ECE 514, Fall 2008

Exam 3: Due 2pm at ECE front desk, December 5, 2008

Name:________________________ 75 mins.; Total 50 pts.

1. (12 pts.) Suppose that X represents the amount of money deposited into a bank account, and Y represents the amount of interest earned after a year. We don't have direct access to X and Y individually, but we can see the total account balance after a year, Z (so that $Z = X + Y$). We wish to estimate the vector $[X, Y]'$ from Z (assuming X and Y are real-valued). Assume that X and Y are independent.

a. Suppse that X and Y are exponentially distributed with mean 1 and $1/2$, respectively. Find the LMMSE estimator. (The answer is a function of Z.)

b. Suppose now that X and Y are Gaussian random variables, where the means are 1 and $1/2$, respectively, and the variances are 1 and $1/4$, respectively. Find the MMSE (conditional mean) estimator. (The answer is a function of Z.)

c. Why can't we apply the ML rule studied in class to parts a or b?

2. (12 pts.) Let $X = [X_1, X_2]'$ represent a random vector, taking values in $\mathbb{R}^2$, with zero mean and correlation matrix R_X. We wish to design a storage system that only stores a single real random variable Y such that (1) $Y = v'X = v_1X_1 + v_2X_2$, (2) the variance of Y is 1, and (3) the reconstruction $\hat{X} = uY$ is such that the mean-squared error $\mathsf{E}[\|\hat{X} - X\|^2]$ is minimized. (Both the vectors $v = [v_1, v_2]'$ and $u = [u_1, u_2]'$ need to be designed.)

a. First fix v and design u (in terms of v). In other words, given v, find the vector u such that $\mathsf{E}[\|\hat{X} - X\|^2]$ is minimized. State explicitly how your answer depends on v and R_X.

b. Having optimized u in terms of v in part a, now find the optimal v. State explicitly how your answer depends on R_X. You may assume that the eigenvalues of R_X are distinct.

3. (13 pts.) Consider the square-wave function $s(t)$, $t \in \mathbb{R}$, taking values in $\{1, -1\}$, such that $s(t) = 1$ for $t \in [0, 1)$, $s(t) = -1$ for $t \in [1, 2)$, and so on. Note that $s(t)$ is periodic with period 2. Let Θ be a random variable uniformly distributed on $[0, 1)$, and consider the random process $X_t = s(t - \Theta)$.

a. For each fixed τ, does $\mathsf{E}[X_t X_{t+\tau}]$ depend on t?

b. For each fixed t, is the function $R(\tau) = \mathsf{E}[X_t X_{t+\tau}]$ periodic (as a function of τ)? If it is, what is its period?

c. Compute $R(\tau)$ for $\tau \in [0, 1)$.

d. Is $\{X_t\}$ WSS? Justify fully.

4. (13 pts.) Consider a continuous-time random process $X = \{X_t : t \in [0, T_X]\}$, where $T_X > 0$. We cannot directly observe X. Instead, we wish to estimate X based on the observed process $Y = \{Y_t : t \in [0, T_Y]\}$, where $T_Y > 0$. The estimator we wish to derive is the linear minimum mean-squared error (Wiener) filter. In other words, we wish to find a function $h : [0, T_X] \times [0, T_Y] \to \mathbb{R}$ (representing the Wiener filter) such that if we define

$$\hat{X}_t = \int_0^{T_Y} h(t, \tau) Y_\tau \, d\tau, \quad t \in [0, T_X],$$

then for all $\tilde{h} : [0, T_X] \times [0, T_Y] \to \mathbb{R}$ with

$$\tilde{X}_t = \int_0^{T_Y} \tilde{h}(t, \tau) Y_\tau \, d\tau, \quad t \in [0, T_X],$$

we have

$$\mathsf{E}\left[\int_0^{T_X} |X_t - \hat{X}_t|^2 \, dt\right] \leq \mathsf{E}\left[\int_0^{T_X} |X_t - \tilde{X}_t|^2 \, dt\right].$$

Assume that both X and Y are zero-mean (*not* necessarily WSS) and we know the autocorrelation functions R_X and R_Y and crosscorrelation function R_{XY}.

a. Write down the *orthogonality principle* for this problem, involving an equation with h analogous to equations (8.11) and (10.33).
   Hint: Think about what inner product is relevant here. For convenience, let M be the subspace of all processes $\tilde{X} = \{\tilde{X}_t : t \in [0, T_X]\}$ as defined above (linear functions of Y).

b. Prove the *orthogonality principle* for this problem: that the equation in part a is sufficient for h to be optimal.
   Hint: Study the arguments in Sections 8.4 and 10.8.

c. Use the equation in part a to derive an equation involving R_Y, R_{XY}, and h (the Wiener filter), analogous to equations (8.9) and (10.36).

d. Write down the value of the minimum mean-squared error $\mathsf{E}[\int_0^{T_X} |X_t - \hat{X}_t|^2 \, dt]$ in terms of h, R_X, and R_{XY}.
   Hint: Look at the equation in Example 8.15. You may directly write down the analog of this involving integrals.