Math 131A Practice Problems

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1 Problem 1.1

Prove $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all positive integers n.

Proof. Our proposition $P_n : 1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$. We consider the base case $n = 1$.

$$1^2 = \tfrac{1}{6}(1)(2)(3) = \tfrac{6}{6} = 1.$$

Since our base case is proved, we assume the proposition to be true for n. We thus prove it for $n+1$.

$$\begin{aligned}
1^2 + 2^2 + \cdots + n^2 + (n+1)^2 &= \frac{1}{6}(n)(n+1)(2n+1) + (n+1)^2 \\
&= \frac{(n+1)(n(2n+1) + 6(n+1))}{6} \\
&= \frac{(n+1)(2n^2 + n + 6n + 6)}{6} \\
&= \frac{(n+1)(2n^2 + 7n + 6)}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6} \\
&= \frac{(n+1)((n+1)+1)(2(n+1)) + 1}{6}
\end{aligned}$$

Thus, our proof is complete. □

2 Problem 1.2

Prove $3 + 11 + \cdots + (8n-5) = 4n^2 - n$ for all positive integers n.

Proof. Our proposition $P_n : 3 + 11 + \cdots + (8n-5) = 4n^2 - n$. We consider the base case $n = 1$.

$$3 = 4(1^2) - 1 = 3$$

Since our base case is true, we assume the proposition to be true for n. Thus, we prove it for $n+1$.

$$\begin{aligned}
3 + 11 + \cdots + (8n-5) + (8n+3) &= 4n^2 - n + 8n + 3 \\
&= 4n^2 + 8n + 4 - (n+1) \\
&= 4(n^2 + 2n + 1) - (n+1) \\
&= 4(n+1)^2 - (n+1)
\end{aligned}$$

Thus, our proof is complete. □

3 Problem 1.6

Prove $(11)^n - 4^n$ is divisible by 7 when n is a positive integer.

Proof. We check our base case $n = 1$.

$$11^1 - 4^1 = 11 - 4 = 7$$

As 7 is divisible by 7, we assume the statement true for n. We thus prove it for $n + 1$.

$$\begin{aligned}(11)^{n+1} - 4^{n+1} &= 11 \cdot 11^n - 4 \cdot 4^n \\ &= 4(11^n - 4^n) + 7(11^n)\end{aligned}$$

We already know $11^n - 4^n$ divisible by 7 and $7(11^n)$ is divisible by 7 because of a constant factor of 7 is divisible by 7. Thus, our proof is complete. □

4 Problem 1.7

Prove $7^n - 6n - 1$ is divisible by 36 for all positive integers n.

Proof. We check our base case $n = 1$.

$$7^1 - 6(1) - 1 = 7 - 6 - 1 = 0$$

0 is divisible by 36, thus we assume the statement true for n.

$$\begin{aligned}7^{n+1} - 6(n+1) - 1 &= 7(7^n) - 6n - 6 - 1 \\ &= 7(7^n - 6n - 1) + 36n\end{aligned}$$

We know $7^n - 6n - 1$ is divisible by 36 and since $36n$ is divisible by n, we have proved $n + 1$, thus our proof is complete. □

5 Problem 1.8

Prove $n^2 > n + 1$ for all integers $n \geq 2$.

Proof. We check our base case $n = 2$.

$$2^2 > 2 + 1,\ 4 > 3$$

We thus assume the statement true for n.

$$\begin{aligned}(n+1)^2 &> n + 2 \\ n^2 + 2n + 1 &> n + 2 \\ 2n + 1 &> 1\end{aligned}$$

Thus, since $2n + 1 > 1$, we have proved $n + 1$, and our proof is complete. □

6 Problem 2.1

Show $\sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{24}, \sqrt{31}$ are not rational numbers.

We shall use the rational root theorem.

$$\begin{aligned} \alpha &= \sqrt{3} \\ \alpha^2 &= 3 \\ \alpha^2 - 3 &= 0 \end{aligned}$$

Now, we check none of $\frac{div c_0}{div c_n}$ is a root. We have $\pm 1, \pm 3$. Since none are solutions of the equation, $\sqrt{3}$ is not a rational number. We can apply the same logic to similar numbers.

7 Problem 2.2

Show $\sqrt[3]{2}$, $\sqrt[7]{5}$, $\sqrt[4]{13}$ are not rational numbers.

We once again follow the rational root theorem.

$$\begin{aligned} \alpha &= \sqrt[3]{2} \\ \alpha^3 - 2 &= 0 \end{aligned}$$

Our roots are $\pm 1, \pm 2$. Since none of the roots are solutions, $\sqrt[3]{2}$ is not a rational number. We can apply the same logic to similar numbers.

8 Problem 2.3

Show $\sqrt{2 + \sqrt{2}}$ is not a rational number.

We will use the rational root theorem.

$$\begin{aligned} \alpha &= \sqrt{2 + \sqrt{2}} \\ \alpha^2 &= 2 + \sqrt{2} \\ \alpha^2 - 2 &= \sqrt{2} \\ (\alpha^2 - 2)^2 &= 2 \\ \alpha^4 - 4\alpha^2 + 4 &= 2 \\ \alpha^4 - 4\alpha^2 + 2 &= 0 \end{aligned}$$

We check that none of the roots are solutions: $\pm 1, \pm 2$. None of the roots are solutions, so $\sqrt{2 + \sqrt{2}}$ is not a rational number.

9 Problem 2.4

Show $\sqrt[3]{5-\sqrt{3}}$ is not a rational number.

We will use the rational root theorem.

$$\begin{aligned}
\alpha &= \sqrt[3]{5-\sqrt{3}} \\
\alpha^3 &= 5-\sqrt{3} \\
\alpha^3 - 5 &= -\sqrt{3} \\
(\alpha^3-5)^2 &= 3 \\
\alpha^6 - 10\alpha^3 + 25 &= 3 \\
\alpha^6 - 10\alpha^3 + 22 &= 0
\end{aligned}$$

We check the roots: $\pm1, \pm2, \pm11, \pm22$. Since none of the roots are solutions, $\sqrt[3]{5-\sqrt{3}}$ is not a rational number.

10 Problem 2.5

Show $[3+\sqrt{2}]^{2/3}$ is not a rational number.

We use the rational root theorem.

$$\begin{aligned}
\alpha &= \sqrt[3]{(3+\sqrt{2})^2} \\
\alpha^3 &= (3+\sqrt{2})^2 \\
\alpha^3 &= 9+2+6\sqrt{2} \\
\alpha^3 - 11 &= 6\sqrt{2} \\
(\alpha^3-11)^2 &= 72 \\
\alpha^6 - 22\alpha^3 + 121 &= 72 \\
\alpha^6 - 22\alpha^3 + 49 &= 0
\end{aligned}$$

We check the roots: $\pm1, \pm7, \pm49$. None are solutions, so $[3+\sqrt{2}]^{2/3}$ is not a rational number.

11 Problem 2.6

Discuss why $4-7b^2$ is rational if b is rational.

Since b is rational, we can describe $b = \frac{p}{q}$ where $p, q \in \mathbb{Z}$. Thus, we can describe $7b^2 = \frac{7p^2}{q^2} = \frac{m}{n}$ where $m, n \in \mathbb{Z}$. Thus, $4-7b^2 = \frac{4n-m}{n} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$. Thus, $4-7b^2$ is rational.

12 Problem 2.7

Show the following irrational-looking expressions are actually rational numbers: $(a)\sqrt{4+2\sqrt{3}}-\sqrt{3}$ and $(b)\sqrt{6+4\sqrt{2}}-\sqrt{2}$.

We express (a) in terms of α.

$$\begin{aligned}
\alpha &= \sqrt{4+2\sqrt{3}}-\sqrt{3} \\
\alpha+\sqrt{3} &= \sqrt{4+2\sqrt{3}} \\
(\alpha+\sqrt{3})^2 &= 4+2\sqrt{3} \\
\alpha^2+2\sqrt{3}\alpha+3 &= 4+2\sqrt{3} \\
\alpha^2+2\sqrt{3}(\alpha-1)-1 &= 0 \\
\alpha^2+2\sqrt{3}\alpha-2\sqrt{3}-1 &= 0 \\
(\alpha-1)(\alpha+1+2\sqrt{3}) &= 0
\end{aligned}$$

Thus, we check our roots $1, -1-2\sqrt{3}$. Since a negative number is not a viable solution, $\alpha = 1$ is a solution, thus $\sqrt{4+2\sqrt{3}}-\sqrt{3}$ is a rational number.
We express (b) in terms of α.

$$\begin{aligned}
\alpha &= \sqrt{6+4\sqrt{2}}-\sqrt{2} \\
\alpha+\sqrt{2} &= \sqrt{6+4\sqrt{2}} \\
(\alpha+\sqrt{2})^2 &= 6+4\sqrt{2} \\
\alpha^2+2\sqrt{2}\alpha+2 &= 6+4\sqrt{2} \\
\alpha^2+2\sqrt{2}\alpha-4-4\sqrt{2} &= 0 \\
(\alpha-2)(\alpha+2+2\sqrt{2}) &= 0
\end{aligned}$$

Thus, we check our roots $2, -2-2\sqrt{2}$. Since a negative number is not a viable solution, $\alpha = 2$ is a solution, thus $\sqrt{6+4\sqrt{2}}-\sqrt{2}$ is a rational number.

13 Problem 2.8

Find all rational solutions of the equation $x^8-4x^5+13x^3-7x+1=0$.

Using the rational root theorem, our only viable solutions are ± 1. For $x=1$, we get

$$1-4+13-7+1=-3+13-7+1=10-7+1=3+1=4$$

Thus, $x=1$ is not a solution. For $x=-1$, we get

$$1+4-13+7+1=5-13+8=0$$

Thus, $x=-1$ is the only rational solution to the equation.

14 Problem 3.1

Which of the properties $A1 - A4, M1 - M4, DL, O1 - O5$ fail for $\mathbb{N}$? Which of these properties fail for $\mathbb{Z}$?

A1. $a + (b + c) = (a + b) + c$ for all a, b, c.
A2. $a + b = b + a$ for all a, b.
A3. $a + 0 = a$ for all a.
A4. For each a, there is an element $-a$ such that $a + (-a) = 0$.
M1. $a(bc) = (ab)c$ for all a, b, c.
M2. $ab = ba$ for all a, b.
M3. $a \cdot 1 = a$ for all a.
M4. For each $a \neq 0$, there is an element a^{-1} such that $aa^{-1} = 1$.
DL. $a(b + c) = ab + ac$ for all a, b, c.
O1. Given a and b, either $a \leq b$ or $b \leq a$.
O2. If $a \leq b$ and $b \leq a$, then $a = b$.
O3. If $a \leq b$ and $b \leq c$, then $a \leq c$.
O4. If $a \leq b$, then $a + c \leq b + c$.
O5. If $a \leq b$ and $0 \leq c$, then $ac \leq bc$.

A3. This fails for $\mathbb{N}$ since $\mathbb{N}$ does not contain the element 0.
A4. This fails for $\mathbb{N}$ since $\mathbb{N}$ does not contain negative numbers.
M4. This fails for $\mathbb{N}$ since $\mathbb{N}$ does not contain fractions.
M4. This fails for $\mathbb{Z}$ since $\mathbb{Z}$ does not contain fractions.

15 Problem 3.2

The commutative law **A2.** was used in the proof of (ii) and (iii) in Theorem 3.1. Where?

Theorem 3.1

The following are consequences of the field properties:
(i) $a + c = b + c$ *implies* $a = b$;
(ii) $a \cdot 0 = 0$ *for all* a;
(iii) $(-a)b = -ab$ *for all* a, b;
(iv) $(-a)(-b) = ab$ *for all* a, b;
(v) $ac = bc$ *and* $c \neq 0$ *imply* $a = b$;
(vi) $ab = 0$ *implies either* $a = 0$ *or* $b = 0$;
for $a, b, c \in \mathbb{R}$.

For (ii), we have

$$\begin{aligned} a \cdot 0 &= a \cdot (0+0) \\ &= a \cdot 0 + a \cdot 0 \\ a \cdot 0 &= a \cdot 0 + 0 \\ &= 0 + a \cdot 0 \\ 0 + a \cdot 0 &= a \cdot 0 + a \cdot 0 \\ a \cdot 0 &= 0 \end{aligned}$$

We used **A2.** in step $3, 4$ where we rearranged $a \cdot 0 + 0 = 0 + a \cdot 0$.
For (iii), we have

$$\begin{aligned} ab + (-a)b &= (a + (-a))b \\ &= 0 \cdot b \\ &= 0 \\ &= ab + (-ab) \\ ab + (-a)b &= ab + (-ab) \\ (-a)b + ab &= (-ab) + ab \\ (-a)b &= -ab \end{aligned}$$

We used **A2.** in step $5, 6$ where we rearranged $ab + (-a)b = (-a)b + ab$ and $ab + (-ab) = (-ab) + ab$.

16 Problem 3.3

Prove (iv) and (v) of Theorem 3.1.

Proof. (iv) $(-a)(-b) = ab$ *for all* a, b;

$$\begin{aligned} (-a)(-b) + (-ab) &= (-a)(-b) + (-a)b \\ &= (-a)(-b + b) \\ &= (-a)(0) \\ &= 0 \\ ab + (-ab) &= 0 \\ (-a)(-b) + (-ab) &= ab + (-ab) \\ (-a)(-b) &= ab \end{aligned}$$

□

Proof. (v) $ac = bc$ *and* $c \neq 0$ *implies* $a = b$;

$$\begin{aligned} a &= a(cc^{-1}) \\ &= (ac)c^{-1} \\ &= (bc)c^{-1} \\ &= b(cc^{-1}) \\ &= b \end{aligned}$$

□

17 Problem 3.4

Prove (v) and (vii) of Theorem 3.2.

Theorem 3.2

The following are consequences of an ordered field:

(i) *If* $a \leq b$, *then* $-b \leq -a$;
(ii) *If* $a \leq b$ *and* $c \leq 0$, *then* $bc \leq ac$;
(iii) *If* $0 \leq a$ *and* $0 \leq b$, *then* $0 \leq ab$;
(iv) $0 \leq a^2$ *for all* a;
(v) $0 < 1$;
(vi) *If* $0 < a$, *then* $0 < a^{-1}$;
(vii) *If* $0 < a < b$, *then* $0 < b^{-1} < a^{-1}$;

for $a, b, c \in \mathbb{R}$.

Proof. (v) $0 < 1$;

We shall prove $0 < 1$ *for* $0 \leq 1$ *and* $0 \neq 1$. $0 \leq a^2$ *for all* a *implies* $0 \leq 1$.

Assume $0 = 1$. *Choose a non-zero real number* a.

$$\begin{aligned} a &= a \cdot 1 \\ &= a \cdot 0 \\ &= 0 \end{aligned}$$

However, since $a \neq 0$, *we have a contradiction. Thus,* $0 \neq 1$. *Thus,* $0 < 1$. □

Proof. (vii) *If* $0 < a < b$, *then* $0 < b^{-1} < a^{-1}$;

$$\begin{aligned} a &< b \\ aa^{-1} &< ba^{-1} \\ 1 &< ba^{-1} \\ bb^{-1} &< ba^{-1} \\ b^{-1}bb^{-1} &< b^{-1}ba^{-1} \\ b^{-1} &< a^{-1} \end{aligned}$$

□

18 Problem 3.5

Show $|b| \leq a$ if and only if $-a \leq b \leq a$. Prove $||a| - |b|| \leq |a - b|$ for all $a, b \in \mathbb{R}$.

Proof. We prove both sides of the claim. Assume $|b| \leq a$. There are two cases.

Case 1: $b \geq 0$. $b \leq a$. Thus, since $0 \leq b$, $0 \leq a$, then $-b \leq -a$. Thus, $-b \leq a$. Therefore, $-a \leq b$. Thus, $-a \leq b \leq a$.

Case 2: $b \leq 0$. We know $0 \leq a$ since $0 \leq |a| \forall a$. Thus, we have $-b \leq a$ so by Theorem 3.2 we have $-a \leq b$. Since $b \leq 0$ and $0 \leq a$, we have $b \leq a$. Thus, $-a \leq b \leq a$.

Assume $-a \leq b \leq a$. Thus, we have $-a \leq b$, and by transitivity $-b \leq a$. We thus have $|b| \leq a$. There are two cases.

Case 1: $b \geq 0$. It is given then that $b \leq a$.

Case 2: $b \leq 0$. Then we have $|b| = -b \leq a$. □

Proof. It is enough to show that $-|a - b| \leq |a| - |b| \leq |a - b|$ by part (a).

$$\begin{aligned}
|b| &= |(b - a) + a| \\
&\leq |b - a| + |a| \\
&= |a - b| + |a| \\
-|a - b| &\leq |a| - |b| \\
|a| &= |(a - b) + b| \\
&\leq |a - b| + |b| \\
|a| - |b| &\leq |a - b|
\end{aligned}$$

Thus, it is sufficient to assume $-|a-b| \leq |a| - |b| \leq |a-b|$, thus $||a| - |b|| \leq |a - b|$ for all $a, b \in \mathbb{R}$. □

19 Problem 3.6

Prove $|a + b + c| \leq |a| + |b| + |c|$ for all $a, b, c \in \mathbb{R}$.

Proof.

$$\begin{aligned}
|(a + b) + c| &\leq |(a + b)| + |c| \\
&\leq |a| + |b| + |c|
\end{aligned}$$

□

20 Problem 3.7

Show $|b| < a$ if and only if $-a < b < a$.

Proof. We prove both sides of the claim. Assume $|b| < a$.

Case 1: $b \geq 0$. Thus, $b < a$. Thus, $0 \leq a$. Therefore, if $0 \leq b$, $-b \leq 0$. We therefore know $-b < a$. By transitivity $-a < b$. Therefore $-a < b < a$.

Case 2: $b \leq 0$. Thus, $-b < a$. We know $0 < a$ since $0 \leq |b| < a$. Since $b \leq 0$ and $0 < a$, we have $b < a$. By transitivity we also have $-a < b$. Therefore $-a < b < a$.

Assume $-a < b < a$. By transitivity we have $-b < a$. There are two cases.

Case 1: $b \geq 0$. Thus, $b < a$ is given.

Case 2: $b \leq 0$. Thus, we have $|b| = -b < a$. □

21 Problem 3.8

Let $a, b \in \mathbb{R}$. Show if $a \leq b_1$ for every $b_1 > b$, then $a \leq b$.

Proof. We consider the contrapositive. If $a > b$ there exists $b_1 > b$ such that $a > b_1$.

$$\begin{aligned} a > b \textit{ implies } a + (-b) &> b + (-b) \\ a - b &> 0 \\ \frac{a}{2} &> \frac{b}{2} \\ \frac{a-b}{2} &> 0 \end{aligned}$$

Now pick $b_1 = b + \frac{a-b}{2}$. We note $b_1 > b$ as if $b_1 \leq b$ then $b + \frac{a-b}{2} \leq b$ *implies* $\frac{a+b}{2} \leq b$. This implies $\frac{a}{2} \leq \frac{b}{2}$ which would be a contradiction, thus $b_1 > b$. Now we claim $b_1 < a$. For contradiction assume $b_1 \geq a$.

$$\begin{aligned} a &\leq \frac{a+b}{2} \\ &= \frac{a}{2} + \frac{b}{2} \\ \frac{a}{2} &\leq \frac{b}{2} \\ a &\leq b \end{aligned}$$

However, since $a > b$, this cannot be true, thus $b_1 < a$. Therefore, we have $b < b_1 < a$. Thus our contrapositive is true, hence our proof is complete. $\square$

22 Problem 4.1

For each set below that is bounded above, list three upper bounds for the set. Otherwise write "NOT BOUNDED ABOVE" or "NBA".

(a) $[0, 1]$

(b) $(0, 1)$

(c) $\{2, 7\}$

(d) $\{\pi, e\}$

(e) $\{\frac{1}{n} : n \in \mathbb{N}\}$

(f) $\{0\}$

(g) $[0, 1] \cup [2, 3]$

(h) $\cup_{n=1}^{\infty}[2n, 2n + 1]$

(i) $\cap_{n=1}^{\infty}[-\frac{1}{n}, 1 + \frac{1}{n}]$

(j) $\{1 - \frac{1}{3^n} : n \in \mathbb{N}\}$

(k) $\{n + \frac{(-1)^n}{n} : n \in \mathbb{N}\}$

(l) $\{r \in \mathbb{Q} : r < 2\}$

(m) $\{r \in \mathbb{Q} : r^2 < 4\}$

(n) $\{r \in \mathbb{Q} : r^2 < 2\}$

(o) $\{x \in \mathbb{R} : x < 0\}$

(p) $\{1, \frac{\pi}{3}, \pi^2, 10\}$

(q) $\{0, 1, 2, 4, 8, 16\}$

(r) $\cap_{n=1}^{\infty}(1 - \frac{1}{n}, 1 + \frac{1}{n})$

(s) $\{\frac{1}{n} : n \in \mathbb{N}$ and n is prime$\}$

(t) $\{x \in \mathbb{R} : x^3 < 8\}$

(u) $\{x^2 : x \in \mathbb{R}\}$

(v) $\{cos(\frac{n\pi}{3}) : n \in \mathbb{R}\}$

(w) $\{sin(\frac{n\pi}{3}) : n \in \mathbb{N}\}$

(a) $2, 3, 4$

(b) $2, 3, 4$

(c) $10, 11, 12$

(d) $4, 5, 6$

(e) $2, 3, 4$

(f) $1, 2, 3$

(g) $4, 5, 6$

(h) NBA

(i) $2, 3, 4$

(j) $2, 3, 4$

(k) NBA

(l) $4, 5, 6$

(m) $5, 6, 7$

(n) $5, 6, 7$

(o) $1, 2, 3$

(p) $12, 13, 14$

(q) $20, 21, 22$

(r) $3, 4, 5$

(s) $2, 3, 4$

(t) $10, 11, 12$

(u) NBA

(v) $2, 3, 4$

(w) $2, 3, 4$

23 Problem 4.5

Let S be a nonempty subset of $\mathbb{R}$ that is bounded above. Prove if sup S belongs to S, then sup S = max S.

We know that

$$s \leq supS \; \forall s \in S$$

If sup S is in S, then we know it is equal to max = s_0 such that

$$s \leq s_0 \; \forall s \in S$$

and $s_0 \in S$. Therefore, max S = sup S.

24 Problem 4.6

Let S be a nonempty bounded subset of $\mathbb{R}$.
(a) Prove $infS \leq supS$.
(b) What can you say about S if $infS = supS$?

We know that $infS \leq S \leq supS$. Thus, by transitivity, we know $infS \leq supS$. If $infS = supS$, we thus know that S only contains one element.

25 Problem 4.7

Let S and T be nonempty bounded subsets of $\mathbb{R}$.
(a) Prove $S \subseteq T$, then $infT \leq infS \leq supS \leq supT$.

(a) If $S \subseteq T$, if $s \in S, s \in T$. Thus, we have that $infT \leq s \leq supT$. Since $infS$ is the greatest lower bound for S, we have $infT \leq infS$. The same applies for sup, giving us $supS \leq supT$. Thus, $infT \leq infS \leq supS \leq supT$.

26 Problem 4.9

Complete the proof that $infS = -sup(-S)$ in Corollary 4.5 by proving 1 and 2.

Corollary 4.5

Every nonempty subset S of $\mathbb{R}$ that is bounded below has a greatest lower bound $infS$.

We first prove $sup(-S)$ exists.

Proof. Let $-S$ be $\{-s : s \in S\}$. Since S is bounded below, we know there is an m in $\mathbb{R}$ s.t. $m \leq s$ for all $s \in S$. By transitivity we have $-s \leq -m$ for all $s \in S$. Thus, our set $-S$ is bounded above by $-m$. Thus, by completeness axiom $sup(-S)$ exists. □

We now prove $infS = -sup(-S)$.

Proof. Let $s_0 = sup(-S)$. We want to prove

$$-s_0 \leq s \quad \text{for all} \quad s \in S.$$

and

$$\text{if } t \leq s \quad \text{for all} \quad s \in S, \quad \text{then } t \leq -s_0.$$

Since s_0 is the supremum of $-S$, we know $-s \leq s_0$ for all $s \in S$. By transitivity we have $-s_0 \leq s$ for all $s \in S$.
If $t \leq s$ for all $s \in S$, then by transitivity we have $-s \leq -t$. Thus, $-t$ is an upper bound of $-S$. Since s_0 is the supremum of $-S$. we have that $s_0 \leq -t$, and by transitivity $t \leq -s_0$. As t is a lower bound for S and $t \leq -s_0$, $-s_0$ is an infinum of S, therefore $infS = -sup(-S)$. □

27 Problem 4.10

Prove that if $a > 0$, then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < a < n$.

Proof. By the Archimedian property we have that for any $a > 0$ and $b > 0$ there exists some $n \in \mathbb{N}$ such that $na > b$. Therefore, setting $b = 1$, we get some $n_1 a > 1, a > \frac{1}{n_1}$ for $n_1 \in \mathbb{N}$. Setting $b = a^2$ for some $n_2 \in \mathbb{N}$ we get $n_2 a > a^2, n_2 > a$. Since we can pick any $n = max\{n_1, n_2\}$ such that $\frac{1}{n} \leq \frac{1}{n_1}$ and $n_2 \leq n$ which implies $\frac{1}{n} < a$ and $a < n$. Therefore $\frac{1}{n} < a < n$. □

28 Problem 4.11

Consider $a, b \in \mathbb{R}$ where $a < b$. Use Denseness of Q 4.7 to show there are infinitely many rationals between a and b.

Denseness of $\mathbb{Q}$ 4.7

If $a, b \in \mathbb{R}$ *and* $a < b$*, then there is a rational* $r \in \mathbb{Q}$ *such that* $a < r < b$.

Proof. We will use induction. Our base case is proved through the denseness of $\mathbb{Q}$. Thus, we assume there exist n distinct rationals between a and b. Thus, we have $a < r_n < r_{n-1} < \cdots < r_1 < b$. Since $a < r_n$, we can find another $a < r_{n+1} < r_n$ through denseness of $\mathbb{Q}$. Thus, our proof is complete. □

29 Problem 4.12

Let $\mathbb{I}$ be the set of real numbers that are not rational; elements of $\mathbb{I}$ are called *irrational numbers.* Prove if $a < b$, then there exists $x \in \mathbb{I}$ such that $a < x < b$.

Proof. We first show that $\{r + \sqrt{2} : r \in \mathbb{Q}\} \subseteq \mathbb{I}$. Assume by contradiction that $r + \sqrt{2} \notin \mathbb{I}$. Then that means

$$\begin{aligned} \frac{p}{q} + \sqrt{2} &= \frac{a}{b} \\ \sqrt{2} &= \frac{a}{b} - \frac{p}{q} \\ \sqrt{2} &= \frac{x}{y} \end{aligned}$$

Thus, we imply that $\sqrt{2} \in \mathbb{Q}$. However, since we know that $\sqrt{2}$ is irrational through the rational root theorem, we have a contradiction. Thus our original statement is true. Now, consider $a < b$. We know $a - \sqrt{2} < b - \sqrt{2}$, and since $r \in \mathbb{Q}$, by denseness of $\mathbb{Q}$ we know there exists $a - \sqrt{2} < r < b - \sqrt{2}$, implying $a < r + \sqrt{2} < b$. Since we know $r + \sqrt{2} \in \mathbb{I}$, our proof is complete. □

30 Problem 4.13

Prove the following are equivalent for all real numbers a, b, c.

(i) $|a - b| < c$

(ii) $b - c < a < b + c$

(iii) $a \in (b - c, b + c)$

Proof. We know $|b| < a$ if and only if $-a < b < a$. Thus, we know $|a-b| < c$ if and only if $-c < a-b < c$ implying $b-c < a < b+c$. Our infinum of a is $b-c$ and supremum of a is $b+c$. Thus, as $a \in (b-c, b+c)$, it is clear to see that a will range but never reach our infinum or supremum. Thus, $b-c < a < b+c$. □

31 Problem 4.14

Let A and B be nonempty bounded subsets of $\mathbb{R}$, and let $A + B$ be the set of all sums $a + b$ where $a \in A$ and $b \in B$.

(a). Prove $\sup(A + B) = \sup A + \sup B$.

(b). Prove $\inf(A + B) = \inf A + \inf B$.

Proof. We know that $a \leq supA$ for all a and $b \leq supB$ for all b. Thus, $a + b \leq supA + supB$. This implies $supA + supB$ is the upper bound of $A + B$ and as we know $a + b \leq sup(A + B)$, we get $sup(A + B) \leq sup(A) + sup(B)$.

We will show that $supA+supB \leq sup(A+B)$. We know $a+b \leq sup(A+B)$, thus $a \leq sup(A+B)-b$ and $b \leq sup(A+B)-a$. This implies $sup(A+B)-b$ is an upper bound for a, thus, $supA \leq sup(A+B)-b$, which implies $b \leq sup(A + B) - sup(A)$. This implies $sup(A + B) - sup(A)$ is an upper bound of b, thus $sup(B) \leq sup(A + B) - sup(A)$, implying $supA + supB \leq sup(A + B)$. Combining the two, we get that $sup(A + B) = supA + supB$. We do the same for inf. □

32 Problem 4.15

Let $a, b \in \mathbb{R}$. Show if $a \leq b + \frac{1}{n}$ for all $n \in \mathbb{N}$, then $a \leq b$.

Proof. Assume by contradiction that if $a \leq b + \frac{1}{n}$ then $a > b$. Thus, $a > b$ implies $a - b > 0$. By the archimedian property, there exists some positive integer n such that $n(a - b) > 1$. This implies $a - b > \frac{1}{n}$. Thus, $a > b + \frac{1}{n}$. However, since this is a contradiction, $a \leq b$. □

33 Problem 4.16

Show $sup\{r \in \mathbb{Q} : r < a\} = a$ for each $a \in \mathbb{R}$.

Proof. Let $x = sup\{r \in \mathbb{Q} : r < a\}$. Since $r < a$, then a is an upper bound of r and therefore $x \leq a$. If $x < a$, then by the denseness of $\mathbb{Q}$ there exists some $r \in \mathbb{Q}$ such that $x < r < a$ which is a contradiction since x is our supremum. Therefore $x \nless a$. $x = a$. □

34 Problem 5.1

Write the following sets in interval notation.
(a). $\{x \in \mathbb{R} : x < 0\}$
(b). $\{x \in \mathbb{R} : x^3 \le 8\}$
(c). $\{x^2 : x \in \mathbb{R}\}$
(d). $\{x \in \mathbb{R} : x^2 < 8\}$

Interval notation is as follows: $(-\infty, 0), (-\infty, 2], [0, \infty), (-2\sqrt{2}, 2\sqrt{2})$.

35 Problem 5.4

Let S be a nonempty subset of $\mathbb{R}$, and let $-S = \{-s : s \in S\}$. Prove $\inf S = -\sup(-S)$.

Proof. We have two cases:

Case 1: $-\infty < infS$. This means S has a lower bound, which was already proved by *Corollary 4.5.*

Case 2: $-\infty = infS$. This means S is unbounded below. Thus, for every $x \in \mathbb{R}$, there exists a $s \in S$ such that $s \le x$, which implies $-x \le -s$, thus the set $-S$ is unbounded above. Thus, $infS = -sup(-S)$. ☐

36 Problem 5.5

Prove $infS \le supS$ for every nonempty subset of $\mathbb{R}$.

Proof. There are four cases:
Case 1: S is bounded. We have proved this in Problem 4.6. *Case 2, 3, 4:* S is unbounded below, above, or both. In all cases, we know that the set $\mathbb{R} \cup (-\infty, \infty)$ satisfies order properties, thus we can claim that $infS \le s \le supS$ for all $s \in S$. ☐

37 Problem 5.6

Let S and T be nonempty subsets of $\mathbb{R}$ such that $S \subseteq T$. Prove $infT \le infS \le supS \le supT$.

Proof. We have previously that $infS \le supS$. Additionally, since $S \subseteq T$, if S is unbounded in anyway, so is T. Since we have the set of $\mathbb{R} \cup (-\infty, \infty)$ satisfying order properties, we can once again apply it similarily like bounded sets. If $S \subseteq T$, if $s \in S$, $s \in T$. Thus, we have that $infT \le infS$ and $supS \le supT$. Therefore we have $infT \le infS \le supS \le supT$. ☐

38 Problem 7.1

Write out the first five terms of the following sequences.
(a). $s_n = \frac{1}{3n+1}$
(b). $b_n = \frac{3n+1}{4n-1}$
(c). $c_n = \frac{n}{3^n}$
(d). $sin(\frac{n\pi}{4})$

(a). $s_1 = \frac{1}{4}, s_2 = \frac{1}{7}, s_3 = \frac{1}{10}, s_4 = \frac{1}{13}, s_5 = \frac{1}{16}$
(b). $b_1 = \frac{4}{3}, b_2 = 1, b_3 = \frac{10}{11}, b_4 = \frac{13}{15}, b_5 = \frac{16}{19}$
(c). $c_1 = \frac{1}{3}, c_2 = \frac{2}{9}, c_3 = \frac{1}{9}, c_4 = \frac{4}{81}, c_5 = \frac{5}{243}$
(d). $\frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}$

39 Problem 7.2

For each sequence in 7.1, determine whether it converges. If it converges, give its limit.

(a). $s_n \to 0$
(b). $b_n \to \frac{3}{4}$
(c). $c_n \to 0$
(d). N/A

40 Problem 8.1

Prove the following:
(a). $lim\frac{(-1)^n}{n} = 0$
(b). $lim\frac{1}{n^{1/3}} = 0$
(c). $lim\frac{2n-1}{3n+2} = \frac{2}{3}$
(d). $lim\frac{n+6}{n^2-6} = 0$

Proof. (a). Let $\varepsilon > 0$. We want to find an N such that $n > N$ implies $|\frac{(-1)^n}{n} - 0| < \varepsilon$.

$$\left|\frac{(-1)^n}{n} - 0\right| < \varepsilon$$
$$\frac{1}{n} < \varepsilon$$
$$\frac{1}{\varepsilon} < n$$

Let $N = \frac{1}{\varepsilon}$. Now, $n > N$ implies $n > \frac{1}{\varepsilon}$, $|\frac{1}{n}| < \varepsilon$, thus $|\frac{(-1)^n}{n} - 0| < \varepsilon$. Thus, $\lim\frac{-1^n}{n} = 0$. ☐

Proof. (b). Let $\varepsilon > 0$. We want to find an N such that $n > N$ implies $|\frac{1}{n^{1/3}} - 0| < \varepsilon$.

$$\left|\frac{1}{n^{1/3}} - 0\right| < \varepsilon$$
$$\frac{1}{n^{1/3}} < \varepsilon$$
$$\frac{1}{\varepsilon^{1/3}} < n$$

Let $N = \frac{1}{\varepsilon^{1/3}}$. Now, $n > N$ implies $n > \frac{1}{\varepsilon^{1/3}}$, $|\frac{1}{n^{1/3}}| < \varepsilon$, thus $|\frac{1}{n^{1/3}} - 0| < \varepsilon$. Thus, $\lim\frac{1}{n^{1/3}} = 0$. ☐

Proof. (c). Let $\varepsilon > 0$. We want to find an N such that $n > N$ implies $|\frac{2n-1}{3n+2} - \frac{2}{3}| < \varepsilon$.

$$\left|\frac{2n-1}{3n+2} - \frac{2}{3}\right| < \varepsilon$$
$$\left|\frac{2n-1}{3n+2} - \frac{2n+4/3}{3n+2}\right| < \varepsilon$$
$$\left|\frac{-7/3}{3n+2}\right| < \varepsilon$$
$$\frac{7}{9n+6} < \varepsilon$$
$$7 < 9n\varepsilon + 6\varepsilon$$
$$7 - 6\varepsilon < 9n\varepsilon$$
$$\frac{7-6\varepsilon}{9\varepsilon} < n$$

Let $N = \frac{7-6\varepsilon}{9\varepsilon}$. Thus, $n > N$ implies $n > \frac{7-6\varepsilon}{9\varepsilon}$, $|\frac{2n-1}{3n+2} - \frac{2}{3}| < \varepsilon$. Thus, $\lim\frac{2n-1}{3n+2} = \frac{2}{3}$. ☐

Proof. (d). Let $\varepsilon > 0$. We want to find an N such that $n > N$ implies $|\frac{n+6}{n^2-6} - 0| < \varepsilon$. Note for $n > 6$ we have $2n > n + 6$. To get $n^2 - 6 > \frac{1}{2}n^2$, we need $n^2 > 12$, therefore $n > 4$. Thus, for $n > 6$ we have

$$\left|\frac{n+6}{n^2-6} - 0\right| < \frac{4n}{n^2} < \varepsilon$$
$$\frac{4}{n} < \varepsilon$$
$$\frac{4}{\varepsilon} < n$$

Let $N = max\{\frac{4}{\varepsilon}, 6\}$. Thus, $n > N$ implies $n > \frac{4}{\varepsilon}$ or $n > 6$ which implies $\frac{4}{\varepsilon} < n$. Thus, $\lim\frac{n+6}{n^2-6} = 0$. ☐

41 Problem 8.2

Determine the limits of the following sequences, and then prove your claims.
(a). $a_n = \frac{n}{n^2+1}$
(b). $b_n = \frac{7n-19}{3n+7}$
(c). $c_n = \frac{4n+3}{7n-5}$
(d). $d_n = \frac{2n+4}{5n+2}$
(e). $s_n = \frac{sin(n)}{n}$

Proof. (a). We note

$$\frac{n}{n^2+1} < \frac{n}{n^2} < \frac{1}{n}$$

As $\frac{1}{n}$ converges to 0, $\lim a_n = 0$. Let $\varepsilon > 0$. We find an N such that $n > N$ implies $|\frac{n}{n^2+1} - 0| < \varepsilon$. We note for all $n \in R, n^2 + 1 > n^2$.

$$\left|\frac{n}{n^2+1} - 0\right| < \frac{1}{n} < \varepsilon$$
$$\frac{1}{\varepsilon} < n$$

Let $N = \frac{1}{\varepsilon}$. Thus, $n > N$ implies $n > \frac{1}{\varepsilon}, \varepsilon > |\frac{n}{n^2+1} - 0|$. Thus, $\lim \frac{n}{n^2+1} = 0$. □

Proof. (b). We have $b_n = \frac{7n-19}{3n+7}$. Our limit $lim_{n\to\infty} \frac{7n-19}{3n+7} = lim_{n\to\infty} \frac{7-\frac{19}{n}}{3+\frac{7}{n}} = \frac{7}{3}$. Let $\varepsilon > 0$. We want to find an N such that $n > N$ implies $|\frac{7n-19}{3n+7} - \frac{7}{3}| < \varepsilon$.

$$\left|\frac{7n-19}{3n+7} - \frac{7}{3}\right| < \varepsilon$$
$$\left|\frac{21n-57}{3(3n+7)} - \frac{7(3n+7)}{3(3n+7)}\right| < \varepsilon$$
$$\left|\frac{-176}{3(3n+17)}\right| < \varepsilon$$
$$\frac{176}{3(3n+17)} < \varepsilon$$
$$\frac{176-51\varepsilon}{9\varepsilon} < n$$

Let $N = \frac{176-51\varepsilon}{9\varepsilon}$. Note that $n > N$ implies $|\frac{7n-19}{3n+7} - \frac{7}{3}| < \varepsilon$. Thus, $\lim b_n = \frac{7}{3}$. □

Proof. (c). We have $c_n = \frac{4n+3}{7n-5}$. Our limit $lim_{n\to\infty} \frac{4n+3}{7n-5} = lim_{n\to\infty} \frac{4+\frac{3}{n}}{7-\frac{5}{n}} = \frac{4}{7}$. Let $\varepsilon > 0$. We want to find an N such that $n > N$ implies $|\frac{4n+3}{7n-5} - \frac{4}{7}| < \varepsilon$.

$$\left|\frac{4n+3}{7n-5} - \frac{4}{7}\right| < \varepsilon$$
$$\left|\frac{7(4n+3)}{7(7n-5)} - \frac{4(7n-5)}{7(7n-5)}\right| < \varepsilon$$
$$\left|\frac{41}{7(7n-5)}\right| < \varepsilon$$
$$\frac{41+35\varepsilon}{49\varepsilon} < n$$

Let $N = \frac{41+35\varepsilon}{49\varepsilon}$. $n > N$ implies $|\frac{4n+3}{7n-5} - \frac{4}{7}| < \varepsilon$, thus $\lim c_n = \frac{4}{7}$. □

Proof. (d). We have $d_n = \frac{2n+4}{5n+2}$. Our limit $lim_{n\to\infty} \frac{2n+4}{5n+2} = lim_{n\to\infty} \frac{2+\frac{4}{n}}{5+\frac{2}{n}} = \frac{2}{5}$. Let $\varepsilon > 0$. We want to find an N such that $n > N$ implies $|\frac{2n+4}{5n+2} - \frac{2}{5}| < \varepsilon$.

$$\left|\frac{2n+4}{5n+2} - \frac{2}{5}\right| < \varepsilon$$
$$\left|\frac{5(2n+4)}{5(5n+2)} - \frac{2(5n+2)}{5(5n+2)}\right| < \varepsilon$$
$$\left|\frac{16}{5(5n+2)}\right| < \varepsilon$$
$$\frac{16-10\varepsilon}{25\varepsilon} < n$$

Let $N = \frac{16-10\varepsilon}{25\varepsilon}$. $n > N$ implies $|\frac{2n+4}{5n+2} - \frac{2}{5}| < \varepsilon$, thus $\lim d_n = \frac{2}{5}$. ☐

Proof. (e). We have $s_n = \frac{sin(n)}{n}$. We note $-1 \le sin(n) \le 1$. Since $-\frac{1}{n}$ and $\frac{1}{n}$ both converge to 0, by the squeeze theorem $\frac{sin(n)}{n}$ converges to 0. Let $\varepsilon > 0$. We want to find an N such that $n > N$ implies $|\frac{sin(n)}{n} - 0| < \varepsilon$. We note for all n, $|\frac{sin(n)}{n}| \le \frac{1}{n}$.

$$\left|\frac{sin(n)}{n} - 0\right| \le \left|\frac{1}{n}\right| < \varepsilon$$
$$\frac{1}{\varepsilon} < n$$

Let $N = \frac{1}{\varepsilon}$. $n > N$ implies $|\frac{sin(n)}{n} - 0| < \varepsilon$, thus $\lim s_n = 0$. ☐

42 Problem 8.3

Let (s_n) be a sequence of nonnegative real numbers, and suppose $\lim s_n = 0$. Prove $\lim \sqrt{s_n} = 0$. This will complete the proof for Example 5.

Proof. We want to prove that $\sqrt{s_n} = 0$. Let $\varepsilon > 0$. We want to find an N such that $n > N$ implies $|\sqrt{s_n} - 0| < \varepsilon$. Additionally, we have that $\varepsilon^2 > 0$, such that $|s_n - 0| < \varepsilon^2$. This implies there exists $|\sqrt{s_n} - 0| < \varepsilon$. Thus, $\lim \sqrt{s_n} = 0$. ☐

43 Problem 8.4

Let (t_n) be a bounded sequence, i.e., there exists M such that $|t_n| \le M$ for all n, and let (s_n) be a sequence such that $\lim s_n = 0$. Prove $\lim (s_n t_n) = 0$.

Proof. We want to prove that $s_n t_n = 0$. Thus, we have that $|s_n t_n - 0| \le |s_n M|$ for all n. Let $\varepsilon > 0$. This implies that $\frac{\varepsilon}{M} > 0$. Thus, there exists an N such that $n > N$ implies $|s_n - 0| < \frac{\varepsilon}{M}$. Therefore $|s_n M - 0| < \varepsilon$. Therefore $\lim s_n M = 0$. Thus, we have $0 \le |s_n t_n - 0| \le |s_n M| < \varepsilon$, so $\lim (s_n t_n) = 0$. ☐

44 Problem 8.5

(a). Consider three sequences $(a_n), (b_n)$, and (s_n) such that $a_n \leq s_n \leq b_n$ for all n and lim a_n = lim $b_n = s$. Prove lim $s_n = s$. This is called the "squeeze lemma".
(b). Suppose (s_n) and (t_n) are sequences such that $|s_n| \leq t_n$ for all n and lim $t_n = 0$. Prove lim $s_n = 0$.

Proof. **(a).** Given $a_n \leq s_n \leq b_n$ for all n and lim a_n = lim $b_n = s$. Thus, we want to prove that lim $s_n = s$. Let $\varepsilon > 0$. We want to find some N such that $n > N$ implies

$$|s_n - s| < \varepsilon$$

Now let $\varepsilon > 0$. Since lim $a_n = s$, we have there exists some N_1 such that $n > N_1$ implies

$$\begin{aligned} |a_n - s| &< \varepsilon \\ -\varepsilon < a_n - s &< \varepsilon \\ s - \varepsilon < a_n &< s + \varepsilon \end{aligned}$$

Since lim $b_n = s$, we have that there exists some N_2 such that $n > N_2$ implies

$$\begin{aligned} |b_n - s| &< \varepsilon \\ -\varepsilon < b_n - s &< \varepsilon \\ s - \varepsilon < b_n &< s + \varepsilon \end{aligned}$$

Thus, let $N = max\{N_1, N_2\}$. We have for some $n > N$ that

$$s - \varepsilon < a_n \leq s_n \leq b_n < s + \varepsilon$$

Thus, this implies $s - \varepsilon < s_n < s + \varepsilon$. Thus, we have that $|s_n - s| < \varepsilon$, therefore lim $s_n = s$. $\square$

Proof. **(b).** Given $|s_n| \leq t_n$, we have that $0 \leq |s_n - 0| \leq |t_n - 0|$. Let $\varepsilon > 0$. We have that there is some N such that $n > N$ implies

$$|t_n - 0| < \varepsilon$$

Thus, this implies that

$$|s_n - 0| < \varepsilon$$

Thus, this implies lim $s_n = 0$. $\square$

45 Problem 8.6

Let (s_n) be a sequence in $\mathbb{R}$.
(a). Prove $\lim s_n = 0$ if and only if $\lim |s_n| = 0$.
(b). Observe that if $s_n = (-1)^n$, then $\lim |s_n|$ exists, but $\lim s_n$ does not exist.

Proof. **(a).** Assume $\lim s_n = 0$. Let $\varepsilon > 0$. There exists some N such that $n > N$ implies

$$||s_n| - 0| < \varepsilon$$

Therefore $\lim |s_n| = 0$. Assume $lim|s_n| = 0$. Let $\varepsilon > 0$. Thus, there exists some N such that $n > N$ implies

$$\begin{aligned} ||s_n| - 0| &< \varepsilon \\ ||s_n|| &< \varepsilon \end{aligned}$$

This implies

$$|s_n - 0| < \varepsilon$$

Thus, $\lim s_n = 0$. □

Proof. **(b).** Suppose by contradiction that $\lim s_n$ exists and $\lim (-1)^n = a$. Let $\varepsilon = 1$. Thus, we have

$$\begin{aligned} |(-1)^n - a| &< 1 \\ |1 - a| &< 1 \end{aligned}$$

We can order with the following:

$$\begin{aligned} 2 &= |1 + 1| \\ &= |1 - a + (-1 - a)| \\ &= |1 - a| + |1 + a| \\ &< 1 + 1 \\ &= 2 \end{aligned}$$

However, since $2 \not< 2$, we have a contradiction. Thus, $\lim (-1)^n$ does not converge. □

46 Problem 8.7

Show the following sequences do not converge.
(a). $cos(\frac{n\pi}{3})$
(b). $s_n = (-1)^n n$
(c). $sin(\frac{n\pi}{3})$

Proof. **(a).** Assume $cos(\frac{n\pi}{3})$ converges to a. Let $\varepsilon = 1$. Thus, we have some N such that $n > N$ implies

$$|cos(\frac{n\pi}{3}) - a| < 1$$

Consider $cos(\frac{n\pi}{3}) = 1$ and $cos(\frac{n'\pi}{3}) = -1$. Thus, we have that

$$\begin{aligned} |1 - a| &< 1 \\ |-1 - a| = |1 + a| &< 1 \end{aligned}$$

Thus, we have the following:

$$\begin{aligned} 2 &= |1 + 1| \\ &= |1 - a + (-1 - a)| \\ &= |1 - a| + |1 + a| \\ &< 1 + 1 \\ &= 2 \end{aligned}$$

Thus, as we have reached a contradiction, $cos(\frac{n\pi}{3})$ does not converge. □

Proof. **(b).** Assume for contradiction that $(-1)^n n$ converges, and $\lim s_n = a$. Let $\varepsilon = 1$. Thus, we have some N such that $n > N$ implies

$$|(-1)^n n - a| < 1$$

Consider n even and $n + 2$. Thus, we have that

$$\begin{aligned} |n - a| &< 1 \\ |n + 2 - a| &< 1 \end{aligned}$$

Thus, we have the following:

$$\begin{aligned} 2 &= |1 + 1| \\ &= |n + 2 - a + (-(n - a))| \\ &= |n + 2 - a| + |n - a| \\ &< 1 + 1 \\ &= 2 \end{aligned}$$

Thus, as we have readched a contradiction, the sequence s_n does not converge. □

Proof. **(c).** Assume by contradiction that $sin(\frac{n\pi}{3})$ converges to a. Let $\varepsilon = \frac{\sqrt{3}}{2}$. Thus, we have some N such that $n > N$ implies

$$|sin(\frac{n\pi}{3}) - a| < \frac{\sqrt{3}}{2}$$

Consider n and n' such that $sin(\frac{n\pi}{3}) = \frac{\sqrt{3}}{2}$ and $sin(\frac{n'\pi}{3}) = -\frac{\sqrt{3}}{2}$.

$$\begin{aligned} \left|\frac{\sqrt{3}}{2} - a\right| &< \frac{\sqrt{3}}{2} \\ \left|-\frac{\sqrt{3}}{2} - a\right| = \left|\frac{\sqrt{3}}{2} + a\right| &< \frac{\sqrt{3}}{2} \end{aligned}$$

Thus, we have the following:

$$\begin{aligned}
\sqrt{3} &= \left|\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right| \\
&= \left|\frac{\sqrt{3}}{2} - a + \left(\frac{\sqrt{3}}{2} + a\right)\right| \\
&= \left|\frac{\sqrt{3}}{2} - a\right| + \left|\frac{\sqrt{3}}{2} + a\right| \\
&< \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \\
&= \sqrt{3}
\end{aligned}$$

Thus, since $\sqrt{3} \nless \sqrt{3}$, by contradiction the sequence $sin(\frac{n\pi}{3})$ does not converge. □

47 Problem 8.9

Let (s_n) be a sequence that converges.
(a). Show that if $s_n \geq a$ for all but finitely many n, then $\lim s_n \geq a$.
(b). Show that if $s_n \leq b$ for all but finitely many n, then $\lim s_n \leq b$.
(c). Conclude that if all but finitely many s_n belong to $[a, b]$, then $\lim s_n$ belongs to $[a, b]$.

Proof. **(a).** Assume by contradiction that $\lim s_n = s < a$. Let $\varepsilon = \frac{a-s}{2}$. Then there exists N such that $n > N$ implies

$$|s_n - s| < \varepsilon = \tfrac{a-s}{2}$$

Thus, this implies that

$$\begin{aligned} s_n - s &< \frac{a-s}{2} \\ 2s_n - 2s &< a - s \\ 2s_n &< a + s \\ s_n < \frac{a+s}{2} &< \frac{a+a}{2} = a \end{aligned}$$

Thus, we have that $s_n < a$ for all $n > N$. However, since the set is an infinite set, we have reached a contradiction. Thus, $\lim s_n \geq a$. □

Proof. **(b).** Assume by contradiction that $\lim s_n = s > b$. Let $\varepsilon = \frac{s-b}{2}$. Then there exists N such that $n > N$ implies

$$|s_n - s| < \varepsilon = \tfrac{s-b}{2}$$

Thus, this implies that

$$\begin{aligned} -\varepsilon &< s_n - s \\ -\frac{s-b}{2} &< s_n - s \\ b - s &< 2s_n - 2s \\ b + s &< 2s_n \\ b = \frac{b+b}{2} < \frac{b+s}{2} &< s_n \end{aligned}$$

Therefore $b < s_n$ for all $n > N$. However, since the set is an infinite set, we have reached a contradiction. Thus $\lim s_n \leq b$. □

Proof. **(c).** If all but finitely many s_n belong to $[a, b]$, then we have that $s_n \geq a$ and $s_n \leq b$ for all but finitely many s_n. By part (a) and (b), we have that $\lim s_n \geq a$ and $\lim s_n \leq b$, therefore $\lim s_n$ belongs to $[a, b]$. □

48 Problem 8.10

Let (s_n) be a convergent sequence, and suppose $\lim s_n > a$. Prove there exists a number N such that $n > N$ implies $s_n > a$.

Proof. Assume $\lim s_n > a$. We have that $\lim s_n = a + \delta$. Let $\varepsilon = \frac{\delta}{2}$. Thus, there exists a number N such that $n > N$ implies

$$\begin{aligned} |s_n - (a+\delta)| &< \varepsilon = \frac{\delta}{2} \\ -\frac{\delta}{2} &< s_n - a - \delta \\ \frac{\delta}{2} &< s_n - a \\ \frac{\delta}{2} + a &< s_n \\ a &< s_n \end{aligned}$$

Thus, we have that $s_n > a$. ☐

49 Problem 9.1

Using the limit Theorems 9.2 – 9.7, prove the following.
(a). $\lim \frac{n+1}{n} = 1$.
(b). $\lim \frac{3n+7}{6n-5} = \frac{1}{2}$
(c). $\lim \frac{17n^5+73n^4-18n^2+3}{23n^5+13n^3} = \frac{17}{23}$

Proof. **(a).** $\lim \frac{n+1}{n} = 1$. We can rewrite the limit to the following: $\lim \frac{1+\frac{1}{n}}{1}$.

$$\begin{aligned} lim\left(\frac{n+1}{n}\right) &= lim\left(\frac{1+\frac{1}{n}}{1}\right) \\ &= \frac{lim(1+\frac{1}{n})}{lim(1)} \\ &= \frac{lim(1)+lim(\frac{1}{n})}{lim(1)} = 1 \end{aligned}$$

☐

Proof. **(b).** $\lim \frac{3n+7}{6n-5} = \frac{1}{2}$.

$$\begin{aligned} lim\left(\frac{3n+7}{6n-5}\right) &= lim\left(\frac{3+\frac{7}{n}}{6-\frac{5}{n}}\right) \\ &= \frac{lim(3)+lim(\frac{7}{n})}{lim(6)-lim(\frac{5}{n})} = \frac{1}{2} \end{aligned}$$

☐

Proof. **(c).** $\lim \frac{17n^5+73n^4-18n^2+3}{23n^5+13n^3} = \frac{17}{23}$.

$$lim\left(\frac{17n^5+73n^4-18n^2+3}{23n^5+13n^3}\right) = \frac{lim(17)}{lim(23)} = \frac{17}{23}$$

☐

50 Problem 9.4

Let $s_1 = 1$ and for $n \geq 1$ let $s_{n+1} = \sqrt{s_1 + 1}$.
(a). List the first four terms of (s_n).
(b). It turns out that (s_n) converges. Assume this fact and prove the limit is $\frac{1}{2}(1 + \sqrt{5})$.

(a). $s_1 = 1, s_2 = \sqrt{2}, s_3 = \sqrt{\sqrt{2} + 1}, s_4 = \sqrt{\sqrt{\sqrt{2} + 1}}$.
(b). Assume s_n converges. Thus, we have for all n that $\lim s_n = s = \lim s_{n+1}$. Thus, $s = \sqrt{s + 1}$. $s^2 = s + 1, s^2 - s - 1 = 0$. Thus, solving the roots of s gives us $s = \frac{1 \pm \sqrt{5}}{2}$. Since we know $s_1 = 1$ and s_n is an increasing sequence by induction, we show that the limit converges to $\frac{1}{2}(1 + \sqrt{5})$.

51 Problem 9.6

Let $x_1 = 1$ and $x_{n+1} = 3x_n^2$ for $n \geq 1$.
(a). Show that if $a = \lim x_n$, then $a = \frac{1}{3}$ or $a = 0$.
(b). Does $\lim x_n$ exist? Explain.
(c). Discuss the apparent contradiction between parts (a) and (b).

(a). Assume $\lim x_n = a$. Thus, we have that $\lim x_{n+1} = \lim 3x_n^2$. $a = 3a^2$. $3a = 1$. Thus, we have that $a = 0$ or $a = \frac{1}{3}$.

Proof. (b). We use induction to prove the following. Our claim is that $x_n > n$ for all $n \geq 2$.
Base Case: $n = 2$. $3 > 2$.
Induction:

$$\begin{aligned} x_n &> n \\ x_n x_n &> n x_n > n \\ 3x_n^2 &> 3n > n + 1 \end{aligned}$$

Thus, $x_n > n$ for all $n \geq 2$. Now Let $\varepsilon > 0$. Let $N = \varepsilon$. $n > N$ implies

$$\begin{aligned} x_n &> n > \varepsilon \\ x_n &> \varepsilon \end{aligned}$$

Thus, $\lim x_n = \infty$, and therefore does not converge. □

(c). The difference is obvious.

52 Problem 9.9

Suppose there exists N_0 such that $s_n \leq t_n$ for all $n > N_0$.
(a). Prove that if $\lim s_n = \infty$, then $\lim t_n = \infty$.

Proof. Let $\varepsilon > 0$. $N_1 = \varepsilon$ such that $n > N_1$ implies

$$s_n > \varepsilon$$

Let $N = max\{N_0, N_1\}$. $n > N$ implies

$$\begin{aligned} t_n &\geq s_n \geq \varepsilon \\ t_n &\geq \varepsilon \end{aligned}$$

Thus, $\lim t_n = \infty$. □

53 Problem 9.10

(a). Show that if $\lim s_n = \infty$ and $k > 0$, then $\lim (ks_n) = \infty$.
(b). Show $\lim s_n = \infty$ if and only if $\lim (-s_n) = -\infty$.
(c). Show that if $\lim s_n = \infty$ and $k < 0$, then $\lim (ks_n) = -\infty$.

Proof. (a). Assume $\lim s_n = \infty$. Let $\varepsilon > 0$. We pick $N = \varepsilon$ such that $n > N$ implies

$$\begin{aligned} s_n &> \frac{\varepsilon}{k} \\ ks_n &> \varepsilon \end{aligned}$$

Thus, $\lim ks_n = \infty$. □

Proof. (b). Assume $\lim s_n = \infty$. Thus, we have $\varepsilon < 0$. Thus we have $N = \varepsilon$ such that $n > N$ implies

$$\begin{aligned} s_n &> -\varepsilon \\ -s_n &< \varepsilon \end{aligned}$$

Thus, $\lim -s_n = -\infty$.
Assume $\lim (-s_n) = -\infty$. Thus, we have $\varepsilon > 0$. We have $N = \varepsilon$ such that $n > N$ implies

$$\begin{aligned} -s_n &< -\varepsilon \\ s_n &> \varepsilon \end{aligned}$$

Thus, $\lim s_n = \infty$. □

Proof. (c). Let $\varepsilon < 0$. We pick $N = \varepsilon$ such that $n > N$ implies

$$\begin{aligned} s_n &> \frac{\varepsilon}{k} \\ ks_n &< \varepsilon \end{aligned}$$

Thus, $\lim ks_n = -\infty$. □

54 Problem 9.11

(a). Show that if $\lim s_n = \infty$ and $\inf\{t_n : n \in \mathbb{N}\} > -\infty$, then $\lim (s_n + t_n) = \infty$.
(b). Show that if $\lim s_n = \infty$ and $\lim t_n > -\infty$, then $\lim (s_n + t_n) = \infty$.
(c). Show that if $\lim s_n = \infty$ and if (t_n) is a bounded sequence, then $\lim (s_n + t_n) = \infty$.

Proof. (a). Define $a = inf\{t_n : n \in \mathbb{N}\}$. We have that $a > -\infty$. Thus we know $s_n + t_n \geq s_n + a$. Thus, pick $\varepsilon > 0$. We pick N such that $n > N$ implies

$$\begin{aligned} s_n &> \varepsilon - a \\ s_n + a &> \varepsilon \\ s_n + t_n &> \varepsilon \end{aligned}$$

Thus, $\lim (s_n + t_n) = \infty$. □

Proof. (b). $\lim t_n > -\infty$ implies that there exists some $a \in \mathbb{R}$ such that $t_n \geq a$. Thus, t_n is bounded below. Thus, our proof follows part (a). □

Proof. (c). Same as (a). □

55 Problem 9.12

Assume all $s_n \neq 0$ and that the limit $L = \lim |\frac{s_{n+1}}{s_n}|$ exists.
(a). Show that if $L < 1$, then $\lim s_n = 0$.
(b). Show that if $L > 1$, then $\lim |s_n| = \infty$.

Proof. (a). Note if $L < 1$, we have $|\frac{s_{n+1}}{s_n}| < 1$, and thus s_n is a decreasing sequence. Assume $L < 1$. Pick $a \in (L, 1)$. Thus, there exists $|\frac{s_{n+1}}{s_n}| < a$. This implies $|s_{n+1}| < a|s_n|$. Since s_n is decreasing, we know that $|s_n| < a^{n-N}|s_N|$ for $n > N$. Thus, for $|a| < 1$ we have that a^n converges to 0. Let $\varepsilon > 0$. Thus, we have M such that $n - N > M$ implies

$$a^{n-N} < \frac{\varepsilon}{|s_N|}$$
$$a^{n-N}|s_N| < \varepsilon$$
$$|s_n - 0| < \varepsilon$$

Thus, $\lim s_n = 0$. □

Proof. (b). Assume $s_n \neq 0$. Thus, we have that $\lim \frac{1}{s_n} = \frac{1}{s}$. Thus, we say that $\lim \frac{s_n}{s_{n+1}} = \frac{1}{L}, \frac{1}{L} < 1$. Thus, we pick $a \in (\frac{1}{L}, 1)$. There exists N such that $n \geq N$ implies

$$\left|\frac{s_n}{s_{n+1}}\right| < a$$
$$\frac{1}{|s_{n+1}|} < \frac{a}{|s_n|}$$
$$\frac{1}{|s_n|} < \frac{a^{n-N}}{|s_N|}$$

Let $\varepsilon > 0$. Thus, we have M such that $n - N > M$ implies

$$a^{n-N} < \varepsilon \cdot |s_N|$$
$$\frac{a^{n-N}}{|s_N|} < \varepsilon$$
$$\left|\frac{1}{|s_N|} - 0\right| < \varepsilon$$

Thus $\lim |s_n| = \infty$. □

56 Problem 9.13

Show $lim_{n\to\infty} a^n = \infty$ if $a > 1$.

Proof. By theorem 9.7 we know that limit $M^{\frac{1}{n}} = 0$. Thus, there exists N such that $n > N$ implies

$$M^{\frac{1}{n}} < a$$
$$M < a^n$$

Thus, $\lim (a^n) = \infty$. □

57 Problem 10.2

Prove Theorem 10.2 for bounded decreasing sequences.

Theorem 10.2

All bounded monotone sequences converge.

Proof. Let s_n be a bounded decreasing sequence. Thus the set $\{s_n : n \in \mathbb{N}\}$ is bounded. Thus, there exists an infinum and let $L = inf\{s_n : n \in \mathbb{N}\}$. Let $\varepsilon > 0$. Thus, there exists some $s_N < L + \varepsilon$ and since s_n is decreasing, $n > N$ implies

$$s_n < L + \varepsilon$$
$$L \leq s_n < L + \varepsilon$$
$$|s_n - L| < \varepsilon$$

Thus, s_n has a limit and converges to L. □

58 Problem 10.5

Prove Theorem 10.4 for unbounded decreasing sequences.

Theorem 10.4

(i) *If* s_n *is an unbounded increasing sequence, then* $\lim s_n = \infty$.
(i) *If* s_n *is an unbounded decreasing sequence, then* $\lim s_n = -\infty$.

Proof. (ii). Let (s_n) be an unbounded decreasing sequence. Let $M > 0$. Since the set $\{s_n : n \in \mathbb{N}\}$ is unbounded and bounded above by s_1, it is unbounded below. Thus, for some M we have $s_N < M$. Thus there exists N such that $n > N$ implies $s_n < s_N < M$. $\lim s_n = -\infty$. □

59 Problem 10.7

Let S be a bounded nonempty subset of $\mathbb{R}$ such that sup$S$ is not in S. Prove there is a sequence (s_n) of points in S such that $\lim s_n = \sup S$.

Proof. Since $supS - \frac{1}{n}$ is not an upper bound of S and S is nonempty, there exists $x \in S$ such that

$$supS - \frac{1}{n} < x < supS$$

Since x can be defined in S, we know that $lim_{n\to\infty}(supS - \frac{1}{n}) = supS$. □

60 Problem 10.9

Let $s_1 = 1$ and $s_{n+1} = (\frac{n}{n+1})s_n^2$ for $n \geq 1$.
(a). Find s_2, s_3, s_4.
(b). Show $\lim s_n$ exists.
(c). Prove $\lim s_n = 0$.

(a). $s_2 = \frac{1}{2}, s_3 = \frac{1}{6}, s_4 = \frac{1}{48}$.

Proof. (b). We claim s_n is decreasing. It is clear $s_2 < s_1$. Thus, we assume $s_{n+1} < s_n$.

$$\begin{aligned} s_{n+2} &= \frac{n+1}{n+2} s_{n+1}^2 \\ &< s_{n+1} \end{aligned}$$

Thus is as $\frac{n+1}{n+2} < 1$ and $s_{n+1} < 1$, thus we can simplify to s_{n+1}. Thus, the sequence is decreasing. □

Proof. (c). We know $\lim s_n = s$. Thus, we have that $s = lim(\frac{n}{n+1}) \cdot lim(s_n)^2 = 1 \cdot s^2$. Thus, we get that $s = 0$ or $s = 1$. Since $s_n < 1$, we get that $s = 0$, therefore $\lim s_n = 0$. □

61 Problem 11.1

Let $a_n = 3 + 2(-1)^n$ for $n \in \mathbb{N}$.
(a). List the first eight terms of the sequence (a_n).
(b). Give a subsequence that is constant [takes a single value]. Specify the selection function σ.

(a). $a_n = 1, 5, 1, 5, 1, 5, 1, 5, 1, 5$
(b). $s_{n_k} = (1, 1, 1, 1, \ldots)$ where $\sigma(k) = 2k - 1$.

62 Problem 11.2

Consider the sequences defined as follows:

$$a_n = (-1)^n,\ b_n = \frac{1}{n},\ c_n = n^2,\ d_n = \frac{6n+4}{7n-3}$$

(a). For each sequence, give an example of a monotone subsequence.
(b). For each sequence, give its set of subsequential limits.
(c). For each sequence, give its lim sup and lim inf.
(d). Which of the sequences converge? diverges to $+\infty$? diverges to $-\infty$?
(e). Which of the sequences is bounded?

(a). $a_n = (1, 1, 1, \ldots)$ where $\sigma(k) = 2k$. The rest are monotonic intrinsically.
(b). $a_n : \{1, -1\}, b_n : \{0\}, c_n : \{\infty\}, d_n : \{6/7\}$
(c). a_n : lim sup = 1, lim inf = -1. The rest are trivial.
(d). Trivial.
(e). a_n, b_n, d_n are all bounded.

63 Problem 11.6

Show every subsequence of a subsequence of a given sequence is itself a subsequence of the given sequence. *Hint:* Define subsequences as in (3) of Definition 11.1.

Definition 11.1

$$t_k = t(k) = s \circ \sigma(k) = s(\sigma(k)) = s(n_k) = s_{n_k} \text{ for } k \in \mathbb{N}. \quad (3)$$

Proof. Let s_n define a sequence and t_k define a subsequence of s_n defined as

$$(t_k)_{k \in \mathbb{N}} = s(\sigma_1(k)) = s(n_k) = s_{n_k}$$

where σ_1 is an increasing function from $\mathbb{N} \to \mathbb{N}$. We now define a_l to be a subsequence of (t_k) as

$$(a_l)_{l \in \mathbb{N}} = \sigma_1(\sigma_2(l)) = \sigma_1(k_l) = n_{k_l}$$

We have that σ is an increasing mapping from $\mathbb{N} \to \mathbb{N}$ so we can redefine a_l as

$$(a_l)_{l \in \mathbb{N}} = s(\sigma(l)) = s(n_{k_l}) = s_{n_{k_l}}$$

Thus, we have that a_l is also a subsequence of s_n. □

64 Problem 11.7

Let (r_n) be an enumeration of the set $\mathbb{Q}$ of all rational numbers. Show there exists a subsequence (r_{n_k}) such that $\lim_{k \to \infty} r_{n_k} = \infty$.

Theorem 11.2(ii)

If the sequence (s_n) is unbounded above, it has a subsequence with limit ∞.

Proof. We use Theorem 11.2(ii) to prove this claim. Pick $M > 0$. By denseness of $\mathbb{Q}$, we know there exists some r_n inside of $(M, M+1)$. Thus, r_n is unbounded above. Therefore, there exists some subsequence of r_n such that $lim_{k \to \infty} r_{n_k} = +\infty$. □

65 Problem 11.8

Use Definition 10.6 and Exercise 5.4 to prove $\lim\inf s_n = -\lim\sup(-s_n)$ for every sequence (s_n).

Definition 10.6

Let s_n be a sequence in $\mathbb{R}$. We define

$$\lim\sup s_n = \lim_{N\to\infty} \sup\{s_n : n > N\}$$

and

$$\lim\inf s_n = \lim_{N\to\infty} \inf\{s_n : n > N\}$$

Exercise 5.4

For any non empty subset S of $\mathbb{R}$,

$$\inf S = -\sup(-S)$$

Proof. We have that

$$inf\{s_n : n > N\} = -sup\{-s_n : n > N\}$$
$$\lim\inf s_n = \lim_{N\to\infty} \inf\{s_n : n > N\} = \lim_{N\to\infty} -\sup\{-s_n : n > N\} = -\lim\sup(-s_n)$$

□

66 Problem 11.9

(a). Show the closed interval $[a, b]$ is a closed set.
(b). Is there a sequence (s_n) such that $(0, 1)$ is its set of subsequential limits?

Proof. (a). Let (r_n) be the enumeration of rational numbers in (a, b). Thus, we have that for any $t \in [a, b]$, there exist an infinite number of rationals in between $(t - \varepsilon, t + \varepsilon)$ for any $\varepsilon > 0$ by denseness of $\mathbb{Q}$. Thus, we can describe t to be a sequential limit of some subsequence of (r_n) where the set of subsequential limits of r_n is $[a, b]$, thus it is a closed set. □

Proof. (b). By Theorem 11.9, we have that for any sequence t_n in $S \cap \mathbb{R}$ where S is the set of subsequential limits, $\lim t_n = t$ where t belongs to S. Thus, we have that S is closed for any s_n. □

67 Problem 11.11

Let S be a bounded set. Prove there is an increasing sequence s_n of points in S such that $\lim s_n = \sup S$.

Proof. If $\sup S$ is in S, it is sufficient to define $s_n = supS$ for all n. Thus, we consider when $\sup S$ is not in S. Thus, we define $s = supS$. By denseness of $\mathbb{Q}$, we have that there exist infinitely many rationals in the interval $(s - \varepsilon, s)$ for all $\varepsilon > 0$. Since we have that s is our $\sup S$, all values within the interval exist in S. Thus, we construct a subsequence of r_n values converging to s. Since a subsequence can only converge to a limit if its sequence is monotonic, we have that $\lim s_n = \sup S$. □

68 Problem 12.1

Let (s_n) and (t_n) be sequences and suppose there exists N_0 such that $s_n \leq t_n$ for all $n > N_0$. Show $\liminf s_n \leq \liminf t_n$ and $\limsup s_n \leq \limsup t_n$.

Proof. We define
$$a_N = \inf\{s_n : n > N\}$$
and
$$b_N = \inf\{t_n : n > N\}$$
Since we have that $s_n \leq t_n$, $n > N_0$ implies
$$a_N \leq b_N$$
Thus,
$$lima_N \leq limb_N$$
Since $\lim a_N = \lim_{N\to\infty} \inf\{s_n : n > N\} = \liminf s_n$ and $\lim b_N = \lim_{N\to\infty} \inf\{t_n : n > N\} = \liminf t_n$, we have
$$\liminf s_n \leq \liminf t_n$$
We can do the same for sup to prove $\limsup s_n \leq \limsup t_n$. □

69 Problem 12.2

Prove $\limsup |s_n| = 0$ if and only if $\lim s_n = 0$.

Proof. Assume $\limsup |s_n| = 0$. Thus, we have that $\limsup s_n = \lim_{N\to\infty} v_N$ where $v_N = sup\{s_n : n > N\}$. Thus, it is true that

$$\begin{aligned} \limsup|s_n| &= -\liminf| - s_n| \\ &= -\liminf|s_n| \\ &= 0 \end{aligned}$$

Thus, it is shown that $\lim |s_n| = 0$, thus implying $\lim s_n = 0$.
Assume $\lim s_n = 0$. Thus, we have that $\limsup s_n = \lim s_n$, thus implying $\limsup |s_n| = 0$. □

70 Problem 12.4

Show lim sup $(s_n + t_n) \leq$ lim sup s_n+ lim sup t_n for bounded sequences (s_n) and (t_n). *Hint:* First show

$$\sup\{s_n + t_n : n > N\} \leq \sup\{s_n : n > N\} + \sup\{t_n : n > N\}$$

Proof. We have that for all $n > N$,

$$s_n \leq sup\{s_n : n > N\}$$

and

$$t_n \leq sup\{t_n : n > N\}$$

Thus, it is true that

$$s_n + t_n \leq sup\{s_n : n > N\} + sup\{t_n : n > N\}$$

Therefore, $sup\{s_n : n > N\} + sup\{t_n : n > N\}$ is an upper bound of $s_n + t_n$. Thus, we have that

$$sup\{s_n + t_n : n > N\} \leq sup\{s_n : n > N\} + sup\{t_n : n > N\}$$

Thus, it is true that

$$\text{lim sup } (s_n + t_n) \leq \text{lim sup } s_n + \text{ lim sup } t_n$$

□

71 Problem 12.5

Prove

$$\text{lim inf } (s_n + t_n) \geq \text{lim inf } s_n + \text{ lim inf } t_n$$

for bounded sequences (s_n) and (t_n).

Proof. We have that

$$s_n \geq inf\{s_n : n > N\}$$

and

$$t_n \geq inf\{t_n : n > N\}$$

Thus,

$$s_n + t_n \geq inf\{s_n : n > N\} + inf\{t_n : n > N\}$$

implying that $inf\{s_n : n > N\} + inf\{t_n : n > N\}$ is a lower bound of $s_n + t_n$. Therefore,

$$inf\{s_n + t_n : n > N\} \geq inf\{s_n : n > N\} + inf\{t_n : n > N\}$$

Thus, it is true that

$$\text{lim inf } (s_n + t_n) \geq \text{lim inf } s_n + \text{ lim inf } t_n$$

□

Proof. Similarily, we know from Problem 12.4 that

$$\text{lim sup } (s_n + t_n) \leq \text{lim sup } s_n + \text{ lim sup } t_n$$

implying

$$-\text{lim sup } (-(s_n + t_n)) \geq -\text{lim sup } (-s_n) - \text{lim sup } (-t_n)$$

implying

$$\text{lim inf } (s_n + t_n) \geq \text{lim inf } (s_n) + \text{ lim inf } (t_n)$$

□

72 Problem 12.10

Prove (s_n) is bounded if and only if lim sup $|s_n| < +\infty$.

Proof. Assume (s_n) to be bounded. Thus, there exists some $M > 0$ such that $|s_n| \leq M$ for all n. Therefore, we have that $sup\{|s_n| : n \in N\} \leq M < +\infty$, implying

$$\text{lim sup } |s_n| < +\infty$$

Assume lim sup $|s_n| < +\infty$. Then there exists some $M > 0$ such that lim sup $|s_n| < M$. Thus, we define a sequence $v_n = \sup\{|s_n| : n > N\}$ that converges to M, so there exists $n > N_1$ implying

$$|sup\{|s_n| : n > N_1\} - M| < 1$$

implying

$$|s_n| < M + 1$$

for all $n > N$. Thus, we pick $N_{max} = \{|s_1|, |s_2|, \ldots, |s_N|, M + 1\}$ such that

$$|s_n| \leq N_{max}$$

implying that (s_n) is bounded. □

73 Problem 12.11

Prove the first inequality of Theorem 12.2.

Theorem 12.2

Let (s_n) be any sequence of nonzero real numbers. Then we have

$$\liminf \left|\frac{s_{n+1}}{s_n}\right| \leq \liminf |s_n|^{1/n} \leq \limsup |s_n|^{1/n} \leq \limsup \left|\frac{s_{n+1}}{s_n}\right|$$

Proof. We will prove the first inequality. Let $\alpha = \liminf |s_n|^{1/n}$ and $L = \liminf |\frac{s_{n+1}}{s_n}|$. We need to prove $\alpha \geq L$. This is obvious if $L = -\infty$, so we assume $L > -\infty$. It is enough to show

$$\alpha \geq L_1 \text{ for any } L_1 < L$$

Since

$$L = \liminf |\tfrac{s_{n+1}}{s_n}| = \lim_{N \to \infty} \inf \{|\tfrac{s_{n+1}}{s_n}| : n > N\} > L_1,$$

there exists a positive integer N such that

$$L_1 < \inf\{|\tfrac{s_{n+1}}{s_n}| : n \geq N\}$$

which implies

$$L_1 < |\tfrac{s_{n+1}}{s_n}| \text{ for } n \geq N$$

Now for $n > N$ we can write

$$|s_n| = |\tfrac{s_n}{s_{n-1}}| \cdot |\tfrac{s_{n-1}}{s_{n-2}}| \cdots |\tfrac{s_{N+1}}{s_N}| \cdot |s_N|$$

As there are $n - (N+1) + 1 = n - N$ fractions, we can simplify to

$$|s_n| > (L_1)^{n-N} |s_N| \text{ for } n > N$$

Since N and L_1 are fixed, $a = L_1^{-N} \cdot |s_N|$ is a positive constant and we get that

$$|s_n| > L_1^n a \text{ for } n > N.$$

Therefore it exists that

$$|s_n|^{1/n} > L_1 a^{1/n} \text{ for } n > N.$$

Since $lima^{1/n} = 1$, we can further simplify to

$$L_1 < |s_n|^{1/n}$$

implying

$$\liminf |\tfrac{s_{n+1}}{s_n}| \leq \liminf |s_n|^{1/n}$$

$\square$

74 Problem 12.12

Let (s_n) be a sequence of nonnegative numbers, and for each n define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$.
(a). Show

$$\text{lim inf } s_n \le \text{lim inf } \sigma_n \le \text{lim sup } \sigma_n \le \text{lim sup } s_n$$

Hint: For the last inequality, show first that $M > N$ implies

$$\sup\{\sigma_n : n > M\} \le \tfrac{1}{M}(s_1 + s_2 + \cdots + s_N) + \sup\{s_n : n > N\}.$$

(b). Show that if lim s_n exists, then lim σ_n exists and lim σ_n = lim s_n.
(c). Give an example where lim σ_n exists, but lim s_n does not exist.

Proof. Note the middle inequality is obvious. Thus, we prove the third inequality first. Let $M, N \in \mathbb{R}$ such that $M > N$ implies

$$\begin{aligned}
\sigma_n &= \frac{1}{n}(s_1, s_2, \ldots, s_N) + \frac{1}{n}(s_{N+1}, s_{N+2}, \ldots, s_n) \\
&\le \frac{1}{M}(s_1, s_2, \ldots, s_N) + \frac{1}{n}(s_{N+1}, s_{N+2}, \ldots, s_n) \\
&\le \frac{1}{M}(s_1, s_2, \ldots, s_N) + \frac{n-N}{n}\sup\{s_n : n > N\} \\
&\le \frac{1}{M}(s_1, s_2, \ldots, s_N) + \sup\{s_n : n > N\}
\end{aligned}$$

Thus, we have that $\frac{1}{M}(s_1, s_2, \ldots, s_N) + \sup\{s_n : n > N\}$ is an upper bound of $\{\sigma_n : n > M\}$, implying

$$\sup\{\sigma_n : n > M\} \le \tfrac{1}{M}(s_1, s_2, \ldots, s_N) + \sup\{s_n : n > N\}$$

Thus, we see that

$$\lim_{M\to\infty} \sup\{s_n : n > N\} = (s_1, s_2, \ldots, s_N) \cdot \lim_{M\to\infty} \tfrac{1}{M} + \sup\{s_n : n > N\}$$

which implies

$$\text{lim sup } \sigma_n = \sup\{s_n : n > N\}$$

Thus, we get that

$$\text{lim sup } \sigma_n \le \text{lim sup } s_n$$

Thus, this implies

$$-\text{lim sup}(-\sigma_n) \ge -\text{lim sup}(-s_n)$$

which implies

$$\text{lim inf } (\sigma_n) \ge \text{lim inf } (s_n)$$

Therefore, our proof is complete as we have shown both the first and third inequality. $\square$

75 Problem 12.13

Let (s_n) be a bounded sequence in $\mathbb{R}$. Let A be the set of $a \in \mathbb{R}$ such that $\{n \in \mathbb{N} : s_n < a\}$ is finite, i.e. all but finitely many s_n are $\geq a$. Let B be the set of $b \in \mathbb{R}$ such that $\{n \in \mathbb{N} : s_n > b\}$ is finite. Prove $\sup A = \liminf s_n$ and $\inf B = \limsup s_n$.

Proof. We will prove $\sup A \geq \liminf s_n$ and $\sup A \leq \liminf s_n$. Consider $m < \liminf s_n$. Thus, there exists $N \in \mathbb{N}$ such that $\inf\{s_n : n > N\} > m$, implying $s_n > n$ for $n > N$. Thus, we have that $\{n : s_n < m\}$ is finite and thus $m \in A$ since $m \in A \forall m < \liminf s_n$. Thus, we have $\sup A \geq \liminf s_n$ For any $a \in A$, we have that the set $\{n : s_n < a\}$ is finite. Thus, there exists $N \in \mathbb{N}$ such that $n > N$ implies $s_n > a$. It follows that $\inf\{s_n : n > N\} \geq a$ which implies $\liminf\{s_n : n > N\} \geq a$. Thus, the set is an upper bound of A and $\sup A \leq \inf\{s_n : n > N\}$, therefore proving $\sup A = \liminf s_n$. The proof for B is similar. $\square$

76 Problem 14.1

Determine which of the following series converge. Justify your answers.
(a). $\sum \frac{n^4}{2^n}$
(b). $\sum \frac{2^n}{n!}$
(f). $\sum_{n=2}^{\infty} \frac{1}{logn}$

(a). We have that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^4 \cdot 2^n}{n^4 \cdot 2^{n+1}} = \lim_{n\to\infty} \frac{(n+1)^4}{2n^4} = \frac{1}{2}$. Thus, as $\frac{1}{2} < 1$, the series converges.
(b). $\lim_{n\to\infty} \frac{2^{n+1} n!}{(n+1)! \cdot 2^n} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$. Thus, the series is convergent.
(f). We have that $\frac{1}{logn} \leq \frac{1}{n}$. Thus, since we know the series $\sum \frac{1}{n}$ is divergent, $\sum \frac{1}{logn}$ is divergent as well.

77 Problem 14.5

Suppose $\sum a_n = A$ and $\sum b_n = B$ where A and B are real numbers. Use limit theorems to quickly prove the following.
(a). $\sum(a_n + b_n) = A + B$.
(b). $\sum ka_n = kA$ for $k \in \mathbb{R}$.
(c). Is $\sum a_n b_n = AB$ a reasonable conjecture? Discuss.

Proof. (a). Let us define $S_n = \sum_{k=1}^{n} a_k$ and $S_n^{'} = \sum_{k=1}^{n} b_k$. Thus, we have that $lim_{n\to\infty} S_n = A$ and $lim_{n\to\infty} S_n^{'} = B$. Let $\varepsilon > 0$. There exists some N_1, N_2 respectively such that $n \geq N_1, N_2$

$$|S_n - A| < \tfrac{\varepsilon}{2},\ |S_n^{'} - B| < \tfrac{\varepsilon}{2}$$

Thus, let us pick $N = max\{N_1, N_2\}$. We have that

$$|S_n - A| < \tfrac{\varepsilon}{2},\ |S_n^{'} - B| < \tfrac{\varepsilon}{2}$$

Thus,

$$\begin{aligned}\left|\sum_k (a_k + b_k - (A+B)\right| &= |(S_n - A) + (S_n^{'} - B)| \\ &\leq |S_n - A| + |S_n^{'} - B| \\ &< \varepsilon\end{aligned}$$

Thus, we have that the series $\sum(a_n + b_n)$ converges to $A + B$. □

Proof. (b). Let us define $S_n = \sum_{k=1}^{n} a_k = A$. Let $\varepsilon > 0$. There exists N such that $n \geq N$ implies

$$|S_n - A| < \tfrac{\varepsilon}{k}$$

Thus,

$$\begin{aligned}\left|\sum_n ka_n - kA\right| &= |kS_n - kA| \\ &\leq k|S_n - A| \\ &< k \cdot \frac{\varepsilon}{k} = \varepsilon\end{aligned}$$

Thus, $\sum_k ka_n$ converges to kA. □

(c). We can simply consider two sequences, $a_n = \frac{1}{2^n}$ and $b_n = \frac{1}{4^n}$. Thus, we have that $\sum a_n = 2$ and $\sum b_n = \frac{4}{3}$. However, we have $\sum \frac{1}{8^n} = \frac{8}{7} \neq \frac{8}{3}$.

78 Problem 14.6

(a). Prove that if $\sum |a_n|$ converges and (b_n) is a bounded sequence, then $\sum a_n b_n$ converges.
(b). Observe that Corollary 14.7 is a special case of part (a).

Corollary 14.7

Absolutely convergent series are convergent.

Proof. (a). Assume $\sum |a_n|$ converges and b_n is a bounded sequence. Thus, we know that $\exists M > 0$ s.t. $\forall n, |b_n| \leq M$. Thus, we have that $|a_n b_n| = |a_n| \cdot |b_n| \leq |a_n| \cdot M$. Thus, we have that since $\sum M|a_n| = M \sum |a_n|$ is bounded, $\sum |a_n b_n|$ converges so $\sum a_n b_n$ converges. □

Proof. (b). We set $b_n = 1 \forall n$. We know b_n is bounded, and $\sum a_n b_n$ simplifies to $\sum a_n$, therefore since $|b_n| \leq 1$, Corollary 14.7 is a special case of (a). □

79 Problem 14.7

Prove that if $\sum a_n$ is a convergent series of nonnegative numbers and $p > 1$, then $\sum a_n^p$ converges.

Proof. We use induction to prove the statement. Given $\sum a_n$ is convergent, we know $\lim_{n\to\infty} a_n = 0$, thus, a_n is bounded. Therefore, there exists a real number $M > 0$ such that

$$a_n < M \text{ for all } n \in \mathbb{N}$$

Thus, we have that $a_n^2 < a_n \cdot M$ which implies $\sum a_n^2$ is convergent. Thus, we assume the statement true for $p = k$ and shall prove $k + 1$. We know $a_n^k < M$ for all $n \in \mathbb{N}$, thus $a_n^{k+1} < M \cdot a_n^k$, so by comparison test our proof is complete. □

80 Problem 14.8

Show that if $\sum a_n$ and $\sum b_n$ are convergent series of nonnegative numbers, then $\sum \sqrt{a_n b_n}$ converges. *Hint*: Show $\sqrt{a_n b_n} \leq a_n + b_n$ for all n.

Proof. We are given that $\sum a_n$ and $\sum b_n$ are convergent series. Thus, $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = 0$. We define $S_n = \sum_{k=1}^n a_k$, $S_n' = \sum_{k=1}^n b_k$, $\lim_{n\to\infty} S_n = l$, and $\lim_{n\to\infty} S_n' = l'$. Thus, let $\varepsilon > 0$. There exists $n > N_1$ and $n > N_2$ such that

$$|S_n - l| < \varepsilon, \quad |S_n' - l'| < \varepsilon$$

Thus, we claim $N = max\{N_1, N_2\}$. Thus, we have that

$$\begin{aligned}
\left| \frac{S_n + S_n'}{2} - \frac{l + l'}{2} \right| &= \frac{1}{2}|S_n - l + S_n' - l'| \\
&\leq \frac{1}{2}|S_n - l| + \frac{1}{2}|S_n' - l'| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

Thus, we have that the series $\frac{1}{2}\sum \frac{a_n+b_n}{2}$ is convergent. Thus, through $AM - GM$, we have that $\sqrt{a_n b_n} \leq \frac{a_n+b_n}{2}$, thus, by comparison test $\sum \sqrt{a_n b_n}$ is convergent. □

81 Problem 15.1

Determine which of the following series converge. Justify your answers.
(a). $\sum \frac{(-1)^n}{n}$
(b). $\sum \frac{(-1)^n n!}{2^n}$

(a). Consider $a_n = \frac{1}{n}$. We know the sequence is a monotone decreasing sequence since

$$n + 1 > n \implies \frac{1}{n+1} < \frac{1}{n} \implies a_{n+1} < a_n$$

Thus, as we have $\lim_{n\to\infty} \frac{1}{n} = 0$, the series is convergent. (b). Consider $a_n = \frac{n!}{2^n}$. By ratio test we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!2^n}{n!2^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{n+1}{2} \right| = \infty.$$

Thus, as the series is divergent, we have that $\sum \frac{(-1)^n n!}{2^n}$ is divergent.

82 Problem 15.7

(a). Prove if (a_n) is a decreasing sequence of real numbers and if $\sum a_n$ converges, then $\lim na_n = 0$.
Hint: Consider $|a_{N+1} + a_{N+2} + \cdots + a_n|$ for suitable N.
(b). Use (a) to give another proof that $\sum \frac{1}{n}$ diverges.

Proof. (a). Since $\sum a_n$ converges, we have that for some $\varepsilon > 0$ there exists $M > 0$ s.t.

$$|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| < \frac{\varepsilon}{2} \text{ for all } n \geq m \text{ and for every } p \in \mathbb{N}$$

Consider $n = m$, Since a_n is a decreasing sequence, we have $a_{m+1} + a_{m+2} + \cdots + a_{m+p} \geq pa_{m+p}$. Set $p = m$. Thus, we have $2ma_{2m} < \varepsilon$. Take $p = m + 1$. Thus, $(m + 1)a_{2m+1} < \frac{\varepsilon}{2}$. It follows that

$$(2m + 1)a_{2m+1} < (2m + 2)a_{2m+1} < \varepsilon$$

Therefore $na_n < \varepsilon$ for all $n \geq 2m$, therefore $\lim_{n\to\infty} na_n = 0$. □

(b). $\frac{1}{n}$ is both monotonically decreasing. If we want $\sum \frac{1}{n}$ to converge, then $\lim_{n\to\infty} n \cdot \frac{1}{n} = 1 \neq 0$ must follow, but since it is false, $\sum \frac{1}{n}$ does not converge.

83 Problem 17.1

Let $f(x) = \sqrt{4 - x}$ for $x \leq 4$ and $g(x) = x^2$ for all $x \in \mathbb{R}$.
(a). Give the domains of $f + g, fg, f \circ g$, and $g \circ f$.
(b). Find the values $f \circ g(0), g \circ f(0), f \circ g(1), g \circ f(1), f \circ g(2)$, and $g \circ f(2)$.
(c). Are the functions $f \circ g$ and $g \circ f$ equal?
(d). Are $f \circ g(3)$ and $g \circ f(3)$ meaningful?

(a). We have that $f + g = \sqrt{4 - x} + x^2$. Thus, the domain is $\{x : x \leq 4\}$ since $\text{dom}(f) \cap \text{dom}(g) = \{x : x \leq 4\}$. The domain of fg is same as well. We have that $f \circ g$ is $f(g(x)) = \sqrt{4 - x^2}$. Thus, this is valid when $4 - x^2 \geq 0, x \leq 2$. Thus, our domain is $\{x : |x| \leq 2\}$. For $g \circ f$, we have that $g(f(x)) = (\sqrt{4 - x})^2$. Thus, we have that our domain is $\{x : x \leq 4\}$.
(d). $f \circ g(3)$ is not meaningful because it does not fall within our domain. $g \circ f(3)$ is meaningful because it falls within our domain. Thus, $g \circ f(3) = g(f(3)) = g(1) = 1$.

84 Problem 17.4

Prove the function $\sqrt{x}$ is continuous on its domain $[0, \infty)$.

Proof. Let x_0 be in the domain $[0, \infty)$ and let $\varepsilon > 0$. Thus, we want to show

$$|f(x) - f(x_0)| < \varepsilon$$

provided $|x - x_0| < \delta$. Thus, we observe

$$\begin{aligned} f(x) - f(x_0) &= \sqrt{x} - \sqrt{x_0} \\ &= (\sqrt{x} - \sqrt{x_0}) \cdot \frac{\sqrt{x} + \sqrt{x_0}}{\sqrt{x} + \sqrt{x_0}} \\ &= \frac{x - x_0}{\sqrt{x} + \sqrt{x_0}} \end{aligned}$$

We observe two cases. Case $1: x_0 = 0$. Thus, we let $\delta = \varepsilon^2$. Then $|x - x_0| < \delta$ implies

$$\begin{aligned} f(x) - f(x_0) &= \frac{x - x_0}{\sqrt{x} + \sqrt{x_0}} \\ &= \frac{x - 0}{\sqrt{x} + 0} \\ &= \sqrt{x} \\ &< \varepsilon \end{aligned}$$

Thus, we have that $|f(x) - f(x_0)| < \varepsilon$. Next, consider when $x_0 > 0$. Thus we let $\delta = \sqrt{x_0}\varepsilon$. Then $|x - x_0| < \delta$ implies

$$\begin{aligned} f(x) - f(x_0) &= \frac{x - x_0}{\sqrt{x} + \sqrt{x_0}} \\ &\leq \frac{|x - x_0|}{\sqrt{x_0}} \\ &= \varepsilon \end{aligned}$$

Thus, we have that $|f(x) - f(x_0| < \varepsilon$. □

85 Problem 17.5

(a). Prove that if $m \in \mathbb{N}$, then the function $f(x) = x^m$ is continuous on $\mathbb{R}$.
(b). Prove every *polynomial function* $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is continuous on $\mathbb{R}$.

Proof. (a). Let $f(x) = x^m$. Suppose we have $\lim_{n\to\infty} x_n = x_0$. Thus, we have that

$$\begin{aligned} \lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} [x_n]^m \\ &= x_0^m = f(x_0) \end{aligned}$$

□

Proof. (b). By Theorem 17.4(i) and (a) we have that the polynomial function is continuous on $\mathbb{R}$. □

86 Problem 17.9

Prove each of the following functions in continuous at x_0 by verifying the $\varepsilon - \delta$ property of Theorem 17.2.
(a). $f(x) = x^2, x_0 = 2$;
(b). $f(x) = \sqrt{x}, x_0 = 0$;
(c). $f(x) = x\sin(\frac{1}{x})$ for $x \neq 0$ and $f(0) = 0, x_0 = 0$;
(d). $g(x) = x^3, x_0$ arbitrary. *Hint:* $x^3 - x_0^3 = (x - x_0)(x^2 + x_0 x + x_0^2)$.

Proof. (a). We want to show that $|f(x) - f(x_0)| < \varepsilon$. Thus, we have that

$$\begin{aligned} |f(x) - f(2)| &= |x^2 - 2^2| \\ &= |(x-2)(x+2)| \\ &= |x-2| \cdot |x+2| \\ &\leq |x-2| \cdot (|x| + |2|) \end{aligned}$$

Thus, let $\varepsilon > 0$. $|x - 2| < \delta$ implies

$$\begin{aligned} |x| - |2| \geq |x-2| &< \delta \\ |x| &< \delta + |2| \\ |x| &< \delta + 2 \end{aligned}$$

Thus, we have that

$$\begin{aligned} |f(x) - f(2)| &= |x-2| \cdot |x+2| \\ &\leq |x-2| \cdot (|x| + |2|) \\ &< |x-2| \cdot (\delta + 4) \end{aligned}$$

For $\delta = 1$, we have $|f(x) - f(2)| < |x-2| \cdot (5) = 5|x-2|$. Thus, setting $5|x-2| < \varepsilon$, we get that $|x-2| < \frac{\varepsilon}{5}$. Thus, we have $\delta = min\{1, \frac{\varepsilon}{5}\}$. Therefore $|x-2| < \delta$ implies $|f(x) - f(2)| < \varepsilon$. Therefore the function is continuous at $x_0 = 2$. □

Proof. (b). We have that

$$\begin{aligned} |f(x) - f(0)| &= |\sqrt{x} - \sqrt{0}| \\ &= |\sqrt{x}| \\ &= \sqrt{x} \end{aligned}$$

Thus, we let $\delta = \varepsilon^2$. Therefore $|x - 0| < \delta$ implies $|f(x) - f(0)| < \varepsilon$. Therefore the function is continuous at $x_0 = 0$. □

Proof. (c). We have that

$$\begin{aligned} |f(x) - f(0)| &= |xsin(\frac{1}{x}) - 0| \\ &= |xsin(\frac{1}{x})| \\ &\leq x \end{aligned}$$

Let $\varepsilon > 0$ and $\delta = \varepsilon$. Thus, $|x - 0| < \delta$ implies $|f(x) - f(0)| < \varepsilon$. □

Proof. (d). We have that

$$\begin{aligned} |g(x) - g(x_0)| &= |x^3 - x_0^3| \\ &= |x - x_0| \cdot |x^2 + xx_0 + x_0^2| \end{aligned}$$

Thus, $|x - x_0| < \delta$ implies

$$|x| - |x_0| \leq |x - x_0| < \delta$$
$$|x| < \delta + |x_0|$$

Thus, we have that

$$\begin{aligned} |g(x) - g(x_0)| &= |x - x_0| \cdot |x^2 + xx_0 + x_0^2| \\ &< |x - x_0| \cdot |(|x_0| + \delta)^2 + (|x_0| + 1)x_0 + x_0^2| \\ &= |x - x_0| \cdot N \end{aligned}$$

Thus, for $\delta = 1$, we have that $|g(x) - g(x_0)| < |x - x_0| \cdot N$. Thus, for $|x - x_0| \cdot N < \varepsilon$, we have $|x - x_0| < \frac{\varepsilon}{N}$. Thus, $\delta = \min\{1, \frac{\varepsilon}{N}\}$ has $|x - x_0| < \delta$ implying $|g(x) - g(x_0)| < \varepsilon$. Therefore $g(x)$ is continuous at x_0 arbitrary. $\square$

87 Problem 17.11

Let f be a real valued function with $\mathrm{dom}(f) \subseteq \mathbb{R}$. Prove f is continuous at x_0 if and only if, for every *monotonic* sequence (x_n) in $\mathrm{dom}(f)$ converging to x_0, we have $\lim f(x_n) = f(x_0)$. *Hint:* Don't forget Theorem 11.4.

Theorem 11.4

Every sequence (x_n) has a monotonic subsequence, and if the sequence is convergent, then its subsequence also converges to the same point.

Proof. ($\Longrightarrow$) Assume f is a continuous function at $x_0 \in \mathrm{dom}(f)$. We want to show for every monotonic sequence x_n in $\mathrm{dom}(f)$ converging to x_0, $\lim_{n\to\infty} f(x_n) = f(x_0)$ holds. By Theorem 17.1, we have this statement to be true for every sequence x_n converging to x_0. By Theorem 11.4 we have that every sequence has a monotonic subsequence x_n and if the sequence is convergent then the subsequence also converges to the same point.
($\Longleftarrow$) Suppose for every monotonic sequence x_n in $\mathrm{dom}(f)$ we have that $\lim_{n\to\infty} f(x_n) = f(x_0)$ is true. We claim $f(x)$ is a discontinuous function. Therefore, for all $\varepsilon, \delta > 0$, there exists $y_0 \in \mathrm{dom}(f)$ such that for all y in $\mathrm{dom}(f)$, $|y - y_0| < \delta$ implies $|f(y) - f(y_0)| \geq \varepsilon$. Let $\varepsilon_0 > 0$ and $\delta_n = \frac{1}{n}$ for $n \in \mathbb{N}$. Thus, we find a sequence y_n such that it satisfies our conditions and y_n is convergent towards y_0. However, by Theorem 11.4, we have a contradiction, therefore implying $f(x)$ is continuous. $\square$

88 Problem 17.14

For each nonzero rational number x, write x as $\frac{p}{q}$ where p, q are integers with no common factors and $q > 0$, and then define $f(x) = \frac{1}{q}$. Also define $f(0) = 1$ and $f(x) = 0$ for all $x \in \mathbb{R}\backslash\mathbb{Q}$. Thus $f(x) = 1$ for each integer, $f(\frac{1}{2}) = f(-\frac{1}{2}) = f(\frac{3}{2}) = \cdots = \frac{1}{2}$, *etc.* Show f is continuous at each point of $\mathbb{R}\backslash\mathbb{Q}$ and discontinuou at each point of $\mathbb{Q}$.

Proof. We have the function $f : \mathbb{R} \to \mathbb{R}$ to be defined as

$$\begin{cases} 1 & x = 0 \\ \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q}] \\ 0 & x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$$

Thus, pick $x_0 \in \mathbb{Q}$ arbitrarily. We want to show f is discontinuous at x_0, therefore we let $\varepsilon, \delta > 0$ such that there exists $x \in \operatorname{dom}(f)$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| > \varepsilon$. Thus, pick $\varepsilon = 2|f(x_0)|$. Thus, by denseness of irrationals, we have that there exists an non-rational real number $x \in (x_0 - \delta, x_0 + \delta)$, which for x, $|x - x_0| > \delta$ and $|f(x) - f(x_0)| = |f(x_0)| < 2|f(x_0)| = \varepsilon$. Thus, we have that f is discontinuous at each point $\mathbb{Q}$.
Next pick $x_0 \in \mathbb{R}\backslash\mathbb{Q}$ arbitrarily. We want to show f is continuous in x_0, therefore we let $\varepsilon, \delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$. Thus, we have that $|f(x) - f(x_0)| = |f(x)| < \varepsilon$. Hence f is continuous at each point $\mathbb{R}\backslash\mathbb{Q}$. $\square$

89 Problem 17.16

The postage-stamp function P is defined by $P(x) = A$ for $0 \le x < 1$ and $P(x) = A + Bn$ for $n \le x < n + 1$. The definition of P means that P takes the value A on the interval $[0, 1)$, the value $A + B$ on the interval $[1, 2)$, etc. Show P is discontinuous at every positive integer.

Proof. We have

$$\begin{cases} A & 0 \le x < 1 \\ A + B & 1 \le x < 2 \\ & . \\ & . \\ A + Bn & n \le x < n + 1 \\ & . \\ & . \end{cases}$$

We claim P is discontinuous. Thus, for $x = 1$, we have that

$$\begin{aligned} \lim_{x \to 1^-} P(x) &= \lim_{x \to 1^-} A \\ &= A \end{aligned}$$

$$\begin{aligned} \lim_{x \to 1^+} P(x) &= \lim_{x \to 1^+} A + B \\ &= A + B \end{aligned}$$

We have proved our base. Thus, we assume P to be discontinuous for n, and therefore prove it for $n + 1$.

$$\begin{aligned} \lim_{x \to (n+1)^-} P(x) &= \lim_{x \to (n+1)^-} A + Bn \\ &= A + Bn \end{aligned}$$

$$\begin{aligned} \lim_{x \to (n+1)^+} P(x) &= \lim_{x \to (n+1)^+} A + Bn \\ &= A + B(n + 1) \end{aligned}$$

Thus, we have P to be discontinuous for $n+1$, therefore our function is discontinuous at every positive integer. □

90 Problem 18.1

Let f be as in Theorem 18.1. Show that if f assumes its maximum at $x_0 \in [a, b]$, then f assumes its minimum at x_0.

Proof. Let f be a continuous real-valued function on a closed interval $[a, b]$. We first show f is bounded. Assume f is not bounded on $[a, b]$. Then for each $n \in \mathbb{N}$ there corresponds an $x_n \in [a, b]$ such that $|f(x_n)| > n$. By the Bolzano-Weierstrass Theorem (x_n) has a subsequence (x_{n_k}) that also converges to some real number x_0. Thus, since x_0 is in $[a, b]$, we have that $\lim_{n\to\infty} f(x_{n_k}) = f(x_0) \neq \infty$. As we have a contradiction, f is bounded. Now we let $M = \sup\{f(x) : x \in [a, b]\}$ where M is finite as previously shown. Thus, we claim that the maximum of $-f$ is $x_0 \in [a, b]$ such that $-f(x_0) \leq -f(x)$, implying $f(x_0) \geq f(x)$. Thus, we have that $f(x)$ is the minimum at x_0. □

91 Problem 18.2

Reread the proof of Theorem 18.1 with $[a, b]$ replaced by (a, b). Where does it break down? Discuss.

We have two cases in which the proof breaks down. *Case* 1*:* When showing a function to be bounded. Thus, take $f(x) = \frac{1}{x}$ for example. We have that it is continuous but unbounded on $(0, 1)$. $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{1}{n} = 0 \notin (0, 1)$. *Case* 2*:* $g(x) = x^2$. We have the function to be continuous and bounded on $(-1, 1)$, but it does not contain a max.

92 Problem 18.4

Let $S \subseteq \mathbb{R}$ and suppose there exists a sequence (x_n) in S converging to a number $x_0 \notin S$. Show there exists an unbounded continuous function on S.

Proof. Define function $f(x_n) = \frac{1}{|x_n - x_0|}$. We have the function to be continuous on S. Thus, let $M > 0$. We have there exists N implying $n > N$ such that $|x_n - x_0| < M$. Thus $\frac{1}{|x_n - x_0|} > M$. Thus $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{1}{|x_n - x_0|} = +\infty$. Therefore $f(x_n)$ is continuous and unbounded on S. □

93 Problem 18.5

(a) Let f and g be continuous functions on $[a, b]$ such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Prove $f(x_0) = g(x_0)$ for at least one x_0 in $[a, b]$.

Proof. (a). Let f and g be continuous functions on $[a, b]$. Let $h(x) = f(x) - g(x)$. Thus, on a, we have $h(a) = f(a) - g(a) \geq 0$. On b, we have $h(b) = f(b) - g(b) \leq 0$. Therefore it is true that $h(b) \leq 0 \leq h(a)$. Therefore by IVT there exists some value x_0 in $[a, b]$ such that $h(x_0) = 0$. Therefore we have that $h(x_0) = f(x_0) - g(x_0) = 0$ implying $f(x_0) = g(x_0)$ on some x_0 in $[a, b]$. □

94 Problem 18.9

Prove that a polynomial function f of odd degree has at least one real root. *Hint:* It may help to consider the cases, $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$.

Proof. Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ where $a_n \neq 0$ and n is odd degree. We observe that $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = x^n(a_n + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n})$ Thus, consider when $a_n > 0$. We have that $\lim_{n\to\infty} f(x) = \infty$ and $\lim_{n\to-\infty} f(x) = -\infty$. Thus, there exists some $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) > 0$ and $f(x_2) < 0$. Thus by IVT we have $f(x_2) < 0 < f(x_1)$ implying there is some point $x \in (x_2, x_1)$ such that $f(x) = 0$. We apply the same logic where $a_n < 0$. □

95 Problem 18.10

Suppose f is continuous on $[0, 2]$ and $f(0) = f(2)$. Prove there exist x, y in $[0, 2]$ such that $|y - x| = 1$ and $f(x) = f(y)$. *Hint:* Consider $g(x) = f(x + 1) - f(x)$ on $[0, 1]$.

Proof. We consider $g(x) = f(x + 1) - f(x)$ which is continuous on $[0, 1]$. Thus, we have that $g(0) = f(1) - f(0) = f(1) - f(2) = -[f(2) - f(1)] = -g(1)$. Thus, we consider three cases. *Case* 1*:* $g(0) = 0$ implies $g(1) = 0$, thus, $x = 0$ is a zero of g. *Case* 2*:* $g(0) > 0$ implies $g(1) < 0$. Thus, by IVT this implies there is some x in $[0, 1]$ such that $g(x) = 0$. *Case* 3*:* $g(0) < 0$ implies $g(1) > 0$, thus by IVT there is some $x \in [0, 1]$ such that $g(x) = 0$. In all cases there is some $x \in [0, 1]$ such that $g(x) = 0$. This implies $g(x) = f(x + 1) - f(x) = 0$ implying $f(x + 1) = f(x)$. We set $y = x + 1$. Thus, $|y - x| = |x + 1 - x| = 1$ and $f(y) = f(x)$. □

96 Problem 19.1

Which of the following continuous functions are uniformly continuous on the specified set? Justify your answers. Use any theorems you wish.
(a). $f(x) = x^{17} \sin x - e^x \cos 3x$ on $[0, \pi]$.
(c). $f(x) = x^3$ on $(0, 1)$.
(d). $f(x) = x^3$ on $\mathbb{R}$.
(e). $f(x) = \frac{1}{x^3}$ on $(0, 1]$.
(f). $f(x) = \sin\frac{1}{x^2}$ on $(0, 1]$.

(a). We note that x^{17}, sin$x$, e^x, and cos3x are all continuous on $[0, \pi]$. Thus, the function is uniformly continuous on $[0, \pi]$.
(c). We note that $\tilde{f}$ is an extension of a function f if

$$\text{dom}(f) \subseteq \text{dom}(\tilde{f}) \text{ and } f(x) = f(\tilde{x})$$

for all $x \in \text{dom}(f)$. Thus, let $f(x) = x^3$. We define

$$f(\tilde{x}) = \begin{cases} 0 & , x = 0 \\ x^3 & , x \in (0,1) \\ 1 & , x = 1 \end{cases}$$

as an extension of f. We note that dom$(f) = (0, 1)$ and dom$(\tilde{f}) = [0, 1]$. From part (b) we know that $\tilde{f}$ is continuous on $[0, 1]$, therefore $f(x) = x^3$ is unformly continuous on $(0, 1)$.

Proof. (d). We claim f is not uniformly continuous on $\mathbb{R}$. Thus, we define $\varepsilon = 1$ for any $\delta > 0$ such that there exists $x, y \in \mathbb{R}$ implying

$$|x - y| < \delta \text{ and } |x^3 - y^3| \geq 1$$

We choose $y = x + \frac{\delta}{2}$. Thus, we have that

$$\begin{aligned} |x^3 - y^3| &= \left|x^3 - (x + \frac{\delta}{2})^3\right| \\ &= \frac{\delta^3}{8} + 3x^2\frac{\delta}{2} + 3x\frac{\delta^2}{2} \\ &> \frac{3}{2}x^2\delta \end{aligned}$$

Thus, we know that

$$\begin{aligned} \frac{3}{2}x^2\delta &= 1 \\ x &= \sqrt{\frac{2}{3\delta}} \end{aligned}$$

Thus, for $\delta > 0$ we have the previous true, implying $|x - y| = |x - x - \frac{\delta}{2}| = \frac{\delta}{2} < \delta$ implying $|x^3 - y^3| > \frac{3}{2}x^2\delta = 1 = \varepsilon$. Thus, $f(x) = x^3$ is not uniformly continuous on $\mathbb{R}$. □

Proof. (e). We claim f is not uniformly continuous on $(0, 1]$. Thus, assume the sequence $s_n = \frac{1}{n}, n \in \mathbb{N}$. We know s_n is Cauchy. Since $f(s_n) = n^3$, it is not Cauchy, therefore f cannot be uniformly continuous on $(0, 1]$ by Theorem 19.4. □

Proof. (f). We claim f is not uniformly continuous on $(0, 1]$. We define the sequence $s_n = \frac{1}{\sqrt{\frac{\pi}{2}+n\pi}}$, where s_n is Cauchy. However, we have that $f(s_n) = sin(\frac{\pi}{2}+n\pi)$ where $f(s_n)$ is not Cauchy. Therefore, f is not uniformly continuous on $(0, 1]$ by Theorem 19.4. □

97 Problem 19.2

Prove each of the following functions is uniformly continuous on the indicated set by directly verifying the $\varepsilon - \delta$ property in Definition 19.1.
(a). $f(x) = 3x + 11$ on $\mathbb{R}$.
(b). $f(x) = x^2$ on $[0, 3]$.
(c). $f(x) = \frac{1}{x}$ on $[\frac{1}{2}, \infty)$.

Proof. (a). We have a function to be uniformly continuous on S if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x_0, x \in S, |x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \varepsilon$$

Let $f(x) = 3x + 11$. We have that $3|x - y| = |3x - 3y| = |(3x + 11) - (3y + 11)| = |f(x) - f(y)|$. Thus, we let $\varepsilon > 0$ and $\delta = \frac{\varepsilon}{3}$. This implies $|f(x) - f(y)| = 3|x - y| < \varepsilon$. □

Proof. (b). Let $f(x) = x^2$. We have that $|x^2 - y^2| = |(x - y)(x + y)| \leq 6|x - y|$. Thus, let $\varepsilon > 0$ and $\delta = \frac{\varepsilon}{6}$. $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. □

Proof. (c). Let $f(x) = \frac{1}{x}$. We have that

$$\begin{aligned}
\left|\frac{1}{x} - \frac{1}{y}\right| &= |\frac{y - x}{xy}| \\
&= \frac{x - y}{xy} \\
&\leq 4|x - y|
\end{aligned}$$

Thus let $\varepsilon > 0$ and $\delta = \frac{\varepsilon}{4}$. $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. □

98 Problem 19.3

Repeat Exercise 19.2 for the following.
(a). $f(x) = \frac{x}{x+1}$ on $[0, 2]$
(b). $f(x) = \frac{5x}{2x-1}$ on $[1, \infty)$.

Proof. (a). Let $f(x) = \frac{x}{x+1}$. We have that

$$\begin{aligned}
\left|\frac{x}{x + 1} - \frac{y}{y + 1}\right| &= \left|\frac{x(y + 1) - y(x + 1)}{(x + 1)(y + 1)}\right| \\
&= \left|\frac{x - y}{(x + 1)(y + 1)}\right| \\
&\leq |x - y|
\end{aligned}$$

Thus, let $\varepsilon > 0$ and $\delta = \varepsilon$. $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. □

Proof. (b). Let $f(x) = \frac{5x}{2x-1}$ on $[1, \infty)$. We have that

$$\begin{aligned}
\left|\frac{5x}{(2x - 1)} - \frac{5y}{2y - 1}\right| &= \left|\frac{5x(2y - 1) - 5y(2x - 1)}{(2x - 1)(2y - 1)}\right| \\
&= \left|\frac{10xy - 5x - 10xy - 5y}{(2x - 1)(2y - 1)}\right| \\
&= \left|\frac{5(x - y)}{(2x - 1)(2y - 1)}\right| \\
&\leq 5|x - y|
\end{aligned}$$

Thus, let $\varepsilon > 0$ and $\delta = \frac{\varepsilon}{5}$. We know the rest. □

99 Problem 19.4

(a). Prove that if f is uniformly continuous on a bounded set S, then f is a bounded function on S. *Hint:* Assume not.

Proof. (a). Let f be uniformly continuous on a bounded set S. Assume f is not a bounded function on S. Thus, this implies there exists a sequence $(x_n) \in S$ such that

$$\lim_{n\to\infty} |f(x_n)| = \infty.$$

Thus, since S is bounded, so is x_n. Therefore there is a convergent subsequence (x_{n_k}). Therefore it is Cauchy as well. However, since the limit converges to ∞, this is a contradiction, thus f is a bounded function on S. □

100 Problem 19.5

Which of the following continuous functions is uniformly continuous on the specified set? Justify your answers, using appropriate theorems.
(a). $\tan x$ on $[0, \frac{\pi}{4}]$.
(b). $\tan x$ on $[0, \frac{\pi}{2})$.
(c). $\frac{1}{x}\sin^2 x$ on $(0, \pi]$.
(d). $\frac{1}{x-3}$ on $(0, 3)$.
(f) $\frac{1}{x-3}$ on $(4, \infty)$.

(a). f is continuous on the closed interval, thus is uniformly continuous on $[0, \frac{\pi}{4}]$.
(b). We claim that f is not uniformly continuous on $[0, \frac{\pi}{2})$. We thus check

$$\lim_{x\to\frac{\pi}{2}^-} f(x) = \infty.$$

Thus, this implies f is unbounded on $[0, \frac{\pi}{2})$.
(c). We define the extension function $\tilde{f}$ of f to be

$$\tilde{f}(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{x}\sin^2 x & x \in (0, \pi] \end{cases}$$

Since f is a continuous function, it is uniformly continuous on the specified set.
(d). We claim f is not uniformly continuous on $(0, 3)$. We check

$$\lim_{x\to 3^-} f(x) = -\infty.$$

Thus, we have that f is not uniformly continuous on $(0, 3)$.
(f). Let $f(x) = \frac{1}{x-3}$. We have that

$$\begin{aligned} \left|\frac{1}{x-3} - \frac{1}{y-3}\right| &= \left|\frac{y-3+3-x}{(x-3)(y-3)}\right| \\ &= \frac{|x-y|}{(x-3)(y-3)} \\ &\le |x-y| \end{aligned}$$

Thus, for $\varepsilon > 0$ let $\delta = \varepsilon$. $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Thus, f is uniformly continuous on $(4, \infty)$.

101 Problem 19.6

(a). Let $f(x) = \sqrt{x}$ for $x \geq 0$. Show f' is unbounded on $(0, 1]$ but f is nevertheless uniformly continuous on $(0, 1]$. Compare with Theorem 19.6.
(b). Show f is uniformly continuous on $[1, \infty)$.

Proof. (a). We first show uniform continuity of f. We consider the function $f(x) = \sqrt{x}$ on $x \in (0, 1]$. We define the extension $\tilde{f}$ to be

$$\tilde{f}(x) = \begin{cases} 0 & x = 0 \\ \sqrt{x} & x \in (0, 1] \end{cases}$$

Thus, as $\tilde{f}$ is continuous on $[0, 1]$, we have that f is uniformly continuous on $(0, 1]$. Next we consider $f'(x) = \frac{1}{2\sqrt{x}}$. We have that

$$\lim_{x \to 0^+} f(x) = \infty.$$

Thus, we have that f' is unbounded but f is still uniformly continuous. □

Proof. (b). Let $f(x) = \sqrt{x}$. We have that

$$\begin{aligned} |\sqrt{x} - \sqrt{y}| &= \left| \frac{x - y}{\sqrt{x} + \sqrt{y}} \right| \\ &\leq \frac{|x - y|}{2} \end{aligned}$$

Thus, let $\varepsilon > 0$ and $\delta = 2\varepsilon$. $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. □

102 Problem 19.8

(a). Use the MVT to prove

$$|\sin x - \sin y| \leq |x - y|$$

for all x, y in $\mathbb{R}$; see the proof of Theorem 19.6.
(b). Show $\sin x$ is uniformly continuous on $\mathbb{R}$.

Proof. (a). Let $f(x) = \sin x$. We want to show that $|\sin x - \sin y| \leq |x - y|$. We have by the MVT that

$$\begin{aligned} f'(c) &= \frac{f(x) - f(c)}{x - c} \\ \cos x &= \frac{\sin x - \sin c}{x - c} \\ \frac{\sin x - \sin c}{x - c} &\leq 1 \\ \sin x - \sin c &\leq x - c \end{aligned}$$

Thus, it is true that $|\sin x - \sin y| \leq |x - y|$. □

Proof. Let $f(x) = \sin x$. We have that

$$\begin{aligned} |\sin x - \sin y| &\leq |x - y| \\ &< \delta = \varepsilon \end{aligned}$$

Thus, let $\varepsilon > 0$ and $\delta = \varepsilon$. $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. □

103 Problem 19.12

Let f be a continuous function on $[a,b]$. Show the function f^* defined by $f^*(x) = \sup\{f(y) : a \leq y \leq x\}$, for $x \in [a,b]$, is an increasing continuous function on $[a,b]$.

Proof. We first show f^* to be increasing. Let $x_1, x_2 \in [a,b]$ such that $x_1 \leq x_2$. Thus, we have that

$$\{f(y) : a \leq y \leq x_1\} \subseteq \{f(y) : a \leq y \leq x_2\}$$

and thus

$$f^*(x_1) = \sup\{f(y) : a \leq y \leq x_1\} \leq \sup\{f(y) : a \leq y \leq x_2\} = f^*(x_2)$$

Thus, f^* is increasing. Next, we show f^* is continuous. Let $\varepsilon, \delta > 0$ such that $|x - x_0| < \delta$ implies $|f^*(x) - f^*(x_0)| < \varepsilon$. Let there be $|f(x) - f(x_0)| < \frac{\varepsilon}{2}$. Thus, we have that

$$\begin{aligned} |f^*(x) - f^*(x_0)| &= |\sup\{f(y) : a \leq y \leq x\} - \sup\{f(y) : a \leq y \leq x_1\}| \\ &\leq |\sup\{f(y) - f(x_0) : \min\{x, x_0\} \leq y \leq \max\{x, x_0\}| \\ &\leq \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

Thus, f^* is continuous. □