Constrained optimization problems

- So far, we have considered unconstrained optimization problems:

$$\text{minimize } f(\boldsymbol{x})$$

  where $f : \mathbb{R}^n \to \mathbb{R}$. The solution $\boldsymbol{x}$ can be anything in $\mathbb{R}^n$.

- We now turn to problems with *constraints*.

- In constrained problems, the solution must lie inside some prespecified set, called the *constraint set* or *feasible set*.

- General constrained optimization problem:

$$\begin{array}{ll} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \Omega, \end{array}$$

  where $\Omega \subset \mathbb{R}^n$ is the constraint set.

- Points in Ω are called *feasible points*.

- Example: $\Omega = \{\boldsymbol{x} : x_i \geq 0,\ i = 1, \ldots, n\}$.

- The desired solution $\boldsymbol{x}^*$ must lie inside Ω.

Linear programming problems (§15.2)

- To begin, we first consider a special class of constrained optimization problems.

- Consider the case where

$$f(\boldsymbol{x}) = \boldsymbol{c}^T\boldsymbol{x},$$

  where $\boldsymbol{c} \in \mathbb{R}^n$ is a given vector. Note that f is a linear function.

- Example: $f(\boldsymbol{x}) = 3x_1 - 4x_2 + x_3$.
  Here, $\boldsymbol{c}^T = [3, -4, 1]$.

- Next, consider

$$\Omega = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \;\&\; \boldsymbol{x} \geq \boldsymbol{0}\},$$

  where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$.

- The notation $\boldsymbol{x} \geq \boldsymbol{0}$ means that every component of $\boldsymbol{x}$ must be ≥ 0.

- Example: $\Omega = \{\boldsymbol{x} : 4x_1 + x_2 = 5,\ x_1, x_2 \geq 0\}$.

  Here,
$$\boldsymbol{A} = [4 \ \ 1], \qquad \boldsymbol{b} = [5].$$

- We can write the problem as
$$\begin{array}{ll} \text{minimize} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{array}$$

- Name: *Linear programming (LP) problem*, or *linear program*.
- Note that the objective function and constraint set are expressed using *linear* functions.
- The constraint set involves equations and inequalities.

- There are other variations that are possible.
- Example:
$$\begin{array}{ll} \text{maximize} & 3x_1 + 2x_2 \\ \text{subject to} & x_1 - x_2 \leq 6 \\ & 4x_2 \geq 2 \\ & x_1 \leq 0. \end{array}$$

- The above problem also involves only linear functions in the objective and constraints (with equations and inequalities).
- We also refer to problems like the above as linear programming problems.

Example: Production scheduling

Woodworking shop:

	Products		Input
Inputs	Table	Chair	Availabilities
Labor	5	6	30
Materials	3	2	12
Production levels	x_1	x_2	
Unit price	1	5	

- Total revenues: $x_1 + 5x_2$.

- Production constraints:
  - Labor: $5x_1 + 6x_2 \leq 30$.
  - Materials: $3x_1 + 2x_2 \leq 12$.
- Physical constraint: $x_1, x_2 \geq 0$.
- Optimization problem:

$$\begin{array}{ll} \text{maximize} & x_1 + 5x_2 \\ \text{subject to} & 5x_1 + 6x_2 \leq 30 \\ & 3x_1 + 2x_2 \leq 12 \\ & x_1, x_2 \geq 0. \end{array}$$

Example: Optimal diet

Nutrition table:

Vitamin	Food type Milk	Food type Eggs	Daily Requirements
V	2	4	40
W	3	2	50
Intake	x_1	x_2	
Unit cost	3	5/2	

- Total cost: $3x_1 + 5x_2/2$.
- Dietary constraints:
  - Vitamin V: $2x_1 + 4x_2 \geq 40$.
  - Vitamin W: $3x_1 + 2x_2 \geq 50$.
- Physical constraint: $x_1, x_2 \geq 0$.
- Optimization problem:

$$\begin{array}{ll} \text{minimize} & 3x_1 + \frac{5}{2}x_2 \\ \text{subject to} & 2x_1 + 4x_2 \geq 40 \\ & 3x_1 + 2x_2 \geq 50 \\ & x_1, x_2 \geq 0. \end{array}$$

Example: taken from Byte Magazine

Consider the following "real-life" problem, taken from *Byte* magazine (Sept. 1980, p. 242):

> "The Whiz-Golly Computer Board Company makes two kinds of video boards: the Ohwow and the Hohum. Each board is handmade by Jim and then tested by Jack. Each Ohwow board takes Jim two days to complete, while he can make one Hohum board each day. Jack can test one Ohwow board in one day, but he needs two days for each Hohum. Like most basement entrepreneurs, Jim and Jack have many other things to do with their time. Jim will not make boards for more than four days a week; Jack will test them for no more than three days a week. If the profit is two dollars for each Ohwow board and three dollars for each Hohum, how many of each should they make per week to obtain the greatest profit?"

- Let x_1 be the number of Ohwow boards and x_2 the number of Hohum boards made per week.
- Constraint for Jim:
$$2x_1 + x_2 \leq 4.$$
Constraint for Jack:
$$x_1 + 2x_2 \leq 3.$$
- Thus, the linear program is
$$\begin{array}{rl} \text{maximize} & 2x_1 + 3x_2 \\ \text{subject to} & 2x_1 + x_2 \leq 4 \\ & x_1 + 2x_2 \leq 3 \\ & x_1, x_2 \geq 0. \end{array}$$

Other typical examples:

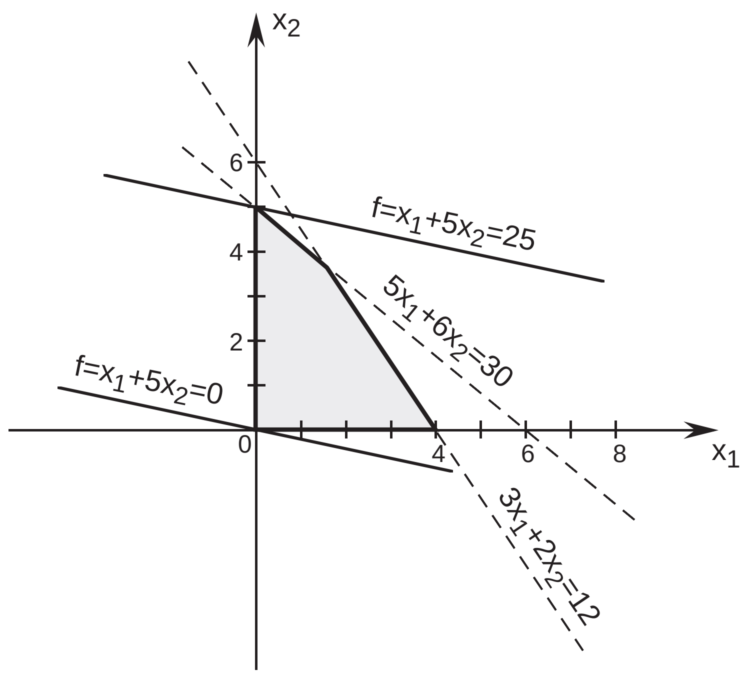
- Linear costs: compound interest, physical laws, etc.
- Linear constraints: physical laws (e.g., Kirchoff's law, Hooke's law, etc.)
- "Linearization" of given optimization problem.
- See examples in §15.2.

Simple solution of LP problem (§15.3)

- Consider the production scheduling example given before:

$$\begin{array}{ll} \text{maximize} & x_1 + 5x_2 \\ \text{subject to} & 5x_1 + 6x_2 \leq 30 \\ & 3x_1 + 2x_2 \leq 12 \\ & x_1, x_2 \geq 0. \end{array}$$

- We draw the constraint set, which is a polyhedron.
- We then draw level sets of the objective function on the same diagram.



- Note that the solution lies on the boundary of the constraint set.
- In fact, unless the level sets happen to be parallel to one of the edges of the constraint set, the solution will lie on a "corner point" (or vertex). Even if the level sets happen to be parallel to one of the edges of the constraint set, a corner point will be *an* optimal solution.
- It turns out that the/a solution of an LP problem (if it exists) *always* lies on a vertex of the constraint set.
- Therefore, instead of looking for candidate solutions everywhere in the constraint set, we need only focus on the vertices.

Standard form LP problems (§15.5)

- For the purpose of analyzing and solving LP problems, we will consider only problems in a particular *standard form*.
- LP problem in *standard form*:

$$\begin{aligned} \text{minimize } \boldsymbol{c}^T\boldsymbol{x} & \\ \text{subject to } \boldsymbol{A}\boldsymbol{x} &= \boldsymbol{b} \\ \boldsymbol{x} &\geq \boldsymbol{0}, \end{aligned}$$

  where $\boldsymbol{A}$ is $m \times n$, $m < n$, rank $\boldsymbol{A} = m$, $\boldsymbol{b} \geq \boldsymbol{0}$.

Examples:

- In standard form:

$$\begin{array}{ll} \text{minimize} & 3x_1 + 5x_2 - x_3 \\ \text{subject to} & x_1 + 2x_2 + 4x_3 = 4 \\ & -5x_1 - 3x_2 + x_3 = 15 \\ & x_1, x_2, x_3 \geq 0. \end{array}$$

- *Not* in standard form:

$$\begin{array}{ll} \text{maximize} & 3x_1 + 5x_2 - x_3 \\ \text{subject to} & x_1 + 2x_2 + 4x_3 \leq -4 \\ & -5x_1 - 3x_2 + x_3 \geq 15 \\ & x_2 \leq 0. \end{array}$$

- All our analyses and algorithms will apply only to standard form LP problems.
- What about other variations of LP problems?
- Any LP problem can be converted into an equivalent standard form LP problem.
- Equivalent $\equiv$ solution to one problem gives us the solution to the other.
- How to convert from given LP problem to an equivalent problem in standard form?
- If problem is a maximization, simply multiply the objective function by -1 to get minimization.
- If $\boldsymbol{A}$ not of full rank, can remove one or more rows.
- If a component of $\boldsymbol{b}$ is negative, say the ith component, multiply the ith constraint by -1 to obtain a positive right-hand side.
- What about inequality constraints?

Converting to standard form: Slack variables

- Suppose we have a constraint of the form:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b.$$

- We can convert the above inequality constraint into the standard equality constraint by introducing a *slack variable*.
- Specifically, the above constraint is equivalent to:

$$\begin{aligned} a_1x_1 + a_2x_2 + \cdots + a_nx_n + x_{n+1} &= b \\ x_{n+1} &\geq 0, \end{aligned}$$

where x_{n+1} is the slack variable.

Converting to standard form: Surplus variables

- Suppose we have a constraint of the form:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq b.$$

- We can convert the above inequality constraint into the standard equality constraint by introducing a *surplus variable*.
- Specifically, the above constraint is equivalent to:

$$\begin{aligned} a_1x_1 + a_2x_2 + \cdots + a_nx_n - x_{n+1} &= b \\ x_{n+1} &\geq 0, \end{aligned}$$

where x_{n+1} is the surplus variable.

Converting to standard form: Nonpositive variable

- Suppose one of the variables (say, x_1) has the constraint

$$x_1 \leq 0.$$

- We can convert the variable into the usual nonnegative variable by changing every occurrence of x_1 by its negative $x_1' = -x_1$.

Example:

- Consider the constraint

$$\begin{aligned} a_1x_1 + a_2x_2 + \cdots + a_nx_n &= b \\ x_1 &\leq 0. \end{aligned}$$

- By introducing $x_1' = -x_1$, we obtain the following equivalent constraint:

$$\begin{aligned} -a_1x_1' + a_2x_2 + \cdots + a_nx_n &= b \\ x_1' &\geq 0. \end{aligned}$$

Converting to standard form: Free variables

- Suppose one of the variables (say, x_1) does not have a nonnegativity constraint (i.e., there is no $x_1 \geq 0$ constraint).
- We can introduce variables $u_1 \geq 0$ and $v_1 \geq 0$ and replace x_1 by $u_1 - v_1$.
- Example: The constraint

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

is equivalent to

$$\begin{aligned} a_1(u_1 - v_1) + a_2x_2 + \cdots + a_nx_n &= b \\ u_1, v_1 &\geq 0. \end{aligned}$$

Converting to standard form: An example

- Consider the LP problem (not in standard form)

$$\begin{array}{ll} \text{maximize} & 3x_1 + 5x_2 - x_3 \\ \text{subject to} & x_1 + 2x_2 + 4x_3 \leq -4 \\ & -5x_1 - 3x_2 + x_3 \geq 15 \\ & x_2 \leq 0, x_3 \geq 0. \end{array}$$

- We first convert the problem into a minimization problem by multiplying the objective function by -1.
- Next, we introduce a slack variable $x_4 \geq 0$ to convert the first inequality constraint into

$$x_1 + 2x_2 + 4x_3 + x_4 = -4.$$

- We then multiply the equation by -1 to make the right hand side positive:

$$-x_1 - 2x_2 - 4x_3 - x_4 = 4.$$

- Next, we introduce a surplus variable $x_5 \geq 0$ to convert the second inequality constraint into

$$-5x_1 - 3x_2 + x_3 - x_5 = 15.$$

- Next, we substitute $x_1 = u_1 - v_1$, with $u_1, v_1 \geq 0$.
- Finally, we replace $x_2 = -x_2'$ in the equations.

- Resulting equivalent LP problem in standard form:

$$\begin{array}{rl} \text{minimize} & -3(u_1 - v_1) + 5x_2' + x_3 \\ \text{subject to} & -(u_1 - v_1) + 2x_2' - 4x_3 - x_4 = 4 \\ & -5(u_1 - v_1) + 3x_2' + x_3 - x_5 = 15 \\ & u_1, v_1, x_2', x_3, x_4, x_5 \geq 0. \end{array}$$

Basic solutions (§15.6)

- Consider LP problem in standard form:

$$\begin{array}{rl} \text{minimize} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}, \end{array}$$

  where $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^m$, $m < n$, rank $\boldsymbol{A} = m$, and $\boldsymbol{b} \geq \boldsymbol{0}$.

- The feasible points are those $\boldsymbol{x}$ that satisfy $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq \boldsymbol{0}$ (i.e., have nonnegative components).

- Since $m < n$, there are infinitely many points $\boldsymbol{x}$ satisfying $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$; they form a "linear variety" (plane) in $\mathbb{R}^n$.

- How to find points $\boldsymbol{x}$ satisfying $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$?

- Simple way of solving linear equations: use elementary row operations.

- Recall three kinds of elementary row operations that we can perform on a given matrix:

  1. Interchanging two rows of the matrix;
  2. Multiplying one row of the matrix by a (nonzero) constant;
  3. Adding a constant times one row to another row.

- Performing an elementary row operation to a matrix is equivalent to multiplying the matrix by some nonsingular matrix (called an *elementary matrix*; see §16.1).

- We can find solutions to $\boldsymbol{Ax} = \boldsymbol{b}$ using elementary row operations.
- First, we form the "augmented matrix" $[\boldsymbol{A}, \boldsymbol{b}]$.
- Assume that the first m columns of $\boldsymbol{A}$ are linearly independent.
- Using elementary row operations, we can reduce the augmented matrix $[\boldsymbol{A}, \boldsymbol{b}]$ into the form

$$\begin{bmatrix} 1 & & & 0 & \cdots & y_1 \\ & 1 & & & \cdots & y_2 \\ & & \ddots & & & \vdots \\ 0 & & & 1 & \cdots & y_m \end{bmatrix}.$$

Name: *canonical augmented matrix*.

- One specific solution to $\boldsymbol{Ax} = \boldsymbol{b}$ is

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{y}_0 \\ \boldsymbol{0} \end{bmatrix},$$

where $\boldsymbol{y}_0 = [y_1, \ldots, y_m]^T$ is the last column of the above canonical augmented matrix.

- The previous specific solution $\boldsymbol{x} = [\boldsymbol{y}_0^T, \boldsymbol{0}^T]^T$ is called a *basic solution* to the equation $\boldsymbol{Ax} = \boldsymbol{b}$.
- In obtaining the above basic solution, we assumed that the first m columns of $\boldsymbol{A}$ are linearly independent.
- Notation: let $\boldsymbol{B}$ be the $m \times m$ matrix corresponding to the first m columns of $\boldsymbol{A}$. We assumed that $\boldsymbol{B}$ is nonsingular.
- We call $\boldsymbol{B}$ the *basis* matrix associated with the above basic solution. The columns of $\boldsymbol{B}$ are called *basic columns*.
- Note that doing elementary row operations to a matrix is equivalent to multiplying the matrix by an invertible matrix.
- The previous elementary row operations transformed the augmented matrix $[\boldsymbol{A}, \boldsymbol{b}] = [\boldsymbol{B}, \boldsymbol{D}, \boldsymbol{b}]$ into canonical form $[\boldsymbol{I}, \boldsymbol{Y}, \boldsymbol{y}_0]$.
- Therefore, $\boldsymbol{y}_0 = \boldsymbol{B}^{-1}\boldsymbol{b}$.
- Write $\boldsymbol{B} = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m]$, $\boldsymbol{y}_0 = [y_1, \ldots, y_m]^T$. Note that

$$y_1\boldsymbol{a}_1 + \cdots + y_m\boldsymbol{a}_m = \boldsymbol{b}.$$

- Hence, the components of $\boldsymbol{y}_0$ are simply the coordinates of $\boldsymbol{b}$ with respect to the basis $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$.

- We could have chosen some other set of m linearly independent (basic) columns of $\boldsymbol{A}$ (not necessarily the first m columns).
- We can similarly do elementary row operations to reduce it into a form where the basic columns have all 0s except one 1.
- These also give rise to basic solutions, except the associated basis is different.
- The associated transformed matrix (after doing elementary row operations) is called the canonical augmented matrix associated with the given basis.

Example (15.9):

- Consider the equation $\boldsymbol{Ax} = \boldsymbol{b}$ with
$$\boldsymbol{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4] = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 1 & -2 & -1 & 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 8 \\ 2 \end{bmatrix},$$
where $\boldsymbol{a}_i$ denotes the ith column of the matrix $\boldsymbol{A}$.
- For the basis $\boldsymbol{B} = [\boldsymbol{a}_1, \boldsymbol{a}_2]$, the basic solution is $\boldsymbol{x} = [6, 2, 0, 0]^T$.
- For the basis $\boldsymbol{B} = [\boldsymbol{a}_2, \boldsymbol{a}_3]$, the basic solution is $\boldsymbol{x} = [0, 2, -6, 0]^T$.
- For the basis $\boldsymbol{B} = [\boldsymbol{a}_3, \boldsymbol{a}_4]$, the basic solution is $\boldsymbol{x} = [0, 0, 0, 2]^T$.

- For the basis $\boldsymbol{B} = [\boldsymbol{a}_1, \boldsymbol{a}_4]$, the basic solution is $\boldsymbol{x} = [0, 0, 0, 2]^T$.
- For the basis $\boldsymbol{B} = [\boldsymbol{a}_2, \boldsymbol{a}_4]$, the basic solution is $\boldsymbol{x} = [0, 0, 0, 2]^T$.
- Note that $\boldsymbol{x} = [3, 1, 0, 1]^T$ is a solution, but it is not basic.
- A basis is an *ordered* list of vectors; $[\boldsymbol{a}_1, \boldsymbol{a}_2]$ and $[\boldsymbol{a}_2, \boldsymbol{a}_1]$ are different bases: they have the same basic solution but different canonical augmented matrices.

- For notational convenience, we usually assume that the basis is the first m columns (for otherwise, we just imagine interchanging the columns appropriately).
- Consider the basic solution
$$\boldsymbol{x} = [b_1', b_2', \ldots, b_m', 0, \ldots, 0]^T.$$
- The variables $x_1, \ldots, x_m$ are called the *basic variables*, while $x_{m+1}, \ldots, x_n$ are called the *nonbasic variables*.
- Note that the value of nonbasic variables is always 0.

Feasible solutions

- Consider the constraints of the LP problem:
  $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$.
- Recall definition of basic solution.
- If $\boldsymbol{x}$ is a basic solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ and it also satisfies $\boldsymbol{x} \geq \boldsymbol{0}$, we call it a *basic feasible solution* (BFS).
- Note that a basic solution $\boldsymbol{x}$ is feasible iff every basic variable is ≥ 0.
- If at least one of the basic variables is $= 0$, we say that the BFS is *degenerate*.
- We henceforth assume nondegeneracy (i.e., all the BFSs we consider are not degenerate). This makes the derivations simpler. The degenerate case can be handled also.

Example (15.9):

- Consider the constraint $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$, with

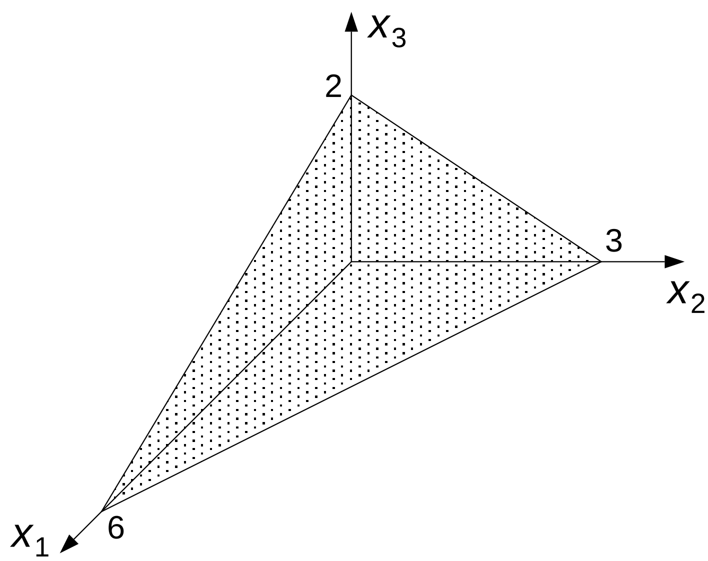
$$\boldsymbol{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4] = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 1 & -2 & -1 & 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}.$$

- $[6, 2, 0, 0]^T$ is a BFS.
- $[0, 0, 0, 2]^T$ is a degenerate BFS.
- $[0, 2, -6, 0]^T$ is a basic solution, but is not feasible.
- $[3, 1, 0, 1]^T$ is a feasible solution, but is not basic.

Geometric view

- Geometrically, a BFS corresponds to a "corner" point (vertex) of the constraint set.
- Example: Consider the constraint $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$, where

$$\boldsymbol{A} = [\,1 \quad 2 \quad 3\,], \qquad b = [6].$$



- The BFSs are: $[6, 0, 0]^T$, $[0, 3, 0]^T$, and $[0, 0, 2]^T$.
- Note that the BFSs are just the vertices of the constraint set.
- In general, we can think (geometrically) of a BFS as a vertex of the constraint set.

- Consider LP problem in standard form:

$$\begin{aligned} \text{minimize } \boldsymbol{c}^T\boldsymbol{x} & \\ \text{subject to } \boldsymbol{A}\boldsymbol{x} &= \boldsymbol{b} \\ \boldsymbol{x} &\geq \boldsymbol{0}. \end{aligned}$$

- A feasible point that minimizes the objective function is called an *optimal* feasible solution.
- A BFS that is also optimal is called an *optimal basic feasible solution*.

Fundamental Theorem of LP (Theorem 15.1)

- Consider an LP problem in standard form.
  1. If there exists a feasible solution, then there exists a BFS;
  2. If there exists an optimal feasible solution, then there exists an optimal BFS.

Or, more informally, ...

- If the feasible set is nonempty, then it has a vertex.
- If the problem has a minimizer (optimal solution), then one of the vertices is a minimizer.

Consequences of FTLP

- Suppose we are given an LP problem. Assume that a solution exists.
- To find the solution, it suffices to look among the set of BFSs.
- There are only a finite number of BFSs; in fact, at most

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}.$$

- Therefore, the FTLP allows us to transform the original problem (infinite number of feasible points) to a problem over a finite number of points.

- The total number of BFSs may of course be very large.
- For example, if $m = 5$ and $n = 50$, $\binom{n}{m} > 2 \times 10^6$.
- The brute force approach of exhaustively comparing all the BFSs is impractical.
- Therefore, we need a more computationally efficient method to find the minimizer among the possible BFSs.
- Simplex method: an organized way of going from one BFS to another to search for the global minimizer.

