

# Voronoi Cells of Varieties

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# Overview

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# Voronoi cells

## Definition

Let  $X$  be a real algebraic variety of codimension  $c$  in  $\mathbb{R}^n$  and  $y$  a smooth point on  $X$ . Its *Voronoi cell* consists of all points whose closest point in  $X$  is  $y$ , i.e.

$$\text{Vor}_X(y) := \left\{ u \in \mathbb{R}^n : y \in \arg \min_{x \in X} \|x - u\|^2 \right\}.$$

The Voronoi cell  $\text{Vor}_X(y)$  is a convex semialgebraic set of dimension  $c$ , living in the normal space  $N_X(y)$  to  $X$  at  $y$ . Its boundary consists of the points in  $\mathbb{R}^n$  that have at least two closest points in  $X$ , including  $y$ .

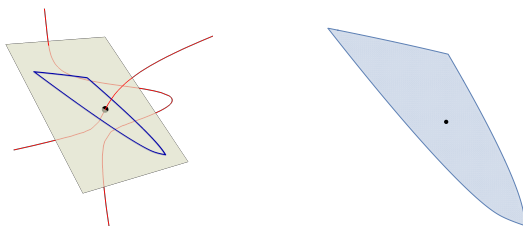


Figure: A quartic space curve and the Voronoi cell in one of its normal planes. 3 / 19

Let  $I = \langle f_1, f_2, \dots, f_m \rangle$  in  $\mathbb{Q}[x_1, \dots, x_n]$  and  $X = V(I) \subset \mathbb{R}^n$ . We will now give equations for the ideal of  $\text{Vor}_X(y)$ . Consider the polynomial ring  $R = \mathbb{Q}[x_1, \dots, x_n, u_1, \dots, u_n]$  where  $u = (u_1, \dots, u_n)$  is an additional unknown point.

## Definition

The *augmented Jacobian* of  $X$  at  $x$  is

$$J_I(x, u) \quad := \quad \begin{bmatrix} u - x \\ (\nabla f_1)(x) \\ \vdots \\ (\nabla f_m)(x) \end{bmatrix}$$

Let  $N_I$  denote the ideal in  $R$  generated by  $I$  and the  $(c+1) \times (c+1)$  minors of the augmented Jacobian  $J_I(x, u)$ , where  $c$  is the codimension of the given variety  $X \subset \mathbb{R}^n$ . The ideal  $N_I$  in  $R$  defines a subvariety of dimension  $n$  in  $\mathbb{R}^{2n}$ , namely the *Euclidean normal bundle* of  $X$ . Let  $N_I(y)$  denote the linear ideal that is obtained from  $N_I$  by replacing the unknown point  $x$  by the given point  $y \in \mathbb{R}^n$ .

## Example

Let  $n = 2$  and  $I = \langle x_1^3 - x_2^2 \rangle$ , so  $X = V(I) \subset \mathbb{R}^2$  is a cubic curve with a cusp at the origin. The ideal of the Euclidean normal bundle of  $X$  is

$$N_I = \langle x_1^3 - x_2^2, \det \begin{pmatrix} u_1 - x_1 & u_2 - x_2 \\ 3x_1^2 & -2x_2 \end{pmatrix} \rangle.$$

For the point  $y = (4, 8)$ , we have  $N_I(y) = \langle u_1 + 3u_2 - 28 \rangle$ .

# Voronoi ideal

Let  $I = \langle f_1, f_2, \dots, f_m \rangle$  in  $\mathbb{Q}[x_1, \dots, x_n]$  and  $X = V(I) \subset \mathbb{R}^n$ .

## Definition

The *critical ideal* of the variety  $X$  at the point  $y$  is

$$C_I(y) = I + N_I + N_I(y) + \langle \|x - u\|^2 - \|y - u\|^2 \rangle.$$

## Definition

The *Voronoi ideal* is the following ideal in  $\mathbb{Q}[u_1, \dots, u_n]$ . It is obtained from the critical ideal by saturation and elimination:

$$\text{Vor}_I(y) = (C_I(y) : \langle x - y \rangle^\infty) \cap \mathbb{Q}[u_1, \dots, u_n].$$

# Voronoi ideal

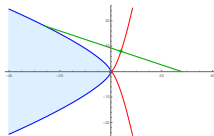
## Example

Let  $n = 2$  and  $I = \langle x_1^3 - x_2^2 \rangle$ , so  $X = V(I) \subset \mathbb{R}^2$  is a cubic curve with a cusp at the origin. Let  $y = (4, 8)$ .

The Voronoi ideal is

$$\text{Vor}_I(y) =$$

$$\langle u_1 - 28, u_2 \rangle \cap \langle u_1 + 26, u_2 - 18 \rangle \cap \langle u_1 + 3u_2 - 28, 27u_2^2 - 486u_2 + 2197 \rangle.$$



**Figure:** The cuspidal cubic is shown in red. The Voronoi cell of a smooth point is a green line segment. The Voronoi cell of the cusp is the convex region bounded by the blue curve.

## Remark

*When discussing degree, we identify the variety  $X$  and  $\delta_{\text{alg}} \text{Vor}_X(y)$  with their Zariski closures in  $\mathbb{P}_{\mathbb{C}}^n$ .*

## Definition

The algebraic boundary of the Voronoi cell  $\text{Vor}_X(y)$  is a hypersurface in the normal space to  $X$  at  $y$ . Its degree  $\delta_X(y)$  is called the *Voronoi degree* of  $X$  at  $y$ .



## Theorem

*Let  $X \subset \mathbb{P}^n$  be a curve of degree  $d$  and geometric genus  $g$  with at most ordinary multiple points as singularities. The Voronoi degree at a general point  $y \in X$  equals*

$$\delta_X(y) = 4d + 2g - 6,$$

*provided  $X$  is in general position in  $\mathbb{P}^n$ .*

# Voronoi degree of curve

## Theorem

*Let  $X \subset \mathbb{P}^n$  be a curve of degree  $d$  and geometric genus  $g$  with at most ordinary multiple points as singularities. The Voronoi degree at a general point  $y \in X$  equals*

$$\delta_X(y) = 4d + 2g - 6,$$

*provided  $X$  is in general position in  $\mathbb{P}^n$ .*

## Example

If  $X$  is a rational curve of degree  $d$ , then  $g = 0$  and hence

$\delta_X(y) = 4d - 6$ . If  $X$  is an elliptic curve, so the genus is  $g = 1$ , then we have  $\delta_X(y) = 4d - 4$ .

## Voronoi degree of curve: Example

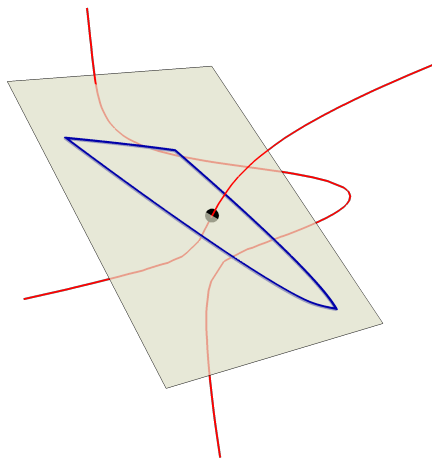


Figure: A space curve with  $d = 4$  and  $g = 1$  has Voronoi degree  $\delta_X(y) = 12$ .

# Voronoi degree of surface

## Theorem

*Let  $X \subset \mathbb{P}^n$  be a smooth surface of degree  $d$ . Then its Voronoi degree equals*

$$\delta_X(y) = 3d + \chi(X) + 4g(X) - 11,$$

*provided the surface  $X$  is in general position in  $\mathbb{P}^n$  and  $y$  is a general point on  $X$ .*

- $\chi(X) := c_2(X)$  is the topological Euler characteristic, which equals the degree of the second Chern class of the tangent bundle
- $g(X)$  is the genus of the curve obtained by intersecting  $X$  with a general smooth quadratic hypersurface in  $\mathbb{P}^n$

# Norms on Space of Matrices

Fix the space  $\mathbb{R}^{m \times n}$  of real  $m \times n$  matrices.

## Frobenius Norm

$$\|U\|_F := \sqrt{\sum_{ij} U_{ij}^2}$$

## Spectral Norm

$\|U\|_2 := \max_i \sigma_i(U)$  which extracts the largest singular value.

# Low Rank Matrix Approximation

## Remark

*Let  $X$  denote the variety in  $\mathbb{R}^{m \times n}$  of real  $m \times n$  matrices of rank  $\leq r$ . Fix a rank  $r$  matrix  $V$  in  $X$ . Let  $U \in \text{Vor}_X(V)$  and let  $U = \Sigma_1 D \Sigma_2$  be its singular value decomposition. Let  $D^{[r]}$  be the matrix that is obtained from  $D$  by replacing all singular values except for the  $r$  largest ones by zero. By the Eckart-Young Theorem, we have  $V = \Sigma_1 \cdot D^{[r]} \cdot \Sigma_2$ .*

## Remark

*The Eckart-Young Theorem works for both norms, so both give the same Voronoi cell  $\text{Vor}_X(V)$ .*

# Voronoi cell of low rank matrix

## Theorem

*Let  $V$  be an  $m \times n$ -matrix of rank  $r$ . The Voronoi cell  $\text{Vor}_X(V)$  is congruent up to scaling to the unit ball in the spectral norm on the space of  $(m - r) \times (n - r)$ -matrices.*

# Norms on Space of Symmetric Matrices

Consider the space  $\mathbb{R}^{\binom{n+1}{2}}$  whose coordinates are the upper triangular entries of a symmetric  $n \times n$  matrix. Let  $X$  be the variety of symmetric matrices of rank  $\leq r$ .

## Remark

*The Frobenius norm and Euclidean norm differ on this space.*

The Frobenius norm on  $\mathbb{R}^{\binom{n+1}{2}}$  is the restriction of the Frobenius norm on  $\mathbb{R}^{n \times n}$  to the subspace of symmetric matrices.

## Example

Let  $n = 2$ . We identify the vector  $(a, b, c)$  with the symmetric matrix  $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ . The Frobenius norm is  $\sqrt{a^2 + 2b^2 + c^2}$ , whereas the Euclidean norm is  $\sqrt{a^2 + b^2 + c^2}$ .



# Low Rank Approximation on the Space of Symmetric Matrices

## Remark

*The Frobenius and Euclidean norms have dramatically different properties with respect to low rank approximation. The Eckart-Young Theorem remains valid for the Frobenius norm on  $\mathbb{R}^{\binom{n+1}{2}}$ , but not for the Euclidean norm.*



**Figure:** The Voronoi cell of a symmetric  $3 \times 3$  matrix of rank 1 is a convex body of dimension 3. It is shown for the Frobenius norm (left) and for the Euclidean norm (right).

## Current Work (Joint with Madeline Brandt)

- Follow up on the approach in MatheMaddies' Ice Cream Map! Study limits of Voronoi decompositions.
- Detect features of variety (reach, medial axis, maximal radius of curvature) from Voronoi decomposition.

Thank you!