We now assume that X_n is positive reccurent, which implies that it has aunique stationary distribution π.
Consider $Y = X$ in constructing Z.
As an exercise, it follows that the coupled chain Z has stationary distribution $\nu = (\nu_{ij} : i, j \in S)$, $\nu_{ij} = \pi_i \pi_j$, and hence Z is also positive reccurent.

Choose $X_0 = i$, $Y_0 = j$, so $Z_0 = (i, j)$. Choose any state $s \in S$ and let

$$T = \min\{n \geq 1 : Z_n = (s,s)\}$$

denote the time of the first passage of Z_n to (s,s). The recurrence of Z_n implies that $P(T < \infty) = 1$ (Exercise).

$\boxed{\#19 \quad 4/3}$

Observation:
Now suppose $m \leq n$ and $X_m = Y_m$. Then X_n and Y_n are identically distributed since the distributions of X_n and Y_n depend only on P and the common value at m. Thus, conditional on $\{T \leq n\}$, X_n and Y_n have the same distribution.

We use this observation and the finiteness of T to show the ultimate distributions of X_n and Y_n are independent of their starting points.

Starting from $Z_0 = (i, j)$,

$$
\begin{aligned}
P_{ik}^{n} &= P(X_n = k) \\
&= P(X_n = k, T \le n) + P(X_n = k, T > n) \\
&= P(Y_n = k, T \le n) + P(X_n = k, T > n)
\end{aligned}
$$

($T \le n \Rightarrow X_n, Y_n$ are identically distributed)

$$
\begin{aligned}
&\le P(Y_n = k) + P(T > n) \\
&= P_{jk}^{n} + P(T > n)
\end{aligned}
$$

Of course, by a symmetric argument,

$$P_{jk}^{n} \le P_{ik}^{n} + P(T > n).$$

Hence,

$$|P_{ik}^{n} - P_{jk}^{n}| \le P(T > n) \longrightarrow 0 \quad \text{as } n \to \infty,$$

for all $i, j, k \in S$.
In other words,

$$(3.4.2) \qquad P_{ik}^{n} - P_{jk}^{n} \longrightarrow 0 \quad \text{as } n \to \infty$$

for all $i, j, k \in S$.

If $\lim_{n\to\infty} P_{ik}^n$ exists, then it is independent of i. We just have to show the limit exists.

We write

$$\underbrace{\pi_k - P_{jk}^n = \sum_i \pi_i (P_{ik}^n - P_{jk}^n)}_{\text{stationary}}$$

π is a p.m.f.

For any finite set $F \subset S$,

$$\sum_i \pi_i \,|P_{ik}^n - P_{jk}^n| \le \sum_{i\in F} |P_{ik}^n - P_{jk}^n| + 2\sum_{i\notin F} \pi_i$$

$$\longrightarrow 2\cdot \sum_{i\notin F} \pi_i \quad \text{as } n\to\infty$$

$$\longrightarrow 0 \quad \text{as } F \uparrow S.$$

So

$$(3.4.3) \quad \pi_k - P_{jk}^n = \sum_i \pi_i (P_{ik}^n - P_{jk}^n) \xrightarrow[n\to\infty]{} 0.$$

Finally, assume that X_n is null recurrent.

If the coupled chain Z_n is transient, then Thm 3.1.2 (3) applied to Z_n implies

$$P(Z_n = (j,j) \mid Z_0 = (i,i)) = (P_{ij}^n)^2 \xrightarrow[n\to\infty]{} 0.$$

which proves the result.

If Z_n is positive recurrent, then starting from $Z_0 = (i,i)$, the time T_{ii}^{Z} of the first return of Z_n to (i,i) is no smaller than the time T_i of the first return of X_n to i. However, $E(T_i) = \infty$ while $E(T_{ii}^{Z}) < \infty$, which is a contradiction.

So, assume Z_n is null recurrent.

The argument leading to (3.4.2) still holds,

and we want to show that $P_{ij}^{n} \to 0$ as $n \to \infty$ for all i,j. If this does not hold, there is a subsequence $n_1, n_2, \ldots$ such that

$$(3.4.4) \qquad P_{ij}^{n_r} \to \alpha_j \quad \text{as } r \to \infty \quad \text{for all } i,j,$$

where the $\{\alpha_j\}$ are not all zero. Note, the α_j are independent of i by (3.4.2).

For a finite set $F \subset S$,

$$\sum_{j \in F} \alpha_j = \lim_{n \to \infty} \sum_{j \in F} P_{ij}^{n_r} \leq 1$$

So $\alpha = \sum_j \alpha_j$ satisfies $0 < \alpha \le 1$, and

$$\sum_{k \in F} P_{ik}^{n_r} P_{kj} \le P_{ij}^{n_r+1} = \sum_k P_{ik} P_{kj}^{n_r}$$

Letting $r \to \infty$, (3.4.2) and bounded convergence implies

$$\sum_{k \in F} \alpha_k P_{kj} \le \sum_k P_{ik} \alpha_j = \alpha_j$$

Letting $F \uparrow S$,

$$\sum_k \alpha_k P_{kj} \le \alpha_j \quad , \text{ each } j \in S.$$

Now equality must hold, since if inequality holds for some j,

$$\sum_k \alpha_k = \sum_{k,j} \alpha_k P_{kj} < \sum_j \alpha_j$$

which is a contradiction. So

$$\sum_k \alpha_k P_{kj} = \alpha_j \quad , \text{ each } j \in S.$$

Thus, $\pi = (\alpha_j / \alpha, \ j \in S)$ is a stationary distribution of X_n. This contradicts the nullity of X_n.

A more general version of Thm 3.4.1 drops the assumption of irreducibility. We do not prove this.

Theorem 3.4.6

For any aperiodic state j of a Markov chain

$$p_{jj}^n \to \frac{1}{\mu_j} \quad as\ n \to \infty$$

If i is any other state,

$$p_{ij}^n \to \frac{f_{ij}}{\mu_j} \quad as\ n \to \infty.$$

Let

$$\tau_{ij}(n) = \frac{1}{n} \sum_{m=1}^{n} p_{ij}^m$$

be the mean proportion of elapsed time up to the n^{th} step during which the chain was in state j starting from state i. If j is aperiodic,

$$\tau_{ij}(n) \to \frac{f_{ij}}{\mu_j} \quad as\ n \to \infty.$$

§3.5 Reversibility

Some physical situations have the symmetry property that they would be the same if time ran in the "reverse" direction.

We consider this for Markov chains.

Let $\{X_n, 0 \le n \le N\}$ be an irreducible positive recurrent Markov chain with probability transition matrix P and stationary distribution π. Suppose further that X_n has distribution π for all n.

Definition 3.5.1

The reversed chain or time reversal Y_n is defined by $Y_n = X_{N-n}$, $0 \le n \le N$.

Theorem 3.5.1

The sequence Y_n is a Markov chain with

$$P(Y_{n+1}=j \mid Y_n = i) = \frac{\pi_j}{\pi_i} P_{ji}$$

Proof

$$P(Y_{n+1} = i_{n+1} \mid Y_n = i_n, Y_{n-1} = i_{n-1}, \ldots, Y_0 = i_0)$$

$$= \frac{P(Y_k = i_k,\ 0 \le k \le n+1)}{P(Y_k = i_k,\ 0 \le k \le n)}$$

$$= \frac{P(X_{N-n-1} = i_{n+1},\ X_{N-n} = i_n, \ldots, X_N = i_0)}{P(X_{N-n} = i_n, \ldots, X_N = i_0)}$$

$$= \frac{\pi_{i_{n+1}} P_{i_{n+1}, i_n} P_{i_n, i_{n-1}} \cdots P_{i_1, i_0}}{\pi_{i_n} P_{i_n, i_{n-1}} \cdots P_{i_1, i_0}}$$

$$= \frac{\pi_{i_{n+1}} P_{i_{n+1}, i_n}}{\pi_{i_n}}$$

- <u>Definition 3.5.2</u> reorder definition <u>next time</u>

#4 The chain is <u>reversible</u> if the probability transition matrices of X_n and its time reversal Y_n are the same,

#2 given in terms of X

(3.5.1) $\quad \pi_i P_{ij} = \pi_j P_{ji} \quad$ all i, j

(3.5.1) are called the <u>detailed balance equations</u>. A transition matrix P and

start #1 a distribution λ are in <u>detailed balance</u> if

$$\lambda_i P_{ij} = \lambda_j P_{ji} \quad \text{all } i, j.$$

#3 An irreducible chain X_n with a stationary distribution π is called reversible in equilibrium if its transition probability matrix is in detailed balance with π.

Being in detailed balance is special

Theorem 3.5.1

Let P be the probability transition matrix of an irreducible chain X_n, and suppose there is a distribution π with $\pi_i P_{ij} = \pi_j P_{ji}$ for all $i, j \in S$. Then, π is a stationary distribution of X_n, and X_n is reversible in equilibrium.

Proof

Suppose π satisfies the assumptions. We have

$$\sum_i \pi_i P_{ij} = \sum_i \pi_j P_{ji}$$

$$= \pi_j \sum_i P_{ji}$$

$$= \pi_j$$

or

$$\pi = \pi P.$$

$\boxed{\#20 \quad 4/8}$

Given a reversible chain, if we observe a sequence of consecutive states, there is no way to tell if the sequence was generated forward or backward in time.

Example 3.5.1

Consider the ON/OFF example in Ex. 2.2.3,

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}, \quad 0 \le p \le 1, \ 0 \le q \le 1.$$

(3.5.1) reads

$$\pi_0 P_{00} = \pi_0 P_{00}$$

$$\pi_0 P_{01} = \pi_1 P_{10}$$

$$\pi_1 P_{10} = \pi_0 P_{01}$$

$$\pi_1 P_{11} = \pi_1 P_{11}$$