ECE 514, Fall 2008

Homework Problems 6

Solutions (version: November 18, 2008, 14:30)

8.4 The (i,i)th entry of AB is given by

$$(AB)_{ii} = \sum_{k=1}^{n} A_{ik} B_{ki}.$$

Similarly, the (i,i)th entry of BA is given by

$$(BA)_{ii} = \sum_{k=1}^{n} B_{ik} A_{ki}.$$

Hence,

$$\begin{aligned}
\mathrm{tr}(AB) &= \sum_{i=1}^{r} (AB)_{ii} \\
&= \sum_{i=1}^{r} \sum_{k=1}^{n} A_{ik} B_{ki} \\
&= \sum_{k=1}^{n} \sum_{i=1}^{r} B_{ki} A_{ik} \\
&= \sum_{k=1}^{n} (BA)_{kk} \\
&= \mathrm{tr}(BA).
\end{aligned}$$

8.8 If $X = [X_{ij}]$, then both sides of the required equation are equal to $\mathsf{E}[X_{11}] + \cdots + \mathsf{E}[X_{nn}]$, where we have used the linearity of expectations and the fact that the expectation of a matrix corresponds to entry-wise expectation.

8.9 We have

$$\begin{aligned}
\mathsf{E}[\|X - \mathsf{E}[X]\|^2] &= \mathsf{E}[(X - \mathsf{E}[X])'(X - \mathsf{E}[X])] \quad \text{which is a scalar} \\
&= \mathrm{tr}(\mathsf{E}[(X - \mathsf{E}[X])'(X - \mathsf{E}[X])]) \\
&= \mathsf{E}[\mathrm{tr}((X - \mathsf{E}[X])'(X - \mathsf{E}[X]))] \quad \text{by part (a)} \\
&= \mathsf{E}[\mathrm{tr}((X - \mathsf{E}[X])(X - \mathsf{E}[X])')] \quad \text{by part (b)} \\
&= \mathrm{tr}(\mathsf{E}[(X - \mathsf{E}[X])(X - \mathsf{E}[X])']) \quad \text{by part (a)} \\
&= \mathrm{tr}(C) \\
&= \sum_{i=1}^{n} \mathsf{var}(X_i).
\end{aligned}$$

Alternatively,

$$\begin{aligned}\mathsf{E}[\|X-\mathsf{E}[X]\|] &= \mathsf{E}\left[\sum_{i=1}^{n}(X_i-\mathsf{E}[X_i])^2\right] \\ &= \sum_{i=1}^{n}\mathsf{E}\left[(X_i-\mathsf{E}[X_i])^2\right] \\ &= \sum_{i=1}^{n}\mathsf{var}(X_i) \\ &= \mathrm{tr}(C).\end{aligned}$$

8.10 We have

$$\begin{aligned}\mathsf{E}[Z] &= \int_1^2 z\frac{3}{7}z^2\,dz = \frac{45}{28} \\ \mathsf{E}[Y] &= \mathsf{E}[\mathsf{E}[Y|Z]] = \mathsf{E}[1/Z] = \int_1^2 \frac{1}{z}\frac{3}{7}z^2\,dz = \frac{9}{14} \\ \mathsf{E}[X] &= E[ZU+Y] = E[Z]E[U]+E[Y] = \frac{9}{14}.\end{aligned}$$

Hence, $\mathsf{E}[[X,Y,Z]'] = [9/14, 9/14, 45/28]'$.

Next,

$$\begin{aligned}\mathsf{E}[YZ] &= \mathsf{E}[Z\mathsf{E}[Y|Z]] = 1 \\ \mathsf{E}[Z^2] &= \frac{93}{35} \\ \mathsf{E}[XZ] &= \mathsf{E}[(ZU+Y)Z] = \mathsf{E}[Z^2]\mathsf{E}[U]+\mathsf{E}[YZ] = 1 \\ \mathsf{E}[Y^2] &= \mathsf{E}[\mathsf{E}[Y^2|Z]] = \mathsf{E}[2/Z^2] = \frac{6}{7} \\ \mathsf{E}[XY] &= \mathsf{E}[ZY]\mathsf{E}[U]+E[Y^2] = \frac{6}{7} \\ \mathsf{E}[X^2] &= \mathsf{E}[(ZU+Y)^2] = \mathsf{E}[Z^2]\mathsf{E}[U^2]+2\mathsf{E}[U]\mathsf{E}[ZY]+\mathsf{E}[Y^2] = \frac{123}{35}.\end{aligned}$$

Hence, the covariance matrix of $[X,Y,Z]'$ is:

$$\begin{aligned}&\begin{bmatrix}\mathsf{E}[X^2]-(\mathsf{E}[X])^2 & \mathsf{E}[XY]-\mathsf{E}[X]\mathsf{E}[Y] & \mathsf{E}[XZ]-\mathsf{E}[X]\mathsf{E}[Z] \\ \mathsf{E}[YX]-\mathsf{E}[Y]\mathsf{E}[X] & \mathsf{E}[Y^2]-(\mathsf{E}[Y])^2 & \mathsf{E}[YZ]-\mathsf{E}[Y]\mathsf{E}[Z] \\ \mathsf{E}[ZX]-\mathsf{E}[Z]\mathsf{E}[X] & \mathsf{E}[ZY]-\mathsf{E}[Z]\mathsf{E}[Y] & \mathsf{E}[Z^2]-(\mathsf{E}[Z])^2\end{bmatrix} \\ &\qquad = \begin{bmatrix}3.10 & 0.444 & -0.0332 \\ 0.444 & 0.444 & -0.0332 \\ -0.0332 & -0.0332 & 0.0742\end{bmatrix}.\end{aligned}$$

8.30 First, note that $m_Y = E[Y] = Gm_X$. Next,

$$\begin{aligned} C_{XY} &= E[(X - m_X)(Y - m_Y)'] \\ &= E[(X - m_X)(GX + W - Gm_X)'] \\ &= E[(X - m_X)(X - m_X)'G' + (X - m_X)W'] \\ &= E[(X - m_X)(X - m_X)']G' + E[(X - m_X)W'] \\ &= C_X G' + C_{XW} \\ &= C_X G'. \end{aligned}$$

Also,

$$\begin{aligned} C_Y &= E[(Y - m_Y)(Y - m_Y)'] \\ &= E[(GX + W - Gm_X)(GX + GW - Gm_X)'] \\ &= E[(G(X - m_X) + W)(G(X - m_X) + W)'] \\ &= GE[(X - m_X)(X - m_X)']G' + E[W(X - m_X)']G' \\ &\quad + GE[(X - m_X)W'] + E[WW'] \\ &= GC_X G' + C'_{XW} G' + GC_{XW} + C_W \\ &= GC_X G' + C_W. \end{aligned}$$

Since we assume that C_Y is invertible (as requested in the question), we can write

$$A = C_{XY} C_Y^{-1} = C_X G'(GC_X G' + C_W)^{-1}.$$

The MMSE estimate of X based on Y is therefore

$$\hat{X} = A(Y - m_Y) + m_X = C_X G'(GC_X G' + C_W)^{-1}(Y - Gm_X) + m_X.$$

Note that $GC_X G'$ and C_W are both positive semidefinite. Hence, C_Y is positive definite (and hence invertible) if either of them are, as written in the remark at the end of the question.

9.1 Suppose $m = 0$ and

$$C = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_1\sigma_2\rho & \sigma_2^2 \end{bmatrix}.$$

In this case, we can see that the variable x in the PDF is in $\mathbb{R}^2$ (i.e., it has two components). We have $\det(C) = \sigma_1^2\sigma_2^2(1 - \rho^2)$ and $\sqrt{\det(C)} = \sigma_1\sigma_2\sqrt{1 - \rho^2}$. Hence,

$$C^{-1} = \frac{1}{\sigma_1^2\sigma_2^2(1 - \rho^2)} \begin{bmatrix} \sigma_2^2 & -\sigma_1\sigma_2\rho \\ -\sigma_1\sigma_2\rho & \sigma_1^2 \end{bmatrix}.$$

From this we obtain

$$x'C^{-1}x = [x_1\ x_2]C^{-1}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{1 - \rho^2}\left(\left(\frac{x_1}{\sigma_1}\right)^2 - 2\rho\frac{x_1 x_2}{\sigma_1\sigma_2} + \left(\frac{x_2}{\sigma_2}\right)^2\right),$$

and the result follows.

9.8 We start with $\mathsf{E}[Y_j]$:

$$\begin{aligned}\mathsf{E}[Y_j] &= \mathsf{E}[X_j] - \mathsf{E}[\overline{X}] \\ &= m - \mathsf{E}\left[\frac{1}{n}\sum_{n=1}^{n} X_i\right] \\ &= m - \frac{1}{n}\sum_{n=1}^{n}\mathsf{E}[X_i] \\ &= m - \frac{1}{n}(nm) \\ &= 0.\end{aligned}$$

Now,

$$\mathsf{E}[\overline{X}Y_j] = \mathsf{E}[\overline{X}(X_j - \overline{X})] = \mathsf{E}[\overline{X}X_j] - \mathsf{E}[\overline{X}^2].$$

We have

$$\begin{aligned}\mathsf{E}[\overline{X}X_j] &= \mathsf{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) X_j\right] \\ &= \frac{1}{n}\sum_{i=1}^{n}\mathsf{E}[X_iX_j] \\ &= \frac{1}{n}\left(\mathsf{E}[X_j^2] + \sum_{i\neq j}\mathsf{E}[X_i]\mathsf{E}[X_j]\right) \\ &= \frac{1}{n}\big((\sigma^2 + m^2) + (n-1)m^2\big) \\ &= \frac{\sigma^2}{n} + m^2.\end{aligned}$$

Also,

$$\begin{aligned}\mathsf{E}[\overline{X}^2] &= \mathsf{E}\left[\overline{X}\frac{1}{n}\sum_{j=1}^{n} X_j\right] \\ &= \frac{1}{n}\sum_{j=1}^{n}\mathsf{E}[\overline{X}X_j] \\ &= \frac{1}{n}\sum_{j=1}^{n}\left(\frac{\sigma^2}{n} + m^2\right) \\ &= \frac{\sigma^2}{n} + m^2.\end{aligned}$$

The result now follows.

9.12 Now, $\varphi_X(\nu) = e^{j\nu' m - \nu' C\nu/2}$. We have

$$\nu' m = [\alpha' \ \beta'] \begin{bmatrix} m_U \\ m_V \end{bmatrix} = \alpha' m_U + \beta' m_V,$$

where $m_U = \mathsf{E}[U]$ and $m_V = \mathsf{E}[V]$. Also,

$$\nu' C \nu = [\alpha' \; \beta'] \begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha' S \alpha + \beta' T \beta.$$

Hence,

$$\begin{aligned} \varphi_X(\nu) &= e^{j(\alpha' m_U + \beta' m_V) - (\alpha' S \alpha + \beta' T \beta)/2} \\ &= e^{(j\alpha' m_U - \alpha' S \alpha/2) + (j\beta' m_V - \beta' T \beta/2)} \\ &= e^{j\alpha' m_U - \alpha' S \alpha/2} e^{j\beta' m_V - \beta' T \beta/2} \\ &= \varphi_U(\alpha)\varphi_V(\beta), \end{aligned}$$

which shows that U and V are independent.