$$\rho(k) = \rho(k) P$$

The components of $\rho(k)$ are nonnegative with sum μ_R. If $\mu_R < \infty$, the vector π with entries

$$\pi_i = \rho_i(k)/\mu_R$$

satisfies $\pi = \pi P$ and has nonnegative entries that sum to 1. π is a stationary distribution.

$\boxed{\#16 \quad 3/25}$

Summarizing so far, when the chain is recurrent and irreducible, there is a solution of $X = XP$ with nonnegative entries. It is an exercise to show that this solution can be taken to have strictly positive entries, and moreover the solution is unique up to a multiplicative factor.

We conclude

<u>Theorem 3.3.7</u>

If the chain is irreducible and recurrent, there is a solution X of $X = XP$ with strictly positive entries that is unique up to a multiplicative factor. The chain is positive if $\sum X_i < \infty$ and null if $\sum X_i = \infty$.

Proof of Theorem 3.3.3

Suppose that π is a stationary distribution of the chain. If all the states are transient, then $P_{ij}^n \to 0$ as $n \to \infty$ for all i, j by Thm 3.1.2.

By (3.3.1),

$$\pi_j = \sum_i \pi_i P_{ij}^n \longrightarrow 0 \quad \text{as } n \to \infty \tag{3.3.4}$$

for all j, which contradicts Defn 3.3.2 (i). Hence, all states are reccurent if we show that we can take the limit in (3.3.4). Let $F \subset S$ be a finite subset of the state space. We can write

$$\sum_i \pi_i P_{ij}^n \le \sum_{i \in F} \pi_i P_{ij}^n + \sum_{i \notin F} \pi_i$$

$$\to \sum_{i \notin F} \pi_i \quad \text{as } n \to \infty \text{ since } F \text{ is finite}$$

$$\to 0 \quad \text{as } F \uparrow S.$$

We next show that the existence of π implies all states are positive and $\pi_i = \mu_i^{-1}$ for all i. Suppose that X_0 has distribution π, $P(X_0 = i) = \pi_i$ for all i. An exercise shows

$$\pi_j \mu_j = \sum_{n=1}^{\infty} P(T_j \geq n \mid X_0 = j) P(X_0 = j)$$
$$= \sum_{n=1}^{\infty} P(T_j \geq n, X_0 = j).$$

But, $P(T_j \geq 1, X_0 = j) = P(X_0 = j)$ and for $n \geq 2$,

$$P(T_j \geq n, X_0 = j) = P(X_0 = j, X_m \neq j, 1 \leq m \leq n-1)$$
$$= P(X_m \neq j, 1 \leq m \leq n-1) - P(X_m \neq j, 0 \leq m \leq n-1)$$
$$= P(X_m \neq j, 0 \leq m \leq n-2) - P(X_m \neq j, 0 \leq m \leq n-1)$$

(by homogeneity)

$$= a_{n-2} - a_{n-1},$$

$a_n = P(X_m \neq j, 0 \leq m \leq n)$

We sum over n to get

$$\pi_j \mu_j = P(X_0 = j) + P(X_0 \neq j) - \lim_{n\to\infty} a_n$$
$$= 1 - \lim_{n\to\infty} a_n$$

Now, $a_n \to P(X_m \neq j \text{ for all } m) = 0$ as $n \to \infty$ since j is recurrent. We have shown that $\mu_j \pi_j = 1$ so $\mu_j = \pi_j^{-1} < \infty$ if $\pi_j > 0$.

To show that $\pi_j > 0$ for all j, suppose $\pi_j = 0$ for some j.

We have

$$0 = \pi_j = \sum_i \pi_i p_{ij}^n \geq \pi p_{ij}^n \qquad \text{for all } i, n$$

hence $\pi_i = 0$ when $i \to j$. The chain is irreducible, $\pi_i = 0$ for all i, which is impossible if $\sum_i \pi_i = 1$. Hence, $\mu_j < \infty$ and all the states are positive.

If π exists, then it is unique and all the states are positive recurrent. If the states are all positive recurrent, then the chain has a stationary distribution with $\pi_i = \mu_i^{-1}$.

Proof of Theorem 3.2.3 (3)

Let C_i be the irreducible closed communication class that contains the positive recurrent state i. Suppose $X_0 \in C_i$, so $X_n \in C_i$ for all n. The results above, Thm 3.3.3 and Thm 3.3.5, imply all the states in C_i are positive.

Examde 3.3.5

Consider the OFF/ON system in Ex. 3.3.2 with

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix}$$

We can compute $\pi = (1/3, 2/3)$, so $\mu_1 = 3, \mu_2 = \frac{3}{2}$.

Example 3.3.6

Consider the Gambler's Ruin in Ex. 3.3.4 where $p < 1/2$. We find that

$$\mu_n = \pi_n^{-1} = \frac{1-p}{1-2p}\left(\frac{1-p}{p}\right)^n, \quad n \geq 0$$

When $p = 1/4$,

$$\mu_n = \frac{3}{2} 3^n, \quad n \geq 0.$$

Theorem 3.3.3 can be used to determine whether or not an irreducible chain is positive recurrent since we can look for a stationary distribution.

We can do something similar for detecting transience.

Theorem 3.3.8 $\boxed{\#17 \quad 3/27}$

Let $s \in S$ be a state of an irreducible chain. The chain is transient if and only if there is a nonzero solution $\{y_j,\ j \in S\}$ of the equations

$$(3.3.5) \qquad y_i = \sum_{\substack{j \in S \\ j \neq s}} P_{ij} y_j \ , \quad i \neq s,$$

with $|y_j| \leq 1$ for all j.

Proof

The chain is transient if and only if s is transient, so suppose s is transient. Define

$$(3.3.6) \quad \tau_i(n) = P(\text{no visit to } s \text{ in the first } n \text{ steps} \mid X_0 = i)$$

$$= P(X_m \neq s,\ 1 \leq m \leq n \mid X_0 = i)$$