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ST 521
HW 3

1.

$$P^n = \begin{pmatrix} 1 & 0 & 0 \\ (1/4)^n & (1/2)^n & (1/4)^n \\ (3/4)^n & 0 & (1/4)^n \end{pmatrix}$$

2. (a) $U_{10} = P_{10} + P_{11}U_{10} + P_{12}U_{20}$
   $U_{20} = P_{20} + P_{21}U_{10} + P_{22}U_{20}$
   or
   $U_{10} = 0.1 + 0.6U_{10} + 0.1U_{20}$
   $U_{20} = 0.2 + 0.3U_{10} + 0.4U_{20}$
   so
   $U_{10} = 8/21, U_{20} = 11/21$

   (b) $\nu_1 = 1 + .6\nu_1 + .1\nu_2$
   $\nu_2 = 1 + .3\nu_1 + .4\nu_2$

$$\begin{pmatrix} .4 & -.1 \\ -.3 & .6 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \nu_1 = \nu_2 = 10/3$$

3. $\nu_i = 1 + p\nu_{i+1} + q\nu_0, i = 0, 1, 2, 3$
   $\nu_4 = 0$
   $\nu_0 = 1 + q\nu_0 + p(1 + q\nu_0 + p(1 + q\nu_0 + p(1 + q\nu_0 + p(\nu_4)))) =$
   $1 + q\nu_0 + p + pq\nu_0 + p^2 + p^2q\nu_0 + p^3 + p^3q\nu_0 = 1 + q\nu_0 + p(1 + q\nu_0) + p^2(1 + q\nu_0) + p^3(1 + q\nu_0) =$
   $(1 + q\nu_0)(1 + p + p^2 + p^3) = \nu_0$

4. Probability P_n of having n boys:
   $P_0 = 1/4 + (3/4 * 1/8) = 11/32, P_1 = (3/4 * 3/8) = 9/32, P_2 = 9/32, P_3 = (3/4 * 1/8) = 3/32$
   $\mu = 9/32 + 2 * 9/32 + 3 * 3/32 = 36/32 > 1.$
   $\phi(s) = 11/32 + 9/32s + 9/32s^2 + 3/32s^3, \phi'(1) = \mu$, so the probability of ultimate extinction is $\frac{11}{32} + \frac{9t}{32} + \frac{9t^2}{32} + \frac{3t^3}{32} = t \Rightarrow 32(11 + 9t + 9t^2 + 3t^3) = 0$, which has solutions at $t = 2.38672, 0.306639 \pm 1.20094i$. But $\phi(s)$ is not convex, so does this analysis hold?

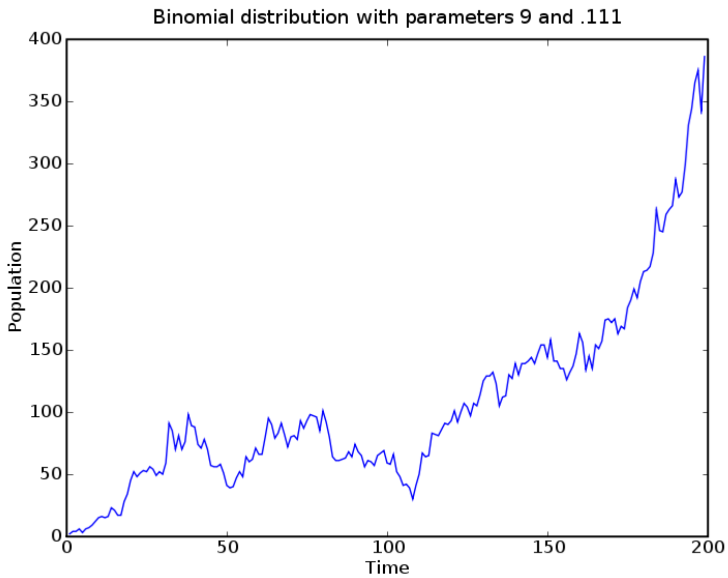
5. $p_k = (1 - c)c^k, 0 < c < 1, k = 0, 1, \ldots.$
   $\phi(t) = (1 - c)\Sigma_{k=1}^{\infty} c^{k-1}t^k = (1 - c)\frac{t}{1-ct}, \mu = |\phi'(1)| = \frac{1-c}{(1-c)^2} = \frac{1}{1-c} > 1.$
   $(1 - c)\frac{t}{1-ct} = t \Rightarrow (1 - c)t = t - ct^2 \Rightarrow -ct + ct^2 = 0 = c(t^2 - t) = ct(t - 1) \Rightarrow \eta = 0$

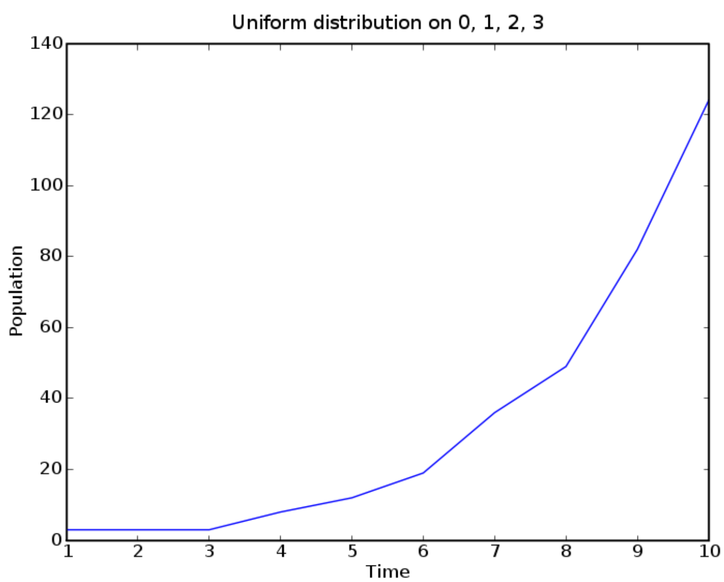
6. $p'(1) = .05 + 2 * .03 + 3 * .07 + 4 * .4 + 5 * .25 + 6 * .05 = 3.47 > 1.$
   $.15 - .95t + .03t^2 + .07t^3 + .4t^4 + .25t^5 + .05t^6 = 0 \Rightarrow \eta = 0.1592928$

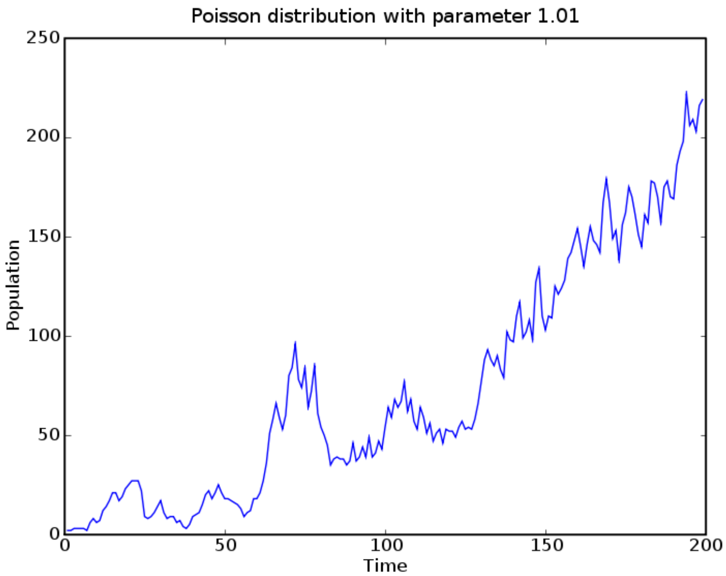
7. If $|\phi'(1)| = 2a + b < 1, \eta = 1$. If $|\phi'(1)| > 1, \eta$ is the smallest nonnegative value of $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ that is < 1.

8. (a) For $\mu \leq 1$ extinction occurs with probability 1, so W is finite.
   (b) $E(W_n) = \Sigma_{k=0}^{n} \mu^k$. If $\mu \geq 1$ then as $n \to \infty, E(W_n) \to \infty$. If $\mu < 1$ then as $n \to \infty, E(W_n) \to \frac{1}{1-\mu}$.

9.


Binomial distribution with parameters 9 and .111
Population
Time
0
50
100
150
200
250
300
350
400
0
50
100
150
200


Uniform distribution on 0, 1, 2, 3
Population
Time
0
20
40
60
80
100
120
140
1
2
3
4
5
6
7
8
9
10



Generated by some python (with appropriate changes for uniform and poisson):

```
from pylab import *
from random import *

def fac(n):
    if n <= 1:
        return 1
    else:
        return n * fac(n - 1)

def comb(n, k):
    return fac(n) / (fac(k) * fac(n - k))

def bernoulli(k, n, p):
    return comb(n, k) * p ** k * (1. - p) ** (n - k)

def poisson(k, l):
    return exp(-l) * l**k / fac(k)

prob = []

n = 9
p = .111
tot = 0
for i in range(n + 1):
    tot += bernoulli(i, n, p)
    prob.append(tot)

while True:
    t = []
    prev = 1
    s = range(1, 200)

    for i in s:
        cur = 0
        for j in range(prev):
            r = random()
            for k in range(len(prob)):
                if r < prob[k]:
                    cur += k
                    break

        t.append(cur)
        prev = cur

    if cur == 0:
        continue

    plot(s, t)
    xlabel('Time')
    ylabel('Population')
    title('Binomial distribution with parameters 9 and .111')
    savefig('binom')
    show()
    break
```