Definition 2.5.2 Branching Process

Let $\{Z_{nj}, n \geq 1, j \geq 1\}$ be i.i.d. nonnegative integer valued random variables with p.m.f. $\{p_k\}$. Below, if a random sum has 0 summands, we assign the sum the value 0. The branching process $\{X_n\}$ is defined

$$
(2.5.1) \quad \begin{cases} X_0 = 1 \\ X_1 = Z_{1, X_0} \\ X_2 = Z_{21} + Z_{22} + \dots + Z_{2, X_1} \\ \vdots \\ X_n = Z_{n,1} + \dots + Z_{n, X_{n-1}} \\ \vdots \end{cases}
$$

Note that $Z_{n,j}$ = numbers of members of the n^{th} generation which are offspring of the j^{th} member of the $(n-1)^{st}$ generation

Note if $X_n = 0$, then $X_{n+1} = 0$, so 0 is absorbing.

Theorem 2.5.2

X_n is a Markov chain.

Proof

Exercise

We have characterized a branching process as a random sum. We use some facts about those.

Definition 2.5.3 Random Sums

Let $Y_1, Y_2, \dots$ be a sequence of i.i.d. r.v.

Let N be a discrete, nonnegative integer valued r.v. independent of $\{Y_i\}$. Let N have p.m.f.

$$p_N(n) = P(N=n), \quad n = 0, 1, 2, \dots$$

Set

$$X = \begin{cases} 0, & N = 0, \\ Y_1 + \dots + Y_N, & N > 0. \end{cases}$$

X is a random sum.

Theorem 2.5.3

Assume that $\{Y_i\}$ and N have finite moments,

$$E(Y_i) = \mu \quad , \quad Var(Y_i) = \sigma^2$$

$$E(N) = \nu \, , \quad Var(N) = \tau^2$$

Then,

(2.5.2) $$E(X) = \mu\nu$$

(2.5.3) $$Var(X) = \nu\sigma^2 + \mu^2\tau^2$$

Proof

$$E(X) = \sum_{n=0}^{\infty} E(X \mid N=n) P_N(n)$$

$$= \sum_{n=1}^{\infty} E(Y_1 + \cdots + Y_N \mid N=n) \, P_N(n)$$

$$= \sum_{n=1}^{\infty} \underbrace{E(Y_1 + \cdots + Y_n)}_{n\mu} P_N(n)$$

$$= \mu \sum_{n=1}^{\infty} n P_N(n) = \mu\nu.$$

Exercise: prove (2.5.3)

We apply this to a branching process.

Let I be a random variable with pmf $\{p_k\}$. We define $\mu = E(I)$, $\sigma^2 = Var(I)$.

Theorem 2.5.4

Let X_n be a branching process with pmf $\{p_k\}$ and assume μ, σ^2 are finite. Let $M(n)$, $V(n)$ be the mean and variance of X_n conditioned on $X_0 = 1$. Then

(2.5.4) $\quad M(n) = \mu^n$

(2.5.5) $\quad V(n) = \sigma^2 \mu^{n-1} \times \begin{cases} n, & \mu = 1, \\ \frac{1-\mu^n}{1-\mu}, & \mu \neq 1. \end{cases}$

Proof

Using (2.5.2) in (2.5.1) gives

$$(2.5.6)\quad \begin{cases} M(n+1) = \mu M(n) \\ V(n+1) = \sigma^2 M(n) + \mu^2 V(n) \end{cases}$$

Now $X_0 = 1$ means $M(0) = 1$, $V(0) = 0$. So (2.5.4) follows immediately. Then we note that $V(1) = \sigma^2$, $V(2) = \sigma^2\mu + \sigma^2\mu^2$, and so on.

The mean population increases geometrically when $\mu > 1$, decreases geometrically when $\mu < 1$, and is constant when $\mu = 1$.

The variance increases or decreases geometrically when $\mu > 1$ or $\mu < 1$, and increases linearly when $\mu = 1$.

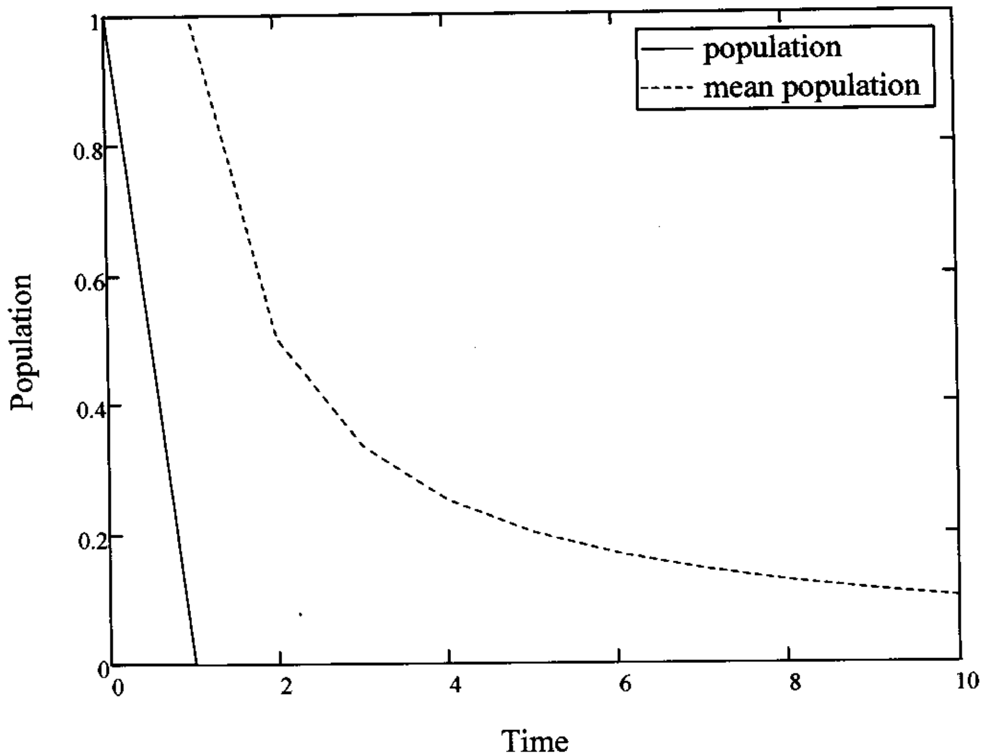
<u>Example 2.5.2</u>

On pages 98, 99 we plot X_n, $M(n)$ for a Poisson distribution for number of offspring

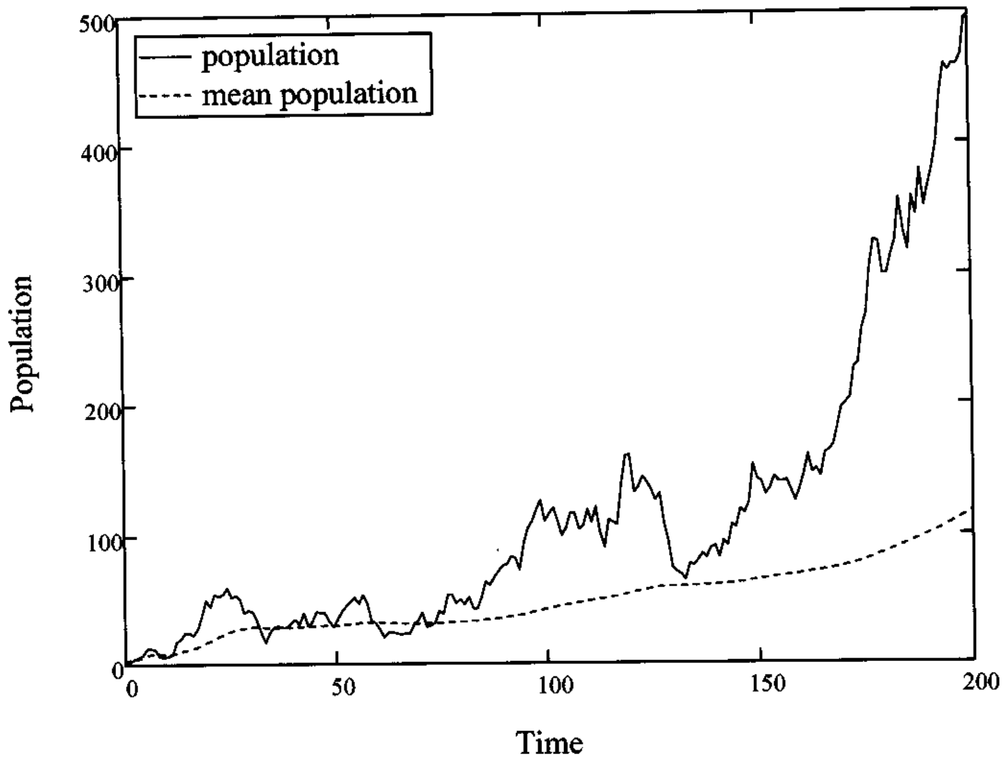
On page 100, we plot X_n, $M(n)$ for a uniform distribution on $\{0, 1, 2\}$ offspring

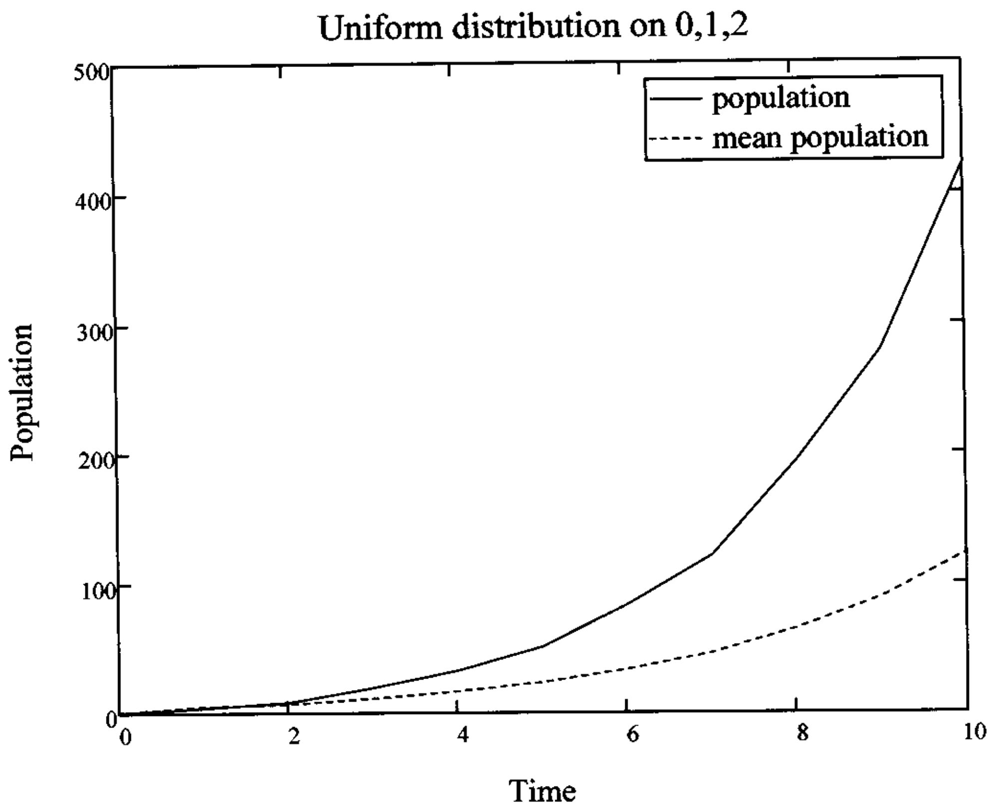
On page 101, we plot X_n, $M(n)$ for a binomial distribution for 9 children.

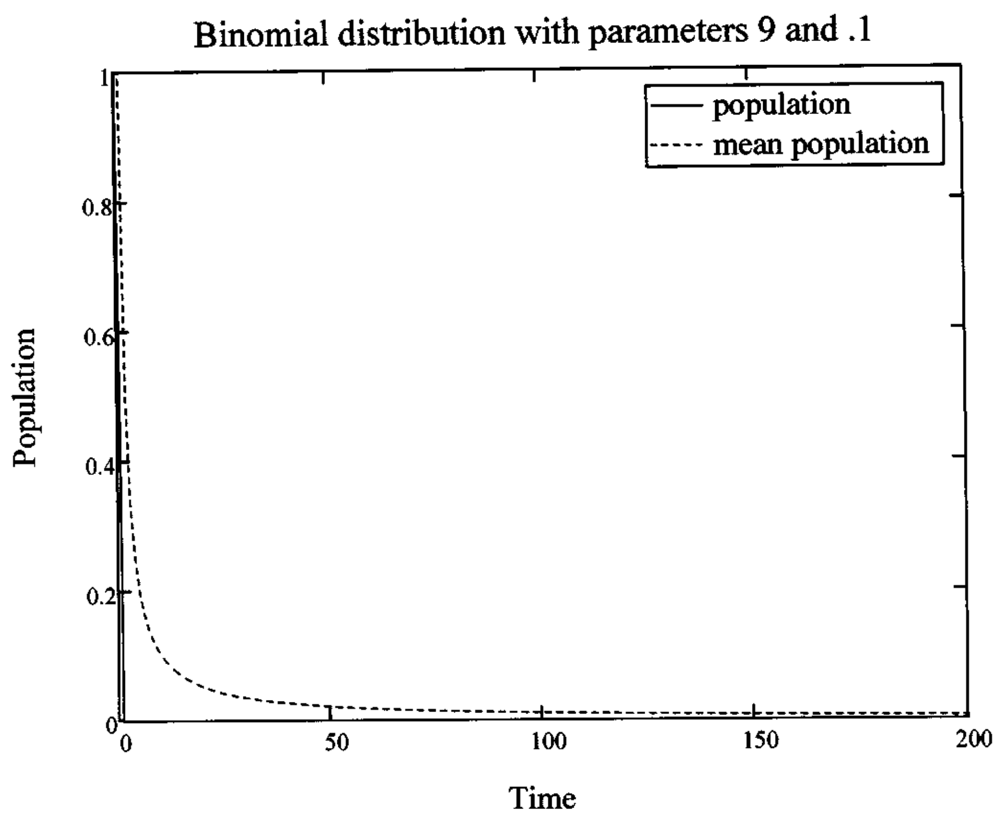
Poisson distribution with parameter .25



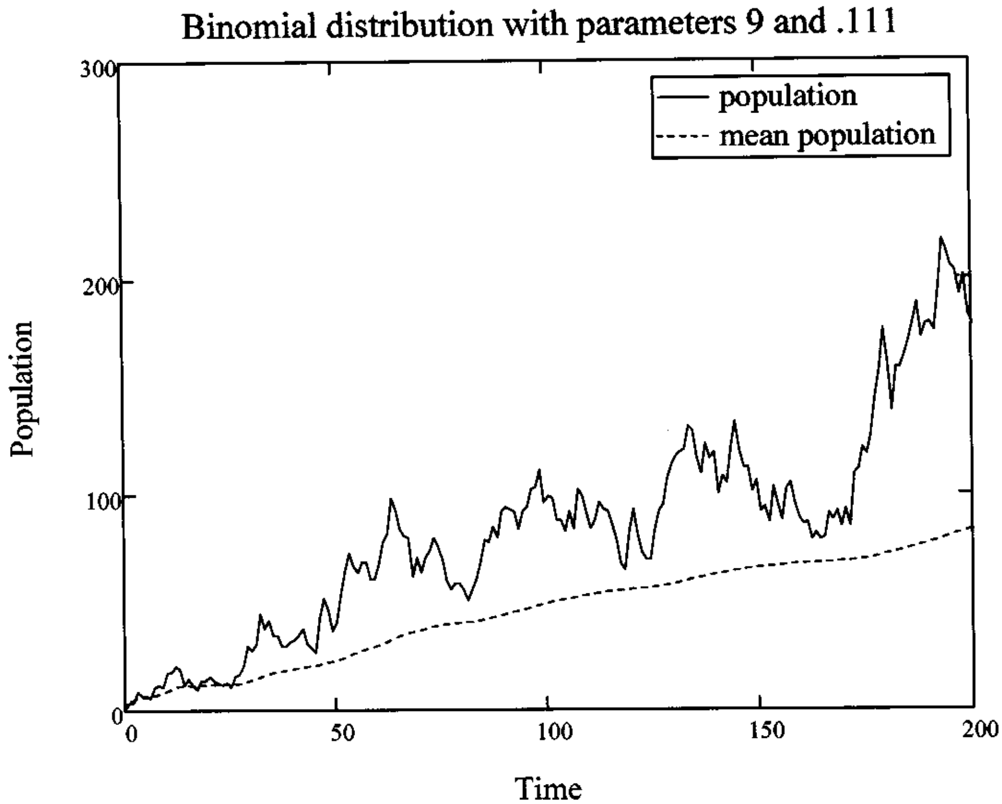
Poisson distribution with parameter 1.01




Uniform distribution on 0,1,2
population
mean population
Population
Time
0
100
200
300
400
500
0
2
4
6
8
10



Binomial distribution with parameters 9 and .1
population
mean population
Population
Time
0
0.2
0.4
0.6
0.8
1
0
50
100
150
200


Binomial distribution with parameters 9 and .111
population
mean population
Population
Time
0
100
200
300
0
50
100
150
200

One of the important dynamical questions is the probability of extinction.

Definition 2.5.4

The random time of extinction N is the first time for which $X_N = 0$.

This is an absorption time since $X_n = 0$ for $n \geq N$.

We let

$$(2.5.7) \quad U_n = P(N \leq n \mid X_0 = 1) = P(X_n = 0 \mid X_0 = 1)$$

be the probability of extinction at or prior to the n^{th} generation, conditioned on $X_0 = 1$.

Theorem 2.5.5

We have

$$(2.5.8) \quad \begin{cases} U_0 = 0 \\ U_1 = P_0 \\ U_n = \sum_{k=0}^{\infty} P_k (U_{n-1})^k, \quad n \geq 2. \end{cases}$$

Proof

The single parent $X_0 = 1$ has $Z_{1, X_0} = k$ offspring.

These in turn generate a population of their own descendants. If the original population dies out in n generations, then each of the k offspring lines of descent must die out in $n-1$ generations, or less.

The k subpopulations generated by the distinct offspring off the original parent are independent of each other and have the same statistical properties as the original generation.

Hence, the probability that any one of the subpopulations dies out in $n-1$ generations or less is U_{n-1}, so the probability that they all die out is $(U_{n-1})^k$. The total law of probability gives (2.5.8).

Example 2.5.3

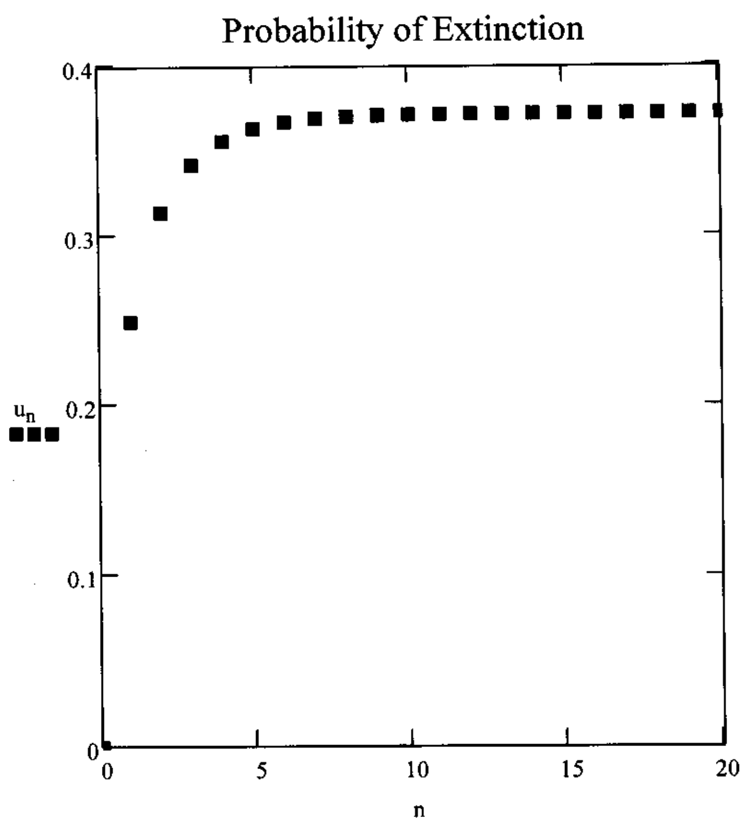
We plot $\{U_n\}$ for a given pmf, see pg. 105

A parent has $\frac{1}{4}$ chance of no offspring

$\frac{1}{8}$ chance of one offspring

$\frac{1}{2}$ chance of two offsprings

$\frac{1}{8}$ chance of three offspring



$u =$

	0
0	0
1	0.25
2	0.314
3	0.343
4	0.357
5	0.364
6	0.368
7	0.37
8	0.371
9	0.372
10	0.372
11	0.372
12	0.372
13	0.372
14	0.372
15	0.372
16	0.372
17	0.372
18	0.372
19	0.372
20	0.372

- Generating functions are very useful for dealing with sums of random variables, recall §1.5.

Recall that Thm 1.5.3 implies that if $Y_1, \ldots, Y_n$ are independent random variables having generating functions $P_{Y_1}, \ldots, P_{Y_n}$ respectively,

then the generating function for $X = Y_1 + \cdots + Y_n$ is

$$(2.5.9) \qquad P_X(s) = P_{Y_1}(s) \cdots P_{Y_n}(s).$$

- From Theorem 1.5.1, if a random variable Y has p.m.f. $\{p_k\}$ and prob. generating function P_Y,

$$(2.5.10) \quad \frac{dP_Y}{ds}(1) = p_1 + 2p_2 + 3p_3 + \cdots = E(Y)$$

and

$$\frac{d^2 P_Y}{ds^2}(1) = 2p_2 + 3\cdot 2 p_3 + 4\cdot 3 p_4 + \cdots = E(Y^2) - E(Y)$$

so

$$(2.5.11) \quad Var(Y) = \left.\frac{d^2 P_Y(s)}{ds^2}\right|_{s=1} + \left.\frac{dP_Y(s)}{ds}\right|_{s=1} - \left(\left.\frac{dP_Y(s)}{ds}\right|_{s=1}\right)^2$$

We next prove a useful theorem, see §9.2 in the text.

Theorem 2.5.6

If $Z_1, Z_2, \ldots$ is a sequence of iid random variables with common generating function P_Z

and if $N \geq 0$ is a random variable independent of the Z_i with probability generating function P_N, then $X = Z_1 + Z_2 + \cdots + Z_N$ has prob. generating function

$$P_X(s) = P_N(P_Z(s)). \tag{2.5.12}$$

Proof

$$P_X(s) = E(s^X) = E(E(s^X | N))$$

(Recall $E(E(X | Y)) = E(X)$)

$$= \sum_n E(s^X | N=n) P(N=n)$$

$$= \sum_n E(s^{Z_1 + \cdots + Z_n}) P(N=n)$$

↓ independence

$$= \sum_n E(s^{Z_1}) \cdots E(s^{Z_n}) P(N=n)$$

$$= \sum_n (P_Z(s))^n P(N=n)$$

$$= P_N(P_Z))$$

Returning to the branching process with population X_n at time n, we assume the offspring distribution $Y \sim \{p_k\}$ has generating function

$$\phi(s) = E(s^{X_1}) = \sum_k p_k s^k$$

We are interested in the generating function ϕ_n of X_n, assuming $X_0 = 1$. Note that ϕ is the generating function for X_1, because $X_0 = 1$.

<u>Theorem 2.5.7</u>

We have

$$(2.5.13) \qquad \phi_{m+n}(s) = \phi_m(\phi_n(s)) = \phi_n(\phi_m(s)) \qquad m, n \geq 0$$

$$(2.5.14) \qquad \phi_n(s) = \phi \circ \phi \circ \cdots \circ \phi(s) = \underbrace{\phi(\phi(\cdots \phi(s))\cdots)}_{n \text{ compositions}}$$

<u>Proof</u>

Each member of the $(m+n)^{th}$ generation has a unique ancestor in the m^{th} generation. So

$$X_{m+n} = Z_1 + \cdots + Z_{X_m}$$

where Z_i is the number of members of the

$(m+n)^{th}$ generation that stem from the i^{th} member of the m^{th} generation.

This is a random sum. The variables are independent and iid with the same distribution as the number X_n of the n^{th} generation offspring of the first individual in the process, by the Markov property.

By Theorem 2.5.6,

$$\phi_{m+n}(s) = \phi_m(\phi_{Z_1}(s))$$

where

$$\phi_{Z_1}(s) = \phi_n(s).$$

Iterating gives (2.5.14).

$\boxed{\#10 \quad 2/21}$

Example 2.5.3

Suppose that $0 \le p \le 1$ and the p.m.f. for the offspring is

$$\{q p^k\}_{k \ge 0}, \quad q = 1 - p.$$

The prob generating function is

$$\phi(s) = q(1 - ps)^{-1}$$

$$\left(\frac{1}{1-a} = 1 + a + a^2 + a^3 + \dots \right)$$