Algebraic Geometry of Curvature

Maddie Weinstein

UC Berkeley - UC San Diego - Stanford

maddie@math.berkeley.edu

with Paul Breiding and Kristian Ranestad

The Reach of an Algebraic Variety

Definition

The **medial axis** of a variety $V \subset \mathbb{R}^n$ is the set $Med(V)$ of all points $u \in \mathbb{R}^n$ such that the minimum distance from V to u is attained by two distinct points. The **reach** τ_V is the infimum of all distances from points on the variety V to points in its medial axis $Med(V)$.

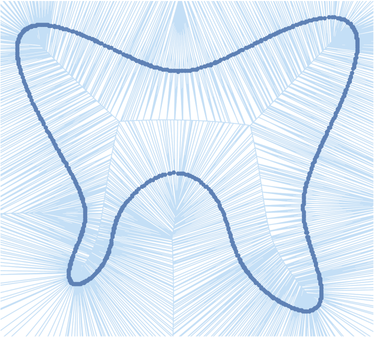


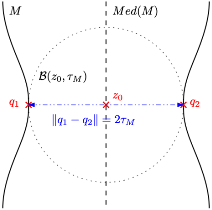
Figure: The medial axis of the quartic butterfly curve can be seen in its Voronoi approximation.

Algebraicity of Reach

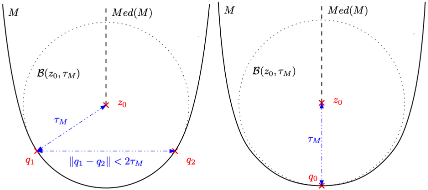
Proposition (Horobet-W. '18)

Let V be a smooth algebraic variety in $\mathbb{R}^n$. Let $f_1, \ldots, f_s \in \mathbb{Q}[x_1, \ldots, x_n]$ with $V = V_{\mathbb{R}}(f_1, \ldots, f_s)$. Then the reach of V is an algebraic number over $\mathbb{Q}$.

Reach, Bottlenecks, and Curvature



(a) A bottleneck.



(b) A non-bottleneck reach attaining pair. (c) Curvature-attaining point.

Figure: The reach of a manifold is attained by a bottleneck, two points on a circular arc, or a point of maximal curvature. Figure and Theorem due to Aamari-Kim-Chazal-Michel-Rinaldo-Wasserman '17.

Bottleneck Degree

Denote by $BND(V)$ the bottleneck degree of $V \subset \mathbb{C}^n$. Under certain conditions, this coincides with twice the number of bottleneck pairs.

Theorem (Di Rocco-Eklund-W. '19)

- *Let $V \subset \mathbb{C}^2$ be a "general" curve of degree d. Then $BND(V) = d^4 - 5d^2 + 4d$.*
- *Let $V \subset \mathbb{C}^3$ be a "general" surface of degree d. Then $BND(V) = d^6 - 2d^5 + 3d^4 - 15d^3 + 26d^2 - 13d$.*
- *For any smooth variety $V \subset \mathbb{P}^n_{\mathbb{C}}$ in "general position," we have an algorithm to express the bottleneck degree in terms of the polar classes of V.*

Building Bridges Between Differential Geometry and Computational Algebraic Geometry

- Curvature is central to the study of differential geometry.
- Curvature is a property of algebraic varieties.
- Properties of algebraic varieties should have defining polynomial equations and degrees!

Algebraic Manifold: Algebraic Variety and Differentiable Manifold

- $f \in \mathbb{R}[x_1, \ldots, x_n]$
- $V = \{x \in \mathbb{C}^n | f(x) = 0\}$ smooth algebraic variety
- $M = V \cap \mathbb{R}^n$ differentiable submanifold of $\mathbb{R}^n$
- M is an **algebraic manifold**

Euclidean Connection and Second Fundamental Form

For any manifold M, let $\mathcal{T}(M)$ denote the set of smooth vector fields on M; this is the space of smooth sections of the tangent bundle TM. For $M \subset \mathbb{R}^n$, let $\mathcal{N}(M)$ denote the space of smooth sections of the normal bundle NM. The **Euclidean connection** $\overline{\nabla}$ on $\mathbb{R}^n$ is a map $\overline{\nabla} : \mathcal{T}(\mathbb{R}^n) \times \mathcal{T}(\mathbb{R}^n) \to \mathcal{T}(\mathbb{R}^n), (X, Y) \mapsto \overline{\nabla}_X Y$ defined as follows:

$$(\overline{\nabla}_X Y)(p) = \sum_{i=1}^{n} X_i(p) \frac{\partial Y}{\partial x_i}(p).$$

In other words, $\overline{\nabla}_X Y$ is the vector field whose components are the directional derivatives of the components of Y in the direction X.
The **second fundamental form** of M is the map II from $\mathcal{T}(M) \times \mathcal{T}(M)$ to $\mathcal{N}(M)$ given by

$$\mathrm{II}(X, Y) := (\overline{\nabla}_X Y)^{\perp}.$$

Principal Curvatures

Let $M \subset \mathbb{R}^3$ be a surface. Fix a point $p \in M$ and vector fields $X, Y \in \mathcal{T}(M)$ such that $X(p)$ and $Y(p)$ form an orthonormal basis of $T_p M$. Let $N(p)$ be a unit vector in $N_p M$. The **principal curvatures** of M at p are the eigenvalues of the symmetric matrix

$$\begin{bmatrix} \mathrm{II}(X,X)(p) \cdot N(p) & \mathrm{II}(X,Y)(p) \cdot N(p) \\ \mathrm{II}(X,Y)(p) \cdot N(p) & \mathrm{II}(Y,Y)(p) \cdot N(p) \end{bmatrix}.$$

If X and Y are selected so that the matrix is diagonal, then $X(p)$ and $Y(p)$ are the **principal directions**, up to a choice of normal vector.

Umbilics and Critical Curvature Points

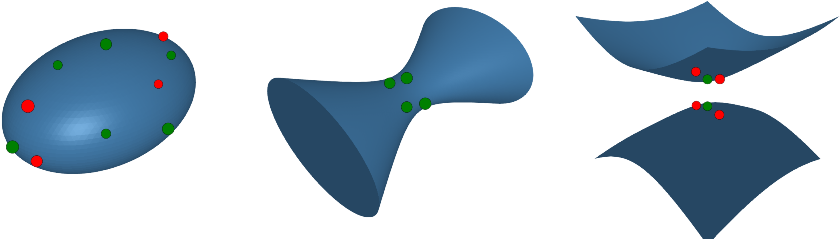


Figure: The pictures show the three quadric surfaces $X_1 = \{x_1^2 + 2x_2^2 + 4x_3^2 = 1\}$ (left picture) and $X_2 = \{-x_1^2 + 2x_2^2 + 4x_3^2 = 1\}$ (picture in the middle) and $X_3 = \{-2x_1^2 - x_2^2 + 4x_3^2 = 1\}$ (right picture). The umbilical points are shown in red and the critical curvature points are shown in green.

Umbilics

Theorem (Salmon 1865)

The degree of the variety of umbilics of a general surface of degree d in $\mathbb{R}^3$ is $10d^3 - 28d^2 + 22d$.

Degree of Critical Curvature

Theorem (Brandt-W. '19)

Let $V \subset \mathbb{R}^2$ *be a smooth, irreducible curve of degree* $d \geq 3$. *Then the degree of critical curvature of* V *is* $6d^2 - 10d$.

Equations for Critical Curvature Locus

The following equations define the locus of pairs (x, u) where $x \in M$ and u is a principal direction at x:

$$f(x_1, \ldots, x_n) = 0,$$

$$\nabla f \cdot u = 0,$$

$$\sum_{i=1}^{n} u_i^2 - 1 = 0,$$

$$\lambda^2(\nabla f \cdot \nabla f) - 1 = 0,$$

$$H_f \cdot u + y_1 u + y_2 \nabla f = 0.$$

The curvature is given by the absolute value of $g(x, u, \lambda) = \lambda u^t \cdot H_f \cdot u$. Using the principle of Lagrange multipliers, we intersect the above locus with the locus defined by the vanishing of the minors of a matrix of partial derivatives of the above equations and partial derivatives of g.

Upper Bound for Critical Curvature Degree

Theorem (Breiding-Ranestad-W.'21)

Let $V \subset \mathbb{R}^3$ be a smooth, irreducible surface of degree $d \geq 3$. There are only finitely many complex critical curvature points of V. An upper bound for their number is given by $\frac{1}{8}(2796d^3 - 6444d^2 + 3696d)$.

Future Work

- Obtain an exact formula, or tighter bound, for the critical curvature degree.
- Formulate systems of polynomial equations for other concepts in differential geometry.

Thank you!

Polar Classes

Example

For a smooth surface $V \subset \mathbb{P}^3$ we have two polar varieties. Let $p \in \mathbb{P}^3$ be a general point and $l \subset \mathbb{P}^3$ a general line. Then $P_1(V,p)$ is the set of points $x \in V$ such that the projective tangent plane $\mathbb{T}_x V \subset \mathbb{P}^3$ contains p. This is a curve on V. Similarly, $P_2(V,l) = \{x \in V : l \subseteq \mathbb{T}_x V\}$, which is finite.

Definition

Let $V \subset \mathbb{P}^n$ be a smooth variety of dimension m. For $j = 0, \ldots, m$ and a general linear space $L \subseteq \mathbb{P}^n$ of dimension $n - m - 2 + j$ the **polar variety** is given by

$$P_j(V,L) = \{x \in V : \dim \mathbb{T}_x V \cap L \geq j - 1\}.$$

For each polar variety $P_j(V,L)$, there is a corresponding **polar class** $[P_j(V,L)] = p_j$ which represents $P_j(V,L)$ up to rational equivalence.

Polar Classes and Chern Classes

$P_j(V, L)$ is either empty or of pure codimension j and

$$p_j = \sum_{i=0}^{j} (-1)^i \binom{m-i+1}{j-i} h^{j-i} c_i(T_X),$$

where $h \in A_{n-1}(X)$ is the hyperplane class.
The polar loci $P_j(V, L)$ are reduced. We have

$$c_j(T_X) = \sum_{i=0}^{j} (-1)^i \binom{m-i+1}{j-i} h^{j-i} p_i.$$

Bottleneck Genericity Assumptions 1/2

Let $V \subset \mathbb{P}^n$ be a variety. Consider the **conormal variety**

$$\mathcal{C}_V = \{(p, q) \in \mathbb{P}^n \times \mathbb{P}^n : p \in V, q \in (\mathbb{T}_p V)^\perp\}$$

and the map

$$f : \mathcal{C}_V \to \mathrm{Gr}(2, n+1) : (p, q) \mapsto \langle p, q \rangle$$

from $\mathcal{C}_V$ to the Grassmannian of lines in $\mathbb{P}^n$ that sends a pair (p, q) to the line spanned by p and q.
The orthogonality relation on $\mathbb{P}^n$ is defined via the **isotropic quadric** $Q \subset \mathbb{P}^n$ given in homogeneous coordinates by $\sum_0^n x_i^2 = 0$. Varieties which are tangent to Q are to be considered degenerate in this context and we say that a smooth projective variety is in **general position** if it intersects Q transversely. Equivalently, a smooth variety $V \subset \mathbb{P}^n$ is in general position if $\mathcal{C}_V$ is disjoint from the diagonal $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$.

Bottleneck Genericity Assumptions 2/2

A smooth variety $V \subset \mathbb{P}^n$ is **bottleneck regular** if

1. V is in general position,
2. V has only finitely many bottlenecks and
3. the differential $df_p : T_p\mathcal{C}_V \to T_{f(p)}G$ of the map f has full rank for all $p \in \mathcal{C}_V$.

If $V \subset \mathbb{P}^n$ is bottleneck regular, then V is equal to the number of bottlenecks of V counted with multiplicity.