EE 514, Fall 2006
Exam 4: Due ECE front desk, 1:45pm, December 6, 2006

Solutions (version: December 6, 2006, 20:30) 75 mins.; Total 50 pts.

1. (14 pts.) Let $X_t = A\cos(\omega t + \Theta)$ ($t \in \mathbb{R}$), where $\omega > 0$ is constant, A is a random variable with exponential distribution, Θ is a uniform random variable on $(-\pi, \pi]$, and A and Θ are independent.

a. Determine if the process is wide-sense stationary (and if so, find its correlation function). Provide a complete argument.

b. Determine if the process is strictly stationary. Provide a complete argument.

Ans.: a. First, by independence, we calculate

$$\mathsf{E}[X_t] = \mathsf{E}[A]\mathsf{E}[\cos(\omega t + \Theta)].$$

Using the argument in Example 10.8, we deduce that $\mathsf{E}[X_t] = 0$. Next, again by independence, we calculate

$$R_X(t_1, t_2) = \mathsf{E}[A^2]\mathsf{E}[\cos(\omega t_1 + \Theta)\cos(\omega t_2 + \Theta)].$$

Again using the argument in Example 10.8, we find that

$$R_X(t_1, t_2) = \frac{\mathsf{E}[A^2]}{2}\cos(\omega(t_1 - t_2)).$$

Hence, the process is WSS.

b. Yes, indeed the process is strictly stationary. The basic idea here is that time-shifting the process by Δ (i.e., replacing t by $t + \Delta$) is is equivalent to using a different random phase shift: $\Theta + \omega\Delta$. But because Θ is uniform on $(-\pi, \pi]$, $\Theta + \omega\Delta$ is uniform on $(-\pi + \omega\Delta, \pi + \omega\Delta]$ (the width of this duration is 2π, regardless of Δ). As a result, the law of $X_{t+\Delta}$ is the same as that of X_t.

To show this rigorously, consider a finite set of times $t_1 < t_2 < \cdots < t_n$ and the associated finite-dimensional distribution function $F_{X_{t_1},\ldots,X_{t_n}}(x_1, \ldots, x_n)$. Now,

$$\begin{aligned}
F_{X_{t_1},\ldots,X_{t_n}}(x_1, \ldots, x_n) &= \mathsf{P}\{X_{t_1} \le x_1, \ldots, X_{t_n} \le x_n\} \\
&= \mathsf{P}\{A\cos(\omega t_1 + \Theta) \le x_1, \ldots, A\cos(\omega t_n + \Theta) \le x_n\} \\
&= \mathsf{E}[\mathsf{P}\{A\cos(\omega t_1 + \Theta) \le x_1, \ldots, A\cos(\omega t_n + \Theta) \le x_n | A\}] \\
&= \mathsf{E}\left[\int_{-\pi}^{\pi} I_C(\theta)\frac{1}{2\pi}d\theta\right],
\end{aligned}$$

where

$$C = \{\theta : A\cos(\omega t_1 + \theta) \le x_1, \ldots, A\cos(\omega t_n + \theta) \le x_n\}$$

and I_C is the indicator function of C. Notice that I_C is a random function (because A appears in its definition), but its sample paths are are periodic with period 2π.

Next, consider the shift Δ, and the associated distribution function

$$\begin{aligned}
&F_{X_{t_1+\Delta},\ldots,X_{t_n+\Delta}}(x_1,\ldots,x_n) \\
&\quad= \mathsf{P}\{X_{t_1+\Delta} \le x_1,\ldots,X_{t_n+\Delta} \le x_n\} \\
&\quad= \mathsf{P}\{A\cos(\omega(t_1+\Delta)+\Theta) \le x_1,\ldots,A\cos(\omega(t_n+\Delta)+\Theta) \le x_n\} \\
&\quad= \mathsf{E}[\mathsf{P}\{A\cos(\omega(t_1+\Delta)+\Theta) \le x_1,\ldots,A\cos(\omega(t_n+\Delta)+\Theta) \le x_n|A\}] \\
&\quad= \mathsf{E}\left[\int_{-\pi}^{\pi} I_D(\theta)\frac{1}{2\pi}d\theta\right],
\end{aligned}$$

where

$$\begin{aligned}
D &= \{\theta : A\cos(\omega(t_1+\Delta)+\theta) \le x_1,\ldots,A\cos(\omega(t_n+\Delta)+\theta) \le x_n\} \\
&= \{\theta : A\cos(\omega t_1+(\omega\Delta+\theta)) \le x_1,\ldots,A\cos(\omega t_n+(\omega\Delta+\theta)) \le x_n\}.
\end{aligned}$$

Note that $I_D(\theta) = I_C(\omega\Delta+\theta)$. Hence, substituting the variable $\tau = \omega\Delta+\theta$ in the integration, we get

$$F_{X_{t_1+\Delta},\ldots,X_{t_n+\Delta}}(x_1,\ldots,x_n) = \mathsf{E}\left[\int_{-\pi+\omega\Delta}^{\pi+\omega\Delta} I_C(\tau)\frac{1}{2\pi}d\tau\right] = F_{X_{t_1},\ldots,X_{t_n}}(x_1,\ldots,x_n),$$

because the integrand above is periodic and the integral is over one period. This shows that the process is strictly stationary.

2. (20 pts.) Recall our discussion of the matched filter. We have the sum of a signal $v(t)$ $(t \in \mathbb{R})$ with a wide-sense stationary process X_t, and we pass this resulting process through an LTI filter with impulse response h. We then sample the output $v_o(t) + Y_t$ of the filter at time 0. The SNR is then given by $v_o(0)^2/E[Y_0^2]$.

Given S_X, find a signal v and an impulse response h such that the SNR is maximized, subject to the constraint that v satisfies $\int_{-\infty}^{\infty} |V(f)|^4\, df \le 1$. Note that, unlike in the standard matched filter problem, our problem here is to design *both* v and h. Whatever the design, it will depend on S_X. For technical reasons, assume that $\int_{-\infty}^{\infty} 1/S_X(f)^p\, df < \infty$ for $p = 1, 2, 3$ and identify when you rely on this assumption.

You may give your answer in terms of $V(f)$ and $H(f)$, which are the Fourier transforms of v and h, respectively.

Ans.: We know that for any v, the h that maximizes the SNR is given by the formula

$$H(f) = \alpha\frac{V(f)^*}{S_X(f)}.$$

The SNR in this case is

$$SNR = \int_{-\infty}^{\infty} \frac{|V(f)|^2}{S_X(f)}\, df.$$

By the Cauchy-Schwarz inequality,

$$SNR^2 \le \int_{-\infty}^{\infty} |V(f)|^4\, df \int_{-\infty}^{\infty} 1/S_X(f)^2\, df.$$

(Because of our assumptions on S_X and the constraint on V, the right-hand side is finite.) So, to maximize the SNR, we deduce that the optimal choice for V satisfies

$$|V(f)|^2 = \beta \frac{1}{S_X(f)}.$$

To find β, we note that the SNR is linear in β, so we need to maximize β. This is given by the value of β for which: $\int_{-\infty}^{\infty} |V(f)|^4 \, df = 1$. Hence, β is given by

$$\beta^2 = 1 / \int_{-\infty}^{\infty} 1/S_X(f)^2 \, df.$$

So a particular choice for $V(f)$ is

$$V(f) = \frac{\sqrt{\beta}}{\sqrt{S_X(f)}}.$$

(Note that because of our assumption on S_X, this $V(f)$ is square integrable, and is therefore legitimately the Fourier transform of some signal v.) The corresponding optimal H (ignoring scaling constants) is

$$H(f) = \frac{1}{S_X(f)^{3/2}}.$$

(Again, because of our assumption on S_X, this $H(f)$ is square integrable, and is therefore legitimately the Fourier transform of some signal h.)

3. (8 pts.) Let b be a random variable uniform on $\{-1, 1\}$, representing a single bit. Next, let $\{C_t\}$ represent a wide-sense stationary zero-mean random process with power spectral density S_C. Consider a communication system in which we communicate the bit b as follows: the transmitter transmits the signal bC_t over time and the receiver receives a noise-corrupted version: $bC_t + N_t$ ($t \in \mathbb{R}$), where N_t represents noise, and is wide-sense stationary with zero mean and power-spectral density S_N, and is independent of b and $\{C_t\}$. We filter this received signal using an LTI filter with transfer function H. Write down a formula for H (in terms of S_C and S_N) such that if Y_t is the output of the filter, then the following is minimized for each fixed t: $\mathsf{E}[(Y_t - bC_t)^2]$.

Ans.: Note that the process $V_t = bC_t$ is WSS. Hence, the received signal $U_t = bC_t + N_t$ is WSS also. Next, because bC_t and N_t are independent, U_t and V_t are J-WSS. Hence, the solution to this problem is the Wiener filter. To find this filter, we first write down (using problem 10.59) $R_{VU} = R_V = R_C$ and $R_U = R_V + R_N = R_C + R_N$. Hence, the filter is given by

$$H = \frac{S_C}{S_C + S_N}.$$

4. (8 pts.) Give an example of a discrete-time random process $\{X_n : n = 1, 2, \ldots\}$ such that $\mathsf{E}[X_n]$ does not depend on n (i.e., the mean sequence is constant over n), $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i$ exists almost surely, but, with nonzero probability,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i \neq \mathsf{E}[X_1].$$

Ans.: Let Y be a random variable such that the mean $\mathsf{E}[Y]$ exists, and $\mathsf{P}\{Y \neq \mathsf{E}[Y]\} > 0$ (e.g., Bernoulli(0.5)). Define the process $\{X_n\}$ by $X_n = Y$ for all n. Hence, $\mathsf{E}[X_n] = \mathsf{E}[Y]$ does not depend on n. Moreover, $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i = Y$, and hence the limit exists a.s. However, by assumption on Y, with nonzero probability we have

$$\mathsf{E}[X_n] = \mathsf{E}[Y] \neq Y = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i.$$