# Math 131A Practice Problems

Maddox Yu

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## 1 Problem 1.1

Prove $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all positive integers $n$.

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*Proof.* Our proposition $P_n : 1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$. We consider the base case $n = 1$.

$$1^2 = \tfrac{1}{6}(1)(2)(3) = \tfrac{6}{6} = 1.$$

Since our base case is proved, we assume the proposition to be true for $n$. We thus prove it for $n+1$.

$$\begin{aligned}
1^2 + 2^2 + \cdots + n^2 + (n+1)^2 &= \frac{1}{6}(n)(n+1)(2n+1) + (n+1)^2 \\
&= \frac{(n+1)(n(2n+1) + 6(n+1))}{6} \\
&= \frac{(n+1)(2n^2 + n + 6n + 6)}{6} \\
&= \frac{(n+1)(2n^2 + 7n + 6)}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6} \\
&= \frac{(n+1)((n+1)+1)(2(n+1)) + 1}{6}
\end{aligned}$$

Thus, our proof is complete. □

## 2 Problem 1.2

Prove $3 + 11 + \cdots + (8n-5) = 4n^2 - n$ for all positive integers $n$.

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*Proof.* Our proposition $P_n : 3 + 11 + \cdots + (8n-5) = 4n^2 - n$. We consider the base case $n = 1$.

$$3 = 4(1^2) - 1 = 3$$

Since our base case is true, we assume the proposition to be true for $n$. Thus, we prove it for $n+1$.

$$\begin{aligned}
3 + 11 + \cdots + (8n-5) + (8n+3) &= 4n^2 - n + 8n + 3 \\
&= 4n^2 + 8n + 4 - (n+1) \\
&= 4(n^2 + 2n + 1) - (n+1) \\
&= 4(n+1)^2 - (n+1)
\end{aligned}$$

Thus, our proof is complete. □

## 3 Problem 1.6

Prove $(11)^n - 4^n$ is divisible by 7 when $n$ is a positive integer.

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*Proof.* We check our base case $n = 1$.

$$11^1 - 4^1 = 11 - 4 = 7$$

As 7 is divisible by 7, we assume the statement true for $n$. We thus prove it for $n + 1$.

$$\begin{aligned}(11)^{n+1} - 4^{n+1} &= 11 \cdot 11^n - 4 \cdot 4^n \\ &= 4(11^n - 4^n) + 7(11^n)\end{aligned}$$

We already know $11^n - 4^n$ divisible by 7 and $7(11^n)$ is divisible by 7 because of a constant factor of 7 is divisible by 7. Thus, our proof is complete. $\square$

## 4 Problem 1.7

Prove $7^n - 6n - 1$ is divisible by 36 for all positive integers $n$.

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*Proof.* We check our base case $n = 1$.

$$7^1 - 6(1) - 1 = 7 - 6 - 1 = 0$$

0 is divisible by 36, thus we assume the statement true for $n$.

$$\begin{aligned}7^{n+1} - 6(n + 1) - 1 &= 7(7^n) - 6n - 6 - 1 \\ &= 7(7^n - 6n - 1) + 36n\end{aligned}$$

We know $7^n - 6n - 1$ is divisible by 36 and since $36n$ is divisible by $n$, we have proved $n + 1$, thus our proof is complete. $\square$

## 5 Problem 1.8

Prove $n^2 > n + 1$ for all integers $n \geq 2$.

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*Proof.* We check our base case $n = 2$.

$$2^2 > 2 + 1,\ 4 > 3$$

We thus assume the statement true for $n$.

$$\begin{aligned}(n + 1)^2 &> n + 2 \\ n^2 + 2n + 1 &> n + 2 \\ 2n + 1 &> 1\end{aligned}$$

Thus, since $2n + 1 > 1$, we have proved $n + 1$, and our proof is complete. $\square$

# 6 Problem 2.1

Show $\sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{24}, \sqrt{31}$ are not rational numbers.

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We shall use the rational root theorem.

$$\begin{aligned} \alpha &= \sqrt{3} \\ \alpha^2 &= 3 \\ \alpha^2 - 3 &= 0 \end{aligned}$$

Now, we check none of $\frac{div c_0}{div c_n}$ is a root. We have $\pm 1, \pm 3$. Since none are solutions of the equation, $\sqrt{3}$ is not a rational number. We can apply the same logic to similar numbers.

# 7 Problem 2.2

Show $\sqrt[3]{2}$, $\sqrt[7]{5}$, $\sqrt[4]{13}$ are not rational numbers.

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We once again follow the rational root theorem.

$$\begin{aligned} \alpha &= \sqrt[3]{2} \\ \alpha^3 - 2 &= 0 \end{aligned}$$

Our roots are $\pm 1, \pm 2$. Since none of the roots are solutions, $\sqrt[3]{2}$ is not a rational number. We can apply the same logic to similar numbers.

# 8 Problem 2.3

Show $\sqrt{2 + \sqrt{2}}$ is not a rational number.

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We will use the rational root theorem.

$$\begin{aligned} \alpha &= \sqrt{2 + \sqrt{2}} \\ \alpha^2 &= 2 + \sqrt{2} \\ \alpha^2 - 2 &= \sqrt{2} \\ (\alpha^2 - 2)^2 &= 2 \\ \alpha^4 - 4\alpha^2 + 4 &= 2 \\ \alpha^4 - 4\alpha^2 + 2 &= 0 \end{aligned}$$

We check that none of the roots are solutions: $\pm 1, \pm 2$. None of the roots are solutions, so $\sqrt{2 + \sqrt{2}}$ is not a rational number.

## 9 Problem 2.4

Show $\sqrt[3]{5-\sqrt{3}}$ is not a rational number.

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We will use the rational root theorem.

$$\begin{aligned}
\alpha &= \sqrt[3]{5-\sqrt{3}} \\
\alpha^3 &= 5-\sqrt{3} \\
\alpha^3 - 5 &= -\sqrt{3} \\
(\alpha^3-5)^2 &= 3 \\
\alpha^6 - 10\alpha^3 + 25 &= 3 \\
\alpha^6 - 10\alpha^3 + 22 &= 0
\end{aligned}$$

We check the roots: $\pm1, \pm2, \pm11, \pm22$. Since none of the roots are solutions, $\sqrt[3]{5-\sqrt{3}}$ is not a rational number.

## 10 Problem 2.5

Show $[3+\sqrt{2}]^{2/3}$ is not a rational number.

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We use the rational root theorem.

$$\begin{aligned}
\alpha &= \sqrt[3]{(3+\sqrt{2})^2} \\
\alpha^3 &= (3+\sqrt{2})^2 \\
\alpha^3 &= 9+2+6\sqrt{2} \\
\alpha^3 - 11 &= 6\sqrt{2} \\
(\alpha^3-11)^2 &= 72 \\
\alpha^6 - 22\alpha^3 + 121 &= 72 \\
\alpha^6 - 22\alpha^3 + 49 &= 0
\end{aligned}$$

We check the roots: $\pm1, \pm7, \pm49$. None are solutions, so $[3+\sqrt{2}]^{2/3}$ is not a rational number.

## 11 Problem 2.6

Discuss why $4-7b^2$ is rational if $b$ is rational.

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Since $b$ is rational, we can describe $b = \frac{p}{q}$ where $p, q \in \mathbb{Z}$. Thus, we can describe $7b^2 = \frac{7p^2}{q^2} = \frac{m}{n}$ where $m, n \in \mathbb{Z}$. Thus, $4-7b^2 = \frac{4n-m}{n} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$. Thus, $4-7b^2$ is rational.

## 12 Problem 2.7

Show the following irrational-looking expressions are actually rational numbers: $(a)\sqrt{4+2\sqrt{3}}-\sqrt{3}$ and $(b)\sqrt{6+4\sqrt{2}}-\sqrt{2}$.

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We express (a) in terms of $\alpha$.

$$\begin{aligned}
\alpha &= \sqrt{4+2\sqrt{3}}-\sqrt{3}\\
\alpha+\sqrt{3} &= \sqrt{4+2\sqrt{3}}\\
(\alpha+\sqrt{3})^2 &= 4+2\sqrt{3}\\
\alpha^2+2\sqrt{3}\alpha+3 &= 4+2\sqrt{3}\\
\alpha^2+2\sqrt{3}(\alpha-1)-1 &= 0\\
\alpha^2+2\sqrt{3}\alpha-2\sqrt{3}-1 &= 0\\
(\alpha-1)(\alpha+1+2\sqrt{3}) &= 0
\end{aligned}$$

Thus, we check our roots $1, -1-2\sqrt{3}$. Since a negative number is not a viable solution, $\alpha = 1$ is a solution, thus $\sqrt{4+2\sqrt{3}}-\sqrt{3}$ is a rational number.
We express (b) in terms of $\alpha$.

$$\begin{aligned}
\alpha &= \sqrt{6+4\sqrt{2}}-\sqrt{2}\\
\alpha+\sqrt{2} &= \sqrt{6+4\sqrt{2}}\\
(\alpha+\sqrt{2})^2 &= 6+4\sqrt{2}\\
\alpha^2+2\sqrt{2}\alpha+2 &= 6+4\sqrt{2}\\
\alpha^2+2\sqrt{2}\alpha-4-4\sqrt{2} &= 0\\
(\alpha-2)(\alpha+2+2\sqrt{2}) &= 0
\end{aligned}$$

Thus, we check our roots $2, -2-2\sqrt{2}$. Since a negative number is not a viable solution, $\alpha = 2$ is a solution, thus $\sqrt{6+4\sqrt{2}}-\sqrt{2}$ is a rational number.

## 13 Problem 2.8

Find all rational solutions of the equation $x^8-4x^5+13x^3-7x+1=0$.

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Using the rational root theorem, our only viable solutions are $\pm 1$. For $x = 1$, we get

$$1-4+13-7+1 = -3+13-7+1 = 10-7+1 = 3+1 = 4$$

Thus, $x = 1$ is not a solution. For $x = -1$, we get

$$1+4-13+7+1 = 5-13+8 = 0$$

Thus, $x = -1$ is the only rational solution to the equation.

# 14 Problem 3.1

Which of the properties $A1 - A4, M1 - M4, DL, O1 - O5$ fail for $\mathbb{N}$? Which of these properties fail for $\mathbb{Z}$?
**A1.** $a + (b + c) = (a + b) + c$ for all $a, b, c$.
**A2.** $a + b = b + a$ for all $a, b$.
**A3.** $a + 0 = a$ for all $a$.
**A4.** For each $a$, there is an element $-a$ such that $a + (-a) = 0$.
**M1.** $a(bc) = (ab)c$ for all $a, b, c$.
**M2.** $ab = ba$ for all $a, b$.
**M3.** $a \cdot 1 = a$ for all $a$.
**M4.** For each $a \neq 0$, there is an element $a^{-1}$ such that $aa^{-1} = 1$.
**DL.** $a(b + c) = ab + ac$ for all $a, b, c$.
**O1.** Given $a$ and $b$, either $a \leq b$ or $b \leq a$.
**O2.** If $a \leq b$ and $b \leq a$, then $a = b$.
**O3.** If $a \leq b$ and $b \leq c$, then $a \leq c$.
**O4.** If $a \leq b$, then $a + c \leq b + c$.
**O5.** If $a \leq b$ and $0 \leq c$, then $ac \leq bc$.

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**A3.** This fails for $\mathbb{N}$ since $\mathbb{N}$ does not contain the element 0.
**A4.** This fails for $\mathbb{N}$ since $\mathbb{N}$ does not contain negative numbers.
**M4.** This fails for $\mathbb{N}$ since $\mathbb{N}$ does not contain fractions.
**M4.** This fails for $\mathbb{Z}$ since $\mathbb{Z}$ does not contain fractions.

## 15 Problem 3.2

The commutative law **A2.** was used in the proof of $(ii)$ and $(iii)$ in Theorem 3.1. Where?

**Theorem 3.1**

*The following are consequences of the field properties:*
(i) $a + c = b + c$ *implies* $a = b$;
(ii) $a \cdot 0 = 0$ *for all* $a$;
(iii) $(-a)b = -ab$ *for all* $a, b$;
(iv) $(-a)(-b) = ab$ *for all* $a, b$;
(v) $ac = bc$ *and* $c \neq 0$ *imply* $a = b$;
(vi) $ab = 0$ *implies either* $a = 0$ *or* $b = 0$;
*for* $a, b, c \in \mathbb{R}$.

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For $(ii)$, we have

$$\begin{aligned} a \cdot 0 &= a \cdot (0+0) \\ &= a \cdot 0 + a \cdot 0 \\ a \cdot 0 &= a \cdot 0 + 0 \\ &= 0 + a \cdot 0 \\ 0 + a \cdot 0 &= a \cdot 0 + a \cdot 0 \\ a \cdot 0 &= 0 \end{aligned}$$

We used **A2.** in step $3, 4$ where we rearranged $a \cdot 0 + 0 = 0 + a \cdot 0$.
For $(iii)$, we have

$$\begin{aligned} ab + (-a)b &= (a + (-a))b \\ &= 0 \cdot b \\ &= 0 \\ &= ab + (-ab) \\ ab + (-a)b &= ab + (-ab) \\ (-a)b + ab &= (-ab) + ab \\ (-a)b &= -ab \end{aligned}$$

We used **A2.** in step $5, 6$ where we rearranged $ab + (-a)b = (-a)b + ab$ and $ab + (-ab) = (-ab) + ab$.

## 16 Problem 3.3

Prove $(iv)$ and $(v)$ of Theorem 3.1.

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*Proof.* $(iv)$ $(-a)(-b) = ab$ *for all* $a, b$;

$$\begin{aligned}
(-a)(-b) + (-ab) &= (-a)(-b) + (-a)b \\
&= (-a)(-b + b) \\
&= (-a)(0) \\
&= 0 \\
ab + (-ab) &= 0 \\
(-a)(-b) + (-ab) &= ab + (-ab) \\
(-a)(-b) &= ab
\end{aligned}$$

□

*Proof.* $(v)$ $ac = bc$ *and* $c \neq 0$ *implies* $a = b$;

$$\begin{aligned}
a &= a(cc^{-1}) \\
&= (ac)c^{-1} \\
&= (bc)c^{-1} \\
&= b(cc^{-1}) \\
&= b
\end{aligned}$$

□

## 17 Problem 3.4

Prove $(v)$ and $(vii)$ of Theorem 3.2.

**Theorem 3.2**

*The following are consequences of an ordered field:*
(i) *If* $a \leq b$, *then* $-b \leq -a$;
(ii) *If* $a \leq b$ *and* $c \leq 0$, *then* $bc \leq ac$;
(iii) *If* $0 \leq a$ *and* $0 \leq b$, *then* $0 \leq ab$;
(iv) $0 \leq a^2$ *for all* $a$;
(v) $0 < 1$;
(vi) *If* $0 < a$, *then* $0 < a^{-1}$;
(vii) *If* $0 < a < b$, *then* $0 < b^{-1} < a^{-1}$;
*for* $a, b, c \in \mathbb{R}$.

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*Proof.* $(v)$ $0 < 1$;
*We shall prove* $0 < 1$ *for* $0 \leq 1$ *and* $0 \neq 1$. $0 \leq a^2$ *for all* $a$ *implies* $0 \leq 1$.
*Assume* $0 = 1$. *Choose a non-zero real number* $a$.

$$\begin{aligned} a &= a \cdot 1 \\ &= a \cdot 0 \\ &= 0 \end{aligned}$$

*However, since* $a \neq 0$, *we have a contradiction. Thus,* $0 \neq 1$. *Thus,* $0 < 1$. □

*Proof.* $(vii)$ *If* $0 < a < b$, *then* $0 < b^{-1} < a^{-1}$;

$$\begin{aligned} a &< b \\ aa^{-1} &< ba^{-1} \\ 1 &< ba^{-1} \\ bb^{-1} &< ba^{-1} \\ b^{-1}bb^{-1} &< b^{-1}ba^{-1} \\ b^{-1} &< a^{-1} \end{aligned}$$

□

## 18 Problem 3.5

Show $|b| \leq a$ if and only if $-a \leq b \leq a$. Prove $||a| - |b|| \leq |a - b|$ for all $a, b \in \mathbb{R}$.

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*Proof.* We prove both sides of the claim. Assume $|b| \leq a$. There are two cases.

*Case 1:* $b \geq 0$. $b \leq a$. Thus, since $0 \leq b$, $0 \leq a$, then $-b \leq -a$. Thus, $-b \leq a$. Therefore, $-a \leq b$. Thus, $-a \leq b \leq a$.

*Case 2:* $b \leq 0$. We know $0 \leq a$ since $0 \leq |a| \forall a$. Thus, we have $-b \leq a$ so by Theorem 3.2 we have $-a \leq b$. Since $b \leq 0$ and $0 \leq a$, we have $b \leq a$. Thus, $-a \leq b \leq a$.

Assume $-a \leq b \leq a$. Thus, we have $-a \leq b$, and by transitivity $-b \leq a$. We thus have $|b| \leq a$. There are two cases.

*Case 1:* $b \geq 0$. It is given then that $b \leq a$.

*Case 2:* $b \leq 0$. Then we have $|b| = -b \leq a$. □

*Proof.* It is enough to show that $-|a - b| \leq |a| - |b| \leq |a - b|$ by part $(a)$.

$$\begin{aligned}
|b| &= |(b - a) + a| \\
&\leq |b - a| + |a| \\
&= |a - b| + |a| \\
-|a - b| &\leq |a| - |b| \\
|a| &= |(a - b) + b| \\
&\leq |a - b| + |b| \\
|a| - |b| &\leq |a - b|
\end{aligned}$$

Thus, it is sufficient to assume $-|a-b| \leq |a|-|b| \leq |a-b|$, thus $||a|-|b|| \leq |a-b|$ for all $a, b \in \mathbb{R}$. □

## 19 Problem 3.6

Prove $|a + b + c| \leq |a| + |b| + |c|$ for all $a, b, c \in \mathbb{R}$.

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*Proof.*

$$\begin{aligned}
|(a + b) + c| &\leq |(a + b)| + |c| \\
&\leq |a| + |b| + |c|
\end{aligned}$$

□

## 20 Problem 3.7

Show $|b| < a$ if and only if $-a < b < a$.

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*Proof.* We prove both sides of the claim. Assume $|b| < a$.

*Case 1:* $b \geq 0$. Thus, $b < a$. Thus, $0 \leq a$. Therefore, if $0 \leq b$, $-b \leq 0$. We therefore know $-b < a$. By transitivity $-a < b$. Therefore $-a < b < a$.

*Case 2:* $b \leq 0$. Thus, $-b < a$. We know $0 < a$ since $0 \leq |b| < a$. Since $b \leq 0$ and $0 < a$, we have $b < a$. By transitivity we also have $-a < b$. Therefore $-a < b < a$.

Assume $-a < b < a$. By transitivity we have $-b < a$. There are two cases.

*Case 1:* $b \geq 0$. Thus, $b < a$ is given.

*Case 2:* $b \leq 0$. Thus, we have $|b| = -b < a$. □

## 21 Problem 3.8

Let $a, b \in \mathbb{R}$. Show if $a \leq b_1$ for every $b_1 > b$, then $a \leq b$.

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*Proof.* We consider the contrapositive. If $a > b$ there exists $b_1 > b$ such that $a > b_1$.

$$\begin{aligned} a > b \textit{ implies } a + (-b) &> b + (-b) \\ a - b &> 0 \\ \frac{a}{2} &> \frac{b}{2} \\ \frac{a-b}{2} &> 0 \end{aligned}$$

Now pick $b_1 = b + \frac{a-b}{2}$. We note $b_1 > b$ as if $b_1 \leq b$ then $b + \frac{a-b}{2} \leq b$ *implies* $\frac{a+b}{2} \leq b$. This implies $\frac{a}{2} \leq \frac{b}{2}$ which would be a contradiction, thus $b_1 > b$. Now we claim $b_1 < a$. For contradiction assume $b_1 \geq a$.

$$\begin{aligned} a &\leq \frac{a+b}{2} \\ &= \frac{a}{2} + \frac{b}{2} \\ \frac{a}{2} &\leq \frac{b}{2} \\ a &\leq b \end{aligned}$$

However, since $a > b$, this cannot be true, thus $b_1 < a$. Therefore, we have $b < b_1 < a$. Thus our contrapositive is true, hence our proof is complete. □

## 22 Problem 4.1

For each set below that is bounded above, list three upper bounds for the set. Otherwise write "NOT BOUNDED ABOVE" or "NBA".

(a) $[0,1]$

(b) $(0,1)$

(c) $\{2,7\}$

(d) $\{\pi,e\}$

(e) $\{\frac{1}{n} : n \in \mathbb{N}\}$

(f) $\{0\}$

(g) $[0,1] \cup [2,3]$

(h) $\cup_{n=1}^{\infty}[2n, 2n+1]$

(i) $\cap_{n=1}^{\infty}[-\frac{1}{n}, 1+\frac{1}{n}]$

(j) $\{1-\frac{1}{3^n} : n \in \mathbb{N}\}$

(k) $\{n+\frac{(-1)^n}{n} : n \in \mathbb{N}\}$

(l) $\{r \in \mathbb{Q} : r < 2\}$

(m) $\{r \in \mathbb{Q} : r^2 < 4\}$

(n) $\{r \in \mathbb{Q} : r^2 < 2\}$

(o) $\{x \in \mathbb{R} : x < 0\}$

(p) $\{1, \frac{\pi}{3}, \pi^2, 10\}$

(q) $\{0,1,2,4,8,16\}$

(r) $\cap_{n=1}^{\infty}(1-\frac{1}{n}, 1+\frac{1}{n})$

(s) $\{\frac{1}{n} : n \in \mathbb{N}$ and $n$ is prime$\}$

(t) $\{x \in \mathbb{R} : x^3 < 8\}$

(u) $\{x^2 : x \in \mathbb{R}\}$

(v) $\{cos(\frac{n\pi}{3}) : n \in \mathbb{R}\}$

(w) $\{sin(\frac{n\pi}{3}) : n \in \mathbb{N}\}$

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(a) $2,3,4$

(b) $2,3,4$

(c) $10,11,12$

(d) $4,5,6$

(e) $2,3,4$

(f) $1,2,3$

(g) $4,5,6$

(h) $NBA$

(i) $2,3,4$

(j) $2,3,4$

(k) $NBA$

(l) $4,5,6$

(m) $5,6,7$

(n) $5,6,7$

(o) $1,2,3$

(p) $12,13,14$

(q) $20,21,22$

(r) $3,4,5$

(s) $2,3,4$

(t) $10,11,12$

(u) $NBA$

(v) $2,3,4$

(w) $2,3,4$

## 23 Problem 4.5

Let $S$ be a nonempty subset of $\mathbb{R}$ that is bounded above. Prove if sup $S$ belongs to $S$, then sup $S$ = max $S$.

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We know that

$$s \leq supS \ \forall s \in S$$

If sup $S$ is in $S$, then we know it is equal to max = $s_0$ such that

$$s \leq s_0 \ \forall s \in S$$

and $s_0 \in S$. Therefore, max $S$ = sup $S$.

## 24 Problem 4.6

Let $S$ be a nonempty bounded subset of $\mathbb{R}$.
(a) Prove $infS \leq supS$.
(b) What can you say about $S$ if $infS = supS$?

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We know that $infS \leq S \leq supS$. Thus, by transitivity, we know $infS \leq supS$. If $infS = supS$, we thus know that $S$ only contains one element.

## 25 Problem 4.7

Let $S$ and $T$ be nonempty bounded subsets of $\mathbb{R}$.
(a) Prove $S \subseteq T$, then $infT \leq infS \leq supS \leq supT$.

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(a) If $S \subseteq T$, if $s \in S, s \in T$. Thus, we have that $infT \leq s \leq supT$. Since $infS$ is the greatest lower bound for $S$, we have $infT \leq infS$. The same applies for $sup$, giving us $supS \leq supT$. Thus, $infT \leq infS \leq supS \leq supT$.

# 26 Problem 4.9

Complete the proof that $infS = -sup(-S)$ in Corollary 4.5 by proving 1 and 2.

**Corollary 4.5**

*Every nonempty subset $S$ of $\mathbb{R}$ that is bounded below has a greatest lower bound $infS$.*

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We first prove $sup(-S)$ exists.

*Proof.* Let $-S$ be $\{-s : s \in S\}$. Since $S$ is bounded below, we know there is an $m$ in $\mathbb{R}$ s.t. $m \leq s$ for all $s \in S$. By transitivity we have $-s \leq -m$ for all $s \in S$. Thus, our set $-S$ is bounded above by $-m$. Thus, by completeness axiom $sup(-S)$ exists. □

We now prove $infS = -sup(-S)$.

*Proof.* Let $s_0 = sup(-S)$. We want to prove

$$-s_0 \leq s \quad \text{for all} \quad s \in S.$$

and

$$\text{if } t \leq s \quad \text{for all} \quad s \in S, \quad \text{then } t \leq -s_0.$$

Since $s_0$ is the supremum of $-S$, we know $-s \leq s_0$ for all $s \in S$. By transitivity we have $-s_0 \leq s$ for all $s \in S$.
If $t \leq s$ for all $s \in S$, then by transitivity we have $-s \leq -t$. Thus, $-t$ is an upper bound of $-S$. Since $s_0$ is the supremum of $-S$. we have that $s_0 \leq -t$, and by transitivity $t \leq -s_0$. As $t$ is a lower bound for $S$ and $t \leq -s_0$, $-s_0$ is an infinum of $S$, therefore $infS = -sup(-S)$. □

# 27 Problem 4.10

Prove that if $a > 0$, then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < a < n$.

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*Proof.* By the Archimedian property we have that for any $a > 0$ and $b > 0$ there exists some $n \in \mathbb{N}$ such that $na > b$. Therefore, setting $b = 1$, we get some $n_1 a > 1, a > \frac{1}{n_1}$ for $n_1 \in \mathbb{N}$. Setting $b = a^2$ for some $n_2 \in \mathbb{N}$ we get $n_2 a > a^2, n_2 > a$. Since we can pick any $n = max\{n_1, n_2\}$ such that $\frac{1}{n} \leq \frac{1}{n_1}$ and $n_2 \leq n$ which implies $\frac{1}{n} < a$ and $a < n$. Therefore $\frac{1}{n} < a < n$. □

# 28 Problem 4.11

Consider $a, b \in \mathbb{R}$ where $a < b$. Use Denseness of $Q$ 4.7 to show there are infinitely many rationals between $a$ and $b$.

**Denseness of** $\mathbb{Q}$ 4.7

*If* $a, b \in \mathbb{R}$ *and* $a < b$, *then there is a rational* $r \in \mathbb{Q}$ *such that* $a < r < b$.

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*Proof.* We will use induction. Our base case is proved through the denseness of $\mathbb{Q}$. Thus, we assume there exist $n$ distinct rationals between $a$ and $b$. Thus, we have $a < r_n < r_{n-1} < \cdots < r_1 < b$. Since $a < r_n$, we can find another $a < r_{n+1} < r_n$ through denseness of $\mathbb{Q}$. Thus, our proof is complete. $\square$

# 29 Problem 4.12

Let $\mathbb{I}$ be the set of real numbers that are not rational; elements of $\mathbb{I}$ are called *irrational numbers.* Prove if $a < b$, then there exists $x \in \mathbb{I}$ such that $a < x < b$.

---

*Proof.* We first show that $\{r + \sqrt{2} : r \in \mathbb{Q}\} \subseteq \mathbb{I}$. Assume by contradiction that $r + \sqrt{2} \notin \mathbb{I}$. Then that means

$$\begin{aligned} \frac{p}{q} + \sqrt{2} &= \frac{a}{b} \\ \sqrt{2} &= \frac{a}{b} - \frac{p}{q} \\ \sqrt{2} &= \frac{x}{y} \end{aligned}$$

Thus, we imply that $\sqrt{2} \in \mathbb{Q}$. However, since we know that $\sqrt{2}$ is irrational through the rational root theorem, we have a contradiction. Thus our original statement is true. Now, consider $a < b$. We know $a - \sqrt{2} < b - \sqrt{2}$, and since $r \in \mathbb{Q}$, by denseness of $\mathbb{Q}$ we know there exists $a - \sqrt{2} < r < b - \sqrt{2}$, implying $a < r + \sqrt{2} < b$. Since we know $r + \sqrt{2} \in \mathbb{I}$, our proof is complete. $\square$

## 30 Problem 4.13

Prove the following are equivalent for all real numbers $a, b, c$.

$(i)$ $|a - b| < c$

$(ii)$ $b - c < a < b + c$

$(iii)$ $a \in (b - c, b + c)$

---

*Proof.* We know $|b| < a$ if and only if $-a < b < a$. Thus, we know $|a-b| < c$ if and only if $-c < a-b < c$ implying $b-c < a < b+c$. Our infinum of $a$ is $b-c$ and supremum of $a$ is $b+c$. Thus, as $a \in (b-c, b+c)$, it is clear to see that $a$ will range but never reach our infinum or supremum. Thus, $b-c < a < b+c$. $\square$

## 31 Problem 4.14

Let $A$ and $B$ be nonempty bounded subsets of $\mathbb{R}$, and let $A + B$ be the set of all sums $a + b$ where $a \in A$ and $b \in B$.

**(a).** Prove $\sup(A + B) = \sup A + \sup B$.

**(b).** Prove $\inf(A + B) = \inf A + \inf B$.

---

*Proof.* We know that $a \leq supA$ for all $a$ and $b \leq supB$ for all $b$. Thus, $a + b \leq supA + supB$. This implies $supA + supB$ is the upper bound of $A + B$ and as we know $a + b \leq sup(A + B)$, we get $sup(A + B) \leq sup(A) + sup(B)$.

We will show that $supA+supB \leq sup(A+B)$. We know $a+b \leq sup(A+B)$, thus $a \leq sup(A+B)-b$ and $b \leq sup(A+B)-a$. This implies $sup(A+B)-b$ is an upper bound for $a$, thus, $supA \leq sup(A+B)-b$, which implies $b \leq sup(A + B) - sup(A)$. This implies $sup(A + B) - sup(A)$ is an upper bound of $b$, thus $sup(B) \leq sup(A + B) - sup(A)$, implying $supA + supB \leq sup(A + B)$. Combining the two, we get that $sup(A + B) = supA + supB$. We do the same for inf. $\square$

## 32 Problem 4.15

Let $a, b \in \mathbb{R}$. Show if $a \leq b + \frac{1}{n}$ for all $n \in \mathbb{N}$, then $a \leq b$.

---

*Proof.* Assume by contradiction that if $a \leq b + \frac{1}{n}$ then $a > b$. Thus, $a > b$ implies $a - b > 0$. By the archimedian property, there exists some positive integer $n$ such that $n(a - b) > 1$. This implies $a - b > \frac{1}{n}$. Thus, $a > b + \frac{1}{n}$. However, since this is a contradiction, $a \leq b$. $\square$

## 33 Problem 4.16

Show $sup\{r \in \mathbb{Q} : r < a\} = a$ for each $a \in \mathbb{R}$.

---

*Proof.* Let $x = sup\{r \in \mathbb{Q} : r < a\}$. Since $r < a$, then $a$ is an upper bound of $r$ and therefore $x \leq a$. If $x < a$, then by the denseness of $\mathbb{Q}$ there exists some $r \in \mathbb{Q}$ such that $x < r < a$ which is a contradiction since $x$ is our supremum. Therefore $x \nless a$. $x = a$. $\square$

## 34 Problem 5.1

Write the following sets in interval notation.
**(a).** $\{x \in \mathbb{R} : x < 0\}$
**(b).** $\{x \in \mathbb{R} : x^3 \leq 8\}$
**(c).** $\{x^2 : x \in \mathbb{R}\}$
**(d).** $\{x \in \mathbb{R} : x^2 < 8\}$

---

Interval notation is as follows: $(-\infty, 0), (-\infty, 2], [0, \infty), (-2\sqrt{2}, 2\sqrt{2})$.

## 35 Problem 5.4

Let $S$ be a nonempty subset of $\mathbb{R}$, and let $-S = \{-s : s \in S\}$. Prove $\inf S = -\sup(-S)$.

---

*Proof.* We have two cases:

*Case 1:* $-\infty < infS$. This means $S$ has a lower bound, which was already proved by *Corollary 4.5.*

*Case 2:* $-\infty = infS$. This means $S$ is unbounded below. Thus, for every $x \in \mathbb{R}$, there exists a $s \in S$ such that $s \leq x$, which implies $-x \leq -s$, thus the set $-S$ is unbounded above. Thus, $infS = -sup(-S)$. □

## 36 Problem 5.5

Prove $infS \leq supS$ for every nonempty subset of $\mathbb{R}$.

---

*Proof.* There are four cases:
*Case 1:* $S$ is bounded. We have proved this in Problem 4.6. *Case 2, 3, 4:* $S$ is unbounded below, above, or both. In all cases, we know that the set $\mathbb{R} \cup (-\infty, \infty)$ satisfies order properties, thus we can claim that $infS \leq s \leq supS$ for all $s \in S$. □

## 37 Problem 5.6

Let $S$ and $T$ be nonempty subsets of $\mathbb{R}$ such that $S \subseteq T$. Prove $infT \leq infS \leq supS \leq supT$.

---

*Proof.* We have previously that $infS \leq supS$. Additionally, since $S \subseteq T$, if $S$ is unbounded in anyway, so is $T$. Since we have the set of $\mathbb{R} \cup (-\infty, \infty)$ satisfying order properties, we can once again apply it similarily like bounded sets. If $S \subseteq T$, if $s \in S$, $s \in T$. Thus, we have that $infT \leq infS$ and $supS \leq supT$. Therefore we have $infT \leq infS \leq supS \leq supT$. □

# 38 Problem 7.1

Write out the first five terms of the following sequences.
**(a).** $s_n = \frac{1}{3n+1}$
**(b).** $b_n = \frac{3n+1}{4n-1}$
**(c).** $c_n = \frac{n}{3^n}$
**(d).** $sin(\frac{n\pi}{4})$

---

**(a).** $s_1 = \frac{1}{4}, s_2 = \frac{1}{7}, s_3 = \frac{1}{10}, s_4 = \frac{1}{13}, s_5 = \frac{1}{16}$
**(b).** $b_1 = \frac{4}{3}, b_2 = 1, b_3 = \frac{10}{11}, b_4 = \frac{13}{15}, b_5 = \frac{16}{19}$
**(c).** $c_1 = \frac{1}{3}, c_2 = \frac{2}{9}, c_3 = \frac{1}{9}, c_4 = \frac{4}{81}, c_5 = \frac{5}{243}$
$(d)$. $\frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}$

# 39 Problem 7.2

For each sequence in 7.1, determine whether it converges. If it converges, give its limit.

---

$(a)$. $s_n \to 0$
$(b)$. $b_n \to \frac{3}{4}$
$(c)$. $c_n \to 0$
$(d)$. $N/A$

## 40 Problem 8.1

Prove the following:
**(a).** $lim \frac{(-1)^n}{n} = 0$
**(b).** $lim \frac{1}{n^{1/3}} = 0$
**(c).** $lim \frac{2n-1}{3n+2} = \frac{2}{3}$
**(d).** $lim \frac{n+6}{n^2-6} = 0$

---

*Proof.* $(a)$. Let $\varepsilon > 0$. We want to find an $N$ such that $n > N$ implies $|\frac{(-1)^n}{n} - 0| < \varepsilon$.

$$\left|\frac{(-1)^n}{n} - 0\right| < \varepsilon$$
$$\frac{1}{n} < \varepsilon$$
$$\frac{1}{\varepsilon} < n$$

Let $N = \frac{1}{\varepsilon}$. Now, $n > N$ implies $n > \frac{1}{\varepsilon}, |\frac{1}{n}| < \varepsilon$, thus $|\frac{(-1)^n}{n} - 0| < \varepsilon$. Thus, $\lim \frac{-1^n}{n} = 0$. □

*Proof.* $(b)$. Let $\varepsilon > 0$. We want to find an $N$ such that $n > N$ implies $|\frac{1}{n^{1/3}} - 0| < \varepsilon$.

$$\left|\frac{1}{n^{1/3}} - 0\right| < \varepsilon$$
$$\frac{1}{n^{1/3}} < \varepsilon$$
$$\frac{1}{\varepsilon^{1/3}} < n$$

Let $N = \frac{1}{\varepsilon^{1/3}}$. Now, $n > N$ implies $n > \frac{1}{\varepsilon^{1/3}}, |\frac{1}{n^{1/3}}| < \varepsilon$, thus $|\frac{1}{n^{1/3}} - 0| < \varepsilon$. Thus, $\lim \frac{1}{n^{1/3}} = 0$. □

*Proof.* $(c)$. Let $\varepsilon > 0$. We want to find an $N$ such that $n > N$ implies $|\frac{2n-1}{3n+2} - \frac{2}{3}| < \varepsilon$.

$$\left|\frac{2n-1}{3n+2} - \frac{2}{3}\right| < \varepsilon$$
$$\left|\frac{2n-1}{3n+2} - \frac{2n+4/3}{3n+2}\right| < \varepsilon$$
$$\left|\frac{-7/3}{3n+2}\right| < \varepsilon$$
$$\frac{7}{9n+6} < \varepsilon$$
$$7 < 9n\varepsilon + 6\varepsilon$$
$$7 - 6\varepsilon < 9n\varepsilon$$
$$\frac{7-6\varepsilon}{9\varepsilon} < n$$

Let $N = \frac{7-6\varepsilon}{9\varepsilon}$. Thus, $n > N$ implies $n > \frac{7-6\varepsilon}{9\varepsilon}$, $|\frac{2n-1}{3n+2} - \frac{2}{3}| < \varepsilon$. Thus, $\lim \frac{2n-1}{3n+2} = \frac{2}{3}$. □

*Proof.* $(d)$. Let $\varepsilon > 0$. We want to find an $N$ such that $n > N$ implies $|\frac{n+6}{n^2-6} - 0| < \varepsilon$. Note for $n > 6$ we have $2n > n + 6$. To get $n^2 - 6 > \frac{1}{2}n^2$, we need $n^2 > 12$, therefore $n > 4$. Thus, for $n > 6$ we have

$$\left|\frac{n+6}{n^2-6} - 0\right| < \frac{4n}{n^2} < \varepsilon$$
$$\frac{4}{n} < \varepsilon$$
$$\frac{4}{\varepsilon} < n$$

Let $N = max\{\frac{4}{\varepsilon}, 6\}$. Thus, $n > N$ implies $n > \frac{4}{\varepsilon}$ or $n > 6$ which implies $\frac{4}{\varepsilon} < n$. Thus, $\lim \frac{n+6}{n^2-6} = 0$. □

## 41 Problem 8.2

Determine the limits of the following sequences, and then prove your claims.
**(a).** $a_n = \frac{n}{n^2+1}$
**(b).** $b_n = \frac{7n-19}{3n+7}$
**(c).** $c_n = \frac{4n+3}{7n-5}$
**(d).** $d_n = \frac{2n+4}{5n+2}$
**(e).** $s_n = \frac{sin(n)}{n}$

---

*Proof.* $(a)$. We note

$$\frac{n}{n^2+1} < \frac{n}{n^2} < \frac{1}{n}$$

As $\frac{1}{n}$ converges to 0, $\lim a_n = 0$. Let $\varepsilon > 0$. We find an $N$ such that $n > N$ implies $|\frac{n}{n^2+1} - 0| < \varepsilon$. We note for all $n \in R, n^2 + 1 > n^2$.

$$\left|\frac{n}{n^2+1} - 0\right| < \frac{1}{n} < \varepsilon$$
$$\frac{1}{\varepsilon} < n$$

Let $N = \frac{1}{\varepsilon}$. Thus, $n > N$ implies $n > \frac{1}{\varepsilon}, \varepsilon > |\frac{n}{n^2+1} - 0|$. Thus, $\lim \frac{n}{n^2+1} = 0$. ☐

*Proof.* $(b)$. We have $b_n = \frac{7n-19}{3n+7}$. Our limit $lim_{n\to\infty} \frac{7n-19}{3n+7} = lim_{n\to\infty} \frac{7-\frac{19}{n}}{3+\frac{7}{n}} = \frac{7}{3}$. Let $\varepsilon > 0$. We want to find an $N$ such that $n > N$ implies $|\frac{7n-19}{3n+7} - \frac{7}{3}| < \varepsilon$.

$$\left|\frac{7n-19}{3n+7} - \frac{7}{3}\right| < \varepsilon$$
$$\left|\frac{21n-57}{3(3n+7)} - \frac{7(3n+7)}{3(3n+7)}\right| < \varepsilon$$
$$\left|\frac{-176}{3(3n+17)}\right| < \varepsilon$$
$$\frac{176}{3(3n+17)} < \varepsilon$$
$$\frac{176-51\varepsilon}{9\varepsilon} < n$$

Let $N = \frac{176-51\varepsilon}{9\varepsilon}$. Note that $n > N$ implies $|\frac{7n-19}{3n+7} - \frac{7}{3}| < \varepsilon$. Thus, $\lim b_n = \frac{7}{3}$. ☐

*Proof.* $(c)$. We have $c_n = \frac{4n+3}{7n-5}$. Our limit $lim_{n\to\infty} \frac{4n+3}{7n-5} = lim_{n\to\infty} \frac{4+\frac{3}{n}}{7-\frac{5}{n}} = \frac{4}{7}$. Let $\varepsilon > 0$. We want to find an $N$ such that $n > N$ implies $|\frac{4n+3}{7n-5} - \frac{4}{7}| < \varepsilon$.

$$\left|\frac{4n+3}{7n-5} - \frac{4}{7}\right| < \varepsilon$$
$$\left|\frac{7(4n+3)}{7(7n-5)} - \frac{4(7n-5)}{7(7n-5)}\right| < \varepsilon$$
$$\left|\frac{41}{7(7n-5)}\right| < \varepsilon$$
$$\frac{41+35\varepsilon}{49\varepsilon} < n$$

Let $N = \frac{41+35\varepsilon}{49\varepsilon}$. $n > N$ implies $|\frac{4n+3}{7n-5} - \frac{4}{7}| < \varepsilon$, thus $\lim c_n = \frac{4}{7}$. ☐

*Proof.* $(d)$. We have $d_n = \frac{2n+4}{5n+2}$. Our limit $lim_{n\to\infty} \frac{2n+4}{5n+2} = lim_{n\to\infty} \frac{2+\frac{4}{n}}{5+\frac{2}{n}} = \frac{2}{5}$. Let $\varepsilon > 0$. We want to find an $N$ such that $n > N$ implies $|\frac{2n+4}{5n+2} - \frac{2}{5}| < \varepsilon$.

$$\left|\frac{2n+4}{5n+2} - \frac{2}{5}\right| < \varepsilon$$
$$\left|\frac{5(2n+4)}{5(5n+2)} - \frac{2(5n+2)}{5(5n+2)}\right| < \varepsilon$$
$$\left|\frac{16}{5(5n+2)}\right| < \varepsilon$$
$$\frac{16-10\varepsilon}{25\varepsilon} < n$$

Let $N = \frac{16-10\varepsilon}{25\varepsilon}$. $n > N$ implies $|\frac{2n+4}{5n+2} - \frac{2}{5}| < \varepsilon$, thus $\lim d_n = \frac{2}{5}$. □

*Proof.* $(e)$. We have $s_n = \frac{sin(n)}{n}$. We note $-1 \leq sin(n) \leq 1$. Since $-\frac{1}{n}$ and $\frac{1}{n}$ both converge to 0, by the squeeze theorem $\frac{sin(n)}{n}$ converges to 0. Let $\varepsilon > 0$. We want to find an $N$ such that $n > N$ implies $|\frac{sin(n)}{n} - 0| < \varepsilon$. We note for all $n$, $|\frac{sin(n)}{n}| \leq \frac{1}{n}$.

$$\left|\frac{sin(n)}{n} - 0\right| \leq \left|\frac{1}{n}\right| < \varepsilon$$
$$\frac{1}{\varepsilon} < n$$

Let $N = \frac{1}{\varepsilon}$. $n > N$ implies $|\frac{sin(n)}{n} - 0| < \varepsilon$, thus $\lim s_n = 0$. □

# 42 Problem 8.3

Let $(s_n)$ be a sequence of nonnegative real numbers, and suppose $\lim s_n = 0$. Prove $\lim \sqrt{s_n} = 0$. This will complete the proof for Example 5.

---

*Proof.* We want to prove that $\sqrt{s_n} = 0$. Let $\varepsilon > 0$. We want to find an $N$ such that $n > N$ implies $|\sqrt{s_n} - 0| < \varepsilon$. Additionally, we have that $\varepsilon^2 > 0$, such that $|s_n - 0| < \varepsilon^2$. This implies there exists $|\sqrt{s_n} - 0| < \varepsilon$. Thus, $\lim \sqrt{s_n} = 0$. □

# 43 Problem 8.4

Let $(t_n)$ be a bounded sequence, i.e., there exists $M$ such that $|t_n| \leq M$ for all $n$, and let $(s_n)$ be a sequence such that $\lim s_n = 0$. Prove $\lim (s_n t_n) = 0$.

---

*Proof.* We want to prove that $s_n t_n = 0$. Thus, we have that $|s_n t_n - 0| \leq |s_n M|$ for all $n$. Let $\varepsilon > 0$. This implies that $\frac{\varepsilon}{M} > 0$. Thus, there exists an $N$ such that $n > N$ implies $|s_n - 0| < \frac{\varepsilon}{M}$. Therefore $|s_n M - 0| < \varepsilon$. Therefore $\lim s_n M = 0$. Thus, we have $0 \leq |s_n t_n - 0| \leq |s_n M| < \varepsilon$, so $\lim (s_n t_n) = 0$. □

## 44 Problem 8.5

**(a).** Consider three sequences $(a_n), (b_n)$, and $(s_n)$ such that $a_n \leq s_n \leq b_n$ for all $n$ and $\lim a_n = \lim b_n = s$. Prove $\lim s_n = s$. This is called the "squeeze lemma".
**(b).** Suppose $(s_n)$ and $(t_n)$ are sequences such that $|s_n| \leq t_n$ for all $n$ and $\lim t_n = 0$. Prove $\lim s_n = 0$.

---

*Proof.* **(a).** Given $a_n \leq s_n \leq b_n$ for all $n$ and $\lim a_n = \lim b_n = s$. Thus, we want to prove that $\lim s_n = s$. Let $\varepsilon > 0$. We want to find some $N$ such that $n > N$ implies

$$|s_n - s| < \varepsilon$$

Now let $\varepsilon > 0$. Since $\lim a_n = s$, we have there exists some $N_1$ such that $n > N_1$ implies

$$\begin{aligned} |a_n - s| &< \varepsilon \\ -\varepsilon < a_n - s &< \varepsilon \\ s - \varepsilon < a_n &< s + \varepsilon \end{aligned}$$

Since $\lim b_n = s$, we have that there exists some $N_2$ such that $n > N_2$ implies

$$\begin{aligned} |b_n - s| &< \varepsilon \\ -\varepsilon < b_n - s &< \varepsilon \\ s - \varepsilon < b_n &< s + \varepsilon \end{aligned}$$

Thus, let $N = max\{N_1, N_2\}$. We have for some $n > N$ that

$$s - \varepsilon < a_n \leq s_n \leq b_n < s + \varepsilon$$

Thus, this implies $s - \varepsilon < s_n < s + \varepsilon$. Thus, we have that $|s_n - s| < \varepsilon$, therefore $\lim s_n = s$. □

*Proof.* **(b).** Given $|s_n| \leq t_n$, we have that $0 \leq |s_n - 0| \leq |t_n - 0|$. Let $\varepsilon > 0$. We have that there is some $N$ such that $n > N$ implies

$$|t_n - 0| < \varepsilon$$

Thus, this implies that

$$|s_n - 0| < \varepsilon$$

Thus, this implies $\lim s_n = 0$. □

## 45 Problem 8.6

Let $(s_n)$ be a sequence in $\mathbb{R}$.
**(a).** Prove $\lim s_n = 0$ if and only if $\lim |s_n| = 0$.
**(b).** Observe that if $s_n = (-1)^n$, then $\lim |s_n|$ exists, but $\lim s_n$ does not exist.

---

*Proof.* **(a).** Assume $\lim s_n = 0$. Let $\varepsilon > 0$. There exists some $N$ such that $n > N$ implies

$$||s_n| - 0| < \varepsilon$$

Therefore $\lim |s_n| = 0$. Assume $lim|s_n| = 0$. Let $\varepsilon > 0$. Thus, there exists some $N$ such that $n > N$ implies

$$\begin{aligned} ||s_n| - 0| &< \varepsilon \\ ||s_n|| &< \varepsilon \end{aligned}$$

This implies

$$|s_n - 0| < \varepsilon$$

Thus, $\lim s_n = 0$. □

*Proof.* **(b).** Suppose by contradiction that $\lim s_n$ exists and $\lim (-1)^n = a$. Let $\varepsilon = 1$. Thus, we have

$$\begin{aligned} |(-1)^n - a| &< 1 \\ |1 - a| &< 1 \end{aligned}$$

We can order with the following:

$$\begin{aligned} 2 &= |1 + 1| \\ &= |1 - a + (-1 - a)| \\ &= |1 - a| + |1 + a| \\ &< 1 + 1 \\ &= 2 \end{aligned}$$

However, since $2 \not< 2$, we have a contradiction. Thus, $\lim (-1)^n$ does not converge. □

# 46 Problem 8.7

Show the following sequences do not converge.
**(a).** $cos(\frac{n\pi}{3})$
**(b).** $s_n = (-1)^n n$
**(c).** $sin(\frac{n\pi}{3})$

---

*Proof.* **(a).** Assume $cos(\frac{n\pi}{3})$ converges to $a$. Let $\varepsilon = 1$. Thus, we have some $N$ such that $n > N$ implies

$$|cos(\tfrac{n\pi}{3}) - a| < 1$$

Consider $cos(\frac{n\pi}{3}) = 1$ and $cos(\frac{n'\pi}{3}) = -1$. Thus, we have that

$$\begin{aligned} |1 - a| &< 1 \\ |-1 - a| = |1 + a| &< 1 \end{aligned}$$

Thus, we have the following:

$$\begin{aligned} 2 &= |1 + 1| \\ &= |1 - a + (-1 - a)| \\ &= |1 - a| + |1 + a| \\ &< 1 + 1 \\ &= 2 \end{aligned}$$

Thus, as we have reached a contradiction, $cos(\frac{n\pi}{3})$ does not converge. □

*Proof.* **(b).** Assume for contradiction that $(-1)^n n$ converges, and $\lim s_n = a$. Let $\varepsilon = 1$. Thus, we have some $N$ such that $n > N$ implies

$$|(-1)^n n - a| < 1$$

Consider $n$ even and $n + 2$. Thus, we have that

$$\begin{aligned} |n - a| &< 1 \\ |n + 2 - a| &< 1 \end{aligned}$$

Thus, we have the following:

$$\begin{aligned} 2 &= |1 + 1| \\ &= |n + 2 - a + (-(n - a))| \\ &= |n + 2 - a| + |n - a| \\ &< 1 + 1 \\ &= 2 \end{aligned}$$

Thus, as we have readched a contradiction, the sequence $s_n$ does not converge. □

*Proof.* **(c).** Assume by contradiction that $sin(\frac{n\pi}{3})$ converges to $a$. Let $\varepsilon = \frac{\sqrt{3}}{2}$. Thus, we have some $N$ such that $n > N$ implies

$$|sin(\tfrac{n\pi}{3}) - a| < \tfrac{\sqrt{3}}{2}$$

Consider $n$ and $n'$ such that $sin(\frac{n\pi}{3}) = \frac{\sqrt{3}}{2}$ and $sin(\frac{n'\pi}{3}) = -\frac{\sqrt{3}}{2}$.

$$\begin{aligned} \left|\frac{\sqrt{3}}{2} - a\right| &< \frac{\sqrt{3}}{2} \\ \left|-\frac{\sqrt{3}}{2} - a\right| = \left|\frac{\sqrt{3}}{2} + a\right| &< \frac{\sqrt{3}}{2} \end{aligned}$$

Thus, we have the following:

$$\begin{aligned}\sqrt{3} &= \left|\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right| \\ &= \left|\frac{\sqrt{3}}{2} - a + \left(\frac{\sqrt{3}}{2} + a\right)\right| \\ &= \left|\frac{\sqrt{3}}{2} - a\right| + \left|\frac{\sqrt{3}}{2} + a\right| \\ &< \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \\ &= \sqrt{3}\end{aligned}$$

Thus, since $\sqrt{3} \nless \sqrt{3}$, by contradiction the sequence $sin(\frac{n\pi}{3})$ does not converge. $\square$

## 47 Problem 8.9

Let $(s_n)$ be a sequence that converges.
**(a).** Show that if $s_n \geq a$ for all but finitely many $n$, then $\lim s_n \geq a$.
**(b).** Show that if $s_n \leq b$ for all but finitely many $n$, then $\lim s_n \leq b$.
**(c).** Conclude that if all but finitely many $s_n$ belong to $[a, b]$, then $\lim s_n$ belongs to $[a, b]$.

---

*Proof.* **(a).** Assume by contradiction that $\lim s_n = s < a$. Let $\varepsilon = \frac{a-s}{2}$. Then there exists $N$ such that $n > N$ implies

$$|s_n - s| < \varepsilon = \tfrac{a-s}{2}$$

Thus, this implies that

$$\begin{aligned} s_n - s &< \frac{a-s}{2} \\ 2s_n - 2s &< a - s \\ 2s_n &< a + s \\ s_n < \frac{a+s}{2} &< \frac{a+a}{2} = a \end{aligned}$$

Thus, we have that $s_n < a$ for all $n > N$. However, since the set is an infinite set, we have reached a contradiction. Thus, $\lim s_n \geq a$. ☐

*Proof.* **(b).** Assume by contradiction that $\lim s_n = s > b$. Let $\varepsilon = \frac{s-b}{2}$. Then there exists $N$ such that $n > N$ implies

$$|s_n - s| < \varepsilon = \tfrac{s-b}{2}$$

Thus, this implies that

$$\begin{aligned} -\varepsilon &< s_n - s \\ -\frac{s-b}{2} &< s_n - s \\ b - s &< 2s_n - 2s \\ b + s &< 2s_n \\ b = \frac{b+b}{2} < \frac{b+s}{2} &< s_n \end{aligned}$$

Therefore $b < s_n$ for all $n > N$. However, since the set is an infinite set, we have reached a contradiction. Thus $\lim s_n \leq b$. ☐

*Proof.* **(c).** If all but finitely many $s_n$ belong to $[a, b]$, then we have that $s_n \geq a$ and $s_n \leq b$ for all but finitely many $s_n$. By part $(a)$ and $(b)$, we have that $\lim s_n \geq a$ and $\lim s_n \leq b$, therefore $\lim s_n$ belongs to $[a, b]$. ☐

# 48 Problem 8.10

Let $(s_n)$ be a convergent sequence, and suppose $\lim s_n > a$. Prove there exists a number $N$ such that $n > N$ implies $s_n > a$.

---

*Proof.* Assume $\lim s_n > a$. We have that $\lim s_n = a + \delta$. Let $\varepsilon = \frac{\delta}{2}$. Thus, there exists a number $N$ such that $n > N$ implies

$$\begin{aligned} |s_n - (a+\delta)| &< \varepsilon = \frac{\delta}{2} \\ -\frac{\delta}{2} &< s_n - a - \delta \\ \frac{\delta}{2} &< s_n - a \\ \frac{\delta}{2} + a &< s_n \\ a &< s_n \end{aligned}$$

Thus, we have that $s_n > a$. □

# 49 Problem 9.1

Using the limit Theorems $9.2 - 9.7$, prove the following.
**(a).** $\lim \frac{n+1}{n} = 1$.
**(b).** $\lim \frac{3n+7}{6n-5} = \frac{1}{2}$
**(c).** $\lim \frac{17n^5+73n^4-18n^2+3}{23n^5+13n^3} = \frac{17}{23}$

---

*Proof.* **(a).** $\lim \frac{n+1}{n} = 1$. We can rewrite the limit to the following: $\lim \frac{1+\frac{1}{n}}{1}$.

$$\begin{aligned} lim\left(\frac{n+1}{n}\right) &= lim\left(\frac{1+\frac{1}{n}}{1}\right) \\ &= \frac{lim(1+\frac{1}{n})}{lim(1)} \\ &= \frac{lim(1)+lim(\frac{1}{n})}{lim(1)} = 1 \end{aligned}$$

□

*Proof.* **(b).** $\lim \frac{3n+7}{6n-5} = \frac{1}{2}$.

$$\begin{aligned} lim\left(\frac{3n+7}{6n-5}\right) &= lim\left(\frac{3+\frac{7}{n}}{6-\frac{5}{n}}\right) \\ &= \frac{lim(3)+lim(\frac{7}{n})}{lim(6)-lim(\frac{5}{n})} = \frac{1}{2} \end{aligned}$$

□

*Proof.* **(c).** $\lim \frac{17n^5+73n^4-18n^2+3}{23n^5+13n^3} = \frac{17}{23}$.

$$lim\left(\frac{17n^5+73n^4-18n^2+3}{23n^5+13n^3}\right) = \frac{lim(17)}{lim(23)} = \frac{17}{23}$$

□

## 50 Problem 9.4

Let $s_1 = 1$ and for $n \geq 1$ let $s_{n+1} = \sqrt{s_1 + 1}$.
$(a)$. List the first four terms of $(s_n)$.
$(b)$. It turns out that $(s_n)$ converges. Assume this fact and prove the limit is $\frac{1}{2}(1 + \sqrt{5})$.

---

$(a)$. $s_1 = 1, s_2 = \sqrt{2}, s_3 = \sqrt{\sqrt{2} + 1}, s_4 = \sqrt{\sqrt{\sqrt{2} + 1}}$.
$(b)$. Assume $s_n$ converges. Thus, we have for all $n$ that $\lim s_n = s = \lim s_{n+1}$. Thus, $s = \sqrt{s+1}$. $s^2 = s + 1, s^2 - s - 1 = 0$. Thus, solving the roots of $s$ gives us $s = \frac{1 \pm \sqrt{5}}{2}$. Since we know $s_1 = 1$ and $s_n$ is an increasing sequence by induction, we show that the limit converges to $\frac{1}{2}(1 + \sqrt{5})$.

## 51 Problem 9.6

Let $x_1 = 1$ and $x_{n+1} = 3x_n^2$ for $n \geq 1$.
$(a)$. Show that if $a = \lim x_n$, then $a = \frac{1}{3}$ or $a = 0$.
$(b)$. Does $\lim x_n$ exist? Explain.
$(c)$. Discuss the apparent contradiction between parts $(a)$ and $(b)$.

---

$(a)$. Assume $\lim x_n = a$. Thus, we have that $\lim x_{n+1} = \lim 3x_n^2$. $a = 3a^2$. $3a = 1$. Thus, we have that $a = 0$ or $a = \frac{1}{3}$.

*Proof.* $(b)$. We use induction to prove the following. Our claim is that $x_n > n$ for all $n \geq 2$.
*Base Case:* $n = 2$. $3 > 2$.
*Induction:*

$$\begin{aligned} x_n &> n \\ x_n x_n &> n x_n > n \\ 3x_n^2 &> 3n > n + 1 \end{aligned}$$

Thus, $x_n > n$ for all $n \geq 2$. Now Let $\varepsilon > 0$. Let $N = \varepsilon$. $n > N$ implies

$$\begin{aligned} x_n &> n > \varepsilon \\ x_n &> \varepsilon \end{aligned}$$

Thus, $\lim x_n = \infty$, and therefore does not converge. □

$(c)$. The difference is obvious.

## 52 Problem 9.9

Suppose there exists $N_0$ such that $s_n \leq t_n$ for all $n > N_0$.
$(a)$. Prove that if $\lim s_n = \infty$, then $\lim t_n = \infty$.

---

*Proof.* Let $\varepsilon > 0$. $N_1 = \varepsilon$ such that $n > N_1$ implies

$$s_n > \varepsilon$$

Let $N = max\{N_0, N_1\}$. $n > N$ implies

$$\begin{aligned} t_n &\geq s_n \geq \varepsilon \\ t_n &\geq \varepsilon \end{aligned}$$

Thus, $\lim t_n = \infty$. □

## 53 Problem 9.10

**(a).** Show that if $\lim s_n = \infty$ and $k > 0$, then $\lim (ks_n) = \infty$.
**(b).** Show $\lim s_n = \infty$ if and only if $\lim (-s_n) = -\infty$.
**(c).** Show that if $\lim s_n = \infty$ and $k < 0$, then $\lim (ks_n) = -\infty$.

---

*Proof.* $(a)$. Assume $\lim s_n = \infty$. Let $\varepsilon > 0$. We pick $N = \varepsilon$ such that $n > N$ implies

$$\begin{aligned} s_n &> \frac{\varepsilon}{k} \\ ks_n &> \varepsilon \end{aligned}$$

Thus, $\lim ks_n = \infty$. □

*Proof.* $(b)$. Assume $\lim s_n = \infty$. Thus, we have $\varepsilon < 0$. Thus we have $N = \varepsilon$ such that $n > N$ implies

$$\begin{aligned} s_n &> -\varepsilon \\ -s_n &< \varepsilon \end{aligned}$$

Thus, $\lim -s_n = -\infty$.
Assume $\lim (-s_n) = -\infty$. Thus, we have $\varepsilon > 0$. We have $N = \varepsilon$ such that $n > N$ implies

$$\begin{aligned} -s_n &< -\varepsilon \\ s_n &> \varepsilon \end{aligned}$$

Thus, $\lim s_n = \infty$. □

*Proof.* $(c)$. Let $\varepsilon < 0$. We pick $N = \varepsilon$ such that $n > N$ implies

$$\begin{aligned} s_n &> \frac{\varepsilon}{k} \\ ks_n &< \varepsilon \end{aligned}$$

Thus, $\lim ks_n = -\infty$. □

## 54 Problem 9.11

**(a).** Show that if $\lim s_n = \infty$ and $\inf\{t_n : n \in \mathbb{N}\} > -\infty$, then $\lim (s_n + t_n) = \infty$.
**(b).** Show that if $\lim s_n = \infty$ and $\lim t_n > -\infty$, then $\lim (s_n + t_n) = \infty$.
**(c).** Show that if $\lim s_n = \infty$ and if $(t_n)$ is a bounded sequence, then $\lim (s_n + t_n) = \infty$.

---

*Proof.* $(a)$. Define $a = inf\{t_n : n \in \mathbb{N}\}$. We have that $a > -\infty$. Thus we know $s_n + t_n \geq s_n + a$. Thus, pick $\varepsilon > 0$. We pick $N$ such that $n > N$ implies

$$\begin{aligned} s_n &> \varepsilon - a \\ s_n + a &> \varepsilon \\ s_n + t_n &> \varepsilon \end{aligned}$$

Thus, $\lim (s_n + t_n) = \infty$. □

*Proof.* $(b)$. $\lim t_n > -\infty$ implies that there exists some $a \in \mathbb{R}$ such that $t_n \geq a$. Thus, $t_n$ is bounded below. Thus, our proof follows part $(a)$. □

*Proof.* $(c)$. Same as $(a)$. □

## 55 Problem 9.12

Assume all $s_n \neq 0$ and that the limit $L = \lim |\frac{s_{n+1}}{s_n}|$ exists.
$(a)$. Show that if $L < 1$, then $\lim s_n = 0$.
$(b)$. Show that if $L > 1$, then $\lim |s_n| = \infty$.

---

*Proof.* $(a)$. Note if $L < 1$, we have $|\frac{s_{n+1}}{s_n}| < 1$, and thus $s_n$ is a decreasing sequence. Assume $L < 1$. Pick $a \in (L, 1)$. Thus, there exists $|\frac{s_{n+1}}{s_n}| < a$. This implies $|s_{n+1}| < a|s_n|$. Since $s_n$ is decreasing, we know that $|s_n| < a^{n-N}|s_N|$ for $n > N$. Thus, for $|a| < 1$ we have that $a^n$ converges to 0. Let $\varepsilon > 0$. Thus, we have $M$ such that $n - N > M$ implies

$$a^{n-N} < \frac{\varepsilon}{|s_N|}$$
$$a^{n-N}|s_N| < \varepsilon$$
$$|s_n - 0| < \varepsilon$$

Thus, $\lim s_n = 0$. □

*Proof.* $(b)$. Assume $s_n \neq 0$. Thus, we have that $\lim \frac{1}{s_n} = \frac{1}{s}$. Thus, we say that $\lim \frac{s_n}{s_{n+1}} = \frac{1}{L}, \frac{1}{L} < 1$. Thus, we pick $a \in (\frac{1}{L}, 1)$. There exists $N$ such that $n \geq N$ implies

$$\left|\frac{s_n}{s_{n+1}}\right| < a$$
$$\frac{1}{|s_{n+1}|} < \frac{a}{|s_n|}$$
$$\frac{1}{|s_n|} < \frac{a^{n-N}}{|s_N|}$$

Let $\varepsilon > 0$. Thus, we have $M$ such that $n - N > M$ implies

$$a^{n-N} < \varepsilon \cdot |s_N|$$
$$\frac{a^{n-N}}{|s_N|} < \varepsilon$$
$$\left|\frac{1}{|s_N|} - 0\right| < \varepsilon$$

Thus $\lim |s_n| = \infty$. □

## 56 Problem 9.13

Show $lim_{n\to\infty} a^n = \infty$ if $a > 1$.

---

*Proof.* By theorem 9.7 we know that limit $M^{\frac{1}{n}} = 0$. Thus, there exists $N$ such that $n > N$ implies

$$M^{\frac{1}{n}} < a$$
$$M < a^n$$

Thus, $\lim (a^n) = \infty$. □

## 57 Problem 10.2

Prove Theorem 10.2 for bounded decreasing sequences.

**Theorem** 10.2

*All bounded monotone sequences converge.*

---

*Proof.* Let $s_n$ be a bounded decreasing sequence. Thus the set $\{s_n : n \in \mathbb{N}\}$ is bounded. Thus, there exists an infinum and let $L = inf\{s_n : n \in \mathbb{N}\}$. Let $\varepsilon > 0$. Thus, there exists some $s_N < L + \varepsilon$ and since $s_n$ is decreasing, $n > N$ implies

$$\begin{aligned} s_n &< L + \varepsilon \\ L \leq s_n &< L + \varepsilon \\ |s_n - L| &< \varepsilon \end{aligned}$$

Thus, $s_n$ has a limit and converges to $L$. □

## 58 Problem 10.5

Prove Theorem 10.4 for unbounded decreasing sequences.

**Theorem 10.4**

(i) *If* $s_n$ *is an unbounded increasing sequence, then* $\lim s_n = \infty$.
(i) *If* $s_n$ *is an unbounded decreasing sequence, then* $\lim s_n = -\infty$.

---

*Proof.* $(ii)$. Let $(s_n)$ be an unbounded decreasing sequence. Let $M > 0$. Since the set $\{s_n : n \in \mathbb{N}\}$ is unbounded and bounded above by $s_1$, it is unbounded below. Thus, for some $M$ we have $s_N < M$. Thus there exists $N$ such that $n > N$ implies $s_n < s_N < M$. $\lim s_n = -\infty$. □

## 59 Problem 10.7

Let $S$ be a bounded nonempty subset of $\mathbb{R}$ such that sup$S$ is not in $S$. Prove there is a sequence $(s_n)$ of points in $S$ such that $\lim s_n = \sup S$.

---

*Proof.* Since $supS - \frac{1}{n}$ is not an upper bound of $S$ and $S$ is nonempty, there exists $x \in S$ such that

$$supS - \frac{1}{n} < x < supS$$

Since $x$ can be defined in $S$, we know that $lim_{n\to\infty}(supS - \frac{1}{n}) = supS$. □

## 60 Problem 10.9

Let $s_1 = 1$ and $s_{n+1} = (\frac{n}{n+1})s_n^2$ for $n \geq 1$.
**(a).** Find $s_2, s_3, s_4$.
**(b).** Show $\lim s_n$ exists.
**(c).** Prove $\lim s_n = 0$.

---

$(a)$. $s_2 = \frac{1}{2}, s_3 = \frac{1}{6}, s_4 = \frac{1}{48}$.

*Proof.* $(b)$. We claim $s_n$ is decreasing. It is clear $s_2 < s_1$. Thus, we assume $s_{n+1} < s_n$.

$$\begin{aligned} s_{n+2} &= \frac{n+1}{n+2} s_{n+1}^2 \\ &< s_{n+1} \end{aligned}$$

Thus is as $\frac{n+1}{n+2} < 1$ and $s_{n+1} < 1$, thus we can simplify to $s_{n+1}$. Thus, the sequence is decreasing. □

*Proof.* $(c)$. We know $\lim s_n = s$. Thus, we have that $s = lim(\frac{n}{n+1}) \cdot lim(s_n)^2 = 1 \cdot s^2$. Thus, we get that $s = 0$ or $s = 1$. Since $s_n < 1$, we get that $s = 0$, therefore $\lim s_n = 0$. □

## 61 Problem 11.1

Let $a_n = 3 + 2(-1)^n$ for $n \in \mathbb{N}$.
**(a).** List the first eight terms of the sequence $(a_n)$.
**(b).** Give a subsequence that is constant [takes a single value]. Specify the selection function $\sigma$.

---

$(a)$. $a_n = 1, 5, 1, 5, 1, 5, 1, 5, 1, 5$
$(b)$. $s_{n_k} = (1, 1, 1, 1, \ldots)$ where $\sigma(k) = 2k - 1$.

## 62 Problem 11.2

Consider the sequences defined as follows:

$$a_n = (-1)^n,\ b_n = \frac{1}{n},\ c_n = n^2,\ d_n = \frac{6n+4}{7n-3}$$

**(a).** For each sequence, give an example of a monotone subsequence.
**(b).** For each sequence, give its set of subsequential limits.
**(c).** For each sequence, give its lim sup and lim inf.
**(d).** Which of the sequences converge? diverges to $+\infty$? diverges to $-\infty$?
**(e).** Which of the sequences is bounded?

---

$(a)$. $a_n = (1, 1, 1, \ldots)$ where $\sigma(k) = 2k$. The rest are monotonic intrinsically.
$(b)$. $a_n : \{1, -1\}, b_n : \{0\}, c_n : \{\infty\}, d_n : \{6/7\}$
$(c)$. $a_n$ : lim sup = 1, lim inf = $-1$. The rest are trivial.
$(d)$. Trivial.
$(e)$. $a_n, b_n, d_n$ are all bounded.

## 63 Problem 11.6

Show every subsequence of a subsequence of a given sequence is itself a subsequence of the given sequence. *Hint:* Define subsequences as in (3) of Definition 11.1.

**Definition** 11.1

$$t_k = t(k) = s \circ \sigma(k) = s(\sigma(k)) = s(n_k) = s_{n_k} \text{ for } k \in \mathbb{N}. \quad (3)$$

---

*Proof.* Let $s_n$ define a sequence and $t_k$ define a subsequence of $s_n$ defined as

$$(t_k)_{k\in\mathbb{N}} = s(\sigma_1(k)) = s(n_k) = s_{n_k}$$

where $\sigma_1$ is an increasing function from $\mathbb{N} \to \mathbb{N}$. We now define $a_l$ to be a subsequence of $(t_k)$ as

$$(a_l)_{l\in\mathbb{N}} = \sigma_1(\sigma_2(l)) = \sigma_1(k_l) = n_{k_l}$$

We have that $\sigma$ is an increasing mapping from $\mathbb{N} \to \mathbb{N}$ so we can redefine $a_l$ as

$$(a_l)_{l\in\mathbb{N}} = s(\sigma(l)) = s(n_{k_l}) = s_{n_{k_l}}$$

Thus, we have that $a_l$ is also a subsequence of $s_n$. □

## 64 Problem 11.7

Let $(r_n)$ be an enumeration of the set $\mathbb{Q}$ of all rational numbers. Show there exists a subsequence $(r_{n_k})$ such that $\lim_{k\to\infty} r_{n_k} = \infty$.

**Theorem 11.2(ii)**

*If the sequence $(s_n)$ is unbounded above, it has a subsequence with limit $\infty$.*

---

*Proof.* We use Theorem 11.2$(ii)$ to prove this claim. Pick $M > 0$. By denseness of $\mathbb{Q}$, we know there exists some $r_n$ inside of $(M, M+1)$. Thus, $r_n$ is unbounded above. Therefore, there exists some subsequence of $r_n$ such that $lim_{k\to\infty} r_{n_k} = +\infty$. □

## 65 Problem 11.8

Use Definition 10.6 and Exercise 5.4 to prove lim inf $s_n = -$lim sup$(-s_n)$ for every sequence $(s_n)$.

**Definition** 10.6

Let $s_n$ be a sequence in $\mathbb{R}$. We define

$$\text{lim sup } s_n = \ \lim_{N\to\infty} \sup\{s_n : n > N\}$$

and

$$\text{lim inf } s_n = \ \lim_{N\to\infty} \inf\{s_n : n > N\}$$

**Exercise 5.4**

For any non empty subset $S$ of $\mathbb{R}$,

$$\inf S = -\sup(-S)$$

---

*Proof.* We have that

$$inf\{s_n : n > N\} = -sup\{-s_n : n > N\}$$
$$\text{lim inf } s_n = \lim_{N\to\infty} \inf\{s_n : n > N\} = \ \lim_{N\to\infty} - \sup\{-s_n : n > N\} = -\text{lim sup } (-s_n)$$

□

## 66 Problem 11.9

$(a)$. Show the closed interval $[a, b]$ is a closed set.
$(b)$. Is there a sequence $(s_n)$ such that $(0, 1)$ is its set of subsequential limits?

---

*Proof.* $(a)$. Let $(r_n)$ be the enumeration of rational numbers in $(a, b)$. Thus, we have that for any $t \in [a, b]$, there exist an infinite number of rationals in between $(t-\varepsilon, t+\varepsilon)$ for any $\varepsilon > 0$ by denseness of $\mathbb{Q}$. Thus, we can describe $t$ to be a sequential limit of some subsequence of $(r_n)$ where the set of subsequential limits of $r_n$ is $[a, b]$, thus it is a closed set. □

*Proof.* $(b)$. By Theorem 11.9, we have that for any sequence $t_n$ in $S \cap \mathbb{R}$ where $S$ is the set of subsequential limits, lim $t_n = t$ where $t$ belongs to $S$. Thus, we have that $S$ is closed for any $s_n$. □

# 67 Problem 11.11

Let $S$ be a bounded set. Prove there is an increasing sequence $s_n$ of points in $S$ such that lim $s_n$ = sup $S$.

---

*Proof.* If sup $S$ is in $S$, it is sufficient to define $s_n = supS$ for all $n$. Thus, we consider when sup $S$ is not in $S$. Thus, we define $s = supS$. By denseness of $\mathbb{Q}$, we have that there exist infinitely many rationals in the interval $(s - \varepsilon, s)$ for all $\varepsilon > 0$. Since we have that $s$ is our sup $S$, all values within the interval exist in $S$. Thus, we construct a subsequence of $r_n$ values converging to $s$. Since a subsequence can only converge to a limit if its sequence is monotonic, we have that lim $s_n$ = sup $S$. □

# 68 Problem 12.1

Let $(s_n)$ and $(t_n)$ be sequences and suppose there exists $N_0$ such that $s_n \leq t_n$ for all $n > N_0$. Show lim inf $s_n \leq$ lim inf $t_n$ and lim sup $s_n \leq$ lim sup $t_n$.

---

*Proof.* We define

$$a_N = \inf\{s_n : n > N\}$$

and

$$b_N = \inf\{t_n : n > N\}$$

Since we have that $s_n \leq t_n$, $n > N_0$ implies

$$a_N \leq b_N$$

Thus,

$$lima_N \leq limb_N$$

Since $\lim a_N = \lim_{N\to\infty} \inf\{s_n : n > N\} = \liminf s_n$ and $\lim b_N = \lim_{N\to\infty} \inf\{t_n : n > N\} = \liminf t_n$, we have

$$\liminf s_n \leq \liminf t_n$$

We can do the same for sup to prove lim sup $s_n \leq$ lim sup $t_n$. □

# 69 Problem 12.2

Prove lim sup $|s_n| = 0$ if and only if lim $s_n = 0$.

---

*Proof.* Assume lim sup $|s_n| = 0$. Thus, we have that lim sup $s_n = \lim_{N\to\infty} v_N$ where $v_N = sup\{s_n : n > N\}$. Thus, it is true that

$$\begin{aligned} \limsup |s_n| &= -\liminf |-s_n| \\ &= -\liminf |s_n| \\ &= 0 \end{aligned}$$

Thus, it is shown that lim $|s_n| = 0$, thus implying lim $s_n = 0$.
Assume lim $s_n = 0$. Thus, we have that lim sup $s_n$ = lim $s_n$, thus implying lim sup $|s_n| = 0$. □

## 70 Problem 12.4

Show lim sup $(s_n + t_n) \leq$ lim sup $s_n$+ lim sup $t_n$ for bounded sequences $(s_n)$ and $(t_n)$. *Hint:* First show

$$\sup\{s_n + t_n : n > N\} \leq \sup\{s_n : n > N\} + \sup\{t_n : n > N\}$$

---

*Proof.* We have that for all $n > N$,

$$s_n \leq sup\{s_n : n > N\}$$

and

$$t_n \leq sup\{t_n : n > N\}$$

Thus, it is true that

$$s_n + t_n \leq sup\{s_n : n > N\} + sup\{t_n : n > N\}$$

Therefore, $sup\{s_n : n > N\} + sup\{t_n : n > N\}$ is an upper bound of $s_n + t_n$. Thus, we have that

$$sup\{s_n + t_n : n > N\} \leq sup\{s_n : n > N\} + sup\{t_n : n > N\}$$

Thus, it is true that

$$\text{lim sup } (s_n + t_n) \leq \text{lim sup } s_n + \text{ lim sup } t_n$$

□

## 71 Problem 12.5

Prove

$$\text{lim inf } (s_n + t_n) \geq \text{lim inf } s_n + \text{ lim inf } t_n$$

for bounded sequences $(s_n)$ and $(t_n)$.

---

*Proof.* We have that

$$s_n \geq inf\{s_n : n > N\}$$

and

$$t_n \geq inf\{t_n : n > N\}$$

Thus,

$$s_n + t_n \geq inf\{s_n : n > N\} + inf\{t_n : n > N\}$$

implying that $inf\{s_n : n > N\} + inf\{t_n : n > N\}$ is a lower bound of $s_n + t_n$. Therefore,

$$inf\{s_n + t_n : n > N\} \geq inf\{s_n : n > N\} + inf\{t_n : n > N\}$$

Thus, it is true that

$$\text{lim inf } (s_n + t_n) \geq \text{lim inf } s_n + \text{ lim inf } t_n$$

$\square$

*Proof.* Similarily, we know from Problem 12.4 that

$$\text{lim sup } (s_n + t_n) \leq \text{lim sup } s_n + \text{ lim sup } t_n$$

implying

$$-\text{lim sup } (-(s_n + t_n)) \geq -\text{lim sup } (-s_n) - \text{lim sup } (-t_n)$$

implying

$$\text{lim inf } (s_n + t_n) \geq \text{lim inf } (s_n) + \text{ lim inf } (t_n)$$

$\square$

## 72 Problem 12.10

Prove $(s_n)$ is bounded if and only if lim sup $|s_n| < +\infty$.

---

*Proof.* Assume $(s_n)$ to be bounded. Thus, there exists some $M > 0$ such that $|s_n| \leq M$ for all $n$. Therefore, we have that $sup\{|s_n| : n \in N\} \leq M < +\infty$, implying

$$\text{lim sup } |s_n| < +\infty$$

Assume lim sup $|s_n| < +\infty$. Then there exists some $M > 0$ such that lim sup $|s_n| < M$. Thus, we define a sequence $v_n = \sup\{|s_n| : n > N\}$ that converges to $M$, so there exists $n > N_1$ implying

$$|sup\{|s_n| : n > N_1\} - M| < 1$$

implying

$$|s_n| < M + 1$$

for all $n > N$. Thus, we pick $N_{max} = \{|s_1|, |s_2|, \ldots, |s_N|, M + 1\}$ such that

$$|s_n| \leq N_{max}$$

implying that $(s_n)$ is bounded. $\square$

## 73 Problem 12.11

Prove the first inequality of Theorem 12.2.

**Theorem** 12.2

*Let $(s_n)$ be any sequence of nonzero real numbers. Then we have*

$$\liminf \left|\frac{s_{n+1}}{s_n}\right| \le \liminf |s_n|^{1/n} \le \limsup |s_n|^{1/n} \le \limsup \left|\frac{s_{n+1}}{s_n}\right|$$

---

*Proof.* We will prove the first inequality. Let $\alpha = \liminf |s_n|^{1/n}$ and $L = \liminf |\frac{s_{n+1}}{s_n}|$. We need to prove $\alpha \ge L$. This is obvious if $L = -\infty$, so we assume $L > -\infty$. It is enough to show

$$\alpha \ge L_1 \text{ for any } L_1 < L$$

Since

$$L = \liminf |\tfrac{s_{n+1}}{s_n}| = \lim_{N\to\infty} \inf \{|\tfrac{s_{n+1}}{s_n}| : n > N\} > L_1,$$

there exists a positive integer $N$ such that

$$L_1 < \inf\{|\tfrac{s_{n+1}}{s_n}| : n \ge N\}$$

which implies

$$L_1 < |\tfrac{s_{n+1}}{s_n}| \text{ for } n \ge N$$

Now for $n > N$ we can write

$$|s_n| = |\tfrac{s_n}{s_{n-1}}| \cdot |\tfrac{s_{n-1}}{s_{n-2}}| \cdots |\tfrac{s_{N+1}}{s_N}| \cdot |s_N|$$

As there are $n - (N+1) + 1 = n - N$ fractions, we can simplify to

$$|s_n| > (L_1)^{n-N} |s_N| \text{ for } n > N$$

Since $N$ and $L_1$ are fixed, $a = L_1^{-N} \cdot |s_N|$ is a positive constant and we get that

$$|s_n| > L_1^n a \text{ for } n > N.$$

Therefore it exists that

$$|s_n|^{1/n} > L_1 a^{1/n} \text{ for } n > N.$$

Since $lima^{1/n} = 1$, we can further simplify to

$$L_1 < |s_n|^{1/n}$$

implying

$$\liminf |\tfrac{s_{n+1}}{s_n}| \le \liminf |s_n|^{1/n}$$

$\square$

## 74 Problem 12.12

Let $(s_n)$ be a sequence of nonnegative numbers, and for each $n$ define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$.
$(a)$. Show

$$\text{lim inf } s_n \le \text{lim inf } \sigma_n \le \text{lim sup } \sigma_n \le \text{lim sup } s_n$$

*Hint:* For the last inequality, show first that $M > N$ implies

$$\sup\{\sigma_n : n > M\} \le \tfrac{1}{M}(s_1 + s_2 + \cdots + s_N) + \sup\{s_n : n > N\}.$$

$(b)$. Show that if lim $s_n$ exists, then lim $\sigma_n$ exists and lim $\sigma_n$ = lim $s_n$.
$(c)$. Give an example where lim $\sigma_n$ exists, but lim $s_n$ does not exist.

---

*Proof.* Note the middle inequality is obvious. Thus, we prove the third inequality first. Let $M, N \in \mathbb{R}$ such that $M > N$ implies

$$\begin{aligned}
\sigma_n &= \frac{1}{n}(s_1, s_2, \ldots, s_N) + \frac{1}{n}(s_{N+1}, s_{N+2}, \ldots, s_n) \\
&\le \frac{1}{M}(s_1, s_2, \ldots, s_N) + \frac{1}{n}(s_{N+1}, s_{N+2}, \ldots, s_n) \\
&\le \frac{1}{M}(s_1, s_2, \ldots, s_N) + \frac{n-N}{n}\sup\{s_n : n > N\} \\
&\le \frac{1}{M}(s_1, s_2, \ldots, s_N) + \sup\{s_n : n > N\}
\end{aligned}$$

Thus, we have that $\frac{1}{M}(s_1, s_2, \ldots, s_N) + \sup\{s_n : n > N\}$ is an upper bound of $\{\sigma_n : n > M\}$, implying

$$\sup\{\sigma_n : n > M\} \le \tfrac{1}{M}(s_1, s_2, \ldots, s_N) + \sup\{s_n : n > N\}$$

Thus, we see that

$$\lim_{M\to\infty} \sup\{s_n : n > N\} = (s_1, s_2, \ldots, s_N)\cdot \lim_{M\to\infty} \tfrac{1}{M} + \sup\{s_n : n > N\}$$

which implies

$$\text{lim sup } \sigma_n = \sup\{s_n : n > N\}$$

Thus, we get that

$$\text{lim sup } \sigma_n \le \text{lim sup } s_n$$

Thus, this implies

$$-\text{lim sup}(-\sigma_n) \ge -\text{lim sup}(-s_n)$$

which implies

$$\text{lim inf } (\sigma_n) \ge \text{lim inf } (s_n)$$

Therefore, our proof is complete as we have shown both the first and third inequality. □

## 75 Problem 12.13

Let $(s_n)$ be a bounded sequence in $\mathbb{R}$. Let $A$ be the set of $a \in \mathbb{R}$ such that $\{n \in \mathbb{N} : s_n < a\}$ is finite, i.e. all but finitely many $s_n$ are $\geq a$. Let $B$ be the set of $b \in \mathbb{R}$ such that $\{n \in \mathbb{N} : s_n > b\}$ is finite. Prove $\sup A = \liminf s_n$ and $\inf B = \limsup s_n$.

---

*Proof.* We will prove $\sup A \geq \liminf s_n$ and $\sup A \leq \liminf s_n$. Consider $m < \liminf s_n$. Thus, there exists $N \in \mathbb{N}$ such that $\inf\{s_n : n > N\} > m$, implying $s_n > n$ for $n > N$. Thus, we have that $\{n : s_n < m\}$ is finite and thus $m \in A$ since $m \in A \forall m < \liminf s_n$. Thus, we have $\sup A \geq \liminf s_n$ For any $a \in A$, we have that the set $\{n : s_n < a\}$ is finite. Thus, there exists $N \in \mathbb{N}$ such that $n > N$ implies $s_n > a$. It follows that $\inf\{s_n : n > N\} \geq a$ which implies $\liminf\{s_n : n > N\} \geq a$. Thus, the set is an upper bound of $A$ and $\sup A \leq \inf\{s_n : n > N\}$, therefore proving $\sup A = \liminf s_n$. The proof for $B$ is similar. □

## 76 Problem 14.1

Determine which of the following series converge. Justify your answers.
$(a)$. $\sum \frac{n^4}{2^n}$
$(b)$. $\sum \frac{2^n}{n!}$
$(f)$. $\sum_{n=2}^{\infty} \frac{1}{logn}$

---

$(a)$. We have that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^4 \cdot 2^n}{n^4 \cdot 2^{n+1}} = \lim_{n\to\infty} \frac{(n+1)^4}{2n^4} = \frac{1}{2}$. Thus, as $\frac{1}{2} < 1$, the series converges.
$(b)$. $\lim_{n\to\infty} \frac{2^{n+1} n!}{(n+1)! \cdot 2^n} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$. Thus, the series is convergent.
$(f)$. We have that $\frac{1}{logn} \leq \frac{1}{n}$. Thus, since we know the series $\sum \frac{1}{n}$ is divergent, $\sum \frac{1}{logn}$ is divergent as well.

# 77 Problem 14.5

Suppose $\sum a_n = A$ and $\sum b_n = B$ where $A$ and $B$ are real numbers. Use limit theorems to quickly prove the following.
**(a).** $\sum(a_n + b_n) = A + B$.
**(b).** $\sum ka_n = kA$ for $k \in \mathbb{R}$.
**(c).** Is $\sum a_n b_n = AB$ a reasonable conjecture? Discuss.

---

*Proof.* $(a)$. Let us define $S_n = \sum_{k=1}^{n} a_k$ and $S_n^{'} = \sum_{k=1}^{n} b_k$. Thus, we have that $lim_{n\to\infty} S_n = A$ and $lim_{n\to\infty} S_n^{'} = B$. Let $\varepsilon > 0$. There exists some $N_1, N_2$ respectively such that $n \geq N_1, N_2$

$$|S_n - A| < \tfrac{\varepsilon}{2},\ |S_n^{'} - B| < \tfrac{\varepsilon}{2}$$

Thus, let us pick $N = max\{N_1, N_2\}$. We have that

$$|S_n - A| < \tfrac{\varepsilon}{2},\ |S_n^{'} - B| < \tfrac{\varepsilon}{2}$$

Thus,

$$\begin{aligned}\left|\sum_k (a_k + b_k - (A + B)\right| &= |(S_n - A) + (S_n^{'} - B)| \\ &\leq |S_n - A| + |S_n^{'} - B| \\ &< \varepsilon\end{aligned}$$

Thus, we have that the series $\sum(a_n + b_n)$ converges to $A + B$. $\square$

*Proof.* $(b)$. Let us define $S_n = \sum_{k=1}^{n} a_k = A$. Let $\varepsilon > 0$. There exists $N$ such that $n \geq N$ implies

$$|S_n - A| < \tfrac{\varepsilon}{k}$$

Thus,

$$\begin{aligned}\left|\sum_n ka_n - kA\right| &= |kS_n - kA| \\ &\leq k|S_n - A| \\ &< k \cdot \frac{\varepsilon}{k} = \varepsilon\end{aligned}$$

Thus, $\sum_k ka_n$ converges to $kA$. $\square$

$(c)$. We can simply consider two sequences, $a_n = \frac{1}{2^n}$ and $b_n = \frac{1}{4^n}$. Thus, we have that $\sum a_n = 2$ and $\sum b_n = \frac{4}{3}$. However, we have $\sum \frac{1}{8^n} = \frac{8}{7} \neq \frac{8}{3}$.

# 78 Problem 14.6

**(a).** Prove that if $\sum |a_n|$ converges and $(b_n)$ is a bounded sequence, then $\sum a_n b_n$ converges.
**(b).** Observe that Corollary 14.7 is a special case of part $(a)$.

**Corollary 14.7**

*Absolutely convergent series are convergent.*

---

*Proof.* (a). Assume $\sum |a_n|$ converges and $b_n$ is a bounded sequence. Thus, we know that $\exists M > 0$ s.t. $\forall n, |b_n| \leq M$. Thus, we have that $|a_n b_n| = |a_n| \cdot |b_n| \leq |a_n| \cdot M$. Thus, we have that since $\sum M|a_n| = M \sum |a_n|$ is bounded, $\sum |a_n b_n|$ converges so $\sum a_n b_n$ converges. □

*Proof.* (b). We set $b_n = 1 \forall n$. We know $b_n$ is bounded, and $\sum a_n b_n$ simplifies to $\sum a_n$, therefore since $|b_n| \leq 1$, Corollary 14.7 is a special case of $(a)$. □

# 79 Problem 14.7

Prove that if $\sum a_n$ is a convergent series of nonnegative numbers and $p > 1$, then $\sum a_n^p$ converges.

---

*Proof.* We use induction to prove the statement. Given $\sum a_n$ is convergent, we know $\lim_{n \to \infty} a_n = 0$, thus, $a_n$ is bounded. Therefore, there exists a real number $M > 0$ such that

$$a_n < M \text{ for all } n \in \mathbb{N}$$

Thus, we have that $a_n^2 < a_n \cdot M$ which implies $\sum a_n^2$ is convergent. Thus, we assume the statement true for $p = k$ and shall prove $k+1$. We know $a_n^k < M$ for all $n \in \mathbb{N}$, thus $a_n^{k+1} < M \cdot a_n^k$, so by comparison test our proof is complete. □

# 80 Problem 14.8

Show that if $\sum a_n$ and $\sum b_n$ are convergent series of nonnegative numbers, then $\sum \sqrt{a_n b_n}$ converges. *Hint*: Show $\sqrt{a_n b_n} \leq a_n + b_n$ for all $n$.

---

*Proof.* We are given that $\sum a_n$ and $\sum b_n$ are convergent series. Thus, $\lim_{n \to \infty} a_n = 0$ and $\lim_{n \to \infty} b_n = 0$. We define $S_n = \sum_{k=1}^n a_k$, $S_n' = \sum_{k=1}^n b_k$, $\lim_{n \to \infty} S_n = l$, and $\lim_{n \to \infty} S_n' = l'$. Thus, let $\varepsilon > 0$. There exists $n > N_1$ and $n > N_2$ such that

$$|S_n - l| < \varepsilon, \quad |S_n' - l'| < \varepsilon$$

Thus, we claim $N = max\{N_1, N_2\}$. Thus, we have that

$$\begin{aligned} \left| \frac{S_n + S_n'}{2} - \frac{l + l'}{2} \right| &= \frac{1}{2} |S_n - l + S_n' - l'| \\ &\leq \frac{1}{2} |S_n - l| + \frac{1}{2} |S_n' - l'| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Thus, we have that the series $\frac{1}{2} \sum \frac{a_n + b_n}{2}$ is convergent. Thus, through $AM - GM$, we have that $\sqrt{a_n b_n} \leq \frac{a_n + b_n}{2}$, thus, by comparison test $\sum \sqrt{a_n b_n}$ is convergent. □

## 81 Problem 15.1

Determine which of the following series converge. Justify your answers.
$(a)$. $\sum \frac{(-1)^n}{n}$
$(b)$. $\sum \frac{(-1)^n n!}{2^n}$

---

$(a)$. Consider $a_n = \frac{1}{n}$. We know the sequence is a monotone decreasing sequence since

$$n + 1 > n \Longrightarrow \frac{1}{n+1} < \frac{1}{n} \Longrightarrow a_{n+1} < a_n$$

Thus, as we have $\lim_{n\to\infty} \frac{1}{n} = 0$, the series is convergent. $(b)$. Consider $a_n = \frac{n!}{2^n}$. By ratio test we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)! 2^n}{n! 2^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{n+1}{2} \right| = \infty.$$

Thus, as the series is divergent, we have that $\sum \frac{(-1)^n n!}{2^n}$ is divergent.

## 82 Problem 15.7

**(a).** Prove if $(a_n)$ is a decreasing sequence of real numbers and if $\sum a_n$ converges, then $\lim na_n = 0$. *Hint*: Consider $|a_{N+1} + a_{N+2} + \cdots + a_n|$ for suitable $N$.
**(b).** Use $(a)$ to give another proof that $\sum \frac{1}{n}$ diverges.

---

*Proof.* $(a)$. Since $\sum a_n$ converges, we have that for some $\varepsilon > 0$ there exists $M > 0$ s.t.

$$|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| < \frac{\varepsilon}{2} \text{ for all } n \geq m \text{ and for every } p \in \mathbb{N}$$

Consider $n = m$, Since $a_n$ is a decreasing sequence, we have $a_{m+1} + a_{m+2} + \cdots + a_{m+p} \geq pa_{m+p}$. Set $p = m$. Thus, we have $2ma_{2m} < \varepsilon$. Take $p = m + 1$. Thus, $(m + 1)a_{2m+1} < \frac{\varepsilon}{2}$. It follows that

$$(2m + 1)a_{2m+1} < (2m + 2)a_{2m+1} < \varepsilon$$

Therefore $na_n < \varepsilon$ for all $n \geq 2m$, therefore $\lim_{n\to\infty} na_n = 0$. □

$(b)$. $\frac{1}{n}$ is both monotonically decreasing. If we want $\sum \frac{1}{n}$ to converge, then $\lim_{n\to\infty} n \cdot \frac{1}{n} = 1 \neq 0$ must follow, but since it is false, $\sum \frac{1}{n}$ does not converge.

## 83 Problem 17.1

Let $f(x) = \sqrt{4 - x}$ for $x \leq 4$ and $g(x) = x^2$ for all $x \in \mathbb{R}$.
**(a).** Give the domains of $f + g, fg, f \circ g$, and $g \circ f$.
**(b).** Find the values $f \circ g(0), g \circ f(0), f \circ g(1), g \circ f(1), f \circ g(2)$, and $g \circ f(2)$.
**(c).** Are the functions $f \circ g$ and $g \circ f$ equal?
**(d).** Are $f \circ g(3)$ and $g \circ f(3)$ meaningful?

---

$(a)$. We have that $f + g = \sqrt{4 - x} + x^2$. Thus, the domain is $\{x : x \leq 4\}$ since $\mathrm{dom}(f) \cap \mathrm{dom}(g) = \{x : x \leq 4\}$. The domain of $fg$ is same as well. We have that $f \circ g$ is $f(g(x)) = \sqrt{4 - x^2}$. Thus, this is valid when $4 - x^2 \geq 0, x \leq 2$. Thus, our domain is $\{x : |x| \leq 2\}$. For $g \circ f$, we have that $g(f(x)) = (\sqrt{4 - x})^2$. Thus, we have that our domain is $\{x : x \leq 4\}$.
$(d)$. $f \circ g(3)$ is not meaningful because it does not fall within our domain. $g \circ f(3)$ is meaningful because it falls within our domain. Thus, $g \circ f(3) = g(f(3)) = g(1) = 1$.

## 84 Problem 17.4

Prove the function $\sqrt{x}$ is continuous on its domain $[0, \infty)$.

---

*Proof.* Let $x_0$ be in the domain $[0, \infty)$ and let $\varepsilon > 0$. Thus, we want to show

$$|f(x) - f(x_0)| < \varepsilon$$

provided $|x - x_0| < \delta$. Thus, we observe

$$\begin{aligned} f(x) - f(x_0) &= \sqrt{x} - \sqrt{x_0} \\ &= (\sqrt{x} - \sqrt{x_0}) \cdot \frac{\sqrt{x} + \sqrt{x_0}}{\sqrt{x} + \sqrt{x_0}} \\ &= \frac{x - x_0}{\sqrt{x} + \sqrt{x_0}} \end{aligned}$$

We observe two cases. Case 1 : $x_0 = 0$. Thus, we let $\delta = \varepsilon^2$. Then $|x - x_0| < \delta$ implies

$$\begin{aligned} f(x) - f(x_0) &= \frac{x - x_0}{\sqrt{x} + \sqrt{x_0}} \\ &= \frac{x - 0}{\sqrt{x} + 0} \\ &= \sqrt{x} \\ &< \varepsilon \end{aligned}$$

Thus, we have that $|f(x) - f(x_0)| < \varepsilon$. Next, consider when $x_0 > 0$. Thus we let $\delta = \sqrt{x_0}\varepsilon$. Then $|x - x_0| < \delta$ implies

$$\begin{aligned} f(x) - f(x_0) &= \frac{x - x_0}{\sqrt{x} + \sqrt{x_0}} \\ &\leq \frac{|x - x_0|}{\sqrt{x_0}} \\ &= \varepsilon \end{aligned}$$

Thus, we have that $|f(x) - f(x_0| < \varepsilon$. □

## 85 Problem 17.5

**(a).** Prove that if $m \in \mathbb{N}$, then the function $f(x) = x^m$ is continuous on $\mathbb{R}$.
**(b).** Prove every *polynomial function* $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is continuous on $\mathbb{R}$.

---

*Proof.* $(a)$. Let $f(x) = x^m$. Suppose we have $\lim_{n\to\infty} x_n = x_0$. Thus, we have that

$$\begin{aligned} \lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} [x_n]^m \\ &= x_0^m = f(x_0) \end{aligned}$$

□

*Proof.* $(b)$. By Theorem 17.4$(i)$ and $(a)$ we have that the polynomial function is continuous on $\mathbb{R}$. □

## 86 Problem 17.9

Prove each of the following functions in continuous at $x_0$ by verifying the $\varepsilon - \delta$ property of Theorem 17.2.
$(a)$. $f(x) = x^2, x_0 = 2$;
$(b)$. $f(x) = \sqrt{x}, x_0 = 0$;
$(c)$. $f(x) = x\sin(\frac{1}{x})$ for $x \neq 0$ and $f(0) = 0, x_0 = 0$;
$(d)$. $g(x) = x^3, x_0$ arbitrary. *Hint:* $x^3 - x_0^3 = (x - x_0)(x^2 + x_0x + x_0^2)$.

---

*Proof.* $(a)$. We want to show that $|f(x) - f(x_0)| < \varepsilon$. Thus, we have that

$$\begin{aligned} |f(x) - f(2)| &= |x^2 - 2^2| \\ &= |(x-2)(x+2)| \\ &= |x-2| \cdot |x+2| \\ &\leq |x-2| \cdot (|x| + |2|) \end{aligned}$$

Thus, let $\varepsilon > 0$. $|x - 2| < \delta$ implies

$$\begin{aligned} |x| - |2| \geq |x-2| &< \delta \\ |x| &< \delta + |2| \\ |x| &< \delta + 2 \end{aligned}$$

Thus, we have that

$$\begin{aligned} |f(x) - f(2)| &= |x-2| \cdot |x+2| \\ &\leq |x-2| \cdot (|x| + |2|) \\ &< |x-2| \cdot (\delta + 4) \end{aligned}$$

For $\delta = 1$, we have $|f(x) - f(2)| < |x - 2| \cdot (5) = 5|x - 2|$. Thus, setting $5|x - 2| < \varepsilon$, we get that $|x - 2| < \frac{\varepsilon}{5}$. Thus, we have $\delta = min\{1, \frac{\varepsilon}{5}\}$. Therefore $|x - 2| < \delta$ implies $|f(x) - f(2)| < \varepsilon$. Therefore the function is continuous at $x_0 = 2$. □

*Proof.* $(b)$. We have that

$$\begin{aligned} |f(x) - f(0)| &= |\sqrt{x} - \sqrt{0}| \\ &= |\sqrt{x}| \\ &= \sqrt{x} \end{aligned}$$

Thus, we let $\delta = \varepsilon^2$. Therefore $|x-0| < \delta$ implies $|f(x)-f(0)| < \varepsilon$. Therefore the function is continuous at $x_0 = 0$. □

*Proof.* $(c)$. We have that

$$\begin{aligned} |f(x) - f(0)| &= |xsin(\frac{1}{x}) - 0| \\ &= |xsin(\frac{1}{x})| \\ &\leq x \end{aligned}$$

Let $\varepsilon > 0$ and $\delta = \varepsilon$. Thus, $|x - 0| < \delta$ implies $|f(x) - f(0)| < \varepsilon$. □

*Proof.* $(d)$. We have that

$$\begin{aligned} |g(x) - g(x_0)| &= |x^3 - x_0^3| \\ &= |x - x_0| \cdot |x^2 + xx_0 + x_0^2| \end{aligned}$$

Thus, $|x - x_0| < \delta$ implies

$$|x| - |x_0| \leq |x - x_0| < \delta$$
$$|x| < \delta + |x_0|$$

Thus, we have that

$$\begin{aligned} |g(x) - g(x_0)| &= |x - x_0| \cdot |x^2 + xx_0 + x_0^2| \\ &< |x - x_0| \cdot |(|x_0| + \delta)^2 + (|x_0| + 1)x_0 + x_0^2| \\ &= |x - x_0| \cdot N \end{aligned}$$

Thus, for $\delta = 1$, we have that $|g(x) - g(x_0)| < |x - x_0| \cdot N$. Thus, for $|x - x_0| \cdot N < \varepsilon$, we have $|x - x_0| < \frac{\varepsilon}{N}$. Thus, $\delta = \min\{1, \frac{\varepsilon}{N}\}$ has $|x - x_0| < \delta$ implying $|g(x) - g(x_0)| < \varepsilon$. Therefore $g(x)$ is continuous at $x_0$ arbitrary. ☐

## 87 Problem 17.11

Let $f$ be a real valued function with $\mathrm{dom}(f) \subseteq \mathbb{R}$. Prove $f$ is continuous at $x_0$ if and only if, for every *monotonic* sequence $(x_n)$ in $\mathrm{dom}(f)$ converging to $x_0$, we have $\lim f(x_n) = f(x_0)$. *Hint:* Don't forget Theorem 11.4.

**Theorem 11.4**

*Every sequence $(x_n)$ has a monotonic subsequence, and if the sequence is convergent, then its subsequence also converges to the same point.*

---

*Proof.* ($\Longrightarrow$) Assume $f$ is a continuous function at $x_0 \in \mathrm{dom}(f)$. We want to show for every monotonic sequence $x_n$ in $\mathrm{dom}(f)$ converging to $x_0$, $\lim_{n\to\infty} f(x_n) = f(x_0)$ holds. By Theorem 17.1, we have this statement to be true for every sequence $x_n$ converging to $x_0$. By Theorem 11.4 we have that every sequence has a monotonic subsequence $x_n$ and if the sequence is convergent then the subsequence also converges to the same point.
($\Longleftarrow$) Suppose for every monotonic sequence $x_n$ in $\mathrm{dom}(f)$ we have that $\lim_{n\to\infty} f(x_n) = f(x_0)$ is true. We claim $f(x)$ is a discontinuous function. Therefore, for all $\varepsilon, \delta > 0$, there exists $y_0 \in \mathrm{dom}(f)$ such that for all $y$ in $\mathrm{dom}(f)$, $|y - y_0| < \delta$ implies $|f(y) - f(y_0)| \geq \varepsilon$. Let $\varepsilon_0 > 0$ and $\delta_n = \frac{1}{n}$ for $n \in \mathbb{N}$. Thus, we find a sequence $y_n$ such that it satisfies our conditions and $y_n$ is convergent towards $y_0$. However, by Theorem 11.4, we have a contradiction, therefore implying $f(x)$ is continuous. ☐

## 88 Problem 17.14

For each nonzero rational number $x$, write $x$ as $\frac{p}{q}$ where $p, q$ are integers with no common factors and $q > 0$, and then define $f(x) = \frac{1}{q}$. Also define $f(0) = 1$ and $f(x) = 0$ for all $x \in \mathbb{R}\backslash\mathbb{Q}$. Thus $f(x) = 1$ for each integer, $f(\frac{1}{2}) = f(-\frac{1}{2}) = f(\frac{3}{2}) = \cdots = \frac{1}{2}, etc.$ Show $f$ is continuous at each point of $\mathbb{R}\backslash\mathbb{Q}$ and discontinuou at each point of $\mathbb{Q}$.

---

*Proof.* We have the function $f : \mathbb{R} \to \mathbb{R}$ to be defined as

$$\begin{cases} 1 & x = 0 \\ \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q}] \\ 0 & x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$$

Thus, pick $x_0 \in \mathbb{Q}$ arbitrarily. We want to show $f$ is discontinuous at $x_0$, therefore we let $\varepsilon, \delta > 0$ such that there exists $x \in \operatorname{dom}(f)$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| > \varepsilon$. Thus, pick $\varepsilon = 2|f(x_0)|$. Thus, by denseness of irrationals, we have that there exists an non-rational real number $x \in (x_0 - \delta, x_0 + \delta)$, which for $x$, $|x - x_0| > \delta$ and $|f(x) - f(x_0)| = |f(x_0)| < 2|f(x_0)| = \varepsilon$. Thus, we have that $f$ is discontinuous at each point $\mathbb{Q}$.
Next pick $x_0 \in \mathbb{R}\backslash\mathbb{Q}$ arbitrarily. We want to show $f$ is continuous in $x_0$, therefore we let $\varepsilon, \delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$. Thus, we have that $|f(x) - f(x_0)| = |f(x)| < \varepsilon$. Hence $f$ is continuous at each point $\mathbb{R}\backslash\mathbb{Q}$. □

## 89 Problem 17.16

The postage-stamp function $P$ is defined by $P(x) = A$ for $0 \leq x < 1$ and $P(x) = A + Bn$ for $n \leq x < n+1$. The definition of $P$ means that $P$ takes the value $A$ on the interval $[0,1)$, the value $A+B$ on the interval $[1,2)$, etc. Show $P$ is discontinuous at every positive integer.

---

*Proof.* We have

$$\begin{cases} A & 0 \leq x < 1 \\ A+B & 1 \leq x < 2 \\ & . \\ & . \\ A+Bn & n \leq x < n+1 \\ & . \\ & . \end{cases}$$

We claim $P$ is discontinuous. Thus, for $x = 1$, we have that

$$\begin{aligned} \lim_{x \to 1^-} P(x) &= \lim_{x \to 1^-} A \\ &= A \end{aligned}$$

$$\begin{aligned} \lim_{x \to 1^+} P(x) &= \lim_{x \to 1^+} A + B \\ &= A + B \end{aligned}$$

We have proved our base. Thus, we assume $P$ to be discontinuous for $n$, and therefore prove it for $n+1$.

$$\begin{aligned} \lim_{x \to (n+1)^-} P(x) &= \lim_{x \to (n+1)^-} A + Bn \\ &= A + Bn \end{aligned}$$

$$\begin{aligned} \lim_{x \to (n+1)^+} P(x) &= \lim_{x \to (n+1)^+} A + Bn \\ &= A + B(n+1) \end{aligned}$$

Thus, we have $P$ to be discontinuous for $n+1$, therefore our function is discontinuous at every positive integer. □

## 90 Problem 18.1

Let $f$ be as in Theorem 18.1. Show that if $f$ assumes its maximum at $x_0 \in [a,b]$, then $f$ assumes its minimum at $x_0$.

---

*Proof.* Let $f$ be a continuous real-valued function on a closed interval $[a,b]$. We first show $f$ is bounded. Assume $f$ is not bounded on $[a,b]$. Then for each $n \in \mathbb{N}$ there corresponds an $x_n \in [a,b]$ such that $|f(x_n)| > n$. By the Bolzano-Weierstrass Theorem $(x_n)$ has a subsequence $(x_{n_k})$ that also converges to some real number $x_0$. Thus, since $x_0$ is in $[a,b]$, we have that $\lim_{n\to\infty} f(x_{n_k}) = f(x_0) \neq \infty$. As we have a contradiction, $f$ is bounded. Now we let $M = \sup\{f(x) : x \in [a,b]\}$ where $M$ is finite as previously shown. Thus, we claim that the maximum of $-f$ is $x_0 \in [a,b]$ such that $-f(x_0) \leq -f(x)$, implying $f(x_0) \geq f(x)$. Thus, we have that $f(x)$ is the minimum at $x_0$. □

## 91 Problem 18.2

Reread the proof of Theorem 18.1 with $[a,b]$ replaced by $(a,b)$. Where does it break down? Discuss.

---

We have two cases in which the proof breaks down. *Case* 1*:* When showing a function to be bounded. Thus, take $f(x) = \frac{1}{x}$ for example. We have that it is continuous but unbounded on $(0,1)$. $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{1}{n} = 0 \notin (0,1)$. *Case* 2*:* $g(x) = x^2$. We have the function to be continuous and bounded on $(-1,1)$, but it does not contain a max.

## 92 Problem 18.4

Let $S \subseteq \mathbb{R}$ and suppose there exists a sequence $(x_n)$ in $S$ converging to a number $x_0 \notin S$. Show there exists an unbounded continuous function on $S$.

---

*Proof.* Define function $f(x_n) = \frac{1}{|x_n - x_0|}$. We have the function to be continuous on $S$. Thus, let $M > 0$. We have there exists $N$ implying $n > N$ such that $|x_n - x_0| < M$. Thus $\frac{1}{|x_n - x_0|} > M$. Thus $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{1}{|x_n - x_0|} = +\infty$. Therefore $f(x_n)$ is continuous and unbounded on $S$. □

## 93 Problem 18.5

$(a)$ Let $f$ and $g$ be continuous functions on $[a,b]$ such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Prove $f(x_0) = g(x_0)$ for at least one $x_0$ in $[a,b]$.

---

*Proof.* $(a)$. Let $f$ and $g$ be continuous functions on $[a,b]$. Let $h(x) = f(x) - g(x)$. Thus, on $a$, we have $h(a) = f(a) - g(a) \geq 0$. On $b$, we have $h(b) = f(b) - g(b) \leq 0$. Therefore it is true that $h(b) \leq 0 \leq h(a)$. Therefore by IVT there exists some value $x_0$ in $[a,b]$ such that $h(x_0) = 0$. Therefore we have that $h(x_0) = f(x_0) - g(x_0) = 0$ implying $f(x_0) = g(x_0)$ on some $x_0$ in $[a,b]$. □

# 94 Problem 18.9

Prove that a polynomial function $f$ of odd degree has at least one real root. *Hint:* It may help to consider the cases, $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$.

---

*Proof.* Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ where $a_n \neq 0$ and $n$ is odd degree. We observe that $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = x^n(a_n + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n})$ Thus, consider when $a_n > 0$. We have that $\lim_{n\to\infty} f(x) = \infty$ and $\lim_{n\to-\infty} f(x) = -\infty$. Thus, there exists some $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) > 0$ and $f(x_2) < 0$. Thus by IVT we have $f(x_2) < 0 < f(x_1)$ implying there is some point $x \in (x_2, x_1)$ such that $f(x) = 0$. We apply the same logic where $a_n < 0$. □

# 95 Problem 18.10

Suppose $f$ is continuous on $[0, 2]$ and $f(0) = f(2)$. Prove there exist $x, y$ in $[0, 2]$ such that $|y - x| = 1$ and $f(x) = f(y)$. *Hint:* Consider $g(x) = f(x + 1) - f(x)$ on $[0, 1]$.

---

*Proof.* We consider $g(x) = f(x + 1) - f(x)$ which is continuous on $[0, 1]$. Thus, we have that $g(0) = f(1) - f(0) = f(1) - f(2) = -[f(2) - f(1)] = -g(1)$. Thus, we consider three cases. *Case* 1*:* $g(0) = 0$ implies $g(1) = 0$, thus, $x = 0$ is a zero of $g$. *Case* 2*:* $g(0) > 0$ implies $g(1) < 0$. Thus, by IVT this implies there is some $x$ in $[0, 1]$ such that $g(x) = 0$. *Case* 3*:* $g(0) < 0$ implies $g(1) > 0$, thus by IVT there is some $x \in [0, 1]$ such that $g(x) = 0$. In all cases there is some $x \in [0, 1]$ such that $g(x) = 0$. This implies $g(x) = f(x + 1) - f(x) = 0$ implying $f(x + 1) = f(x)$. We set $y = x + 1$. Thus, $|y - x| = |x + 1 - x| = 1$ and $f(y) = f(x)$. □

# 96 Problem 19.1

Which of the following continuous functions are uniformly continuous on the specified set? Justify your answers. Use any theorems you wish.
$(a)$. $f(x) = x^{17} \sin x - e^x \cos 3x$ on $[0, \pi]$.
$(c)$. $f(x) = x^3$ on $(0, 1)$.
$(d)$. $f(x) = x^3$ on $\mathbb{R}$.
$(e)$. $f(x) = \frac{1}{x^3}$ on $(0, 1]$.
$(f)$. $f(x) = \sin\frac{1}{x^2}$ on $(0, 1]$.

---

$(a)$. We note that $x^{17}$, sin$x$, $e^x$, and cos$3x$ are all continuous on $[0, \pi]$. Thus, the function is uniformly continuous on $[0, \pi]$.
$(c)$. We note that $\tilde{f}$ is an extension of a function $f$ if

$$\text{dom}(f) \subseteq \text{dom}(\tilde{f}) \text{ and } f(x) = f(\tilde{x})$$

for all $x \in \text{dom}(f)$. Thus, let $f(x) = x^3$. We define

$$f(\tilde{x}) = \begin{cases} 0 & , x = 0 \\ x^3 & , x \in (0,1) \\ 1 & , x = 1 \end{cases}$$

as an extension of $f$. We note that dom$(f) = (0, 1)$ and dom$(\tilde{f}) = [0, 1]$. From part $(b)$ we know that $\tilde{f}$ is continuous on $[0, 1]$, therefore $f(x) = x^3$ is unformly continuous on $(0, 1)$.

*Proof.* $(d)$. We claim $f$ is not uniformly continuous on $\mathbb{R}$. Thus, we define $\varepsilon = 1$ for any $\delta > 0$ such that there exists $x, y \in \mathbb{R}$ implying

$$|x - y| < \delta \text{ and } |x^3 - y^3| \geq 1$$

We choose $y = x + \frac{\delta}{2}$. Thus, we have that

$$\begin{aligned} |x^3 - y^3| &= \left| x^3 - (x + \frac{\delta}{2})^3 \right| \\ &= \frac{\delta^3}{8} + 3x^2\frac{\delta}{2} + 3x\frac{\delta^2}{2} \\ &> \frac{3}{2}x^2\delta \end{aligned}$$

Thus, we know that

$$\begin{aligned} \frac{3}{2}x^2\delta &= 1 \\ x &= \sqrt{\frac{2}{3\delta}} \end{aligned}$$

Thus, for $\delta > 0$ we have the previous true, implying $|x - y| = |x - x - \frac{\delta}{2}| = \frac{\delta}{2} < \delta$ implying $|x^3 - y^3| > \frac{3}{2}x^2\delta = 1 = \varepsilon$. Thus, $f(x) = x^3$ is not uniformly continuous on $\mathbb{R}$. $\square$

*Proof.* $(e)$. We claim $f$ is not uniformly continuous on $(0, 1]$. Thus, assume the sequence $s_n = \frac{1}{n}, n \in \mathbb{N}$. We know $s_n$ is Cauchy. Since $f(s_n) = n^3$, it is not Cauchy, therefore $f$ cannot be uniformly continuous on $(0, 1]$ by Theorem 19.4. $\square$

*Proof.* $(f)$. We claim $f$ is not uniformly continuous on $(0, 1]$. We define the sequence $s_n = \frac{1}{\sqrt{\frac{\pi}{2}+n\pi}}$, where $s_n$ is Cauchy. However, we have that $f(s_n) = sin(\frac{\pi}{2}+n\pi)$ where $f(s_n)$ is not Cauchy. Therefore, $f$ is not uniformly continuous on $(0, 1]$ by Theorem 19.4. $\square$

## 97 Problem 19.2

Prove each of the following functions is uniformly continuous on the indicated set by directly verifying the $\varepsilon - \delta$ property in Definition 19.1.
$(a)$. $f(x) = 3x + 11$ on $\mathbb{R}$.
$(b)$. $f(x) = x^2$ on $[0, 3]$.
$(c)$. $f(x) = \frac{1}{x}$ on $[\frac{1}{2}, \infty)$.

---

*Proof.* $(a)$. We have a function to be uniformly continuous on $S$ if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x_0, x \in S, |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$$

Let $f(x) = 3x + 11$. We have that $3|x - y| = |3x - 3y| = |(3x + 11) - (3y + 11)| = |f(x) - f(y)|$. Thus, we let $\varepsilon > 0$ and $\delta = \frac{\varepsilon}{3}$. This implies $|f(x) - f(y)| = 3|x - y| < \varepsilon$. □

*Proof.* $(b)$. Let $f(x) = x^2$. We have that $|x^2 - y^2| = |(x - y)(x + y)| \le 6|x - y|$. Thus, let $\varepsilon > 0$ and $\delta = \frac{\varepsilon}{6}$. $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. □

*Proof.* $(c)$. Let $f(x) = \frac{1}{x}$. We have that

$$\begin{aligned} \left|\frac{1}{x} - \frac{1}{y}\right| &= |\frac{y - x}{xy}| \\ &= \frac{x - y}{xy} \\ &\le 4|x - y| \end{aligned}$$

Thus let $\varepsilon > 0$ and $\delta = \frac{\varepsilon}{4}$. $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. □

## 98 Problem 19.3

Repeat Exercise 19.2 for the following.
$(a)$. $f(x) = \frac{x}{x+1}$ on $[0, 2]$
$(b)$. $f(x) = \frac{5x}{2x-1}$ on $[1, \infty)$.

---

*Proof.* $(a)$. Let $f(x) = \frac{x}{x+1}$. We have that

$$\begin{aligned} \left|\frac{x}{x + 1} - \frac{y}{y + 1}\right| &= \left|\frac{x(y + 1) - y(x + 1)}{(x + 1)(y + 1)}\right| \\ &= \left|\frac{x - y}{(x + 1)(y + 1)}\right| \\ &\le |x - y| \end{aligned}$$

Thus, let $\varepsilon > 0$ and $\delta = \varepsilon$. $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. □

*Proof.* $(b)$. Let $f(x) = \frac{5x}{2x-1}$ on $[1, \infty)$. We have that

$$\begin{aligned} \left|\frac{5x}{(2x - 1)} - \frac{5y}{2y - 1}\right| &= \left|\frac{5x(2y - 1) - 5y(2x - 1)}{(2x - 1)(2y - 1)}\right| \\ &= \left|\frac{10xy - 5x - 10xy - 5y}{(2x - 1)(2y - 1)}\right| \\ &= \left|\frac{5(x - y)}{(2x - 1)(2y - 1)}\right| \\ &\le 5|x - y| \end{aligned}$$

Thus, let $\varepsilon > 0$ and $\delta = \frac{\varepsilon}{5}$. We know the rest. □

## 99 Problem 19.4

$(a)$. Prove that if $f$ is uniformly continuous on a bounded set $S$, then $f$ is a bounded function on $S$. *Hint:* Assume not.

---

*Proof.* $(a)$. Let $f$ be uniformly continuous on a bounded set $S$. Assume $f$ is not a bounded function on $S$. Thus, this implies there exists a sequence $(x_n) \in S$ such that

$$\lim_{n\to\infty} |f(x_n)| = \infty.$$

Thus, since $S$ is bounded, so is $x_n$. Therefore there is a convergent subsequence $(x_{n_k})$. Therefore it is Cauchy as well. However, since the limit converges to $\infty$, this is a contradiction, thus $f$ is a bounded function on $S$. $\square$

## 100 Problem 19.5

Which of the following continuous functions is uniformly continuous on the specified set? Justify your answers, using appropriate theorems.
$(a)$. $\tan x$ on $[0, \frac{\pi}{4}]$.
$(b)$. $\tan x$ on $[0, \frac{\pi}{2})$.
$(c)$. $\frac{1}{x}\sin^2 x$ on $(0, \pi]$.
$(d)$. $\frac{1}{x-3}$ on $(0, 3)$.
$(f)$ $\frac{1}{x-3}$ on $(4, \infty)$.

---

$(a)$. $f$ is continuous on the closed interval, thus is uniformly continuous on $[0, \frac{\pi}{4}]$.
$(b)$. We claim that $f$ is not uniformly continuous on $[0, \frac{\pi}{2})$. We thus check

$$\lim_{x\to\frac{\pi}{2}^-} f(x) = \infty.$$

Thus, this implies $f$ is unbounded on $[0, \frac{\pi}{2})$.
$(c)$. We define the extension function $\tilde{f}$ of $f$ to be

$$\tilde{f}(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{x}\sin^2 x & x \in (0, \pi] \end{cases}$$

Since $f$ is a continuous function, it is uniformly continuous on the specified set.
$(d)$. We claim $f$ is not uniformly continuous on $(0, 3)$. We check

$$\lim_{x\to 3^-} f(x) = -\infty.$$

Thus, we have that $f$ is not uniformly continuous on $(0, 3)$.
$(f)$. Let $f(x) = \frac{1}{x-3}$. We have that

$$\begin{aligned} \left|\frac{1}{x-3} - \frac{1}{y-3}\right| &= \left|\frac{y-3+3-x}{(x-3)(y-3)}\right| \\ &= \frac{|x-y|}{(x-3)(y-3)} \\ &\le |x-y| \end{aligned}$$

Thus, for $\varepsilon > 0$ let $\delta = \varepsilon$. $|x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Thus, $f$ is uniformly continuous on $(4, \infty)$.

## 101 Problem 19.6

$(a)$. Let $f(x) = \sqrt{x}$ for $x \geq 0$. Show $f'$ is unbounded on $(0,1]$ but $f$ is nevertheless uniformly continuous on $(0,1]$. Compare with Theorem 19.6.
$(b)$. Show $f$ is uniformly continuous on $[1,\infty)$.

---

*Proof.* $(a)$. We first show uniform continuity of $f$. We consider the function $f(x) = \sqrt{x}$ on $x \in (0,1]$. We define the extension $\tilde{f}$ to be

$$\tilde{f}(x) = \begin{cases} 0 & x = 0 \\ \sqrt{x} & x \in (0,1] \end{cases}$$

Thus, as $\tilde{f}$ is continuous on $[0,1]$, we have that $f$ is uniformly continuous on $(0,1]$. Next we consider $f'(x) = \frac{1}{2\sqrt{x}}$. We have that

$$\lim_{x\to 0^+} f(x) = \infty.$$

Thus, we have that $f'$ is unbounded but $f$ is still uniformly continuous. □

*Proof.* $(b)$. Let $f(x) = \sqrt{x}$. We have that

$$\begin{aligned} |\sqrt{x} - \sqrt{y}| &= \left| \frac{x-y}{\sqrt{x}+\sqrt{y}} \right| \\ &\leq \frac{|x-y|}{2} \end{aligned}$$

Thus, let $\varepsilon > 0$ and $\delta = 2\varepsilon$. $|x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. □

## 102 Problem 19.8

$(a)$. Use the MVT to prove

$$|\sin x - \sin y| \leq |x-y|$$

for all $x, y$ in $\mathbb{R}$; see the proof of Theorem 19.6.
$(b)$. Show $\sin x$ is uniformly continuous on $\mathbb{R}$.

---

*Proof.* $(a)$. Let $f(x) = \sin x$. We want to show that $|\sin x - \sin y| \leq |x-y|$. We have by the MVT that

$$\begin{aligned} f'(c) &= \frac{f(x) - f(c)}{x-c} \\ \cos x &= \frac{\sin x - \sin c}{x-c} \\ \frac{\sin x - \sin c}{x-c} &\leq 1 \\ \sin x - \sin c &\leq x - c \end{aligned}$$

Thus, it is true that $|\sin x - \sin y| \leq |x-y|$. □

*Proof.* Let $f(x) = \sin x$. We have that

$$\begin{aligned} |\sin x - \sin y| &\leq |x-y| \\ &< \delta = \varepsilon \end{aligned}$$

Thus, let $\varepsilon > 0$ and $\delta = \varepsilon$. $|x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. □

## 103 Problem 19.12

Let $f$ be a continuous function on $[a, b]$. Show the function $f^*$ defined by $f^*(x) = \sup\{f(y) : a \le y \le x\}$, for $x \in [a, b]$, is an increasing continuous function on $[a, b]$.

---

*Proof.* We first show $f^*$ to be increasing. Let $x_1, x_2 \in [a, b]$ such that $x_1 \le x_2$. Thus, we have that

$$\{f(y) : a \le y \le x_1\} \subseteq \{f(y) : a \le y \le x_2\}$$

and thus

$$f^*(x_1) = \sup\{f(y) : a \le y \le x_1\} \le \sup\{f(y) : a \le y \le x_2\} = f^*(x_2)$$

Thus, $f^*$ is increasing. Next, we show $f^*$ is continuous. Let $\varepsilon, \delta > 0$ such that $|x - x_0| < \delta$ implies $|f^*(x) - f^*(x_0)| < \varepsilon$. Let there be $|f(x) - f(x_0)| < \frac{\varepsilon}{2}$. Thus, we have that

$$\begin{aligned} |f^*(x) - f^*(x_0)| &= |\sup\{f(y) : a \le y \le x\} - \sup\{f(y) : a \le y \le x_1\}| \\ &\le |\sup\{f(y) - f(x_0) : \min\{x, x_0\} \le y \le \max\{x, x_0\}| \\ &\le \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

Thus, $f^*$ is continuous. □

## 104 Problem 20.17

Show that if $\lim_{x \to a^+} f_1(x) = \lim_{x \to a^+} f_3(x) = L$ and if $f_1(x) \le f_2(x) \le f_3(x)$ for all $x$ in some interval $(a, b)$, then $\lim_{x \to a^+} f_2(x) = L$. This is called the squeeze lemma.

---

*Proof.* We first check when $L$ is finite. We have that $\lim_{x \to a^+} f(x) = L$ implies for each $\varepsilon > 0$ there exists $\delta > 0$ such that $a < x < a + \delta$ implies $|f(x) - L| < \varepsilon$. Thus, we apply the same for $\delta_1$ and $\delta_3$ to get some $\delta = \min\{\delta_1, \delta_3\}$ such that $a < x < a + \delta$ implies $L - \varepsilon < f_2(x) < L + \varepsilon$. Thus, we have that $\lim_{x \to a^+} f_2(x) = L$. When $L = \infty, -\infty$, the proof is left to the reader. □

## 105 Problem 28.13

Show that $f$ is defined on an open interval containing $a$, if $g$ is defined on an open interval containing $f(a)$, and if $f$ is continuous at $a$, then $g \circ f$ is defined on an open interval containing $a$.

---

*Proof.* We have $\forall \varepsilon > 0, \exists \delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$. Thus, $x \in \mathrm{dom}(f)$, $f(x) \in \mathrm{dom}(g)$, $x \in \mathrm{dom}(g \circ f)$. □