$$\sum_i \pi_i P_{ij} = \sum_i \pi_j P_{ji}$$

$$= \pi_j \sum_i P_{ji}$$

$$= \pi_j$$

or

$$\pi = \pi P.$$

$\boxed{\#20 \quad 4/8}$

Given a reversible chain, if we observe a sequence of consecutive states, there is no way to tell if the sequence was generated forward or backward in time.

Example 3.5.1

Consider the ON/OFF example in Ex. 2.2.3,

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}, \quad 0 \le p \le 1, \; 0 \le q \le 1.$$

(3.5.1) reads

$$\pi_0 P_{00} = \pi_0 P_{00}$$

$$\pi_0 P_{01} = \pi_1 P_{10}$$

$$\pi_1 P_{10} = \pi_0 P_{01}$$

$$\pi_1 P_{11} = \pi_1 P_{11}$$

or

$$\pi_0(1-p) = \pi_0(1-p)$$
$$\pi_0 p = \pi_1 q \qquad \Rightarrow \qquad \pi_0 p = \pi_1 q$$
$$\pi_1 q = \pi_0 p$$
$$\pi_1(1-q) = \pi_1(1-q)$$

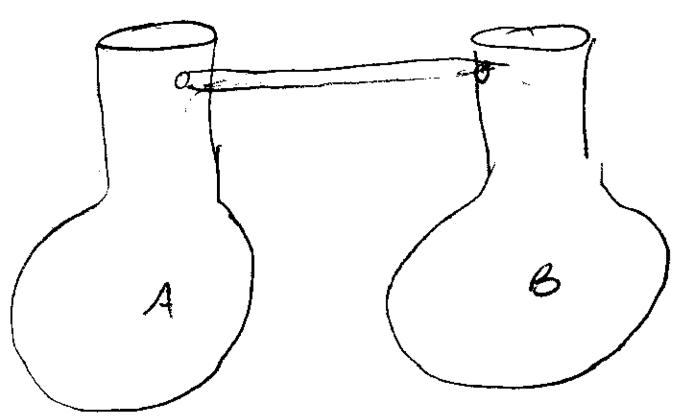
Recall the stationary distribution is

$$\pi_0 = \frac{q}{p+q}, \quad \pi_1 = \frac{p}{p+q}$$

and so $\pi_1 q = \pi_0 p$ as required for reversibility.

Example 3.5.2 Ehrenfest Model of Diffusion

Two containers A, B are placed near each other and gas is allowed through a small tube connecting them.



A total of M molecules is distributed between the two. At each time, one molecule picked uniformly at random, passes through the apeture. Let X_n be the number of molecules in A after n time steps. X_n is a Markov chain and

$$P = \begin{pmatrix} 0 & 1 & 0 & \cdots & & \\ \frac{1}{m} & 0 & 1-\frac{1}{m} & 0 & \cdots & \\ 0 & \frac{2}{m} & 0 & 1-\frac{2}{m} & \cdots & \\ \vdots & & \ddots & \ddots & \ddots & \\ & & & & & \\ 0 & 0 & \cdots & & 0 & 1 & 0 \end{pmatrix}$$

or

$$\left.\begin{aligned} P_{i,i+1} &= 1-\frac{i}{m} \\ P_{i,i-1} &= \frac{i}{m} \end{aligned}\right) \quad 0 \le i \le m,$$

It seems reasonable that this would be reversible. If we look for solutions of (3.5.1), we find

$$\pi_i = \binom{m}{i}\left(\frac{1}{2}\right)^m$$

and this is also a stationary distribution.

§3.6 Chains with finitely many states

As discussed, the theory is much simpler when the chain has a finite state space.

By Thm 3.2.6, if X_n is irreducible and the state space is finite, then X_n is positive recurrent. Hence,

<u>Theorem 3.6.1</u>

An irreducible aperiodic chain with a finite state space has a stationary distribution which is also a limit distribution.

It is also possible to prove some important properties of the probability transition matrix. For example,

<u>Theorem 3.6.2</u> <u>Perron-Frobenius</u>

If P is the transition probability matrix of an irreducible chain with period d and a finite state space then

(1) $\lambda_0 = 1$ is an eigenvalue of P

(2) the d complex roots

$$\lambda_1 = e^{2\pi i/d}, \ \lambda_2 = e^{2\pi i \cdot \frac{2}{d}}, \ \ldots, \ \lambda_d = e^{2\pi i \frac{d-1}{d}}$$

$((\lambda_m)^d = 1$ for all $m)$
are eigenvalues of P

(3) the remaining eigenvalues $\lambda_{d+1}, \ldots, \lambda_N$ satisfy $|\lambda_m| < 1$.

Using this theorem, it is possible to analyze the long time behavior of the chain.

Example 3.6.1

Suppose the chain is aperiodic and P has distinct eigenvalues. Then there is a B such that

$$P = B^{-1} \Lambda B = B^{-1} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix} B$$

and
$$P^n = B^{-1} \Lambda^n B$$

Since $|\lambda_i|^n \to 0$ as $n \to \infty$ for all $i \geq 1$,

$$\lim_{n\to\infty} P^n = B^{-1}\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & & \cdots & \\ 0 & & \ddots & & \\ \vdots & & & & \\ 0 & & & \cdots & 0 \end{pmatrix} B$$

If the eigenvalues are not distinct, then we have to use the notion of generalized eigenvectors.

§3.7 Branching Processes

Unfortunately, the preceeding theory does not prove useful for many important Markov chains.

For example, we consider branching processes §2.5, page 93, X_n, where as usual $X_0 = 1$.

If there is a strictly positive probability that each family is empty, $P(X_n = 0) > 0$, then 0 is an absorbing state. Hence, 0 is a positive recurrent state, while all other states must be transient (exercise).

The chain is not irreducible, but there is a unique stationary distribution, namely

$$\pi_0 = 1, \quad \pi_i = 0, \quad i \geq 1.$$

This tells nothing interesting about the behavior of the process.

The difficulty is that the process may behave in a number of qualitatively different ways depending, for example, on whether or not it becomes extinct.

We can approach this problem by studying the behavior conditional on the occurrence on some event, e.g. extinction or the value of some random variable such as the total number $\sum_i X_i$ of offspring.

Recall that we let p_k denote the offspring distribution, $p_k = P(X_1 = k)$, and let

$$\phi(s) = E(s^{X_1})$$

be the probability generating function of p_k.

We let $N = \inf\{n : X_n = 0\}$ be the time of extinction, where $N = \infty$ if $X_n \neq 0$ for all n. If $N = \infty$, the process grows without bound, while if $N < \infty$, the process never becomes very large and eventually becomes extinct.

Recall that the probability of ultimate

extinction $P(N<\infty)$ is the smallest nonnegative root of $s=\phi(s)$.

We let

$$E_n = \{n < N < \infty\}$$

denote the event that extinction occurs at some time larger than n. We study the distribution of X_n conditional on E_n. We let

$$P_{0,j}(n) = P(X_n = j \mid E_n)$$

be the conditional probability that $X_n = j$ given the future extinction of X_n. We try to compute

$$\pi_{0,j} = \lim_{n\to\infty} P_{0,j}(n),$$

if the limit exists. To avoid some trivial cases, we assume

$$0 < p_0 + p_1 < 1, \quad p_0 > 0,$$

which implies that $0 < P(E_n) < 1$ and the probability of ultimate extinction is in $(0,1]$.

Theorem 3.7.1

If $E(X_1) < \infty$, then $\pi_{0,j} = \lim_{n\to\infty} P_{0,j}(n)$

exists. The generating function

$$G(s) = \sum_j \Pi_{0,j} s^j$$

satisfies

(3.7.1) $$G(\eta^{-1}\Phi(s\eta)) = \alpha G(s) + 1 - \alpha,$$

where η is the probability of ultimate extinction and $\alpha = \Phi'(\eta)$.

If $\mu = E(X_1) \le 1$, then $\eta \equiv 1$ and $\alpha = \mu$, so (3.7.1) becomes

(3.7.2) $$G(\Phi(s)) = \mu G(s) + 1 - \mu.$$

For all μ, $G'(\eta) \le 1$, and $G'(\eta) = 1 \Leftrightarrow \mu = 1$.

<u>Outline of proof</u>

For $0 \le s < 1$, set

$$G_n(s) = E(s^{X_n} \mid E_n) = \sum_j P_{0,j}(n) s^j$$

$$= \sum_j s^j \frac{P(X_n = j, E_n)}{P(E_n)} = \frac{\Phi_n(s\eta) - \Phi_n(0)}{\eta - \Phi_n(0)}$$

where $\Phi_n(s) = E(s^{X_n})$. This holds because

$$P(X_n = j, E_n) = P(X_n = j) \cdot \eta^j, \quad j \geq 1.$$

(exercise)

and

$$P(E_n) = P(N < \infty) - P(N \leq n)$$

extinction

$$= \eta - \Pi_n(0)$$

(exercise: show claim)

If we define

$$H_n(s) = \frac{\eta - \phi_n(s)}{\eta - \phi_n(0)}$$

then

$$G_n(s) = 1 - H_n(s\eta)$$

We can show that $H_n(s) \geq H_{n-1}(s)$ for $s < \eta$, and hence the limit

$$\lim_{n\to\infty} H_n(s\eta) = H(s\eta)$$

exists. This means the limit

$$\lim_{n\to\infty} G_n(s) = G(s)$$

exists as well for $0 \leq s < 1$, and

$$G(s) = 1 - H(s\eta), \quad 0 \leq s < 1.$$

Hence, π_{0j} exists for all j. It also follows that

$$H(\phi(s)) = \phi'(\eta)\, H(s), \quad 0 \le s < \eta,$$

and the rest follows.

Theorem 3.7.2

(1) If $E(X_1) \neq 1$, then $\sum_j \pi_{0,j} = 1$

(2) If $E(X_1) = 1$, then $\pi_{0,j} = 0$ for all j.

Proof

$\mu = E(X_1) = 1 \Leftrightarrow \phi'(\eta) = 1$. If $\mu \neq 1$,

$\phi'(\eta) \neq 1$, and $\lim_{s \uparrow \eta} H(s) = 0 \Rightarrow \lim_{s \uparrow 1} G(s) = 1$,

or $\sum_j \pi_{0,j} = 1$. If $\mu = 1$, $\phi'(\eta) = 1$, and

$G(\phi(s)) = G(s)$. But, $G(s) > s$ for all $s < 1$,

so $G(s) = G(0) = 0$ for $s < 1$, and $\pi_{0,j} = 0$.

As long as $\mu \neq 1$, the distribution of X_n conditional on future extinction converges to some limit $\{\pi_{0,j}\}$, which is a proper distribution, as $n \to \infty$.

The critical process with $\mu = 1$ is more difficult to study.

§3.8 Review

We analyzed the behavior of Markov chains on three time scales:

(a) short

(b) intermediate

(c) asymptotic at infinity – long time

The analysis for each scale is specialized to that scale.

§3.8.1 Short time scale

The key observation is that the transition of a Markov chain from one state to the next is determined by the transition probability matrix P with entries $P_{ij} = P(X_{n+1}=j \mid X_n = i) = P(X_1 = j \mid X_0 = i)$. This follows from the Chapman-Kolmogorov equations, which says that we can compute the n-step transition probability matrix (Defn 2.2.1) by computing powers P^n

§3.8.2 Intermediate Time Scale Analysis

We first noted that some Markov chains have states that are special with respect to behavior in time, most notably absorbing states.

Once the chain enters an absorbing state, it never leaves. The question becomes: how long does it take to enter an absorbing state?

We introduced the notion of the time to absorption

$$T = \min_n \{ X_n = \text{absorbing state } j \,/\, X_0 = \text{initial state} \}$$

and the probability of entering an absorbing state

$$P(X_n = \text{absorbing state } j \,/\, X_0 = \text{initial state}).$$

If T is finite, then we are discussing intermediate time scale behavior.

We also noted that for finite state Markov chains, eigenvalue decomposition of the transition probability matrix can indicate a lot.

Later in §3.6, we continue this direction and describe how the Perron-Froebenius theorem actually allows us to describe the intermediate and long time scale behavior completely.

We continued the analysis of finite state space chains by using the canonical form of the probability transition matrix (2.3.1) and introducing the notions of the mean number of the mean number of visits to state k before absorbtion starting from state i $\{W_{ik}\}$ and the mean time to absorbtion starting from state i $\{V_i\}$ and deriving equations for these (2.3.3), (2.3.4), (2.3.5).

§ 3.8.3 Long time analysis

We began the long time analysis by observing that there are different kinds of long time behavior with some behaviors [illegible] easily [illegible]ed. That lead us into classifying states and connections between states.

In all cases, we described the long time behavior in probabilistic terms, i.e. how much time will the chain spend in state i as time passes? The goal of the classification is to find properties that enable us to make meaningful statements about such questions. The first classification was based on considering whether or not a chain returns to its initial state in finite time, the finite time of arrival. A recurrent state has the property that the probability of a finite first return time is 1. If that probability is less than one, the state is transient. There is a positive probability of no return in a transient state.

We addressed this question by finding relations between the transition probabilities and the probabilities that the first visit to state j starting from state i occurs in n steps (Thm 3.1.1, Thm 3.1.2) Thm 3.1.2 shows that we can decide this question by looking the convergence properties of series of terms expressing the probability of first returns.

We then realized that recurrent states can be further classified using the mean recurrence time

(Defn 3.1.6). Positive recurrent states have finite mean recurrence times, while null recurrent states have infinite mean recurrence time. We can decide this by looking at the behavior of the probability of recurrence over n steps for large n (Thm 3.1.6)

We made another classification of states. The period of a state is the g.c.d. of times through which it is possible to return to the state. when the period is 1, the state is aperiodic

Continuing with the classification of chains, we defined the notion of state i communicating with state j when there is a positive probability of getting to j starting from i. We can determine this from the n step transition probabilites. Two states intercommunicate if they communicate with each other, and we defined communication classes. We saw that the key properties, eg periodicity, transientivity, recurrence are shared by all the members in a class. We defined the closed set to be those which have the property that the chain never leaves once it goes in such a the set and the irreducible sets by the property that all the states in such a set intercommunicate.

The big result (Thm 3.2.5) is that we can partition the state space uniquely into the union of a transient class together with a collection of irreducible closed sets of recurrent states. We can't really talk about the long time behavior for states in the transient class, and we can consider the long time behavior of the chain with respect to any particular member of the set of irreducible, closed sets of recurrent states by restricting the chain to one of those sets, since it never leaves once it is in one of those.

In this classification, we noted that finite state spaces are special (Thm 3.2.6).

Finally, we returned to the long time behavior of a chain. We observed that there are at least two natural questions

(1) What are the long-time proportions of times spent in each state up to some long time n?

This leads to the idea of a stationary distribution.

(2) If we look at many realizations of a chain at a long time n, what proportion is spent in each state i.

This leads to the idea of a limit distribution

The material discusses how to determine each kind of distribution, stationary and limit, and then when these are the same.

A stationary distribution, if it exists, satisfies the system of equations $\{\pi = \pi P,\ \sum \pi_i = 1\}$. The main result (Theorem 3.3.3) says that an irreducible chain has a stationary distribution iff all the states are positive recurrent, in which case the coefficients of the stationary distribution are the reciprocal of the mean recurrence times.

To prove this theorem, we used (Defn. 3.3.3) the mean number of visits to state i between successive visits to state k and the time of first return to a state.

It turned out that we could come up with an analogous condition for determining if a state of an irreducible chain is transient (Thm 3.3.8), where s is transient if there is a nonzero solution $\{y_j, j \neq s\}$ of (3.3.5) $y_i = \sum_{j \neq s} p_{ij} y_j$, $i \neq s$, with $|y_j| \leq 1$ for all j. The important observation is that these equations have the form $\tilde{y} = \tilde{P} \tilde{y}$ as opposed to $\pi = \pi P$, where "~" means take out one row and column from P.

The main limit theorem for limiting distributions (Thm 3.4.1) says that for an aperiodic irreducible chain, $P_{ij}^n \to \frac{1}{\mu_j}$ $n \to \infty$ all i, j, μ_j = mean recurrence time. This result is more general than the previous discussion, because it applies to transient, null recurrent and positive recurrent states (though for different reasons for all three).

We also saw that it is possible to extend the result to periodic chains in some sense.

The summarizing main result (Thm 3.4.2 says that an ergodic (aperiodic, positive recurrent, irreducible) Markov chain has the property that it has both stationary and limiting distributions and these are equal.