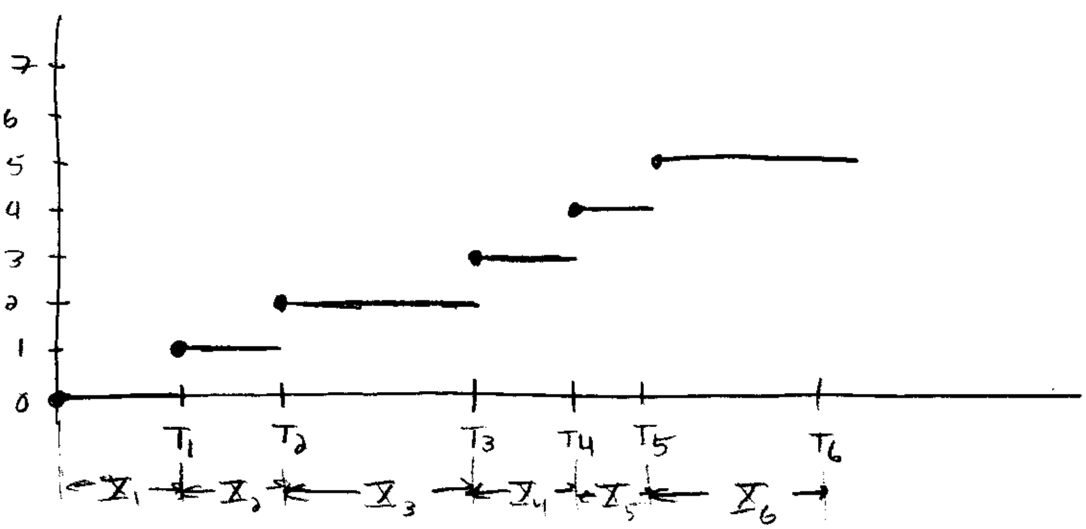
(4.1.7) $$X_n = T_n - T_{n-1}$$

If we know N, we can compute $X_1, X_2, \ldots$

Vice versa, if we know the entire collection $\{X_i\}$, then

(4.1.8) $$T_n = \sum_{i=1}^{n} X_i, \quad N(t) = \max_{T_n \le t} n$$

Here is an illustration



#22 4/15

Theorem 4.1.2

The random variables $X_1, X_2, \ldots$ are i.i.d. with exponential distribution with parameter λ.

Proof

Consider X_1:

$$P(X_1 > t) = P(N(t) = 0) = e^{-\lambda t}$$

so X_1 is exponentially distributed.

Conditional on X_1,

$$P(X_2 > t \mid X_1 = t_1)$$
$$= P(\text{no arrival in } (t_1, t_1 + t] \mid X_1 = t_1)$$

The event $\{X_1 = t_1\}$ is related to arrivals in $[0, t_1]$ whereas the event $\{\text{no arrival in } (t_1, t_1 + t]\}$ relates to arrivals after t_1. These are independent by Defn 4.1.1(d), so

$$P(X_2 > t \mid X_1 = t_1) = P(\text{no arrival in } (t_1, t_1 + t]) = e^{-\lambda t}$$

So X_2 is independent of X_1 and has the same distribution. Generally,

$$P(X_{n+1} > t \mid X_1 = t_1, \ldots, X_n = t_n) = P(\text{no arrival in } (T, T+t])$$

with $T = t_1 + \cdots + t_n$.

Induction proves the theorem

These two points of view are equivalent in some sense:

Theorem 4.1.3

The process N constructed by (4.1.8) from a sequence $\{X_1, X_2, \dots\}$ is a Poisson process iff the $\{X_i\}$ are i.i.d. exponentially distributed.

Proof

exercise

Given such a sequence $\{X_n\}$, we can determine the distribution of $N(t)$ easily. We can prove that $T_n = \sum_{i=1}^{n} X_i$ has the gamma distribution, and $N(t)$ is determined by the observation

$$N(t) \geq j \iff T_j \leq t$$

so

$$P(N(t)=j) = P(T_j \leq t < T_{j+1}) = P(T_j \leq t) - P(T_{j+1} \leq t)$$

$$= \frac{(\lambda t)^j}{j!} e^{-\lambda t}. \quad \text{(details skipped)}$$

Example 4.1.2

Defects occur along an undersea cable. We let position denote "time" and the defects occur according to a Poisson process of rate $\lambda = .1$ /mile.

(a) What is the probability of no defects in the first two miles?

(b) Given there are no defects in the first two miles, what is the conditional probability of no defects between miles two and three?

(a) $P(N(2)=0) = \frac{(.1\times 2)^0}{0!} e^{-.1\times 2} \approx .819$

(b) $N(3)-N(2)$ and $N(2)-N(0)=N(2)$ are independent, hence

$$P(N(3)-N(2)=0 \mid N(2)=0) = P(N(3)-N(2)=0)$$

$$= P(N(1)=0) = e^{-.1} \approx .905$$

Example 4.1.3

Customers arrive at a store according to a

Poisson process of rate $\lambda = 4/\text{hour}$. From the opening time of 9:00 am, what is the probability that exactly one customer has arrived at 9:30am and a total of five by 11:30am?

We want $P(N(1/2)=1, N(5/2)=5)$. Since $N(5/2)-N(1/2)$ and $N(1/2)$ are independent

$$P(N(1/2)=1,\ N(5/2)=5) = P(N(1/2)=1,\ N(5/2)-N(1/2)=4)$$

$$= \frac{e^{-4\cdot\frac{1}{2}}(4\cdot\frac{1}{2})^1}{1!} \cdot \frac{e^{-4\cdot 2}(4\cdot 2)^4}{4!}$$

$$\approx .0155.$$

§ 4.2 Birth Processes

We now consider more sophisticated examples in which λ may vary with time.

Example 4.2.1

The Poisson process describes the emissions from uranium 235 since it has a half life of $7 \cdot 10^8$ years and decays slowly. It is not a good model for the emissions of strontium 92, which has a half life of 2.7 hours. In this case, the rate depends on the amount already detected.

<u>Definition 4.2.1</u>

A <u>birth process</u> with <u>intensities</u> $\lambda_0, \lambda_1, \ldots$ is a process $\{N(t), t \geq 0\}$ with values in $S = \{0, 1, 2, \ldots\}$ such that

(a) $N(0) \geq 0$

(b) If $s < t$, $N(s) \leq N(t)$

(c) $$P(N(t+h) = n+m \mid N(t) = n) = \begin{cases} \lambda_n h + o(h), & m = 1, \\ o(h), & m > 1, \\ 1 - \lambda_n h + o(h), & m = 0. \end{cases}$$

(d) If $s < t$, then conditional on the value of $N(s)$, the increment $N(t)-N(s)$ is independent of the times of arrivals prior to s.

Example 4.2.2

A Poisson process is a birth process with $\lambda_n=\lambda$ for all n.

Example 4.2.3

The simple or Yule birth process has $\lambda_n = n\lambda$. This models the growth of a population in which living individuals give birth independently of one another, each giving birth to a new individual with probability $\lambda h + o(h)$ in $(t, t+h)$. No individuals die. The number M of births in $(t, t+h)$ satisfies

$$P(M=m \mid N(t)=n) = \binom{n}{m}\underbrace{(\lambda h)^m}_{\text{give birth}}\underbrace{(1-\lambda h)^{n-m}}_{\text{do not give birth}} + o(h)$$

$$= \begin{cases} 1 - n\lambda h + o(h), & m = 0, \\ n\lambda h + o(h), & m = 1, \\ o(h), & m > 1. \end{cases}$$

Example 4.2.4

Simple birth with immigration, we have $\lambda_n = n\lambda + \nu$, ν = constant immigration rate.

We now try to analyze a birth process N with intensities $\lambda_0, \lambda_1, \ldots$ in the same way we treated the Poisson process.

Definition 4.2.2

The transition probabilities are

$$P_{ij}(t) = P(N(s+t) = j \mid N(s) = i)$$
$$= P(N(t) = j \mid N(0) = i)$$

If we suppose the intensities are positive (birth process), we can use the same argument used for the Poisson process. We condition on $N(t)$

$$P(N(t+h) = j \mid N(s) = i)$$

$$= \sum_{k=0}^{\infty} P(N(t+h) = j \mid N(t) = k)\, P(N(t) = k \mid N(s) = i)$$

or

$$P_{ij}(t+h) = \sum_{k=0}^{\infty} P(N(t+h) = j \mid N(t) = k)\, P_{ik}(t)$$

From Defn 4.2.1(c),

$$P_{ij}(t+h) = \lambda_{j-1} h\, P_{ij-1}(t) + (1 - \lambda_j h)\, P_{ij}(t) + o(h)$$

$$\frac{P_{ij}(t+h) - P_{ij}(t)}{h} = \lambda_{j-1} P_{ij-1}(t) - \lambda_j P_{ij}(t) + o(h)$$

Taking the limit as $h \to 0$ yields

<u>Theorem 4.2.1</u> <u>Forward System</u>

With $\lambda_{-1} = 0$ and $P_{ij}(0) = \delta_{ij}$, the transition probabilities for the birth process satisfy the forward equations

(4.2.1) $P_{ij}'(t) = \lambda_{j-1} P_{ij-1}(t) - \lambda_j P_{ij}(t)$ for $j \geq i$.

Alternatively, we could have conditioned on

$N(h)$ to derive

Theorem 4.2.2 Backward System

With $P_{ij}(0)=\delta_{ij}$, the transition probabilities for the birth process satisfy the backward equations

Review
1) Birth process
2) transition prob.
3) F'ward and B'ward system.

$$(4.2.2) \quad P_{ij}'(t) = \lambda_i P_{i+1\,j}(t) - \lambda_i P_{ij}(t), \quad j \ge i,$$

$\boxed{\#23 \quad 4/22}$

Theorem 4.2.3

The forward system has a unique solution that also satisfies the backward system.

Proof

First note that

$$(4.2.3) \qquad P_{ij}(t) = 0 \quad \text{if } j < i$$

We solve the forward problem with $j=i$,

$$P_{ii}'(t) = \cancel{\lambda_{i-1} P_{i\,i-1}(t)}^{\,0} - \lambda_i P_{ii}(t),$$

to obtain

$$(4.2.4) \qquad P_{ii}(t) = e^{-\lambda_i t}$$