Metric Algebraic Geometry

Maddie Weinstein

University of Puget Sound

mweinstein@pugetsound.edu

Overview

1. Background on Algebraic Varieties and Metric Algebraic Geometry
2. The Reach of an Algebraic Variety
   - Reach: Function of Bottlenecks and Curvature
   - Bottlenecks
   - Curvature
3. Voronoi Cells
   - From Point Sets to Positive Dimension
   - Application: Low-Rank Matrix Approximation

Background on Algebraic Varieties and Metric Algebraic Geometry

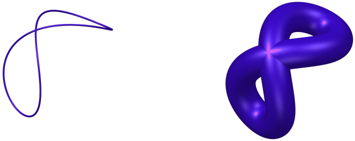


Figure: A curve and a surface in $\mathbb{R}^3$. The curve is the zero set of the polynomials $x_1^2 + x_2^2 + x_3^2 - 4$ and $(x_1 - 1)^2 + x_2^2 - 1$. The surface is defined by a degree 10 polynomial.

An **algebraic variety** (a.k.a. variety) is the zero set of a system of polynomial equations.

Degree of an Algebraic Variety

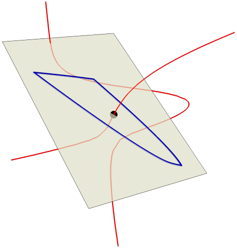
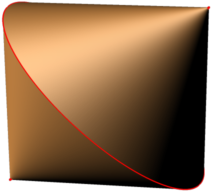
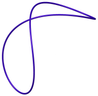
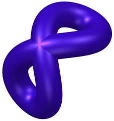
- The degree of a variety defined by one polynomial is the degree of the polynomial.
- For varieties defined by more than one polynomial, it is harder to determine the degree. The field of **intersection theory** studies these questions.
- Degree is a proxy for computational complexity.
- In $\mathbb{R}^n$, a variety of degree d intersects a linear space of complementary dimension in at most d points. For a consistent count, we turn to **complex projective space**.

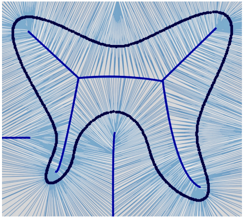
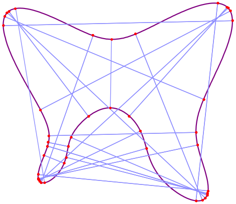
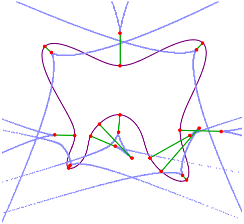
Projective Space

- We switch from $\mathbb{R}^n$ to $\mathbb{C}^n$ when we want every polynomial of degree d to have d roots.
- We switch from **affine space** to **projective space** when we want every pair of lines to intersect.
- Example: $\mathbb{P}^2_{\mathbb{R}} =$ $\mathbb{R}^2 \cup \{\text{one point at } \infty \text{ for each equivalence class of parallel lines}\}$
- *For an introduction for students, I recommend the textbook* **Ideals, Varieties, and Algorithms** *by Cox, Little, and O'Shea.*

What is metric algebraic geometry?

Metric algebraic geometry is concerned with properties of **real algebraic varieties** that depend on a **metric**. Results can be applied to **distance optimization**, **algebraic statistics**, and the study of the **geometry of data** with **nonlinear models**.

The Reach of an Algebraic Variety

Guaranteeing Persistent Homology: The Reach of an Algebraic Variety

Theorem (Partha Niyogi, Stephen Smale, Shmuel Weinberger 2006)

Let M be a compact submanifold of $\mathbb{R}^N$ of dimension k with **reach** *τ. Let $\bar{x} = \{x_1, \ldots, x_n\}$ be a set of n points drawn in independent and identically distributed fashion according to the uniform probability measure on M. Let $0 < \epsilon < \frac{\tau}{2}$. Let $U = \bigcup_{x \in \bar{x}} B_\epsilon(x)$ be a corresponding random open subset of $\mathbb{R}^N$. Let $\beta_1 = \frac{vol(M)}{(\cos^k(\theta_1))vol(B^k_{\epsilon/4})}$ and $\beta_2 = \frac{vol(M)}{(\cos^k(\theta_2))vol(B^k_{\epsilon/8})}$ where $\theta_1 = \arcsin(\frac{\epsilon\tau}{8})$ and $\theta_2 = \arcsin(\frac{\epsilon\tau}{16})$. Then for all*

$$n > \beta_1 \left(\log(\beta_2) + \log\left(\frac{1}{\delta}\right) \right)$$

the homology of U equals the homology of M with high confidence (probability $> 1 - \delta$).

TL;DR

The **reach** *(to be defined on the next slide) of an algebraic variety determines the number of sample points required for persistent homology to work.*

The Reach of an Algebraic Variety

Definition

The **medial axis** of a variety $V \subset \mathbb{R}^n$ is the set of all points $u \in \mathbb{R}^n$ such that the minimum distance from V to u is attained by two distinct points. The **reach** τ_V is the infimum of all distances from points on the variety V to points in its medial axis.

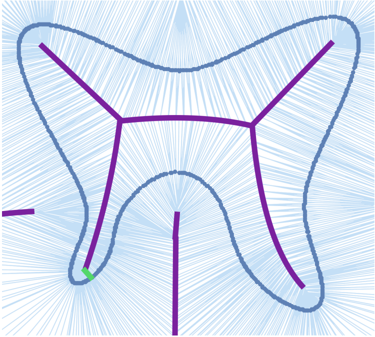


Figure: The medial axis (purple) and reach (green) of the quartic butterfly curve can be seen in its Voronoi approximation.

Algebraicity of Reach

Proposition (Horobet-W.)

Let V be a smooth algebraic variety in $\mathbb{R}^n$. Let $f_1, \ldots, f_s \in \mathbb{Q}[x_1, \ldots, x_n]$ with $V = V_{\mathbb{R}}(f_1, \ldots, f_s)$. Then the reach of V is an algebraic number over $\mathbb{Q}$.

This theorem is an invitation to use algebraic geometry to find the reach of a variety.

Reach, Bottlenecks, and Curvature

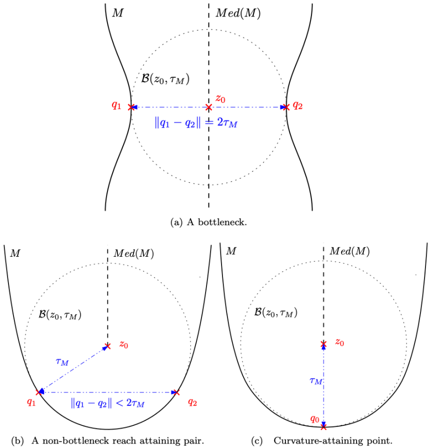


Figure: The reach of a manifold is attained by a bottleneck, two points on a circular arc, or a point of maximal curvature. Figure and Theorem due to Aamari-Kim-Chazal-Michel-Rinaldo-Wasserman '17.

Bottlenecks

Definition

Let $V \subset \mathbb{R}^n$ be a smooth variety. A line is **orthogonal** to V if it is orthogonal to the tangent space $T_x V \subset \mathbb{R}^n$ at x. The **bottlenecks** of V are pairs (x, y) of distinct points $x, y \in V$ such that the line spanned by x and y is orthogonal to V at both points.

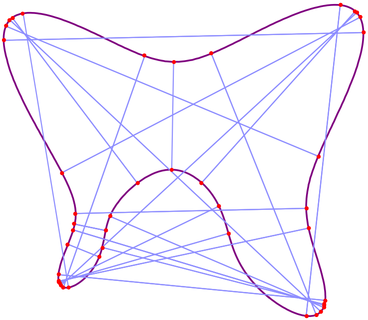


Figure: The real bottleneck pairs of the quartic butterfly curve.

Bottlenecks as Distance Optimization

Remark

Bottlenecks are the critical points of the squared distance function

$$\mathbb{R}^n \times \mathbb{R}^n : (x, y) \mapsto ||x - y||^2,$$

subject to the constraints $x, y \in V$ *as well as the non-triviality condition* $x \neq y$.

Bottleneck Degree

Denote by $BND(V)$ the bottleneck degree of $V \subset \mathbb{C}^n$. Under certain conditions, this coincides with twice the number of bottleneck pairs.

Theorem (Di Rocco-Eklund-W.)

- *Let $V \subset \mathbb{C}^2$ be a "general" curve of degree d. Then* $BND(V) = d^4 - 5d^2 + 4d$.
- *Let $V \subset \mathbb{C}^3$ be a "general" surface of degree d. Then* $BND(V) = d^6 - 2d^5 + 3d^4 - 15d^3 + 26d^2 - 13d$.
- *For any smooth variety $V \subset \mathbb{P}^n_{\mathbb{C}}$ in "general position," we have an algorithm to express the bottleneck degree in terms of the polar classes of V.*

Curvature

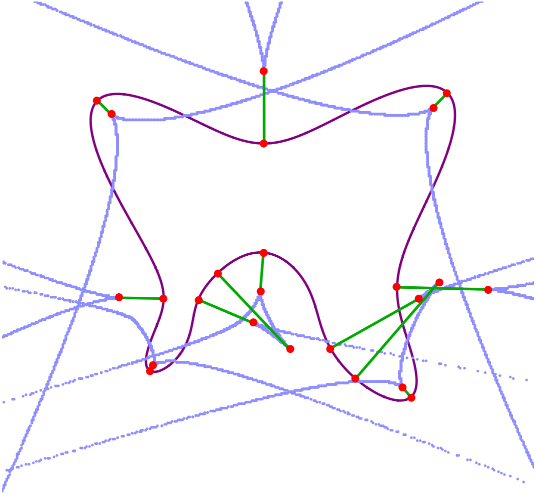


Figure: The real points of critical curvature on the butterfly curve (purple) joined by green line segments to their centers of curvature. These give cusps on the evolute (light blue).

Degree of Critical Curvature

Theorem (Salmon, 1852)

Let $V \subset \mathbb{R}^2$ be a general algebraic curve of degree d. Then the number of complex critical curvature points of V counted with multiplicity is $6d^2 - 10d$.

Theorem (Breiding-Ranestad-W.)

Let $V \subset \mathbb{R}^3$ be a general algebraic surface of degree d. Then V has isolated complex critical curvature points. An upper bound for their number is $\frac{1}{8}(2796d^3 - 6444d^2 + 3696d)$.

A formula for hypersurfaces in $\mathbb{R}^n$ can be computed using the same process given in our proof. However, we do not have a proof that a general hypersurface in $\mathbb{R}^n$ for $n \geq 4$ has isolated complex critical curvature points.

Voronoi Cells

Voronoi Diagrams of Point Clouds

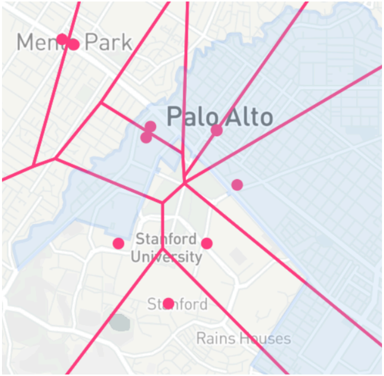


Figure: A Voronoi diagram of the bookstores near Stanford University. Created using program by Rodion Chachura.

Voronoi Cells of Varieties

Definition

Let V be a real algebraic variety of codimension c in $\mathbb{R}^n$ and y a smooth point on V. Its *Voronoi cell* consists of all points whose closest point in V is y, i.e.

$$\mathrm{Vor}_V(y) \ := \ \left\{ u \in \mathbb{R}^n : y \in \operatorname*{arg\,min}_{x \in V} \|x - u\|^2 \right\}.$$

The Voronoi cell $\mathrm{Vor}_V(y)$ is a convex semialgebraic set of dimension c, living in the normal space $N_V(y)$ to V at y. Its boundary consists of the points in $\mathbb{R}^n$ that have at least two closest points in V, including y.

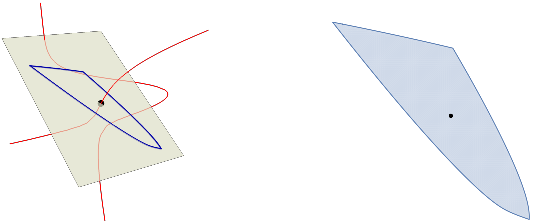


Figure: A quartic space curve and the Voronoi cell in one of its normal planes.

Voronoi Convergence

Theorem (Brandt-W.)

As sampling density increases, the Voronoi diagrams of a point sample of a variety "converge" to the Voronoi decomposition of the variety.

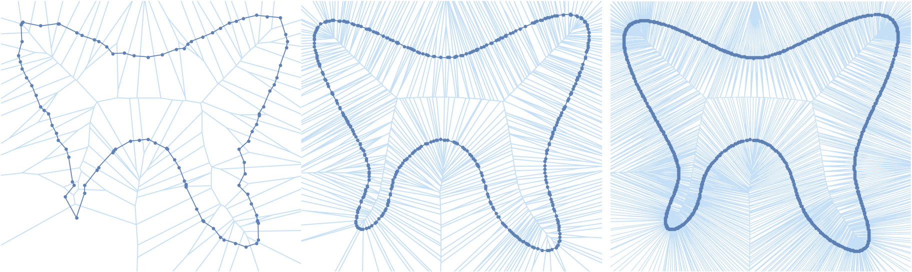


Figure: Voronoi cells of 101, 441, and 1179 points sampled from the quartic butterfly curve: $x^4 - x^2y^2 + y^4 - 4x^2 - 2y^2 - x - 4y + 1 = 0$.

Voronoi Degrees

Remark

When discussing degree, we identify the variety V *and* $\delta_{alg}\mathrm{Vor}_V(y)$ *with their Zariski closures in* $\mathbb{P}^n_{\mathbb{C}}$*.*

Definition

The algebraic boundary of the Voronoi cell $\mathrm{Vor}_V(y)$ is a hypersurface in the normal space to V at y. Its degree $\delta_V(y)$ is called the *Voronoi degree* of V at y.

Voronoi Degrees

Theorem (Cifuentes-Ranestad-Sturmfels-W.)

- *Let $V \subset \mathbb{P}^n$ be a curve of degree d and geometric genus g with at most ordinary multiple points as singularities. The Voronoi degree at a general point $y \in V$ equals*

$$\delta_V(y) \;=\; 4d + 2g - 6,$$

  provided V is in general position in $\mathbb{P}^n$.

- *Let $V \subset \mathbb{P}^n$ be a smooth surface of degree d. Then its Voronoi degree equals*

$$\delta_V(y) \;=\; 3d + \chi(V) + 4g(V) - 11,$$

  provided the surface V is in general position in $\mathbb{P}^n$ and y is a general point on V, where $\chi(V) := c_2(V)$ is the topological Euler characteristic and $g(V)$ is the genus of the curve obtained by intersecting V with a general smooth quadratic hypersurface in $\mathbb{P}^n$.

Application: Low-Rank Matrix Approximation

The study of Voronoi cells can be applied to low-rank matrix approximation.

Norms on Space of Matrices

Fix the space $\mathbb{R}^{m \times n}$ of real $m \times n$ matrices.

Frobenius Norm

$\|U\|_F := \sqrt{\sum_{ij} U_{ij}^2}$

Spectral Norm

$\|U\|_2 := \max_i \sigma_i(U)$ which extracts the largest singular value.

Low-Rank Matrix Approximation

Remark

Let X denote the variety in $\mathbb{R}^{m \times n}$ of real $m \times n$ matrices of rank $\leq r$. Fix a rank r matrix V in X. Let $U \in \mathrm{Vor}_X(V)$ and let $U = \Sigma_1 \, D \, \Sigma_2$ be its singular value decomposition. Let $D^{[r]}$ be the matrix that is obtained from D by replacing all singular values except for the r largest ones by zero. By the Eckart-Young Theorem, we have $V = \Sigma_1 \cdot D^{[r]} \cdot \Sigma_2$.

Remark

The Eckart-Young Theorem works for both norms, so both give the same Voronoi cell $\mathrm{Vor}_X(V)$.

Norms on Space of Symmetric Matrices

Consider the space $\mathbb{R}^{\binom{n+1}{2}}$ whose coordinates are the upper triangular entries of a symmetric $n \times n$ matrix. Let X be the variety of symmetric matrices of rank $\leq r$.

Remark

The Frobenius norm and Euclidean norm differ on this space.

The Frobenius norm on $\mathbb{R}^{\binom{n+1}{2}}$ is the restriction of the Frobenius norm on $\mathbb{R}^{n \times n}$ to the subspace of symmetric matrices.

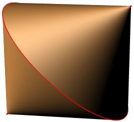
Example

Let $n = 2$. We identify the vector (a, b, c) with the symmetric matrix $\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$. The Frobenius norm is $\sqrt{a^2+2b^2+c^2}$, whereas the Euclidean norm is $\sqrt{a^2+b^2+c^2}$.

Low-Rank Approximation on the Space of Symmetric Matrices

Remark

The Frobenius and Euclidean norms have dramatically different properties with respect to low-rank approximation of symmetric matrices. The Eckart-Young Theorem is valid for the Frobenius norm on $\mathbb{R}^{\binom{n+1}{2}}$ *but not for the Euclidean norm.*



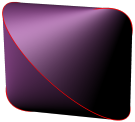


Figure: The Voronoi cell of a symmetric 3×3 matrix of rank 1 is a convex body of dimension 3. It is shown for the Frobenius norm (left) and for the Euclidean norm (right).

Thank you!

Polar Classes

An m-dimensional variety has $m+1$ polar varieties defined by exceptional tangent loci.

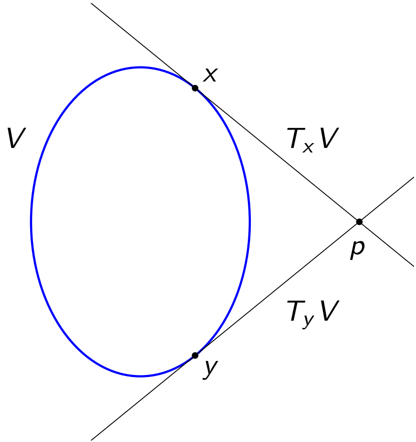


Figure: Polar locus of a point p with respect to an ellipse V: $P_1(V,p) = \{x, y\}$.

Polar Classes

Example

For a smooth surface $V \subset \mathbb{P}^3$ we have two polar varieties. Let $p \in \mathbb{P}^3$ be a general point and $l \subset \mathbb{P}^3$ a general line. Then $P_1(V, p)$ is the set of points $x \in V$ such that the projective tangent plane $\mathbb{T}_x V \subset \mathbb{P}^3$ contains p. This is a curve on V. Similarly, $P_2(V, l) = \{x \in V : l \subseteq \mathbb{T}_x V\}$, which is finite.

Definition

Let $V \subset \mathbb{P}^n$ be a smooth variety of dimension m. For $j = 0, \ldots, m$ and a general linear space $L \subseteq \mathbb{P}^n$ of dimension $n - m - 2 + j$ the **polar variety** is given by

$$P_j(V, L) = \{x \in V : \dim \mathbb{T}_x V \cap L \geq j - 1\}.$$

For each polar variety $P_j(V, L)$, there is a corresponding **polar class** $[P_j(V, L)] = p_j$ which represents $P_j(V, L)$ up to rational equivalence.

Polar Classes and Chern Classes

$P_j(V, L)$ is either empty or of pure codimension j and

$$p_j = \sum_{i=0}^{j} (-1)^i \binom{m-i+1}{j-i} h^{j-i} c_i(T_X),$$

where $h \in A_{n-1}(X)$ is the hyperplane class.
The polar loci $P_j(V, L)$ are reduced. We have

$$c_j(T_X) = \sum_{i=0}^{j} (-1)^i \binom{m-i+1}{j-i} h^{j-i} p_i.$$

Bottleneck Genericity Assumptions 1/2

Let $V \subset \mathbb{P}^n$ be a variety. Consider the **conormal variety**

$$\mathcal{C}_V = \{(p, q) \in \mathbb{P}^n \times \mathbb{P}^n : p \in V, q \in (\mathbb{T}_p V)^\perp\}$$

and the map

$$f : \mathcal{C}_V \to \mathrm{Gr}(2, n+1) : (p, q) \mapsto \langle p, q \rangle$$

from $\mathcal{C}_V$ to the Grassmannian of lines in $\mathbb{P}^n$ that sends a pair (p, q) to the line spanned by p and q.

The orthogonality relation on $\mathbb{P}^n$ is defined via the **isotropic quadric** $Q \subset \mathbb{P}^n$ given in homogeneous coordinates by $\sum_0^n x_i^2 = 0$. Varieties which are tangent to Q are to be considered degenerate in this context and we say that a smooth projective variety is in **general position** if it intersects Q transversely. Equivalently, a smooth variety $V \subset \mathbb{P}^n$ is in general position if $\mathcal{C}_V$ is disjoint from the diagonal $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$.

Bottleneck Genericity Assumptions 2/2

A smooth variety $V \subset \mathbb{P}^n$ is **bottleneck regular** if

1. V is in general position,
2. V has only finitely many bottlenecks and
3. the differential $df_p : T_p\mathcal{C}_V \to T_{f(p)}G$ of the map f has full rank for all $p \in \mathcal{C}_V$.

If $V \subset \mathbb{P}^n$ is bottleneck regular, then V is equal to the number of bottlenecks of V counted with multiplicity.

Voronoi Genericity Assumptions

Let $V \subset \mathbb{P}^n$ be a smooth projective variety defined over $\mathbb{R}$. We assume that $y \in V$ is a general point, and that we fixed an affine space $\mathbb{R}^n \subset \mathbb{P}^n$ containing y such that the hyperplane at infinity $\mathbb{P}^n \setminus \mathbb{R}^n$ is in general position with respect to V.

Example

For example, let V be the twisted cubic curve in $\mathbb{P}^3$, with affine parameterization $t \mapsto (t, t^2, t^3)$. Here $g = 0$ and $d = 3$, so the expected Voronoi degree is 6.
However, for $V \subset \mathbb{R}^3$ defined by the two equations $\mathbf{f} = (x_1^2 - x_2, x_1x_2 - x_3)$, we can compute that algebraic boundary of $Vor_V(0)$ is given by the quartic curve $27u_3^4 + 128u_2^3 + 72u_2u_3^2 - 160u_2^2 - 35u_3^2 + 66u_2 = 9$ in the plane $u_1 = 0$. So $\delta_V(y) = 4$.
This is explained by the fact that the plane at infinity in $\mathbb{P}^3$ intersects the curve V in a triple point. After a general linear change of coordinates in $\mathbb{P}^3$, which amounts to a linear fractional transformation in $\mathbb{R}^3$, we correctly find $\delta_V(y) = 6$.

Convergence Theorem: Voronoi Version

Definition

Given a point $x \in \mathbb{R}^n$ and a closed set $B \subset \mathbb{R}^n$, define

$$d_w(x, B) = \inf_{b \in B} d(x, b).$$

A sequence $\{B_\nu\}_{\nu \in \mathbb{N}}$ of compact sets is *Wijsman convergent* to B if for every $x \in \mathbb{R}^n$, we have that

$$d_w(x, B_\nu) \to d_w(x, B).$$

Theorem (Brandt-W.)

Let X be a compact curve in $\mathbb{R}^2$ and $\{A_\epsilon\}_{\epsilon \searrow 0}$ a sequence of finite subsets of X containing all singular points of X such that every point of X is within distance ϵ of some point in A_ϵ. Then every Voronoi cell of X is the Wijsman limit of a sequence of Voronoi cells of $\{A_\epsilon\}_{\epsilon \searrow 0}$.

Convergence Theorem: Delaunay Version

Theorem (Brandt-W.)

Let X be a compact curve in $\mathbb{R}^2$ and $\{A_\epsilon\}_{\epsilon \searrow 0}$ a sequence of finite subsets of X containing all singular points of X such that no point of X is more than distance ϵ from every point in A_ϵ. If X is not tangent to any circle in four or more points, then every maximal Delaunay cell is the Hausdorff limit of a sequence of Delaunay cells of $\{A_\epsilon\}_{\epsilon \searrow 0}$.

Delaunay Cells

Definition

Let $B(p, r)$ denote the open disc with center $p \in \mathbb{R}^n$ and radius $r > 0$. We say this disc is *inscribed* with respect to X if $X \cap B(p, r) = \emptyset$ and we say it is *maximal* if no disc containing $B(p, r)$ shares this property. Given an inscribed disc B of an algebraic variety $X \subset \mathbb{R}^n$, the *Delaunay cell* $Del_X(B)$ is $conv(\overline{B} \cap X)$.

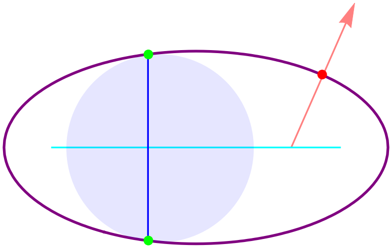


Figure: The dark blue line segment is a Delaunay cell defined by the light blue maximally inscribed circle with center $(-3/8, 0)$ and radius $\sqrt{61}/8$.

Duality of Delaunay and Voronoi Cells

Definition

Let $X \subset \mathbb{R}^2$ be a finite point set. A *Delaunay triangulation* is a triangulation $DT(X)$ of X such that no point of X is inside the circumcircle of any triangle of $DT(X)$.

Remark

The circumcenters of triangles in $DT(X)$ are the vertices in the Voronoi diagram of X.

Hausdorff Convergence

The *Hausdorff distance* of two compact sets B_1 and B_2 in $\mathbb{R}^n$ is defined as

$$d_h(B_1, B_2) := \sup \left\{ \sup_{x \in B_1} \inf_{y \in B_2} d(x, y), \sup_{y \in B_2} \inf_{x \in B_1} d(x, y) \right\}.$$

If an adversary gets to put your ice cream on either set B_1 or B_2 with the goal of making you go as far as possible, and you get to pick your starting place in the opposite set, then $d_h(B_1, B_2)$ is the farthest the adversary could make you walk in order for you to reach your ice cream.

Definition

A sequence $\{B_\nu\}_{\nu \in \mathbb{N}}$ of compact sets is *Hausdorff convergent* to B if $d_h(B, B_\nu) \to 0$ as $\nu \to \infty$.