Definition 3.2.1

We say state i communicates with state j, written $i \to j$, if the chain may visit state j with positive probability, having started at state i. Equivalently, $i \to j$ if $P_{ij}^{m} > 0$ for some $m \geq 0$. If $i \to j$ and $j \to i$, we say i and j intercommunicate.

Note: if $i \to j$, $j \to i$ is possible but not certain.

Example 3.2.1

Consider the roulette wheel in Ex. 2.1.6. All nonzero states intercommunicate, and 0 only communicates with itself.

$\boxed{\#13 \quad 3/6}$

Recall defn
recurrent 115
transient
mean recurrence time 127
null/positive 128

Theorem 3.2.1

If $i \neq j$, then $i \to j \iff f_{ij} > 0$.

Proof

Exercise.

By definition, $i \leftrightarrow i$ since $P_{ii}^{0} = 1$.

In general, if we fix a state i, we can find all the states j that intercommunicate with i.

Definition 3.2.2

All the states that intercommunicate with a given state form a communication class. If i is in a communicating class, then i intercommunicates with all members of the class, so the members of the class all intercommunicate with each other.

Definition 3.2.3

An equivalence relation $\sim$ on a set S is an operation on pairs of set elements satisfying

(1) $a \sim a, \quad a \in S$
(2) $a \sim b \Rightarrow b \sim a, \quad a, b \in S$
(3) $a \sim b, b \sim c \Rightarrow a \sim c, \quad a, b, c \in S$

The remarks above show

Theorem 3.2.2

$\longleftrightarrow$ is an equivalence relation.

Example 3.2.2

In the roulette wheel Ex. 3.2.1, there are two communication classes, $\{0\}$ and $\{1, 2, \ldots, 38\}$.

Example 3.2.3

In the genotype example Ex 2.1.7, each state forms its own class, so there are three, $\{AA\}$, $\{Aa\}$, $\{aa\}$.

Example 3.2.4

In the ON/OFF system, Ex 2.2.3, with $0 < p < 1$, $0 < q < 1$, there is just one class, $\{ON, OFF\}$.

Theorem 3.2.3

If $i \leftrightarrow j$,

(1) i is transient if and only if j is transient

(2) i and j have the same period

(3) i is null recurrent if and only if j is null recurrent.

Proof

(1) If $i \leftrightarrow j$, there are $m, n \geq 0$ such that

$$\alpha = P_{ij}^{m} P_{ji}^{n} > 0.$$

By the Chapman-Kolmogorov equations (2.2.1)

$$P_{ii}^{m+r+n} \geq P_{ij}^{m} P_{jj}^{r} P_{ji}^{n} = \alpha P_{jj}^{r}$$

for any integer $r \geq 0$. Summing over r, we conclude that

$$\sum_{r=0}^{\infty} P_{ii}^{r} < \infty \Rightarrow \sum_{r=0}^{\infty} P_{jj}^{r} < \infty.$$

The argument holds with j and i reversed. By Theorem 3.1.2 we conclude (1) holds.

(2) is an exercise

(3) will be proved below.

Definition 3.2.4

A set $\mathcal{C}$ of states in the state space S is closed if $P_{ij} = 0$ for all $i \in \mathcal{C}$, $j \notin \mathcal{C}$. A closed set containing exactly one state is called absorbing.

Once a Markov chain takes a value in a closed set, it never leaves the set.

Example 3.2.5
Consider the Markov chain with state space $\{0,1,2\}$ and

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1/4 & 1/2 & 1/4 \\ 1 & 0 & 0 \end{pmatrix}$$

$\{0,2\}$ forms a closed set.

Definition 3.2.5

A set C of states in the state space S is <u>irreducible</u> if $i \leftrightarrow j$ for all $i, j \in C$.

The communication classes of a Markov chain are irreducible.

Because of Theorem 3.2.3, it makes sense to define

Definition 3.2.6

An irreducible set C is periodic, transient, or null recurrent if all the states in C have these properties.

Definition 3.2.7

If the entire state space is irreducible, we say the Markov chain is irreducible.

Theorem 3.2.4

In an irreducible Markov chain, either all states are transient or all states are recurrent.

We have the idea that some states are grouped together because they intercommunicate. Can all states be put into communication classes? Not quite, but we have

__Theorem 3.2.5__ __Decomposition Theorem__

The state space S can be partitioned uniquely as

$$S = T \cup C_1 \cup C_2 \cup C_3 \cup \cdots$$

where T is the set of transient states and $\{C_i\}$ are irreducible closed sets of recurrent states.

Consider starting a realization of the chain with value X_0. If X_0 is in some irreducible closed set of recurrent states C_i, the subsequent values will be in C_i and never leave. Hence we may as well consider the state space to be C_i.

If the initial value X_0 is transient, then the subsequent values are all transient, or the chain eventually takes a value in some C_k, from which it never leaves.

Proof

Let $\{C_i\}$ be the recurrent equivalence classes of $\leftrightarrow$. We only need to show each C_r is closed. Suppose on the contrary, there are $i \in C_r$, $j \notin C_r$ with $P_{ij} > 0$.

Now $j \not\to i$, hence

$$P(X_n \text{ never returns to } i) \geq P(X_n \text{ reaches } j \text{ which does not communicate with } i)$$

$$P(X_n \neq i \text{ for } n \geq 1 \mid X_0 = i) \geq P(X_1 = j \mid X_0 = i) > 0.$$

This is a contradiction of the assumption that i is recurrent.

Markov chains with finite state spaces are special in several ways. For example, it is not possible to remain in the transient states forever.

Theorem 3.2.6

If the state space S is finite, then at least one state is recurrent and all recurrent states are positive.

Proof

Assume all states are transient. We have

$$1 = \sum_{j \in S} P_{ij}^n$$

Since the sum is finite, we can take the limit as $n \to \infty$ through the sum,

$$1 = \lim_{n\to\infty} \sum_{j\in S} P_{ij}^n = \sum_{j\in S} \lim_{n\to\infty} P_{ij}^n = 0$$

by Thm 3.1.2 (3), yielding a contradiction.

The same argument works for the closed set of all null recurrent states, should this set be nonempty. (Exercise).

Every time a chain visits a transient state, there is a chance it will never return. In a finite state space, this can happen only if there is some other state that can be reached, but there is no path back.
In an infinite state space, there is enough "room" for states to be transient even if they communicate with each other.

This is a useful result.

Theorem 3.2.7

Suppose the state space is finite. A state i is transient if and only if there is another state j such that $i \to j$ but $j \not\to i$.

Proof

Exercise

Example 3.2.6

Consider a Markov chain with state space $\{0,1,2,3,4\}$ and

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/2 & 1/8 & 3/8 \end{pmatrix} = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$$

There are two classes $\overset{C_1}{\{0,1\}}$ and $\overset{C_2}{\{2,3,4\}}$ that are both closed. The two classes are both irreducible. Hence, they must contain recurrent positive states. Here $S = C_1 \cup C_2$.

Example 3.2.7

Consider a random walk with

$$P \quad \begin{array}{c} \\ \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ q & 0 & p & 0 & \cdots & \cdots & 0 \\ 0 & q & 0 & p & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ & & & & & & 0 \\ 0 & & \cdots & 0 & q & 0 & p \\ 0 & \cdots & \cdots & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \text{states} \\ 0 \\ 1 \\ 2 \\ \vdots \\ \\ \\ N \end{array}$$

where $q+p=1$, $0 \le q, p \le 1$. There are three classes $\{0\}$, $\{1,2,.., N-1\}$, $\{N\}$. We have

$$\{1,2,..,N-1\} \to \{0\}$$

$$\{1,2,...,N-1\} \to \{N\}$$

But, $\{0\} \not\to \{1,2,..,N-1\}$
$\{N\} \not\to \{1,2,..,N-1\}$

$\{0\}$ and $\{N\}$ are absorbing. Here, $T=\{1,2,.., N-1\}$, $C_1=\{0\}$, $C_2=\{N\}$.

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Example 3.2.8

Consider a Markov chain on $S=\{0,1,2,3,4,5\}$ with

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

$\{0,1\}$ and $\{4,5\}$ are irreducible and closed, therefore contain recurrent positive states.