Notes - 24 April

Theorem 4.2.8 - Weak Markov Property - $N(t)$ is a birth process. T is a fixed time. Conditional on the event $\{N(T) = i\}$ the evolution of the process after time T is independent of the evolution prior to T. It is useful to let T be a random variable. This theorem as stated cannot hold for all possilbe random times T. For example, if T "looks into the future" the conclusion cannot hold.

Definition 4.2.7 - A random time T is a stopping time for the process $N(t)$ if for all $t \geq 0$, the indicator function of the event $\{T \leq t\}$ is a function of the values $\{N(s), s \leq t\}$ of the process up to time t. We can decide if T has occured by t by examining the values of N up to time t.

Example 4.2.5 - The arrival times $T_1, \ldots,$ are stopping times. $I_{T_n=t} = \{1$ for $N(s) \neq n, N(t) = n, s < t, 0$ else. For fixed k, the k^{th} time a process N visits a state n is a stopping time. The last time a process visits a state n is not a stopping time.

Example 4.2.6 - $T_4 - 2$ and $\frac{1}{2}(T_1 + T_2)$ are not stopping times (why?).

For stopping times we can prove:

Theorem 4.2.9 - Strong Markov Property - Let $N(t)$ be a birth process and T a stopping time. Let A be an event that depends on $\{N(s), s > t\}$ and B an event that depends on $\{N(s), s < t\}$. Then (4.2.9) $P(A|N\Pi) = i, B) = P(A|N(T) = i)$. Read outline of proof in notes.

§ 4.3 - More on Poisson Processes

While Poisson processes are special among continuous time processes, they also occur frequently in "nature". Part of the reason is the Law of Rare Events.

Law of Rare Events - This says that in a situation in which a certain event can occur in any of a large number of possibilities but where the probability it occurs in any given possibility is small, then the total number of occurrences follows approximately a Poisson distribution.

Consider a large number N of Bernoulli trials. The probability of success in each trial is p. Let $X_{N,p} =$ number of successes in the N trials. (4.3.1) $P(X_{N,p} = k) = \frac{N!}{k!(N-k)!}p^k(1-p)^{N-k}, k = 0, 1, \ldots, N$. We consider the limit as $N \to \infty$ and $p \to 0$ so that $\mu = Np$ is fixed. $P(X_{N,p} = k) = N(N-1)\cdots(N-k+1)\frac{p^k(1-p)^N}{k!(1-p)^k}$. Substitute $p = \frac{\mu}{N} \Rightarrow P(X_{N,p} = k) = 1 * (1 - \frac{1}{N})\cdots(1 - \frac{k-1}{N})\frac{\mu^k(1-\frac{\mu}{N})^N}{k!(1-\frac{\mu}{N})^k}$. So $N \to \infty (p \to 0). 1 * (1 - \frac{1}{N})\cdots(1 - \frac{k-1}{N}) \to 1, (1 - \frac{\mu}{N})^N \to e^{-\mu}, (1 - \frac{\mu}{N}^k) \to 1$.

Theorem 4.3.1 - Law of Rare Events - If $X_{N,p}$ is the number of successes in N Bernoulli trials with probability of success p, then (4.3.2) $\lim_{N\to\infty, pN=\mu} P(X_{N,p} = k) = \frac{\mu^k e^{-\mu}}{k!}$. (Poisson distribution with parameter μ.) (4.3.2) can be used to compute Binomial probabilities.

Suppose the probability of success varies with each trial. Let $Y_1, Y_2, \ldots,$ be independent Bernoulli r.v. with $P(Y_i = 1) = P_i, P(Y_i = 0) = 1 - P_i$. $S_n = Y_1 + \cdots + Y_N =$ number of successes in N trials. $P(S_N = k) = \Sigma_{(k)}\Pi_{i=1}^N P_i^{Y_i}(1-P_i)^{N-Y_i}$ where $\Sigma_{(k)}$ is the sum over all 0, 1 valued Y_is that sum $Y_1 + \cdots + Y_N = k$.

Theorem 4.3.2 - Law of Rare Events, v.2 - (4.3.3) $|P(S_N = k) - \frac{\mu^k e^{-\mu}}{k!}| \leq \Sigma_{i=1}^N P_i^2$ with $\mu = P_1 + \cdots + P_N$. If the probabilities of success in the trials are small, we obtain a good approximation. Proof: see text, pg. 285.

Recall Theorem 4.1.1 - Let $N(t)$ be a Poisson process. Then (4.1.1) $P(N(t) = j) = \frac{(\lambda t)^j}{j!}e^{-\lambda t}, j = 0, 1, \ldots$ ($\lambda =$ rate of the process).

Alternate Proof (using the Law of Rare Events. Divide $[0, t]$ in to n subintervals of equal length $h = \frac{t}{n}$, setting $Y_i = \{1$ if there is an event in $((i-1)\frac{t}{n}, i\frac{t}{n}], 0$ otherwise. $S_n = Y_1 + \cdots + Y_n$ counts the number of subintervals that contain at least one event and $P_i = P(Y_i = 1) = \lambda\frac{t}{n} + o(\frac{t}{n})$ by definition of a Poisson process. Using (4.3.) $|P(S_n = k) - \frac{\mu^k e^{-\mu}}{k!}| \leq n(\frac{\lambda t}{n} + o(\frac{t}{n}))^2 = \frac{(\lambda t)^2}{n} + 2\lambda + o(\frac{t}{n}) + no((\frac{t}{n})^2)$. Now $no(\frac{t}{n}) = t\frac{o(\frac{t}{n})}{\frac{t}{n}} = t\frac{o(h)}{h} \to_{n\to\infty, h\to 0} 0$. S_n differs from $N(t)$ only if one of the subintervals has more than one event. $P(N(t) \neq S_n) = \Sigma_{i=1}^n P($number of events in $((i-1)\frac{t}{n}, i\frac{t}{n}] \geq 2) \leq n * o(\frac{t}{n})$ by assumption. So $\lim_{n\to\infty} P(N(t) \neq S_n) = 0$.

Theorem 4.3.3 - The arrival time T_n has the gamma distribution with pdf (4.3.4) $f_{T_n}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!}e^{-\lambda t}, n = 1, 2, \ldots, t \geq 0$. Proof: the event $\{T_n \leq t\}$ occurs iff there are atleast n events in $(0, t]$. $N(t)$ is a Poisson distribution with parameter mean λt. So the cdf of T_n is $F_{T_n}(t) = P(T_n \leq t) = P(N(t) \geq n) = \Sigma_{k=n}^\infty \frac{(\lambda t)^k e^{-\lambda t}}{k!} = 1 - \Sigma_{k=0}^{n-1}\frac{(\lambda t)^k e^{-\lambda t}}{k!}$. Differentiating, [some long messy computation, see notes] $= \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}$.

Theorem 4.3.4 - For $0 < s < t, 0 \leq k \leq n$, (4.3.5) $P(N(s) = k|N(t) = n) = \frac{n!}{k!(n-k)!}(\frac{s}{t})^k(1 - \frac{s}{t})^{n-k}$. Proof in notes.