Chapter 4 Continuous Time Markov Chains

Roughly speaking, continuous time chains stay in each state a random time that is a continuous random variable that may depend on the state. The state of the chain at time t is denoted $X(t)$, where $0 \le t < \infty$. Depending on the underlying random mechanism, $X(t)$ may or may not be a Markov process.

§4.1 The Poisson Process

We begin by discussing a relatively simple but extremely important example.

Example 4.1.1

We use a Geiger counter to observe the emission of particles from a radioactive source. If we switch on the counter at time zero, the reading $N(t)$ for $t \ge 0$ is the outcome of an apparently random process with certain properties

(a) $N(0) = 0$, $N(t) \in \{0, 1, 2, \dots\}$

(b) If $s < t$, $N(s) \le N(t)$ (Monotonicity) but other characteristics are more difficult to determine.

We might make the conjecture that in a time period $(t, t+h)$, the likelihood of an emission is proportional to h for all sufficiently small h.

Definition 4.1.1

A Poisson process with intensity λ is a process $N = \{N(t), t \geq 0\}$ taking values in $S = \{0, 1, 2, 3, \ldots\}$ such that

(a) $N(0) = 0$

(b) $s < t \Rightarrow N(s) \leq N(t)$

(c) $P(N(t+h) = n+m \mid N(t) = n) = \begin{cases} \lambda h + o(h), & m = 1, \\ o(h), & m > 1, \\ 1 - \lambda h + o(h), & m = 1, \end{cases}$

(d) If $s < t$, the number $N(t) - N(s)$ of emissions in $(s, t]$ is independent of the times of emissions that occured in $(0, s]$.

Definition 4.1.2

$N(t)$ in Defn 4.1.1 is called the number of arrivals or occurences or events or emissions at time t. N is called a counting process

This is is one of the simplest examples of a continuous time process.

Remarkably, these assumptions imply a lot about the distribution of $N(t)$.

Theorem 4.1.1

$N(t)$ has the Poisson distribution with parameter λt, ie.

$$(4.1.1) \quad P(N(t)=j) = \frac{(\lambda t)^j}{j!} e^{-\lambda t}, \quad j=0,1,2,\ldots$$

Proof

We condition $N(t+h)$ on $N(t)$ to obtain

$$P(N(t+h)=j) = \sum_i P(N(t)=i) P(N(t+h)=j \mid N(t)=i)$$

$$= \sum_i P(N(t)=i)\, P((j-i) \text{ arrivals in } (t,t+h])$$

$$= P(N(t)=j-1)\, P(\text{one arrival in } (t,t+h))$$
$$+ P(N(t)=j)\, P(\text{no arrivals in } (t,t+h))$$
$$+ o(h)$$

So $P_j(t) = P(N(t)=j)$ satisfies

$$P_j(t+h) = \lambda h P_{j-1}(t) + (1-\lambda h) P_j(t) + o(h), \quad j \neq 0$$

and

$$P_0(t+h) = (1-\lambda h) P_0(t) + o(h)$$

We subtract $P_j(t)$ from each side of these equations, divide by h, and let $h \downarrow 0$ assuming that $P_j(t)$ is a smooth function to obtain

(4.1.2) $$P_j'(t) = \lambda P_{j-1}(t) - \lambda P_j(t), \quad j \neq 0$$

and

(4.1.3) $$P_0'(t) = -\lambda P_0(t).$$

The "boundary" or "initial" condition is

(4.1.4) $$P_j(0) = \delta_{j0} = \begin{cases} 1, & j=0, \\ 0, & j \neq 0. \end{cases}$$

These are interesting differential-difference equations and there are several ways to seek solutions.

Induction

We solve (4.1.3) + (4.1.4) to get $P_0(t) = e^{-\lambda t}$. We substitute this into (4.1.2) with $j=1$, where it becomes a forcing term, and we get

$$P_1(t) = \lambda t e^{-\lambda t}.$$

Iteration yields

$$P_j(t) = \frac{(\lambda t)^j}{j!} e^{-\lambda t}.$$

Generating Functions

Define the generating function

$$G(s,t) = \sum_{j=0}^{\infty} P_j(t) s^j$$

Multiply (4.1.2) by s^j and sum over j

$$\frac{\partial G}{\partial t} = \lambda(s-1)G \quad \text{(skipped details!)}$$

with boundary conditions

$$G(s,0) = 1.$$

The solution is

(4.1.5) $$G(s,t) = e^{\lambda(s-1)t} = e^{-\lambda t}\sum_{j=0}^{\infty}\frac{(\lambda t)^j}{j!}s^j.$$

There is an alternative formulation.

<u>Definition 4.1.3</u>

Let $T_0, T_1, T_2, \ldots$ be given by

(4.1.6) $$T_0 = 0, \quad T_n = \inf_t \{N(t) = n\}$$

T_n is the <u>arrival</u> or <u>waiting</u> <u>time</u> for the n^{th} event. The <u>interarrival</u> or <u>sojourn</u> <u>times</u> are the random variables $X_1, X_2, \ldots$ given by

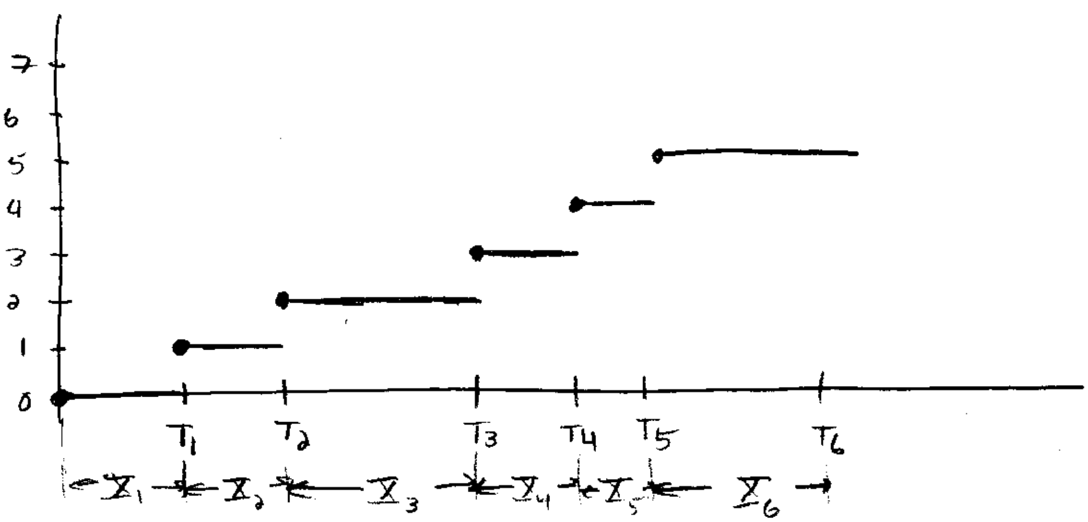
(4.1.7) $\qquad X_n = T_n - T_{n-1}$

If we know N, we can compute $X_1, X_2, \ldots$

Vice versa, if we know the entire collection $\{X_i\}$, then

(4.1.8) $\qquad T_n = \sum_{i=1}^{n} X_i, \qquad N(t) = \max_{T_n \le t} n$

Here is an illustration



#22 4/15

Theorem 4.1.2

The random variables $X_1, X_2, \ldots$ are i.i.d. with exponential distribution with parameter λ.

Proof

Consider X_1:

$$P(X_1 > t) = P(N(t) = 0) = e^{-\lambda t}$$

so X_1 is exponentially distributed.