Notes - 11 Mar

State i is recurrent: $P(X_n = i \text{ some } n \geq 1|X_0 = i) = 1$. i is transient: $P(X_n = i \text{ some } n \geq 1|X_0 = i) < 1$. $f_{ij}(n) = P(X_1 \neq j, X_2 \neq j, \ldots, X_n \neq j, X_n = j|X_0 = i). f_{ij} = \Sigma_{n=1}^{\infty} f_{ij}(n)$. j is recurrent $\iff f_{jj} = 1$.

Theorem - j is recurrent if $\Sigma_n P_{jj}^n = \infty$. j is transient if $\Sigma_n P_{jj}^n < \infty$.

$T_j = \min\{n \geq 1 : X_n = j\}$. time of first visit to $j(X_0 = i).P(T_i = \infty|X_0 = i) > 0 \iff i$ is transient. $\mu_i = E(T_i|X_0 = i) = \{\Sigma_{n=1}^{\infty} f_{ii}(n)$ for i recurrent, ∞ for i transient. Recurrent state is null if $\mu_i = \infty$. Recurrent state is positive if $\mu_i < \infty$.

Theorem - Recurrent state is null iff $P_{ii}^n \to 0$ as $n \to \infty$.

etc, more review...

New stuff:

Example 3.2.8 - $S = \{0, 1, 2, 3, 4, 5\}$.

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

. $\{0, 1\}, \{4, 5\}$ irreducible and closed. Therefore contain positive recurrent states. 2, 3 are transient: $2 \to 3 \to 5$. But return to 2 or 3 from 5 is impossible. $T = \{2, 3\}, C_1 = \{0, 1\}, C_2 = \{4, 5\}$. All states have period 1 since $P_{ii} > 0$ for all i (all entries on diagonal > 0). 0, 1, 4, 5 are ergodic. We can compute $f_0(1) = P_{00} = 1/2, f_{00}(n) = P_{01}(P_{11})^{n-2}P_{10} = 1/2(3/4)^{n-2}1/4, n \geq 2.\mu_0 = \Sigma_n f_{00}(n) * n = 3$.

Example 3.2.9 - Success Runs - $S = \{0, 1, \ldots\}$. P = (q0 p0 0 ... & q1 0 p1 0 ... & q2 0 0 p2 0 ... & ...). $q_{ii}p_i \geq 0, q_i + p_i = 1$ for all i. This is a success run chain. Intuition: assume $p_i = p$ for all i. We attempt independent Bernoulli trials with probability p of success. We count the number of successful trials in a row. If we had n successes in a row, we can extend the run to $n + 1$ if we have success on the next trial or we start over with a run of 0 if we fail in the next trial. This gives the row (q(0th) 0 ... p((n+1)st) 0 ...). We assume $0 < p_i < 1$ for all i so the chain is irreducible. This means state i is recurrent iff state 0 is recurrent. We have $f_{00}(1) = q_0$, and for $n \geq 2, f_{00}(n) = P(X_1 = 1, X_2 = 2, \ldots, X_{n-1} = n - 1, X_n = 0|X_0 = 0) = P_0 P_1 P_2 \ldots P_{n-2} * q_{n-1}$. Set $U_n = \Pi_{i=0}^n P_i, n \geq 0$ since $q_{n-1} = 1 - P_{n-1}, f_{00}(n) = U_{n-2} - U_{n-1} = \Pi_{i=0}^{n-2} P_i(1 - P_{n-1})$. So $\Sigma_{n=1}^{N+1} f_{00}(n) = q_0 + (U_0 - U_1) + \cdots + (U_{N-1} - U_N) = q_0 + U_0 - U_N = 1 - U_N$. 0 is recurrent iff $U_N = \Pi_{i=0}^N Pi \to 0$ as $N \to \infty$. L'Hopital's rule implies that if $0 < P_i < 1$ for all $i, U_N = \Pi_{i=0}^N P_i \to 0 \iff \Sigma_{i=0}^{\infty}(1 - P_i) = \infty.\Pi_{i=0}^{\infty} P_i > 0 \iff \Sigma_{i=0}^{\infty}(1 - P_i) < \infty$. 0 is recurrent iff $\Sigma_{i=0}^{\infty}(1 - P_i) = \infty$, or the P_i's cannot be too close to 1. If $P_i = 1 - (1/2)^i$, not recurrent. P_i constant then recurrent. (Chapter IV, section §3 in text.)

§ 3.3 - Stationary distributions and the limit theorem

We consider behavior as $n \to \infty$. Does the distribution of X_n converge to something?

Example 3.3.1 - ON/OFF system - ex 2.2.3 - P = $(1 - p$, p & q, $1 - q)$. P^n as before. $0 < p < 1, 0 < q < 1, P^n \to \frac{1}{p+q}*$(q p & q p) [as before]. We choose the initial state X_0 according to the probabilities $P(X_0 = 0) = \nu_0, P(X_0 = 1) = \nu_1 = 1 - \nu_0$.

Definition 3.3.1 - An initial distribution is a probability distribution for the initial state of a Markov chain. The probability distribution of X_1, conditioned on X_0 is $P(X_1 = j|X_0) = P_{0j}\nu_0 + P_{ij}\nu_1, j = 0, 1$. Matrix notation $(P(X_1 = 0|X_0)P(X_1 = 1|X_0)) = \nu p$. Suppose we take $\nu_0 = \frac{q}{q+p}, \nu_1 = \frac{p}{q+p}$. If we compute, $P(X_1 = 0) = (1 - p)\frac{q}{q+p} + q\frac{p}{q+p} = \frac{q}{p+q} = \nu_0$ and $P(X_1 = 1) = \nu_1$. In matrix notation $\nu = \nu p$. That particular initial distribution does not change over time.

Definition 3.3.2 - Let S = state space, The vector Π is a stationary distribution if $\Pi = (\Pi_i)_{i \in S}$ satisfies (1) $\Pi_i \geq 0$ all i, $\Sigma_{i \in S}\Pi_i = 1$, (2) $\Pi = \Pi P(\Pi_j = \Sigma_{i \in S}\Pi_i P_{ij}$ all $j \in S)$. P = probability transition matrix. These are also called invariant distributions and equilibrium distributions.

Theorem 3.3.1 - If Π is a stationary distribution, (3.3.1) $\Pi P^n = \Pi$ for all $n \geq 0$. If X_0 has distribution Π, then so does X_n for $n \geq 0$. Proof: exercise.

Aside: long time behavior of ODE's $\dot{y} = f(y)$. Stead-state/equilibrium solutions $f(y_s) = 0 \Rightarrow y_s$ constant, $\dot{y}_s = f(y_s) = 0$.

We assume the chain is irreducible and explore the existence of stationary distributions.

Example 3.3.2 - Consider ex 3.3.1 (ON/OFF), $\Pi = \Pi P = (\Pi_0 \Pi_1)(1-p$, p & q, $1-q$)= $(\Pi_0 \Pi_1) \Rightarrow \Pi_1 = p/q \Pi_0$ (1st equation), $\Pi_1 = p/q \Pi_0$ (2nd equation), $\Pi_0 + \Pi_1 = 1 \Rightarrow \Pi = (\frac{q}{p+q}, \frac{p}{p+q})$.