$$\sum_{k=0}^{\infty} \sum_{n=k}^{\infty} a_k b_{n-k} S^n$$
$$= \sum_{k=0}^{\infty} a_k S^k \sum_{n=k}^{\infty} b_{n-k} S^{n-k}$$
$$= A(s) B(s).$$

Example 1.5.3

If X, Y are independent, $X \sim Pois(\lambda)$, $Y \sim Pois(\mu)$, then

$$P_{X+Y}(s) = P_X(s) P_Y(s) = e^{\lambda(s-1)} e^{\mu(s-1)}$$
$$= e^{(\lambda+\mu)(s-1)}.$$

We conclude that $X+Y \sim Pois(\lambda+\mu)$.

$\boxed{\text{Start \#4} \quad 1/31/08}$

Example 1.5.4

Let X be the number of failures necessary to obtain r successes in repeated independent Bernoulli trials. The density of X is called the negative binomial distribution. We can represent X as a sum. We let $\{X_1, \dots, X_r)$ be iid rv with geometric distribution

$g(k,p) = (1-p)^k p \quad \left(E(X_i) = (1-p)/p\right).$

We set $X = X_1 + \dots + X_r$

(Exercise: convince yourself this is right.)

Exercise: Verify that

$$P_{X_i}(s) = \frac{p}{1-(1-p)s}$$

This means

$$P_X(s) = \prod_{i=1}^{r} P_{X_i}(s) = (P_{X_1}(s))^r = \left(\frac{p}{1-(1-p)s}\right)^r$$

We also know that

$$\sum_{k=0}^{\infty} P(X=k)s^k = \left(\frac{p}{1-(1-p)s}\right)^r$$

We have to expand the function on the right as a power series and then identify terms to get $\{P(X=k)\}$.

The Binomial Theorem states

$$(1+t)^a = \sum_{k=0}^{\infty} \binom{a}{k} t^k \quad , \ |t|<1,$$

so

$$\left(\frac{p}{1-(1-p)s}\right)^r = p^r(1-(1-p)s)^{-r} = p^r \sum_{k=0}^{\infty} \binom{-r}{k}(-1)^k(1-p)^k s^k$$

so

$$P(X = k) = (-1)^k \binom{-r}{k} p^r (1-p)^k.$$

Chapter 2 Markov Chains

Definition 2.0.1
A *stochastic process* is a family of random variables X_t, t in a suitable *index set* T.
A *discrete time process* corresponds to $T = \{0, 1, 2, 3, \ldots\}$. A *continuous time process* has T equal to an interval, typically $[0, \infty)$.

Definition 2.0.2
The *state space* of a stochastic process is the range of the random variables.

Stochastic processes are distinguished by their index set, state space, and the dependence relations between the variables.

§2.1 Markov Chains

The first sort of process that we study has the defining characteristic that it has no memory of what happened in the past, e.g. only the current state of the process influences where it goes next. These are called Markov processes. If the state space is countable or finite, they are called Markov chains.

Definition 2.1.1

A *Markov chain* X_t is a discrete time stochastic process with finite or countable state space that satisfies the *Markov condition*

$$(2.1.1)\quad P(X_{n+1}=j \mid X_0=i_0, \ldots, X_{n-1}=i_{n-1}, X_n=i)$$

$$= P(X_{n+1}=j \mid X_n=i)$$

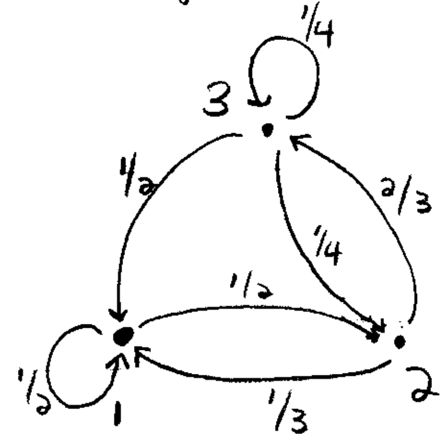
for all $n=0,1,2,\ldots$ and states $i_0, i_1, \ldots, i_{n-1}, i, j$.

Without loss of generality, we assume the state space is a subspace of the integers.

Example 2.1.1

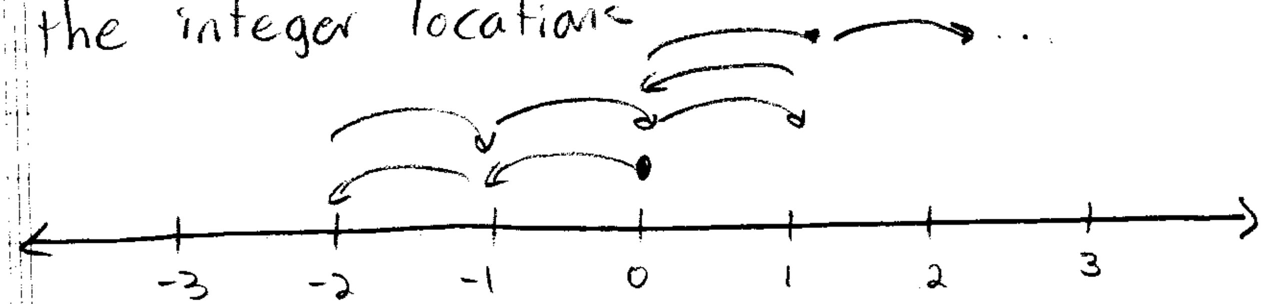
Consider a process with state space $\{1,2,3\}$. If X_n is 1, then in the next time step, it has probability
$1/2$ of moving to state 2 and $1/2$ of remaining at 1. If X_n is 2, then in the next time step it has a probability of $1/3$ of moving to 1 and $2/3$ to 3. If X_n is 3, then in the next step, it has a probability of $1/2$ to move to state 1, $1/4$ to move to 2, and $1/4$ to remain at 3. We can describe

this with a graph



Example 2.1.2 Random Walk

We consider a particle that can move back and forth on the x-axis, inhabiting the integer locations



At each time index, it occupies one place. At time 0, we assume the particle is at X_0. At each subsequent time $1, 2, 3, \dots$, it moves to a new position by either moving one step to the right or to the left. We let the probability of moving to the right to be p and to the left $q = 1-p$. The sequence of moves are assumed to be independent.

The random walk is <u>symmetric</u> if $p = 1/2$.

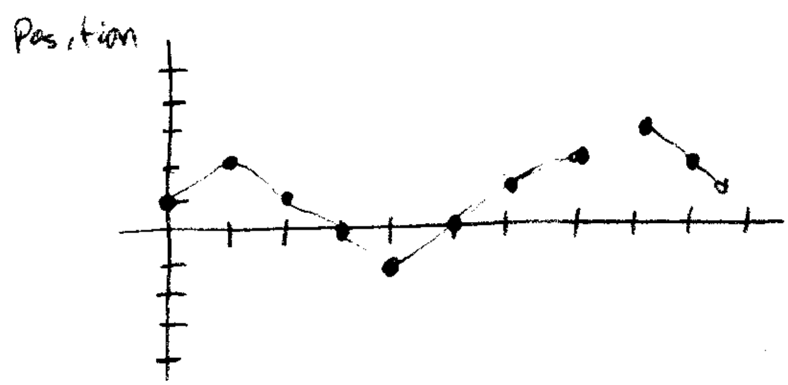
We let X_n denote the position of the particle after n moves, with X_0 = initial position. We have

$$(2.1.2) \qquad X_n = X_0 + \sum_{k=1}^{n} B_k ,$$

where $\{B_k\}$ is a sequence of Bernoulli random variables with values $\{-1, 1\}$ with probabilities $\{q, p\}$ respectively.

We can plot the path of a sequence

$\{(n, X_n)\}$



Note, by (2.1.2), if X_m is known, X_{m+n} depends only on the jumps $B_{m+1}, \ldots, B_{m+n}$, and not on $X_0, \ldots, X_{n-1}$. So this is a Markov chain.

Theorem 2.1.1

The Markov property is equivalent to the following conditions

(2.1.3) $P(X_{n+1}=j \mid X_{n_1}=i_{n_1}, \ldots, X_{n_k}=i_{n_k})$

$$= P(X_{n+1}=j \mid X_{n_k}=i_{n_k})$$

for all $0 \leq n_1 < n_2 < \ldots < n_k \leq n$ and states $j, i_{n_1}, \ldots, i_{n_k}$.

(2.1.4) $P(X_{m+n}=j \mid X_0=i_0, \ldots, X_m=i_m)$

$$= P(X_{m+n}=j \mid X_m=i_m)$$

for all $m, n \geq 0$, $i_0, \ldots, i_m$, j in the state space.

Proof

Exercise

Definition 2.1.2

The probability of X_{n+1} being in state j given X_n is in state i is called the one step transition probability, and is denoted

$$P_{ij}^{n,n+1} = P(X_{n+1}=j \mid X_n=i)$$

Example 2.1.3

In Ex. 2.1.1, we have

$$p_{11}^{n,n+1} = \frac{1}{2} \qquad p_{12}^{n,n+1} = \frac{1}{2} \qquad p_{13}^{n,n+1} = 0$$

$$p_{21}^{n,n+1} = 1/3 \qquad p_{22}^{n,n+1} = 0 \qquad p_{23}^{n,n+1} = 2/3$$

$$p_{31}^{n,n+1} = 1/2 \qquad p_{32}^{n,n+1} = 1/4 \qquad p_{33}^{n,n+1} = 1/4$$

for all n.

In general, the transition probabilities can vary with time.

Definition 2.1.3

A Markov chain is homogeneous if

$$P(X_{n+1} = j \mid X_n = i) = P(X_1 = j \mid X_0 = i)$$

for all $n \geq 0$.

This means $p_{ij}^{n,n+1}$ is independent of n.

Example 2.1.4

The random walk in Ex. 2.1.2 is homogeneous.

We restrict ourselves to homogeneous chains for now.

Definition 2.1.4

The transition matrix P is the matrix of transition probabilities

$$P = (P_{ij}), \quad P_{ij} = P(X_{n+1} = j \mid X_n = i)$$

P has the dimensions of the state space.

Example 2.1.5

In Ex. 2.1.1, Ex 2.1.3

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/3 & 0 & 2/3 \\ 1/2 & 1/4 & 1/4 \end{pmatrix}$$

Example 2.1.6

A roulette wheel has 38 numbered slots for the ball to rest. During each spin, we bet \$1 on the ball resting in an odd spot. We start with \$10 and record our fortune after each spin. The chances of winning are 18/38 and losing 20/38.

We let $\{X_n\}$ be the fortune at spin n, X_n clearly depends only on X_{n-1}.

The state space is $\{0,1,2,\ldots\}$. If the chain is in state i, it jumps to states $i-1$, $i+1$ with probability

$$P_{i,i-1} = \frac{20}{38}, \quad P_{i,i+1} = \frac{18}{38}$$

respectively. We have $P_{ii} = 0$ for $i > 1$, and $P_{i,k} = 0$ when $|i-k| > 1$. 0 is a special state, because we cannot bet further. We set $P_{0,0} = 1$. The matrix is

$$P = \begin{pmatrix} 1 & 0 & 0 & \cdots & \\ 20/38 & 0 & 18/38 & 0 & \cdots \\ 0 & 20/38 & 0 & \ddots & \\ \vdots & 0 & 20/38 & \ddots & \\ 0 & \vdots & 0 & \ddots & \\ & & \vdots & & \end{pmatrix}$$

and it is infinite.

Example 2.1.7

Genes occur in pairs where one copy is inherited from the mother and one from the father. Suppose a certain gene has two variants, or alleles, A and a. An individual can have one of these genotypes

$$\{AA, Aa, aa\}$$

Suppose a plant is crossed with itself, then one offspring is crossed with itself, and so on. This yields a Markov chain. The state space is $\{AA, Aa, aa\}$. The Markov property holds because an offspring's genotype depends only on the parent plant, not the grandparent.

Clearly, genotypes AA and aa can only produce offspring with the same genotypes. For genotype Aa,

plant is crossed with itself

$A^1 a^1$
$A^2 a^2$

$A^1 A^2$ probability $1/4$
$A^1 a^2$, $A^2 a^1$ probability $1/2$
$a^1 a^2$ probability $1/4$

The transition matrix is

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{pmatrix}$$

$\boxed{\#5 \quad 2/5}$

Example 2.1.8

For the random walk in Ex. 2.1.2, the state space

$$S = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

The transition probabilities are

$$P_{ij} = \begin{cases} p & , \; j = i+1, \\ 1-p, & , \; j = i-1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$P = \begin{pmatrix} \ddots & & & & & & & \mathbf{0} \\ & 0 & 1-p & 0 & p & 0 & & \\ & & 0 & 1-p & 0 & p & 0 & \\ \mathbf{0} & & & 0 & 1-p & 0 & p & 0 \ldots \\ & & & & & & & \ddots \end{pmatrix}$$