$$P(X_n = i \text{ for some } n \geq 1 \mid X_0 = i) = 1,$$

which says that the probability of returning to state i having started in i is 1.

If

$$P(X_n = i \text{ for some } n \geq 1 \mid X_0 = i) < 1,$$

i is called <u>transient</u>.

$\boxed{\text{start \#11 2/26}}$

A recurrent state has the property that the chain returns to the initial state in finite time. For a transient state, there is a positive probability of no return.

<u>Example 3.1.1</u>

Consider the roulette wheel in Ex. 2.1.6. State 0 is trivilly recurrent since if $X_0 = 0, X_1 = 0, X_2 = 0, \ldots$. State $i \neq 0$ has the property that if we jump to 0, then we cannot return to i. Hence, for $i \neq 0$,

$$P(X_n = i \text{ some } n \geq 1 \mid X_0 = i) < 1,$$

and i is transient.

<u>Example 3.1.2</u>

Consider the genotype example Ex 2.1.7. We assume $X_0 = Aa$. If $X_1 = 0$ or $X_1 = 2$ there can be no return to X_0, hence

$$P(X_n = Aa \text{ some } n \geq 1 \mid X_0 = Aa) = P(X_1 = Aa \mid X_0 = Aa) = 1/2.$$

Aa is transient.

Example 3.1.3

Consider the off/on system in Ex 2.2.3 with

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}.$$

Assume $0 < p, q < 1$.

Consider $X_0 = 0$:

$$X_0 = 0 \xrightarrow{Pr = (1-p)} X_1 = 0$$

$$\xrightarrow{Pr = p} X_1 = 1 \xrightarrow{Pr = q} X_2 = 0$$

$$\xrightarrow{Pr = 1-q} X_2 = 1 \xrightarrow{Pr = q} X_3 = 0$$

$$\xrightarrow{Pr = 1-q} X_3 = 1 \longrightarrow \dots$$

$$\longrightarrow \dots$$

Each step is independent, hence

$$P(X_n = 0 \text{ some } n \geq 1 \mid X_0 = 0)$$

$$(1-p) + p \cdot q + p \cdot (1-q) \cdot q + p \cdot (1-q)^2 \cdot q + \cdots$$

$$= (1-p) + pq\left(1 + (1-q) + (1-q)^2 + \cdots\right)$$

$$= (1-p) + pq \frac{1}{1-(1-q)} = (1-p) + pq \frac{1}{q} = 1.$$

So 0 is recurrent. Now recall that in the limit of large time, X_n has probability $q/(p+q)$ of ending up in state 0. Even if $q \ll p$ and we are not likely to end up in state 0, with probability 1, we see $X_n = 0$ for some $n \geq 1$.

Set up as computational experiments

We are interested in

Definition 3.1.2

The *first passage time* from state i to state j is the smallest time it takes to move from state i to state j. In general, we are interested in the mean first passage times.

Definition 3.1.3

Define
$$f_{ij}(n) = P\left(X_1 \neq j, X_2 \neq j, \ldots, X_{n-1} \neq j, X_n = j \mid X_0 = i\right)$$

to be the probability that the *first* visit to state j starting from state i takes place in the n^{th} step. We set

$$(3.1.1) \qquad f_{ij} = \sum_{n=1}^{\infty} f_{ij}(n)$$

to be the probability that the chain ever visits j starting from i.

Note: j is recurrent iff $f_{jj} = 1$.

We look for a criterion for recurrence in terms of the n step transition probabilities.

Recall our convention (pg 57), P = prob. transition matrix with entries (p_{ij}), and P^n is the n step prob. transition matrix with entries (p_{ij}^n).

<u>Definition 3.1.4</u>

We define the prob. generating functions

$$P_{ij}(s) = \sum_{n=0}^{\infty} p_{ij}^n s^n$$

$$F_{ij}(s) = \sum_{n=0}^{\infty} f_{ij}(n) s^n,$$

where we set

$p_{ij}^0 = \delta_{ij}$, $f_{ij}(0) = 0$, for all i, j. Note

$f_{ij} = F_{ij}(1)$. We usually assume $|s| < 1$ so that $P_{ij}(s)$ is guaranteed to converge.

Theorem 3.1.1

(1) $P_{ii}(s) = 1 + F_{ii}(s) P_{ii}(s)$

(2) $P_{ij}(s) = F_{ij}(s) P_{jj}(s)$, $i \neq j$.

Proof

Fix $i, j \in S$. Set

$$A_m = \{ X_m = j \}$$

and

$$B_m = \{ X_k \neq j \text{ for } 1 \leq k < m, X_m = j \}$$

A_m is the event that X_m hits state j at the m^{th} step. It may have visited j earlier. B_m is the event that the first visit to j after 0 is at time m.

The B_m are disjoint, so

$$P(A_m \mid X_0 = i) = \sum_{k=1}^{m} P(A_m \cap B_k \mid X_0 = i)$$

$A_m \cap B_k =$ event where X_m visits state j at the k^{th} and m^{th} steps.

The Markov property implies

$$P(A_m \cap B_k \mid X_0 = i) = P(A_m \mid B_k, X_0 = i)\, P(B_k \mid X_0 = i)$$

Markov Property

$$= P(A_m \mid X_k = j)\, P(B_k \mid X_0 = i)$$

initial condition for chain for times larger than k

This means

$$(3.1.2) \quad P(A_m \mid X_0 = i) = p_{ij}^m = \sum_{k=1}^{m} p_{jj}^{m-k} f_{ij}(k), \quad m = 1, 2, \ldots$$

We multiply by s^m, $|s|<1$, and sum over $m \geq 0$ to obtain

$$P_{ij}(s) - \delta_{ij} = F_{ij}(s) P_{jj}(s).$$

Theorem 3.1.2

(1) State j is recurrent if $\sum_n P_{jj}^n = \infty$

and if this holds then $\sum_n P_{ij}^n = \infty$ for all i such that $f_{ij} > 0$.

(2) State j is transient if $\sum_n P_{jj}^n < \infty$

and if this holds then $\sum_n P_{ij}^n < \infty$ for all i

(3) If j is transient, then $P_{ij}^n \to 0$ as $n \to \infty$ for all i.

Recall that P_{jj}^n is the probability of revisiting state j starting at j in the n^{th} step. So

A condition on $\sum_n P_{jj}^n$ is a condition on the size of the probabilities of revisiting j on any step.

$\sum_n P_{ij}^n$ has to do with the probabilities of visiting state j from state i on any step n.

Proof

We show that j is recurrent if and only if $\sum_n P_{jj}^n = \infty$. We know

$$P_{jj}(s) = \frac{1}{1 - F_{jj}(s)}, \quad |s| < 1.$$

As $s \uparrow 1$, $P_{jj}(s) \longrightarrow \infty$ if and only if $f_{jj} = F_{jj}(1) =$

We now use Abel's theorem

Theorem 3.1.3

Let $G(s) = \sum_{j=0}^{\infty} a_j s^j$. If $a_j \geq 0$ for all j and $G(s)$ is finite for $|s|<1$, then

$\lim_{s \uparrow 1} G(s) = \sum_{j=0}^{\infty} a_j$ (whether the sum is finite or infinite).

This implies

$$\lim_{s \uparrow 1} P_{jj}(s) = \sum_n P_{jj}^n,$$

which shows the claim. The results in the theorem follow (exercise).

Example 3.1.4

Consider the genotype example Ex 2.1.7, where from Ex 2.2.1

$$P^n = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2}(1-(\frac{1}{2})^n) & (\frac{1}{2})^n & \frac{1}{2}(1-(\frac{1}{2})^n) \\ 0 & 0 & 1 \end{pmatrix}$$

For Aa,

$$\sum_n P_{11}^n = \sum_n (\tfrac{1}{2})^n = 1 < \infty$$

so Aa is transient.

Example 3.1.5

Consider the off/on system in Ex 2.2.3,

$$P^n = \frac{1}{p+q}\begin{pmatrix} q & p \\ q & p \end{pmatrix} + \frac{(1-p-q)^n}{p+q}\begin{pmatrix} p & -p \\ -q & q \end{pmatrix}$$

For state 0

$$\sum_n P_{00}^n = \sum_n \left(\frac{q}{p+q} + \frac{(1-p-q)^n}{p+q} p \right) = \infty$$

so 0 is recurrent. $\boxed{\#12 \quad 3/4}$

Example 3.1.6 Random Walk

We consider the simple random walk in Ex. 2.2.2,

$$X_n = X_0 + \sum_{k=1}^{n} B_k,$$

$\{B_k\}$ i.i.d. Bernoulli variables with $P(B_k = 1) = p$, $P(B_k = -1) = 1-p = q$.

Consider any state j. We note that

$P_{jj}^{2n-1} = 0$ for $n = 1, 2, \ldots$ since we cannot return to j in an odd number of steps.