Theorem 4.5.1 Chapman-Kolmogorov Equations

(4.5.8) $$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s)$$

This says that in order to move from state i to state j in $[t, t+s]$, $X(t)$ moves $i \xrightarrow[t]{} k \xrightarrow[s]{} j$ in some way.

We also need to give

Definition 4.5.2

The probability distribution for the initial state is

$$q_i = P(X(0) = i), \quad i = 0, 1, 2, \ldots$$

It follows

Theorem 4.5.2

(4.5.9) $$P(X(t) = n) = \sum_{i=0}^{\infty} q_i P_{in}(t).$$

We now calculate the distribution of the sojourn times, $\{S_i\}$,

S_i = sojourn time of $X(t)$ in state i.

We set

$$G_i(t) = P(S_i \geq t)$$

By the Markov property, as $h \downarrow 0$,

$$G_i(t+h) = G_i(t)\, G_i(h) = G_i(t)\left(P_{ii}(h) + o(h)\right)$$

$$= G_i(t)\left(1 - (\lambda_i + \mu_i)h\right) + o(h)$$

leading to

$$\frac{G_i(t+h) - G_i(t)}{h} = -(\lambda_i + \mu_i)\, G_i(t) + o(1)$$

and

(4.5.10) $\quad G_i'(t) = -(\lambda_i + \mu_i)\, G_i$

Since $G_i(0) = 1$, $\quad G_i(t) = e^{-(\lambda_i + \mu_i)t}$.

<u>Theorem 4.5.3</u>
The sojourn times are exponentially distributed with mean $(\lambda_i + \mu_i)^{-1}$.

To make this rigorous, we would have to show

$$G_i(h) = P_{ii}(h) + o(h)$$

and G is differentiable.

We use another fact that seems intuitively justified

Theorem 4.5.4

Given a transition occurs at time t, the probability that this transition is to state $i+1$ from i is $\lambda_i/(\lambda_i+\mu_i)$ and to state $i-1$, $\mu_i/(\lambda_i+\mu_i)$.

This gives a very nice description of a birth-death process. The process $X(t)$ sojourns in a given state i for a random length of time whose distribution function is an exponential distribution with parameter $(\lambda_i+\mu_i)$. When leaving state i, the process enters state $i+1$ or $i-1$ with probabilities $\lambda_i/(\lambda_i+\mu_i)$, $\mu_i/(\lambda_i+\mu_i)$ respectively.

This is like a random walk with randomly occurring transition times.

We use this viewpoint to compute realizations.

Assume $X(0) = i$, the particle spends a random length of time, exponentially distributed with parameter $(\lambda_i + \mu_i)$ in state i, then moves with probability $\lambda_i/(\lambda_i+\mu_i)$ to $i+1$ and $\mu_i/(\lambda_i+\mu_i)$ to $i-1$.

We choose a value t_i from the exponential distribution with parameter $(\lambda_i + \mu_i)$ for the sojourn time in i. Then we toss a coin with probability of heads,

$$p_i = \frac{\lambda_i}{\lambda_i+\mu_i},$$

If a head appears, we move to $i+1$, otherwise to $i-1$.

Continuing, if the particle enters $i+1$, we use the exp. distribution with parameter $(\lambda_i + \mu_i)$ to fix the sojourn time in that state. If it enters $i-1$, we use $\lambda_{i-1} + \mu_{i-1}$.

Thus, we construct a sample path by sampling from exponential and Bernoulli distributions.

The question is whether or not this produces a valid realization of the birth-death process. It turns out that we can assign a probability measure to the generated realizations in such a way that $P_{ij}(t)$ is determined satisfying (4.5.7)-(4.5.8).

This is important because there are actually several stochastic processes that satisfy (4.5.2)-(4.5.6), (4.5.7), and (4.5.8), i.e. several Markov processes with the same infinitesimal generator.

<u>Theorem 4.5.5</u>

For birth-death processes with $\lambda_0 > 0$, a sufficient condition that there is a unique Markov process with transition probability function $P_{ij}(t)$ satisfying (4.5.7), (4.5.8) is

$$(4.5.10) \quad \sum_{n=0}^{\infty} \frac{1}{\lambda_n \theta_n} \sum_{k=0}^{n} \theta_k = \infty$$

where

$$\theta_0 = 1, \; \theta_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}, \; n = 1, 2, 3, \ldots$$

As with pure birth processes, $P_{ij}(t)$ satisfies differential equations. Starting from (4.5.8)

$$P_{ij}(t+h) = P_{i\,i-1}(h)\,P_{i-1\,j}(t) + P_{ii}(h)\,P_{ij}(t)$$

(4.5.11) $$+\; P_{i\,i+1}(h)\,P_{i+1\,j}(t)$$

$$+ \sum_{k \neq i-1, i, i+1} P_{ik}(h)\,P_{kj}(t)$$

Using (4.5.2) - (4.5.4),

$$\sum_{k \neq i-1, i, i+1} P_{ik}(h)\,P_{kj}(t) \leq \sum_{k \neq i-1, i, i+1} P_{ik}(h)$$

$$\leq 1 - \left(P_{ii}(h) + P_{i\,i-1}(h) + P_{i\,i+1}(h)\right)$$

$$= 1 - \left(1 - (\lambda_i + \mu_i)h + \mu_i h + \lambda_i h + o(h)\right)$$

$$= o(h)$$

So

$$P_{ij}(t+h) = \mu_i h\,P_{i-1\,j}(t) + \left(1 - (\lambda_i + \mu_i)h\right)P_{ij}(t)$$

$$+\; \lambda_i h\,P_{i+1\,j}(t) \;+ o(h)$$

$$P_{ij}(t) - (\lambda_i + \mu_i)h\,P_{ij}(t)$$

Moving $P_{ij}(t)$ to the left, dividing by h, and letting $h \downarrow 0$, yields

$$P_{ij}'(t) = \mu_i P_{i-1j}(t) - (\lambda_i+\mu_i) P_{ij}(t) + \lambda_i P_{i+1j}(t)$$

Treating the special case of state 0, we obtain

Theorem 4.5.6

Under appropriate assumptions, $P_{ij}(t)$ satisfy the backward Kolmogorov differential equations

(4.5.12)
$$\begin{cases} P_{0j}'(t) = -\lambda_0 P_{0j}(t) + \lambda_0 P_{1j}(t) \\ P_{ij}'(t) = \mu_i P_{i-1j}(t) - (\lambda_i+\mu_i) P_{ij}(t) + \lambda_i P_{i+1j}(t), \ i \geq 1, \\ P_{ij}(0) = \delta_{ij} \end{cases}$$

To derive (4.5.12), we decomposed $(0, t+h)$ into $(0,h)$ and $(h, t+h)$ and examined the transitions in these intervals separately.

If instead we divide $(0,t+h)$ into $(0,t)$ and $(t,t+h)$ (switching t and h in (4.5.11)) and assume in addition

(4.5.13)
$$\frac{P_{kj}(h)}{h} = o(1) \qquad k \neq j, j-1, j+1$$

where the $O(1)$ term tends to 0 and is uniformly bounded with respect to k for fixed j as $h \to 0$, which allows us to prove

$$\sum_{k \neq i-1, i, i+1} P_{ik}(t) P_{kj}(h) = o(h),$$

then

Theorem 4.5.7

Under appropriate assumptions $P_{ij}(t)$ satisfy the forward Kolmogorov differential equations

(4.5.14)
$$\begin{cases} P_{i0}'(t) = -\lambda_0 P_{i0}(t) + \mu_1 P_{i1}(t) \\ P_{ij}'(t) = \lambda_{j-1} P_{ij-1}(t) - (\lambda_j + \mu_j) P_{ij}(t) + \mu_{j+1} P_{ij+1}(t) \quad j \geq 1 \\ P_{ij}(0) = \delta_{ij} \end{cases}$$

Example 4.5.1

A birth-death process is a linear growth process if

$$\lambda_n = \lambda \cdot n + a, \quad \lambda > 0 \text{ fixed}$$

$$\mu_n = \mu \cdot n, \quad \mu > 0 \text{ fixed}$$

The rates grow proportionally with the population, and we also allow for an infinitesimal rate of increase due to a constant "immigration" source as determined by a.

We obtain

$$P_{i0}'(t) = -a P_{i0}(t) + \mu P_{i1}(t)$$

$$P_{ij}'(t) = (\lambda(j-1) + a) P_{ij-1}(t)$$

$$- ((\lambda+\mu) j + a) P_{ij}(t)$$

$$+ \mu (j+1) P_{ij+1}(t) \quad , j \geq 1$$

If we multiply by j and sum, the expected value

$$E(X(t)) = M(t) = \sum_{j=1}^{\infty} j P_{ij}(t)$$

satisfies

$$(4.5.15) \quad \begin{cases} M'(t) = a + (\lambda-\mu) M(t) \\ M(0) = i, \end{cases}$$

when $X(0) = i$. The solution is

$$M(t) = at + i, \quad \lambda = \mu, \tag{4.5.16}$$

$$M(t) = \frac{a}{\lambda-\mu}\left(e^{(\lambda-\mu)t} - 1\right) + i e^{(\lambda-\mu)t}, \lambda \neq \mu \tag{4.5.17}$$

We can derive an equation for the variance similarly.

It is interesting to note that

$$M(t) \longrightarrow \begin{cases} \infty & ; \lambda \geq \mu \\ \dfrac{a}{\mu-\lambda}, & \lambda < \mu \end{cases}$$

This indicates there is a limiting distribution if $\lambda < \mu$.

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Example 4.5.2

Consider a Markov process on $S = \{0,1\}$ where the infinitesimal matrix is

$$A = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}.$$

The process alternates between 0 and 1 with the sojourn times in 0 i.i.d. with exponential distribution with parameter α