ECE/MATH 520, Spring 2008

Homework Problems 6

Solutions (version: April 22, 2008, 10:47)

17.3 Consider the following linear program:

$$\begin{aligned} \text{maximize} \quad & 2x_1 + 3x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \le 4 \\ & 2x_1 + x_2 \le 5 \\ & x_1, x_2 \ge 0 \end{aligned}$$

a. Use the simplex method to solve the above problem.

b. Write down the dual of the above linear program (or the problem converted to standard form), and solve the dual.

Ans.: a. We first transform the problem into standard form:

$$\begin{aligned} \text{minimize} \quad & -2x_1 - 3x_2 \\ \text{subject to} \quad & x_1 + 2x_2 + x_3 = 4 \\ & 2x_1 + x_2 + x_4 = 5 \\ & x_1, x_2, x_3, x_4 \ge 0. \end{aligned}$$

The initial tableau is:

$$\begin{array}{ccccc} 1 & 2 & 1 & 0 & 4 \\ 2 & 1 & 0 & 1 & 5 \\ -2 & -3 & 0 & 0 & 0 \end{array}$$

We now pivot about the $(1, 2)$th element to get:

$$\begin{array}{ccccc} 1/2 & 1 & 1/2 & 0 & 2 \\ 3/2 & 0 & -1/2 & 1 & 3 \\ -1/2 & 0 & 3/2 & 0 & 6 \end{array}$$

Pivoting now about the $(2, 1)$th element gives:

$$\begin{array}{ccccc} 0 & 1 & 2/3 & -1/3 & 1 \\ 1 & 0 & -1/3 & 2/3 & 2 \\ 0 & 0 & 4/3 & 1/3 & 7 \end{array}$$

Thus, the solution to the standard form problem is $x_1 = 2$, $x_2 = 1$, $x_3 = 0$, $x_4 = 0$. The solution to the original problem is $x_1 = 2$, $x_2 = 1$.

b. The dual to the standard form problem is

$$
\begin{aligned}
\text{maximize} \quad & 4\lambda_1 + 5\lambda_2 \\
\text{subject to} \quad & \lambda_1 + 2\lambda_2 \le -2 \\
& 2\lambda_1 + \lambda_2 \le -3 \\
& \lambda_1, \lambda_2 \le 0.
\end{aligned}
$$

From the discussion before Example 17.4, it follows that the solution to the dual is $\boldsymbol{\lambda}^T = \boldsymbol{c}_I^T - \boldsymbol{r}_I^T = [-4/3, -1/3]$.

The solution to the dual of the original problem is similar to the above, except with the sign of the variables changed. The solution to this dual is $[4/3, 1/3]$.

17.6 Consider the linear program

$$
\begin{aligned}
\text{minimize} \quad & x_1 + \cdots + x_n, \qquad x_1, \ldots, x_n \in \mathbb{R} \\
\text{subject to} \quad & a_1 x_1 + \cdots + a_n x_n = 1 \\
& x_1, \ldots, x_n \ge 0,
\end{aligned}
$$

where $0 < a_1 < a_2 < \cdots < a_n$.

a. Write down the dual to the above problem, and find a solution to the dual in terms of $a_1, \ldots, a_n$.
b. State the Duality Theorem, and use it to find a solution to the primal problem above.
c. Suppose that we apply the simplex algorithm to the primal problem. Show that if we start at a nonoptimal initial basic feasible solution, the algorithm terminates in one step if and only if we use the rule where the next nonbasic column to enter the basis is the one with the most negative relative cost coefficient.

Ans.: a. The dual (asymmetric form) is

$$
\begin{aligned}
\text{maximize} \quad & \lambda \\
\text{subject to} \quad & \lambda a_i \le 1, \ i = 1, \ldots, n.
\end{aligned}
$$

We can write the constraint as

$$\lambda \le \min\{1/a_i : i = 1, \ldots, n\} = 1/a_n.$$

Therefore, the solution to the dual problem is

$$\lambda = 1/a_n.$$

b. Duality Theorem: If the primal problem has an optimal solution, then so does the dual, and the optimal values of their respective objective functions are equal.

By the duality theorem, the primal has an optimal solution, and the optimal value of the objective function is $1/a_n$. The only feasible point in the primal with this objective function value is the basic feasible solution $[0, \ldots, 0, 1/a_n]^T$.

c. Suppose we start at a nonoptimal initial basic feasible solution, $[0, \ldots, 1/a_i, \ldots, 0]^T$, where $1 \leq i \leq n-1$. The relative cost coefficient for the qth column, $q \neq i$, is

$$r_q = 1 - \frac{a_q}{a_i}.$$

Since $a_n > a_j$ for any $j \neq n$, r_q is the most negative relative cost coefficient if and only if $q = n$.

17.9 Consider an LP problem in standard form. Suppose that $\boldsymbol{x}$ is a feasible solution to the problem. Show that if there exist $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ such that

$$\begin{aligned} \boldsymbol{A}^T\boldsymbol{\lambda} + \boldsymbol{\mu} &= \boldsymbol{c} \\ \boldsymbol{\mu}^T\boldsymbol{x} &= 0 \\ \boldsymbol{\mu} &\geq 0, \end{aligned}$$

then $\boldsymbol{x}$ is an optimal feasible solution to the LP problem, and $\boldsymbol{\lambda}$ is an optimal feasible solution to the dual. The above are called the *Karush-Kuhn-Tucker optimality conditions for LP*, which are discussed in detail in Chapters 20 and 21.
Note: For this question, you don't have to read Chapters 20 or 21; you need to use only the results of Chapter 17 (specifically, complementary slackness).

Ans.: To prove the result, we use the Complementary Slackness Theorem. Since $\boldsymbol{\mu} \geq \boldsymbol{0}$, we have $\boldsymbol{A}^T\boldsymbol{\lambda} = \boldsymbol{c} - \boldsymbol{\mu} \leq \boldsymbol{c}$. Hence, $\boldsymbol{\lambda}$ is a feasible solution to the dual. Now, $(\boldsymbol{c} - \boldsymbol{A}^T\boldsymbol{\lambda})^T\boldsymbol{x} = \boldsymbol{\mu}^T\boldsymbol{x} = 0$. Therefore, by the Complementary Slackness Theorem, $\boldsymbol{x}$ and $\boldsymbol{\lambda}$ are optimal for their respective problems.

19.6a Consider the problem

$$\begin{aligned} \text{minimize} \quad & 2x_1 + 3x_2 - 4, \qquad x_1, x_2 \in \mathbb{R} \\ \text{subject to} \quad & x_1 x_2 = 6. \end{aligned}$$

a. Use the Lagrange multiplier theorem to find all possible local minimizers and maximizers.

Ans.: a. We can represent the problem as

$$\begin{array}{ll} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & h(\boldsymbol{x}) = \mathbf{0}, \end{array}$$

where $f(\boldsymbol{x}) = 2x_1 + 3x_2 - 4$, and $h(\boldsymbol{x}) = x_1x_2 - 6$. We have $Df(\boldsymbol{x}) = [2, 3]$, and $Dh(\boldsymbol{x}) = [x_2, x_1]$. Note that **0** is not a feasible point. Therefore, any feasible point is regular. If $\boldsymbol{x}^*$ is a local extremizer, then by the Lagrange multiplier theorem, there exists $\lambda^* \in \mathbb{R}$ such that $Df(\boldsymbol{x}^*) + \lambda^* Dh(\boldsymbol{x}^*) = \mathbf{0}^T$, or

$$\begin{aligned} 2 + \lambda^* x_2^* &= 0 \\ 3 + \lambda^* x_1^* &= 0. \end{aligned}$$

Solving, we get two possible extremizers: $\boldsymbol{x}^{(1)} = [3, 2]^T$, with corresponding Lagrange multiplier $\lambda^{(1)} = -1$, and $\boldsymbol{x}^{(2)} = -[3, 2]^T$, with corresponding Lagrange multiplier $\lambda^{(2)} = 1$.

19.10 Consider the problem:

$$\begin{array}{lll} \text{maximize} & ax_1 + bx_2, & x_1, x_2 \in \mathbb{R} \\ \text{subject to} & x_1^2 + x_2^2 = 2 & \end{array}$$

where $a, b \in \mathbb{R}$. Show that if $[1, 1]^T$ is a solution to the problem, then $a = b$.

Ans.: Note that the point $[1, 1]^T$ is a regular point. Applying the Lagrange multiplier theorem gives

$$\begin{aligned} a + 2\lambda^* &= 0 \\ b + 2\lambda^* &= 0. \end{aligned}$$

Hence, $a = b$.

19.11a Consider the problem:

$$\begin{array}{lll} \text{minimize} & x_1x_2 - 2x_1, & x_1, x_2 \in \mathbb{R} \\ \text{subject to} & x_1^2 - x_2^2 = 0. & \end{array}$$

a. Apply the Lagrange multiplier theorem directly to the problem to show that if a solution exists, it must be either $[1, 1]^T$ or $[-1, 1]^T$.

Ans.: a. Denote the solution by $[x_1^*, x_2^*]^T$. The Lagrange conditions for this problem have the form

$$\begin{aligned} x_2^* - 2 + 2\lambda^* x_1^* &= 0 \\ x_1^* - 2\lambda^* x_2^* &= 0 \\ (x_1^*)^2 - (x_2^*)^2 &= 0. \end{aligned}$$

From the first and third equations it follows that $x_1^*, x_2^* \neq 0$. Then, combining the first and second equations, we obtain

$$\lambda^* = \frac{2 - x_2^*}{2x_1^*} = \frac{x_1^*}{2x_2^*}$$

which implies that $2x_2^* - (x_2^*)^2 = (x_1^*)^2$. Hence, $x_2^* = 1$, and by the third Lagrange equation, $(x_1^*)^2 = 1$. Thus, the only two points satisfying the Lagrange conditions are $[1, 1]^T$ and $[-1, 1]^T$. Note that both points are regular.

19.15a Consider the sequence $\{x_k\}$, $x_k \in \mathbb{R}$, generated by the recursion

$$x_{k+1} = ax_k + bu_k, \quad k \geq 0 \qquad (a, b \in \mathbb{R},\ a, b \neq 0)$$

where $u_0, u_1, u_2, \ldots$ is a sequence of "control inputs," and the initial condition $x_0 \neq 0$ is given. We wish to find values of control inputs u_0 and u_1 such that $x_2 = 0$, and the average input energy $(u_0^2 + u_1^2)/2$ is minimized. Denote the optimal inputs by u_0^* and u_1^*.

a. Find expressions for u_0^* and u_1^* in terms of a, b, and x_0.

Ans.: a. By simple manipulations, we can write

$$x_2 = a^2 x_0 + abu_0 + bu_1.$$

Therefore, the problem is

$$\begin{array}{ll} \text{minimize} & \dfrac{1}{2}(u_0^2 + u_1^2) \\ \text{subject to} & a^2 x_0 + abu_0 + bu_1 = 0. \end{array}$$

Alternatively, we may use a vector notation: writing $\boldsymbol{u} = [u_0, u_1]^T$, we have

$$\begin{array}{ll} \text{minimize} & f(\boldsymbol{u}) \\ \text{subject to} & h(\boldsymbol{u}) = 0, \end{array}$$

where $f(\boldsymbol{u}) = \frac{1}{2}\|\boldsymbol{u}\|^2$, and $h(\boldsymbol{u}) = a^2 x_0 + [ab, b]\boldsymbol{u}$. Since the vector $\nabla h(\boldsymbol{u}) = [ab, b]^T$ is nonzero for any $\boldsymbol{u}$, then any feasible point is regular. Therefore, by the Lagrange multiplier theorem, there exists $\lambda^* \in \mathbb{R}$ such that

$$\begin{aligned} u_0^* + \lambda^* ab &= 0 \\ u_1^* + \lambda^* b &= 0 \\ a^2 x_0 + abu_0^* + bu_1^* &= 0. \end{aligned}$$

We have three linear equations in three unknowns, that upon solving yields

$$u_0^* = -\frac{a^3 x_0}{b(1 + a^2)}, \qquad u_1^* = -\frac{a^2 x_0}{b(1 + a^2)}.$$