Notes - 3 April

(Review) Theorem 3.4.1 - irreducible, aperiodic MC. $P^n_{ij} \to_{n\to\infty} \frac{1}{\mu_j}$ for all i, j. μ_j is mean recurrent time.

Theorem 3.4.4 - An ergodic MC has both stationary and limiting distributions and these are equal.

Recurrent cases in proof: X_n, Y_n have common S, P. $Z_n = (X_n, Y_n) \in S \times S, Z_n$ M.C. $P_{ij,kl} = P_{ik}P_{jl}$. Assume X_n positive recurrent. Π is unique stationary distribution. (With Y_n "=" X_n), Z_n has stationary distribution $\nu = (\nu_{ij}, i, j \in S), \nu_{ij} = \Pi_i\Pi_j$. Z_n is positive recurrent. Choose $X_0 = i, Y_0 = j, Z_0 = (i, j)$. Choose $s \in S$. $T = \min\{n \geq 1 : Z_n = (s, s,)\}$. Z_n recurrent $\Rightarrow P(T < \infty) = 1$.

Observation: Suppose $m \leq n$ and $X_m = Y_m$. Then X_n and Y_n are identically distributed. Thus conditional on $\{T \leq n\}, X_n$ and Y_n have the same distribution. We use this observation and the fact that T is finite to prove that in large time, the distributions of X_n and Y_n are independent of the initial values.

Computation: Start from $Z_0 = (i, j).P^n_{ik} = P(X_n = k) = P(X_n = k, T \leq n) + P(X_n = k, T > n) = P(Y_n = k, T \leq n) + P(Y_n = k, T > n) \leq P(Y_n = k) + P(T > n) = P^n_{jk} + P(T > n)$. The symmetric argument implies $P^n_{jk} \leq P^n_{ik} + P(T > n)$. Hence, $|P^n_{ik} - P^n_{jk}| \leq P(T > n) \to_{n\to\infty} 0$ (T finite!), for all $i, j, k \in S$. So (3.4.2) $P^n_{ik} - P^n_{jk} \to 0$ as $n \to \infty$ for all $i, j, k \in S$. If $\lim_{n\to\infty} P^n_{ik}$ exists, then it is independent of i. We show the limit exists. We write $\Pi_k - P^n_{jk} = \Sigma_i\Pi_i(P^n_{ik} - P^n_{jk})$ (we can write this because Π is stationary (Π_k, P^n_{ik}) and a pmf (P^n_{jk}). For any finite set $F \in S, \Sigma_{i\in S}\Pi_i|P^n_{ik} - P^n_{jk}| \leq \Sigma_{i\in F}|P^n_{ik} - P^n_{jk}| + 2\Sigma_{i\notin F}\Pi_i$. As $n \to \infty$, this converges to $2\Sigma_{i\notin F}\Pi_i$. This converges to 0 as $F \uparrow S$. So (3.4.3) $\Pi_k - P^n_{jk} = \Sigma_i\Pi_i(P^n_{ik} - P^n_{jk}) \to_{n\to\infty} 0$. Read in notes about when X_n is null recurrent, 185-187.

More general version of theorem 3.4.1 (no proof given) that drops irreducibility. Theorem 3.4.6 - For any aperiodic state j of a Markov chain, $P^n_{jj} \to \frac{1}{\mu_j}$ as $n \to \infty$. If i is any other state, $P^n_{ij} \to \frac{f_{ij}}{\mu_j}$ as $n \to \infty$. More to the theorem in notes.

§3.5 Reversibility

Some physical situations have the property that observations of the system taken at some times look the same if time runs forward or backward. Let X_n be a Markov chain, $\{X_n, 0 \leq n \leq N\}$ irreducible, positive recurrent Markov chain, prob transition matrix P and stationary distribution Π.

Definition 3.5.1 - The reversed chain or time reversal Y_n is $Y_n = X_{N-n}, 0 \leq n \leq N$.

Theorem 3.5.1 - Y_n is a Markov chain with $P(Y_{n+1} = j|Y_n = i) = \frac{\Pi_j}{\Pi_i}P_{ji}$. Proof: $P(Y_{n+1} = i_{n+1}|Y_n = i_n, \ldots, Y_0 = i_0) = \frac{P(Y_k=i_j, 0\leq k\leq n+1)}{P(Y_k=i_k, 0\leq k\leq n)} = \frac{P(X_{N-n-1}=i_{n+1}, X_{N-n}=i_n, \ldots, X_N=i_0)}{P(X_{N-n}=i_n, \ldots, X_N=i_0)} = \frac{\Pi_{i_{n+1}}P_{i_{n+1},i_n}P_{i_n,i_{n-1}}, \ldots, P_{i_1,i_0}}{\ldots} = \frac{\Pi_{i_{n+1}}P_{in+1,i_n}}{\Pi_{i_n}}$.

Definition 3.5.2 - The chain is reversible if the probability transition matricies of X_n and its time reversal Y_n are the same, (3.5.1) $\Pi_iP_{ij} = \Pi_jP_{ji}$ for all i, j. (3.5.1) are called the detailed balance equations. A transition matrix P and a probability distribution λ are in detailed balance if $\lambda_iP_{ij} = \lambda_jP_{ji}$ for all i, j. An irreducible chain X_n with a stationary distribution Π is reversible in equilibrium if its probability transition matrix is in detailed balance with Π.

Theorem 3.5.1 - Let P be the probability transition matrix of an irreducible chain X_n and suppose there is a distribution Π with $\Pi_iP_{ij} = \Pi_jP_{ji}$ for all $i, j \in S$. Then Π is the stationary distribution of X_n and X_n is reversible in equilibrium. Proof: $\Sigma_i\Pi_iP_{ij} = \Sigma_i\Pi_jP_{ji} = \Pi_j\Sigma_iP_{ji} = \Pi_j$ or $\Pi = \Pi P$.