Duality (Chap. 17)

An Example of Duality

- Recall the optimal diet example:

Vitamin	Food type: Milk	Food type: Eggs	Daily Requirements
V	2	4	40
W	3	2	50
Intake	x_1	x_2	
Unit cost	3	5/2	

- LP problem is:

$$\begin{array}{ll} \text{minimize} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}, \end{array}$$

  where

$$\boldsymbol{c}^T = [3, 5/2], \qquad \boldsymbol{b} = \begin{bmatrix} 40 \\ 50 \end{bmatrix}, \qquad \boldsymbol{A} = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}.$$

- Easy to check that the solution is: $x_1 = 15$, $x_2 = 5/2$, with a minimal cost of $51\frac{1}{4}$.

- Consider now the related problem faced by a health food store owner who sells vitamin pills (vitamins V and W).

- The store owner needs to set the (unit) prices of vitamins V and W: λ_1 and λ_2.

- Since the nutritional requirements for vitamins V and W are 40 and 50, respectively, the store's total daily revenue is

$$40\lambda_1 + 50\lambda_2.$$

- Naturally, the store owner wants to maximize his revenue.

- To be competitive, he cannot set his prices to be higher than the price of obtaining the nutritional equivalent from milk and eggs.

- To compete well with milk prices, the prices must satisfy:

$$2\lambda_1 + 3\lambda_2 \leq 3.$$

- Similarly, for eggs, we have

$$4\lambda_1 + 2\lambda_2 \le \frac{5}{2}.$$

- The health food store owner's LP problem is:

$$\begin{array}{ll} \text{maximize} & 40\lambda_1 + 50\lambda_2 \\ \text{subject to} & 2\lambda_1 + 3\lambda_2 \le 3 \\ & 4\lambda_1 + 2\lambda_2 \le \dfrac{5}{2} \\ & \lambda_1, \lambda_2 \ge 0. \end{array}$$

- If we solve the above problem, we find that the optimal revenue is $51\frac{1}{4}$.

- Why is the maximal revenue for the store owner the same as the minimal cost for the consumer?

- In matrix notation, the health food store owner's LP problem can be written as

$$\begin{array}{ll} \text{maximize} & \boldsymbol{\lambda}^T \boldsymbol{b} \\ \text{subject to} & \boldsymbol{\lambda}^T \boldsymbol{A} \le \boldsymbol{c}^T \\ & \boldsymbol{\lambda} \ge \boldsymbol{0}, \end{array}$$

  where $\boldsymbol{A}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are exactly as before.

- Notice that the health food store owner's LP problem can be deduced from the customer's LP problem by making the following substitutions:

$$\begin{array}{rcl} \text{minimize} & \to & \text{maximize} \\ \ge & \to & \le \\ \boldsymbol{c} & \to & \boldsymbol{b} \\ \boldsymbol{b} & \to & \boldsymbol{c} \\ \boldsymbol{A} & \to & \boldsymbol{A}^T. \end{array}$$

- LP problems related as above are called *dual* problems.

Duality in LP

- Duality, the study of dual LP problems, is fundamentally very important.

- The solution to one gives information about the solution to the other.

- Duality can be used to improve the performance of the simplex algorithm (Primal-Dual algorithm).
- Duality is useful in the design of new algorithms (e.g., Karmarkar's algorithm, Chap. 18).
- Duality is used in sensitivity analysis (how much will the solution to an LP problem change if we slightly change the numbers in the problem data?).
- Duality is used to derive *transposition theorems* (see Exercises 17.10–17.12).
- Duality is the basis for studying *matrix games*.
- Duality has a more general counterpart.

Dual LP problems (§17.1)

- Given: an LP of the form

$$\begin{aligned} \text{minimize } \boldsymbol{c}^T\boldsymbol{x} & \\ \text{subject to } \boldsymbol{A}\boldsymbol{x} &\geq \boldsymbol{b}, \\ \boldsymbol{x} &\geq \boldsymbol{0}. \end{aligned}$$

  We refer to the above as the *primal* problem.

- Define the corresponding *dual* problem as

$$\begin{aligned} \text{maximize } \boldsymbol{\lambda}^T\boldsymbol{b} & \\ \text{subject to } \boldsymbol{\lambda}^T\boldsymbol{A} &\leq \boldsymbol{c}^T, \\ \boldsymbol{\lambda} &\geq \boldsymbol{0}. \end{aligned}$$

- The above pair of related LP problems is called the *symmetric form of duality*.
- The diet example given previously is of this form.
- Note that the primal and dual problems are related via

$$\begin{aligned} \text{minimize} &\to \text{maximize} \\ \geq &\to \leq \\ \boldsymbol{c} &\to \boldsymbol{b} \\ \boldsymbol{b} &\to \boldsymbol{c} \\ \boldsymbol{A} &\to \boldsymbol{A}^T. \end{aligned}$$

- Consider now an LP in the form (primal):

$$\begin{aligned} \text{minimize } & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to } \boldsymbol{A}\boldsymbol{x} &= \boldsymbol{b}, \\ \boldsymbol{x} &\geq \mathbf{0}. \end{aligned}$$

- The corresponding dual is

$$\begin{aligned} \text{maximize } & \boldsymbol{\lambda}^T\boldsymbol{b} \\ \text{subject to } \boldsymbol{\lambda}^T\boldsymbol{A} &\leq \boldsymbol{c}^T. \end{aligned}$$

- The above pair of related LP problems is called the *asymmetric form of duality*.

Some remarks on duality

- The dual of the dual problem is the primal problem.
- We can derive the asymmetric form of duality from the symmetric form (see §17.1).
- Both primal and dual problems are defined by the same data $\boldsymbol{A}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

Properties of dual problems (§17.2)

- We now discuss some fundamental properties of dual LP problems. All these properties hold for both symmetric and asymmetric forms.
- Weak Duality Lemma (17.1): Suppose that $\boldsymbol{x}$ and $\boldsymbol{\lambda}$ are feasible solutions to primal and dual LP problems, respectively. Then, $\boldsymbol{c}^T\boldsymbol{x} \geq \boldsymbol{\lambda}^T\boldsymbol{b}$.
- Proof: simple algebra.

Interpretation of weak duality lemma:

- The objective function value of any feasible solution to one problem is a bound for the optimal objective function value for the other.
- The primal strives to minimize, the dual strives to maximize.
- maximum of dual $\leq$ minimum of primal
- If one problem is unbounded, then the other has no feasible solution.

- Theorem (17.1): Suppose that $\boldsymbol{x}$ and $\boldsymbol{\lambda}$ are feasible solutions to the primal and dual, respectively. If $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{\lambda}^T\boldsymbol{b}$, then $\boldsymbol{x}$ and $\boldsymbol{\lambda}$ are optimal solutions to their respective problems.
- Proof: Simple consequence of Weak Duality Lemma.

Important question:

- Is it possible that maximum (dual) $<$ minimum (primal)?

Duality Theorem (17.2): If the primal problem has an optimal solution, then so does the dual, and the optimal values of their respective objective functions are equal.

Proof:

- Consider standard form primal with asymmetric dual form.
- Suppose primal has an optimal solution. By FTLP, it has an optimal BFS.
- Let $\boldsymbol{B}$ be the basis matrix (for convenience, assume it consists of the first m columns of $\boldsymbol{A}$).
- Define $\boldsymbol{\lambda}^T = \boldsymbol{c}_B^T \boldsymbol{B}^{-1}$.
- Claim: $\boldsymbol{\lambda}$ is feasible in the dual.
- To see this, note that
$$\boldsymbol{\lambda}^T \boldsymbol{A} = \boldsymbol{c}_B^T \boldsymbol{B}^{-1}[\boldsymbol{B}, \boldsymbol{D}] = [\boldsymbol{c}_B^T, \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{D}].$$
- But, $\boldsymbol{r}_D^T = \boldsymbol{c}_D^T - \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{D} \geq \boldsymbol{0}$ (because optimal).
- Hence, $\boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{D} \leq \boldsymbol{c}_D^T$.
- Thus,
$$\boldsymbol{\lambda}^T \boldsymbol{A} \leq [\boldsymbol{c}_B^T, \boldsymbol{c}_D^T] = \boldsymbol{c}^T,$$
which means that $\boldsymbol{\lambda}$ is feasible in the dual.
- Claim: $\boldsymbol{\lambda}$ is optimal in the dual.
- To see this, note that
$$\boldsymbol{\lambda}^T \boldsymbol{b} = \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{b} = \boldsymbol{c}^T \begin{bmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix} = \boldsymbol{c}^T \boldsymbol{x}$$
where $\boldsymbol{x}$ is the optimal BFS.
- Hence, by previous theorem, $\boldsymbol{\lambda}$ is optimal.

Summary:

- Primal unbounded $\Rightarrow$ dual infeasible.
- Primal bounded $\Rightarrow$ dual bounded, no gap.
- Primal infeasible $\Rightarrow$ dual is either unbounded or infeasible.

Duality and simplex method

- If we use the simplex method to solve the primal, we can extract the solution of the dual from the final canonical tableau.
- The last row contains $\boldsymbol{r}_D^T = \boldsymbol{c}_D^T - \boldsymbol{\lambda}^T\boldsymbol{D}$, where $\boldsymbol{\lambda}^T = \boldsymbol{c}_B^T\boldsymbol{B}^{-1}$.
- Therefore, we can solve for $\boldsymbol{\lambda}$ from the linear equation

$$\boldsymbol{\lambda}^T\boldsymbol{D} = \boldsymbol{c}_D^T - \boldsymbol{r}_D^T.$$

- If $\boldsymbol{D}$ is not of full rank, we can append additional equations.
- Combining the previous equation with $\boldsymbol{\lambda}^T\boldsymbol{B} = \boldsymbol{c}_B^T$, we obtain

$$\boldsymbol{\lambda}^T\boldsymbol{A} = \boldsymbol{c}^T - \boldsymbol{r}^T.$$

- Note that $\boldsymbol{r}^T = \boldsymbol{c}^T - \boldsymbol{\lambda}^T\boldsymbol{A}$ is the final RCC vector.

Example: (extracted from Example 17.4)

- Consider the primal LP:

$$\begin{aligned} \text{minimize } & -2x_1 - 5x_2 - x_3 \\ \text{subject to } & 2x_1 - x_2 + 7x_3 + x_4 &=& \; 6 \\ & x_1 + 3x_2 + 4x_3 + x_5 &=& \; 9 \\ & 3x_1 + 6x_2 + x_3 + x_6 &=& \; 3 \\ & x_1, \dots, x_6 &\geq& \; 0. \end{aligned}$$

- The dual is (from asymmetric dual form):

$$\begin{aligned} \text{maximize} \quad & 6\lambda_1 + 9\lambda_2 + 3\lambda_3 \\ \text{subject to} \quad & 2\lambda_1 + \lambda_2 + 3\lambda_3 \leq -2 \\ & -\lambda_1 + 3\lambda_2 + 6\lambda_3 \leq -5 \\ & 7\lambda_1 + 4\lambda_2 + \lambda_3 \leq -1 \\ & \lambda_1, \lambda_2, \lambda_3 \leq 0. \end{aligned}$$

- Using the simplex method to solve the primal LP, we get the following final simplex tableau:

$$\begin{array}{cccccccc} & 15/43 & 0 & 1 & 6/43 & 0 & 1/43 & 39/43 \\ & -74/43 & 0 & 0 & -21/43 & 1 & -25/43 & 186/43 \\ & 19/43 & 1 & 0 & -1/43 & 0 & 7/43 & 15/43 \\ \boldsymbol{r}^T & 24/43 & 0 & 0 & 1/43 & 0 & 36/43 & 114/43 \end{array}$$

- To find the solution to the dual, we use the equation $\boldsymbol{\lambda}^T \boldsymbol{D} = \boldsymbol{c}_D^T - \boldsymbol{r}_D^T$:

$$[\lambda_1, \lambda_2, \lambda_3] \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix} = [-2, 0, 0] - \left[\frac{24}{43}, \frac{1}{43}, \frac{36}{43}\right].$$

- Solving the above, we get

$$\boldsymbol{\lambda}^T = \left[-\frac{1}{43}, 0, -\frac{36}{43}\right].$$

- Exercise: try solving the dual directly (see Example 17.2).

Complementary slackness

Theorem (17.3): The feasible solutions $\boldsymbol{x}$ and $\boldsymbol{\lambda}$ to a dual pair of problems are optimal if and only if

1. $(\boldsymbol{c}^T - \boldsymbol{\lambda}^T \boldsymbol{A})\boldsymbol{x} = 0$; and
2. $\boldsymbol{\lambda}^T(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}) = 0$.

Interpretation:

- Assume asymmetric form.
- Recall the construction of $\boldsymbol{\lambda}$ given before.
- Then, the RCC vector is $\boldsymbol{r}^T = \boldsymbol{c}^T - \boldsymbol{\lambda}^T \boldsymbol{A}$.
- The *complementary slackness theorem* says that $\boldsymbol{r}^T \boldsymbol{x} = 0$.
- Since both $\boldsymbol{r}$ and $\boldsymbol{x}$ are $\geq \boldsymbol{0}$, if a component of one is > 0, the corresponding component of the other must be 0.
- If $x_i > 0$, then $r_i = 0$ (i.e., the RCC for a basic variable must be 0).
- If $r_i > 0$, then $x_i = 0$ (i.e., the value of a nonbasic variable must be 0).
- The complementary slackness condition is related to the *Karush-Kuhn-Tucker* condition (see later).

How difficult is the feasibility problem?

- Consider the LP problem: Given $\boldsymbol{A}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, find $\boldsymbol{x}$ such that $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$, and $\boldsymbol{c}^T\boldsymbol{x}$ is minimized.
- We already have "machinery" to solve the above problem.
- Consider an apparently simpler problem: Given $\boldsymbol{A}$ and $\boldsymbol{b}$, find $\boldsymbol{x}$ such that $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$.
- Name: *feasibility problem*.
- How much simpler is the feasibility problem than the LP problem?

- We can certainly use our machinery for LP to solve the feasibility problem.
- It turns out that the feasibility problem is *as difficult as the LP problem*.
- Specifically, if we have an algorithm that can solve the feasibility problem, then that algorithm can be used to solve LP problems!
- To establish the above result, we use duality.

- Consider an LP problem

$$\begin{array}{rl} \text{minimize} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

and the corresponding dual problem

$$\begin{array}{rl} \text{maximize} & \boldsymbol{\lambda}^T\boldsymbol{b} \\ \text{subject to} & \boldsymbol{\lambda}^T\boldsymbol{A} \leq \boldsymbol{c}^T \\ & \boldsymbol{\lambda} \geq \boldsymbol{0}. \end{array}$$

- By our previous theorem, if we can find feasible points $\boldsymbol{x}$ and $\boldsymbol{\lambda}$ for the primal and dual, respectively, such that $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{\lambda}^T\boldsymbol{b}$, then $\boldsymbol{x}$ and $\boldsymbol{\lambda}$ are optimal for their respective problems.
- Specifically, we want to find $\boldsymbol{x}$ and $\boldsymbol{\lambda}$ such that

$$\begin{aligned} \boldsymbol{c}^T\boldsymbol{x} &= \boldsymbol{b}^T\boldsymbol{\lambda} \\ \boldsymbol{A}\boldsymbol{x} &\geq \boldsymbol{b} \\ \boldsymbol{A}^T\boldsymbol{\lambda} &\leq \boldsymbol{c} \\ \boldsymbol{x} &\geq 0 \\ \boldsymbol{\lambda} &\geq 0. \end{aligned}$$

- We can rewrite the previous set of relations as

$$\begin{bmatrix} \boldsymbol{c}^T & -\boldsymbol{b}^T \\ -\boldsymbol{c}^T & \boldsymbol{b}^T \\ -\boldsymbol{A} & \boldsymbol{0} \\ -\boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}^T \\ \boldsymbol{0} & -\boldsymbol{I}_m \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{\lambda} \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ -\boldsymbol{b} \\ \boldsymbol{0} \\ \boldsymbol{c} \\ \boldsymbol{0} \end{bmatrix}.$$

- The above is just a feasibility problem!
- The apparently simpler problem of feasibility is actually deceivingly difficult.
- We don't have to feel bad when we use the simplex algorithm to find an initial BFS (phase I)!

