Suppose for the sake of argument, that A is the quickest of 3 players.

⇒ A knows his hat cannot be black because if it were black, then B would know his hat cannot be black, otherwise C would not have his hand raised. So, B would have concluded that his own hat was red and would have said so.

⇒ Player C could have reasoned in the same way, if A's hat had been black, and would have spoken up.

⇒ Since neither B nor C has spoken up, A concludes that his own hat is red, says so, and wins the prize.

Problem :→ (Games)

A game is played by two players. They begin with a pile of 30 chips, all the same. For his or her move, a player may remove 1 to 6 chips. The player who removes the last chip wins. What strategy can the first player use so that he will always win?

⇒ Call first player A and the second player B. We wish to devise a strategy for A so that he/she can surely win.

⇒ Clearly A would like to be left, on his last move, with a pile of 6 or fewer chips.

⇒ Thus, on the preceding move, B should be faced with a number of chips such that, after B selects his chips, A will be left with 6 or fewer.

⇒ If B is faced with 8 chips, then he will pick just 1, leaving A with 7. (Not good for A). Same is true if B is faced with 9 chips or even more.

⇒ Best for A would be if B is faced with 7 chips. A wants B to be faced with 7 chips on the second-to-last move.

⇒ Fourth-to-last move, A would like B to be faced with 14 chips.

⇒ Sixth-to last move, A would like B to be faced with 21 chips.

⇒ Eight-to last move A would like B to be faced with 28 chips.

⇒ Hence for his first move, A removes two chips, leaving B with 28.

Problem :→

A game is played on a board consisting of eight adjacent squares as shown in figure in next page. The initial position for the three pieces is shown in figure. A legal move is to move one piece to the left by one square. A piece can be moved on top of another piece or off of another piece. The goal is to move all 3 pieces to the square at the far left. The player who makes the last move wins. What is the winning strategy for the first player.

⇒ A game is played on a board consisting of eight adjacent squares.

⇒ Pieces move only forward (left) and never backward (right). Number three pieces 1, 2 and 3.

⇒ Piece 1 will be moved a total of three times on its journey to far left.

⇒ Piece 2 will be moved a total of five times.

⇒ Piece 3 will be moved a total of seven times.

⇒ Total number of legal moves in any game is 3+5+7 = 15. Any game will have the same odd number of moves.

⇒ Of those moves, A will make moves number 1, 3, 5, 7, 9, 11, 13 and 15. In other words, A will always make the last move ∴ A will win

The Tower of Hanoi :→

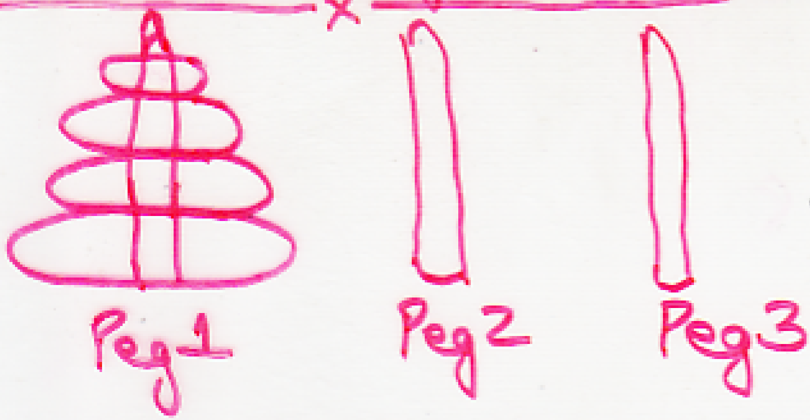


Figure illustrates the setup for the famous "Tower of Hanoi" puzzle. Notice that there are four discs, of increasing size, on the leftmost post. The goal is to move all the discs, in the same configuration, to the right most post. The rules are that any disc that is on top of any pile can be lifted and moved to any other post. However, at no time are we allowed to place a larger disc atop a smaller disc.

What is a strategy for moving all four discs to the rightmost post?

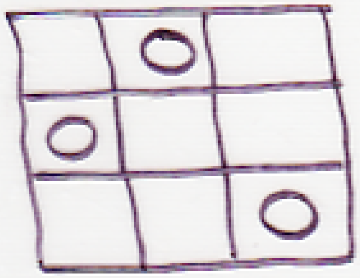
⇒ ~~D1 — P3~~
~~D2 → P2~~
~~D1 → P~~

D1 — P2
D2 — P3
D1 — P3
D3 — P2
D1 — ~~P3~~ P1
D2 — P2
D1 — P2
D4 — P3

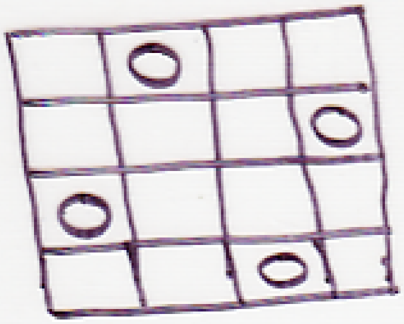
~~P1~~ D1 → P3
D2 → P1
D1 → P1
D3 → P3
D1 → ~~P3~~ P2
~~[illegible]~~
~~[illegible]~~
D2 — P3
D1 — P3

Suppose that a pseudo-chekerboard is k squares by k squares, instead of 8X8. Is it possible to put k checkers on the board so that no two are in the same row, no two are in the same column, and none is on either of the diagonals?

⇒ Try 3X3

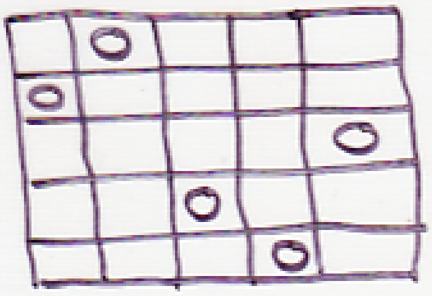


⇒ Try 4X4

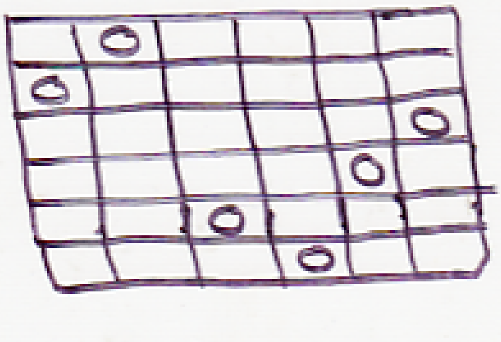


(3,1), (1,2), (4,3) and (2,4)

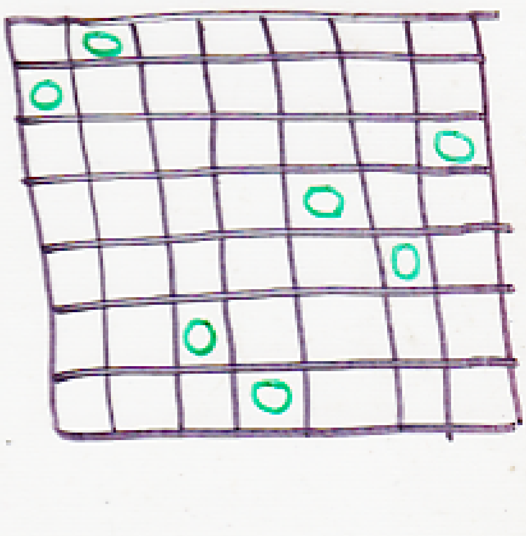
⇒ Try 5X5



⇒ Try 6X6

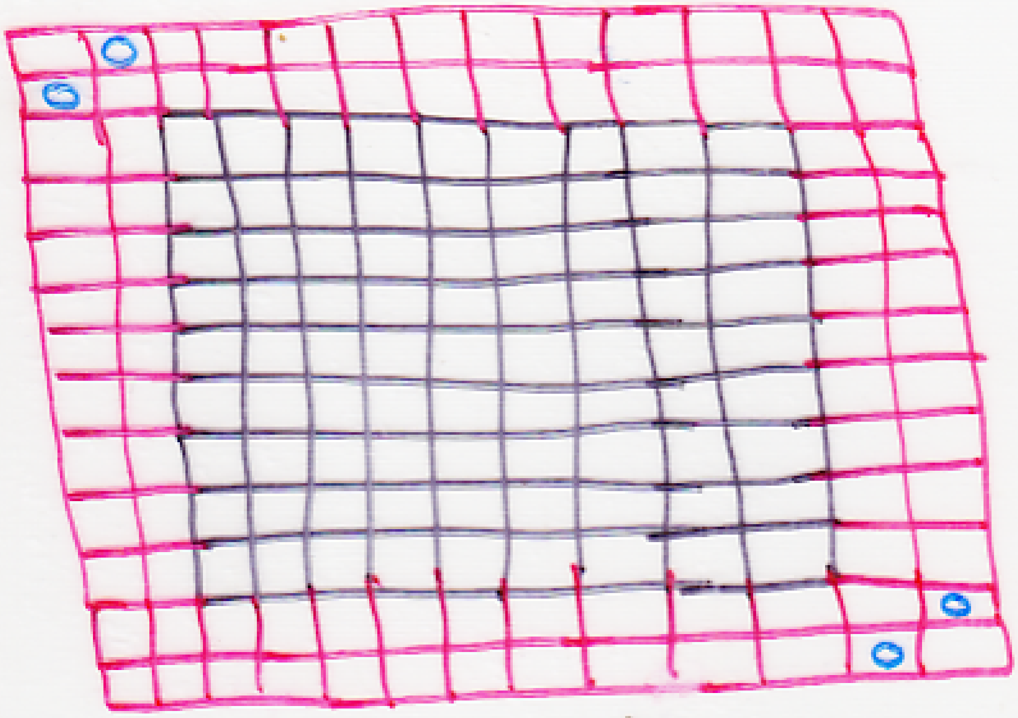


7X7



⇒ Try 7X7

Now, we check the following assertion: if we can solve the problem for a $K \times K$ board then we can use that solution to solve the problem for a $(K+4) \times (K+4)$ board.



⇒ Add checkers at positions $(1,2), (2,1), (K+3, K+4)$, and $(K+4, K+3)$

⇒ Note that the result of the last paragraph can now be applied beginning with the 4×4 board.

⇒ This gives solutions for $K = 4, 8, 12, 16$ etc.

⇒ Next we can apply the result of last para beginning with the 5×5 board.
This gives solutions for $K = 5, 9, 13, 17 \ldots$

⇒ Similarly, 6×6 gives solutions for
$K = 6, 10, 14, 18, \ldots$

⇒ Finally 7×7 gives solutions for

$K = 7, 11, 15, 19$ etc.

Hence we have constructed solutions for all $K \geq 4$.

Problem :→ Consider the diagram in Figure ~~below~~ in next Page. It shows various letters which can ~~be~~ used to spell out the sentence "WAS IT A CAT I SAW?". The question is this: in how many different ways, beginning on an edge, can you march along, from letter to letter, in the diagram and spell out this sentence.

Faulty attempt :→

Since ~~the~~ sends ends with "SAW," and hence with a 'W', therefore it must terminate on an edge. Likewise, the sentence begins with "WAS," so begins with a 'W', and so must start on an edge. There are 24 distinct W's running around the edge. Each version of the sentence must begin at one of these W's and end at one of these W's. All different beginnings and endings are possible. Therefore, the number of different ways to trace out the sentence is 24×24 or 576 ways. This count falls far short

W

W A W

W A S A W

W A S I S A W

W A S I T I S A W

W A S I T A T I S A W

W A S I T A C A T I S A W

W A S I T A T I S A W

W A S I T I S A W

W A S I S A W

W A S A W

W A W

W

⇒ We have overlooked the fact that the sentence is a palindrome, it reads the same forward as backwards. What one then notices is that one needs to

(i) begin at a square on the edge

(ii) proceed to the center, spelling out 'WAS IT A C', and then proceed back out to the edge, spelling out 'AT I SAW'

⇒ Counting all the branch rules, there are 252 different ways to begin at an edge square and proceed to the center

⇒ There ~~are~~ are, of course, just as many ways to begin at the center and go back out of the edge. A complete path would consist of one of the first followed by one of the second.

⇒ Thus, the total number of possibilities is $252^2 = 63504$

Problem :→ A lattice point in three dimensional space ($\mathbb{R}^3$) is one with all integer coordinates. Take any nine different lattice points in $\mathbb{R}^3$. Explain why one of the (thirty six) line segments connecting two of them must have midpoint that is a lattice point.

⇒ It is not automatic for the midpoint of a segment connecting two lattice points to be a lattice point. For eg:→

$A=(0,0,0)$ and $B=(1,1,1)$ are lattice points but their midpoint $C=(1/2, 1/2, 1/2)$, which is not a lattice point.

Let $A=(a,b,c)$ and $A'=(a',b',c')$

$$\text{Mid-point} = \left(\frac{a+a'}{2}, \frac{b+b'}{2}, \frac{c+c'}{2}\right)$$

In order for these coordinates to be whole numbers, it must be that

$a+a'$ is even

$b+b'$ is even

$c+c'$ is even

Then a, a' must either both be even or both be odd. Likewise for b, b' and c, c'. This is determining future.

Let e stand for "even" and o stand for "odd". Then the possible parities for a lattice point in space are

(o,o,o)	(e,o,o)
(o,o,e)	(e,o,e)
(o,e,o)	(e,e,o)
(o,e,e)	(e,e,e)

⇒There are a total of eight possibilities. But in problem, we are given nine lattice points. Thus, two of them must have the same parity. By the preceding reasoning, the midpoint of those two points must also be a lattice point.

Problem :→

Consider the letters in the display

$$\begin{array}{r} LETS \\ WAVE \\ \hline LATER \end{array}$$

This represents an addition problem. Different letters stand for different digits (chosen from 0, 1, 2, ..., 9). Two occurrences of the same letter (such as A) stand for the same digit. The problem is to identify all the digits.

⇒We begin with the leftmost L in LATER. This L arises from the carrying operation of arithmetic. Since L and W that we added to get this leftmost L can each be no greater than 9, there is no way that the leftmost L could be anything other than a 1. Thus L = 1 in both occurrences.

⇒ Now W could only be an 8 or 9. But W cannot be 8, since then A must be zero (0) and there will have to be carrying from the addition of E and A to bump L+W = 1+8 up to 10. With A zero (0), this will force E to be 9, and we would have to carry from the addition of T and V so that E would not equal T. But even that wouldn't help, because after the carrying the result would be that T=0 and 0 is already taken. So, W cannot equal 8; it must equal 9.

```
   1 E T S
   9 O V E
 ---------
 1 O T E R
```

Now, whatever E is, T will have to be one greater (from carrying) so that T will be unequal to E. But T+V must yield E again. How could this be unless V is 9? But V cannot be 9 because 9 is taken. So, V must be equal to 8, the addition of S and E will have to force a carry.

$$\begin{array}{r} 1\,E\,T\,S \\ 9\,0\,8\,E \\ \hline 1\,0\,T\,E\,R \end{array}$$

Notice that T cannot be 2 because the E is 1 and 1 is taken. If T=3 then E is 2 and we have

$$\begin{array}{r} 1\,2\,3\,S \\ 9\,0\,8\,2 \\ \hline 1\,0\,3\,2\,R \end{array}$$

Since 9 and 8 are taken, then S cannot be greater than 7. But then the addition in the rightmost column does not result in a carry and nothing works.

If T=4 then E=3

$$\begin{array}{r} 1\,3\,4\,S \\ 9\,0\,8\,3 \\ \hline 1\,0\,4\,3\,R \end{array}$$

If S=7 then R=0 and 0 is taken; if S=6 then there is no carry in the rightmost column. So, we cannot allow T=4. The possibility of T=5 is eliminated in the same fashion

⇒ Let T=6, then E=5 and we have

$$\begin{array}{r} 1\,5\,6\,S \\ 9\,0\,8\,5 \\ \hline 1\,0\,6\,5\,R \end{array}$$

⇒ Now S=7 and R=2 is viable choice and everything works. We have solved our puzzle as

$$\begin{array}{r} 1567 \\ 9085 \\ \hline 10652 \end{array}$$

⇒ T=7 cannot work; T=8 is not an option because 8 is taken

If T=7, E=6

$$\begin{array}{r} 167S \\ 9086 \\ \hline 1076R \end{array}$$

Then S cannot be 7 or 6, the option for S is 5. Then R=1 which is taken

Consider the division problem

```
           *53
    **9 ) 6*8***
          **2
          ----
           *9**
           **4*
           ----
            **4*
            ****
            ----
```

in which a number of the digits have been obliterated. It turns out that each of the missing digits is uniquely determined by the information provided. Solve for the missing digits.

Rename *'s with letters for easy reference

```
          a53
      ___________
 bc9 ) 6d8efg ........ First line
       hij2 .......... Second line
       ____
        k9lm ......... Third line
        n04p ......... Fourth line
        ____
          qr45 ....... Fifth line
           tuvw ...... Sixth line
           ____
```

⇒ Sixth line

⇒ ∵ 3·bc9 results in numbers that end with 7. ~~and~~ with 2 to be carried

⇒ ∴ w=7 and c=4 so that the next step of multiplication 3·bc9 gives a 4 in the second-to-last place (i.e. v=4)

⇒ s must equal w ∴ s=7

⇒ Fourth line

∵ 5·bc9 results in numbers that end with 5 ~~and carry~~ ∴ p=5.

```
          a53
      ___________
 b48 ) 6d8efg ........ First line
       hij2 .......... Second line
       ____
        k9lm ......... Third line
        n045 ......... Fourth line
        ____
          qr47 ....... Fifth line
          tu47 ....... Sixth line
          ____
```

```
       a53
b49)6d8efg . . . . . First line
    hij2 . . . . . . Second line
    ----
     kglm . . . . . .Third line
     no45 . . . . . .Fourth line
     ----
       qr47 . . . . .Fifth line
       tu47 . . . . .Sixth line
       ----
```

⇒ clearly, on third line $m=9$ $(\because 9-5=4)$ and therefore, $f=9$

⇒ a must be 8 so that the last digit of second line turns out to be a 2. Then $8\cdot b49 = *\overset{3}{*}92$ which makes $j=9$.

```
       853
b49)6d8e9g . . . . . First line
    hi92 . . . . . . Second line
    ----
     kgl9 . . . . . .Third line
     no45 . . . . . .Fourth line
     ----
       qr47 . . . . .Fifth line
       tu47 . . . . .Sixth line
       ----
```

⇒ Notice that $g=7$ and h must turn out to be a 6. Thus, b is either a 7 or an 8. If $b=8$ then $hi92 = 6792$ and $no45 = 4245$

```
       853
849)6d8e97 . . . . . First line
    6792 . . . . . . Second line
    ----
     kgl9 . . . . . .Third line
     4245 . . . . . .Fourth line
     ----
       qr47 . . . . .Fifth line
       tu47 . . . . .Sixth line
       ----
```

⇒ Now there is a problem as k must turn out to be 4. This means that d is a 2. But if this is a case then second line will be greater than the First line. Hence $b \neq 8$ and b must equal 7.

$$\begin{array}{rl}
853 & \\
749\overline{)6d8e97} & \text{—— First Line} \\
5992 & \text{---- Second Line} \\
\hline
k9l9 & \text{---- Third Line} \\
3745 & \text{---- Fourth Line} \\
\hline
qr47 & \text{---- Fifth Line} \\
2247 & \text{---- Sixth Line} \\
\hline
\end{array}$$

Now, we have complete divisor and the quotient. Hence

$$\text{Dividend} = \text{Divisor} \times \text{Quotient}$$
$$= 749 \times 853$$
$$= 638{,}897$$

$\therefore d = 3,\ e = 8$ First Line

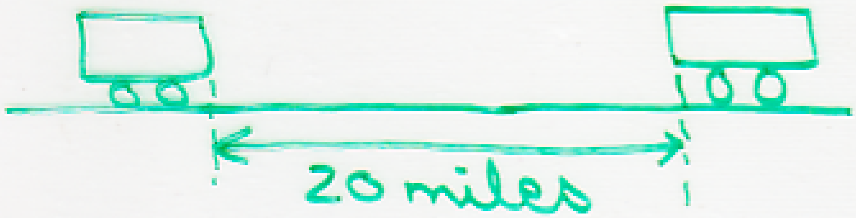
$\therefore k = 3,\ l = 6$ Third Line

$\therefore q = 2,\ r = 2$ Fifth Line

<u>Surprise Problem</u> :→

Two train engines are approaching each other. They begin at a distance of 20 miles, and are each traveling at a rate of 10 miles per hour. At the instant they begin, a fly takes off from the front of one train at a rate of 15 miles per hour. When it meets the front of the other train, it immediately turns around and heads back towards the first train. It continues back and forth until it is crushed when the two trains meet.

What is the total linear distance traveled by the fly on its zig-zag journey before it meets its end?



⇒ How far does each train travel before the crash?

⇒ They are separated by 20 miles and each travels at the constant speed of 10 miles per hour.

⇒ Each travels 10 miles up to the point of impact, so each travels for one hour. But, during that one hour, the fly travels 15 miles.

Imagine a steel band strapped tightly around the equator of the earth. Such a band would be about 25,000 miles long. Now suppose that we lengthen the band just enough so that it stands uniformly one foot off the surface of the earth (but still forms a continuous circular loop). How long will it be now? [Assume in solving this problem that the earth is spherical and that the band strapped tightly about the equator forms a circle.]

Let R and C be the radius and circumference in feet of the earth at the equator respectively. Then

$$C = 2\pi R$$

$\Rightarrow$ Our goal is to increase the radius by 1. Then, new radius is

$$R' = R + 1$$ and new circumference

is $$C' = 2\pi(R') = 2\pi(R+1)$$

$\Rightarrow$ Amount of steel band that must be added to make the band stand one foot off the surface of the earth is given by

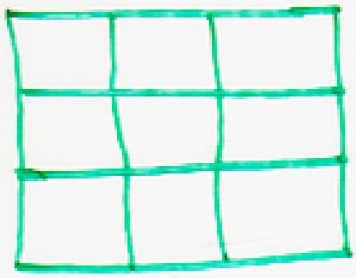
$$\begin{aligned} C' - C &= 2\pi(R+1) - 2\pi R \\ &= 2\pi R + 2\pi - 2\pi R \\ &= 2\pi \end{aligned}$$

$\Rightarrow$ We must add 2π feet, or about $6 \cdot 28318$ feet, to the band to achieve our goal.

Recreational Math :→

Magic Squares :→

Imagine a 3X3 array of squares, as shown in Figure. The challenge is to put the integers 1 to 9, one in each square, so that each row and each column adds upto the same number.



⇒ Determine the common sum S must be.

⇒ If we add each number in each row then the following is true

(i) Since there are three rows, we will obtained the common sum S three different times. So our grand total is 3S. And we have counted each square exactly once. These contain the numbers 1 through 9, just as we have prescribed.

Thus, we have

$$3S = [1+2+\dots+9] = \frac{9\cdot 10}{2} \qquad \left[\because S_n = \frac{n(n+1)}{2}\right]$$

$$S = \frac{45}{3} = 15$$

So, we need to place the numbers 1 to 9 into the array so that each row and of the earth (but still forms a continuous circular loop). How long will it be now? [Assume in solving this problem that the earth is spherical and that the band strapped tightly about the equator forms a circle.]

Let R and C be the radius and circumference in feet of the earth at the equator respectively. Then

$$C = 2\pi R$$

$\Rightarrow$ Our goal is to increase the radius by 1. Then, new radius is

$$R' = R + 1$$ and new circumference is $$C' = 2\pi(R') = 2\pi(R+1)$$

$\Rightarrow$ Amount of steel band that must be added to make the band stand one foot off the surface of the earth is given by

$$\begin{aligned} C' - C &= 2\pi(R+1) - 2\pi R \\ &= 2\pi R + 2\pi - 2\pi R \\ &= 2\pi \end{aligned}$$

$\Rightarrow$ We must add 2π feet, or about 6.28318 feet, to the band to achieve our goal.

each column adds to 15.

⇒ Put 9 in the center (Extreme Value)

⇒ This puts severe limits on what can go to the left and right of 9, or what can go above and below it. The choices are

$1+5=6$

$2+4=6$

$3+3=6$ → No use as we can use only one 3.

	5	
4	9	2
	1	

⇒ We cannot put 3 in the bottom row, for 1+3 is too small; there is no third number that will make 15. So, put 3 in upper row; say in the upper left hand corner.

⇒ This forces the upper right to be 7, which forces the lower right to be 6, which forces the lower left to be 8.

3	5	7
4	9	2
8	1	6

⇒ The best magic squares not only have all the rows and all the columns summing to the same number, but also the diagonals sum to the same number. 3X3 magic square in which that is true is shown below.

I

8	1	6
3	5	7
4	9	2

For now, assume this is obtained by hit and trial.

⇒ Suppose that in above magic square we bump each number "up" (in the vertical direction) one square. This empties the bottom row. But it also pushes the first row off the top; so we rotate it down to the bottom. The result is shown below

II

3	5	7
4	9	2
8	1	6

⇒ The result is still a magic square, with the same magic number of 15.

⇒ By same token, push everything one unit to the right.

For I

6	8	1
7	3	5
2	4	9

7	3	5
2	4	9
6	8	1

For II

⇒ The result is still a magic square with magic number of 15.

⇒ Shift each number in Fig II one square to the right and one square up. This is a diagonal shift.

8	1	6
3	5	7
4	9	2

3x3 Magic Square

23	6	19	2	15
10	18	1	14	22
17	5	13	21	9
4	12	25	8	16
11	24	7	20	3

5x5 Magic Square

46	15	40	9	34	3	28
21	39	8	33	2	27	45
38	14	32	1	26	44	20
13	31	7	25	43	19	37
30	6	24	49	18	36	12
5	23	48	17	42	11	29
22	47	16	41	10	35	4

7X7 Magic Square

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

4X4 Magic Square

64	63	3	4	5	6	58	57
56	55	11	12	13	14	50	49
17	18	46	45	44	43	23	24
25	26	38	37	36	35	31	32
33	34	30	29	28	27	39	40
41	42	22	21	20	19	47	48
16	15	51	52	53	54	10	9
8	7	59	60	61	62	2	1

8X8 Magic Square

Result

2	4	9
6	8	1
7	3	5

⇒ A quick check shows it is still a magic square.

⇒ Now, we build a 3X3 magic square by beginning in the location a_{12}. Place a 1 there. The rows and columns in the picture have 3 elements each, so 3 seems to be a natural "period" for this problem.

⇒ Beginning with a_{12}, we lay out 1,2,3 along a diagonal.

	1	

	1	
3		
		2

⇒ We laid out 1,2,3 along a diagonal that moves up and to the right.

⇒ Now we look at diagonals that move up and to the left.

⇒ We lay out numbers on those diagonals using the natural period 3.

⇒ Beginning with 1 at a_{12}, we next lay a 4 at a_{31} and then a 7 at a_{23}. Next, beginning with 2 at a_{33}, we lay 5 at a_{22} and an 8 at a_{11}. Finally, beginning with 3 at a_{21}, we next lay a 6 at a_{13} and then a 9 at a_{32}.

8	1	6
3	5	7
4	9	2

Magic square

Problem

Use the ideas developed so far to produce a 5X5 magic square

⇒ We have a 5X5 tableau and we have placed a 1 in the square a_{13}.

⇒ Now we lay out the numbers 1, 2, 3, 4, 5 along a diagonal moving up and to the right.

⇒ Next we lay out diagonals that proceed in the opposite direction - moving up and to the left - and with the numbers having period 5.

		1		

		1		
	5			
4				
				3
			2	

Up → Right

	18	1	13	22
17	5	13	21	9
4	12		8	18
11		7		3
23	6	18	2	

Up → Left

10	18	1	14	22
17	5	13	21	9
4	12	25	8	16
11	24	7	20	3
23	6	19	2	15

Up → left

⇒ This is indeed a magic square. A magic number is 65.

$$5S = (1+2+\ldots\ldots+25)$$
$$= \frac{25 \cdot 26}{2}$$
$$\Rightarrow S = \frac{5 \cdot 26}{2} = 65$$

What modification can you make to the method so that it will still work in the 4X4 case?

⇒ Place the numbers 1 through 16 into 4X4 tableau in their natural order.

⇒ For each element in either of the diagonals replace it with its complementary number. Hence, by a complementary number, we mean the number, that added to it, makes 17. For eg, the complementary number to 6 is 11 and the complementary number to 13 is 4.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

Magic Square with magic number 34.

$$4S = (1+2+\ldots+16)$$
$$= \frac{16 \cdot 17}{2}$$
$$\Rightarrow S = 34$$

Problems Involving Weighings

① Suppose that you have 9 pearls. They all look the same, but 8 of have equal weight and one is different. The odd pearl is either lighter or heavier; you do not know which. The only equipment that you have at hand is a balance scale. How can you use the scale to find the odd pearl in just three weighings?

⇒ Divide the 9 pearls into three groups of three with the names G_1, G_2 and G_3

⇒ Weigh G_1 against G_2

i) If they happen to balance, then all six pearls in G_1 and G_2 are control pearls. The odd pearl is one of the pearls in G_3

ii) If they do not balance, then all the pearls in G_3 are control pearls. The odd pearl is either in G_1 or G_2 but we do not know which.

Case (i)

Weigh G_1 against G_3. Of course they will not balance. G_3 will either be lighter or heavier. Make a note of which. Say that, for the sake of specificity, G_3 is heavier. That means that the odd pearl is heavier than the others, and it lies in G_3. Now, select any two pearls from G_3 and weigh them against each other. If they balance, then the odd pearl is the third pearl in G_3 and it is heavier.

If they do not balance, then the heavier of the two is the odd pearl.

Case (ii)

⇒ ~~If~~ Make a note of whether G_1 or G_2 is heavier. For specificity, say that G_1 is heavier. Now weigh G_1 against G_3. If they balance, then the odd pearl is in G_2 and it is lighter. ~~If they do not balance then the lighter of the two is the odd pearl~~. Pick any two pearls from G_2 and weigh them against each other. If they balance then the odd pearl is the third one from G_2 and it is lighter. If they do not balance then the lighter of the two is the odd pearl.

⇒ If instead G_1 and G_3 do not balance, then the only possibility is that G_1 is heavier than G_3 (otherwise, there would be three weight categories, which is impossible). So, the odd pearl is in G_1 and it is heavier. For the last step, weigh any two pearls from G_1 against each other and proceed as in the earlier cases.

Suppose now that you have 12 pearls, all appearing the same but with one having an odd weight. You do not know whether the odd pearl is heavier or lighter. ~~How~~ many weighings are needed to find the odd pearl?

⇒ Divide the twelve pearls into three groups of 4. Call them G_1, G_2, G_3. At first step, weigh G_1 against G_2.

① If they happen to balance, then all eight pearls in G_1 and G_2 are control pearls. The odd pearl is one of the pearls in G_3

② If they do not balance, then all the pearls in G_3 are control pearls. The odd pearl is either in G_1 or G_2 but we do not know which.

Case ①

⇒ Take any 3 pearls from G_1 and weigh them against any 3 pearls from G_3. If they balance, then the odd pearl is the remaining pearl from G_3. Weigh that last pearl against one of the pearls from G_1 will tell whether the odd perl is heavy or light.

⇒ If they do not balance, then the odd pearl will be among the three selected from G_3 and we all know whether it is lighter or heavier (since the pearls from G_1 are control pearls). Now a third weighing, as usual, will pin down the odd pearl from among those three that we selected from G_3.

Case ②

Suppose for specificity that G_1 is heavier and G_2 lighter. Give the pearls in G_1 names a, b, c, d and give the pearls in G_2 the names a', b', c', d'. For the second weighing, we weigh $\{a, b, a'\}$ against $\{c, d, b'\}$

ⓐ If they balance, then the odd pearl is one of c', d'. Of course c', d' come from light side, so we know that the odd pearl is light. For third weighing, we weigh c' against d'. The odd pearl is the lighter of the two.

ⓑ If they do not balance, then say that $\{a, b, a'\}$ is heavier. This must mean that c, d are control pearls and so is a', or else the balance would be the other way. Thus, the odd pearl is either a, b or b'. Finally weigh a against b. If they balance then the odd pearl is b' and it is light. If they do not balance then the odd pearl is the heavier of the two (since a and b come from G_1).

ⓒ The case that $\{c, d, b'\}$ is heavier is handled just as in sub-case (b).

You have 80 pearls. One is lighter than all the others. Find the odd pearl in just four weighings on a balance scale.

⇒ divide the pearls into three groups of 27, 27 and 26.

Weigh 27 against 27. If they balance, then the odd pearl is among the 26, and it is light. If they do not balance, then the odd pearl is in the lighter of two groups.

In the first case (that the scale balanced), divide the remaining 26 pearls into groups of 9, 9 and 8. Weigh 9 against 9. And so forth and so on. You can see that after narrowing to 9 we will then narrow to 3 and then we are home.

If 27 against 27 do not balance, then we focus on the lighter group of 27. We divide that into 3 groups of 9 and weigh one group of 9 against another. If the balance then we devolve upon the third group of 9; if they do not balance then we choose the lighter group of 9. Then we narrow down to 3 and so forth.

You have 24 marbles, all of which appear to be the same. However, a certain number of them are made of glass and a certain number are made of quartz. The glass marbles are heavier. All the glass marbles weigh the same and all the quartz marbles weigh the same. How many weighings, using a balance scale, would be required to determine the number of glass marbles and the number of quartz marbles?

⇒ Pick up two marbles and test them against each other. There are now two possibilities.

① The marbles do not balance

⇒ One the heavier is glass and one the lighter is quartz. Now put those two marbles together on one side of the balance scale. Pick up two more marbles and put them on the other side. If the two new marbles are heavier, then they are both glass. If the two new marbles are lighter, then they are both quartz. If the two new marbles balance, then one is quartz and one is glass. In any of these three events, we can count how many glass and how many quartz marbles there are among the two new candidates. So, we set those two new marbles aside and mark our tally on a piece of paper. Then we put two more marbles on the balance scale and weigh them against our first two. We keep going in this fashion. We see that all the marbles will be weighed and counted after $1+\frac{22}{2}=12$ weighings.

② The marbles balance

⇒ Either both are glass or both are quartz. We use those two together as a test pair. Pick up another pair of marbles and weigh them against the first two. If the scale balances, then we have two more marbles of the same kind (either glass or quartz) but we do not know which just yet. Keep going until you find a pair that does not balance. Say that it is the k^{th} pair. If the k^{th} pair is heavier, then we may conclude that the test pair, and all the pairs up to that point, are quartz. If the k^{th} pair is lighter, then we may conclude that the test pair, and all the pairs up to that point are glass.

⇒ Suppose that the k^{th} pair is heavier. Now separate the two marbles in this kth pair and weigh these two marbles one against the other. If they balance then they are both glass. If they do not balance, then one of them is glass and you know which one it is. In any event, pick out the glass marble from the Kth pair and one of the quantity marbles from the original test pair. Use these two to form a new test pair and now proceed as in the first case to test the rest of the pairs of marbles. Altogether, we have used

$$1+(K-1)+1+\frac{(24-2K)}{2}=13 \text{ weighings.}$$

Algebra :→

① Show that if n is a positive integer then n^3-n is always divisible by 3.

$$n^3-n=n(n^2-1)=n(n-1)(n+1)=(n-1)n(n+1)$$

⇒ These factors are three integers in succession. Therefore, one of them is a multiple of 3. As a result, n^3-n is a multiple of 3.

② Show that if n is a positive integer then n^5-n is always divisible by 5.

$$n^5-n=n(n^4-1)=n(n^2-1)(n^2+1)=n(n-1)(n+1)(n^2+1)$$

⇒ If n is an integer ending with one of the digits 0, 1, 4, 5, 6, 9 then one of $(n-1)$, n, $(n+1)$ is divisible by 5. So, the product is divisible by 5.

⇒ If n is an integer ending with 2, 3, 7 or 8, then n^2 ends with 4 or 9 hence n^2+1 is divisible by 5. Therefore, the product is divisible by 5. Hence n^5-n is divisible by 5.

Verify the combinatorial identity

$$\binom{k}{m}+\binom{k}{m+1}=\binom{k+1}{m+1}$$

$$\text{L.H.S.}=\binom{k}{m}+\binom{k}{m+1}$$

$$=\frac{k!}{m!(k-m)!}+\frac{k!}{(m+1)!(k-m-1)!}$$

$$=\frac{(m+1)k!}{(m+1)!(k-m)!}+\frac{(k-m)k!}{(m+1)!(k-m)!}$$

$$=\frac{\cancel{mk!}+k!+kk!-\cancel{mk!}}{(m+1)!(k-m)!}$$

$$=\frac{(k+1)k!}{(m+1)!(k-m)!}$$

$$=\frac{(k+1)!}{(m+1)!((k+1)-(m+1))!}$$

$$=\binom{k+1}{m+1}$$

Verify the binomial formula

$$(a+b)^k=a^k+\binom{k}{1}a^{k-1}b+\binom{k}{2}a^{k-2}b^2+\dots\binom{k}{k-2}a^2b^{k-2}$$

$$+\binom{k}{k-1}ab^{k-1}+b^k$$

Basis Step :→ When $k=1$

$$a+b=a+b \text{ (True)}$$

When $k=2$

$$(a+b)^2=a^2+\binom{2}{1}a^{2-1}b+b^2=a^2+\binom{2}{1}ab+b^2$$

$$=a^2+2ab+b^2 \text{ (True)}$$

Inductive Step :→ Assume that the given formula is true for k and prove that it is true for $k+1$.

Multiply both sides by $(a+b)$ in the above formula

$$(a+b)^K(a+b) = (a+b)\Big[a^K + \binom{K}{1}a^{K-1}b + \binom{K}{2}a^{K-2}b^2 + \dots$$
$$\binom{K}{(K-2)}a^2b^{K-2} + \binom{K}{(K-1)}ab^{K-1} + b^K\Big]$$

$$(a+b)^{K+1} = a^{K+1} + \left[1+\binom{K}{1}\right]a^Kb + \left[\binom{K}{1}+\binom{K}{2}\right]a^{K-1}b^2$$
$$+\left[\binom{K}{2}+\binom{K}{3}\right]a^{K-2}b^3 + \dots\dots$$
$$+\left[\binom{K}{K-2}+\binom{K}{K-1}\right]a^2b^{K-1} + \left[\binom{K}{K-1}+\binom{K}{K}\right]ab^K$$
$$+ b^K$$

From previous results

$$(a+b)^{K+1} = a^{K+1} + (K+1)a^Kb + \binom{K+1}{2}a^{K-1}b^2 +$$
$$\binom{K+1}{3}a^{K-2}b^3 + \dots + \binom{K+1}{(K-1)}a^2b^{K-1}$$
$$+\binom{K+1}{K}ab^K + b^K$$

$$\left[\because \left[1+\binom{K}{1}\right] = (K+1)\right]$$

Which is greater?

$K = (1+0.000001)^{1,000,000}$ or 2?

Expand K using binomial theorem

$$K = (1+0.00001)^{1,000,000} = 1^{1,000,000}$$
$$+$$
$$1,000,000 \cdot 1^{999,999} \cdot 0.000001$$
$$+ \text{positive terms}$$
$$= 1 + 1 + \text{positive terms}$$
$$> 2$$

Hence $K > 2$

Which is greater?

1000^{1000} or 1001^{999} ?

Use binomial theorem

$$1001^{999} = [1000+1]^{999}$$

$$= 1000^{999} + 999 \cdot 1000^{998} \cdot 1$$

$$+ \binom{999}{2} \cdot 1000^{997} \cdot 1^{2} + \binom{999}{3} \cdot 1000^{996} \cdot 1^{3}$$

$$+ \ldots + \binom{999}{997} \cdot 1000^{2} \cdot 1^{997} + \binom{999}{998} \cdot 1000 \cdot 1^{998} + 1$$

$$< \underbrace{1000^{999} + 1000^{999} + \ldots + 1000^{999}}_{1000 \text{ times}} = 1000^{1000}$$

Hence 1000^{1000} is the greater number

Assume that k is a positive integer. Calculate

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{(k-1) \cdot k} + \frac{1}{k \cdot (k+1)}$$

Sum up for $k = 1, 2, 3, 4$ and see whether there is a pattern. Call the sum S_k

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{2}{3}$$

$$S_3 = \frac{3}{4}$$

$$S_4 = \frac{4}{5}$$

$\Rightarrow$ Plainly there is a pattern. so, verify using induction

The only terms that do not occur twice are the first and last terms.

$$2(1+3)+3(2+4)+4(3+5)+\ldots+n((n-1)+(n+1))$$
$$=2[1\cdot2+2\cdot3+3\cdot4+\ldots+n(n+1)]-1\cdot2-n(n+1)$$

$$\Rightarrow 2\cdot4+3\cdot6+4\cdot8+\ldots+n\cdot2n=2T_n-2-n(n+1)$$

$$\Rightarrow 2[2^2+3^2+4^2+\ldots+n^2]=2T_n-(n^2+n+2)$$

We have,

$$1^2+2^2+3^2+\ldots+n^2=\frac{2n^3+3n^2+n}{6}$$

Hence

$$2^2+3^2+\ldots+n^2=\frac{2n^3+3n^2+n}{6}-1^2$$
$$=\frac{2n^3+3n^2+n-6}{6}$$

Putting this in the above equation

$$\cancel{2}\left[\frac{2n^3+3n^2+n-6}{\cancel{6}\,3}\right]=2T_n-(n^2+n+2)$$

$$2T_n=\frac{2n^3+3n^2+n-6}{3}+(n^2+n+2)$$
$$=\frac{2n^3+3n^2+n-\cancel{6}+3n^2+3n+\cancel{6}}{3}$$
$$=\frac{2n^3+6n^2+4n}{3}=\frac{n(2n^2+6n+4)}{3}$$
$$=\frac{2n(n^2+3n+2)}{3}$$

$$\Rightarrow T_n=\frac{n(n+1)(n+2)}{3}$$

Assume S_j is true

$$S_j = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{(j-1)\cdot j} + \frac{1}{j(j+1)} = \frac{j}{j+1}$$

Add $\frac{1}{(j+1)(j+2)}$ to both sides

$$S_{j+1} = \frac{j}{j+1} + \frac{1}{(j+1)(j+2)} = \frac{j+1}{j+2}$$

Second Method :→

$$S_K = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{(K-1)\cdot K} + \frac{1}{K\cdot(K+1)}$$

$$= \left[\frac{1}{1} - \frac{1}{2}\right] + \left[\frac{1}{2} - \frac{1}{3}\right] + \left[\frac{1}{3} - \frac{1}{4}\right] + \dots + \left[\frac{1}{K} - \frac{1}{K+1}\right]$$

$$S_K = 1 - \frac{1}{K+1} = \frac{K}{K+1}$$

Calculate the sum

$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \dots n(n+1)$

Call the sum T_n

$T_1 = 2$

$T_2 = 8$

$T_3 = 20$

$T_4 = 40$

$T_5 = 70$

No pattern is apparent. Try

$T_n = 2(1+3) + 3(2+4) + 4(3+5) + \dots + n((n-1)+(n+1))$. $\boxed{*}$

Notice that the term $2\cdot 3$; $3\cdot 4$; .... occurs twice

MISCELLANY :→

① There are two married couples that need to cross a river. A small boat is available that will hold just two people at a time. The males involved are quite jealous. No woman can be left with a man unless her husband is also present. There are no other constraints. How can these four people cross the river? What is the fewest number of trips possible?

⇒ Let us denote the people by H_1, H_2 for the husbands and W_1, W_2 for the corresponding wives.

⇒ H_1 and W_1 cross the river on the first trip.

⇒ H_1 returns.

⇒ H_1 and H_2 cross the river.

⇒ H_2 returns.

⇒ H_2 picks up his wife W_2 and then crosses the river.

② A general needs to take his troops across the river. He spies two boys with a small boat. He commanders both the boat and the boys. Unfortunately, the boat will only hold two boys or one soldier. Yet he determines a method for getting his troops across. What could it be?

⇒ Notice that the number of soldiers is not given, suggesting that the problem is independent of the number of soldiers.

⇒ For the first trip, there is no sense to send one soldier. For all he could do would be to go across and then either
(i) stay and leave everyone else stranded
or (ii) just row back.

⇒ For the first trip, the two boys go. One boy returns. Then one soldier goes across. Then the boy on the far side of the river can take the boat back.

⇒ Now matters are just as at the start, except that one soldier has been transferred across the river.

⇒ So, now the two boys cross again. One boy stays on the far shore (with the lone soldier) and the other returns with the boat. Now a second soldier can go across. The boy returns with the boat. Now both boys are on the starting shore with the remaining soldiers.

⇒ Clearly, this process can be repeated indefinitely to get all the soldiers, and the commander, across.

③ A railroad engine can just couple with, then either pull or push, either one or two cars. Of course the two cars can be coupled to each other as well.

At a certain junction, the railroad track is configured as in Figure next slide. Note that the portion of track between γ and δ in figure is a dead end and can hold either just one car or just the engine. However, the portion of the track to the left of α and the portion of the track to the right of β have no restrictions and can hold any number of cars.

How can the engine reverse the positions of the black car and the white car (that is, put the black car on the right track b/w β and γ and the white car on the left track b/w α and γ) and return to its original position facing right? This should be done in at most 10 moves. Here, a move consists either of having the engine go to some point and couple with a car, or having the engine pull a car to some point and uncouple.

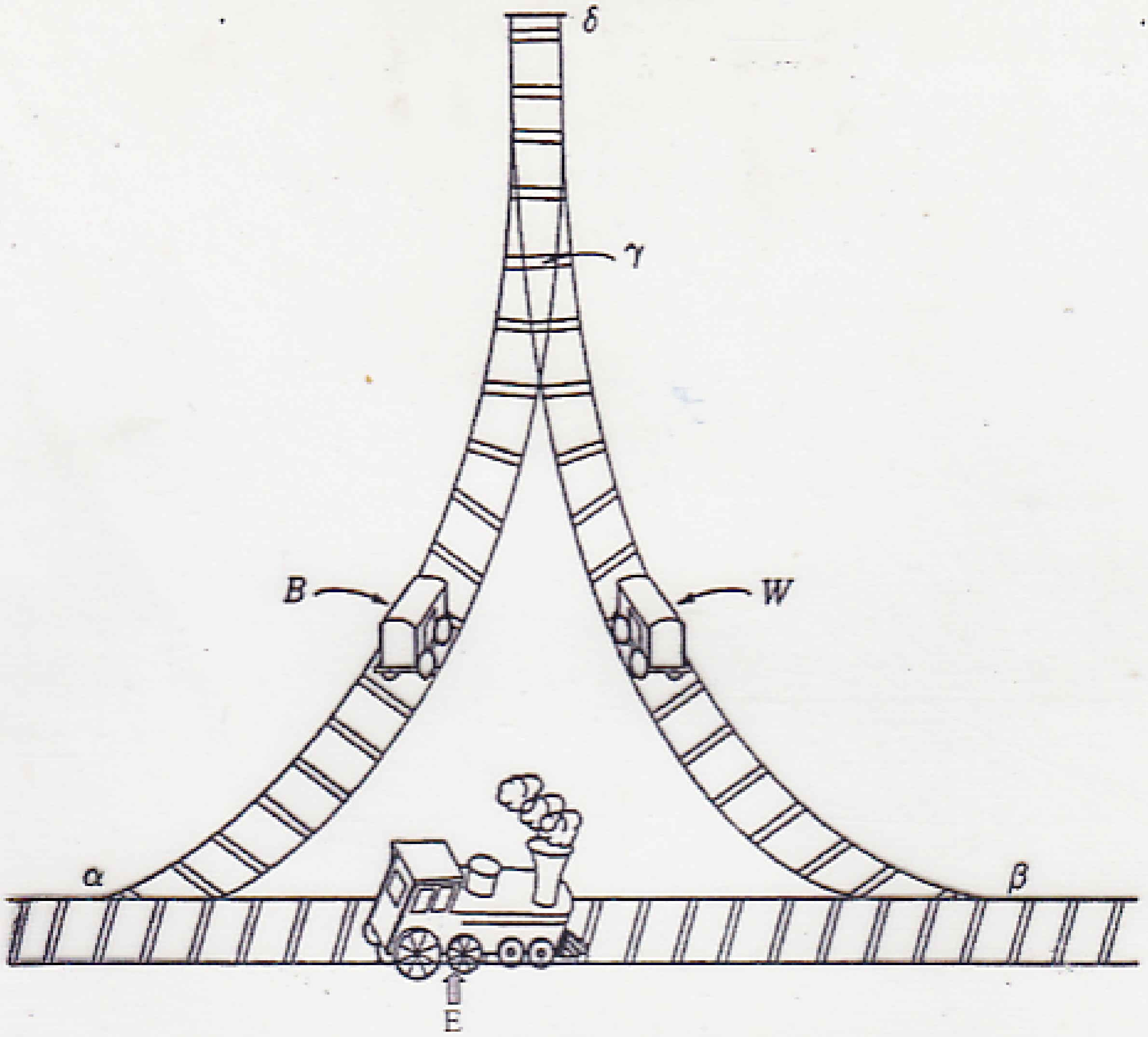
⇒ Call the white car W, the black car B, and the engine E.

(i) E drives past β, backs into $\beta\delta$, and couples with the white car.

(ii) E backs the white car into $\gamma\delta$, uncouples, and moves out along $\beta\delta$.

(iii) E moves past β, backs along $\alpha\beta$, goes forward into $\alpha\gamma$, and couples its front end on the black car.

(iv) E pushes B forward, couples it on W, and backs out past α.



δ
γ
B
W
α
β
E

(v) E pushes both cars forward until W is midway between α and β, then uncouples W.

(vi) E backs past α, then pushes B into $\alpha\gamma$ and up into $\gamma\delta$, where it uncouples B.

(vii) E backs down $\alpha\gamma$ past α, then moves forward on $\alpha\beta$ and couples with W.

(viii) E backs on $\alpha\beta$ past α, then goes forward on $\alpha\gamma$, pushing W to the center of $\alpha\gamma$, where it uncouples W.

(ix) E backs down $\alpha\gamma$, past α, moves forward on $\alpha\beta$ past β, backs into $\beta\gamma$ all the way to B, and couples with B.

(x) E pulls B forward, decouples in the middle of $\beta\gamma$, continues forward past β, and then backs down $\alpha\beta$ to the midpoint of $\alpha\beta$.

ASSIGMENT x SOLUTION

① Calculate the sum of the first K odd integers.

We know,

$$1+2+3+\dots+K = \frac{K^2+K}{2}$$

We want to find:

$$1+3+5+7+\dots+(2K-1)$$

The first integer $1 = 2\cdot1-1$

The second integer $3 = 2\cdot2-1$

The third integer $5 = 2\cdot 3 - 1$

⋮

The kth integer $= 2\cdot k - 1$

$\Rightarrow$ Hence the last term is $2k-1$ in the above equation.

$\Rightarrow$ Adding all numbers from 1 to $(2k-1)$ and then subtracting the even numbers in this sum gives the sum of the odd numbers from 1 to $(2k-1)$

$$[1+2+3+4+\dots+(2k-1)] - [2+4+6+\dots+(2k-2)]$$

$$\Rightarrow [1+2+3+4+\dots+(2k-1)] - 2[1+2+3+\dots+(k-1)]$$

$$\Rightarrow \left[\frac{(2k-1)^2+2k-1}{2}\right] - 2\left[\frac{(k-1)^2+(k-1)}{2}\right]$$

$$\Rightarrow \left[\frac{(2k-1)(2k-1+1)}{2}\right] - 2\left[\frac{(k-1)(k-1+1)}{2}\right]$$

$$\Rightarrow \frac{2k(2k-1)}{2} - \frac{2k(k-1)}{2}$$

$$\Rightarrow \frac{2k}{2}[2k-1-k+1]$$

$$\Rightarrow k[k]$$

$$\Rightarrow k^2.$$

Hence the required sum is k^2.

② How many zeroes end the number $2^{300}\cdot 5^{600}\cdot 4^{400}$?

$$2^{300}\cdot 5^{600}\cdot 4^{400} = 2^{300}\cdot 5^{600}\cdot (2^2)^{400}$$

$$= 2^{300}\cdot 5^{600}\cdot 2^{800} = 2^{600}\cdot 5^{600}\cdot 2^{500}$$

$$= (2\cdot 5)^{600}\cdot 2^{500}$$

$$= 10^{600}\cdot 2^{500}$$ (Ends with 600 zeroes)

③ How many digits are used to number the pages of a book having 100 pages - numbered from 1 to 100?

Between 1 and 100, we have

9 numbers with 1 digit = 9 digits
90 numbers with 2 digits = 180 digits
1 number with 3 digits = 3 digits

192 digits

④ A watermelon weighs 500 pounds. It turns out that 99% of the weight of the watermelon is due to water. After the watermelon has sat in a drying room for a while, it turns out that it is only 98% water by weight. How much does it weigh now?

99% of 500 pounds = 495 pounds of water

⇒ So, we have 5 pounds that are not water.

⇒ After the watermelon has sat in a drying room for a while, this 5 pounds constitute 2%

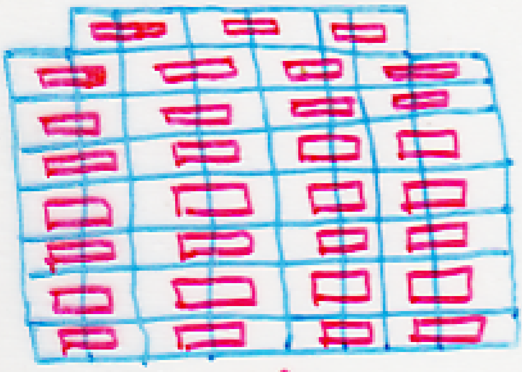
2% is 5 pounds

1% is $\frac{5}{2}$ pounds

100% is $\frac{5}{2} \times 100 = 250$ pounds

Hence it weighs 250 pounds now.

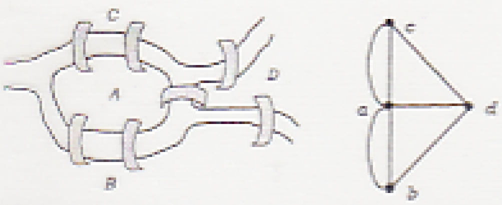
(5) Examine the example from the text about tiling the bathroom floor. What happens if the two omitted squares are in adjacent corners of the bathroom instead of opposite corners?



Possible

Euler Paths and Circuits

- The Seven bridges of Königsberg

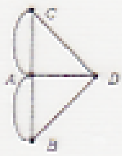


Euler Paths and Circuits

- An *Euler path* is a path using every edge of the graph G exactly once.
- An *Euler circuit* is an Euler path that returns to its start.

Does this graph have an Euler circuit?

No.

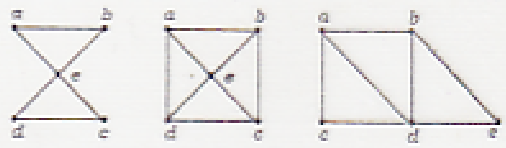


Necessary and Sufficient Conditions

- How about multigraphs?
- A connected multigraph has a Euler circuit iff *each of its vertices has an even degree.*
- A connected multigraph has a Euler path but not an Euler circuit iff *it has exactly two vertices of odd degree.*

Example

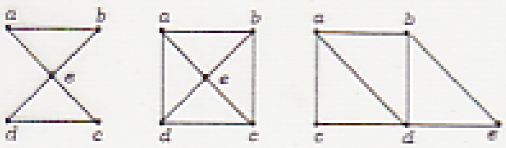
- Which of the following graphs has an Euler *circuit*?



yes (a, e, c, d, e, b, a) | no | no

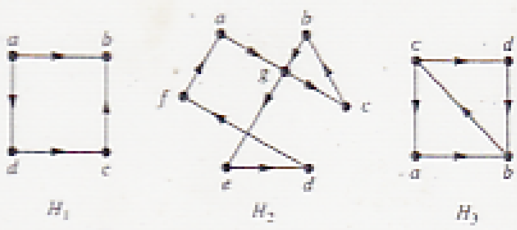
Example

- Which of the following graphs has an Euler *path*?



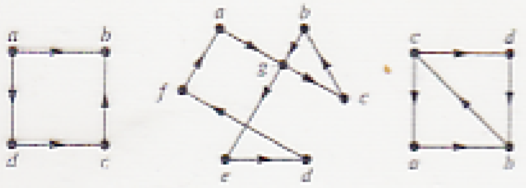
yes (a, e, c, d, e, b, a) | no | yes (a, c, d, e, b, d, a, b)

Euler Circuit in Directed Graphs



H_1 NO | H_2 (a, g, c, b, g, e, d, f, a) | H_3 NO

Euler Path in Directed Graphs



H_1 NO

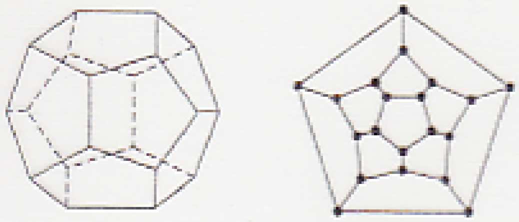
H_2 (a, g, c, b, g, e, d, f, a)

H_3 (c, a, b, c, d, b)

Hamilton Paths and Circuits

- A *Hamilton path* in a graph G is a path which visits every vertex in G exactly once.
- A *Hamilton circuit* is a Hamilton path that returns to its start.

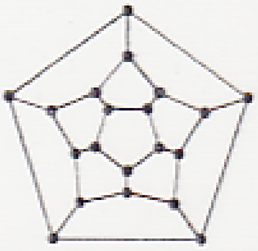
Hamilton Circuits



(a) (b)

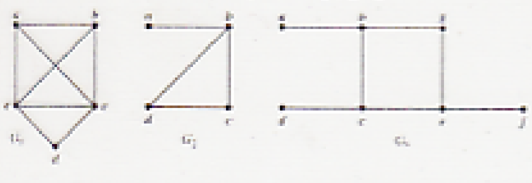
Is there a circuit in this graph that passes through each vertex exactly once?

Hamilton Circuits



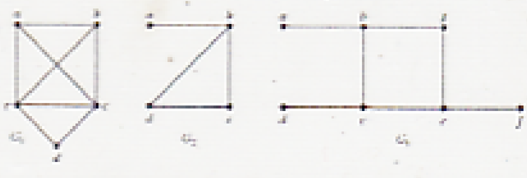
Yes; this is a circuit that passes through each vertex exactly once.

Finding Hamilton Circuits



Which of these three figures has a Hamilton circuit? Or, if no Hamilton circuit, a Hamilton path?

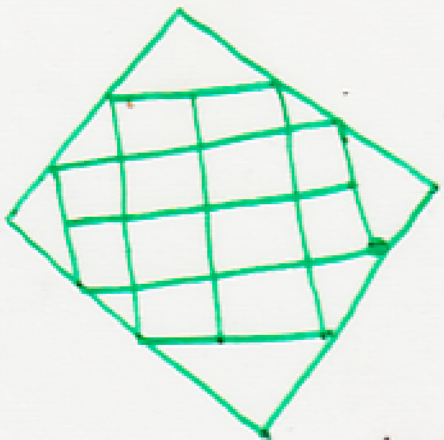
Finding Hamilton Circuits



- G_1 has a Hamilton circuit: a, b, c, d, e, a
- G_2 does not have a Hamilton circuit, but does have a Hamilton path: a, b, c, d
- G_3 has neither.

Impossible Problems :→

Explain why the line drawing in the figure below cannot be traced with a single, continuous, non-overlapping stroke of the pen.



⇒ The above figure contains two types of nodes

i) those which have an even number of edges and

ii) those which have an odd number of edges.

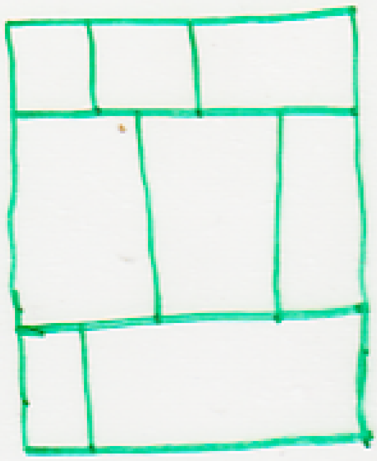
⇒ If the "trace" is to enter a node and then leave it (with no overlaps allowed) then there must be an even number of edges emanating from that node.

⇒ If instead there are an odd number of edges emanating then either

ⓐ the trace will not begin at that node but, at some time, the trace will arrive at that node and never leave or

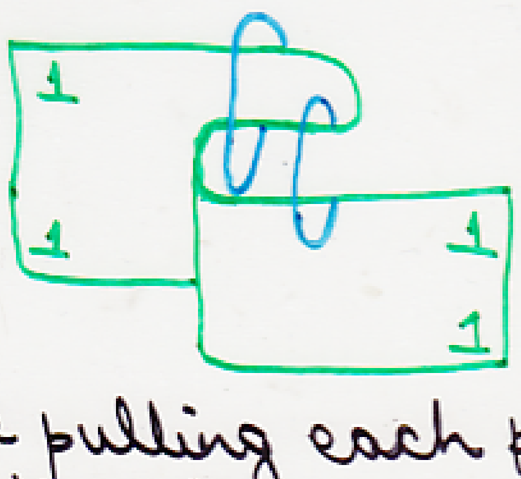
ⓑ the trace will begin at that node but not end there

⇒ The drawing in above figure has four nodes from which an odd number of edges emanate. The trace must either begin or end at each one of these. That is clearly impossible.

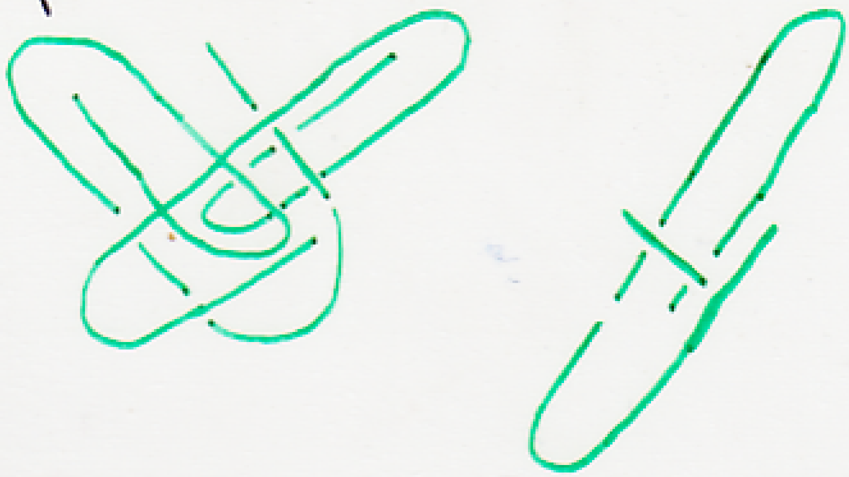


REAL LIFE PROBLEMS

① Examine the configuration of two paper clips and a dollar bill shown in Figure below. If the two ends of the dollar bill are pulled apart sharply, then the paper clips will spring free of the dollar bill and will be linked. Explain why they become linked.

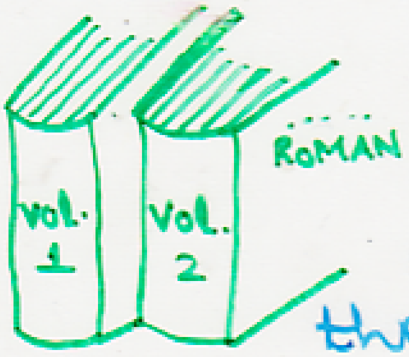


⇒ You are pulling each paper clip b/w two lines of the other paper clip, thus linking them. The final jerk of the paper tosses the front paper clip over the top of the dollar bill to join the other paper clip in back, thus freeing the linked clips.



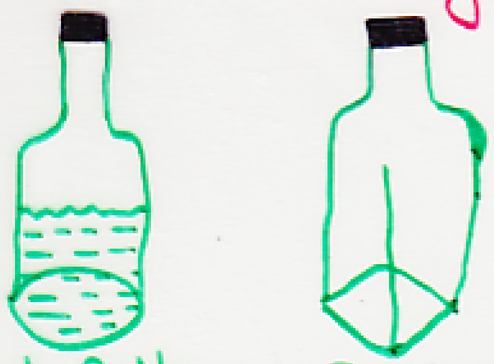
~~② Look at Figure below. Notice that Volume 1 is on the left and volume 2 is on~~

② Two volume of Gibbon's The Rise and Fall of the Roman Empire sit in order on the shelf. The cover material of these books is 1/8" inch thick. The 500 pages in each volume are 2" thick. A small worm bores a hole and crawls from the very first page of Volume 1 to the very last page of Volume 2. How far does the worm crawl?



Notice that Volume 1 is on the left and Volume 2 is on the right. Notice that the first page of Volume 1 is just left of center and the last page of Volume 2 is just to the right of center. The worm only has to bore from the inside front cover of Volume 1 to the outside (b/w the two books) and then bore into the back cover of volume 2. The worm only bores through two book covers, and no pages. So, the worm travels a total of 1/4".

③ Say that a bottle has a round or square, flat, bottom and it has straight sides. The bottle is partly full (about half) of liquid. It is tapered at the top (as bottles usually are) and has a screw-on cap. How can you accurately determine the volume of the bottle if you are equipped with only a ruler?



Round Bottom   Square Bottom

⇒ Use the ruler to measure the base. Calculate the area. ~~Calculate~~ Call that A. Now measure the height h of the water. Then volume of the water in the bottle is $V = A \cdot h$

⇒ Now turn the bottle upside down. We see that the water fills up the (difficult to measure) neck, and that the part not filled by water is cylindrical. Now, measure the height h' of the air column

inside the bottle and above the water. The volume of that air column is
$V' = A \cdot h'$

⇒ The total volume inside the bottle is then just $V = V + V'$.

④ A new car is equipped with three fuel saving devices. Device A, by itself, saves 25% on fuel; device B, by itself, saves 45% on fuel; and device C, by itself, saves 30% on fuel.

Now suppose that the three devices are used together, and that they act independently. Will the combination save $25+45+30=\underline{1}00$ percent on fuel? Probably not. What is the correct answer.

⇒ Device A results in using only .75 of the fuel that the machine would use without any devices. Device B applied on top of device A results in .55 of that fuel being used to achieve the same mileage and device C on top of those two results in .70 of that fuel being used to achieve the same mileage.

All told, equipped with devices A, B and C, we need only use $.75 \times .55 \times .70$

as much fuel to achieve the same mileage as the car without any fuel saving devices. That is, we need only use .28875 as much fuel. Thus, in total, we are saving .71125 or 71.125% on fuel consumption.

⑤ A martini is made by mixing k parts gin with 1 part vermouth. Gin is usually 40% alcohol while vermouth is 20% alcohol. A martini is said to be "dry" if it contains relatively little vermouth. For instance if $k=15$ then the martini is said to be dry. If instead $k=5$ then the martini is said to be "sweet".

Some people refuse to drink dry martinis, as they claim it makes them drunk very quickly. Others prefer dry martinis because they taste better. Shed some light on this discussion by calculating the amount of alcohol in a dry martini vs. a sweet martini as described in the last paragraph.

Sweet Martini : $k=5$

⇒ It is made up of 6 parts - 5 of them gin and 1 of them vermouth.

Total alcohol parts contain in sweet martini
$= 40\%$ of $5 + 20\%$ of $1 = 2 + 0.2 = 2.2$ parts

Hence in sweet martini, we have

altogether 2.2 out of 6 parts alcohol. Thus, the percentage of alcohol in a "sweet martini" is 2.2/6 or 36.67% alcohol.

Similarly, for dry martini, $K = 16$

⇒ It is made up of 16 parts - 15 of them gin and 1 of them vermouth.

Total alcohol parts contain in dry martini
$= 40\%$ of $15 + 20\%$ of $1 = 6 + 0.2 = 6.2$ parts.

⇒ Hence in dry martini, we have altogether 6.2 out of 16 parts alcohol. Thus, the percentage of alcohol in a "dry" martini" is $\frac{6.2}{16}$ or 38.75% alcohol.

⇒ Difference in alcohol content b/w the sweet and dry martini is about 2.09%

Statistics Problem :→

⇒ It is claimed that the average child has no time to go to school. For the child spends 8 hours per day, or one third of his/her time sleeping. Based on a 365 day year, that's 121.67 days sleeping.

Also the child spends three hours per day eating. That's a total of 45 days in the year spent eating.

Also the child spends 90 days taking summer vacation.

Also the child spends 21 days on christmas and easter holiday

Finally, the child has each Saturday and Sunday off. That's a total of 104 days
⇒ In short, we have (rounding to whole days) accounted for

$$122+45+90+21+104=382 \text{ days}$$

of the year taken up by ordinary child-like activities. This is already more that the 365 days that are known to comprise a year. We conclude that there is certainly no time for the child to attend the school. What is wrong with this reasoning?

⇒ During the 90 days that the child has summer vacation, the child is also eating and sleeping. Thus, those hours have been counted twice. Likewise, the child eats and sleeps on weekends.

② A certain small business shows the following data at the end of a certain fiscal year. There are 4 owners and 120 employees. Each owner receives \$100,000 per year in annual salary. Each employee is paid \$12,000 per year in annual wages. There is a total of \$240,000 profit, to be divided among the owners. Set up two different statistical models which could be used to generate a report on the finances of this company.

⇒ The most simple-minded report (and perhaps the most truthful) is this

① Each employee receives \$12,000 - that is all.

② Each owner receives \$100,000 in salary and \$60,000 in profits for a total of \$160,000.

⇒ Each owner is to be remunerated at a rate that is about 13.333 times the rate at which each employee is remunerated. Moreover, each owner receives profits equal to 500% of the wages of a typical employee. These numbers makes for bad public relations, from the point of view of management. There are a great many ways to do book keeping.

⇒ Suppose that we say that \$160,000 of the profits will be used as salary bonuses for the owners. Thus, each owner will receive \$100,000 in regular salary, \$40,000 in salary bonus (for a total of \$140,000 salary for each owner) and \$20,000 in profits. Look how advantageously this may be reported

$$\text{average salary} = \frac{120 \cdot 12{,}000 + 4 \cdot 140{,}000}{120 + 4} = \$16{,}129.03$$

And the profit per owner is now \$20,000.

⇒ The average salary is 4/3 of what each worker makes - but that is reasonable because the owners will (and should) make more than the workers. And each owner ~~makes~~ receives \$20,000 in profit. But that makes a total of \$80,000 in profits; compare that with \$2 million dollars paid in salaries.

⇒ We see that a very small total profit is being made as compared to the amount of salaries paid.

⇒ Clearly, this is a company that is primarily interested in the welfare of its employees, and only just a little in the wealth of its owners

⇒ Imagine that a certain election has three candidates: A, B, C. Now we describe the three voting systems as they would apply to these candidates

Method (i) is the simplest, and perhaps the most flawed: the candidates with the greatest number of votes wins. (Plurality method).

In Method (ii), each voter ranks the candidates first, second, and third. The "average rank" of each candidate is calculated, and the candidate with the greatest average wins. (Borda method)

In Method (iii), each voter ranks the candidates first, second, and third. If no candidate gets a clear majority of firsts, then the candidate with the fewest firsts is eliminated. Each of that candidate's votes is given to the candidate whom each voter ranked second (Hare method).

⇒ Imagine an election involving three candidates A, B, C. There are 33 voters. So that we may treat all voting method at once, we suppose that every voter votes, and that every voter gives a linear ranking to 3 candidates

Ranking	No. of Voters
ABC	10
ACB	4
BAC	2
BCA	7
CAB	3
CBA	7

Method (i) Fourteen voters prefer A, 9 voters prefer B and 10 voters prefer C. The candidates with the greatest number of votes is A. Thus, A wins the election by the plurality method.

Method (ii) (Borda method)

⇒ we assign a candidate 3 points for each voter who ranks him first, 2 points for each votes who ranks him second, and 1 point for each voter who ranks him third

Total number of points for A

$= (10\times 3) + (4\times 3) + (2\times 2) + (3\times 2) + (7\times 1) + (7\times 1) = 66$

Total number of points for B

$= (10\times 2) + (4\times 1) + (2\times 3) + (7\times 3) + (3\times 1) + (7\times 2) = 68$

Total number of points for C

$= (10\times 1) + (4\times 2) + (2\times 1) + (7\times 2) + (3\times 3) + (7\times 3) = 64$

Plainly, A's average rank is 66/33, B's average rank is 68/33 and C's average rank is 64/33. With this method, B wins.

Method (iii) Hare method

⇒ Plainly no candidates has a majority of first place votes. We eliminate B, because B has the fewest first place votes. Two of B's first place votes rank A second, so we give those two votes to A. Seven of B's first place votes rank C second, so we give those seven votes to B. Thus, A ends up with 14+2=16 votes, and C ends up with 10+7=17 votes. As a result C wins.

Inequalities :→

① If a and b are positive real numbers then show that

$$ab \le \frac{a^2+b^2}{2}$$

$$\Rightarrow \quad 2ab \le a^2+b^2$$

$$\Rightarrow \quad 0 \le a^2-2ab+b^2$$

$$\Rightarrow \quad 0 \le (a-b)^2$$

i.e., $(a-b)^2 \ge 0$

$$\Rightarrow \quad a^2-2ab+b^2 \ge 0$$

$$\Rightarrow \quad a^2+b^2 \ge 2ab$$

dividing both sides by 2

$$\frac{a^2+b^2}{2} \ge ab$$

$$\therefore \quad ab \le \frac{a^2+b^2}{2}$$

② Prove that

$$2 < \frac{1}{\log_2 \pi} + \frac{1}{\log_5 \pi}$$

We know, $\log_a b = \frac{\ln b}{\ln a}$

$$\therefore 2 < \frac{1}{\ln\pi/\ln 2} + \frac{1}{\ln\pi/\ln 5}$$

$$\therefore 2 < \frac{\ln 2}{\ln\pi} + \frac{\ln 5}{\ln\pi}$$

$\Rightarrow 2\ln\pi \leq \ln 2 + \ln 5$

$\Rightarrow \ln\pi^2 \leq \ln(2\cdot 5)$

$\Rightarrow \ln\pi^2 \leq \ln 10$

$\Rightarrow e^{\ln\pi^2} \leq e^{\ln 10}$

$\Rightarrow \pi^2 \leq 10$

$\Rightarrow \pi^2 < 10$

Hence the given inequality is true.

③ Show that $|\cos x + \sin x| \leq \sqrt{2}$

$$|\cos x + \sin x| = \sqrt{(\cos x + \sin x)^2}$$
$$= \sqrt{\cos^2 x + \sin^2 x + 2\sin x\cos x}$$
$$= \sqrt{1 + 2\sin x \cos x}$$
$$= \sqrt{1 + \sin 2x}$$

$\Rightarrow$ The greatest that $\sin 2x$ can be is 1.
That yields $|\cos x + \sin x| \leq \sqrt{2}$

④ Which is greater, $\sin(\cos x)$ or $\cos(\sin x)$?

$$\cos\left(\cos x + \frac{\pi}{2}\right) = \cos(\cos x)\cos\frac{\pi}{2} - \sin(\cos x)\sin\frac{\pi}{2}$$
$$= -\sin(\cos x)$$

Hence

$$\cos(\sin x) - \sin(\cos x) = \cos(\sin x) + \cos\left(\cos x + \frac{\pi}{2}\right) \quad \text{——— } (*)$$

From trigonometry

$$\cos X + \cos Y = 2\cdot\cos\left(\frac{X+Y}{2}\right)\cdot\cos\left(\frac{X-Y}{2}\right) \quad \ldots\ldots (**)$$

[Hint :→ $\cos X = \cos\left(\frac{X+Y}{2} + \frac{X-Y}{2}\right)$ and $\cos Y = \cos\left(\frac{X+Y}{2} - \frac{X-Y}{2}\right)$. Then apply the usual sum/difference formula for cosine and add the results]

Apply $(**)$ to the RHS of $(*)$ to obtain

$$\cos(\sin x) - \sin(\cos x)$$
$$= 2\cdot\cos\left(\frac{\sin x + \cos x + \pi/2}{2}\right)\cdot\cos\left(\frac{\sin x - \cos x - \pi/2}{2}\right)$$
$$= 2\cdot\cos\left(\frac{\sin x + \cos x + \pi/2}{2}\right)\cdot\cos\left(\frac{\cos x - \sin x + \pi/2}{2}\right) \quad -(***)$$

We know $|\sin x + \cos x| \leq \sqrt{2}$ and since $\pi \approx 3.141$ and $\sqrt{2} \approx 1.414$, we can be sure that

$$0 < \left|\frac{\sin x + \cos x + \pi/2}{2}\right| \leq \frac{1.415 + 1.58}{2} < 1.5 < \frac{\pi}{2}$$

Show that $|\cos x - \sin x| \leq \sqrt{2}$ with equality only if $\sin 2x = -1$.

$$|\cos x - \sin x| = \sqrt{(\cos x - \sin x)^2} = \sqrt{\cos^2 x - 2\sin x\cos x + \sin^2 x}$$
$$= \sqrt{1 - \sin 2x}$$

Clearly, the greatest $\sin 2x$ can be is 1. But if that is the case

$$|\cos x - \sin x| = 0$$

If $\sin 2x = -1$, then

$$|\cos x - \sin x| = \sqrt{1-(-1)} = \sqrt{2}.$$

Hence

$|\cos x - \sin x| \leq \sqrt{2}$ with equality only if $\sin 2x = -1$.

Continuing the previous problem, we can prove that

$$0 < \left| \frac{\cos x - \sin x + \pi/2}{2} \right| < \frac{\pi}{2}$$

Now, $\cos\omega$ is positive when $0 < |\omega| < \frac{\pi}{2}$. Thus, we may conclude that the two factors on the right of (***) are +ve. Therefore, $\cos(\sin x) - \sin(\cos x)$ is always +ve, for any value of the argument x.

In conclusion,

$$\sin(\cos x) < \cos(\sin x)$$

for any real number x.

Trigonometry and Related Ideas

① Suppose that α is an angle and that $\tan(\alpha/2)$ is rational. Verify that then $\sin\alpha$ and $\cos\alpha$

are both rational.

We know that,

$$1+\tan^2\frac{\alpha}{2} = \frac{\cos^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}} + \frac{\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}}$$

$$= \frac{1}{\cos^2\frac{\alpha}{2}}$$

$\because$ Since $\tan\frac{\alpha}{2}$ is rational, LHS is rational and hence so is the right. It follows that $\cos^2\frac{\alpha}{2}$ is rational.

Now, $\cos\alpha = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} = \cos^2\frac{\alpha}{2} - \left[1-\cos^2\frac{\alpha}{2}\right]$

$$= 2\cos^2\frac{\alpha}{2} - 1$$

$\Rightarrow$ Hence RHS is rational and so is LHS and $\cos\alpha$.

We have, $\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\sin 2\frac{\alpha}{2}}{\cos 2\frac{\alpha}{2}}$

$$\tan\alpha = \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}}$$

Dividing both Numerator and Denominator in RHS by $\cos^2\frac{\alpha}{2}$

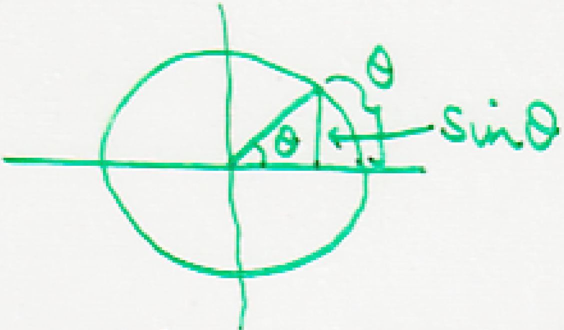
$$\tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}} = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}$$

⇒ Each component of the RHS of this last identity is rational.

$\therefore \tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$ is rational

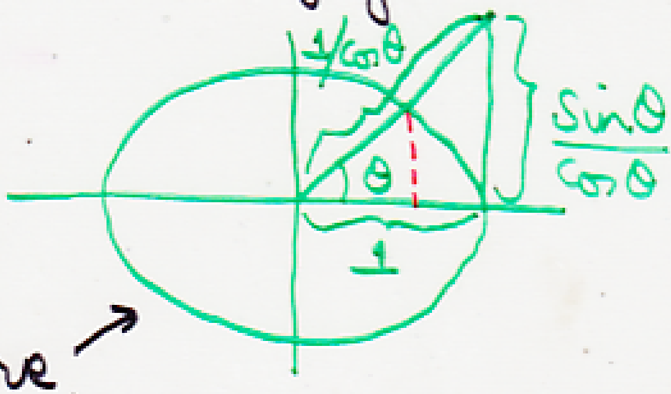
$\because$ $\cos\alpha$ is rational, $\sin\alpha$ is also rational.

② If θ is a positive, acute angle, measured in radians, then show that $\tan\theta > \theta$



Here $\sin\theta < \theta$

⇒ The standard setup for the trigonometry of the angle θ. From figure $\sin\theta < \theta$.



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In above figure,

⇒ Base of the Δ has length 1. Using similar Δ's, we see hypotenuse has length $1/\cos\theta$ and the height is $\sin\theta/\cos\theta$

⇒ The height of this Δ is greater that the length of the arc of the circle that is subtended. But the length of the arc is θ.

Thus, we conclude

$$\frac{\sin\theta}{\cos\theta} > \theta$$

i.e. $\tan\theta > \theta$.

③ Suppose that θ be any angle. Explain why $\cos\frac{\alpha}{2}\cos\frac{\alpha}{4}\cos\frac{\alpha}{8} = \frac{\sin\alpha}{8\sin(\alpha/8)}$

We know, $\sin 2\theta = 2\sin\theta\cos\theta$

Rewriting the above eq^n:

$$\cos\frac{\alpha}{2}\cos\frac{\alpha}{4}\cos\frac{\alpha}{8}\left(2\sin\frac{\alpha}{8}\right) = \frac{\sin\alpha}{4}$$

$$\Rightarrow \cos\frac{\alpha}{2}\cos\frac{\alpha}{4}\sin 2\frac{\alpha}{8} = \frac{\sin\alpha}{4}$$

$$\Rightarrow \cos\frac{\alpha}{2}\cos\frac{\alpha}{4}\sin\frac{\alpha}{4} = \frac{\sin\alpha}{4}$$

Multiplying both sides by 2

$$\Rightarrow \cos\frac{\alpha}{2}\, 2\cos\frac{\alpha}{4}\sin\frac{\alpha}{4} = 2\frac{\sin\alpha}{4} = \frac{2\sin\alpha}{4}$$

$$\Rightarrow \cos\frac{\alpha}{2}\sin 2\frac{\alpha}{4} = \frac{\sin\alpha}{2}$$

$$\Rightarrow \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} = \frac{\sin\alpha}{2} = \frac{\sin\alpha}{2}$$

Again multiplying both sides by 2

$$\Rightarrow \cancel{\sin}\ 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} = \frac{2\sin\alpha}{2}$$

$$\Rightarrow \sin 2\frac{\alpha}{2} = \sin\alpha \Rightarrow \sin\alpha = \sin\alpha \text{ (True)}$$

④ How many solutions are there to the equation

$$\tan x = \tan(x+10^\circ)\cdot\tan(x+20^\circ)\cdot\tan(x+30^\circ)$$

with $0 < x < 60^\circ$? ----- $(*)$

⇒ The function tangent is a strictly increasing function when the argument is between 0° and 90°. Therefore, the left hand side of $(*)$ is strictly increasing. Also each factor on RHS is increasing for $0<x<60^\circ$, so the entire function on RHS is strictly increasing

Let $f(x) = \tan x$ and

$$g(x) = \tan(x+10^\circ)\cdot\tan(x+20^\circ)\cdot\tan(x+30^\circ)$$

For $x=0^\circ$, $\left.\begin{array}{l} f(0)=0 \\ g(0)=0.037 \end{array}\right\} g(0)>f(0)$

For $x=7^\circ$, $\left.\begin{array}{l} f(7)=0.1227 \\ g(7)=0.0183 \end{array}\right\} g(7)<f(7)$

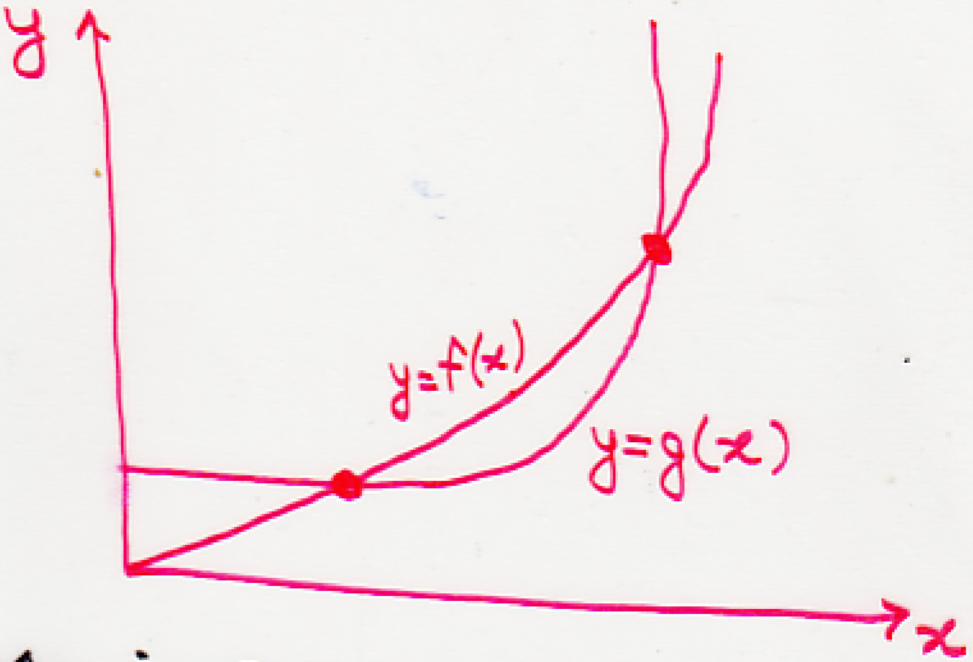
Hence there must be a value where f equals g. Hence $(*)$ has at least one solution

For $x=45^\circ$, $\left.\begin{array}{l} f(45^\circ)=1 \\ g(45^\circ)=11.43 \end{array}\right\} g(45^\circ)>f(45^\circ)$

For $x=59^\circ$, $\left.\begin{array}{l} f(59^\circ)= \cancel{1.732}\ 1.664 \\ g(59^\circ)=767.801 \end{array}\right\} g(59^\circ)>f(59^\circ)$

Hence the graph of g is above f in the interval, $45^\circ < x < 60^\circ$

Hence at $x=0°$, g is above f, at $x=7°$, g is below f and again between $45°<x<60°$ g is above f. This can only happen if the graphs cross at least twice



QUESTION :→ Magic square 5X5 with diagonals also having the same sum

⇒ Move in diagonal direction

⇒ Put the new number exactly below the old number if diagonal is already occupied.

	18	25	2	9	
17	24	1	8	15	17
23	5	7	14	16	23
4	6	13	20	22	4
10	12	19	21	3	10
11	18	25	2	9	