Notes - 22 April

HW 6 - question 2 does have a limiting distribution.

(Recall we are talking about birth processes.)

Definition 4.2.1 - $N(t)$ (continuous-time birth process). Assume: (a) $N(0) \geq 0$, (b) $s < t \Rightarrow N(s) \leq N(t)$, (c) $P(N(t+h) = n+m|N(t) = m) = \{\lambda_n h + O(h)$ for $m = 1, O(h)$ for $m > 1, 1 - \lambda_n h + O(h)$ for $m = 0$, (d) If $s < t$ then (conditional on the value of $N(s)$, the increment $N(t) - N(s)$ is independent of the times of arrivals prior to s.

Transition probabilities: $P_{ij}(t) = P(N(t) = j|N(0) = i) = P(N(s+t) = j|N(s) = i), s \leq t$.

Forward System - Assume $\lambda_{-1} = 0, P_{ij}(0) = \delta_{ij}$, then (4.2.1) $P'_{ij}(t) = \lambda_{j-1}P_{ij-1}(t) - \lambda_j P_{ij}(t)$.

Backward System - (4.2.2) $P'_{ij}(t) = \lambda_i, P_{i+1j}(t) - \lambda_i P_{ij}(t), j \geq i$. (Yes, that is P_{i+1j}. I asked in class and verified it. I don't understand why...)

Theorem 4.2.3 - The forward system has a unique solution that also satisfies the backward system. Proof: First note that (4.2.3) $P_{ij}(t) = 0$ if $j < i$. We solve the forward problem with $j = i$, so $P'_{ii}(t) = \lambda_{i-1}P_{ii-1}(t) - \lambda_i P_{ii}(t)$ (first term after equals goes to zero), so (4.2.4) $P_{ii}(t) = e^{-\lambda_i t}$. We substitute into the forward equation with $j = i+1$ to find that $P_{ii+1}(t)$ exists (using standard ODE theory). By induction, we conclude the solution of the forward system exists and is unique. We use the Laplace transform: $\widehat{P_{ij}}(\theta) = \int_0^\infty e^{-\theta t} P_{ij}(t)dt, P_{ij}(t) \widehat{\rightarrow} = \widehat{P_{ij}}(\theta)$. This transforsm derivatives with respect to t to products in the θ variable domain. If we transform both sides of the forward equation, we get $(\theta + \lambda_j)\widehat{P_{ij}}(\theta) = \delta_{ij} + \lambda_{j-1}\widehat{P_{ij-1}}(\theta)$. This difference equation can be solved: (4.2.5) $\widehat{P_{ij}}(\theta) = \frac{1}{\lambda_j}\frac{\lambda_i}{\theta+\lambda_i}\frac{\lambda_{i+1}}{\theta+\lambda_{i+1}} \cdots \frac{\lambda_j}{\theta+\lambda_j}, j \geq i$. Using the inverse Laplace transform gives $P_{ij}(t)$.

To show the claim about the backward equations, we also take the Laplace transform in the same way but now with the backward equation to find that any solution (call it Π—it may be P, may not be) with $\widehat{\Pi_{ij}}(\theta) = \int_0^\infty e^{-\theta t}\Pi_{ij}(t)dt$ satisfies $(\theta + \lambda_j)\widehat{\Pi_{ij}}(\theta) = \delta_{ij} + \lambda_i \widehat{\Pi_{i+1j}}(\theta)$. Note that $\widehat{P}_{ij}$ satisfies this equation, but so can other functions. (TeX note: the widehat change here is meaningless.)

Theorem 4.2.4 - If $\{P_{ij}(t)\}$ is the unique solution of the forward system then any solution $\{\Pi_{ij}\}$ of the backwards system satisfies $P_{ij}(t) \leq \Pi_{ij}(t)$ for all t. Proof not given.

Observe: if (4.2.6) $\Sigma_j P_{iJ}(t) \equiv 1$ (for all t), then Thm 4.2.4 would imply that $\{P_{ij}\}$ is the unique solution of the backward system that is a probability distribution. However (4.2.6) may not hold.

Definition 4.2.5 - An explosion occurs if the birth rates λ_n increase sufficiently quickly that there is a positive probability that the process N can pass through all finite states in finite (bounded) time.

Definition 4.2.6 - Let $T_\infty = \lim_{n\to\infty} T_n$ be the limit of the arrival times. We say N is honest if $P(T_\infty = \infty) = 1$ and dishonest otherwise.

Theorem 4.2.5 - (4.2.6) holds (for each $i\{P_{ij}(t)\}$ is a probability distribution in j) if and only if N is honest. Proof: (4.2.6) is equivalent to $P(T_\infty > t) = 1$, any t, why? Exercise.

Theorem 4.2.6 - N is honest $\iff \Sigma_n \lambda_n^{-1} = \infty$. This says that if the birth rates are sufficiently small (increase sufficiently slowly) then N is honest. $\Sigma a_n = \infty, a_n$ decreases sufficiently slowly, e.g. $\Sigma_n \frac{1}{n} = \infty$. If the λ_n increase sufficiently quickly that $\Sigma_n \lambda_n^{-1}$ converges, then N is dishonest. We can think of the deficit as $1 - \Sigma_j P_{ij}(t)$ as the probability $P(T_\infty \leq t)$ of escaping to infinity at time t starting from state i.

Theorem 4.2.6 follows from:

Theorem 4.2.7 - Let $X_1, X_2, \ldots$ be independent random variables with X_n having the exponential distribution with parameter λ_{n-1} and let $T_\infty = \Sigma_n X_n$. Then $P(T_\infty < \infty) = \{0$ for $\Sigma_n \lambda_n^{-1} = \infty, 1$ for $\Sigma_n \lambda_n^{-1} < \infty$. Proof in notes. When the rates vary, the situation becomes more complicated.

Last topic in this section: what does the condition (d) mean? Recall that a sequence of random variables $\{X_n, n \geq 0\}$ satisfies the Markov property if, conditional on the event $\{X_n = i\}$ events related to the collection $\{X_m, m > n\}$ are independent of events related to the collection $\{X_m, m < n\}$.

Theorem 4.2.8 - Weak Markov Property - Let $N(t)$ be a birth process and T a fixed time. Conditional on the event $\{N(T) = i\}$ the evolution of the process after T is independent of the evolution before T. Proof: definition 4.2.1(a).