EE/M 520, Spring 2006

Final Exam: May 10, 2006

Solutions (version: May 11, 2006, 9:56) 120 mins. (total 70 pts.)

Part 1: Read each statement *carefully* and circle true (T) or false (F).

Example:

0. Any optimization problem has a global minimizer. T **F**

Score: 1 pt. for correct answer, -0.5 pts. for incorrect answer, 0 pts. for no answer.

1. Given $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $m < n$, the linear equation $\boldsymbol{Ax} = \boldsymbol{b}$ has at least one solution. T **F**

2. Let $\boldsymbol{x}^*$ be a global minimizer of the problem: minimize $f(\boldsymbol{x})$ subject to $\boldsymbol{x} \in \Omega$. If $\Omega \subset \Omega'$ and $\boldsymbol{x}^* \in \Omega'$, then $\boldsymbol{x}^*$ is a global minimizer of the problem: minimize $f(\boldsymbol{x})$ subject to $\boldsymbol{x} \in \Omega'$. T **F**

3. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are distinct elements of a convex set $\Omega \subset \mathbb{R}^n$, then the vector $\boldsymbol{d} = (\boldsymbol{y} - \boldsymbol{x})/2$ is a feasible direction at $\boldsymbol{x}$. **T** F

4. Consider the problem: minimize $f(\boldsymbol{x})$ subject to $\|\boldsymbol{x}\| \leq 1$, where $f \in \mathcal{C}^1$. If $\nabla f(\boldsymbol{0}) \neq \boldsymbol{0}$, then $\boldsymbol{0}$ is not a local minimizer. **T** F

5. If $\boldsymbol{x}^*$ is the unique local minimizer for an optimization problem, then it is the global minimizer. T **F**

6. Given $f : \mathbb{R}^n \to \mathbb{R}$, $f \in \mathcal{C}^2$, suppose that $\nabla f(\boldsymbol{x}^*) = \boldsymbol{0}$ and $\boldsymbol{F}(\boldsymbol{x}^*) \geq 0$ for some $\boldsymbol{x}^* \in \mathbb{R}^n$. Then, $\boldsymbol{x}^*$ is a local minimizer of f. T **F**

7. If the steepest descent algorithm is applied to the problem of minimizing a quadratic function, it converges in one step. T **F**

8. Given $\boldsymbol{Q} \in \mathbb{R}^{n \times n}$, $\boldsymbol{Q} = \boldsymbol{Q}^T > 0$, if the nonzero vectors $\boldsymbol{d}^{(0)} \in \mathbb{R}^n$ and $\boldsymbol{d}^{(1)} \in \mathbb{R}^n$ are not $\boldsymbol{Q}$-conjugate, then $\boldsymbol{d}^{(0)} + \alpha \boldsymbol{d}^{(1)} = \boldsymbol{0}$ for some scalar α. T **F**

9. It is possible to find a function $f \in \mathcal{C}^2$, with bounded inverse Hessian, for which Newton's method has order of convergence of 1 for some initial condition. T **F**

10. If $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $m < n$, has full rank, then so does $\boldsymbol{A}^T \boldsymbol{A}$. T **F**

11. If $\boldsymbol{x} = [x_1, x_2, x_3]^T$ is a basic solution to $\boldsymbol{Ax} = \boldsymbol{b}$ and x_1 is a basic variable, then $x_1 \geq 0$. T **F**

12. The relative cost coefficient corresponding to a basic variable is zero. **T** F

13. If $\boldsymbol{x}$ is an optimal feasible solution to a linear programming problem in standard form, then $\boldsymbol{x}$ is a basic feasible solution. T F

14. Consider a linear programming problem in standard form with $\boldsymbol{A} \in \mathbb{R}^{m \times n}$. If $\boldsymbol{x}^* \in \mathbb{R}^n$ is an optimal solution to the problem, then at least m components of $\boldsymbol{x}^*$ are 0. T F

15. Given a dual pair of linear programming problems, if $\boldsymbol{x}$ and $\boldsymbol{\lambda}$ are feasible in the primal and dual, respectively, and $\boldsymbol{x}^T(\boldsymbol{c} - \boldsymbol{A}^T\boldsymbol{\lambda}) = \boldsymbol{\lambda}^T(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}) = 0$, then $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{\lambda}^T\boldsymbol{b}$. T F

16. If a problem has a local minimizer that is not a global minimizer, then the problem is not a convex optimization problem. T F

17. Consider the constraint set $\Omega = \{\boldsymbol{x} : \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}\}$, where $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^m$, $m < n$. If $\boldsymbol{x}^*$ is a regular point of Ω, then the nullspace of $D\boldsymbol{h}(\boldsymbol{x}^*)$ has dimension m. T F

18. Let $f : \Omega \to \mathbb{R}$ be a convex function. If $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are local minimizers of f, then $(\boldsymbol{x}_1 + \boldsymbol{x}_2)/2$ is a global minimizer of f. T F

19. If $h : \mathbb{R}^n \to \mathbb{R}$ is convex, then $\Omega = \{\boldsymbol{x} : h(\boldsymbol{x}) = 0\}$ is convex. T F

20. Given a convex optimization problem with the usual equality and inequality constraints, if a point does not satisfy the Karush-Kuhn-Tucker conditions, then it is not a local minimizer. T F

Part 2: Answer all questions fully. Use both sides of the sheet.

1. (8 pts.) Consider a pair of primal and dual linear programming problems (either in symmetric or asymmetric form). Identify which of the following situations are possible (depending on the particular primal-dual pair) and which are impossible (regardless of the primal-dual pair). In each case, justify your answer (citing results like the Weak Duality Lemma and the Duality Theorem whenever needed).

i. The primal has a feasible solution, and the dual has no feasible solution.

ii. The primal has an optimal feasible solution, and the dual has no optimal feasible solution.

iii. The primal has a feasible solution but no optimal feasible solution, and the dual has an optimal feasible solution.

iv. Both the primal and the dual have no feasible solution.

Ans.: i. Possible. This situation arises if the primal is unbounded, which by the Weak Duality Lemma implies that the dual has no feasible solution.

ii. Impossible, because the Duality Theorem requires that if the primal has an optimal feasible solution, then so does the dual.

iii. Impossible, because the Duality Theorem requires that if the dual has an optimal feasible solution, then so does the primal. Also, the Weak Dual Lemma requires that if the primal is unbounded (i.e., has a feasible solution but no optimal feasible solution), then the dual must have no feasible solution.

iv. Possible. For example, consider the symmetric primal-dual pair as follows. The primal is

$$\begin{array}{ll} \text{minimize} & [1, -2]\boldsymbol{x} \\ \text{subject to} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \boldsymbol{x} \geq \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{array}$$

The problem has no feasible solution, because the constraints require that $x_1 - x_2 \geq 2$ and $x_1 - x_2 \leq 1$. The dual is

$$\begin{array}{ll} \text{maximize} & \boldsymbol{\lambda}^T \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ \text{subject to} & \boldsymbol{\lambda}^T \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \leq [1, -2] \\ & \boldsymbol{\lambda} \geq \boldsymbol{0}. \end{array}$$

The dual also has no feasible solution, because the constraints require that $\lambda_1 - \lambda_2 \leq 1$ and $\lambda_1 - \lambda_2 \geq 2$.

2. (12 pts.) Consider a matrix $\boldsymbol{A}$ with the property that $\boldsymbol{A}^T\boldsymbol{A}$ has eigenvalues ranging from 1 to 20 (i.e., the smallest eigenvalue is 1 and the largest is 20). Let $\boldsymbol{x}$ be a vector such that $\|\boldsymbol{x}\| = 1$, and let $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$. Use Lagrange multiplier methods to find the range of values that $\|\boldsymbol{y}\|$ can take. *Hint:* What is the largest value that $\|\boldsymbol{y}\|$ can take? What is the smallest value that $\|\boldsymbol{y}\|$ can take?

Ans.: Consider the problem

$$\begin{array}{ll} \text{minimize} & \|\boldsymbol{A}\boldsymbol{x}\|^2 \\ \text{subject to} & \|\boldsymbol{x}\|^2 = 1. \end{array}$$

The optimal objective function value of this problem is the smallest value that $\|\boldsymbol{y}\|^2$ can take. The above can be solved easily using Lagrange multipliers. The Lagrange conditions are

$$\begin{aligned} \boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{A} - \lambda\boldsymbol{x}^T &= \boldsymbol{0}^T \\ 1 - \boldsymbol{x}^T\boldsymbol{x} &= 0. \end{aligned}$$

The first equation can be rewritten as $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$, which implies that λ is an eigenvalue of $\boldsymbol{A}^T\boldsymbol{A}$. Moreover, premultiplying by $\boldsymbol{x}^T$ yields $\boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}^T\boldsymbol{x} = \lambda$, which indicates that the Lagrange multiplier is equal to the optimal objective function value. Hence, the range of values that $\|\boldsymbol{y}\| = \|\boldsymbol{A}\boldsymbol{x}\|$ can take is 1 to $\sqrt{20}$.

3. (18 pts.) Consider the linear programming problem

$$\begin{array}{ll} \text{minimize} & ax_1 + bx_2 \\ \text{subject to} & cx_1 + dx_2 = e \\ & x_1, x_2 \geq 0, \end{array}$$

where $a, b, c, d, e \in \mathbb{R}$ are all nonzero constants. Suppose $\boldsymbol{x}^*$ is an optimal basic feasible solution to the problem.

a. Write down the Karush-Kuhn-Tucker condition involving $\boldsymbol{x}^*$ (specifying clearly the number of Lagrange and KKT multipliers).

b. Is $\boldsymbol{x}^*$ regular? Explain.

c. Find the tangent space $T(\boldsymbol{x}^*)$ (defined by the active constraints) for this problem.

d. Assume that the relative cost coefficients of all nonbasic variables are strictly positive. Does $\boldsymbol{x}^*$ satisfy the second order sufficient condition? Explain.

Ans.: a. We have one scalar equality constraint with $h(\boldsymbol{x}) = [c, d]^T\boldsymbol{x} - e$ and two scalar inequality constraints with $\boldsymbol{g}(\boldsymbol{x}) = -\boldsymbol{x}$. Hence, there exists $\boldsymbol{\mu}^* \in \mathbb{R}^2$ and $\lambda^* \in \mathbb{R}$ such that

$$\begin{aligned} \boldsymbol{\mu}^* &\geq 0 \\ a + c\lambda^* - \mu_1^* &= 0 \\ b + d\lambda^* - \mu_2^* &= 0 \\ \boldsymbol{\mu}^{*T}\boldsymbol{x}^* &= 0 \\ cx_1^* + dx_2^* &= e \\ \boldsymbol{x}^* &\geq 0. \end{aligned}$$

b. Because $\boldsymbol{x}^*$ is a basic feasible solution, and the equality constraint precludes the point $\mathbf{0}$, exactly one of the inequality constraints is active. The vectors $\nabla h(\boldsymbol{x}^*) = [c, d]^T$ and $\nabla g_1 = [1, 0]^T$ are linearly independent. Similarly, the vectors $\nabla h(\boldsymbol{x}^*) = [c, d]^T$ and $\nabla g_2 = [0, 1]^T$ are linearly independent. Hence, $\boldsymbol{x}^*$ must be regular.

c. The tangent space is given by

$$\begin{aligned} T(\boldsymbol{x}^*) &= \{\boldsymbol{y} \in \mathbb{R}^n : Dh(\boldsymbol{x}^*)\boldsymbol{y} = \mathbf{0},\ Dg_j(\boldsymbol{x}^*)\boldsymbol{y} = 0,\ j \in J(\boldsymbol{x}^*)\} \\ &= \mathcal{N}(\boldsymbol{M}), \end{aligned}$$

where $\boldsymbol{M}$ is a matrix with the first row equal to $Dh(\boldsymbol{x}^*) = [c, d]$, and the second row is either $Dg_1 = [1, 0]$ or $Dg_2 = [0, 1]$. But, as we have seen in part b, rank $\boldsymbol{M} = 2$ Hence, $T(\boldsymbol{x}^*) = \{\boldsymbol{0}\}$.

d. Recall that we can take $\boldsymbol{\mu}^*$ to be the relative cost coefficient vector (i.e., the KKT conditions are satisfied with $\boldsymbol{\mu}^*$ being the relative cost coefficient vector). If the relative cost coefficients of all nonbasic variables are strictly positive, then $\mu_j^* > 0$ for all $j \in J(\boldsymbol{x}^*)$. Hence, $\tilde{T}(\boldsymbol{x}^*, \boldsymbol{\mu}^*) = T(\boldsymbol{x}^*) = \{\boldsymbol{0}\}$, which implies that $\boldsymbol{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) > 0$ on $\tilde{T}(\boldsymbol{x}^*, \boldsymbol{\mu}^*)$. Hence, the SOSC is satisfied.

4. (12 pts.) Let $\Omega \subset \mathbb{R}^n$ be a nonempty closed convex set, and $\boldsymbol{z} \in \mathbb{R}^n$ a given point such that $\boldsymbol{z} \notin \Omega$. Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & \|\boldsymbol{x} - \boldsymbol{z}\| \\ \text{subject to} & \boldsymbol{x} \in \Omega. \end{array}$$

Does this problem have an optimal solution? If so, is it unique? Whatever your assertion, prove it. *Hint:* (i) If $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are optimal solutions, what can you say about $\boldsymbol{x}_3 = (\boldsymbol{x}_1 + \boldsymbol{x}_2)/2$? (ii) The triangle inequality states that $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$, with equality holding if and only if $\boldsymbol{x} = \alpha \boldsymbol{y}$ for some $\alpha \geq 0$ (or $\boldsymbol{x} = 0$ or $\boldsymbol{y} = 0$).

Ans.: We claim that a solution exists, and that it is unique. To prove the first claim, choose $\varepsilon > 0$ such that there exists $\boldsymbol{x} \in \Omega$ satisfying $\|\boldsymbol{x} - \boldsymbol{z}\| < \varepsilon$. Consider the modified problem

$$\begin{array}{ll} \text{minimize} & \|\boldsymbol{x} - \boldsymbol{z}\| \\ \text{subject to} & \boldsymbol{x} \in \Omega \bigcap \{\boldsymbol{y} : \|\boldsymbol{y} - \boldsymbol{z}\| \leq \varepsilon\}. \end{array}$$

If this modified problem has a solution, then clearly so does the original problem. The objective function here is continuous, and the constraint set is closed and bounded. Hence, by Weierstrass's Theorem, a solution to the problem exists.

Let f be the objective function. Next, we show that f is convex (and hence the problem is a convex optimization problem). Let $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $\alpha \in (0, 1)$. Then,

$$\begin{aligned} f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) &= \|\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) - \boldsymbol{z}\| \\ &= \|\alpha(\boldsymbol{x} - \boldsymbol{z}) + (1-\alpha)(\boldsymbol{y} - \boldsymbol{z})\| \\ &\leq \alpha\|\boldsymbol{x} - \boldsymbol{z}\| + (1-\alpha)\|\boldsymbol{y} - \boldsymbol{z})\| \\ &= \alpha f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{y})\|, \end{aligned}$$

which shows that f is convex.

To prove uniqueness, let $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ be solutions to the problem. Then, by convexity, $\boldsymbol{x}_3 = (\boldsymbol{x}_1 + \boldsymbol{x}_2)/2$ is also a solution. But

$$\begin{aligned}
\|\boldsymbol{x}_3 - \boldsymbol{z}\| &= \left\|\frac{\boldsymbol{x}_1 + \boldsymbol{x}_2}{2} - \boldsymbol{z}\right\| \\
&= \left\|\frac{\boldsymbol{x}_1 - \boldsymbol{z}}{2} + \frac{\boldsymbol{x}_2 - \boldsymbol{z}}{2}\right\| \\
&\leq \frac{1}{2}(\|\boldsymbol{x}_1 - \boldsymbol{z}\| + \|\boldsymbol{x}_2 - \boldsymbol{z}\|) \\
&= \|\boldsymbol{x}_3 - \boldsymbol{z}\|,
\end{aligned}$$

from which we conclude that the triangle inequality above holds with equality, implying that $\boldsymbol{x}_1 - \boldsymbol{z} = \alpha(\boldsymbol{x}_2 - \boldsymbol{z})$ for some $\alpha \geq 0$. Because $\|\boldsymbol{x}_1 - \boldsymbol{z}\| = \|\boldsymbol{x}_2 - \boldsymbol{z}\| = \alpha\|\boldsymbol{x}_1 - \boldsymbol{z}\|$, we have $\alpha = 1$. From this, we obtain $\boldsymbol{x}_1 = \boldsymbol{x}_2$, which proves uniqueness.