26.19 Covariant differentiation

For Cartesian tensors we noted that the derivative of a scalar is a (covariant) vector. This is also true for *general* tensors, as may br shown by considering the differential of a scalar

$$d\phi = \frac{\partial \phi}{\partial u^i} du^i.$$

Since the du^i are the components of a contravariant vector and $d\phi$ is a scalar, we have by the quotient law, discussed in section 26.7, that the quantities $\partial\phi/\partial u^i$ must form the components of a covariant vector. As a second example, if the contravariant components in Cartesian coordinates of a vector $\mathbf{v}$ are v^i, then the quantities $\partial v^i/\partial x^j$ form the components of a second-order tensor.

However, it is straightforward to show that in non-Cartesian coordinates differentiation of the components of a general tensor, other than a scalar, with respect to the coordinates does *not* in general result in the components of another tensor.

► *Show that, in* general *coordinates, the quantities $\partial v^i/\partial u^j$ do not form the components of a tensor.*

We may show this directly by considering

$$\begin{aligned}\left(\frac{\partial v^i}{\partial u^j}\right)' = \frac{\partial v'^i}{\partial u'^j} &= \frac{\partial u^k}{\partial u'^j}\frac{\partial v'^i}{\partial u^k} \\ &= \frac{\partial u^k}{\partial u'^j}\frac{\partial}{\partial u^k}\left(\frac{\partial u'^i}{\partial u^l}v^l\right) \\ &= \frac{\partial u^k}{\partial u'^j}\frac{\partial u'^i}{\partial u^l}\frac{\partial v^l}{\partial u^k} + \frac{\partial u^k}{\partial u'^j}\frac{\partial^2 u'^i}{\partial u^k \partial u^l}v^l.\end{aligned} \tag{26.84}$$

The presence of the second term on the right-hand side of (26.84) shows that the $\partial v^i/\partial x^j$ do not form the components of a second-order tensor.This term arises because the 'transformation matrix'$[\partial u'^i/\partial u^j]$ changes as the position in space at which it is evaluated is changed.This is not true in Cartesian coordinates, for which the second term vanishes and $\partial v^i/\partial x^j$ is a second-order tensor.◄

We may,however, use the Christoffel symbols discussed in the previous section to define a new *covariant* derivative of the components of a tensor that does result in the components of another tensor.

Let as first consider the derivative of a vector $\mathbf{v}$ with respect to the coordinates.Writing the vector in terms of its contravariant components $\mathbf{v} = v^i\mathbf{e}_i$,we find

$$\frac{\partial \mathbf{v}}{\partial u^j} = \frac{\partial v^i}{\partial u^j}\mathbf{e}_i + v^i\frac{\partial \mathbf{e}_i}{\partial u^j}, \tag{26.85}$$

where the second term arises because,in general, the basis vectors $\mathbf{e}_i$ are not constant(this term vanishes in Cartesian coordinates).Using (26.75) we write

$$\frac{\partial \mathbf{v}}{\partial u^j} = \frac{\partial v^i}{\partial u^j}\mathbf{e}_i + v^i\Gamma^k{}_{\mathrm{ij}}\mathbf{e}_k.$$

Since i and k are dummy indices in the last term on the right-hand side,we may interchange the to obtain

$$\frac{\partial \mathbf{v}}{\partial u^j} = \frac{\partial v^i}{\partial u^j}\mathbf{e}_i + v^k\Gamma^i{}_{\mathrm{kj}}\mathbf{e}_i = \left(\frac{\partial v^i}{\partial u^j} + v^k\Gamma^i{}_{\mathrm{kj}}\right)\mathbf{e}_i. \tag{26.86}$$

The reason for the interchanging the dummy indices, as shown in (26.86), is that we may now factor out $\mathbf{e}_i$.The quantity in parentheses is called the *covariant derivative*, for which the standard notation is

$$v^i{}_{;\mathrm{j}} = \frac{\partial v^i}{\partial u^j} + \Gamma^i{}_{\mathrm{kj}}v^k, \tag{26.87}$$

the semicolon subscript denoting covariant differentiation. A similar short-hand notation also exists for the partial derivatives, a comma being used for these instead of a semicolon; for example, $\partial v^i/\partial u^j$ is donated by $v^i{}_{,\,\mathrm{j}}$.In Cartesian coordinates all the $\Gamma^i{}_{\mathrm{kj}}$ are zero,and so the covariant derivative reduces to the simple partial derivative $\partial v^i/\partial u^j$.

Using the short-hand semicolon notation, the derivative of a vector may be written in the very compact form

$$\frac{\partial \mathbf{v}}{\partial u^j} = v^i{}_{;\,\mathrm{j}}\mathbf{e}_i$$

and,by the quotient rule (section 26.7), it is clear that the $v^i{}_{;\,\mathrm{j}}$ are the (mixed) components of a second-order tensor.This may also be verified directly, using the transformation properties of $\partial v^i/\partial u^j$ and $\Gamma^i{}_{\mathrm{kj}}$ given in (26.84) and (26.78) respectively.

In general, we may regard the $v^i{}_{;\mathrm{j}}$ as the mixed components of a second-order tensor called the covariant derivative of $\mathbf{v}$ and denoted by $\mathbf{Vv}$.In Cartesian coordinates,the components of this tensor are just $\partial v^i/\partial u^j$.

► *Calculate $v^i{}_{;i}$ in cylindrical polar coordinates.*

Contracting(26.87) we obtain

$$v^i{}_{;\mathrm{i}} = \frac{\partial v^i}{\partial u^i} + \Gamma^i{}_{\mathrm{ki}}v^k.$$

Now from (26.83) we have

$$\begin{aligned}
\Gamma^i{}_{1\mathrm{i}} &= \Gamma^1{}_{11} + \Gamma^2{}_{12} + \Gamma^3{}_{13} = 1/p,\\
\Gamma^i{}_{2\mathrm{i}} &= \Gamma^1{}_{21} + \Gamma^2{}_{22} + \Gamma^3{}_{23} = 0,\\
\Gamma^i{}_{3\mathrm{i}} &= \Gamma^1{}_{31} + \Gamma^2{}_{32} + \Gamma^3{}_{33} = 0,
\end{aligned}$$

and so

$$v^i{}_{;\mathrm{i}} = \frac{\partial v^\rho}{\partial \rho} + \frac{\partial v^\phi}{\partial \phi} + \frac{\partial v^z}{\partial z} + \frac{1}{\rho}v^\rho$$
$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho v^\rho) + \frac{\partial v^\phi}{\partial \phi} + \frac{\partial v^z}{\partial z}.$$

This result is identical to the expression for the divergence of a vector field in cylindrical polar coordinates given in section 10.9. This is discussed further in section 26.20.◄

So far we have considered only the covariant derivative of the contravariant components v^i of a vector.The corresponding result for the covariant components v_i may be found in a similar way, by considering the derivative of $\mathbf{v} = v_i\mathbf{e}^i$ and using (26.77) to obtain

$$v_{\mathrm{i;j}} = \frac{\partial v_i}{\partial u^j} - \Gamma^k{}_{\mathrm{ij}}v_k. \qquad (26.88)$$

Comparing the expressions (26.87) and (26.88) for the covariant derivative of the contravariant and covariant components of a vector respectively, we see that there are some similarities and some differences. It may help to remember that the index with respect to which the covariant derivative is taken (j in this case), is also the last subscript on the Christoffel symbol; the remaining indices can then be arranged in only one way without raising or lowering them. It only remains to note that for a covariant index (subscript) the Christoffel symbol carries a minus sign, whereas for a contravariant index (superscript) the sign is positive.

Following a similar procedure to that which led to equation (26.87), we may obtain expressions for the covariant derivatives of higher-order tensors.

► *By considering the derivative of the second-order tensor* $\boldsymbol{T}$ *with respect to the coordinate* u^k*, find an expression for the covariant derivative* $T^{ij}{}_{;k}$ *of its contravariant components.*

Expressing $\mathbf{T}$ in terms of its contravariant components, we have

$$\frac{\partial \mathbf{T}}{\partial u^k} = \frac{\partial}{\partial u^k}(T^{ij}\mathbf{e}_i \otimes \mathbf{e}_j)$$
$$= \frac{\partial T^{ij}}{\partial u^k}\mathbf{e}_i \otimes \mathbf{e}_j + T^{ij}\frac{\partial \mathbf{e}_i}{\partial u^k} \otimes \mathbf{e}_j + T^{ij}\mathbf{e}_i \otimes \frac{\partial \mathbf{e}_j}{\partial u^k}.$$

Using (26.75), we can rewrite the derivatives of the basis vectors in terms of Christoffel symbols to obtain

$$\frac{\partial \mathbf{T}}{\partial u^k} = \frac{\partial T^{ij}}{\partial u^k}\mathbf{e}_i \otimes \mathbf{e}_j + T^{ij}\Gamma^l{}_{\mathrm{ik}}\mathbf{e}_l \otimes \mathbf{e}_j + T^{ij}\mathbf{e}_i \otimes \frac{\partial \mathbf{e}_j}{\partial u^k}.$$

Interchanging the dummy indices i and l in the second term and j and l in the third term on the right-hand side, this becomes

$$\frac{\partial \mathbf{T}}{\partial u^k} = \left(\frac{\partial T^{ij}}{\partial u^k} + \Gamma^i{}_{\mathrm{lk}}T^{lj} + \Gamma^j{}_{\mathrm{lk}}T^{il}\right)\mathbf{e}_i \otimes \mathbf{e}_j,$$

where the expression in parentheses is the required covariant derivative

$$T^{ij}{}_{;\mathrm{k}} = \frac{\partial T^{ij}}{\partial u^k} + \Gamma^i{}_{\mathrm{lk}} T^{lj} + \Gamma^j{}_{\mathrm{lk}} T^{il}. \tag{26.89}$$

Using (26.89), the derivative of the sensor $\mathbf{T}$ with respect to u^k can now be written in terms of its contravariant component as

$$\frac{\partial \mathbf{T}}{\partial u^k} = T^{ij}{}_{;\mathrm{k}} \mathbf{e}_i \otimes \mathbf{e}_j. \blacktriangleleft$$

Results similar to (26.89) may be obtained for the the covariant derivatives of the mixed and covariant components of a second-order tensor. Collecting these results together, we have

$$\begin{aligned} T^{ij}{}_{;\mathrm{k}} &= T^{ij}{}_{,\mathrm{k}} + \Gamma^i{}_{\mathrm{lk}} T^{lj} + \Gamma^j{}_{\mathrm{lk}} T^{il}, \\ T^i{}_{\mathrm{j;k}} &= T^i{}_{\mathrm{j,k}} + \Gamma^i{}_{\mathrm{lk}} T^l{}_{\mathrm{j}} - \Gamma^l{}_{\mathrm{jk}} T^i{}_{\mathrm{l}}, \\ T_{\mathrm{ij;k}} &= T_{\mathrm{ij,k}} - \Gamma^l{}_{\mathrm{ik}} T_{\mathrm{lj}} - \Gamma^l{}_{\mathrm{jk}} T_{\mathrm{il}}, \end{aligned}$$

where we have used the comma notation for partial derivatives. The position of the indices in these expressions is very systematic: for each contravariant index (superscript) on the LHS we add a term on the RHS containing a Christoffel symbol with a plus sign, and for every covariant index (subscript) we add a corresponding term with a minus sign. This is extended straightforwardly to tensors with an arbitrary number of contravariant and covariant indices.

We note that the quantities $T^{ij}{}_{;\mathrm{k}}, T^i{}_{\mathrm{j;k}}$ and $T_{\mathrm{ij;k}}$ are the components of the *some* third-order tensor $\nabla \mathbf{T}$ with respect to different tensor bases, i.e.

$$\nabla \mathbf{T} = T^{ij}{}_{;\mathrm{k}} \mathbf{e}_i \otimes {}_{\mathrm{e}j} \otimes \mathbf{e}^k = T^i{}_{\mathrm{j;k}} \mathbf{e}_i \otimes {}_{\mathrm{e}}{}^j \otimes \mathbf{e}^k = T_{\mathrm{ij;k}} \mathbf{e}^i \otimes {}_{\mathrm{e}}{}^j \otimes \mathbf{e}^k.$$

We conclude this section by considering briefly the covariant derivative of a scalar. The covariant derivative differs from the simple partial derivative with respect to the coordinates only because the basis vectors of the coordinate system change with position in space (hence for Cartesian coordinates there is no difference). However, a scalar ϕ does not depend on the basis vectors at all and so its covariant derivative must be the same as its partial derivative, i.e.

$$\phi; j = \frac{\partial \phi}{\partial u^j} = \phi, j. \tag{26.90}$$