Homework 1, SOLUTIONS

Problem #1

1. "Worst-case scenario" corresponds to $\hat{p} = 0.5$, because it yields **highest possible standard error** value of sample proportion $\hat{p}$ when estimating population proportion p (it maximizes $SE(\hat{p}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$)). Calculating sample size n that will yield a specific confidence level for "worst-case scenario" guarantees that this sample size n will yield **at least** that confidence level for the actual real scenario.

```
prop.sample.size <- function(m, c.lvl=0.95){
  alpha <- 1-c.lvl
  samp.size <- 0.5^2*qnorm(1-alpha/2)^2/m^2
  return(samp.size)
}
```

2. Use your *prop.sample.size*() from part 1 to do **exercise** 8.50 from the Agresti book.

```
# (a)
prop.sample.size(0.10)
```

```
## [1] 96.03647
```

```
# (b)
prop.sample.size(0.05)
```

```
## [1] 384.1459
```

```
# (c)
prop.sample.size(0.05, 0.99)
```

```
## [1] 663.4897
```

d. As the margin decreases from 0.10 down to 0.05, hence the required precision increases, the sample size needed increases (96 to 384). As the requested confidence level increases (0.95 to 0.99), the sample size needed also increases (384 to 663).

3.

```
mean.sample.size <- function(m, c.lvl=0.95, s){
  alpha <- 1-c.lvl
  samp.size <- s^2*qnorm(1-alpha/2)^2/m^2
  return(samp.size)
}
```

Presuming that distribution is approximately bell-shaped, $\approx 100\%$ of values are contained within 3 standard deviations of the mean. Hence, standard deviation is $\approx range/6 = 120,000/6 = 20,000$.

```
mean.sample.size(1000, c.lvl=0.99, s=20000)
```

```
## [1] 2653.959
```

So, to to estimate the mean annual income of Native Americans in Onondaga County, New York, correct to within 1000 with probability 0.99, we need to sample about 2654 subjects.

4. For a **two-sample proportion test**, presuming equal sample sizes ($n_1 = n_2 \equiv n$), proceed to derive the mathematical formula for the sample sizes needed to achieve
   - a desired margin of error m
   - for a given confidence level $1 - \alpha$.

Do it in similar fashion to how it was derived in class for **one-sample proportion** test. How is the obtained formula different from the case of **one-sample proportion** test?

$$z_{1-\alpha/2}\sqrt{(\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2})} = m \implies \{n_1 = n_2 \equiv n\} \implies z_{1-\alpha/2}\sqrt{\frac{\hat{p}_1(1-\hat{p}_1) + \hat{p}_2(1-\hat{p}_2)}{n}} = m \implies$$

$$z^2_{1-\alpha/2}(\hat{p}_1(1-\hat{p}_1) + \hat{p}_2(1-\hat{p}_2)) = n\ m^2 \implies n = \frac{z^2_{1-\alpha/2}(\hat{p}_1(1-\hat{p}_1) + \hat{p}_2(1-\hat{p}_2))}{m^2} \implies$$

$$\{\text{"worst case scenario"}, \hat{p}_1 = \hat{p}_2 = 0.5\} \implies 2 \times \frac{z^2_{1-\alpha/2} \times 0.5^2}{m^2}$$

So it is twice the sample size per group needed, compared to the one-sample case.

Problem #2

1. a. As the sample size n increases, both the H_0 and H_a curves get "tighter", with their variance decreasing due to the $\frac{p(1-p)}{n}$ formula. That leads to Type II error decreasing, as the $\hat{p}$ is much more likely to land near the alternative value of $p_a = 0.5$ rather than in the "fail to reject H_0" region near $p_0 = 0.33$.
   b. $n \approx 85$ or so.
   c. It will increase, because with $p_a = 0.4$ (as opposed to $p_a = 0.5$) it is much more likely for $\hat{p}$ to land closer to $p_0 = 0.33$, leading to falsely accepting the H_0 hypothesis (hence, Type II error). App confirms this, showing (for the case of $n = 50, \alpha = 0.05$)
      - $P(\text{Type II error}) = 29\%$ for $p_a = 0.5$,
      - $P(\text{Type II error}) = 81\%$ for $p_a = 0.4$.
   d. It increases, because the higher significance level $\alpha = P(\text{Type I error})$ indicates higher chance of Type I error - which is always in a trade-off with lower Type II error, hence higher power of the test. App confirms this, showing (for the case of $n = 50, p_a = 0.5$)
      - Power $= 0.71$ for $\alpha = 0.05$,
      - Power $= 0.80$ for $\alpha = 0.10$.

2. Proceed to

   a. Let $p = \{\text{true cure rate of new cancer treatment}\}$, then

$$H_0 : p = 0.28, \quad vs \quad H_a : p \neq 0.28$$

b. It means, given that the true cure rate of your new cancer treatment differs from the old treatment by 10-points, your significance test at $\alpha = 0.05$ will be able to correctly detect it (by rejecting the H_0 hypothesis) with probability 0.77.

c. Setting $p.0 = 0.28, p.a = 0.38$ (difference of 10% with the null value of 28%), $n = 100, alpha = 0.05$ in the code, we get $P(\text{Type II error}) = 0.40 \implies \text{Power} = 0.60$.

d. Setting $p.0 = 0.28, p.a = 0.38, n = 100$, trying out various values of α, seeing which will lead to $P(\text{Type II error}) = 0.15$. We get to that value at about $\alpha \approx 0.265$.

e. Setting $p.0 = 0.28, p.a = 0.38, alpha = 0.05$, trying out various values of n, seeing which will lead to $P(\text{Type II error}) = 0.15$. We get to that value at about $n = 191$.

f.
- Lower significance level $\implies$ lower $P(\text{Type I error})$ $\implies$ lower chance of falsely rejecting H_0. Hence, lower chance of falsely claiming that new treatment is better.
- Higher power $\implies$ lower $P(\text{Type II error})$ $\implies$ lower chance of falsely "sticking" with H_0. Hence, lower chance of falsely claiming that new treatment is no better.
- If we state that new treatment is better, when it is actually **no different** (Type I error), then the consequences of adopting that new treatment won't be all that "perilous to humanity". It will cost extra money to roll it out, but as far as the people's health is concerned, we will get about the same effect as the old treatment.
- If we falsely state that new treatment is no better than the old one (Type II error), then the consequences of sticking with the old treatment might be harmful to people's health (as we don't implement the new treatment that is truly more beneficial).

g. You can increase the significance level α, sacrificing the higher chance of committing Type I error (which has less drastic consequences) for lower chance of committing Type II error (which has really harmful consequences), hence higher power.

Problem #3

Provided only upon a student's impassioned request.