Notes - 10 April

Chapter 4 - Continuous Time Markov Chains

Continuous time chains stay in each state a random time that is a continuous random variable that may depend on the state. $X(t)$ = state at time $t, t \geq 0$. $X(t)$ may or may not be a Markov process.

§ 4.1 - The Poisson Process

Example 4.1.1 - We use a Geiger counter to observe the emission of particles from a radioactive source. If we switch on the counter at time zero, the count $N(t)$ is the outcome of an apparently random process. Observations: (a) $N(0) = 0, N(t) \in \{0, 1, 2, 3, \dots\}$, (b) If $s < t, N(s) \leq N(t)$. We conjecture a continuity assumption: in a time period $(t, t+h)$ the probability of an emission is proportional to h for small h.

Definition 4.1.1 - A Poisson process with intensity λ is a process $N = \{N(t), t \geq 0\}$ taking values in $S = \{0, 1, 2, 3, \dots\}$ such that (a) $N(0) = 0$, (b) $S < t \Rightarrow N(s) \leq N(t)$, (c) $P(N(t+h) = n+m | N(t) = n) = \{\lambda h + O(h)$ for $m = 1, O(h)$ for $m > 1, 1 - \lambda h + O(h)$ for $m = 0$. $O(h)$ means an expression $A(h)$ such that $\lim_{h \to 0} \frac{|A(h)|}{h} \to 0$, (d) If $s < t$, the number $N(t) - N(s)$ of emissions in $(s, t]$ is independent of the times of emissions in $(0, s]$ (this is pretty much the Markov condition).

Definition 4.1.2 - $N(t)$ = the number of arrivals or occurences or events or emissions at time t. N is called a counting process.

Theorem 4.1.1 - $N(t)$ has the Poisson distribution with parameter λt, i.e., (4.1.1) $P(N(t) = j) = \frac{(\lambda t)^j}{j!} e^{-\lambda t}, j = 0, 1, 2, \dots$.

Proof: We condition $N(t+h)$ on $N(t)$. $P(N(t+h) = j) = \Sigma_i P(N(t) = i) P(N(t+h) = j | N(t) = j) = \Sigma_i P(N(t) = i) P((j-i)$ arrivals in $(t, t+h]) = P(N+t) = j-1) P(1$ arrival in $(t, t+h =) + P(N(t) = j) P($no arrivals in $(t, t+h]) + O(h)$. Set $P_j(t) = P(N(t) = j), P_j(t+h) = \lambda h P_{j-1}(t) + (1-\lambda h) P_j(t) + O(h), P_0(t+h) = (1-\lambda h) P_0(t) + O(h), j \neq 0$. Assuming $P_j(t)$ is a smooth function, we subtract $P_j(t)$ from each side, divide by h and let $h \downarrow 0$. $\lim_{h \to 0} \frac{P_j(t+h) - P_j(t)}{h}$, etc. (4.1.2) $P_j'(t) = \lambda P_{j-1}(t) - \lambda P_j(t), j \neq 0$. (4.1.3) $P_0'(t) = -\lambda P_0(t)$. The initial condition is (4.1.4) $P_j(0) = \delta_{j0} = \{1$ for $j = 0, 0$ for $j \neq 0$. This is a big system of equations that we need to solve to go further.

Two approaches: (4.1.3) + (4.1.4) together yield $P_0(t) = e^{-\lambda t}$. We substitute into (4.1.2) with $j = 1, P_1'(t) = \lambda e^{-\lambda t} - \lambda P_1(t). P_1'(t) + \lambda P_1(t) = \lambda e^{-\lambda t}, e^{\lambda t} P_1'(t) + e^{\lambda t} \lambda P_1(t) = e^{\lambda t} \lambda e^{-\lambda t} \Rightarrow \frac{d}{dt}(e^{\lambda t} P_1(t) = \lambda, e^{\lambda t} P_1(t) = \lambda t + c \Rightarrow P_1(t) = \lambda t e^{-\lambda t}$. Iteration yields $P_j(t) = \frac{(\lambda t)^j}{j!} e^{-\lambda t}$.

Second approach: Define a generating function $G(s, t) = \Sigma_{j=0}^{\infty} P_j(t) s^j$. Multiply (4.1.2) by s^j and sum (some details) $\Rightarrow \frac{\nabla G}{\nabla t} = \lambda(s-1) G, G(s, 0) = 1$. The solution is (4.1.5) $G(s, t) = e^{\lambda(s-1)t} = e^{-\lambda t} \Sigma_{j=0}^{\infty} \frac{(\lambda t)^j}{j!} s^j$. Reformulation of the process: important for computational purposes.

Definition 4.1.3 - Let $T_0, T_1, T_2, \dots$ be given by (4.1.6) $T_0 = 0, T_n = \inf_t \{N(t) = n\}$ ($\inf = \min$). T_n is the arrival or waiting time for the nth event. The interarrival or sojourn times $X_1, X_2, \dots$ are given by (4.1.7) $X_n = T_n - T_{n-1}$. If we know N, we can compute $X_1, X_2, \dots$. Vice versa if we know the entire collection of sojourn times $\{X_n\}$ then (4.1.8) $T_n = \Sigma_{i=1}^n X_i, N(t) = \max_{T_n \leq t} n$. See picture in notes.

Theorem 4.1.2 - The random variables $X_1, X_2, \dots$ are i.i.d. with exponential distribution with parameter λ.