Lecture XI
Quadratic Programming

A. Introduction

- Consider the following **quadratic programming [QP] problem**:

$$\min_{\mathbf{x}} \quad \mathbf{c}^T\mathbf{x} + \tfrac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x}$$

assume positive definite

$$\text{subject to: } \mathbf{A}\mathbf{x} \geq \mathbf{b}$$
$$\mathbf{x} \geq \mathbf{0}$$

or subject to:

$$\mathbf{g}(\mathbf{x}) \quad \begin{array}{l} \boxed{-\mathbf{A}\mathbf{x} + \mathbf{b}} \leq \mathbf{0} \\ \boxed{-\mathbf{I}\mathbf{x}} \leq \mathbf{0} \end{array}$$

m+n constraints

Lagrangian Function

$$L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\pi}) = \mathbf{c}^T\mathbf{x} + \tfrac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x}$$
$$+ \boldsymbol{\lambda}^T\left[-\mathbf{A}\mathbf{x} + \mathbf{b}\right] + \boldsymbol{\pi}^T\left[-\mathbf{I}\mathbf{x}\right]$$

- Since $\mathbf{Q}$ is positive definite, KKT conditions are necessary and sufficient:

KKT Conditions

$\exists\ \mathbf{x}$ is a global minimum IFF $\exists\ \boldsymbol{\lambda} \in (E^m)^+$ and $\exists\ \boldsymbol{\pi} \in (E^n)^+$ such that:

1) $\mathbf{A}\mathbf{x} \geq \mathbf{b};\ \mathbf{x} \geq \mathbf{0}$ $\left[\text{feasibility}\right]$
2) $\boldsymbol{\lambda}^T\left[-\mathbf{A}\mathbf{x} + \mathbf{b}\right] = \boldsymbol{\pi}^T\left[-\mathbf{I}\mathbf{x}\right] = 0$ $\left[\text{complementary slackness}\right]$
3) $\mathbf{c} + \mathbf{Q}\mathbf{x} - \mathbf{A}^T\boldsymbol{\lambda} - \boldsymbol{\pi} = \mathbf{0}$ $\left[\text{stationarity}\right]$

KKT Conditions (cont.)

$$\mathbf{Ax} - \mathbf{s} = \mathbf{b}$$
$$\mathbf{Qx} - \mathbf{A}^T\boldsymbol{\lambda} - \boldsymbol{\pi} = -\mathbf{c}$$
$$\boldsymbol{\lambda}^T\mathbf{s} = \boldsymbol{\pi}^T\mathbf{x} = 0$$
$$\mathbf{x},\ \mathbf{s},\ \boldsymbol{\lambda},\ \boldsymbol{\pi} \geq \mathbf{0}$$

- If we can find $\mathbf{x},\ \mathbf{s},\ \boldsymbol{\lambda},\ \boldsymbol{\pi}$ that satisfy all these conditions, then we have solved the **original problem**!

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Matrix Form

$$\begin{bmatrix} \boldsymbol{\pi} \\ \mathbf{s} \end{bmatrix} = \begin{bmatrix} \mathbf{Q} & -\mathbf{A}^T \\ \mathbf{A} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{bmatrix} + \begin{bmatrix} \mathbf{c} \\ -\mathbf{b} \end{bmatrix}$$

$$\boldsymbol{\lambda}^T\mathbf{s} = \boldsymbol{\pi}^T\mathbf{x} = 0$$
$$\mathbf{x},\ \mathbf{s},\ \boldsymbol{\lambda},\ \boldsymbol{\pi} \geq \mathbf{0}$$

or…

$$\mathbf{y} = \mathbf{Mz} + \mathbf{q}$$

Called "**fixed point**" problem—useful in many areas such as game theory

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Matrix Form (cont.)

$$\mathbf{y} = \begin{bmatrix} \boldsymbol{\pi} \\ \mathbf{s} \end{bmatrix} \quad \mathbf{M} = \begin{bmatrix} \mathbf{Q} & -\mathbf{A}^T \\ \mathbf{A} & \mathbf{0} \end{bmatrix}$$

$$\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{bmatrix} \quad \mathbf{q} = \begin{bmatrix} \mathbf{c} \\ -\mathbf{b} \end{bmatrix}$$

$$\boldsymbol{\lambda}^T\mathbf{s} = \boldsymbol{\pi}^T\mathbf{x} = 0$$
$$\mathbf{x},\ \mathbf{s},\ \boldsymbol{\lambda},\ \boldsymbol{\pi} \geq \mathbf{0}$$

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B. Lemke's Algorithm

- **Complementary pivot algorithm**

1. Let $\mathbf{y}$ be basic and $\mathbf{z}$ nonbasic (i.e., = 0)
2. If $\begin{bmatrix} \mathbf{c} \\ -\mathbf{b} \end{bmatrix} \geq \mathbf{0}$ **STOP!**

   Optimum found since all conditions satisfied.

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Lemke's Algorithm (cont.)

Otherwise, replace the most negative component of $\mathbf{y}$ with appropriate component of $\mathbf{z}$ that maintains complementary slackness.

3. Repeat until $\mathbf{q} \geq \mathbf{0}$.

Notice that **basic variables** can include the Lagrange multipliers; because of complementary slackness, ½ of variables = 0.

Lemke's Algorithm (cont.)

- It can be proved the Lemke's algorithm converges to a KKT point (may not be global) in finite no. of iterations if:

1. $\mathbf{Q}$ is positive semidefinite and $\mathbf{c} = 0$
2. $\mathbf{Q}$ is positive definite (global optimum)
3. $\mathbf{Q}$ has nonnegative elements with positive diagonal elements

Lemke's Algorithm (cont.)

- Complementary pivot algorithms for quadratic programming (QP) **almost as efficient** as simplex method for LP.
- Successive quadratic programming (SQP) algorithms are some of the most **powerful** generalized nonlinear programming algorithms available today—involve successive quadratic approximations of general problem.

C. Example

$$\min \; -2x_1 - 6x_2 + x_1^2 - 2x_1x_2 + 2x_2^2$$

$$\text{subject to: } \; x_1 + x_2 \leq 2$$

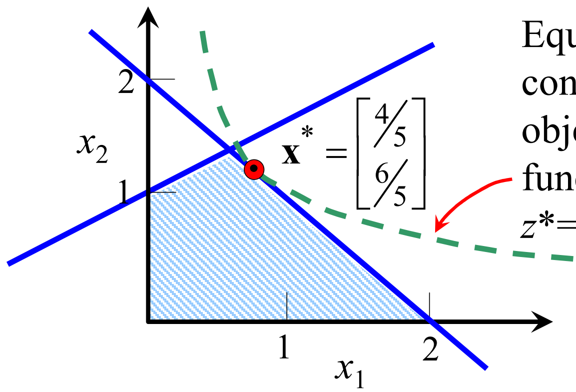
$$-x_1 + 2x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

$$\nabla f(\mathbf{x}) = \begin{bmatrix} 2x_1 - 2x_2 - 2 \\ -2x_1 + 4x_2 - 6 \end{bmatrix} \quad \mathbf{Q} = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}$$

$$\mathbf{c}^{\mathrm{T}} = \begin{bmatrix} -2 & -6 \end{bmatrix}$$

Example (cont.)



Equi-valued contour of objective function z^*=-7.2

Example (cont.)

$$\begin{aligned} -x_1 - x_2 &\geq -2 \\ x_1 - 2x_2 &\geq -2 \end{aligned} \qquad \mathbf{A} = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} 2 & -2 & 1 & -1 \\ -2 & 4 & 1 & 2 \\ -1 & -1 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{bmatrix} \qquad \mathbf{q} = \begin{bmatrix} \mathbf{c} \\ -\mathbf{b} \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \\ 2 \\ 2 \end{bmatrix}$$

Example (cont.)

$$\begin{aligned} \pi_1 &= 2x_1 - 2x_2 + \lambda_1 - \lambda_2 - 2 + z_0 \\ \pi_2 &= -2x_1 + 4x_2 + \lambda_1 + 2\lambda_2 - 6 + z_0 \\ s_1 &= -x_1 - x_2 + 2 + z_0 \\ s_2 &= x_1 - 2x_2 + 2 + z_0 \end{aligned}$$

We add an **artificial variable**

Start with $x_1 = x_2 = \lambda_1 = \lambda_2 = 0$

To maintain nonnegativity, set $z_0 = 6$

[we want to drive z_0 to zero!]

Example (cont.)

Pivot :

$$\begin{aligned} \pi_1 &= 4x_1 - 6x_2 - 3\lambda_2 + \pi_2 + 4 \\ z_0 &= 2x_1 - 4x_2 - \lambda_1 - 2\lambda_2 + \pi_2 + 6 \\ s_1 &= x_1 - 5x_2 - \lambda_1 - 2\lambda_2 + \pi_2 + 8 \\ s_2 &= 3x_1 - 6x_2 - \lambda_1 - 2\lambda_2 + \pi_2 + 8 \end{aligned}$$

z_0 becomes **basic** and π_2 becomes **nonbasic**

Note: we must always maintain complementary slackness

$$\lambda_1 s_1 = \lambda_2 s_2 = \pi_1 x_1 = \pi_2 x_2 = 0$$

Example (cont.)

Now, look at row 2:

$$z_0 = 2x_1 - 4x_2 - \lambda_1 - 2\lambda_2 + \pi_2 + 6$$

Since we are trying to drive z_0 to zero, we look at variables with **negative** coefficients in this row: i.e. $x_2, \lambda_1,$ and λ_2.

However, since s_1 and s_2 already basic, we **cannot** make λ_1 or λ_2 basic without violating complementary slackness!

Example (cont.)

Since π_2 is nonbasic, we can **increase** x_2, which will reduce z_0. But, **how much** can we increase x_2?

Note: if we had a coefficient of **+4** on x_2, then the problem would have **no solution** (i.e., either infeasible or **Q** is not positive definite)

Example (cont.)

$$\left.\begin{aligned}
\pi_1 &= -6x_2 + 4 \quad [\mathbf{4/6}]^* \\
z_0 &= -4x_2 + 6 \quad [6/4] \\
s_1 &= -5x_2 + 8 \quad [8/5] \\
s_2 &= -6x_2 + 8 \quad [8/6]
\end{aligned}\right\} \textbf{Ratio test}$$

If the z_0 row would have won the ratio test, we would be done!

So, π_1 leaves the basis; $x_2 = 4/6 = 2/3$;

new $z_0 = -4\left(4/6\right) + 6 = \mathbf{10/3}$

Example (cont.)

So, we pivot in row 1, column 2:

Pivot :

$$\begin{aligned}
x_2 &= \tfrac{2}{3}x_1 - \tfrac{1}{2}\lambda_2 - \tfrac{1}{6}\pi_1 + \tfrac{1}{6}\pi_2 + \tfrac{2}{3} \\
z_0 &= -\tfrac{2}{3}x_1 - \lambda_1 + \tfrac{2}{3}\pi_1 + \tfrac{1}{3}\pi_1 + \tfrac{10}{3} \\
s_1 &= -\tfrac{7}{3}x_1 - \lambda_1 + \tfrac{1}{2}\lambda_2 + \tfrac{5}{6}\pi_1 + \tfrac{1}{6}\pi_2 + \tfrac{14}{3} \\
s_2 &= -x_1 - \lambda_1 + \lambda_2 + \pi_1 + 4
\end{aligned}$$

Since x_2 has become basic and π_1 became nonbasic, then x_1 can become basic

Example (cont.)

Again, look at row 2:

$$z_0 = -\tfrac{2}{3}x_1 - \lambda_1 + \tfrac{2}{3}\pi_1 + \tfrac{1}{3}\pi_2 + \tfrac{10}{3}$$

$$\left.\begin{aligned}
x_2 &= \tfrac{2}{3}x_1 + \tfrac{2}{3} & & \\
z_0 &= -\tfrac{2}{3}x_1 + \tfrac{10}{3} & & \left[\tfrac{10}{3}\Big/\tfrac{2}{3}\right] \\
s_1 &= -\tfrac{7}{3}x_1 + \tfrac{14}{3} & & \left[\tfrac{\mathbf{14}}{\mathbf{3}}\Big/\tfrac{\mathbf{7}}{\mathbf{3}}\right]* \\
s_2 &= -x_1 + 4 & & [4/1]
\end{aligned}\right\}\textbf{Ratio test}$$

x_1 now basic; s_1 becomes nonbasic; **pivot**

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Example (cont.)

$$\begin{aligned}
x_2 &= -\tfrac{2}{7}\lambda_1 - \tfrac{5}{14}\lambda_2 - \tfrac{2}{7}s_1 + \tfrac{1}{14}\pi_1 + \tfrac{3}{14}\pi_2 + 2 \\
z_0 &= -\tfrac{5}{7}\lambda_1 - \tfrac{1}{7}\lambda_2 + \tfrac{2}{7}s_1 + \tfrac{3}{7}\pi_1 + \tfrac{2}{7}\pi_2 + 2 \\
x_1 &= -\tfrac{3}{7}\lambda_1 + \tfrac{3}{14}\lambda_2 - \tfrac{3}{7}s_1 + \tfrac{5}{14}\pi_1 + \tfrac{1}{14}\pi_2 + 2 \\
s_2 &= -\tfrac{4}{7}\lambda_1 + \tfrac{11}{14}\lambda_2 + \tfrac{3}{7}s_1 + \tfrac{9}{14}\pi_1 + \tfrac{1}{14}\pi_2 + 2
\end{aligned}$$

z_0 now reduced to 2; z_0 can now be reduced by increasing λ_1 or λ_2; but s_2 basic so cannot increase λ_2

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Example (cont.)

$$\left.\begin{aligned}
x_2 &= -\tfrac{2}{7}\lambda_1 + 2 & & \left[2\Big/\tfrac{2}{7}\right] \\
z_0 &= -\tfrac{5}{7}\lambda_1 + 2 & & \left[\mathbf{2}\Big/\tfrac{\mathbf{5}}{\mathbf{7}}\right]* \\
x_1 &= -\tfrac{3}{7}\lambda_1 + 2 & & \left[2\Big/\tfrac{3}{7}\right] \\
s_2 &= -\tfrac{4}{7}\lambda_1 + 2 & & \left[2\Big/\tfrac{4}{7}\right]
\end{aligned}\right\}\textbf{Ratio test}$$

So, z_0 finally finally selected to go nonbasic.

We are done!

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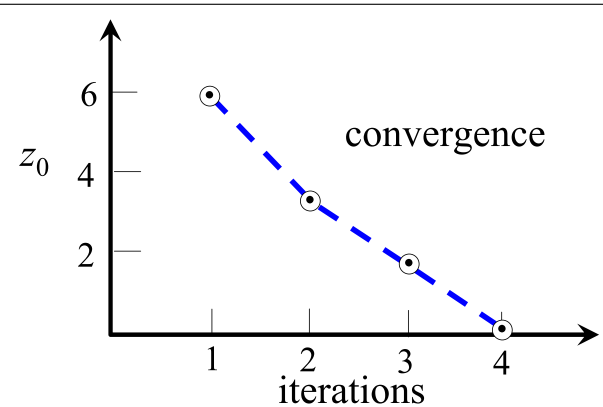
Example (cont.)



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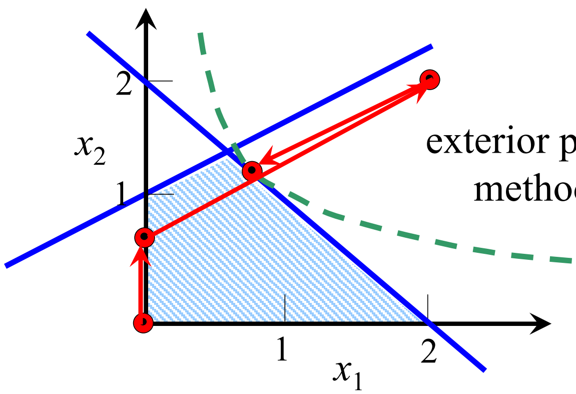
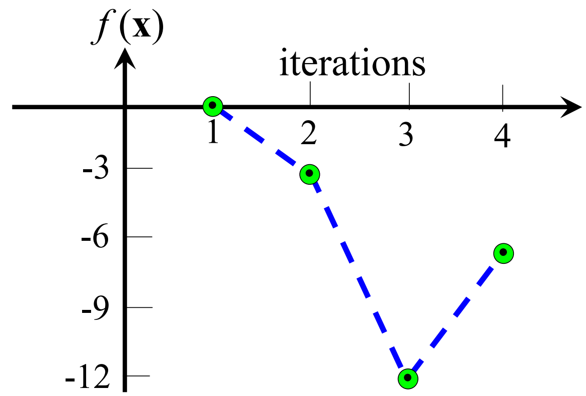
Example (cont.)



Example (cont.)



Example (cont.)

Could perform another pivot to move λ_1 on left side and z_0 on right side.

Rather than doing last pivot, use KKT theory.

Since $\lambda_1 > 0$, then $s_1 = 0$

or $x_1 + x_2 = 2$

Since $\pi_1 = \pi_2 = \lambda_2 = 0$

$$\left.\begin{aligned} 2x_1 - 2x_2 + \lambda_1 - 2 = 0 \\ -2x_1 + 4x_2 + \lambda_1 - 6 = 0 \end{aligned}\right\} i.e. \nabla L_x = \mathbf{0}$$

Example (cont.)

Solving simultaneously gives:

$$x_1^* = \frac{4}{5} \quad x_2^* = \frac{6}{5} \quad \lambda_1^* = \frac{14}{5}$$

all other variables = 0