since we can look for a stationary distribution.

We can do something similar for detecting transience.

Theorem 3.3.8 $\boxed{\#17 \quad 3/27}$

Let $s \in S$ be a state of an irreducible chain. The chain is transient if and only if there is a nonzero solution $\{y_j, j \in S\}$ of the equations

$$(3.3.5) \quad y_i = \sum_{\substack{j \in S \\ j \neq s}} P_{ij} y_j \quad , \quad i \neq s,$$

with $|y_j| \le 1$ for all j.

Proof

The chain is transient if and only if s is transient, so suppose s is transient. Define

$$(3.3.6) \quad \tau_i(n) = P(\text{no visit to } s \text{ in the first } n \text{ steps} \mid X_0 = i)$$

$$= P(X_m \neq s,\ 1 \le m \le n \mid X_0 = i)$$

Then,

$$\tau_i(1) = \sum_{j \neq s} P_{ij} \qquad (X_1 \neq s)$$

and
$$\tau_i(n+1) = \sum_{j \neq s} P_{ij}\, \tau_i(n)$$

Furthermore, $\tau_i(n) \geq \tau_i(n+1)$, so

$$\tau_i = \lim_{n \to \infty} \tau_i(n) = P(\underset{\sim}{\text{no}} \text{ visit to } s \text{ ever} \mid X_0 = i)$$

$$= 1 - f_{is}$$

(" prob. the chain ever visits j from i)

satisfies (3.3.5) (Exercise: prove this). Also,

$\tau_i > 0$ for some i, since otherwise $f_{is} = 1$ for

all $i \neq s$ and this would imply

$$f_{ss} = \underset{(X_1 = s)}{P_{ss}} + \sum_{i \neq s} \underset{(X_1 = i \neq s)}{P_{si} f_{is}} = \sum_i P_{si} = 1,$$

by conditioning on X_1, which contradicts the transience of s.

Conversely, let Y satisfy (3.3.5) with $|Y_i| \leq 1$. Then,

$$|Y_i| \leq \sum_{j \neq s} P_{ij} |Y_j| \leq \sum_{j \neq s} P_{ij} = \tau_i(1)$$

which implies

$$|Y_i| \le \sum_{j \ne s} P_{ij} \tau_j(1) = \tau_i(2),$$

and so on. So, $|Y_i| \le \tau_i(n)$ for all n. Let $n \to \infty$ to show that $\tau_i = \lim_{n \to \infty} \tau_i(n) > 0$ for some i, which implies S is transient (Exercise).

It follows directly

Theorem 3.3.9

An irreducible chain is recurrent if and only if the only bounded solution of (3.3.5) is the zero solution.

Example 3.3.7

Consider the gambler's ruin problem Ex. 3.1.12 with

$$P = \begin{pmatrix} q & p & 0 & \cdots & & \\ q & 0 & p & 0 & \cdots & \\ 0 & q & 0 & p & 0 & \cdots \\ 0 & 0 & q & 0 & p & 0 \cdots \\ & & & \ddots & \ddots & \ddots \\ & 0 & & & & \end{pmatrix} \qquad q + p = 1$$

Set $\gamma = p/q$.

(1) If $q < p$, choose $S = 0$ to test Thm 3.3.8.
The equations (3.3.5) read

$$y_0 = P_{01} y_1 = p\, y_1$$
$$y_1 = P_{02} y_2 = p\, y_2$$
$$y_2 = q\, y_1 + p\, y_3$$
$$\vdots$$

If we set $y_j = 1 - \gamma^{-j}$, then y solves these equations, so the chain is transient.

(2) We can solve the equation $\pi = \pi P$ to see there is a stationary distribution with $\pi_j = \gamma^j (1-\gamma)$ if and only if $q > p$. The chain is positive recurrent if and only if $q > p$.

Example 3.3.8

Consider the discrete queuing example Ex. 2.1.10 with

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$$P = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & & \\ a_0 & a_1 & a_2 & \cdots & & \\ 0 & a_0 & a_1 & a_2 & \cdots & \\ 0 & 0 & a_0 & a_1 & a_2 & \cdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots \end{pmatrix},$$

where $a_i \geq 0$, $\sum_i a_i = 1$, and $S = \{0, 1, 2, \ldots\}$. The system $\pi = \pi P$ gives

$$\begin{aligned}
\pi_0 &= \pi_0 a_0 + \pi_1 a_0 \\
\pi_1 &= \pi_0 a_1 + \pi_1 a_1 + \pi_2 a_0 \\
\pi_2 &= \pi_0 a_2 + \pi_1 a_2 + \pi_2 a_1 + \pi_3 a_0 \\
\pi_3 &= \pi_0 a_3 + \pi_1 a_3 + \pi_2 a_2 + \pi_3 a_1 + \pi_4 a_0 \\
&\vdots
\end{aligned}$$

The i^{th} column of P is $\begin{pmatrix} a_i \\ a_{i-1} \\ \vdots \\ a_0 \\ 0 \\ \vdots \end{pmatrix}$, so for $i \geq 0$,

$$(3.3.7) \qquad \pi_i = a_0 \pi_{i+1} + \sum_{j=1}^{i+1} \pi_j a_{i+1-j}$$

We can solve (3.3.7) using generating functions.

Set

$$\pi(t) = \sum_{i=0}^{\infty} \pi_i t^i$$

We multiply (3.3.7) by t^i and sum

(3.3.8) $$\Pi(t) = \sum_{i=0}^{\infty} \pi_i t^i = \pi_0 \sum_{i=0}^{\infty} a_i t^i + \sum_{i=0}^{\infty} \sum_{j=1}^{i+1} \pi_j a_{i+1-j} t^i.$$

We switch the sums in the last term. Note, $1 \le j \le i+1$ implies $i \ge j-1$ and $j \ge 1$. If we set $A(t) = \sum_{i=0}^{\infty} a_i t^i$ the right-hand side of (3.3.8) is

$$\pi_0 A(t) + \sum_{j=1}^{\infty} \pi_j t^{j-1} \sum_{i=j-1}^{\infty} a_{i-j+1} t^{i-j+1}$$

$$= \pi_0 A(t) + t^{-1} \left(\sum_{j=1}^{\infty} \pi_j t^j \right) A(t)$$

$$= \pi_0 A(t) + t^{-1} (\Pi(t) - \pi_0) A(t),$$

or

$$\Pi(t) = \pi_0 A(t)(1-t^{-1}) + t^{-1} \Pi(t) A(t).$$

Solving

$$\Pi(t) = \pi_0 A(t) (1-t^{-1}) / (1-t^{-1} A(t))$$

$$= \frac{\pi_0 A(t)}{\frac{(1-t^{-1} A(t))}{(1-t^{-1})}} = \frac{\pi_0 A(t)}{\frac{1-t^{-1}+t^{-1}-t^{-1}A(t)}{1-t^{-1}}}$$

or

$$(3.3.9) \qquad \Pi(t) = \frac{\Pi_0 A(t)}{1 - \frac{1-A(t)}{1-t}}.$$

Now the question is when is it possible to specify Π_0 so $\Pi(1) = \sum_{k=0}^{\infty} \Pi_k = 1$? This would imply a stationary distribution exists.

In (3.3.8), we let $t \uparrow 1$ to get

$$\Pi(1) = \sum_{k=0}^{\infty} \Pi_k .$$

We let

$$\lim_{t \uparrow 1} \frac{1-A(t)}{1-t} = A'(1) = \gamma = \sum_{k=0}^{\infty} k\, a_k$$

be the mean number of arrivals per service interval. (Since $\{a_k\}$ is a probability distribution, $A(1) = 1$, which is crucial to the limit above.) Hence, taking $t \uparrow 1$ on the right in (3.3.9) yields

$$\frac{\Pi_0 A(1)}{1 - \lim_{\uparrow 1} \frac{1-A(t)}{1-t}} = \frac{\Pi_0}{1-\gamma}.$$

We can choose π_0 so $\pi(1) = \sum_k \pi_k = 1$ if and only if $0 < \gamma < 1$, and then $\pi_0 = 1 - \gamma$.

We conclude the queuing chain is positive recurrent if and only if $\gamma = \sum_{k=0}^{\infty} k a_k < 1$, which says that the mean number of arrivals does not overwhelm the facility.

We next consider the case $\gamma > 1$. We show that when $\gamma > 1$, (3.3.5) has a nonzero solution y with $0 \le y_i \le 1$ for all i. We choose $s = 0$, so (3.3.5) reads

$$(3.3.10) \quad \begin{cases} y_1 = \sum_{i=1}^{\infty} a_i y_i \\ y_2 = a_0 y_1 + a_1 y_2 + \cdots \\ y_3 = a_0 y_2 + a_1 y_3 + \cdots \\ \vdots \\ y_n = \sum_{i=0}^{\infty} a_i y_{i+n-1} \qquad , n \ge 2 \end{cases}$$

Thinking of branching processes, we try $y_i = 1 - t^i$, $0 < t < 1$. From the equation for y_n,

$$1 - t^n = \sum_{i=0}^{\infty} a_i (1 - t^{i+n-1})$$

$$= 1 - \left(\sum_{i=0}^{\infty} a_i t^i\right) t^{n-1}$$

With $A(t) = \sum_{i=0}^{\infty} a_i t^i$,

$$t^n = A(t)\, t^{n-1}$$

or

$$t = A(t).$$

From the branching process analysis, we know that $t = A(t)$ has a solution with $0 < t < 1$ when $\gamma > 1$. Hence, $\gamma > 1$ implies the chain is transient.

We next argue that if the chain is transient then (3.3.10) has a nonzero solution.

If the chain is transient, then for each $j = 0, 1, 2, \ldots$, there is a last visit. Hence, there

is a last visit to any finite set $\{0,1,2,\dots,M\}$.
This is implies there is an $n_0 = n_0(M)$ such that for $n \geq n_0$, $X_n > M$. So, $X_n \to \infty$ as $n \to \infty$.

If we define C_{n+1} to be the number of arrivals in the service period $(n, n+1)$, then

$$P(C_{n+1} = k) = a_k$$

$$E(C_{n+1}) = \rho$$

$$X_{n+1} = \max\{X_n - 1, 0\} + C_{n+1}$$

For $n \geq n_0$,

$$X_{n+1} = (X_n - 1) + C_{n+1}$$

we have a sufficiently large population, that emptying the queue can't happen!

and if $N \geq n_0$,

$$\sum_{n=n_0}^{N} (X_{n+1} - X_n) = -(N - n_0) + \sum_{n=n_0}^{N} C_{n+1}$$

This gives

$$X_{N+1} - X_{n_0} = -(N - n_0) + \sum_{n=n_0+1}^{N+1} C_n$$

and

$$\mathbb{X}_{n+1} - \sum_{n=1}^{N+1} (C_n - 1) = \mathbb{X}_{n_0} + n_0 - \sum_{n=1}^{n_0} C_n .$$

(verify)

The right-hand side is constant, so $\mathbb{X}_{N+1} \to \infty$ implies

$$\sum_{n=1}^{N+1} (C_n - 1) \to \infty$$

It is an exercise to show that a sum of i.i.d. random variables with mean μ converges to $+\infty$ $\Leftrightarrow$ $\mu > 0$ or $\rho > 1$.

§3.4 Limit Theorems

Now we explore the link between a stationary distribution and the behavior of P_{ij}^n as $n \to \infty$.

Example 3.4.1

Consider the ON/OFF system in Ex 2.2.3, with

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \qquad 0 \le p \le 1,\ 0 \le q \le 1$$

When $0 < p, q < 1$, $P^n \to \frac{1}{p+q}\begin{pmatrix} q & p \\ q & p \end{pmatrix}$. We also know there is a stationary distribution.

However, suppose $p = q = 1$, so the system is always changing states. The stationary distribution satisfies

$$(\pi_0, \pi_1) = (\pi_0, \pi_1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

or $\pi_0 = \pi_1 = 1/2$. We can also compute the n-step transition probabilities, e.g.

$$P_{00}^n = \begin{cases} 0 & , n \text{ even}, \\ 1 & , n \text{ odd}, \end{cases}$$

and similarly with the other three coefficients. There is no limiting behavior in this case. Also note that both states are periodic with period 2.