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a) $K(x;z) = K_1(x;z) + K_2(x;z)$

For a given set $\{x_1, x_2, x_3 \dots x_n\}$, K_1 & K_2 be $n \times n$ Gram Matrix associated with k_1, k_2.

The Gram Matrix associated with

$c_1 K_1 + c_2 K_2$ is $c_1 K_1 + c_2 K_2$ for $c_1 = c_2 = 1$

$\Rightarrow K_1 + K_2 = K$

K is a positive semi definite matrix as $\forall v \in R^n$

$$\left.\begin{array}{l} v^T(c_1K_1 + c_2K_2)v = c_1(v^TK_1v) + c_2(v^TK_2v) \geq 0 \end{array}\right\} \text{as } v^TK_1v \geq 0,\ v^TK_2v \geq 0 \text{ \& } K_1 \text{ \& } K_2 \text{ are semi definite.}$$

And hence K is a valid kernel.

b) $K(x;z) = K_1(x;z) - K_2(x;z)$

Similar to solution a) for $c_1 = 1$ & $c_2 = -1$

$c_1 k_1 + c_2 k_2$ is $c_1 K_1 + c_2 K_2$

$\Rightarrow K = K_1 - K_2$

Not a Kernel.

Let $K_2 = 2K_1$, then

$\forall z\ z^T G_2 = z^T(G_1 - 2G_1)z = -z^T G_1 z \leq 0$

c) $K(x;z) = a K_1(x;z)$

By construction, the Gram Matrix is given by

$$K = \alpha K_1$$

which implies that $\forall a \in R^n,\ a^T K a = \alpha a^T K_1 a \geq 0$

due to the positivity of α & the validity of K_1. Hence a kernel

d) Not a kernel, $K(x;z) = -aK_1(x;z)$

For $a \geq 0$, we have $\forall z - z^T G_1 z \leq 0$. Hence not a kernel.

e) $K(x;z) = K_1(x;z) K_2(x;z)$

Kernel. K_1 is a kernel, thus $\exists \phi^{(1)}$ $K_1(x;z) = \phi^{(1)}(x)^T \phi^{(1)}(z) = \Sigma_i \phi_i^{(1)}(x) \phi_i^{(1)}(z)$

Similarly K_2 is a kernel, thus $\exists \phi^{(2)}$ $K_2(x,z) = \phi^{(2)}(x)^T \phi^{(2)}(z) = \Sigma_j \phi_j^{(2)} \phi_j^{(2)}(z)$

$$K(x;z) = K_1(x;z) \cdot K_2(x;z)$$
$$= \phi_i^{(1)}(x) \phi_i^{(1)}(z) \quad \phi_j^{(2)}(x) \phi_j^{(2)}(z)$$
$$= \phi_i^{(1)}(x) \phi_j^{(2)}(x) \quad \phi_i^{(1)}(z) \phi_j^{(2)}(z)$$
$$= \psi_{ij}(x) \quad \psi_{ij}(z)$$

We see that K can be written in the form $K(x,z) = \psi(x)^T \psi(z)$, So it is a Kernel

f) $K(x;z) = f(x) f(z)$

$\Rightarrow$ Kernel.

Let us assume $\phi(x) = f(x)$. Since $f(x)$ is a scalar, we have $K(x,z) = \phi(x)^T \phi(z)$. Hence a kernel.

g) $K(x;z) = K_3(\phi(x); \phi(z))$

$\Rightarrow$ Kernel. Since K_3 is a kernel, the matrix G_3 obtained for any finite set $\{x^{(1)}, \dots x^{(m)}\}$ is positive semidefinite., & so it is also ~~the~~ positive semidefinite for the sets $\{\phi(x^{(1)}), \dots \phi(x^{(m)})\}$

h) $K(x,z) = p(K_1(x;z))$, where p is a polynomial.

- Valid Kernel

The polynomial P is a linear combination of powers of the kernel K_1 with positive coefficients. Since the powers of K_1 are products of K_1 by itself & thus valid kernels, their linear combination is also a valid kernel.