ECE/MATH 520, Spring 2008

Homework Problems 5

Solutions (version: April 15, 2008, 7:19)

15.1 Convert the following linear programming problem to *standard form*:

$$\begin{array}{ll} \text{maximize} & 2x_1 + x_2 \\ \text{subject to} & 0 \leq x_1 \leq 2 \\ & x_1 + x_2 \leq 3 \\ & x_1 + 2x_2 \leq 5 \\ & x_2 \geq 0 \end{array}$$

Ans.:

$$\begin{aligned} \text{minimize}\, -2x_1 - x_2 & \\ \text{subject to}\, x_1 + x_3 &= 2 \\ x_1 + x_2 + x_4 &= 3 \\ x_1 + 2x_2 + x_5 &= 5 \\ x_1, \ldots, x_5 &\geq 0 \end{aligned}$$

15.5 A cereal manufacturer wishes to produce 1000 pounds of a cereal that contains exactly 10% fiber, 2% fat, and 5% sugar (by weight). The cereal is to be produced by combining four items of raw food material in appropriate proportions. These four items have certain combinations of fiber, fat, and sugar content, and are available at various prices per pound, as shown below:

Item	1	2	3	4
% fiber	3	8	16	4
% fat	6	46	9	9
% sugar	20	5	4	0
Price/lb.	2	4	1	2

The manufacturer wishes to find the amounts of each of the above items to be used to produce the cereal in the least expensive way. Formulate the problem as a linear programming problem. What can you say about the existence of a solution to this problem?

Ans.: Let $x_i \geq 0$, $i = 1, \ldots, 4$, be the weight in pounds of item i to be used. Then, the total weight is $x_1 + x_2 + x_3 + x_4$. To satisfy the percentage content of fiber, fat, and sugar, and the total weight of 1000, we need

$$\begin{aligned}
3x_1 + 8x_2 + 16x_3 + 4x_4 &= 10(x_1 + x_2 + x_3 + x_4) \\
6x_1 + 46x_2 + 9x_3 + 9x_4 &= 2(x_1 + x_2 + x_3 + x_4) \\
20x_1 + 5x_2 + 4x_3 + 0x_4 &= 5(x_1 + x_2 + x_3 + x_4) \\
x_1 + x_2 + x_3 + x_4 &= 1000
\end{aligned}$$

The total cost is $2x_1 + 4x_2 + x_3 + 2x_4$. Therefore, the problem is:

$$\begin{aligned}
\text{minimize} \quad & 2x_1 + 4x_2 + x_3 + 2x_4 \\
\text{subject to} \quad & -7x_1 - 2x_2 + 6x_3 - 6x_4 = 0 \\
& 4x_1 + 44x_2 + 7x_3 + 7x_4 = 0 \\
& 15x_1 - x_3 - 5x_4 = 0 \\
& x_1 + x_2 + x_3 + x_4 = 1000 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}$$

Alternatively, we could have simply replaced $x_1 + x_2 + x_3 + x_4$ in the first three equality constraints above by 1000, to obtain:

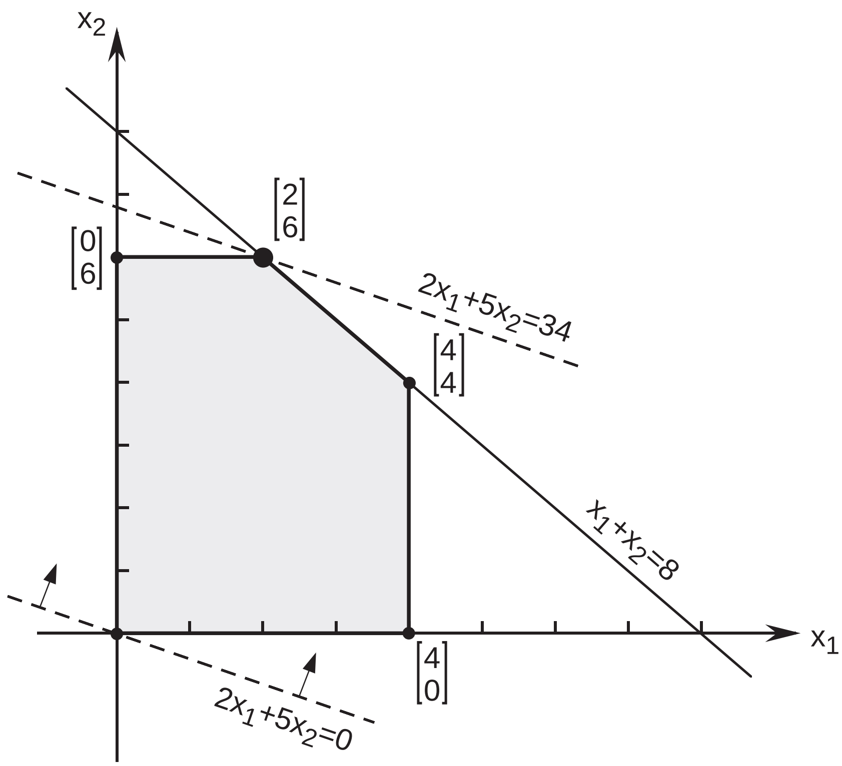
$$\begin{aligned}
3x_1 + 8x_2 + 16x_3 + 4x_4 &= 10000 \\
6x_1 + 46x_2 + 9x_3 + 9x_4 &= 2000 \\
20x_1 + 5x_2 + 4x_3 + 0x_4 &= 5000 \\
x_1 + x_2 + x_3 + x_4 &= 1000.
\end{aligned}$$

Note that the only vector satisfying the above linear equations is $[179, -175, 573, 422]^T$, which is not feasible. Therefore, the constraint does not have any any feasible points, which means that the problem does not have a solution.

15.8 Solve the following linear program graphically:

$$\begin{aligned}
\text{maximize} \quad & 2x_1 + 5x_2 \\
\text{subject to} \quad & 0 \leq x_1 \leq 4 \\
& 0 \leq x_2 \leq 6 \\
& x_1 + x_2 \leq 8
\end{aligned}$$

Ans.: In the figure below, the shaded region corresponds to the feasible set. We then translate the line $2x_1 + 5x_2 = 0$ across the shaded region until the line just touches the region at one point, and the line is as far as possible from the origin. The point of contact is the solution to the problem. In this case, the solution is $[2, 6]^T$, and the corresponding cost is 34.



16.2 Use the simplex method to solve the following linear program:

$$\begin{array}{rl}\text{maximize} & x_1 + x_2 + 3x_3 \\ \text{subject to} & x_1 + x_3 = 1 \\ & x_2 + x_3 = 2 \\ & x_1, x_2, x_3 \geq 0\end{array}$$

Ans.: The problem in standard form is:

$$\begin{array}{rl}\text{minimize} & -x_1 - x_2 - 3x_3 \\ \text{subject to} & x_1 + x_3 = 1 \\ & x_2 + x_3 = 2 \\ & x_1, x_2, x_3 \geq 0.\end{array}$$

We form the tableau for the problem:

$$\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ -1 & -1 & -3 & 0 \end{array}$$

Performing necessary row operations, we obtain a tableau in canonical form:

$$\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & 3 \end{array}$$

We pivot about the $(1,3)$th element to get:

$$\begin{array}{cccc} 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 4 \end{array}$$

The reduced cost coefficients are all nonnegative. Hence, the current basic feasible solution is optimal: $[0,1,1]^T$. The optimal cost is 4.

16.3 Consider the linear program:

$$\begin{array}{rl} \text{maximize} & 2x_1 + x_2 \\ \text{subject to} & 0 \leq x_1 \leq 5 \\ & 0 \leq x_2 \leq 7 \\ & x_1 + x_2 \leq 9 \end{array}$$

Convert the problem to standard form and solve it using the simplex method.

Ans.: The problem in standard form is:

$$\begin{array}{rl} \text{minimize} & -2x_1 - x_2 \\ \text{subject to} & x_1 + x_3 = 5 \\ & x_2 + x_4 = 7 \\ & x_1 + x_2 + x_5 = 9 \\ & x_1, \ldots, x_5 \geq 0. \end{array}$$

We form the tableau for the problem:

$$\begin{array}{cccccc} 1 & 0 & 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 & 0 & 7 \\ 1 & 1 & 0 & 0 & 1 & 9 \\ -2 & -1 & 0 & 0 & 0 & 0 \end{array}$$

The above tableau is already in canonical form, and therefore we can proceed with the Simplex procedure. We first pivot about the $(1,1)$th element, to get

$$\begin{array}{cccccc}
1 & 0 & 1 & 0 & 0 & 5 \\
0 & 1 & 0 & 1 & 0 & 7 \\
0 & 1 & -1 & 0 & 1 & 4 \\
0 & -1 & 2 & 0 & 0 & 10
\end{array}$$

Next, we pivot about the $(3,2)$th element to get

$$\begin{array}{cccccc}
1 & 0 & 1 & 0 & 0 & 5 \\
0 & 0 & 1 & 1 & -1 & 3 \\
0 & 1 & -1 & 0 & 1 & 4 \\
0 & 0 & 1 & 0 & 1 & 14
\end{array}$$

The reduced cost coefficients are all nonnegative. Hence, the optimal solution to the problem in standard form is $[5,4,0,3,0]^T$. The corresponding optimal cost is -14.

16.9a,c Consider a standard form linear programming problem, with

$$\boldsymbol{A} = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}, \qquad \boldsymbol{c} = \begin{bmatrix} 6 \\ c_2 \\ 4 \\ 5 \end{bmatrix}.$$

Suppose that we are told that the relative cost coefficient vector corresponding to some basis is $\boldsymbol{r}^T = [0,1,0,0]$.

a. Find an optimal feasible solution to the given problem.

c. Find c_2.

Ans.: a. By inspection of $\boldsymbol{r}^T$, we conclude that the basic variables are x_1, x_3, x_4, and the basis matrix is

$$\boldsymbol{B} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Since $\boldsymbol{r}^T \geq \boldsymbol{0}^T$, the basic feasible solution corresponding to the basis $\boldsymbol{B}$ is optimal. This optimal basic feasible solution is $[8,0,9,7]^T$.

c. We have $\boldsymbol{r}_D^T = \boldsymbol{c}_D^T - \boldsymbol{\lambda}^T \boldsymbol{D}$, where $\boldsymbol{r}_D^T = [1]$, $\boldsymbol{c}_D^T = [c_2]$, $\boldsymbol{\lambda}^T = [5,6,4]$, and $\boldsymbol{D} = [2,1,3]^T$. We get $1 = c_2 - 10 - 6 - 12$, which yields $c_2 = 29$.

16.10 Consider the linear programming problem:

$$\begin{array}{rl} \text{minimize} & c_1x_1 + c_2x_2 \\ \text{subject to} & 2x_1 + x_2 = 2 \\ & x_1, x_2 \geq 0, \end{array}$$

where $c_1, c_2 \in \mathbb{R}$. Suppose that the problem has an optimal feasible solution that is not basic.

a. Find all basic feasible solutions.

b. Find all possible values of c_1 and c_2.

c. At each basic feasible solution, compute the relative cost coefficients for all nonbasic variables.

Ans.: a. There are two basic feasible solutions: $[1, 0]^T$ and $[0, 2]^T$.

b. The feasible set in $\mathbb{R}^2$ for this problem is the line segment joining the two basic feasible solutions $[1, 0]^T$ and $[0, 2]^T$. Therefore, if the problem has an optimal feasible solution that is not basic, then all points in the feasible set are optimal. For this, we need

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

where $\alpha \in \mathbb{R}$.

c. Since all basic feasible solutions are optimal, the relative cost coefficients are all zero.