Notes - 06 Mar

Theorem 3.2.1 - If $i \neq j$ then $i \to j$ iff $f_{ij} > 0$. Proof: exercise. Note, $i \leftrightarrow i$ since $P^0_{ii} = 1$. If we fix i, we could search for all states j that intercommunicate with it.

Def 3.2.2 - All the states that intercommunicate with a given state form a communication class. $i \leftrightarrow j, j \leftrightarrow k$, so $i \leftrightarrow k$.

Definition 3.2.3 - An equivalence relation $\sim$ on a set S is an operation on pairs of elements satisfying (1) $a \sim a, a \in S$, (2) $a \sim b \Rightarrow b \sim a, a, b \in S$, (3) $a \sim b, b \sim c \Rightarrow a \sim c$.

Theorem 3.2.2 - $\leftrightarrow$ is an equivalence relation.

Example 3.2.2 - In the roulette wheel, ex 3.2.1, there are two communication classes $\{0\}, \{1, 2, \ldots, 38\}$.

Example 3.2.3 - In the genotype example, ex 2.1.7, each state forms its own class, $\{AA\}, \{aa\}, \{Aa\}$.

Example 3.2.4 - In the ON/OFF system, ex 2.2.3, with $0 < p < 1, 0 < q < 1$, there is one class: $\{ON, OFF\}$.

Theorem 3.2.3 - If $i \leftrightarrow j$, (1) i is transient iff j is transient, (2) i and j have the same period, (3) i is null recurrent iff j is null recurrent.

Proof - (1) if $i \leftrightarrow j$, there are $m, n \geq 0$ such that $\alpha = P^m_{ij} P^n_{ji} > 0$. By the Chapman-Kolmogorov equations (2.2.1) $P^{m+r+n}_{ii} \geq P^m_{ij} P^r_{jj} P^n_{ji}$ for any inteteg $r > 0$. Summing over r, $\Sigma^\infty_{r=0} P^r_{ii} < \infty \Rightarrow \Sigma^\infty_{r=0} P^r_{jj} < \infty$. The argument holds with j and i reversed. By thm 3.1.2, (1) holds. (2) excercise. (3) will be proved below.

Definition 3.2.4 - A set C of states in the state space S is closed if $P_{ij} = 0$ for all $i \in C$ and $j \notin C$. A closed set with one element is called absorbing. Once a Markov chain takes a value in a closed set, it never leaves.

Example 3.2.5 - Consider the chain with $S = \{0, 1, 2\}$. $\{0, 2\}$ forms a closed set.

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1/4 & 1/2 & 1/4 \\ 1 & 0 & 0 \end{pmatrix}$$

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Definition 3.2.5 - A set C of states in the state space S is irreducible if $i \leftrightarrow j$ for all $i, j \in C$. The communication classes of a Markov chain are irreducible.

Because of theorem 3.2.3, Definition 3.2.6 - An irreducible set C is periodic, transient, or null recurrent if all or any of the states in C have these properties.

Definition 3.2.7 - If the entire state space is irreducible, we say the Markov chain is irreducible.

Theorem 3.2.4 - In an irreducible Markov chain, either all states are transient or all states are recurrent.

Theorem 3.2.5 - Decomposition Theorem - The state space S can be partitioned uniquely as $S = T \cup C_1 \cup C_2 \cup \ldots, T = \{\text{transient states}\}, \{C_i\} =$ irreducible, closed sets of recurrent states. Lots of words in notes.

Proof: Let $\{C_j\}$ be the recurrent equivalence classes of intercommunication ($\leftrightarrow$). We only need to show that each C_r is closed. Suppose on the contrary that $i \in C_r, j \notin C_r$, and $P_{ij} > 0, j \not\to i$, so $P(X_n$ never returns to $i) \geq P(X_n \text{reaches } j), P(X_n \neq i \text{ for } n \geq 1 | X_0 = i) \geq P(X_1 = j | X_0 = i) > 0$. This contradicts the assumption that i is recurrent.

Markov chains with finite state spaces are special. For example, it is impossible to stay in transient states for all time.

Theorem 3.2.6 - If the state space is finite, then at least one state is recurrent and all recurrent states are positive.

Proof - assume all states are transient. $1 = \Sigma_{j \in S} P^n_{ij}$ (finite sum). So, $1 = \lim_{n \to \infty} \Sigma_{j \in S} P^n_{ij} = \Sigma_{j \in S} \sim_{n \to \infty} P^n_{ij} = 0$ (by thm 3.1.2(3)). (in general $\lim_{n \to \infty} \Sigma^\infty_{i=0} a_i(n) \neq \Sigma^\infty_{i=0} \lim_{n \to \infty} a_i(n)$). That is a contradiction. The same argument works for the closed set of all null recurrent states (exercise).

Theorem 3.2.7 - Suppose the state space is finite. i is transient $\iff$ there is a state j with $i \to j$ but $j \not\to i$.

Example 3.2.6 - Consider a Markov chain with $S = \{0, 1, 2, 3, 4\}$.

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/2 & 1/8 & 3/8 \end{pmatrix}$$

$C_1 = \{0, 1\}, C_2 = \{2, 3, 4\}$ are both closed. They are both irreducible. They must contain positive recurrent states. $S = C_1 \cup C_2$.

Example 3.2.7 - Consider random walk with (1 0 0 ... & q 0 p 0 ... & 0 q 0 p 0 ... & 0 0 q 0 p 0 ... & ... & ... 0 q 0 p & 0 ... 0 0 1), states 0, ..., N.

Three classes $C_1 = \{0\}, T = \{1, 2, \ldots, N-1\}, C_2 = \{N\}$.$\{1, 2, \ldots, N-1\} \to \{0\}$, but $\{0\} \not\to \{1, 2, \ldots, N-1\}$.$\{1, 2, \ldots, N-1\} \to \{N\}$, but $\{N\} \not\to \{1, 2, \ldots, N-1\}$.$\{0\}$ and $\{N\}$ are absorbing.