HH	HH	HT	TT

$\frac{15}{43} = 0.3488$ (close to 1/3)

43 pairs in which one member is a head. Of those 15 consists of two heads. Probability of both flips being a head, given that one is head $\frac{15}{43}$

⑤ A woman goes to visit the house of some friends whom she has not seen in many years. She knows that, besides the two married adults in the household, there are two children of different ages. But she does not know their gender.

When she knocks on the door of the house, a boy answers. He says "I am the oldest child in this family. My sibling is in the back room asleep." What is the probability that the other child is a boy?

All possible pairs of children, in the order (oldest, youngest) are

(B,B), (B,G), (G,B), (G,G)

The probability that the sleeping child is a boy is equal to the probability that the sleeping child is a girl. Hence P = 0.5

31 pairs have "heads" (boy) as the first, or eldest, entry. Of those, 15 have heads as the entry and 16 have tails as the second entry. P(Boy) = 0.4839 and P(Girl) = 0.5161

⑥ You hand a friend a standard deck of 52 playing cards face down. You ask him to divide the deck into three sub-decks, using simple cuts, and to place them face down on the table. Then you say "I'll bet you even money that one of those three top cards is a face card" (here a face card is a jack, queen, or king). Would your friend be wise to accept the bet?

Friend Thinking :→ Only 12 face cards in a deck of 52. The chances of selecting a face card are therefore $\frac{12}{52} \approx .2308$.

The bet is in friends favour and he should take it.

⇒ But you and your friend are selecting 3 cards from a deck of 52 cards. Now, there are $\binom{52}{3}$ ways to select three cards from among 52.

⇒ How many ways there are of not choosing a face card?

⇒ If we are going to select three cards, none of which is a face card, then we must select three cards from among the $40 = 52 - 12$ that are not face cards. The number of ways of doing this is $\binom{40}{3}$. Thus, the probability of our failing to select a face card is

$$\frac{\binom{40}{3}}{\binom{52}{3}} \approx 0.44706$$

Therefore, the probability of getting a face card is $P = 1 - 0.44706 = 0.55294$

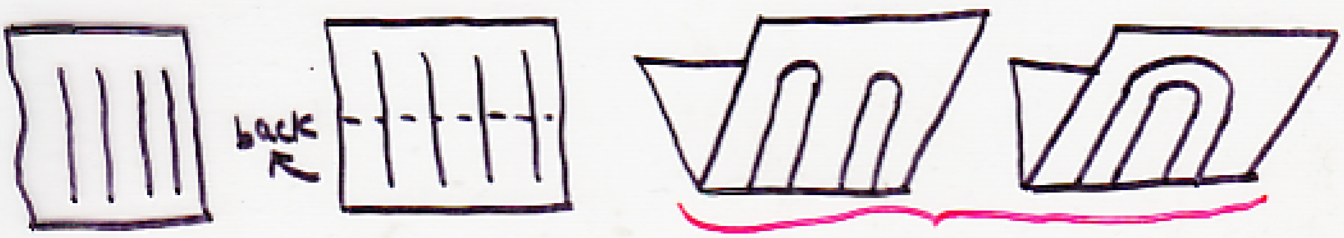
Good Bet For You.

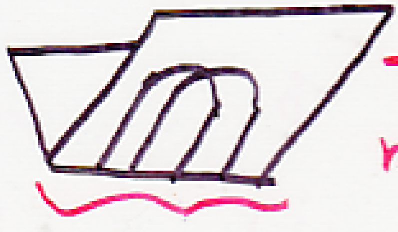
(7) You draw four parallel lines on a piece of paper. You fold the paper along the dotted line indicated in Figure.

You explain to a friend that you will, on one half of the paper, connect the lines in two pairs. [There are three different ways to do this]. You do not show the friend what you have done. With your work face down on the table, you then invite the friend to do as you did: connect the four remaining loose ends in two pairs.

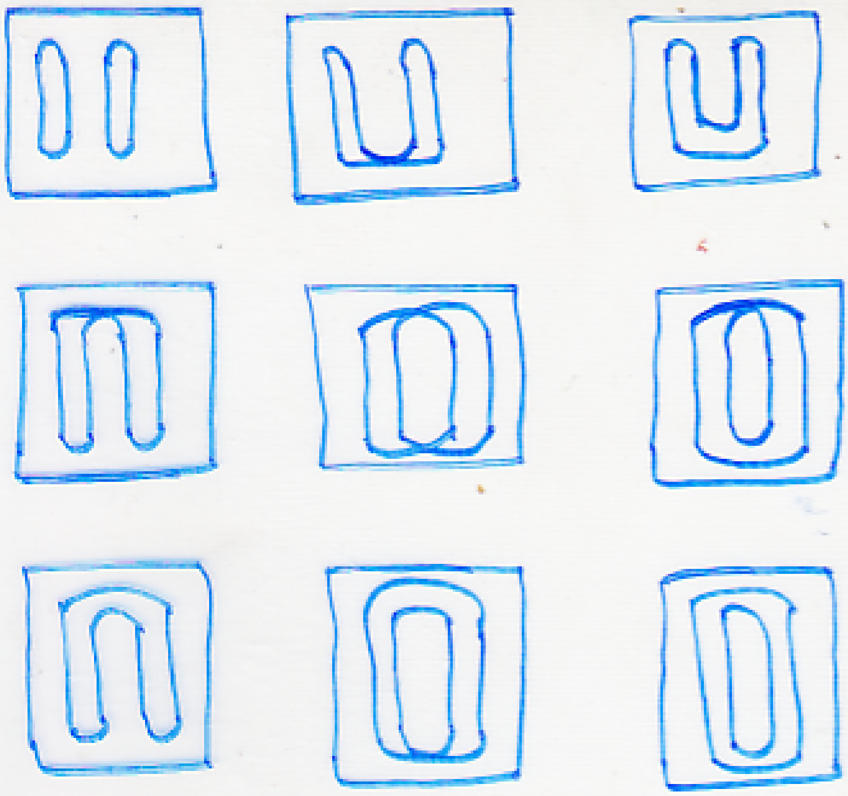
Then you place a bet with the friend: if the resulting figure, when the paper is unfolded, is a continuous loop then you win the bet; if the resulting figure is instead two disjoint loops, then the friend wins the bet.

The bet is even money. Is your friend wise to take this bet?





Three ways that you might do it.



⇒ Out of nine possible configurations,
6 single continuous loops
3 consists of two disjoint loops
Your probability of winning is $6/9 = 2/3$

Logic Problems :→

Six people, named A, B, C, D, E, F, are in the dining car of a train. They are one each from New York City, Chicago, Tulsa, St. Louis, Milwaukee, and Atlanta. The following facts are known:

① A and the man from New York City are physicians.

② E and the woman from Chicago are teachers.

③ The person from Tulsa and C are engineers.

④ B and F are veterans of the Gulf war, but the person from Tulsa has never served in the military.

⑤ The person from Milwaukee is older than A.

⑥ The person from Atlanta is older than C.

⑦ At St. Louis, B and the man from New York get off.

⑧ At San Francisco, C and the man from Milwaukee gets off.

Match the names of the people with their professions and their cities.

	A	B	C	D	E	F
New York	X	X	X	#	X	*
Chicago	X	*	X	#	X	#
Tulsa	X	X	X	*	X	X
St. Louis	#	#	*	#	#	#
Milwaukee	X	#	X	#	*	#
Atlanta	*	#	X	#	#	#

⇒ Put 'x' in a box when that connection is impossible

⇒ ① guarantees that A is not the man from NY. Put 'x' in the A column opposite New York

⇒ ⑦ guarantees that B is not from New York

⇒ ① and ② taken together imply that A, who is a physician, cannot be from Chicago.

⇒ The other X's come from similar reasoning

⇒ Having entered all these x's, we see
(i) C can only be from St. Louis
⇒ Put * in the C column opposite St. Louis to indicate that a city has been matched with a person.
⇒ Eliminate St. Louis for the other 5 people by marking with #.
(ii) After C, the only possible city for A is Atlanta.
⇒ Then, E is from Milwaukee, B from Chicago, F from New York and finally D from Tulsa.
⇒ Finally ①-③ connect six initials or cities to professions.

① A is from Atlanta and is a physician
② B " " Chicago " " " teacher
③ C " " St. Louis " " an engineer
④ D " " Tulsa " " an engineer
⑤ E " " Milwaukee " " a teacher
⑥ F " " New York " " " physician

Problem :→
Three people stand in a circle with their eyes closed. A hat is placed on each of their heads. Each hat is either red or black in color and all three players know this. They all open their eyes simultaneously, and each player [illegible] [illegible] who sees a red hat is to

raise a hand. The first player to then be able to correctly identify the color of his/her own hat will win a prize.

With this setup, what will happen if two hats are red and one is black?

⇒ Suppose the players are A, B and C. Say that C wears the black hat.

⇒ Since there are two red hats, all three players raise their hands. Player A sees that C is wearing a black hat. She reasons that she cannot be wearing a black hat; for if she were, then B would not have his hand raised. Thus, A concludes that she must be wearing a red hat. Player B can reason similarly. Thus, either A or B, whoever is quickest, will speak up and win.

⇒ Player C must lose. He sees that both A and B are wearing red hats and both have their hands raised. Player C realizes that he could have either a red or black hat.

⇒ If we assume that each player wears a red hat.

⇒ Obviously, all 3 players will raise hands because each will see a red hat (indeed two red hats).