EE 514, Fall 2005
Exam 4: Due 10am, December 9, 2005

Solutions (version: December 6, 2005, 16:0) Total 50 pts.

1. (20 pts.) Let Y be a Bernoulli random variable with $\mathsf{P}\{Y = 0\} = \mathsf{P}\{Y = 1\} = 1/2$. Consider the discrete-time random process given by

$$X_n = (-1)^{n+Y}, \quad n = 1, 2, \ldots .$$

a. Is $\{X_n\}$ wide-sense stationary?

b. Is $\{X_n\}$ 2nd order strictly stationary?

c. Is $\{X_n\}$ strictly stationary?

d. Is $\{X_n\}$ an i.i.d. sequence?

In each case, justify your answer fully.

Ans.: Before we begin, first consider X_n for some fixed n. Notice that X_n is a discrete random variable on $\{-1, 1\}$. Moreover, $X_n = 1$ if and only if $n + Y$ is even, which holds with probability $1/2$. Hence, the pmf of X_n is given by $\mathsf{P}\{X_n = 1\} = \mathsf{P}\{X_n = -1\} = 1/2$ (which implies that the sequence $\{X_n\}$ is identically distributed).

a. For each $n \in \{1, 2, \ldots\}$, $\mathsf{E}[X_n] = 0$, which does not depend on n. Moreover, for each $n, m \in \{1, 2, \ldots\}$,

$$\mathsf{E}[X_n X_{n+m}] = \mathsf{E}[(-1)^{n+Y}(-1)^{n+m+Y}] = (-1)^{2n}(-1)^m \mathsf{E}[(-1)^{2Y}] = (-1)^m,$$

which does not depend on n. Hence, $\{X_n\}$ is WSS.

b. For each $n, m \in \{1, 2, \ldots\}$ and $B = (b_1, b_2)$ with $b_1, b_2 \in \{-1, 1\}$,

$$\mathsf{P}\{(X_n, X_{n+m}) = B\} = \mathsf{P}\{X_n = b_1, X_{n+m} = b_2\} = \mathsf{P}\{(-1)^{n+Y} = b_1, (-1)^{n+m+Y} = b_2\}.$$

If $b_1 = b_2$, then the above is equal to 0 if m is odd, and $1/2$ if m is even. Similarly, if $b_1 \neq b_2$, then the above is equal to $1/2$ if m is odd, and 0 if m is even. In either case, the probability does not depend on n. Hence, $\{X_n\}$ is 2nd order strictly stationary.

c. We can do the analysis here in a way similar to part b, but this requires examining stationarity at arbitrary orders, which is tedious. Instead, we will argue as follows. For any fixed $m \in \{0, 1, \ldots\}$, consider the process $\{X_{n+m}, n = 1, 2, \ldots\}$. Notice that $X_{n+m} = (-1)^{n+m+Y} = (-1)^{n+Z_m}$, where $Z_m = m + Y$. But Z_m has the same distribution as Y (which does not depend on m). Moreover, the law of $\{X_{n+m}, n = 1, 2, \ldots\}$ depends only on the distribution of Z_m. Hence, the law of $\{X_{n+m}, n = 1, 2, \ldots\}$ does not depend on m. This means that $\{X_n\}$ is strictly stationary.

d. We have already seen that $\{X_n\}$ is identically distributed. But $\mathsf{P}\{X_1 = 1, X_2 = 1\} = 0$, which is not equal to $\mathsf{P}\{X_1 = 1\}\mathsf{P}\{X_2 = 1\} = 1/4$. Hence, $\{X_n\}$ is not i.i.d.

2. (6 pts.) Consider the real-valued function s given by $s(t) = 1$ if $|t| \leq 1$ and $s(t) = 0$ if $|t| > 1$.

a. Is the function given by $R(\tau) = s(\tau)$, $\tau \in \mathbb{R}$ the correlation function of some wide-sense stationary random process? Explain fully.

b. Is the function given by $S(f) = s(f)$, $f \in \mathbb{R}$ the power spectral density of some wide-sense stationary random process? Explain fully.

Ans.: a. The Fourier transform of R is $S(f) = 2\text{sinc}(2f)$, which is negative for some intervals of f (e.g., $(0.5, 1)$). Hence, S is not a valid power spectral density, which implies that R is not the correlation function of some wide-sense stationary random process.

b. In this case, S is real, even, and nonnegative everywhere. Hence, S is the power spectral density of some wide-sense stationary random process.

3. (6 pts.) Consider the matched filter solution discussed in the book:

$$H(f) = \alpha \frac{V(f)^* e^{-j2\pi f t_0}}{S_X(f)}.$$

But suppose that for all $f \in [1, 2]$, we have $S_X(f) = 0$. In this case, the above solution is undefined. What should we do to design the filter?

Ans.: We can set H to be a bandpass filter in the frequency band $[1, 2]$. If we do, then $P_Y = 0$ (i.e., the filtered noise is zero a.s.). So provided $v_o(t_0) \neq 0$, the SNR is infinite.

4. (18 pts.) Let X be a real-valued random variable with zero mean and $\mathsf{E}[X^2] = \sigma_X^2$. We cannot directly observe X. Instead, all we can observe is $Y_t = X + W_t$, $t \geq 0$, where W_t is a zero mean WSS process with correlation function R_W, and is independent of X. To estimate X based on Y_t, we use the linear estimator

$$\hat{X}_T = \int_0^T h(\theta) Y_{T-\theta}\, d\theta.$$

In other words, we filter Y_t through a causal LTI filter with impulse response h, and sample the output at time T. The goal is to design h such that the mean-squared error of the estimator is minimized.

a. Write down the orthogonality principle for this design problem.

b. Use part a to derive a condition involving the optimal h that looks something like: for all $\tau \in [0, T]$,

$$a = \int_0^T h(\theta)(b + c(\tau - \theta))\, d\theta,$$

where a and b are constants and c is some function.

c. Suppose W_t is white noise with power spectral density σ_W^2. Find the optimal h in this case, and write down an expression for $\hat{X}_T$ in terms of Y_t, $t \in [0, T]$.

Ans.: a. If

$$\mathsf{E}\left[(X - \hat{X}_T)\int_0^T \tilde{h}(\theta)Y_{T-\theta}\,d\theta\right] = 0$$

for every $\tilde{h}$, then h is optimal (i.e., minimizes $\mathsf{E}[(X-\hat{X}_T)^2]$).

b. Based the equation in part a, we get

$$\begin{aligned} 0 &= \int_0^T \tilde{h}(\theta)\mathsf{E}[(X-\hat{X}_T)Y_{T-\theta}]\,d\theta \\ &= \int_0^T \tilde{h}(\theta)(\mathsf{E}[XY_{T-\theta}] - \mathsf{E}[\hat{X}_T Y_{T-\theta}])\,d\theta. \end{aligned}$$

Now,

$$\begin{aligned} \mathsf{E}[XY_{T-\theta}] &= \mathsf{E}[X(X+W_{T-\theta})] \\ &= \mathsf{E}[X^2] + \mathsf{E}[XW_{T-\theta}] \\ &= \mathsf{E}[X^2] + \mathsf{E}[X]\mathsf{E}[W_{T-\theta}] \quad \text{by independence} \\ &= \sigma_X^2. \end{aligned}$$

Also,

$$\begin{aligned} &\mathsf{E}[\hat{X}_T Y_{T-\theta}] \\ &= \mathsf{E}\left[\int_0^T h(\eta)(X+W_{T-\eta})\,d\eta(X+W_{T-\theta})\right] \\ &= \int_0^T h(\eta)(\mathsf{E}[X^2] + \mathsf{E}[XW_{T-\theta}] + \mathsf{E}[W_{T-\eta}X] + \mathsf{E}[W_{T-\eta}W_{T-\theta}])\,d\eta \\ &= \int_0^T h(\eta)(\mathsf{E}[X^2] + \mathsf{E}[X]\mathsf{E}[W_{T-\theta}] + \mathsf{E}[W_{T-\eta}]\mathsf{E}[X] + \mathsf{E}[W_{T-\eta}W_{T-\theta}])\,d\eta \quad \text{by independence} \\ &= \int_0^T h(\eta)(\sigma_X^2 + R_W(\theta-\eta))\,d\eta. \end{aligned}$$

Hence, for all $\tilde{h}$,

$$\int_0^T \tilde{h}(\theta)\left(\sigma_X^2 - \int_0^T h(\eta)(\sigma_X^2 + R_W(\theta-\eta))\,d\eta\right)\,d\theta = 0.$$

Setting $\tilde{h}(\theta) = \sigma_X^2 - \int_0^T h(\eta)(\sigma_X^2 + R_W(\theta-\eta))\,d\eta$, we get

$$\int_0^T \left(\sigma_X^2 - \int_0^T h(\eta)(\sigma_X^2 + R_W(\theta-\eta))\,d\eta\right)^2\,d\theta = 0.$$

This implies that for all $\tau \in [0,T]$,

$$\sigma_X^2 = \int_0^T h(\eta)(\sigma_X^2 + R_W(\tau-\eta))\,d\eta.$$

c. By assumption, $R_W(\tau) = \sigma_W^2 \delta(\tau)$. Then, the condition in part b reduces to: for all $\tau \in [0, T]$,

$$\begin{aligned} \sigma_X^2 &= \int_0^T h(\eta)(\sigma_X^2 + \sigma_W^2 \delta(\tau - \eta))\, d\eta \\ &= \sigma_X^2 \int_0^T h(\eta)\, d\eta + \sigma_W^2 h(\tau). \end{aligned}$$

This implies that $h(\tau)$ is constant over $[0, T]$. Call this constant c. To find c, we substitute c into the equation above:

$$\sigma_X^2 = \sigma_X^2 T c + \sigma_W^2 c,$$

which implies that

$$c = \frac{\sigma_X^2}{\sigma_X^2 T + \sigma_W^2}.$$

So

$$\hat{X}_T = \frac{\sigma_X^2}{\sigma_X^2 T + \sigma_W^2} \int_0^T Y_\theta\, d\theta.$$