absorption. We let W_i denote this mean corresponding to $X_0 = i$,

$$W_i = E\left(\sum_{n=0}^{T-1} g(X_n) \mid X_0 = i\right)$$

Example 2.3.3

If we let $g(i) = 1$, then

$$\sum_{n=0}^{T-1} g(X_n) = \sum_{n=0}^{T-1} 1 = T.$$

This provides motivation for the label "rate" - if we sum the rate, we get total time.

— start #8 2/14 —

Example 2.3.4

For a transient state k, define

$$g(i) = \begin{cases} 1, & i = k, \\ 0, & i \neq k. \end{cases}$$

This gives $W_i = W_{ik}$, the mean number of visits to state k $(0 \leq k \leq r)$ before absorption.

Note: the sum $\sum_{n=0}^{T-1} g(X_n)$ always includes $g(X_0) = g(i)$.

If a transition is made from state i to a transient state j, the sum includes future terms as well. The Markov property implies that this future sum proceeding from j has expected value W_j. Weighting this by the transition probability P_{ij} and using the total law of probability, we get

$$(2.3.3) \qquad W_i = g(i) + \sum_{j=0}^{r-1} P_{ij} W_j \quad , i = 0,1,..r-1$$

(recall $0,1,..,r-1$ are the transient states)

Exercise: explain this

<u>Example 2.3.5</u>

From Ex. 2.3.3, $g(i) = 1$ for all i implies $V_i = E(T \mid X_0 = i)$, and $\{V_i\}$ satisfies

(2.3.4) $$V_i = 1 + \sum_{j=0}^{r-1} P_{ij} V_j \quad , \; i=0,1,..,r-1$$

<u>Example 2.3.6</u>

From Ex. 2.3.4, when $g(i) = \begin{cases} 1, & i=k, \\ 0, & i\neq k, \end{cases} = \delta_{ik}$

W_{ik} solves

(2.3.5) $$W_{ik} = \delta_{ik} + \sum_{j=0}^{r-1} P_{ij} W_{jk} \quad , \; i=0,1,..,r-1.$$

<u>Example 2.3.7</u>

A Markov chain has transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ .1 & .4 & .4 & .1 \\ .2 & .1 & .6 & .1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for states $\{0,1,2,3\}$. Starting in state 1, determine the probability that the chain is absorbed into state 0 and the mean time until absorption.

The matrix is not exactly in the form (2.3.1), but we just have to recognize that 0 and 3 are absorbing states and 1, 2 are transient states. With $U_{i0} = P(\text{absorption into } 0 \mid X_0 = i)$, $i = 1, 2$, (2.3.2) reads

$$U_{10} = P_{10} + P_{11} U_{10} + P_{12} U_{20}$$

$$U_{20} = P_{20} + P_{21} U_{10} + P_{22} U_{20}$$

or

$$U_{10} = .1 + .4 U_{10} + .1 U_{20}$$

$$U_{20} = .2 + .1 U_{10} + .6 U_{20}$$

so

$$\begin{pmatrix} .6 & -.1 \\ -.1 & .4 \end{pmatrix} \begin{pmatrix} U_{10} \\ U_{20} \end{pmatrix} = \begin{pmatrix} .1 \\ .2 \end{pmatrix} \Rightarrow U_{10} = \frac{6}{23},\ U_{20} = \frac{13}{23}$$

Next, (2.3.4) reads

$$v_1 = 1 + \cancel{.1 v_0}^{0} + .4 v_1 + .1 v_2 + \cancel{.4 v_3}^{0}$$

$$v_2 = 1 + \cancel{.2 v_0}^{0} + .1 v_1 + .6 v_2 + \cancel{.1 v_3}^{0}$$

or

$$\begin{pmatrix} .6 & -.1 \\ -.1 & .4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow v_1 = \frac{50}{23}, \; v_2 = \frac{70}{23}$$

We emphasize that this analysis uses the finite state nature of the Markov process. Having finite state simplifies a number of issues, as we will see!

§2.4 Gambler's Ruin

In Ex. 2.1.2, we introduced random walk, and in Ex 2.1.9, we generalized random walk to allow a particle to have different probabilities to move to different positions and even the possibility to remain at a given position. The transition matrix for that case is

$$\begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & q_{-1} & r_{-1} & p_{-1} & & 0 & \\ & & q_0 & r_0 & p_0 & & \\ & 0 & & q_1 & r_1 & p_1 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}$$

with

$$\begin{cases} 0 < p_i < 1,\ 0 < q_i < 1,\ 0 \le r_i < 1, \\ p_i + q_i + r_i = 1 \end{cases}, \quad i = -\infty, \dots, \infty.$$

<u>Definition 2.4.1</u>

A simple random walk has $r_i = 0$ and $p_i = p$, $q_i = q$ for all i.

We use the notion of random walk to describe many situations.

<u>Example 2.4.1</u> <u>Gambler's Ruin</u>

We consider a game with two people A, B who have a total fortune of #N between them. At each step i, player A has a chance p_i of winning #1, q_i of losing #1, and r_i of drawing, where $0 < p_i, q_i < 1$, $0 \le r_i < 1$, $p_i + q_i + r_i = 1$. If player A's fortune drops to 0, the game stops, and if A's fortune reaches N, the game stops. The state of A having fortune 0 is "gambler's ruin." Note, if A has #k, then B has #N-k, and it is only interesting to start with $0 < k < N$.

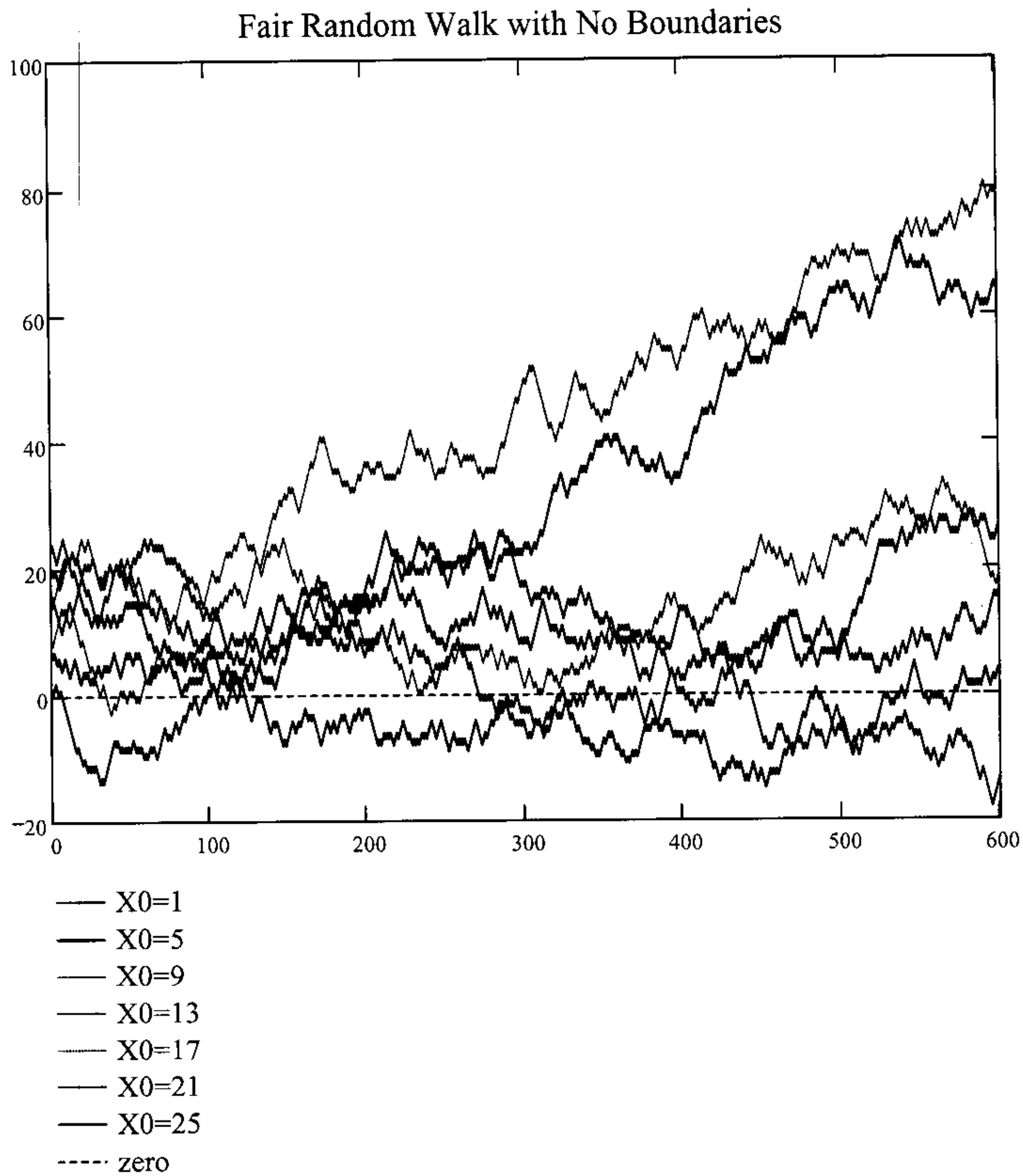
We let X_n = fortune of A at time n. X_n is clearly a Markov chain with transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & & & 0 \\ q_1 & r_1 & p_1 & 0 & \cdots & & & \vdots \\ 0 & q_2 & r_2 & p_2 & 0 & \cdots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & & \\ 0 & \cdots & & & 0 & q_{N-1} & r_{N-1} & p_{N-1} \\ 0 & \cdots & & & & 0 & 0 & 1 \end{pmatrix}$$

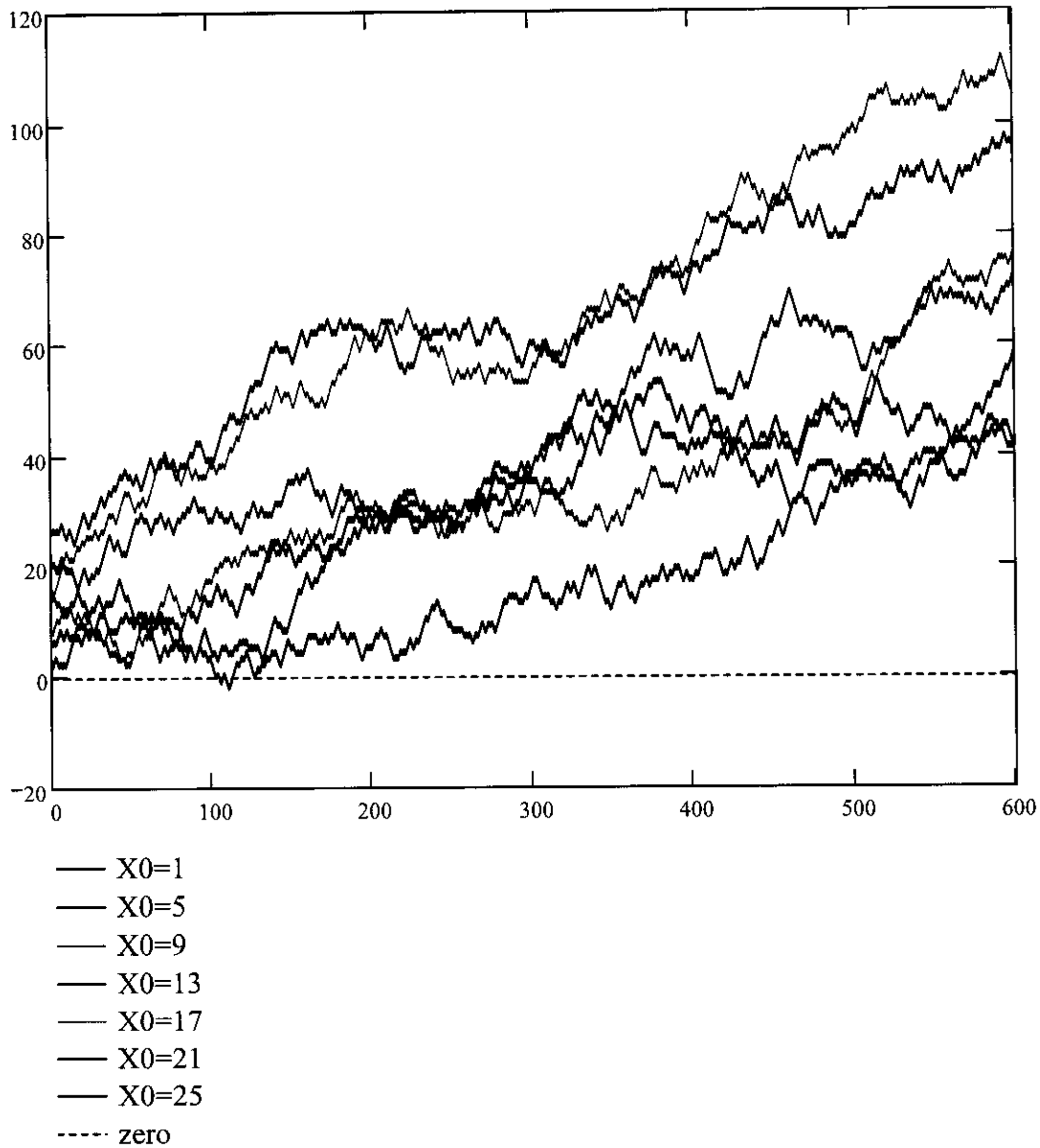
The states $k=0$ and $k=N$ are absorbing.

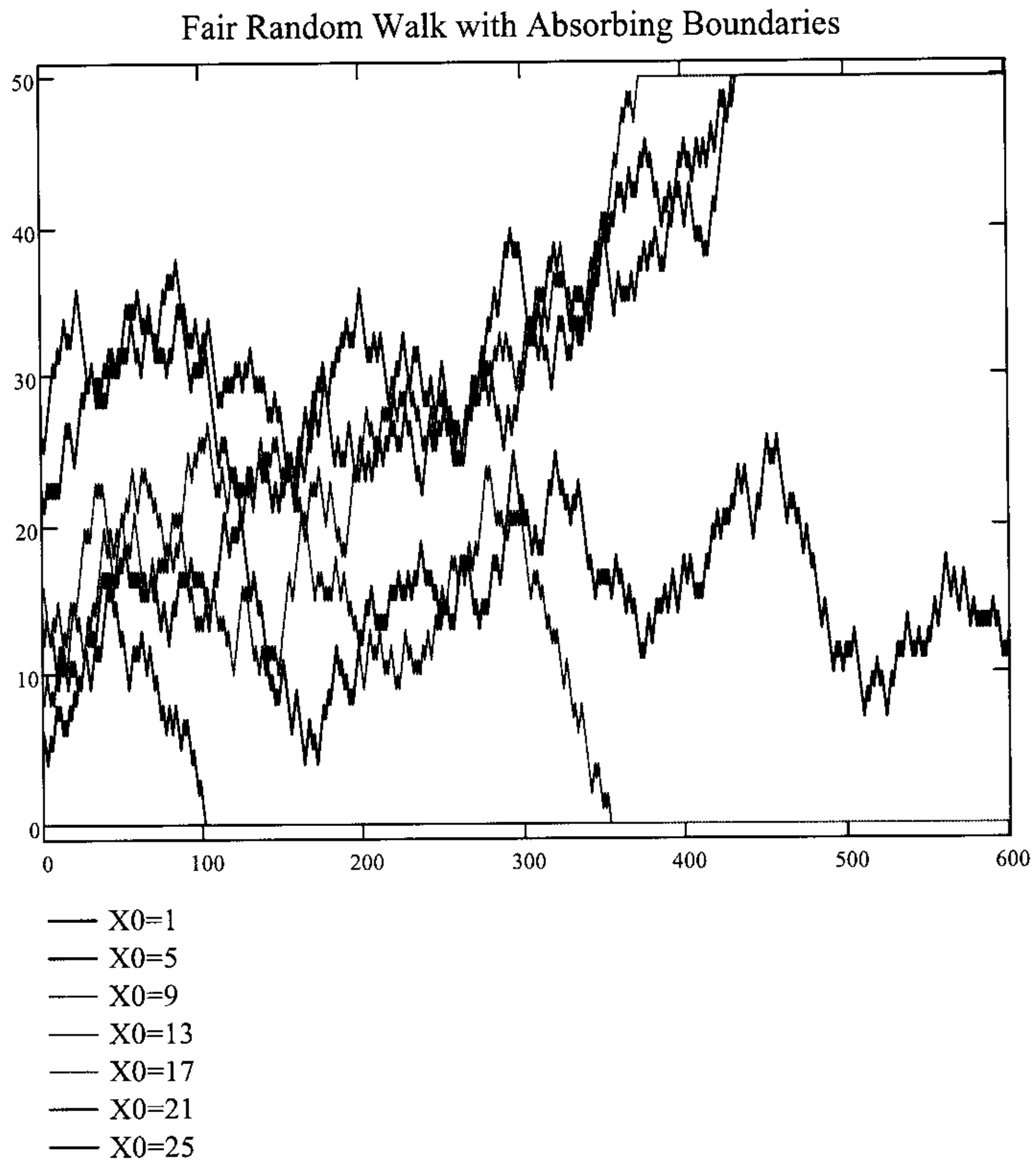
Example 2.4.2

See pages 87-89 for some examples


Fair Random Walk with No Boundaries
100
80
60
40
20
0
−20
0
100
200
300
400
500
600
X0=1
X0=5
X0=9
X0=13
X0=17
X0=21
X0=25
zero

Biased Random Walk with p=6/11 and No Boundaries




Fair Random Walk with Absorbing Boundaries
50
40
30
20
10
0
0
100
200
300
400
500
600
X0=1
X0=5
X0=9
X0=13
X0=17
X0=21
X0=25

We can address some intermediate time results using (2.3.2), (2.3.3).

Example 2.4.3

The probability of ruin for player A in Ex. 2.4.1 is $U_i = U_{i0}$ in (2.3.2) satisfying

(2.4.1) $U_i = P_i U_{i+1} + r_i U_i + q_i U_{i-1}, \quad i = 1, 2, .., N-1$

together with boundary conditions

$$U_0 = 1, \quad U_N = 0$$

It is possible to find a solution in some cases.

Example 2.4.4

Consider $r_i = 0$, $P_i = P$, $q_i = q$ for all i, so $q = 1 - p$. (2.4.1) becomes

(2.4.2) $\begin{cases} U_i = p U_{i+1} + q U_{i-1}, & i = 1, 2 .., N-1 \\ U_0 = 1, \ U_N = 0 \end{cases}$

We look for a solution of the form

$$U_i = \theta^i$$

This gives

$$\theta^i = p\,\theta^{i+1} + q\,\theta^{i-1}$$

and if $\theta \neq 0$

$$\theta = p\,\theta^2 + q$$

This has roots

$$\theta_1 = 1, \quad \theta_2 = q/p$$

If $p \neq 1/2$, the roots are distinct. The general solution has the form

$$U_i = A_1\,\theta_1^{\,i} + A_2\,\theta_2^{\,i}$$

for constants A_1, A_2. Now

$$U_0 = 1 = A_1 + A_2$$

$$U_N = 0 = A_1 + A_2\left(\frac{q}{p}\right)^N$$

(2.4.3) $$U_i = \frac{\left(\frac{q}{p}\right)^i - \left(\frac{q}{p}\right)^N}{1-\left(\frac{q}{p}\right)^N}, \quad p \neq 1/2,\ 0 < i \leq N,$$

If $p = 1/2$, $\theta_1 = \theta_2 = 1$, and the solution is

(2.4.4) $$U_i = 1 - \frac{i}{N}, \quad 0 < i < N.$$

For the mean time to absorption, (2.3.4) becomes

(2.4.5) $$V_i = 1 + pV_{i+1} + qV_{i-1} = p(1+V_{i+1}) + q(1+V_{i-1}), \quad 1 < i < N$$

The boundary conditions are $V_0 = V_N = 0$.

The solution turns out to be

(2.4.6) $$V_i = \begin{cases} \frac{1}{q-p}\left(i - N\left(\frac{1-(q/p)^i}{1-(q/p)^N}\right)\right), & p \neq 1/2, \\ i(N-i), & p = 1/2. \end{cases}$$

We can change the game to have different boundary conditions.

Example 2.4.5

Suppose in Ex. 2.4.1, player A has a backer who guarantee's A's losses. There is no ruin when A's earnings reach zero, but A cannot incur negative amounts. This gives $r_0 + p_0 = 1$. We let $q_0 = r_0$ in an abuse of notation, so

$$P = \begin{pmatrix} q_0 & p_0 & 0 & \cdots & & & 0 \\ q_1 & r_1 & p_1 & 0 & \cdots & & \\ 0 & q_2 & r_2 & p_2 & 0 & \cdots & \\ \vdots & & & \ddots & & & \\ 0 & \cdots & & & q_{N-1} & r_{N-1} & p_{N-1} \\ & & & & 0 & 0 & 1 \end{pmatrix}$$

In the case that $p_i = p$, $q_i = q$ for all i, the absorbing times will again satisfy (2.4.5) but the boundary conditions are different. We still have $\nu_N = 0$, but since 0 is *not* absorbing, ν_0 satisfies

$$\nu_0 = 1 + q\nu_0 + p\nu_1 \Rightarrow p\nu_0 = 1 + p\nu_1$$

We can now determine the solution (Exercise).

§2.5 Simple Branching Processes

We now consider another important example of stochastic processes.

The simple branching process is a model for the evolution of a population. We start at time 0 with a progenitor. The progenitor splits into k offspring with probability p_k, where $\{p_k\}$ is a p.m.f., and dies simultaneously. We assume each offspring reproduces in the same way independently, producing a random number of offspring with probability $\{p_k\}$. The process continues until extinction — when no members of a generation produce offspring.

We let X_n = population at time n.

Definition 2.5.1

X_n is called a branching process.

Theorem 2.5.1
X_n is a Markov chain.

Proof
Exercise

Example 2.5.1 Neutron Chain Reaction

A nucleus is split by a chance collision with a neutron. The resulting fission yields a random number of new neutrons. These may hit other nuclei and cause further fission.