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ST 521
HW 6

1.

$$\begin{aligned}
&\mu_0 = f_{00}(1) + f_{00}(2) + \cdots + f_{00}(\infty)\\
&f_{00}(1) = q\\
&f_{00}(2) = pq\\
&f_{00}(3) = p^2q\\
&f_{00}(4) = p^3q\\
&f_{00}(5) = p^4\\
&f_{00}(n) = 0 \text{ for } n \geq 6\\
&\mu_0 = q(1+p+p^2+p^3)+p^4 = (1-p)+(p-p^2)+(p^2-p^3)+(p^3-p^4)+p^4 = 1 = \mu_0 = \tfrac{1}{\mu_0}\\
&\mu_i, i \geq 1 = 0 \text{ since } f_{ii} = \infty, i \geq 1
\end{aligned}$$

So, the limiting distribution $q = (1, 0, 0, 0, 0)$.

2. Find the stationary distribution Π:

$$\begin{aligned}
&\Pi_0 = q_1\Pi_1\\
&\Pi_1 = \Pi_0 + q_2\Pi_2\\
&\Pi_2 = p_1\Pi_1 + q_3\Pi_3\\
&\Pi_3 = p_2\Pi_2 + q_4\Pi_4
\end{aligned}$$

$$\begin{aligned}
&\Pi_0 P_{01} = \Pi_1 P_{10} \Rightarrow \Pi_0 = q_1\Pi_1\\
&\Pi_1 P_{12} = \Pi_2 P_{21} \Rightarrow p_1\Pi_1 = q_2\Pi_2\\
&\Pi_{N-2} P_{N-2,N-1} = \Pi_{N-1} P_{N-1,N-2} \Rightarrow p_{N-2}\Pi_{N-2} = q_{N-1}\Pi_{N-1}\\
&\Pi_N P_{N,N-1} = \Pi_{N-1} P_{N-1,N} \Rightarrow p_{N-1}\Pi_N = \Pi_{N-1}
\end{aligned}$$

3.

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & \dots & \\ 1/3 & 1/3 & 1/3 & 0 & \dots \\ \vdots & \ddots & & & \end{pmatrix}$$

$$\begin{aligned}
&\Pi_0 = \Pi_1 = \tfrac{1}{2}\Pi_0 + \tfrac{1}{3}\Pi_1 + \tfrac{1}{4}\Pi_2 + \dots\\
&\Pi_2 = \tfrac{1}{3}\Pi_1 + \tfrac{1}{4}\Pi_2 + \dots\\
&\tfrac{1}{2}\Pi_0 = \tfrac{1}{3}\Pi_1 + \tfrac{1}{4}\Pi_2 + \dots\\
&\tfrac{1}{6}\Pi_0 = \tfrac{1}{4}\Pi_2 + \tfrac{1}{5}\Pi_3 + \dots
\end{aligned}$$

4. By theorem 4.1.1, need to show $\frac{(\alpha t)^j}{j!}e^{-\alpha t} + \frac{(\beta t)^j}{j!}e^{-\beta t} = \frac{((\alpha+\beta)t)^j}{j!}e^{-(\alpha+\beta)t}$:

$$\frac{(\alpha t)^j}{j!}e^{-\alpha t} + \frac{(\beta t)^j}{j!}e^{-\beta t} = (\alpha t)^j e^{-\alpha t} + (\beta t)^j e^{-\beta t} = ((\alpha+\beta)t)^j e^{-(\alpha+\beta)t} = \frac{(\alpha t+\beta t)^j}{j!}e^{-\alpha t}e^{-\beta t}$$

5. $P(N(t) = 1, 3, 5, \cdots) = \lambda e + \frac{(\lambda 3)^j}{6}e^{-\lambda 3} + \frac{(\lambda 5)^j}{120}e^{-\lambda 5} + \cdots$. Computationally, as $\lambda \to \infty$ (although it converges pretty fast at about $\lambda = 3$), $P(N(t) = 1, 3, 5, \cdots) \to 0.5$.