N(h) to derive

<u>Theorem 4.2.2</u> <u>Backward System</u>

With $P_{ij}(0) = \delta_{ij}$, the transition probabilities for the birth process satisfy the backward equations

Review
1) Birth process
2) transition prob.
3) F'ward and B'ward system.

(4.2.2) $P_{ij}'(t) = \lambda_i P_{i+1\,j}(t) - \lambda_i P_{ij}(t), \quad j \geq i,$

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<u>Theorem 4.2.3</u>

The forward system has a unique solution that also satisfies the backward system.

<u>Proof</u>

First note that

(4.2.3) $P_{ij}(t) = 0 \quad \text{if } j < i$

We solve the forward problem with $j = i$,

$$P_{ii}'(t) = \cancel{\lambda_{i-1} P_{i\,i-1}(t)}^{\,0} - \lambda_i P_{ii}(t),$$

to obtain

(4.2.4) $P_{ii}(t) = e^{-\lambda_i t}$

We substitute into the forward equation with $j=i+1$ to find $P_{i,i+1}(t)$. By induction, we can conclude the forward system has a unique solution.

We use the Laplace transform

$$\hat{P}_{ij}(\theta) = \int_0^\infty e^{-\theta t} P_{ij}(t)\,dt .$$

If we transform both sides of the forward system, we obtain

$$(\theta+\lambda_j)\hat{P}_{ij}(\theta) = \delta_{ij} + \lambda_{j-1}\hat{P}_{i,j-1}(\theta).$$

differentiation becomes multiplication

This difference equation can be solved

(4.2.5) $$\hat{P}_{ij}(\theta) = \frac{1}{\lambda_j}\frac{\lambda_i}{\theta+\lambda_i}\frac{\lambda_{i+1}}{\theta+\lambda_{i+1}}\cdots\frac{\lambda_j}{\theta+\lambda_j}, \quad j \geq i$$

We take the inverse Laplace transform to find $P_{ij}(t)$.

To show the claim about the backward equations, we take the Laplace transform to discover that any solution $\pi_{ij}(t)$ with

$$\hat{\pi}_{ij}(\theta) = \int_0^\infty e^{-\theta t}\, \pi_{ij}(t)\, dt$$

satisfies

$$(\theta + \lambda_j)\, \hat{\pi}_{ij}(\theta) = \delta_{ij} + \lambda_i \hat{\pi}_{i+1\,j}(\theta),$$

and $\hat{P}_{ij}$ does this

Now, interestingly the backward system may not have a unique solution.

We can show

Theorem 4.2.4

If $\{P_{ij}(t)\}$ is the unique solution of the forward system, then any solution $\{\pi_{ij}\}$ of the backward system satisfies

$$P_{ij}(t) \leq \pi_{ij}(t) \qquad \text{all } t.$$

Proof: not given.

Now, if

$$(4.2.6) \qquad \sum_j P_{ij}(t) \equiv 1$$

then Theorem 4.2.4 would imply that $\{P_{ij}\}$ is the unique solution of the backward system that is a proper distribution.

However, (4.2.6) may not hold.

Definition 4.2.5

An explosion occurs if the birth rates λ_n increase sufficiently quickly that there is a positive probability that the process N can pass through all finite states in bounded time.

Definition 4.2.6

Let $T_\infty = \lim_{n \to \infty} T_n$ be the limit of the arrival times. N is honest if $P(T_\infty = \infty) = 1$ and dishonest otherwise.

Theorem 4.2.5

(4.26) holds $\Leftrightarrow$ N is honest.

Proof

(4.26) is equivalent to $P(T_\infty > t) = 1$.
(why)

Theorem 4.2.6

N is honest $\iff \sum_n \lambda_n^{-1} = \infty$.

This theorem says that if the birth rates are sufficiently 'small', then $N(t)$ is almost surely finite.
But, if they are sufficiently large that $\sum_n \lambda_n^{-1}$ converges, then births occur so frequently that there is positive probability of infinitely many births occurring in a finite interval of time, so $N(t)$ may actually reach ∞.

We can think of the deficit.

$$1 - \sum_j P_{ij}(t)$$

as the probability $P(T_\infty \leq t)$ of escaping to infinity at t starting from i.

Theorem 4.2.6 is an immediate consequence of

Theorem 4.2.7

Let $X_1, X_2, X_3, \ldots$ be independent random variables with X_n having the exponential distribution with parameter λ_{n-1} and let $T_\infty = \sum_n X_n$. Then,

$$P(T_\infty < \infty) = \begin{cases} 0 & \text{if } \sum_n \lambda_n^{-1} = \infty \\ 1 & \text{if } \sum_n \lambda_n^{-1} < \infty. \end{cases}$$

Proof

Since $\{X_n\}$ are nonnegative random variables with finite expectations, we can apply monotone convergence to the partial sums $T_N = \sum_{n=1}^{N} X_n$ to conclude

$$E(T_\infty) = E\left(\sum_{n=1}^{\infty} X_n\right) = \sum_{n=1}^{\infty} E(X_n) = \sum_{n=1}^{\infty} \frac{1}{\lambda_{n-1}}.$$

If $\sum_n \lambda_n^{-1} < \infty$, $E(T_\infty) < \infty \Rightarrow P(T_\infty = \infty) = 0$.

We now work with the bounded random variable $e^{-T_\infty} = \lim_{N \to \infty} e^{-T_N}$. By montone convergence

$$E(e^{-T_\infty}) = E\left(\prod_{n=1}^{\infty} e^{-X_n}\right)$$

$$= \lim_{N\to\infty} E\left(\prod_{n=1}^{N} e^{-X_n}\right)$$

$$= \lim_{N\to\infty} \prod_{n=1}^{N} E(e^{-X_n}) \quad \text{(independence)}$$

$$= \lim_{N\to\infty} \prod_{n=1}^{N} \frac{1}{1+\lambda_{n-1}^{-1}}$$

$$= \left(\prod_{n=1}^{\infty} (1+\lambda_{n-1}^{-1})\right)^{-1}$$

The last product is ∞ if $\sum_n \lambda_n^{-1} = \infty$

$\Rightarrow E(e^{-T_\infty}) = 0.$ Since $e^{-T_\infty} \geq 0$, $P(T_\infty = \infty)$

$= P(e^{-T_\infty} = 0) = 1.$

The point is that if we allow the birth rates to vary in time, special care is needed if the rates increase with time.

We are also interested in the Markov properties of birth processes and specifically the Poisson

Poisson process.

Recall that a sequence of random variables $\{X_n, n \geq 0\}$ satisfies the Markov property if, conditional on the event $\{X_n = i\}$, events related to the collection $\{X_m, m > n\}$ are independent of events related to $\{X_m, m < n\}$.

Birth processes have a similar property.

Theorem 4.2.8 Weak Markov Property

Let $N(t)$ be a birth process and T a fixed time. Conditional on the event $\{N(t) = i\}$, the evolution of the process after time T is independent of the evolution prior to T.

Proof

This is a direct consequence of Defn 4.2.1 (d).

The property is "weak" because T is a fixed constant.

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It turns out to be useful to allow T to be a random variable. But, the analogous conclusion cannot hold for all random T, since if T "looks into the future" as well