where $q+p=1$, $0 \le q, p \le 1$. There are three classes $\{0\}$, $\{1, 2, .., N-1\}$, $\{N\}$. We have

$$\{1, 2, .., N-1\} \to \{0\}$$

$$\{1, 2, ..., N-1\} \to \{N\}$$

But, $\{0\} \not\to \{1, 2, .., N-1\}$
$\{N\} \not\to \{1, 2, .., N-1\}$

$\{0\}$ and $\{N\}$ are absorbing. Here, $T = \{1, 2 .., N-1\}$, $C_1 = \{0\}$, $C_2 = \{N\}$.

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Example 3.2.8

Consider a Markov chain on $S = \{0, 1, 2, 3, 4, 5\}$ with

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

$\{0, 1\}$ and $\{4, 5\}$ are irreducible and closed, therefore contain recurrent positive states.

States 2 and 3 are transient because $2 \to 3 \to 5$ but return to 2 or 3 from 5 is impossible. We have

$$T = \{2,3\}, \quad C_1 = \{0,1\}, \quad C_2 = \{4,5\}.$$

All states have period 1 since $p_{ii} > 0$ for all i. Hence, 0, 1, 4, 5 are ergodic.

We can compute

$$f_{00}(1) = p_{00} = 1/2$$
$$f_{00}(n) = p_{01}\,(p_{11})^{n-2}\,p_{10} = \frac{1}{2}\left(\frac{3}{4}\right)^{n-2}\frac{1}{4}, \quad n \geq 2$$

and

$$\mu_0 = \sum_n f_{00}(n) \cdot n = 3$$

<u>Example 3.2.9</u> (Success Runs)

Consider a Markov chain on $S = \{0,1,2,\dots\}$ with

$$P = \begin{pmatrix} q_0 & p_0 & 0 & \cdots & & & \\ q_1 & 0 & p_1 & 0 & \cdots & & \\ q_2 & 0 & 0 & p_2 & 0 & \cdots & \\ q_3 & 0 & 0 & 0 & p_3 & 0 & \cdots \\ \vdots & & & & & & \end{pmatrix}$$

with $q_i, p_i \geq 0$, $q_i + p_i = 1$, all i. This is called a <u>success</u> <u>run</u> <u>chain</u>.

Consider the case that $p_i = p$ for all i and think of a situation where we attempt independent Bernoulli trials with probability p and we are counting the number of successful trials in a row. If we have had n successes in a row, we can extend the run to $n+1$ if we have success on the next trial or start over with a run of 0 if we fail on the next trial. This gives the row

$$\begin{array}{cccccc} 0 & & n & n+1 & & \\ q & 0 \ldots & 0 & p & 0 & \ldots \end{array}$$

We assume $0 < p_i < 1$ for all i, so the chain is irreducible. This means state $i > 0$ is recurrent if and only if 0 is recurrent.

We have

$$f_{00}(1) = q_0$$

[review defn pg 118]

and for $n \geq 2$

$$\begin{aligned} f_{00}(n) &= P(X_1 = 1, X_2 = 2, \ldots, X_{n-1} = n-1, X_n = 0 \mid X_0 = 0) \\ &= p_0 p_1 \ldots p_{n-2} \cdot q_{n-1} \end{aligned}$$

We set

$$u_n = \prod_{i=0}^{n} p_i, \quad n \geq 0.$$

Since $q_{n-1} = 1 - p_{n-1}$,

$$\begin{aligned} f_{00}(n) &= u_{n-2} - u_{n-1} \\ &= \prod_{i=0}^{n-2} p_i - \prod_{i=0}^{n-1} p_i \\ &= \prod_{i=0}^{n-2} p_i (1 - p_{n-1}) \end{aligned}$$

So

$$\begin{aligned} \sum_{n=1}^{N+1} f_{00}(n) &= q_0 + (u_0 - u_1) + (u_1 - u_2) + \cdots + (u_{N-1} - u_N) \\ &= q_0 + u_0 - u_N = 1 - u_N \end{aligned}$$

Hence, 0 is recurrent if and only if

$$u_N = \prod_{i=0}^{N} p_i \to 0 \quad \text{as } N \to \infty.$$

We can use L'Hopital's rule to show that if $0 < p_i < 1$ for $i \geq 0$, then

$$u_N = \prod_{i=0}^{N} p_i \to 0 \iff \sum_{i=0}^{\infty} (1 - p_i) = \infty$$

and
$$\prod_{i=0}^{\infty} p_i > 0 \iff \sum_{i=0}^{\infty}(1-p_i) < \infty.$$

This means 0 is recurrent if $\sum_{i=0}^{\infty}(1-p_i) = \infty$,

or in other words, the p_i's cannot be too close to 1.

You can read this material in Ch. IV, §3.

§3.3 Stationary distributions and the limit theorem

We now consider the behavior of a Markov chain after a long time has elapsed. The sequence $\{X_n\}$ cannot converge to some particular state in general of course. But, we might hope that the distribution of X_n might converge to something. This will happen under certain restrictions.

Example 3.3.1

Consider the ON/OFF system Ex. 2.2.3 with

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix},$$

we showed that

$$P^n = \frac{1}{p+q}\begin{pmatrix} q & p \\ q & p \end{pmatrix} + \frac{(1-p-q)^n}{p+q}\begin{pmatrix} p & -p \\ -q & q \end{pmatrix}$$

When $0<p<1,\ 0<q<1,$

$$P^n \longrightarrow \frac{1}{p+q}\begin{pmatrix} q & p \\ q & p \end{pmatrix}.$$

Suppose we choose the initial state X_0 according to the probabilities

$$P(X_0=0) = \nu_0$$
$$P(X_0=1) = \nu_1 = 1-\nu_0$$

(We can think of X_0 as being the result of some previous computations with the Markov chain.)

Definition 3.3.1

An initial distribution is a probability distribution for the initial state of a Markov chain.

The probability distribution of X_1 conditioned on X_0 is

$$P(X_1=j\,|\,X_0) = P_{0j}\nu_0 + P_{1j}\nu_1 \quad , \ j=0,1.$$

In matrix notation,

$$\big(P(X_1=0\,|\,X_0),\ P(X_1=1\,|\,X_0)\big) = \nu P$$

Suppose we take

$$\nu_0 = \frac{q}{q+p}, \quad \nu_1 = \frac{p}{q+p}$$

We find that

$$P(X_1 = 0) = (1-p)\frac{q}{q+p} + q\frac{p}{p+q} = \frac{q}{p+q} = \nu_0$$

and likewise

$$P(X_1 = 1) = \nu_1$$

In matrix notation,

$$\nu = \nu P$$

This says that this distribution does not change over time.

<u>Definition 3.3.2</u>

Let S be the state space. The vector π is a <u>stationary distribution</u> for the chain if $\pi = (\pi_j)_{j \in S}$ satisfies

(1) $\pi_j \geq 0$ all j, $\sum_{j \in S} \pi_j = 1$ (π is a p.m.f.)

(2) $\pi = \pi P$ ($\pi_j = \sum_{i \in S} \pi_i P_{ij}$ for all $j \in S$), where

P is the probability transition matrix.

Stationary distributions are also called invariant distributions or equilibrium distributions. Motivation for the names is provided by

Theorem 3.3.1

If π is a stationary distribution of a Markov chain with probability transition matrix P then

$$(3.3.1) \qquad \pi P^n = \pi \qquad \text{for all } n \geq 0$$

If X_0 has distribution π, then X_n also has distribution π.

Proof

exercise.

Given the discussion following the decomposition theorem, Thm 3.2.5, we now assume the chain is irreducible and explore the existence of stationary distributions.

The intuition behind a stationary distribution is that π_j describes the proportion of time that is spent in time j in the long run. Note this is an interesting connection between the distribution of the values at a given time and what is observed over a long time.

Example 3.3.2

Consider the ON/OFF Example again (Ex. 3.3.1)

The equation $\pi = \pi P$ reads

$$(\pi_0 \ \pi_1)\begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} = (\pi_0 \ \pi_1)$$

(The eigenvector equation for rows with eigenvalue 1).

This yields

$$(1-p)\pi_0 + q\pi_1 = \pi_0$$

$$\Rightarrow \qquad \pi_1 = p/q \, \pi_0$$

The second equation is

$$p\pi_0 + (1-q)\pi_1 = \pi_1$$

which also gives

$$\pi_1 = p/q \, \pi_0 .$$

We also have $\pi_0 + \pi_1 = 1$,

$$\pi_0 \left(1 + \frac{p}{q}\right) = 1$$

$$\Rightarrow \qquad \pi = \left(\frac{q}{p+q}, \frac{p}{p+q}\right)$$

This agrees with what we discovered with a lucky guess in E.3.3.1. This proves there is only one stationary distribution in this case.