# Voronoi Cells of Varieties in Various Distances

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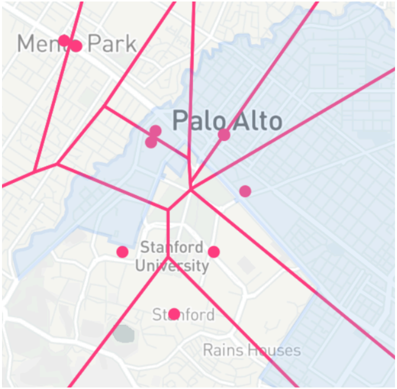


# Outline

- Background
  - What is a variety?
  - What is a Voronoi decomposition?
- My Research
  - What is a Voronoi cell of a variety?
  - What happens if you change the notion of distance?
- Student Research Invitation
- Videos
  - Sweet Voronoi Diagrams with Pumpkin $\pi$ : `https://www.youtube.com/shorts/QHRku-PmFB8`
  - MatheMaddies' Ice Cream Map: `https://youtu.be/YxMsVByhk34`

# Varieties

An **variety** in $\mathbb{R}^n$ is the zero set of a system of polynomial equations.

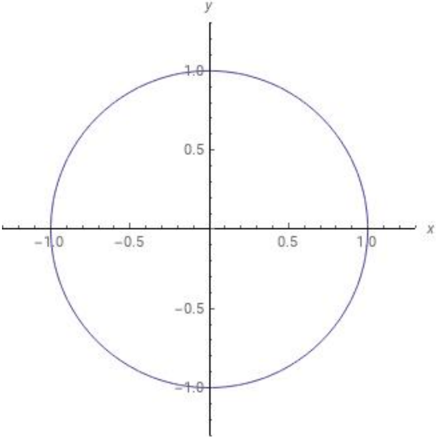
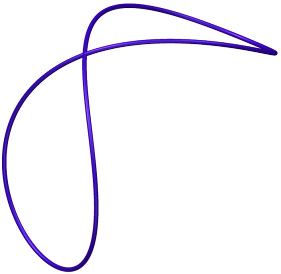



Figure: The circle is the set of points $(x, y) \in \mathbb{R}^2$ such that $x^2 + y^2 - 1 = 0$. That is, the circle is the *zero set* of $x^2 + y^2 - 1$.

# Examples of Varieties in $\mathbb{R}^3$



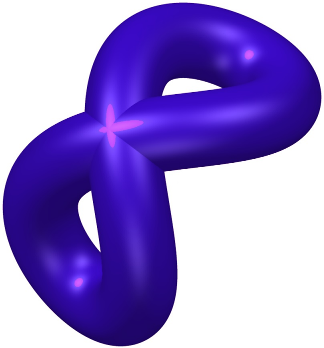


Figure: A curve (left) and a surface (right) in $\mathbb{R}^3$. The curve is the zero set of the polynomials $x^2 + y^2 + z^2 - 4$ and $(x-1)^2 + y^2 - 1$. The surface is defined by a polynomial of degree 10.

# Voronoi Decomposition of a Set of Points

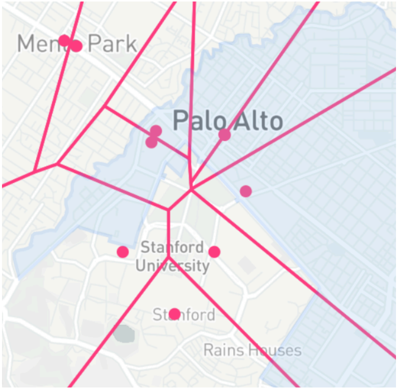



Figure: A Voronoi decomposition of the bookstores near Stanford University. Created using program by Rodion Chachura.

# Real-World Applications of Voronoi Decompositions

- Catchment zones for schools in Victoria, Australia
- John Snow's study of the 1854 Broad Street cholera outbreak in Soho, England
- Modeling the shape of epithelial cells (cells that line the surfaces of our skin, intestines, and blood vessels)

Figure: Dragonfly wings. Credit: Joi Ito, Flickr.

# Voronoi Cells of Varieties

In joint work with Diego Cifuentes, Kristian Ranestad, and Bernd Sturmfels, we extended the concept of a Voronoi decomposition from sets of points to varieties.

## Definition

Let $V$ be a variety in $\mathbb{R}^n$ and $y$ a point on $V$. Its *Voronoi cell* consists of all points in $\mathbb{R}^n$ whose closest point in $V$ is $y$.

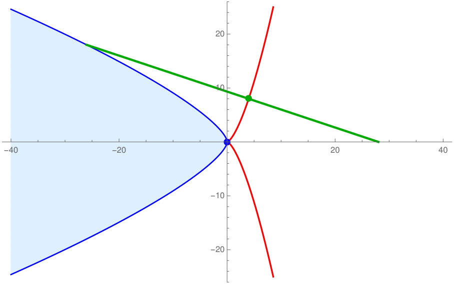



Figure: The cuspidal cubic is shown in red. The Voronoi cell of a smooth point is a green line segment. The Voronoi cell of the cusp is the convex region bounded by the blue curve.

# Voronoi Cells of a Curve in $\mathbb{R}^3$

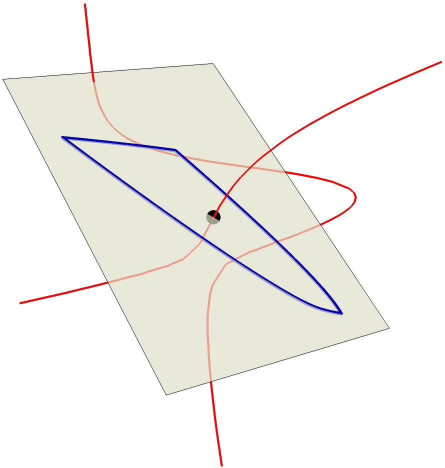


Figure: A space curve (red) and the Voronoi cell of a chosen point (black) inside its normal plane (gray).

# Voronoi Convergence

**Theorem (Madeline Brandt-MW)**

*As sampling density increases, the Voronoi decompositions of a point sample of a variety "converge" to the Voronoi decomposition of the variety.*

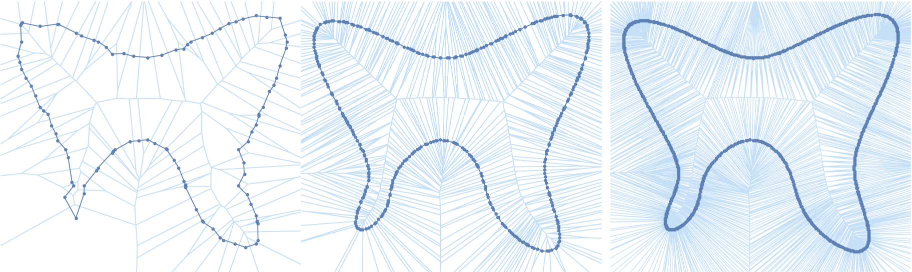


Figure: Voronoi cells of 101, 441, and 1179 points sampled from the quartic butterfly curve: $x^4 - x^2y^2 + y^4 - 4x^2 - 2y^2 - x - 4y + 1 = 0$.

MatheMaddies' Ice Cream Map: `https://youtu.be/YxMsVByhk34`

# Low-Rank Matrix Approximation and Voronoi Cells

- Low-rank matrix approximation: Suppose we have some matrix $M$ in $\mathbb{R}^{m \times n}$. What matrix $M_r$ of rank less than or equal to $r$ is the "best approximation" to $M$?
- "Best approximation" can mean "shortest distance" for some notion of distance.
- The set of all matrices in $\mathbb{R}^{m \times n}$ of rank less than or equal to $r$ forms a variety $V_r$.
- So, low-rank matrix approximation asks: What is the nearest point to $M$ on the variety $V_r$?
- Voronoi cells answer the inverse question: Given some rank $r$ matrix $M_r$, what are all the matrices for which $M_r$ is the best rank $r$ approximation?

# Distances on Space of Symmetric Matrices

Consider the space $\mathbb{R}^{\binom{n+1}{2}}$ whose coordinates are the upper triangular entries of a symmetric $n \times n$ matrix.

**Remark**

*The Frobenius distance and Euclidean distance differ on this space.*

**Example**

Let $n = 2$. We identify the vector $(a, b, c)$ with the symmetric matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$. The Frobenius distance to the origin is $\sqrt{a^2+2b^2+c^2}$, whereas the Euclidean distance to the origin is $\sqrt{a^2+b^2+c^2}$.

# Low-Rank Approximation on the Space of Symmetric Matrices

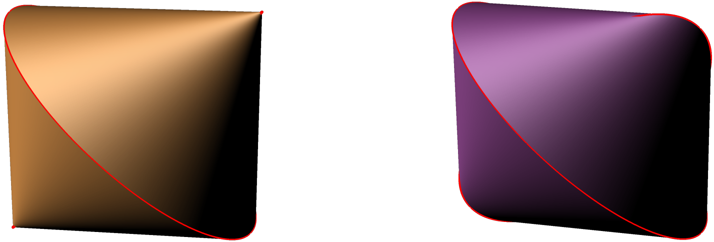


Figure: The Voronoi cell of a symmetric $3\times3$ matrix of rank 1 is a convex body of dimension 3. It is shown for the Frobenius distance (left) and for the Euclidean distance (right).

# Polyhedral Distances

A *polyhedral distance* in $\mathbb{R}^2$ is a distance in which the set of all points at some fixed distance from the origin is a polygon.

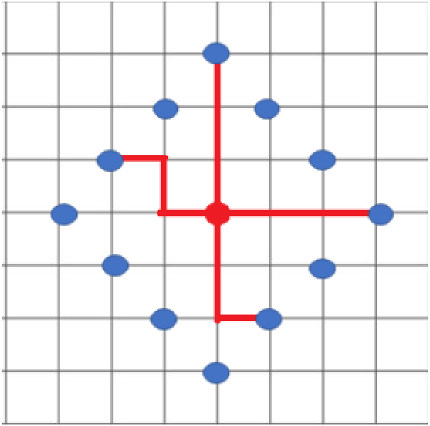


Figure: The set of all points at "taxicab distance" 3 from the origin in $\mathbb{R}^2$ is a square. Credit: Stephanie Glen, StatisticsHowTo.com.

# Distance Optimization in Polyhedral Distances

In current work with Eliana Duarte, Nidhi Kaihnsa, Julia Lindberg, and Angélica Torres, we are studying Voronoi cells and other distance optimization problems with respect to polyhedral distances. Our work has applications to statistics.

Figure: The Voronoi cells of a point set with respect to the taxicab distance. Credit: Balu Ertl, Wikimedia Commons.

# Maximum Likelihood Estimation

- Suppose we have a bag with 100 marbles. Some are red, some are blue, and some are green.
- We know that there are twice as many red marbles as blue marbles.
- We now pull a marble out of the bag, record its color, and replace it. We do this 10 times. We get 4 red marbles, 3 blue marbles, and 3 green marbles.
- What is the most likely number of marbles of each color in the bag of 100 marbles?

# Logarithmic Voronoi Spectrahedra for Linear Concentration Models

In current work with Yulia Alexandr, Maggie Regan, and Libby Taylor, we are studying Voronoi cells where the notion of distance comes from maximum likelihood estimation.

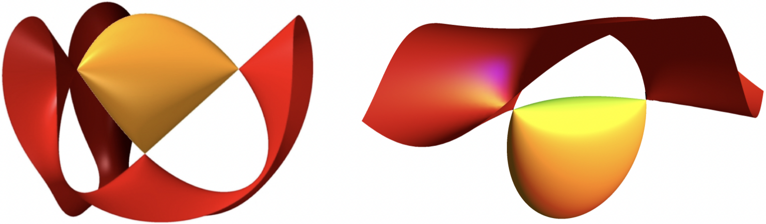


Figure: Two combinatorial types of logarithmic Voronoi spectrahedra, taken from: J. Ottem, K. Ranestad, B. Sturmfels, and C. Vinzant. Quartic spectrahedra. *Math. Program.*, 151(2):585-612, 2015.

# Student Research Invitation: Voronoi Cells in Your Distance

- What distance is interesting to you? Does your distance have applications?
- What do Voronoi cells look like with respect to your distance?
- Can you find polynomial equations to describe the Voronoi cells?
- What are the degrees of the polynomial equations that describe the Voronoi cells?

Thank you!

# Convergence Theorem: Voronoi Version

## Definition

Given a point $x \in \mathbb{R}^n$ and a closed set $B \subset \mathbb{R}^n$, define

$$d_w(x, B) = \inf_{b \in B} d(x, b).$$

A sequence $\{B_\nu\}_{\nu \in \mathbb{N}}$ of compact sets is *Wijsman convergent* to $B$ if for every $x \in \mathbb{R}^n$, we have that

$$d_w(x, B_\nu) \to d_w(x, B).$$

## Theorem (Brandt-W.)

*Let $X$ be a compact curve in $\mathbb{R}^2$ and $\{A_\epsilon\}_{\epsilon \searrow 0}$ a sequence of finite subsets of $X$ containing all singular points of $X$ such that every point of $X$ is within distance $\epsilon$ of some point in $A_\epsilon$. Then every Voronoi cell of $X$ is the Wijsman limit of a sequence of Voronoi cells of $\{A_\epsilon\}_{\epsilon \searrow 0}$.*

# Convergence Theorem: Delaunay Version

**Theorem (Brandt-W.)**

*Let $X$ be a compact curve in $\mathbb{R}^2$ and $\{A_\epsilon\}_{\epsilon \searrow 0}$ a sequence of finite subsets of $X$ containing all singular points of $X$ such that no point of $X$ is more than distance $\epsilon$ from every point in $A_\epsilon$. If $X$ is not tangent to any circle in four or more points, then every maximal Delaunay cell is the Hausdorff limit of a sequence of Delaunay cells of $\{A_\epsilon\}_{\epsilon \searrow 0}$.*

# Delaunay Cells

## Definition

Let $B(p, r)$ denote the open disc with center $p \in \mathbb{R}^n$ and radius $r > 0$. We say this disc is *inscribed* with respect to $X$ if $X \cap B(p, r) = \emptyset$ and we say it is *maximal* if no disc containing $B(p, r)$ shares this property. Given an inscribed disc $B$ of an algebraic variety $X \subset \mathbb{R}^n$, the *Delaunay cell* $Del_X(B)$ is $conv(\overline{B} \cap X)$.

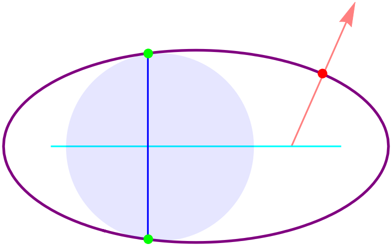


Figure: The dark blue line segment is a Delaunay cell defined by the light blue maximally inscribed circle with center $(-3/8, 0)$ and radius $\sqrt{61}/8$.

# Duality of Delaunay and Voronoi Cells

## Definition

Let $X \subset \mathbb{R}^2$ be a finite point set. A *Delaunay triangulation* is a triangulation $DT(X)$ of $X$ such that no point of $X$ is inside the circumcircle of any triangle of $DT(X)$.

## Remark

*The circumcenters of triangles in $DT(X)$ are the vertices in the Voronoi diagram of $X$.*

# Hausdorff Convergence

The *Hausdorff distance* of two compact sets $B_1$ and $B_2$ in $\mathbb{R}^n$ is defined as

$$d_h(B_1, B_2) := \sup \left\{ \sup_{x \in B_1} \inf_{y \in B_2} d(x, y), \sup_{y \in B_2} \inf_{x \in B_1} d(x, y) \right\}.$$

If an adversary gets to put your ice cream on either set $B_1$ or $B_2$ with the goal of making you go as far as possible, and you get to pick your starting place in the opposite set, then $d_h(B_1, B_2)$ is the farthest the adversary could make you walk in order for you to reach your ice cream.

## Definition

A sequence $\{B_\nu\}_{\nu \in \mathbb{N}}$ of compact sets is *Hausdorff convergent* to $B$ if $d_h(B, B_\nu) \to 0$ as $\nu \to \infty$.