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ST 521
HW 5

1. (a) $S = \{0, 1, 2, 3, 4\}$

$$P = \begin{pmatrix} 0 & 1/3 & 2/3 & 0 & 0 \\ 1/2 & 0 & 1/6 & 1/3 & 0 \\ 0 & 1/2 & 0 & 1/6 & 1/3 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(b) Write down our initial equations:

$$\begin{aligned} \Pi_0 &= \tfrac{1}{2}\Pi_1 \\ \Pi_1 &= \tfrac{1}{3}\Pi_0 + \tfrac{1}{2}\Pi_2 \\ \Pi_2 &= \tfrac{2}{3}\Pi_0 + \tfrac{1}{6}\Pi_1 + \tfrac{3}{4}\Pi_3 \\ \Pi_3 &= \tfrac{1}{3}\Pi_1 + \tfrac{1}{6}\Pi_2 + \Pi_4 \\ \Pi_4 &= \tfrac{1}{3}\Pi_2 + \tfrac{1}{4}\Pi_3 \end{aligned}$$

Write everything in terms of Π_0:

$$\begin{aligned} \Pi_0 &= \Pi_0 \\ \Pi_1 &= 2\Pi_0 \\ \Pi_2 &= \tfrac{10}{3}\Pi_0 \\ \Pi_3 &= \tfrac{28}{9}\Pi_0 \\ \Pi_4 &= \tfrac{17}{9}\Pi_0 \end{aligned}$$

Double checking with $\Pi_4 = \frac{1}{3}\Pi_2 + \frac{1}{4}\Pi_3$ yields $\frac{17}{9}$ on both sides, so things look correct so far. Now sum everything and solve: $\Pi_0 + 2\Pi_0 + \frac{10}{3}\Pi_0 + \frac{28}{9}\Pi_0 + \frac{17}{9}\Pi_0 = 1 \Rightarrow \frac{34}{3}\Pi_0 = 1 \Rightarrow \Pi_0 = \frac{3}{34}$. So, $\Pi = (3/34, 3/17, 5/17, 14/51, 1/6)$.

(c) There is a 3/34 chance she will find it empty and a 1/6 chance she will find it full.

2. (a) For states 0, 1, 2, 3 the smallest number of steps is 4. For states 4, 5 the smallest number of steps is 6.

(b) $d(i) = 2$ for all i.

(c)

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So:

$$\begin{aligned} \Pi_0 &= \tfrac{1}{2}\Pi_3 + \Pi_5 \\ \Pi_1 &= \Pi_0 \\ \Pi_2 &= \Pi_1 \\ \Pi_3 &= \Pi_2 \\ \Pi_4 &= \tfrac{1}{2}\Pi_3 \\ \Pi_5 &= \Pi_4 \end{aligned}$$

Write things in terms of Π_0:

$$\begin{aligned} \Pi_0 &= \Pi_1 = \Pi_2 = \Pi_3 \\ \Pi_4 &= \Pi_5 = \tfrac{1}{2}\Pi_0 \end{aligned}$$

Hence $5\Pi_0 = 1 \Rightarrow \Pi_0 = 1/5$. Thus $\Pi = (1/5, 1/5, 1/5, 1/5, 1/10, 1/10)$.

3. (a) All states are recurrent, so $\mu_{10} = \Sigma_{n=1}^{\infty} nP(X_1 \neq 10, \dots X_{n-1} \neq 10, X_n = 10 | X_0 = 10)$. For $n \leq 10$, the probability is zero. For $n > 10, f_{10,10}(n) = (\frac{1}{2})^n$. So we get $\Sigma_{n=11}^{\infty} n(\frac{1}{2})^n$. The summation approaches 0.01171875, which doesn't make sense, so I probably did something wrong.

   (b) The expected number of times the chain visits state 9 before it is back to state 10 is 2. Since every time we hit state 9 we have a 1/2 chance of moving to state 10, it follows that, on average, each visit to state 10 has been proceeded by 2 visits to state 9.

4. η_j is a recursive definition. For the terminating case, $\eta_j = 1$, if $X_0 = j$ and $j \in A$, then $X_n \in A$ for $n = 0$, hence $T_A = 0$, and so $T_A < \infty$. For the recursive case, we multiply each successive probability of proceeding to the next state until we hit a state in A, then (when we are in a state in A) we drop out a 1 from the terminating case. So, if $T_A = \infty$, there must be no way to get to a state in A from state j, hence the p_{jk} terms would keep piling up, making that term negligibly close to 0, thus T_A would approach infinity, so $P(T_A < \infty | X_0 = j)$ approaches 0 as well.