Poisson process.

Recall that a sequence of random variables $\{X_n, n \geq 0\}$ satisfies the Markov property if, conditional on the event $\{X_n = i\}$, events related to the collection $\{X_m, m > n\}$ are independent of events related to $\{X_m, m < n\}$.

Birth processes have a similar property.

Next time: expand the discussion below

Theorem 4.2.8 Weak Markov Property

Let $N(t)$ be a birth process and T a fixed time. Conditional on the event $\{N(T) = i\}$, the evolution of the process after time T is independent of the evolution prior to T.

Proof

This is a direct consequence of Defn 4.2.1 (d).

The property is "weak" because T is a fixed constant.

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It turns out to be useful to allow T to be a random variable. But, the analogous conclusion cannot hold for all random T, since if T "looks into the future" as well as the past, then information about the past may be relevant to the future.

We have to restrict the class of random times, here is a useful example.

Definition 4.2.7

A random time T is a stopping time for the process $N(t)$ if for all $t \geq 0$, the indicator function of the event $\{T \leq t\}$ is a function of the values $\{N(s), s \leq t\}$ of the process up to time t.

In plain words, we can decide whether or not T has occurred by time t knowing only the values of the process up to time t.

Example 4.2.5

The arrival times $T_1, T_2, \ldots$ are stopping times.

$$I_{T_n = j} = \begin{cases} 1 & \text{if } N(s) \neq n,\ s < T_n \\ 0 & \text{otherwise} \end{cases}$$

For a fixed k, the k^{th} time N visits state n is a stopping time.

and (4.2.9) follows.

To treat more general B,

$$P(A \mid N(T)=i, B) = E(I_A \mid N(T)=i, B)$$

$$= E(E(I_A \mid N(T)=i, B, H) \mid N(T)=i, B))$$

$H = \{N(s) : s \le T\}$. $\quad = P(A \mid N(t)=i)$

But,

$$E(I_A \mid N(t)=i, B, H) = P(A \mid N(T)=i)$$

arguing as above.

§4.3 More on Poisson Processes

We have investigated two continuous time processes so far, the Poisson process and a natural generalization to birth processes.

Before continuing to consider other processes, we return to explore further properties of the Poisson process.

While Poisson processes are special, they still with surprising frequency in nature. Part of the reason is the Law of Rare

Events. We begin by exploring the connection between the two.

Informally, the Law of Rare Events says that where a certain event may occur in any of a large number of possibilities, but where the probability that the event occurs in any given possibility is small, then the total number of events that occur follows approximately a Poisson distribution.

To be precise, consider a large number N of independent Bernoulli trials, where the probability p of success on each trial is small and constant from trial to trial. Let $X_{N,p}$ be the number of successes in N trials, where $X_{N,p}$ follows a binomial distribution

$$(14.3.1)\quad P(X_{N,p} = k) = \frac{N!}{k!\,(N-k)!}\, p^k (1-p)^{N-k}, \quad k = 0,1,\ldots,N$$

We now consider the limit as $N \to \infty$ and $p \to 0$, while we keep $\mu = Np > 0$ constant. We have

$$P(X_{N,p} = k) = N(N-1)\cdots(N-k+1)\,\frac{p^k (1-p)^N}{k!\,(1-p)^k}$$

Substituting $p = \mu/N$,

$$P(X_{N,p}=k) = N(N-1)\cdots(N-k+1)\,\frac{\left(\frac{\mu}{N}\right)^k\left(1-\frac{\mu}{N}\right)^N}{k!\,\left(1-\frac{\mu}{N}\right)^k}$$

$$= 1\cdot\left(1-\frac{1}{N}\right)\cdot\cdots\left(1-\frac{k-1}{N}\right)\,\frac{\mu^k\left(1-\frac{\mu}{N}\right)^N}{k!\,\left(1-\frac{\mu}{N}\right)^k}$$

We let $N\to\infty$ (so $p\to 0$) and observe

$$1\cdot\left(1-\frac{1}{N}\right)\cdot\cdots\cdot\left(1-\frac{k-1}{N}\right)\to 1$$

$$\left(1-\frac{\mu}{N}\right)^N\to e^{-\mu}$$

$$\left(1-\frac{\mu}{N}\right)^k\to 1$$

so

<u>Theorem 4.3.1</u> <u>Law of Rare Events</u>

If $X_{N,p}$ is the number of successes in N Bernoulli trials with a probability of success p, then if $\mu = Np$ is constant,

$$\lim_{\substack{N\to\infty\\ pN=\mu}} P(X_{N,p}=k) = \frac{\mu^k e^{-\mu}}{k!} \qquad (4.3.2)$$

which is the Poisson Distribution with parameter μ.

<u>Example 4.3.1</u>

A large number of cars pass through an intersection on any given day, while the chance of an accident is small. We might expect the number of accidents on a given day to be approximately Poisson distributed.

Before computers, (4.3.2) was actually used to compute binomial probabilities for large N. We don't do this now, but a slight change makes things interesting.

Suppose the probability of success varies from trial to trial. Let $I_1, I_2, \ldots$ be independent Bernoulli random variables with

$$P(I_i = 1) = p_i, \quad P(I_i = 0) = 1 - p_i$$

and set $S_N = I_1 + \cdots + I_N$ be the number of successes in N trials. Now

$$P(S_N = k) = \sum_{(k)} \prod_{i=1}^{N} p_i^{y_i} (1-p_i)^{N-y_i}$$

where $\sum\limits_{(k)}$ denotes the sum over all 0,1-valued y_i's such that $y_1 + \cdots + y_N = k$. This is not very easy to evaluate.
It turns out

<u>Theorem 4.3.2</u> Law of Rare Events, Version 2

$$(4.3.3) \qquad \left| P(S_N = k) - \frac{\mu^k e^{-\mu}}{k!} \right| \le \sum_{i=1}^{N} p_i^2, \quad \mu = p_1 + \cdots + p_N.$$

This is most useful when $\sum_{i=1}^{N} p_i^2$ is small.

Proof

see your text, pg. 285

Now we connect this discussion to Poisson processes.

Recall

Theorem 4.1.1

Let $N(t)$ be a Poisson process. Then,

$$(4.1.1) \quad P(N(t)=j) = \frac{(\lambda t)^j}{j!} e^{-\lambda t}, \quad j=0,1,2,\dots$$

Alternate proof

Divide the interval $[0,t]$ into n subintervals of equal length $h = t/n$, and set

$$I_i = \begin{cases} 1, & \text{if there is an event in } \left((i-1)\frac{t}{n}, i\frac{t}{n}\right], \\ 0, & \text{otherwise} \end{cases}$$

$S_n = I_1 + \dots + I_n$ counts the number of subintervals that contain at least one event, and

$$p_i = P(I_i = 1) = \lambda \frac{t}{n} + o\left(\frac{t}{n}\right)$$

by the assumed properties of a Poisson process.

Using (4.3.3),

$$\left| P(S_n = k) - \frac{\mu^k e^{-\mu}}{k!} \right| \le n \left(\frac{\lambda t}{n} + o\left(\frac{t}{n}\right) \right)^2$$

$$= \frac{(\lambda t)^2}{n} + 2\lambda t\, o\left(\frac{t}{n}\right) + n\, o\left(\left(\frac{t}{n}\right)^2\right),$$

where $\mu = \sum_{i=1}^{n} p_i = \lambda t + n\, o\left(\frac{t}{n}\right)$

Now $o(h) = o\left(\frac{t}{n}\right)$ has order smaller than h and

$$n\, o\left(\frac{t}{n}\right) = t \, \frac{o(t/n)}{t/n} = t \, \frac{o(h)}{h} \xrightarrow[\substack{n \to \infty \\ (h \to 0)}]{} 0.$$

Letting $n \to \infty$,

$$\lim_{n \to \infty} P(S_n = k) = \frac{\mu^k e^{-\mu}}{k!}, \quad \mu = \lambda t.$$

Finally, we observe that S_n differs from $N(t)$ only if one of the subintervals contains two or more events. But

$$P(N(t) \neq S_n) = \sum_{i=1}^{n} P\left(\text{number of events in } \left((i-1)\tfrac{t}{n}, i\tfrac{t}{n}\right] \ge 2\right)$$

$$\le n \cdot o\left(\frac{t}{n}\right) \quad \text{by assumption}$$

So $\lim\limits_{n \to \infty} P(N(t) \neq S_n) = 0.$

We now discover other distributions that are associated with Poisson processes.

Theorem 4.3.3

The arrival time T_n has the gamma distribution with probability density function

$$(4.3.4) \qquad f_{T_n}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t}, \quad n = 1, 2, \ldots, \; t \geq 0$$

Proof

The event $\{T_n \leq t\}$ occurs if and only if there are at least n events in $(0, t]$. Since $N(t)$ has the Poisson distribution with mean λt, we obtain the c.d.f. of T_n via

$$F_{T_n}(t) = P(T_n \leq t) = P(N(t) \geq n)$$

$$= \sum_{k=n}^{\infty} \frac{(\lambda t)^k e^{-kt}}{k!} = 1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

Differentiating

$$f_{T_n}(t) = \frac{d}{dt}\left(1 - e^{-\lambda t}\left(1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \cdots + \frac{(\lambda t)^{n-1}}{(n-1)!}\right)\right)$$

$$= \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t}.$$

Theorem 4.3.4

For $0 < s < t$, $0 \le k \le n$,

$$(4.3.5) \quad P(N(s)=k \mid N(t)=n) = \frac{n!}{k!(n-k)!} \left(\frac{s}{t}\right)^k \left(1-\frac{s}{t}\right)^{n-k}$$

Proof

$$P(N(s)=k \mid N(t)=n) = \frac{P(N(s)=k,\ N(t)=n)}{P(N(t)=n)}$$

$$= \frac{P(N(s)=k,\ N(t)-N(s)=n-k)}{P(N(t)=n)}$$

$\downarrow$ homogeneity

$$= \frac{\left(e^{-\lambda s} \frac{(\lambda s)^k}{k!}\right)\left(e^{-\lambda(t-s)} \frac{(\lambda(t-s))^{n-k}}{(n-k)!}\right)}{\frac{e^{-\lambda t}(\lambda t)^n}{n!}}$$

$$= \frac{n!}{k!(n-k)!} \frac{s^k (t-s)^{n-k}}{t^n}$$

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The next result says that conditioned on a fixed total number of events in an interval, the times of occurence of