Metric Algebraic Geometry

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Outline

- Relevant background on algebraic varieties
  - degree
  - projective space
- Reach
  - Significance in topological data analysis
  - Definition
  - Function of bottlenecks and curvature
- Bottlenecks
- Curvature

Background on Algebraic Varieties

Varieties

An **algebraic variety** (a.k.a. variety) is the zero set of a system of polynomial equations.

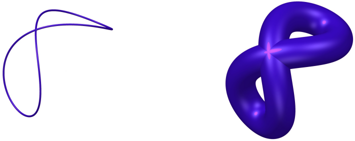


Figure: A curve and a surface in $\mathbb{R}^3$. The curve is the zero set of the polynomials $x_1^2 + x_2^2 + x_3^2 - 4$ and $(x_1 - 1)^2 + x_2^2 - 1$. The surface is defined by a degree 10 polynomial.

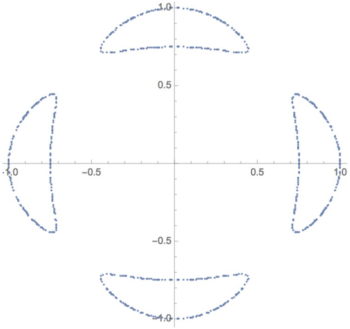
Degree of an Algebraic Variety

- The degree of a variety defined by one polynomial is the degree of the polynomial.
- For varieties defined by more than one polynomial, it is harder to determine the degree. The field of **intersection theory** studies these questions.
- Degree is a proxy for computational complexity.
- In $\mathbb{R}^n$, a variety of degree d intersects a linear space of complementary dimension in at most d points. For a consistent count, we turn to **complex projective space**.

Projective Space

- We switch from $\mathbb{R}^n$ to $\mathbb{C}^n$ when we want every polynomial of degree d to have d roots.
- We switch from **affine space** to **projective space** when we want every pair of lines to intersect.
- Example: $\mathbb{P}^2_{\mathbb{R}} =$ $\mathbb{R}^2 \cup \{\text{one point at } \infty \text{ for each equivalence class of parallel lines}\}$
- *For a more thorough introduction for students, I recommend the textbook* **Ideals, Varieties, and Algorithms** *by Cox, Little, and O'Shea.*

Topological Data Analysis



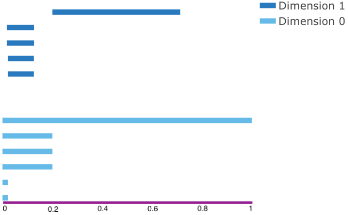


Figure: Persistent homology barcodes for the Trott curve in $\mathbb{R}^2$.

- We draw a ball of radius r around each point and count the number of "holes" in the resulting union of balls.
- The number of light blue bars at a given radius is the number of connected components of the union of balls of that radius.
- The number of dark blue bars at a given radius is the number of two dimensional holes in the union of balls of that radius.

Guaranteeing Persistent Homology: The Reach of an Algebraic Variety

Theorem (Partha Niyogi, Stephen Smale, Shmuel Weinberger 2006)

Let M be a compact submanifold of $\mathbb{R}^N$ of dimension k with **reach** τ*. Let $\bar{x} = \{x_1, \ldots, x_n\}$ be a set of n points drawn in independent and identically distributed fashion according to the uniform probability measure on M. Let $0 < \epsilon < \frac{\tau}{2}$. Let $U = \bigcup_{x \in \bar{x}} B_\epsilon(x)$ be a corresponding random open subset of $\mathbb{R}^N$. Let $\beta_1 = \frac{vol(M)}{(\cos^k(\theta_1))vol(B^k_{\epsilon/4})}$ and $\beta_2 = \frac{vol(M)}{(\cos^k(\theta_2))vol(B^k_{\epsilon/8})}$ where $\theta_1 = \arcsin(\frac{\epsilon\tau}{8})$ and $\theta_2 = \arcsin(\frac{\epsilon\tau}{16})$. Then for all*

$$n > \beta_1 \left(\log(\beta_2) + \log\left(\frac{1}{\delta}\right)\right)$$

the homology of U equals the homology of M with high confidence (probability $> 1 - \delta$).

Highlight

The **reach** *(to be defined on the next slide) of an algebraic variety determines the number of sample points required for persistent homology to work.*

The Reach of an Algebraic Variety

Definition

The **medial axis** of a variety $V \subset \mathbb{R}^n$ is the set of all points $u \in \mathbb{R}^n$ such that the minimum distance from V to u is attained by two distinct points. The **reach** of V is the infimum of all distances from points on the variety V to points in its medial axis.

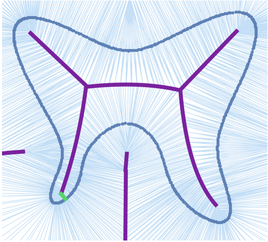


Figure: The medial axis (purple) and reach (green) of the butterfly curve can be seen in its Voronoi approximation.

Algebraicity of Reach

Proposition (Horobet-W.)

Let V be a smooth algebraic variety in $\mathbb{R}^n$. Let $f_1, \ldots, f_s \in \mathbb{Q}[x_1, \ldots, x_n]$ with $V = V_{\mathbb{R}}(f_1, \ldots, f_s)$. Then the reach of V is an algebraic number over $\mathbb{Q}$.

Highlight

This theorem is an invitation to use algebraic geometry to find the reach of a variety.

Reach: Function of Bottlenecks and Curvature

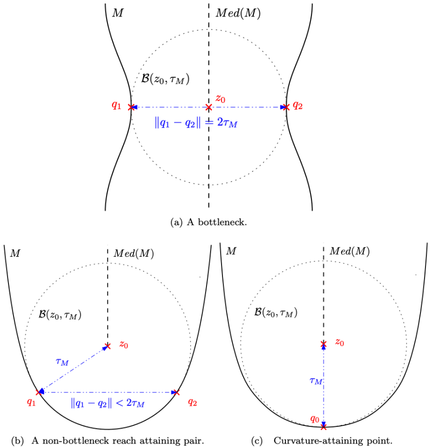


Figure: The reach of a manifold is attained by a bottleneck, two points on a circular arc, or a point of maximal curvature. Figure and Theorem due to Aamari-Kim-Chazal-Michel-Rinaldo-Wasserman '17.

Bottlenecks

Definition

Let $V \subset \mathbb{R}^n$ be a smooth variety. A line is **orthogonal** to V if it is orthogonal to the tangent space $T_x V \subset \mathbb{R}^n$ at x. The **bottlenecks** of V are pairs (x, y) of distinct points $x, y \in V$ such that the line spanned by x and y is orthogonal to V at both points.

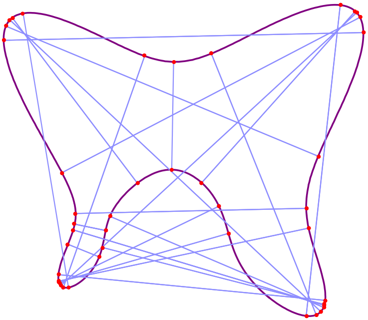


Figure: The real bottleneck pairs of the butterfly curve.

Bottlenecks as Distance Optimization

Remark

Bottlenecks are the critical points of the squared distance function

$$\mathbb{R}^n \times \mathbb{R}^n : (x, y) \mapsto ||x - y||^2,$$

subject to the constraints $x, y \in V$ *as well as the non-triviality condition* $x \neq y$.

Bottleneck Degree

Denote by $BND(V)$ the bottleneck degree of $V \subset \mathbb{C}^n$. Under certain conditions, this coincides with twice the number of bottleneck pairs.

Theorem (Di Rocco-Eklund-W.)

- *Let $V \subset \mathbb{C}^2$ be a "general" curve of degree d. Then* $BND(V) = d^4 - 5d^2 + 4d$.
- *Let $V \subset \mathbb{C}^3$ be a "general" surface of degree d. Then* $BND(V) = d^6 - 2d^5 + 3d^4 - 15d^3 + 26d^2 - 13d$.
- *For any smooth variety $V \subset \mathbb{P}^n_{\mathbb{C}}$ in "general position," we have an algorithm to express the bottleneck degree in terms of the polar classes of V.*

Building Bridges Between Differential Geometry and Computational Algebraic Geometry

- Curvature is central to the study of differential geometry.
- Curvature is a property of algebraic varieties.
- Properties of algebraic varieties should have defining polynomial equations and degrees!

Curvature and the Evolute

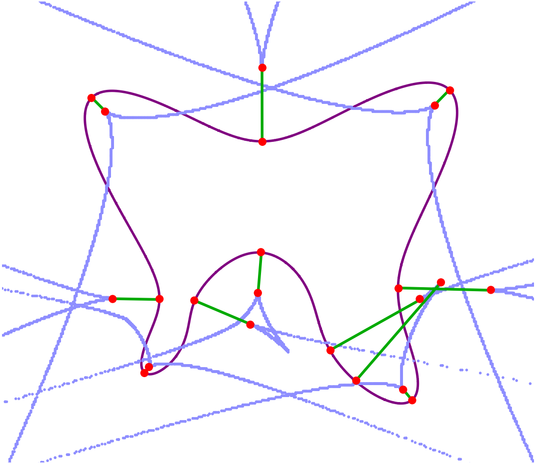


Figure: The real points of critical curvature on the butterfly curve (purple) joined by green line segments to their centers of curvature. These give cusps on the evolute (light blue).

Algebraic Manifold: Algebraic Variety and Differentiable Manifold

- $f \in \mathbb{R}[x_1, \ldots, x_n]$
- $V = \{x \in \mathbb{C}^n | f(x) = 0\}$ smooth algebraic variety (hypersurface)
- $M = V \cap \mathbb{R}^n$ differentiable submanifold of $\mathbb{R}^n$
- M is an **algebraic manifold**

Euclidean Connection and Second Fundamental Form

For any manifold M, let $\mathcal{T}(M)$ denote the set of smooth vector fields on M; this is the space of smooth sections of the tangent bundle TM. For $M \subset \mathbb{R}^n$, let $\mathcal{N}(M)$ denote the space of smooth sections of the normal bundle NM. The **Euclidean connection** $\overline{\nabla}$ on $\mathbb{R}^n$ is a map $\overline{\nabla} : \mathcal{T}(\mathbb{R}^n) \times \mathcal{T}(\mathbb{R}^n) \to \mathcal{T}(\mathbb{R}^n), (X, Y) \mapsto \overline{\nabla}_X Y$ defined as follows:

$$(\overline{\nabla}_X Y)(p) = \sum_{i=1}^{n} X_i(p) \frac{\partial Y}{\partial x_i}(p).$$

In other words, $\overline{\nabla}_X Y$ is the vector field whose components are the directional derivatives of the components of Y in the direction X.
The **second fundamental form** of M is the map II from $\mathcal{T}(M) \times \mathcal{T}(M)$ to $\mathcal{N}(M)$ given by

$$\mathrm{II}(X, Y) := (\overline{\nabla}_X Y)^{\perp}.$$

Principal Curvatures

Let $M \subset \mathbb{R}^3$ be a surface. Fix a point $p \in M$ and vector fields $X, Y \in \mathcal{T}(M)$ such that $X(p)$ and $Y(p)$ form an orthonormal basis of T_pM. Let $N(p)$ be a unit vector in N_pM. The **principal curvatures** of M at p are the eigenvalues of the symmetric matrix

$$\begin{bmatrix} \mathrm{II}(X,X)(p) \cdot N(p) & \mathrm{II}(X,Y)(p) \cdot N(p) \\ \mathrm{II}(Y,X)(p) \cdot N(p) & \mathrm{II}(Y,Y)(p) \cdot N(p) \end{bmatrix}.$$

If X and Y are selected so that the matrix is diagonal, then $X(p)$ and $Y(p)$ are the **principal directions**, up to a choice of normal vector.

Umbilics

A point $p \in M$ is called an **umbilic** if all of the principal curvatures at p are equal. At an umbilic, the best second-order approximation of the manifold is a sphere.

Theorem (Salmon, 1865)

The degree of the variety of umbilics of a general surface of degree d in $\mathbb{R}^3$ is $10d^3 - 28d^2 + 22d$.

Definition of Critical Curvature

A point $p \in M$ is called a **point of critical curvature** if there exists a principal curvature c at p such that the gradient of c vanishes in the tangent direction of the unit normal bundle.

Equations for Critical Curvature Locus

The following equations define the locus of pairs (x, u) where $x \in M$ and u is a principal direction at x:

$$f(x_1, \ldots, x_n) = 0,$$

$$\nabla f \cdot u = 0,$$

$$\sum_{i=1}^{n} u_i^2 - 1 = 0,$$

$$\lambda^2(\nabla f \cdot \nabla f) - 1 = 0,$$

$$H_f \cdot u + y_1 u + y_2 \nabla f = 0.$$

The curvature is given by $g(x, u, \lambda) = \lambda u^t \cdot H_f \cdot u$. Using the principle of Lagrange multipliers, we intersect the above locus with the locus defined by the vanishing of the minors of a matrix of partial derivatives of the above equations and partial derivatives of g.

Upper Bound for Critical Curvature Degree

Theorem (Breiding-Ranestad-W.)

Let $V \subset \mathbb{R}^3$ be a general algebraic surface of degree d. Then X has isolated complex critical curvature points. An upper bound for their number is given by $\frac{1}{8}(2796d^3 - 6444d^2 + 3696d)$.

d	$\frac{1}{8}(2796d^3 - 6444d^2 + 3696d)$	actual number
2	498	18
3	3573	≥ 456
4	11328	≥ 1808

A formula for hypersurfaces in $\mathbb{R}^n$ can be computed using the same process given in our proof. However, we do not have a proof that a general hypersurface in $\mathbb{R}^n$ for $n \geq 4$ has isolated complex critical curvature points.

Thank you!

Polar Classes

An m-dimensional variety has $m+1$ polar varieties defined by exceptional tangent loci.

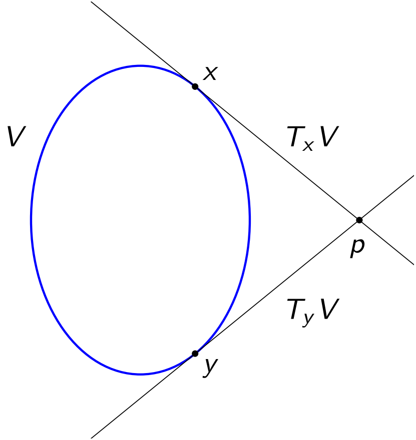


Figure: Polar locus of a point p with respect to an ellipse V: $P_1(V, p) = \{x, y\}$.

Polar Classes

Example

For a smooth surface $V \subset \mathbb{P}^3$ we have two polar varieties. Let $p \in \mathbb{P}^3$ be a general point and $l \subset \mathbb{P}^3$ a general line. Then $P_1(V, p)$ is the set of points $x \in V$ such that the projective tangent plane $\mathbb{T}_x V \subset \mathbb{P}^3$ contains p. This is a curve on V. Similarly, $P_2(V, l) = \{x \in V : l \subseteq \mathbb{T}_x V\}$, which is finite.

Definition

Let $V \subset \mathbb{P}^n$ be a smooth variety of dimension m. For $j = 0, \ldots, m$ and a general linear space $L \subseteq \mathbb{P}^n$ of dimension $n - m - 2 + j$ the **polar variety** is given by

$$P_j(V, L) = \{x \in V : \dim \mathbb{T}_x V \cap L \geq j - 1\}.$$

For each polar variety $P_j(V, L)$, there is a corresponding **polar class** $[P_j(V, L)] = p_j$ which represents $P_j(V, L)$ up to rational equivalence.

Polar Classes and Chern Classes

$P_j(V, L)$ is either empty or of pure codimension j and

$$p_j = \sum_{i=0}^{j} (-1)^i \binom{m-i+1}{j-i} h^{j-i} c_i(T_X),$$

where $h \in A_{n-1}(X)$ is the hyperplane class.
The polar loci $P_j(V, L)$ are reduced. We have

$$c_j(T_X) = \sum_{i=0}^{j} (-1)^i \binom{m-i+1}{j-i} h^{j-i} p_i.$$

Bottleneck Genericity Assumptions 1/2

Let $V \subset \mathbb{P}^n$ be a variety. Consider the **conormal variety**

$$\mathcal{C}_V = \{(p,q) \in \mathbb{P}^n \times \mathbb{P}^n : p \in V, q \in (\mathbb{T}_p V)^\perp\}$$

and the map

$$f : \mathcal{C}_V \to \mathrm{Gr}(2, n+1) : (p,q) \mapsto \langle p, q \rangle$$

from $\mathcal{C}_V$ to the Grassmannian of lines in $\mathbb{P}^n$ that sends a pair (p,q) to the line spanned by p and q.
The orthogonality relation on $\mathbb{P}^n$ is defined via the **isotropic quadric** $Q \subset \mathbb{P}^n$ given in homogeneous coordinates by $\sum_0^n x_i^2 = 0$. Varieties which are tangent to Q are to be considered degenerate in this context and we say that a smooth projective variety is in **general position** if it intersects Q transversely. Equivalently, a smooth variety $V \subset \mathbb{P}^n$ is in general position if $\mathcal{C}_V$ is disjoint from the diagonal $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$.

Bottleneck Genericity Assumptions 2/2

A smooth variety $V \subset \mathbb{P}^n$ is **bottleneck regular** if

1. V is in general position,
2. V has only finitely many bottlenecks and
3. the differential $df_p : T_p\mathcal{C}_V \to T_{f(p)}G$ of the map f has full rank for all $p \in \mathcal{C}_V$.

If $V \subset \mathbb{P}^n$ is bottleneck regular, then V is equal to the number of bottlenecks of V counted with multiplicity.

Voronoi Genericity Assumptions

Let $V \subset \mathbb{P}^n$ be a smooth projective variety defined over $\mathbb{R}$. We assume that $y \in V$ is a general point, and that we fixed an affine space $\mathbb{R}^n \subset \mathbb{P}^n$ containing y such that the hyperplane at infinity $\mathbb{P}^n \setminus \mathbb{R}^n$ is in general position with respect to V.

Example

For example, let V be the twisted cubic curve in $\mathbb{P}^3$, with affine parameterization $t \mapsto (t, t^2, t^3)$. Here $g = 0$ and $d = 3$, so the expected Voronoi degree is 6.
However, for $V \subset \mathbb{R}^3$ defined by the two equations $\mathbf{f} = (x_1^2 - x_2, x_1x_2 - x_3)$, we can compute that algebraic boundary of $Vor_V(0)$ is given by the quartic curve $27u_3^4 + 128u_2^3 + 72u_2u_3^2 - 160u_2^2 - 35u_3^2 + 66u_2 = 9$ in the plane $u_1 = 0$. So $\delta_V(y) = 4$.
This is explained by the fact that the plane at infinity in $\mathbb{P}^3$ intersects the curve V in a triple point. After a general linear change of coordinates in $\mathbb{P}^3$, which amounts to a linear fractional transformation in $\mathbb{R}^3$, we correctly find $\delta_V(y) = 6$.

Convergence Theorem: Voronoi Version

Definition

Given a point $x \in \mathbb{R}^n$ and a closed set $B \subset \mathbb{R}^n$, define

$$d_w(x, B) = \inf_{b \in B} d(x, b).$$

A sequence $\{B_\nu\}_{\nu \in \mathbb{N}}$ of compact sets is *Wijsman convergent* to B if for every $x \in \mathbb{R}^n$, we have that

$$d_w(x, B_\nu) \to d_w(x, B).$$

Theorem (Brandt-W.)

Let X be a compact curve in $\mathbb{R}^2$ and $\{A_\epsilon\}_{\epsilon \searrow 0}$ a sequence of finite subsets of X containing all singular points of X such that every point of X is within distance ϵ of some point in A_ϵ. Then every Voronoi cell of X is the Wijsman limit of a sequence of Voronoi cells of $\{A_\epsilon\}_{\epsilon \searrow 0}$.

Convergence Theorem: Delaunay Version

Theorem (Brandt-W.)

Let X be a compact curve in $\mathbb{R}^2$ and $\{A_\epsilon\}_{\epsilon \searrow 0}$ a sequence of finite subsets of X containing all singular points of X such that no point of X is more than distance ϵ from every point in A_ϵ. If X is not tangent to any circle in four or more points, then every maximal Delaunay cell is the Hausdorff limit of a sequence of Delaunay cells of $\{A_\epsilon\}_{\epsilon \searrow 0}$.

Delaunay Cells

Definition

Let $B(p, r)$ denote the open disc with center $p \in \mathbb{R}^n$ and radius $r > 0$. We say this disc is *inscribed* with respect to X if $X \cap B(p, r) = \emptyset$ and we say it is *maximal* if no disc containing $B(p, r)$ shares this property. Given an inscribed disc B of an algebraic variety $X \subset \mathbb{R}^n$, the *Delaunay cell* $Del_X(B)$ is $conv(\overline{B} \cap X)$.

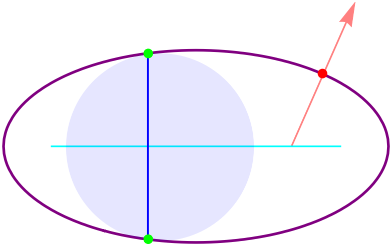


Figure: The dark blue line segment is a Delaunay cell defined by the light blue maximally inscribed circle with center $(-3/8, 0)$ and radius $\sqrt{61}/8$.

Duality of Delaunay and Voronoi Cells

Definition

Let $X \subset \mathbb{R}^2$ be a finite point set. A *Delaunay triangulation* is a triangulation $DT(X)$ of X such that no point of X is inside the circumcircle of any triangle of $DT(X)$.

Remark

The circumcenters of triangles in $DT(X)$ are the vertices in the Voronoi diagram of X.

Hausdorff Convergence

The *Hausdorff distance* of two compact sets B_1 and B_2 in $\mathbb{R}^n$ is defined as

$$d_h(B_1, B_2) := \sup\left\{\sup_{x\in B_1}\inf_{y\in B_2} d(x,y), \sup_{y\in B_2}\inf_{x\in B_1} d(x,y)\right\}.$$

If an adversary gets to put your ice cream on either set B_1 or B_2 with the goal of making you go as far as possible, and you get to pick your starting place in the opposite set, then $d_h(B_1, B_2)$ is the farthest the adversary could make you walk in order for you to reach your ice cream.

Definition

A sequence $\{B_\nu\}_{\nu\in\mathbb{N}}$ of compact sets is *Hausdorff convergent* to B if $d_h(B, B_\nu) \to 0$ as $\nu \to \infty$.