$(m+n)^{th}$ generation that stem from the i^{th} member of the m^{th} generation.

This is a random sum. The variables are independent and iid with the same distribution as the number X_n of the n^{th} generation offspring of the first individual in the process, by the Markov property.

By Theorem 2.5.6,

$$\phi_{m+n}(s) = \phi_m(\phi_{Z_1}(s))$$

where

$$\phi_{Z_1}(s) = \phi_n(s).$$

Iterating gives (2.5.14).

$\boxed{\#10 \quad 2/21}$

<u>Example 2.5.3</u>

Suppose that $0 \le p \le 1$ and the p.m.f. for the offspring is

$$\{q p^k\}_{k \ge 0}, \quad q = 1-p.$$

The prob generating function is

$$\phi(s) = q(1-ps)^{-1}$$

$$\left(\frac{1}{1-a} = 1 + a + a^2 + a^3 + \dots \right)$$

- Each family size is one less than a geometric variable.

By induction,

$$\phi_n(s) = \begin{cases} \dfrac{n-(n-1)s}{n+1-ns}, & p=q=1/2, \\[2ex] \dfrac{q\left(p^n-q^n-ps(p^{n-1}-q^{n-1})\right)}{p^{n+1}-q^{n+1}-ps(p^n-q^n)}, & p\neq q. \end{cases}$$

We can use this to discuss the probability of ultimate extinction. We have

- $$\{\text{ultimate extinction}\} = \bigcup_n \{X_n=0\}.$$

Moreover, $A_n = \{X_n=0\}$ satisfies $A_n \subseteq A_{n+1}$.

Thus, $A = \bigcup_{n=1}^{\infty} A_n = \lim_{n\to\infty} A_n$ and $P(A) = \lim_{n\to\infty} P(A_n)$.

Therefore,

$$P(X_n=0) \xrightarrow[n\to\infty]{} P(\text{Ultimate extinction})$$

$$= \begin{cases} 1, & p\le q, \\ q/p, & p>q. \end{cases}$$

- In this example, extinction occurs with probability 1 if $\mu = E(X_1) = p/q = 1/(q/p)$ satisfies $\mu = E(X_1) \le 1$.

This seems well motivated: if $E(X_n) = E(X_1)^n \leq 1$ then $X_n = 0$ at some point. This turns out to be a general fact.

Theorem 2.5.8

Let ϕ be the prob. generating function for the offspring distribution. Then

$$P(\text{ultimate extinction}) = \lim_{n\to\infty} P(X_n = 0) = \eta,$$

where η is the smallest nonnegative fixed point of the equation

$$(2.5.15) \qquad \phi(s) = s$$

If $|\phi'(1)| < 1$, then $\eta = 1$. If $|\phi'(1)| > 1$, then $\eta < 1$. If $|\phi'(1)| = 1$, then $\eta = 1$ as long as the offspring distribution has positive variance.

Proof

Let $\eta_n = P(X_n = 0)$. Theorem 2.5.7 implies

$$(2.5.16) \qquad \eta_n = \phi_n(0) = \phi(\phi_{n-1}(0)) = \phi(\eta_{n-1}),$$

using the notation in that theorem. From the discussion above, we know that $\eta_n \uparrow \eta$ as $n \to \infty$. Since Φ is continuous, taking the limit in (2.5.16) shows $\eta = \Phi(\eta)$.

Suppose ζ is a nonnegative root of $s = \Phi(s)$, we show $\eta \le \zeta$. Φ is nondecreasing on $[0,1]$ (why?) so

$$\eta_1 = \Phi(0) \le \Phi(\zeta) = \zeta.$$

But,

$$\eta_2 = \Phi(\eta_1) \le \Phi(\zeta) = \zeta$$

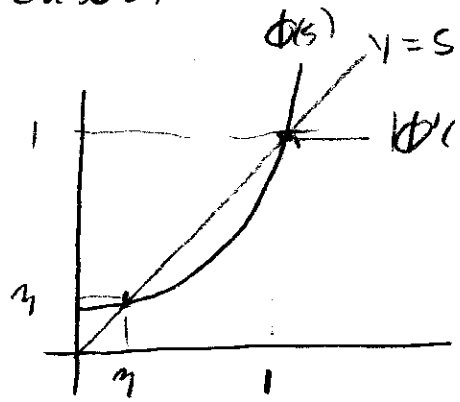
and by induction $\eta \le \zeta$.

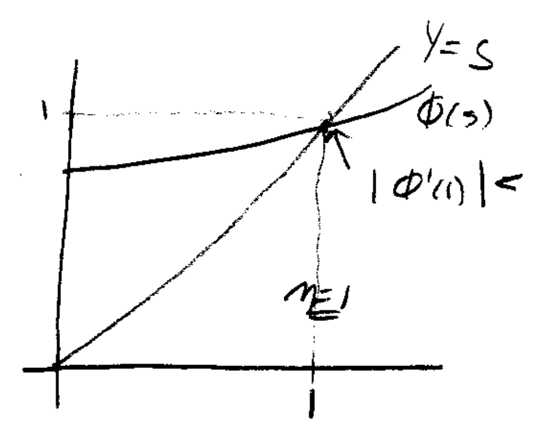
To prove the rest, we use the fact that Φ is convex, since

$$\Phi''(s) = E\left(X_1 (X_1 - 1) s^{X_1 - 2}\right) \ge 0 \quad \text{if } s \ge 0.$$

Hence, Φ is convex and nondecreasing on $[0,1]$ and $\Phi(1) = 1$. The two curves $y = s$ and $y = \Phi(s)$ have two intersection points in $[0,1]$, η and 1. If $|\Phi'(1)| < 1$, $\eta = 1$. If $|\Phi'(1)| > 1$, $\eta \ne 1$.

Two cases:





Example 2.5.4

An individual has a contagious disease, which he passes to others, who in turn passes it to yet more people, and so on.

We assume each individual is infected and contagious only for a brief period, and during this period, each infected person interacts with a number of people that has a Poisson distribution with mean 10, and for each contact, the probability of infection is p.

For what p does the disease die out and for $p = .2$ what is the probability that the disease dies out?

This is a branching process in which the "offspring" are the people infected by an individual.

The disease will eventually die out if $p < .1$
The prob gen. function of a Poisson process with mean μ is $e^{\mu(s-1)}$, so we have to solve

$$s = e^{2 \times 10 (s-1)} \Rightarrow s \approx .2$$

Example 2.5.5

Suppose $P_0 = 1/5$, $P_1 = 1/2$, $P_2 = 3/10$, $P_i = 0$, $i \geq 3$,

for the offspring distribution for a Markov chain X_n. We have

$$\mu = \frac{1}{2} + 2\,\frac{3}{10} = \frac{11}{10} > 1.$$

We have $\Phi(s) = \frac{1}{5} + \frac{1}{2}s + \frac{3}{10}s^2$, and we can verify $\Phi'(1) = \mu$. The probability of ultimate extinction is

$$\frac{1}{5} + \frac{t}{2} + \frac{3t^2}{10} = t.$$

Here, $\frac{1}{5} - \frac{t}{2} + \frac{3t^2}{10} = \frac{1}{10}(3t-2)(t-1) = 0$,

and the roots are $\eta = \frac{2}{3}$ and 1.

Example 2.5.6

Lotka assumed a geometric distribution to fit the male offspring of a human male population. Suppose the number of sons a male has in a lifetime has distribution

$$p_0 = \frac{1}{2}, \quad p_k = \left(\frac{3}{5}\right)^{k-1} \frac{1}{5} \quad k = 1, 2, \ldots$$

(geometric prob. distribution)

Now $\sum_{k=1}^{\infty} p_k = 1/2$, and

$$\phi(t) = \frac{1}{2} + \frac{1}{5} \sum_{k=1}^{\infty} \left(\frac{3}{5}\right)^{k-1} t^k$$

$$= \frac{1}{2} + \frac{1}{5} \left(\frac{t}{1 - 3t/5} \right)$$

We have $\mu = |\Phi'(1)| = \frac{1/5}{(1-3/5)^2} = \frac{5}{4} > 1.$

The fixed points:

$$\frac{1}{2} + \frac{t}{5-3t} = t \implies 6t^2 - 11t + 5 = 0$$

$$\implies \eta = 5/6.$$

Chapter 3 Longtime Analysis for Markov Chains

We now consider the behavior of Markov chains over long time intervals. There are various ways to consider the issue.

§3.1 Classification of States

We can think of the process of a Markov chain as the motion of a "particle" that jumps between the states of the state space S at each time.

The first question we consider is whether or not a particle returns to its starting point within some (possibly infinite) time. It suffices to consider the distribution of the length of time until the article returns the first time, since other times of return are independent copies of this by the Markov property.

Of course, the Markov chain may not return.

Let X_n be a Markov chain with state space S.

Definition 3.1.1

A state i is persistent or recurrent if

$$P(X_n = i \text{ for some } n \geq 1 \mid X_0 = i) = 1,$$

which says that the probability of returning to state i having started in i is 1.

If
$$P(X_n = i \text{ for some } n \geq 1 \mid X_0 = i) < 1,$$

i is called <u>transient</u>.

start #11 2/26

A recurrent state has the property that the chain returns to the initial state in finite time. For a transient state, there is a positive probability of no return.

<u>Example 3.1.1</u>

Consider the roulette wheel in Ex. 2.1.6. State 0 is trivially recurrent since if $X_0 = 0, X_1 = 0, X_2 = 0, \ldots$. State $i \neq 0$ has the property that if we jump to 0, then we cannot return to i. Hence, for $i \neq 0$,
$$P(X_n = i \text{ some } n \geq 1 \mid X_0 = i) < 1,$$
and i is transient.

<u>Example 3.1.2</u>