Now we may argue as for the sojourn time formulation to show the "inter-jump" times of $M$ are independent and have the exponential distribution.

Hence, $M$ is a Poisson process with intensity $\lambda$.

$\boxed{\#26 \quad 4/31}$

## §4.4 Death Processes

This is the complement of a birth process. It moves through states $N$, $N-1, \ldots, 0$, leading to absorption into $0$, or extinction.

### Definition 4.4.1

A death process $X(t)$ with death parameters $\mu_1, \ldots, \mu_N$, $\mu_i > 0$, is a process with state space $\{0, 1, 2, \ldots, N\}$ such that

(1) $X(0) = N$

(2) $s < t \Rightarrow X(s) \geq X(t)$

(3)

$$P\{X(t+h) = k-m \mid X(t) = k\} = \begin{cases} \mu_k h + o(h), & m = 1, \\ 1 - \mu_k h + o(h), & m = 0, \\ o(h), & m > 1. \end{cases}$$

(4) If $s<t$, then conditional on $X(s)$, the increment $X(t) - X(s)$ is independent of all deaths prior to $s$.

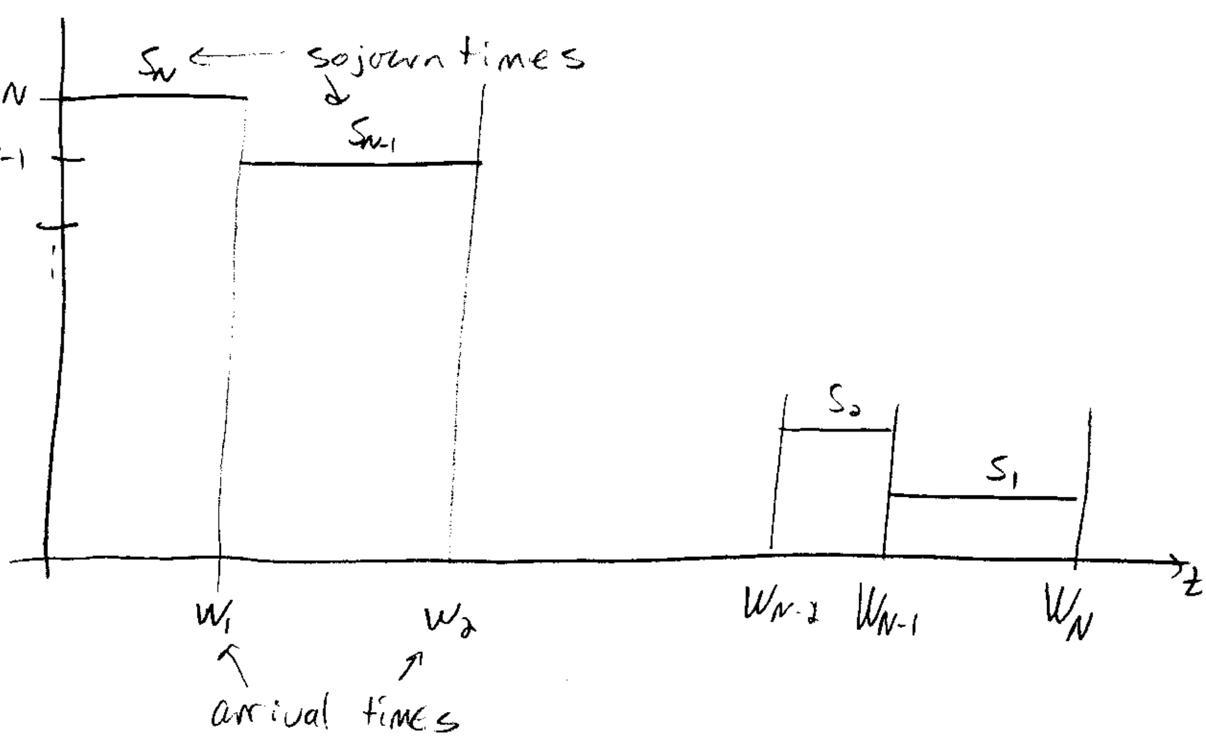
## Theorem 4.4.1

The sojourn time in state $k$ is exponentially distributed with parameter $\mu_k$ and the sojourn times are independent.

Proof

Homework

A typical realization




We usually assign $\mu_0 = 0$.

Theorem 4.4.2

If the $\{\mu_i\}$ are distinct, then the transition probabilities

$$P_n(t) = P(X(t) = n \mid X(0) = N)$$

satisfy

$$P_N(t) = e^{-\mu_N t}$$

$$(4.4.1) \qquad P_n(t) = \mu_{n+1}\mu_{n+2}\cdots\mu_N\left(A_{n,n}e^{-\mu_n t} + \cdots + A_{N,n}e^{-\mu_N t}\right)$$

where

$$A_{k,n} = \frac{1}{(\mu_N-\mu_k)\cdots(\mu_{k+1}-\mu_k)(\mu_{k-1}-\mu_k)\cdots(\mu_n-\mu_k)}$$

$(\mu_k-\mu_k)$ is skipped!

Example 4.4.1 Linear death process

We assume the death rates are proportional to the population,

$$\mu_k = \alpha k,$$

$\alpha$ is the individual death rate. Then

$$A_{nn} = \frac{1}{\alpha^{N-n-1}(N-n)(N-n-1)\cdots 2\cdot 1}$$

$$A_{n+1,n} = \frac{1}{\alpha^{N-n-1}(N-n-1)\cdots 2\cdot 1\cdot(-1)}$$

$\vdots$

$$A_{kn} = \frac{1}{\alpha^{N-n-1}(-1)^{k-n}(N-k)!\,(k-n)!}$$

so

$$P_n(t) = \frac{N!}{n!\,(N-n)!}\, e^{-n\alpha t}\left(1-e^{-\alpha t}\right)^{N-n}, \quad n = 0,1,2,\ldots,N$$

Let $T$ be the time of extinction,

$$T = \min_{t\geq 0} \mathbb{X}(t) = 0$$

Since $T \leq t \Rightarrow \mathbb{X}(t) = 0$, the c.d.f. of $T$ is

$$F_T(t) = P(T \leq t) = P(\mathbb{X}(t) = 0)$$

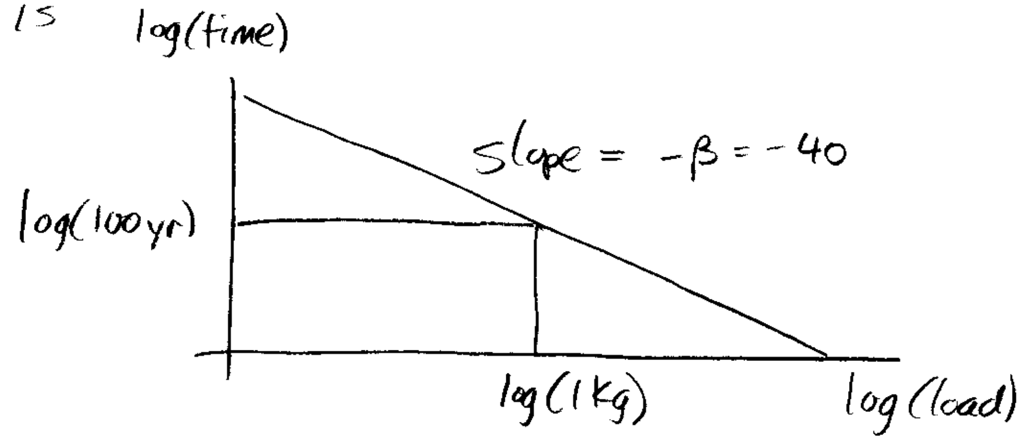
$$= P_0(t) = (1-e^{-\alpha t})^N, \; t \geq 0$$

Example 4.4.2

We consider the failure of a

cable consisting of a lot of parallel fibers under tension. The cable is required to support a load of 1000 Kg for 100 years. The issue is determining how many fibers are required.

The thin fibers are subject to failure after some time when placed under a constant load. The experimental connection is




$$\log_{10} \mu_T = 2 - 40 \log_{10} I$$

$\mu_T$ = mean life

$I$ = load

A naive analysis might suggest that achieving the 100 year design target requires using 1000 fibers, with the rationalization based on using average properties for so many fibers. We will see this does not work.

We assume (which fits bad observations) that the probability distribution for the failure time $T$ of a single fiber under a time-varying load $\ell(t)$ is

$$P(T \le t) = 1 - e^{-\int_0^t K(\ell(s))\,ds} \quad , t \ge 0$$

The failure rate $r(t) = K(\ell(t))$ determines the probability that a fiber under load $\ell(t)$ will fail during the interval $(t, t+\Delta t]$ with probability

$$P(t < T < t+\Delta t \mid T > t) = K(\ell(t))\Delta t + o(\Delta t)$$

The function $K(\ell(s))$ describes how changes in load affect the failure probability. We assume a power law relation

$$K(\ell) = \ell^{\beta}/A \quad , \quad A, \beta > 0 \text{ constant}$$

Assuming this, under a constant load $\ell(t) = \ell$, the single fiber failure time is exponentially distributed with mean

$$\mu_T = E(T|\ell) = \frac{1}{K(\ell)} = A\ell^{-\beta}$$

This gives a linear log-log plot, as described above, with $\beta = 40$, $A = 100$.

We place $N$ of these fibers in a cable and subject the cable to a load $NL$, where $L$ is the nominal load per fiber.

We assume that the cable failure time equals the the failure time of the last fiber.

The number of intact fibers $X(t)$ evolves as a pure death process with parameters

$$\mu_k = k\,K(NL/k), \quad k = 1, 2, \ldots, N$$

To show this:
Given $X(t) = k$ surviving fibers of time $t$ and assuming the load $NL$ is

shared equally, then each has a load $NL/k$ and a failure rate $K(NL/k)$. Since there are $k$ survivors in the cable, the cable failure rate is

$$\mu_k = k\, K(NL/k)$$

The cable failure time is waiting time $W_N$. We can use (4.4.1) to express

$$P(W_N \le t) = P(X(t) = 0) = P_0(t)$$

interms of $\mu_1, \mu_2, \ldots, \mu_N$,

$$(4.4.2) \quad E(W_N) = A L^{-\beta} \sum_{k=1}^{N} \left(\frac{k}{N}\right)^{\beta-1} \frac{1}{N}$$

For large $N$,

$$\sum_{k=1}^{N} \left(\frac{k}{N}\right)^{\beta-1} \frac{1}{N} \approx \int_0^1 x^{\beta-1}\,dx = \frac{1}{\beta}$$

This is about 4% accurate for $N = 1000$, $\beta = 40$. We find

$$E(W_N) \approx \frac{A}{\beta L^{\beta}}$$

to be compared with the average fiber life of

$$\mu_T = \frac{A}{L^\beta}.$$

Thus, a cable only lasts about $\frac{1}{\beta}$ as long as an average fiber under an equivalent load!

In our case, the cable as specified would only last 2.5 years, not 100.

To compute the right number of fibers $\hat{N}$, we set

$$\frac{A}{L^\beta} = \frac{A}{\beta\,(NL/\hat{N})^\beta}$$

which corresponds to decreasing the nominal load per fiber from $L$ to $\hat{L} = NL/\hat{N}$. This gives

$$\hat{N} = N\beta^{1/\beta}$$

or $\approx 1097$ fibers.

## §4.5 Birth and Death Processes

Now we let $X(t)$ increase and decrease. If at time $t$, the process is in state $n$, then after a random sojourn time, it may move to either of the neighboring states $n-1$ or $n+1$.

This is a continuous time extension of a random walk.

Formally:

Definition 4.5.1

A birth-death process $X(t)$ is a Markov process on the state space $S = \{0, 1, 2, \ldots\}$ with stationary transition probabilities described by the transition probability function

$$(4.5.1) \quad P_{ij}(t) = P(X(t+s) = j \mid X(s) = i), \quad \text{for all } s \geq 0,$$

where

$$(4.5.2) \quad P_{i\,i+1}(h) = \lambda_i h + o(h) \qquad h \downarrow 0, \; i \geq 0$$

$$(4.5.3) \quad P_{i\,i-1}(h) = \mu_i h + o(h) \qquad h \downarrow 0, \; i \geq 0$$

$$(4.5.4) \quad P_{i\,i}(h) = 1 - (\lambda_i + \mu_i) h + o(h) \qquad h \downarrow 0, \; i \geq 0$$

$$(4.5.5) \quad P_{ij}(0) = \delta_{ij}$$

(4.5.6) $\mu_0 = 0, \ \lambda_0 > 0, \quad \mu_i, \lambda_i > 0, \ i = 1, 2, \dots$

The $o(h)$ terms may depend on $i$.

The matrix

$$A = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & \cdots & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \cdots & \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & 0 & \cdots \\ 0 & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & & \end{pmatrix}$$

is the <u>infinitesimal generator</u>, $\lambda_i$ and $\mu_i$ are called the infinitesimal <u>birth</u> and <u>death</u> <u>rates</u>.

Since the $p_{ij}$ are probabilities

(4.5.7) $\quad p_{ij}(t) \geq 0, \quad \sum_{j=0}^{\infty} p_{ij}(t) \leq 1$

We may also derive