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EG 510
HW 7

1a. Problem:

$$\begin{array}{ll} \text{maximize} & z = 3x_1 + 13x_2 + 13x_3 \\ \text{subject to} & x_1 + x_2 \le 7 \\ & x_1 + 3x_2 + 2x_3 \le 15 \\ & 2x_2 + 3x_3 \le 9 \\ & x_1, x_2, x_3 \ge 0 \end{array}$$

with optimal basis $\{x_1, x_2, x_3\}$ and

$$B^{-1} = \begin{pmatrix} 5/2 & -3/2 & 1 \\ -3/2 & 3/2 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

First convert to standard form:

$$\begin{array}{ll} \text{minimize} & \hat{z} = -3x_1 - 13x_2 - 13x_3 \\ \text{subject to} & x_1 + x_2 + x_4 = 7 \\ & x_1 + 3x_2 + 2x_3 + x_5 = 15 \\ & 2x_2 + 3x_3 + x_6 = 9 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \ge 0 \end{array}$$

Calculate solution from given optimal basis:

$$\mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} 5/2 & -3/2 & 1 \\ -3/2 & 3/2 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 7 \\ 15 \\ 9 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} = \mathbf{x}_B$$

$$\hat{z} = -3(4) - 13(3) - 13(1) = -64$$

Calculate optimum dual variables:

$$\mathbf{c}_B^T\mathbf{B}^{-1} = \begin{pmatrix} -3 & -13 & -13 \end{pmatrix} \begin{pmatrix} 5/2 & -3/2 & 1 \\ -3/2 & 3/2 & -1 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = \mathbf{y}^T$$

1b. Tableau, calculated with new constraint, and row 2 multiplied by -1:

| **basic** | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | **b** |
|---|---|---|---|---|---|---|---|
| $-z$ | 0 | 0 | 0 | $-40$ | 37 | $-29$ | 0 |
| $x_1$ | 1 | 0 | 0 | 5/2 | $-3/2$ | 1 | 23/2 |
| $x_2$ | 0 | $-1$ | 0 | 3/2 | $-3/2$ | 1 | 9/2 |
| $x_3$ | 0 | 0 | 1 | 1 | $-1$ | 1 | 6 |

Pivot $x_4$ in, $x_2$ out:

| **basic** | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | **b** |
|---|---|---|---|---|---|---|---|
| $-z$ | 0 | 80/3 | 0 | 0 | 3 | 7/3 | $-120$ |
| $x_1$ | 1 | 5/3 | 0 | 0 | 1 | $-2/3$ | 4 |
| $x_4$ | 0 | $-2/3$ | 0 | 1 | $-1$ | 2/3 | 3 |
| $x_3$ | 0 | 2/3 | 1 | 0 | 0 | 1/3 | 3 |

1c. Calculate range:

$$\mathbf{B}^{-1}\{\mathbf{b}+\delta\} = \begin{pmatrix} 5/2 & -3/2 & 1 \\ -3/2 & 3/2 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 7+x \\ 15 \\ 9 \end{pmatrix} = \begin{pmatrix} 4+2.5\delta \\ 3-1.5\delta \\ 1+\delta \end{pmatrix} \Rightarrow \begin{matrix} \delta \geq -1.6 \\ \delta \leq 2 \\ \delta \geq -1 \end{matrix} \quad \Rightarrow -1 \leq \delta \leq 2$$

1d. Tableau:

| **basic** | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | **b** |
|---|---|---|---|---|---|---|---|
| $-z$ | 0 | 0 | 0 | $-43/2$ | $49/2$ | $-12$ | 0 |
| $x_1$ | 1 | 0 | 0 | $5/2$ | $-3/2$ | 1 | 4 |
| $x_2$ | 0 | 1 | 0 | $-3/2$ | $3/2$ | $-1$ | 3 |
| $x_3$ | 0 | 0 | 1 | 1 | $-1$ | 1 | 1 |

$x_4$ enters, $x_3$ leaves:

| **basic** | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | **b** |
|---|---|---|---|---|---|---|---|
| $-z$ | 0 | 0 | $43/2$ | 0 | 3 | 9.5 | $43/2$ |
| $x_1$ | 1 | 0 | $-5/2$ | 0 | 1 | $-3/2$ | $3/2$ |
| $x_2$ | 0 | 1 | $3/2$ | 0 | 0 | $1/2$ | $9/2$ |
| $x_4$ | 0 | 0 | 1 | 1 | $-1$ | 1 | 1 |

1e. Calculate range:

$$\begin{pmatrix} 4 & 3 & 1 \end{pmatrix} \geq \begin{pmatrix} \delta & 0 & 0 \end{pmatrix} \begin{pmatrix} 5/2 & -3/2 & 1 \\ -3/2 & 3/2 & -1 \\ 1 & -1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 4 & 3 & 1 \end{pmatrix} \geq \begin{pmatrix} 2.5\delta & -1.5\delta & \delta \end{pmatrix} \Rightarrow$$

$$1.6 \geq \delta, -2 \geq \delta, 1 \geq \delta \Rightarrow \delta < -2$$

1f. Calculate new reduced costs:

$$\mathbf{B}^{-1}\mathbf{c}_B = \begin{pmatrix} 5/2 & -3/2 & 1 & 2 \\ -3/2 & 3/2 & -1 & -1 \\ 1 & -1 & 1 & 5 \end{pmatrix} \begin{pmatrix} -3 \\ -13 \\ -13 \\ -5 \end{pmatrix} = \begin{pmatrix} -11 \\ 3 \\ -28 \end{pmatrix}$$

Since the second cost is positive, the basis is not optimal.

1g. $4-3+2(1) = 3 \leq 10$, hence constraint is inactive. Solution remains the same, with slack variable $x_7 = 7$.