Example 3.1.5

Consider the off/on system in Ex 2.2.3,

$$P^n = \frac{1}{p+q}\begin{pmatrix} q & p \\ q & p \end{pmatrix} + \frac{(1-p-q)^n}{p+q}\begin{pmatrix} p & -p \\ -q & q \end{pmatrix}$$

For state 0

$$\sum_n P_{00}^n = \sum_n \left(\frac{q}{p+q} + \frac{(1-p-q)^n}{p+q} p \right) = \infty$$

so 0 is recurrent.

$\boxed{\#12 \quad 3/4}$

Example 3.1.6 Random Walk

We consider the simple random walk in Ex. 2.2.2,

$$X_n = X_0 + \sum_{k=1}^{n} B_k,$$

$\{B_k\}$ i.i.d. Bernoulli variables with $P(B_k = 1) = p$, $P(B_k = -1) = 1-p = q$.

Consider any state j. We note that

$P_{jj}^{2n-1} = 0$ for $n = 1, 2, \ldots$ since we cannot return to j in an odd number of steps.

To return in $2n$ steps, we must have n steps in one direction and n steps in the other direction. This has probability

$$(3.1.3) \qquad P_{jj}^{2n} = \binom{2n}{n} p^n (1-p)^n = \frac{(2n)!}{n!\,n!} \left(p(1-p) \right)^n$$

Convergence of the series $\sum_n P_{jj}^{2n}$ is not affected if we drop a finite number of terms in the beginning of the series.

We use an asymptotic large n approximation for $n!$ to judge the size of these coefficients.

Stirling's formula says

$$(3.1.4) \qquad n! \sim n^n \sqrt{n}\, e^{-n} \sqrt{2\pi} \quad , n \text{ large},$$

which means

$$\lim_{n \to \infty} \frac{n!}{n^n \sqrt{n}\, e^{-n} \sqrt{2\pi}} = 1.$$

We can substitute the approximation into (3.1.3) for n large and not affect the convergence or divergence of $\sum_n P_{jj}^{2n}$. We find that

$$P_{jj}^{2n} \approx \frac{(4p(1-p))^n}{\sqrt{\pi n}}.$$

If $p = 1/2$, $P_{jj}^{2n} \approx \frac{1}{\sqrt{\pi n}}$ and $\sum_{n=1}^{\infty} P_{jj}^{n}$ is infinite. So <u>any</u> state j is recurrent when $p = 1/2$.

If $p \neq 1/2$, then $4p(1-p) < 1$, and $\sum_{n=1}^{\infty} P_{jj}^{n} < \infty$. Hence any state j is transient.

Note that Thm 3.1.3 implies any state is either recurrent or transient. It follows easily

<u>Theorem 3.1.4</u>

The number of times $N(i)$ that a Markov chain visits its starting point i, satisfies

$$P(N(i) = \infty) = \begin{cases} 1, & i \text{ is recurrent,} \\ 0, & i \text{ is transient} \end{cases}$$

Proof

After any return to i, a subsequent return is guaranteed if and only if $f_{ii}=1$.

We now consider another classification.

Definition 3.1.5

Let

$$T_j = \min\{n \geq 1 : X_n = j\}$$

be the time of the first visit to j, where $T_j = \infty$ if j is never visited. (T_j depends on X_0 of course).

It follows

Theorem 3.1.5

$P(T_i = \infty \mid X_0 = i) > 0$ if and only if i is transient

When i is transient, $E(T_i \mid X_0 = i) = \infty$. What about recurrent states?

Definition 3.1.6

The mean recurrence time μ_i of a state i is

$$\mu_i = E(T_i \mid X_0 = i) = \begin{cases} \sum_{n=1}^{\infty} n f_{ii}(n), & i \text{ is recurrent,} \\ \infty, & i \text{ is transient.} \end{cases}$$

Note that μ_i may be infinite even for a recurrent state i.

Definition 3.1.7

A recurrent state i is called null if $\mu_i = \infty$ and nonnull or positive if $\mu_i < \infty$.

We prove below

Theorem 3.1.6

A recurrent state is null if and only if $P_{ii}^n \to 0$ as $n \to \infty$ and if this holds, $P_{ji}^n \to 0$ for all j

Proof

This will fall out from our analysis later, so we delay the proof.

Example 3.1.7

Consider the genotype example Ex. 3.1.4. AA and aa are recurrent. We have for 0 = "aa"

$$f_{00}(1) = 1$$
$$f_{00}(n) = 0, \quad n > 1.$$
$$\Rightarrow f_{00} = 1.$$

Similarly for AA.
The mean recurrence time in both cases is 1, so these states are positive.

Example 3.1.8
For simple random walk, Ex 3.1.6, when $p = 1/2$,

good exercise

$P_{jj}^{n} \approx \frac{1}{\sqrt{\pi n}} \to 0$ as $n \to \infty$. So simple random walk is null recurrent when $p = 1/2$.

The next classification of states is a little more complicated. Recall that for a simple random walk, Ex 3.1.6, a return to state i is possible only with an even number of steps. This may be 2, 4, 6, 8, ..., all divisible by 2.

Definition 3.1.8
The greatest common divisor of a set of integers $\{n_1, n_2, \dots\}$, written g.c.d. $(n_1, n_2, \dots)$, is

the largest integer m such that m divides $n_1, n_2, \ldots,$ all without remainder.

<u>Example 3.1.9</u>

$$g.c.d.(2,4,6,8) = 2$$
$$g.c.d.(2,3,5) = 1$$
$$g.c.d.(9,12,18) = 3$$
$$g.c.d.(36,30,48) = 6$$

<u>Definition 3.1.9</u>

The <u>period</u> $d(i)$ of state i is defined

$$d(i) = g.c.d.\{n : P_{ii}^n > 0\},$$

or the g.c.d. of times through which there is a positive probability of returning to state i.

If $d(i) = 1$, i is called <u>aperiodic</u>.
If $d(i) > 1$, i is called <u>periodic</u>.

<u>Example 3.1.10</u>

Consider the OFF/ON system in Ex. 3.1.5. If $0 < p < 1$, $0 < q < 1$, then $P_{00}, P_{11}, P_{01}, P_{10}$ are all strictly between 0 and 1. Hence, $d(i) = 1$ for $i = 0$ or 1.

Suppose $p = q = 1$. Then, $P_{00}^n > 0$ when n is even but $P_{00}^n = 0$ when n is odd, so $d(0) = 2$.

<u>Example 3.1.11</u>

Simple random walk is periodic with period $d(i)=2$ when $p=\frac{1}{2}$.

<u>Example 3.1.12</u>

Consider Gambler's Ruin §2.4, modified so

- A has \$1 initially
- A has a backer that guarantees A's losses (Ex. 2.4.5)
- B is infinitely wealthy

We assume a simple version with $r_1=r_2=\cdots=$ $p_0=p_1=\cdots=p$, $q_0=q_1=\cdots$. The probability transition matrix is

$$P=\begin{pmatrix} q & p & & & \\ q & 0 & p & & 0 \\ & q & 0 & p & \\ & 0 & & \ddots & \ddots \end{pmatrix}$$

(Exercise).

We see that

$$P_{11}^{1}=0$$
$$P_{11}^{2}>0$$
$$P_{11}^{3}>0$$
$$\vdots$$

Hence, $d(i)=1$, since the g.c.d $(2,3)=1$.

We do not have to check further.

Definition 3.1.10

If all states of a Markov chain are aperiodic, we call the chain aperiodic.

Definition 3.1.11

A state is called ergodic if it is recurrent, positive, and aperiodic.

Example 3.1.13

Consider a branching process. State 0 is absorbing, and once there, a chain never leaves. We have $P_{00}^{n} = 1$ for all n, and 0 is recurrent. Clearly, $\mu_0 = 1$. (Compute this using the formulas for f_{ii}!). Hence, 0 is positive. It is also aperiodic. So 0 is ergodic. All other states are transient.

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§ 3.2 Classification of Chains

Next, we consider the relations between states of a Markov chain.