Notes - 04 Mar

Recall - Long time analysis - Classification of states. Recurrent state: $P(X_n = i \text{ for some } n \geq 1 | X_0 = i) = 1$. Transient state: $P(X_n = i \text{ for some } n \geq 1 | X_0 = i) < 1$. First passage time: the smallest time it takes to go from state i to state j. We are interested in mean first passage time.

Theorem 3.1.2 - (1) j is recurrent if $\Sigma_n P_{jj}^n = \infty$. (2) j is transient if $\Sigma_n P_{jj}^n < \infty$. (3) If j is transient, then $P_{ij}^n \to 0$ as $n \to \infty$ for all i.

Example 3.1.6 - Random Walk. We consider the simple random walk in ex 2.2.2. $X_n = X_0 + \Sigma_{k=1}^n B_k$, $\{B_k\}$ are i.i.d. Bernoulli variables. $P(B_k = 1) = p, P(B_k = -1) = 1-p = q$. Consider state j. $P_{jj}^{2n-1} = 0$ for n = 1, 2, 3, ... (2n-1 = odd numbers, n = 1, 2, 3, ...) To return in 2n steps: we must take n steps in one direction and then n in other direction. This has probability (3.1.3) $P_{jj}^{2n} = \binom{2n}{n} p^n (1-p)^n = \frac{(2n)!}{n!n!}(p(1-p))^n$. We approximate these terms by an expression valid for n large then consider the sum of the approximations. Deciding if a series converges or not is not affected if we drop a finite number of terms from the beginning of the series. We use an asympotic expression for n! valid for n large.

Stirling formula - (3.1.4) $n! \sim n^n \sqrt{n} e^{-n} \sqrt{2\pi}$ n large, which means $\lim_{n\to\infty} \frac{n!}{n^n \sqrt{n} e^{-n} \sqrt{2\pi}} = 1$. We can substitute (3.1.4) into the series $\Sigma_n P_{jj}^{2n}$ without affecting convergence/divergence. $P_{jj}^{2n} \sim \frac{(4p(1-p))^n}{\sqrt{\pi n}}$. When $p = 1/2, P_{jj}^{2n} \sim \frac{1}{\sqrt{\pi n}}, \Sigma_n P_{jj}^n = \infty$. Any state is recurrent when $p = 1/2$. If $p \neq 1/2, 4p(1-p) < 1, \Sigma_n P_{jj}^n < \infty$. Any state is transient when $p \neq 1/2$. Note: thereom 3.1.3 implies that any state is either recurrent or transient.

Theorem 3.1.4 - the number of times $N(i)$ that a Markov chain visits its starting point i satisfies $P(N(i) = \infty) = \{1$ if i is recurrent, 0 if i is transient. Proof - After any return to i, a subsequent return is guaranteed iff $f_{ii} = 1$.

Another classification -

Definition 3.1.5 - Let $T_j = min\{n \geq i : X_n = j\}$ be the time of the first visit to state j where $T_j = \infty$ if X_n never visits j. (T_j depends on X_0.)

Theorem 3.1.5 - $P(T_i = \infty | X_0 = i) > 0$ iff i is transient. When i is transient, $E(T_i | X_0 = i) = \infty$. What about reurrent states?

Definition 3.1.6 - The mean recurrence time μ_i of a state i is: $\mu_i = E(T_i | X_0 = i) = \{\Sigma_{n=1}^\infty n f_{ii}(n)$ for i recurrent, ∞ for i transient. μ_i may be infinite when i is recurrent.

Definition 3.1.7 - A recurrent state i is null if $\mu_i = \infty$ and positive if $\mu_i < \infty$.

Theorem 3.1.6 - A recurrent state is null iff $P_{ii}^n \to 0$ as $n \to \infty$ and if this holds, $P_{ji}^n \to 0$ for all j. Proof later.

Example 3.1.7 - Consider the genotype example 3.1.4. AA and aa are recurrent (0 = "aa"). $f_{00}(1) = 1, f_{00}(n) = 0, n > 1, \Rightarrow f_{00} = 1$. These states are positive.

Example 3.1.8 - For simple random walk, ex 3.1.6, when $p = 1/2, P_{jj}^n \approx \frac{1}{\sqrt{\pi n}} \to 0$ as $n \to \infty$. So any state in a simple random walk with $p = 1/2$ is null recurrent.

The last classification of states we discuss: recall in the simple random walk, the chain can return only with an even number of steps, 2, 4, 6, /dots all divisible by 2.

Definition 3.1.8 - The greatest common divisor of a set of integers $\{n_1, n_2, \dots\}$ written g.c.d.$(n_1, n_2, \dots)$ is the largest integer m such that m divides $n_1, n_2, \dots$ all without remainder.

Example 3.1.9 - gcd(2, 4, 6, 8) = 2. gcd(2, 3, 5) = 1.

Definition 3.1.9 - The period d(i) of state i is $d(i) = \gcd\{n : P_{ii}^n > 0\}$. If d(i) = 1, i is called aperiodic. If d(i) > 1, i is called periodic.

Ex 3.1.10 - Consider the OFF/ON system in ex 3.1.5. If $0 < p < 1, 0 < q < 1$, then $P_{00}, P_{01}, P_{10}, P_{11}$ are all strictly between 0 and 1. Hence $d(i) = 1$ for $i = 0, 1$. Suppose $p = q = 1$, then $P_{00}^n > 0$ for n even, $P_{00}^n = 0$ for n odd. d(0) = 2.

Example 3.1.11 - Simple random walk is periodic with d(i) = 2 when $p = 1/2$.

Ex 3.1.12 - Consider gambler's ruin in §2.4, modified so A has \$1 initially, A has a backer that guarantees A's losses (ex 2.4.5), B is infinitely wealthy. We assume $r_1 = r_2 = \cdots = 0, p_0 = p_1 = \cdots = p, q_1 = q_2 = \cdots = q$. Exercise: P = (q p 0 ... & 0 q 0 p 0 ... & 0 0 q 0 p 0 ...). $P_{11}^1 = 0, P_{11}^2 > 0, P_{11}^3 > 0, d(i) = 1$, single gcd(2, 3) = 1.

Definition 3.1.10 - If all the states of a Markov chain are aperiodic, we call the chain aperiodic.

Definition 3.1.11 - A state is ergodic if it is recurrent, positive, and aperiodic.

Ex 3.1.13 - Consider a branching process. 0 is absorbing and once there a chain never leaves. So $P_{00}^n = 1$ for all n, and 0 is recurrent. Using the formulas for $f_{ii}, \mu_0 = 1$, 0 is positive, 0 is aperiodic, so 0 is ergodic. All other states are transient.

§3.2 - Classification of Chains

Definition 3.2.1 - State i communicates with state j $i \to j$ if the chain may visit j with positive probability having started in i. $i \to j \Leftrightarrow P_{ij}^m > 0$ some m. If $i \to j$ and $j \to i$, then i and j intercommunicate, $i \leftrightarrow j$.

Example 3.2.1 - In the roulette wheel in ex 2.1.6, all nonzero states intercommunicate, and 0 only communicates with itself.