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EG 520
HW 4

12.3 a. Start with three linear quations:

$$5.0 = \frac{1}{2}g1.0^2 = 0.5g$$
$$19.5 = \frac{1}{2}g2.0^2 = 2.0g$$
$$44.0 = \frac{1}{2}g3.0^2 = 4.5g$$

So,

$$\boldsymbol{A} = \begin{bmatrix} 0.5 \\ 2.0 \\ 4.5 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 5.0 \\ 19.5 \\ 44.0 \end{bmatrix}, \boldsymbol{x} = [g]$$

$$\boldsymbol{x}^* = [g^*] = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{b} = [9.7755] \Rightarrow g = 9.7755$$

b. Given a new data point $(t, s) = 4.0, 78.5$, use the recursive formula:

$$\boldsymbol{P}_0 = (\boldsymbol{A}^T\boldsymbol{A})^{-1} = [0.040816], \boldsymbol{g}^{(0)} = [9.7755], \boldsymbol{a}_1 = \frac{1}{2}4.0^2 = [8.0], b_1 = 78.5$$

Hence,

$$\boldsymbol{P}_1 = \boldsymbol{P}_0 - \frac{\boldsymbol{P}_0\boldsymbol{a}_1\boldsymbol{a}_1^T\boldsymbol{P_0}}{1 + \boldsymbol{a}_1^T\boldsymbol{P}_0\boldsymbol{a}_1} = 0.011299$$

and

$$\boldsymbol{x}^{(1)} = \boldsymbol{x}^{(0)} + \boldsymbol{P}_1\boldsymbol{a}_1(b_1 - \boldsymbol{a}_1^T\boldsymbol{x}^{(0)}) = 9.80226$$

12.9 a. The system of linear equations is:

$$y_i = \sin(\omega t_i + \theta) \Rightarrow \arcsin y_i = \omega t_i + \theta, i = 1, \ldots, p$$

So,

$$\boldsymbol{A} = \begin{bmatrix} t_1 & 1 \\ \vdots & 1 \\ t_p & 1 \end{bmatrix}, \boldsymbol{x} = \begin{bmatrix} \omega \\ \theta \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} \arcsin y_1 \\ \vdots \\ \arcsin y_p \end{bmatrix}$$

b.

$$\boldsymbol{A}^T\boldsymbol{A} = p\begin{bmatrix} \overline{T^2} & \overline{T} \\ \overline{T} & 1 \end{bmatrix}, \boldsymbol{x}^* = \begin{bmatrix} \omega^* \\ \theta^* \end{bmatrix} = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{b} = \frac{p}{\overline{T^2} - \overline{T}^2}\begin{bmatrix} 1 & -\overline{T} \\ -\overline{T} & \overline{T^2} \end{bmatrix}\begin{bmatrix} \overline{TY} \\ \overline{Y} \end{bmatrix}$$

So,

$$\omega = \frac{p}{\overline{T^2} - \overline{T}^2}(-\overline{Y}\,\overline{T} + \overline{TY}), \theta = \frac{p}{\overline{T^2} - \overline{T}^2}(\overline{Y}\,\overline{T^2} - \overline{TY}\,\overline{T})$$

12.11

12.13 Start by writing the linear equations:

$$\begin{aligned} &x_0 = 0 \\ &x_1 = 1 = ax_0 + b = b \\ &x_2 = 2 = ax_1 + b = a + b \\ &x_3 = 8 = ax_2 + b = 2a + b \end{aligned}$$

So,

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}, \boldsymbol{x} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Hence,

$$\boldsymbol{x}^* = \begin{bmatrix} a^* \\ b^* \end{bmatrix} = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{b} = \begin{bmatrix} 3.5 \\ 0.16666 \end{bmatrix}$$

12.14 Start by writing the linear equations:

$$\begin{aligned} x_0 &= h_0 = 0 \\ x_1 &= h_1 = ax_0 + bu_0 = b \\ x_2 &= h_2 = ax_1 + bu_1 = ah_1 \\ x_3 &= h_3 = ax_2 + bu_2 = ah_2 \end{aligned}$$

So,

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ h_1 & 0 \\ h_2 & 0 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}, \boldsymbol{x} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Hence,

$$\boldsymbol{x}^* = \begin{bmatrix} a^* \\ b^* \end{bmatrix} = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{b} = \frac{1}{h_1^2 + h_2^2}\begin{bmatrix} 1 & 0 \\ 0 & h_1^2 + h_2^2 \end{bmatrix}\boldsymbol{A}^T\boldsymbol{b} = \frac{1}{h_1^2 + h_2^2}\begin{bmatrix} h_3h_2 + k_2h_1 \\ h_1(h_1^2 + h_2^2) \end{bmatrix}$$

Thus,

$$\begin{aligned} a &= \tfrac{h_3h_2 + h_2h_1}{h_1^2 + h_2^2} \\ b &= h_1 \end{aligned}$$

22.6a

$$\boldsymbol{A} = [\ 1 \quad 1\ ], \boldsymbol{b} = [1] \Rightarrow \boldsymbol{A}^T(\boldsymbol{A}\boldsymbol{A}^T)^{-1}\boldsymbol{b} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \boldsymbol{x}^*$$