ECE 514, Fall 2008

Homework Problems 7

Solutions (version: November 18, 2008, 13:55)

10.1 We have

$$\begin{aligned} m_X(t) &= E[X_t] \\ &= E[g(t,Z)] \\ &= g(t,1)p_1 + g(t,2)p_2 + g(t,3)p_3 \\ &= a(t)p_1 + b(t)p_2 + c(t)p_3 \end{aligned}$$

and

$$\begin{aligned} R_X(t_1,t_2) &= E[X_{t_1}X_{t_2}] \\ &= E[g(t_1,Z)g(t_2,Z)] \\ &= g(t_1,1)g(t_2,1)p_1 + g(t_1,2)g(t_2,2)p_2 + g(t_1,3)g(t_2,3)p_3 \\ &= a(t_1)a(t_2)p_1 + b(t_1)b(t_2)p_2 + c(t_1)c(t_2)p_3. \end{aligned}$$

10.4 In general, $R_X(t_1,t_2) = E[X_{t_1}X_{t_2}^*]$. Starting with the observation in the question,

$$\begin{aligned} 0 &\le E\left[\left|\sum_{i=1}^n c_i X_{t_i}\right|^2\right] \\ &= E\left[\left(\sum_{i=1}^n c_i X_{t_i}\right)\left(\sum_{k=1}^n c_k X_{t_k}\right)^*\right] \\ &= \sum_{i=1}^n \sum_{k=1}^n c_i E[X_{t_i}X_{t_k}^*]c_k^* \\ &= \sum_{i=1}^n \sum_{k=1}^n c_i R_X(t_i,t_k)c_k^*. \end{aligned}$$

10.16 The mean function of Y is

$$E[Y_n] = E[X_n - X_{n-1}] = E[X_n] - E[X_{n-1}] = 0,$$

which does not depend on n. The correlation function of Y is

$$\begin{aligned} E[Y_nY_m] &= E[(X_n - X_{n-1})(X_m - X_{m-1})] \\ &= E[X_nX_m - X_nX_{m-1} - X_{n-1}X_m + X_{n-1}X_{m-1}] \\ &= R_X(n-m) - R_X(n-m+1) - R_X(n-m-1) + R_X(n-m), \end{aligned}$$

which depends on n and m only through their difference. Hence, $\{Y_n\}$ is WSS.

10.40 The transfer function is given by (using the table entry for the Cauchy characteristic function)

$$H(f) = \pi e^{-2\pi|f|}.$$

So,

$$S_Y(f) = S_X(f)|H(f)|^2 = (\mathcal{N}_0/2)\pi^2 e^{-4\pi|f|}.$$

Again using the table we get

$$R_Y(\tau) = \frac{\mathcal{N}_0 \pi^2}{2} \frac{2/\pi}{4+\tau^2}. = \frac{\mathcal{N}_0 \pi}{4+\tau^2}.$$

Hence,

$$E[Y_{t+1/2} Y_t] = R_Y(1/2) = \frac{\mathcal{N}_0 \pi}{4+(1/2)^2} = \frac{4\mathcal{N}_0 \pi}{17}.$$

10.54 (a) We need to show that the corresponding Fourier transform is real, even, and nonnegative (see Section 10.6, p. 417). First, it is easy to see that R is real. Next, we show that R is even:

$$\begin{aligned} R(-\tau) &= \int_{-\infty}^{\infty} R_0(\theta) R_0(-\tau-\theta)\, d\theta \\ &= \int_{-\infty}^{\infty} R_0(-\theta) R_0(\tau+\theta)\, d\theta \quad \text{because } R_0 \text{ is even} \\ &= \int_{-\infty}^{\infty} R_0(\alpha) R_0(\tau-\alpha)\, d\alpha \\ &= R(\tau). \end{aligned}$$

This shows that its Fourier transform is real and even. Finally, because $R = R_0 * R_0$, we have $S(f) = S_0(f)^2$. Also, because R_0 is real and even, $S_0(f)$ is real. Hence, $S(f) = S_0(f)^2 \geq 0$.

(b) Given $R_0(\tau) = I_{[-T,T]}(\tau)$, taking Fourier transforms yields

$$S_0(f) = 2T \frac{\sin 2\pi f T}{2\pi f T}.$$

Hence,

$$S(f) = 4T^2 \left(\frac{\sin 2\pi f T}{2\pi f T} \right)^2$$

and

$$R(\tau) = 2T \left(1 - \frac{|\tau|}{2T} \right) I_{[-2T,2T]}(\tau).$$

10.56 Using the table in the book cover (for Gaussian), we have

$$V(f) = 2\sqrt{\pi} e^{-(2\pi f)^2}.$$

Hence,

$$\begin{aligned} H(f) &= \alpha \frac{V(f)^* e^{-j2\pi f t_0}}{S_X(f)} \\ &= \alpha 2\sqrt{\pi} e^{-(2\pi f)^2/2} e^{-j2\pi f t_0}. \end{aligned}$$

Again using the table in the book cover, we have

$$h(t) = \alpha\sqrt{2} e^{-(t-t_0)^2/2}.$$

10.59 Assuming that $\{V_t\}$ and $\{X_t\}$ are uncorrelated, we have $R_U = R_V + R_X$ and $R_{VU} = R_V$. Hence, $H = S_V/(S_V + S_X)$.