Notes - 25 Mar

Decomposition Theorem - S = state space. Π is a stationary distribution. If (1) $\Pi_j \geq 0, \Sigma_{j \in S} \Pi_j = 1$, (!!!) (2) $\Pi = \Pi P, \Pi$ is a LEFT eigenvector of the prob transition matrix. This implies $\Pi = \Pi P^n, n \geq 1$. Does Π exist? (Thm 3.3.2:) Finite dimension state space $\Rightarrow$ theory of eigenvectors.

Theorem 3.3.3 (no assumption of finite state) - An irreducible chain has a stationary distribution $\Pi \iff$ all the states are positive recurrent. In this case, Π is unique, $\Pi_i = \frac{1}{\mu_i}, i \in S$. Let $\rho_i(k)$ = mean number of visits to state i between successive visits to state k.

Theorem (3.3.3) - $\mu_k = \Sigma_{i \in S} \rho_i(k)$.

Theorem - If $\rho(k) = (\rho_i(k))$, if the communication class of k is irreducible and recurrent, then $\rho_i(k) < \infty$ for all i and $\rho(k) = \rho(k)P$.

Theorem - Every positive recurrent irreducible Markov chain has a stationary distribution.

– Start of the real lecture –

Theorem 3.3.7 - If the chain is irreducible and recurrent there is a solution of $x = xP$ with strictly positive entries that is unique up to a multiplicative factor. The chain is positive if $\Sigma X_i < \infty$ and null if $\Sigma X_i = \infty$. Now we are ready to prove theorem 3.3.3.

Proof of Theorem 3.3.3 - Suppose Π is a stationary distribution. If all the states are transient then $P^n_{ij} \to 0$ as $n \to \infty$ by Thm 3.1.2. By (3.3.1), (3.3.4) $\Pi_j = \Sigma_i \Pi_i P^n_{ij} \to 0, n \to \infty$ for all j. This contradicts defn 3.3.2(1), hence all states are recurrent. (More argument in notes.) We next show that all states are positive and $\Pi_i = \mu_i^{-1}$ for all i. Suppose X_0 has distribution Π. Exercise: $\Pi_j \mu_j = \Sigma_{n=1}^{\infty} P(T_j \geq n | X_0 = j) P(X_0 = j) = \Sigma_{n=1}^{\infty} P(T_j \geq n, X_0 = j)$. $P(T_j \geq 1, X_0 = j) = P(X_0 = j), n \geq 2, P(T_j \geq n | X_0 = j) = P(X_0 = j, X_m \neq j, 1 \leq m \leq n-1) = P(X_m \neq j, 1 \leq m \leq n-1) - P(X_m \neq j, 0 \leq m \leq n-1) = P(X_m \neq j, 0 \leq m \leq n-2) - P(X_m \neq j, 0 \leq m \leq n-1) = a_{n-2} - a_{n-1}, a_n = P(X_m \neq j, 0 \leq m \leq n)$. Sum over $n, \Pi_j \mu_j = P(X_0 = j) + P(X_0 \neq j) - \lim_{n \to \infty} a_n = 1 - \lim_{n \to \infty} a_n$. $a_n \to P(X_m \neq j$ for all $m) = 0$ since j is recurrent. We have shown that $\mu_j \Pi_j = 1$ so $\mu_j = \Pi_j^{-1} < \infty$ if $\Pi_j < 0$. To show $\Pi_j > 0$ for all j, assume some $\Pi_j = 0$. $0 = \Pi_j = \Sigma_{j(\mathrm{mightbeiinsteadofj})} \Pi_i P^n_{ij} \geq \Pi_i P^n_{ij}$ for all i, n. This means $\Pi_i = 0$ when $j \to i$. The chain is irreducible so $\Pi_i = 0$ for all i. This contradicts $\Sigma_i \Pi_i = 1$. Hence $\mu_j < \infty$ for all j, and all the states are positive. The other direction follows from the intermediate steps.

Proof of Thm 3.2.3 (3) is given on page 163.

Example 3.3.5 - OFF/ON system: ex 3.3.2 - P = (1/2 1/2 & 1/4 3/4). We can compute $\Pi = (1/3, 2/3) \Rightarrow \mu_1 = 3, \mu_2 = 3/2$. (What is new here is that we can now compute the mean recurrence times, μ_i.)

Example 3.3.6 - Consider Gambler's Ruin in ex 3.3.4 where $P < 1/2$. P = (1-p p 0 ... & 1-p 0 p 0 ... & 0 1-p 0 p 0 ... & ...). Using the formula for Π_n computed there, $\mu_n = \Pi_n^{-1} = \frac{1-p}{1-2p}(\frac{1-p}{p})^n, n \geq 0, p = 1/4 \Rightarrow \mu_n = \frac{3}{2} 3^n, n \geq 0$.

Theorem 3.3.3 can be used to determine if an irreducible chain is positive recurrent.

Theorem 3.3.8 - Let $s \in S$ be a state of an irreducible chain. The chain is transient iff there is a nonzero solution $\{Y_i, i \in S\}$ of the equations (3.3.5) $Y_i = \Sigma_{j \in S, j \neq s} P_{ij} Y_j, i \neq s$, with $|Y_j| \leq 1$ for all j.