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## 1

## 2

### 2.1

### 2.2

### 2.3

$X_n$ :  $S = 0, 1, \dots, r-1, r, \dots, N$  ( $0 - r-1$  transient,  $r - N$  absorbing).

Recall from last time:  $g(i)$  = "rate" to transient state  $i$ . Let  $W_i = E(\sum_{n=0}^{T-1} g(X_n) | X_0 = i)$ .  $g(i)=1 \Rightarrow \sum_{n=0}^{T-1} g(X_n) = T$ .

ex 2.3.4 - if  $g(i) = 1$  if  $i=k$ ,  $0$  if  $i \neq k$ ,  $k$  = transient state, gives  $W_i = W_{ik}$ , mean number of visits to state  $k$  before being absorbed.

Note:  $\sum_{n=0}^{T-1} g(X_n)$  always includes  $g(X_0) = g(i)$ .

If a transition is made from state  $i$  to a transient state  $j$ , the sum will include some future terms. The Markov property implies the future sum proceeding from state  $j$  has value  $W_j$ .

Using the total law of probability, (2.3.3)  $W_i = g(i) + \sum_{j=0}^{r-1} P_{ij} W_j, i = 0, 1, 2, \dots, r-1$ .

exercise: explain this.

Example 2.3.5 - In ex 2.3.3,  $g(i) = 1$  for all  $i$ . This implies that  $\nu_i = E(T | X_0 = i) = W_i$ , and (2.3.4)  $\nu_i = 1 + \sum_{j=0}^{r-1} P_{ij} \nu_j, i = 0, 1, \dots, r-1$ .

Ex 2.3.6 - in ex 2.3.4, (2.3.5)  $W_{ik} = \delta_{ik} + \sum_{j=0}^{r-1} P_{ij} W_{jk}, i = 0, 1, \dots, r-1$ .

Example 2.3.7 - A Markov chain has  $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ .1 & .4 & .4 & .1 \\ .2 & .1 & .6 & .1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ ,  $S=0, 1, 2, 3$ . Start in state 1. Determine probability of being absorbed into state 0, and mean time until absorption. 0, 3 absorbing states. 1, 2 transient. With  $U_{i0} = P(\text{absorption into } 0 | X_0 = i)$ . (2.3.2) gives:  $U_{10} = P_{10} + P_{11}U_{10} + P_{12}U_{20}$ .  $U_{20} = P_{20} + P_{21}U_{10} + P_{22}U_{20}$ .  $U_{10} = .1 + .4U_{10} + .1U_{20}$ .  $U_{20} = .2 + .1U_{10} + .6U_{20}$ .  $\Rightarrow U_{10} = 6/23, U_{20} = 13/23$ . (2.3.4)  $\nu_1 = 1 + .4\nu_1 + .1\nu_2$ .  $\nu_2 = 1 + .1\nu_1 + .6\nu_2$ .  $\Rightarrow \nu_1 = 50/23, \nu_2 = 70/23$ .

## 2.4 Gambler's Ruin

ex 2.1.2, 2.1.9 introduced random walks.  $X_n$  = location. Def 2.4.1 - if  $r_i = 0, p_i = p, q_i = q (q = 1 - p)$ , this is a simple random walk.

ex 2.4.1 - gambler's ruin - We have a game for two people A, B. Total fortune of A and B is \$N. At each step i, A has probability  $p_i$  of winning \$1,  $q_i$  of losing \$1, and  $r_i$  of drawing.  $0 < p_i, q_i < 1, 0 \leq r_i < 1, p_i + q_i + r_i = 1$ . If A's fortune reaches 0 or N, game stops. Let  $X_n$  = fortune of A at time n.  $X_n$  is a Markov chain.  $P = (1, 0, 0, \dots \& q_1, r_1, p_1, 0, \dots \& 0, q_2, r_2, p_2, 0, \dots \& \dots \& 0, \dots, q_{N-1}, r_{N-1}, p_{N-1} \& 0, \dots, 0, 1)$ . States  $k=0, N$  are absorbing.

We address some intermediate time questions using (2.3.2), (2.3.4).

ex 2.4.3 - the probability of ruin for player A starting with \$i in ex 2.4.1 is  $U_i = U_{i0}$  in (2.3.2), (2.4.1)  $U_i = p_i U_{i+1} + r_i U_i + q_i U_{i-1}, i = 1, 2, \dots, N-1$ , with boundary conditions  $U_0 = 1, U_N = 0$ . We can find an explicit formula for the solution in some cases.

ex 2.4.4 - assume  $r_i = 0, p_i = p, q_i = q, q = 1 - p$ . (2.4.1) becomes (2.4.2)  $U_i = p U_{i+1} + q U_{i-1}$  for  $1 \leq i \leq N-1, U_0 = 1, U_N = 0$ . We look for a solution in the form  $U_i = \Theta^i$ .  $\Theta^i = p \Theta^{i+1} + q \Theta^{i-1}, \Theta \neq 0 \Rightarrow \Theta = p \Theta^2 + q$ . This has roots  $\Theta_1 = 1, \Theta_2 = q/p$ . If  $p \neq 1/2, \Theta_1 \neq \Theta_2$ . The general solution is a linear combination  $U_i = A_1 \Theta_1^i + A_2 \Theta_2^i$ .  $A_1, A_2$  are constants. Using the boundary conditions,  $U_0 = 1 = A_1 + A_2, U_N = 0 = A_1 + A_2 \frac{q}{p}^N \Rightarrow (2.3.4) U_i = \frac{(q/p)^i - (q/p)^N}{1 - (q/p)^N}, p \neq 1/2, 0 < i < N$ . If  $p = 1/2, \Theta_1 = \Theta_2 = 1$ , (2.4.4)  $U_i = 1 - i/N, 0 < i < N$ . For mean time to absorption, (2.3.4) becomes (2.4.5)  $\nu_i = 1 + p \nu_{i+1} + q \nu_{i-1}, \nu_0 = \nu_N = 0 \Rightarrow (2.3.6) \nu_i = \frac{1}{q-p} (i - N (\frac{1-(q/p)^i}{1-(q/p)^N}))$  when  $p \neq 1/2, i(N-i)$  when  $p = 1/2$ .

ex 2.4.5 - Suppose in ex 2.4.1 that A has a backer that guarantees A's losses. There is no ruin when A's fortune reaches 0. We can let  $P = (q_0, p_0, 0, \dots \& q_1, r_1, p_1, \dots, \& \dots)$ . If  $p_i = p, q_i = q$ , for all i, the absorbing times satisfy (2.4.5) again, but  $\nu_N = 0, \nu_0 : p \nu_0 = 1 + p \nu_1. q_0 = 0, r_0 + p_0 = 1$ .

## 2.5 Simple Branching Processes

Model for evolution of a population. We start at time 0 with a progenitor. At the first time, the progenitor splits into k offspring with probability  $p_k$ , where  $p_k$  is a pmf, and then dies immediately. We assume the offspring reproduce in the same way. Process continues until extinction - when a generation produces no offspring. Let  $X_n$  = population at time n.

Def 2.5.1,  $X_n$  is a branching process.

Theorem 2.5.1,  $X_n$  is a Markov chain.

Example 2.5.1 - Neutron Chain Reaction - A nucleus is split by a chance collision with a neutron and it releases a random number of new neutrons. These may hit other nuclei and cause further fission.

Definition 2.5.2 - Branching Process - Let  $\{Z_{nj} n \geq 1, j \geq 1\}$  be iid non-negative integer-value random variables with pmf  $p_k$ . Below, if a random sum has zero summands we give it the value zero. The branching process  $X_n$  is defined (2.5.1)  $X_0 = 1, X_1 = Z_{1X_0}, X_2 = Z_{21} + Z_{22} + \dots + Z_{2X_1}, \dots, X_n = Z_{n1} + Z_{n2} + \dots + Z_{nX_{n-1}}$ .  $Z_{nj}$  = number of members in nth generation who are offspring of the jth member of the n-1st generation. Note: if  $X_n = 0$ , then  $X_{n+1} = 0$ , so 0 is absorbing.

Theorem 2.5.2 -  $X_n$  is a Markov chain. Proof: exercise.

The branching process is defined in terms of random sums. We develop a few more useful facts.

Definition 2.5.3 - Random Sums - Let  $Y_1, Y_2, \dots$  be a sequence of iid. r.v. Let  $N$  be a non-negative, integer-valued r.v. independent of  $\{Y_j\}$ . Let  $N$  have pmf  $p_N(n) = p(N = n), n = 0, 1, 2, \dots$ . Set  $X = 0$  for  $N = 0, Y_1 + \dots + Y_N$  for  $N > 0$ .  $X$  is a random sum.

Theorem 2.5.3 - Assume that  $\{Y_i\}$  and  $N$  have finite moments.  $E(Y_i) = \mu, Var(Y_i) = \sigma^2, E(N) = \nu, Var(N) = \tau^2$ . Then (2.5.2)  $E(X) = \mu\nu$ . (2.5.3)  $Var(X) = \nu\sigma^2 + \mu^2\tau^2$ . Proof:  $E(X) = \sum_{n=0}^{\infty} E(X|N=n)p_N(n) = \sum_{n=0}^{\infty} E(Y_1 + \dots + Y_N|N=n)p_N(n) = \sum_{n=0}^{\infty} E(Y_1 + \dots + Y_n)p_N(n) = \mu \sum_{n=0}^{\infty} n p_N(n)$ . (2.5.3) is exercise.

Little notation: if  $Y$  is a random variable with pmf  $\{p_k\}$ , we define  $\mu = E(Y), \sigma^2 = Var(Y)$ .

Theorem 2.5.4 - Let  $X_n$  be a branching process with pmf  $\{P_k\}$  and assume  $\mu, \sigma^2$  are finite. Let  $M_{(n)}, V_{(n)}$  be the mean and variance of  $X_n$  conditioned on  $X_0 = 1$ . (2.5.4)  $M(n) = \mu^n$ . (2.5.5)  $V(n) = \sigma^2 \mu^{n-1} * \{n \text{ for } \mu = 1, \frac{1-\mu^n}{1-\mu} \text{ for } \mu \neq 1$ . Proof: using (2.5.2) in (2.5.1) gives (2.5.6)  $\{M(n+1) = \mu M(n), V(n+1) = \sigma^2 M(n) + \mu^2 V(n), X_0 = 1, M(0) = 1, V(0) = 0$ . Now iterate.  $M(n+1) = \mu M(n) = \mu \mu M(n-1) = \mu^3 M(n-2) = \dots$ . This gives the result.

The mean population increases geometrically if  $\mu > 1$ , decreases geometrically if  $\mu < 1$ , and is fixed if  $\mu = 1$ . The variance increases or decreases geometrically if  $\mu > 1$  or  $\mu < 1$ . It increases linearly if  $\mu = 1$ .

The probability of extinction:

Definition 2.5.4 - The random time of extinction  $N$  is the first time for which  $X_N = 0$ . This is an absorption time. We let (2.5.7)  $U_n = P(N \leq n | X_0 = 1) = P(X_n = 0 | X_0 = 1)$  be the probability of extinction at or prior to the  $n$ th generation, conditioned on  $X_0 = 1$ .

Theorem 2.5.5 - We have (2.5.8)  $U_0 = 0, U_1 = P_0, U_n = \sum_{k=0}^{\infty} P_k(U_{n-1})^k, n \geq 2$ . Proof: the single parent  $X_0$  has  $Z_{1,X_0} = k$  offspring. These offspring in turn have more offspring. If the original population dies out in  $n$  generations, then each of these  $k$  offspring lines die out in  $n - 1$  generations, or less. The  $k$  offspring lines are independent of each other and have the same statistics as the original generation. The probability that one of the  $k$  offspring lines dies out is  $U_{n-1}$ . So the probability that they all die out is  $(U_{n-1})^k$ . The total law of probability gives (2.5.8).

Example 2.5.3 -  $P_0(1/4), P_1(1/8), P_2(1/2), P_3(1/8), P_4(0), \dots$

Recall §1.5. Recall Thm 1.5.3: if  $Y_1, \dots, Y_n$  are indep r.v. having prob. generating functions  $P_{Y_1}, \dots, P_{Y_n}$ , then the generating function for  $X = Y_1 + \dots + Y_n$  is (2.5.9)  $P_X(s) = P_{Y_1}(s) \dots P_{Y_n}(s)$ .

Thm 1.5.1: if a r.v.  $Y$  has pmf  $\{P_k\}$  and prob. generating function  $P_Y$ , (2.5.10)  $\frac{dP_Y(1)}{ds} = P_1 + 2P_2 + 3P_3 + \dots = E(Y)$ . (2.5.11)  $Var(Y) = \frac{d^2 P_Y(s)}{ds^2} \big|_{s=1} + \frac{dP_Y(s)}{ds} \big|_{s=1} - (\frac{dP_Y(s)}{ds} \big|_{s=1})^2$ . See § 9.2 in text.

Theorem 2.5.6 - If  $Z_1, Z_2, \dots$  is a sequence of iid r.v. with common generating function  $P_Z$  and if  $N > 0$  is an integer-valued non-negative r.v. indep of  $\{Z_i\}$  with prob generating function  $P_N$ , then  $X = Z_1 + \dots + Z_N$  has prob. generating function (2.5.12)  $P_X(s) = P_N(P_Z(s))$ . Proof:  $P_X(s) = E(s^X) = E(E(s^X | N)) = \sum_n E(s^X | N = n) P(N = n) = \sum_n E(s^{Z_1 + \dots + Z_n}) P(N = n) = \sum_n E(s^{Z_1}) \dots E(s^{Z_n}) P(N = n) = \sum_n (P_Z(s))^n P(N = n) = P_N(P_Z(s))$ .

Returning to a branching process  $X_n$ , we assume the offspring pmf  $\{P_k\}$  has generating function  $\phi(s) = E(s^{X_1}) = \sum_k P_k s^k$ . We want the prob. generating function  $\phi_n$  for  $X_n$ .

Theorem 2.5.7 - We have (2.5.13)  $\phi_{m+n}(s) = \phi_m(\phi_n(s)) = \phi_n(\phi_m(s))$ . (2.5.14)  $\phi_n(s) = \phi(\phi(\phi(\dots \phi(s)))) \dots$  ( $n$  compositions). Proof: Every member of the  $(m+n)$ th generation has a unique ancestor in the  $m$ th generation. So  $X_{m+n} = Z_1 + \dots + Z_{X_m}$ ,  $Z_i$  = number of members of the  $(m+n)$ th generation that descend from the  $i$ th member of the  $m$ th generation. The  $Z_i$  are iid r.v. with the same distribution as  $X_n$ , by the Markov property. By theorem 2.5.6,  $\phi_{m+n}(s) = \phi_m(\phi_{Z_1}(s)), \phi_{Z_1}(s) = \phi_n(s)$ .

$X_n$  branching process.  $\{P_k\}$  offspring distribution.  $\phi(s) = E(s^{X_1}) = \sum_{k=0}^{\infty} P_k s^k$ . We are interested in  $\phi_n(s)$  = prob. gen. function of  $X_n$ .

Fixed point theory (aside, not in notes). Solve the problem given a function  $g$ , determine  $x$  such that  $g(x) = x$  [fixed point problem,  $x$  = fixed point].

Define  $f(s) = g(s) - s$  then  $f(x) = 0$  iff  $g(x) = x$ . root problem connected to fixed point problems.

Fixed point iteration. Given an initial guess  $X_0$ , let  $X_1 = g(X_0), X_2 = g(X_1), \dots$ . Under certain conditions (Banach, ...),  $X_n \rightarrow X$ .

Example 2.5.3 - Suppose  $0 \leq p \leq 1$  and pmf for the offspring is  $\{qp^k\}_{k \geq 0}, q = 1 - p$ . The prob. gen. function is  $\phi(s) = q(1 - ps)^{-1}$ .  $\sum_{k=0}^{\infty} qp^k s^k = q \sum_{k=0}^{\infty} (ps)^k, |ps| < 1$ . Each family size is one less than a geometric variable. By induction,  $\phi_n(s) = \begin{cases} \frac{n-(n-1)s}{n+1-ns} \text{ for } p = q = 1/2, \\ \frac{q(p^n - q^n - ps(p^{n-1} - q^{n-1}))}{p^{n+1} - q^{n+1} - ps(p^n - q^n)} \text{ for } p \neq q. \end{cases}$

We have  $\{\text{ultimate extinction}\} = U_n \{X_n = 0\}$ . Moreover,  $A_n = \{X_n = 0\}$ , satisfies  $A_n < A_{n+1} < \dots$ . Thus,  $A = \bigcup_{n=1}^{\infty} A_n = \lim_{n \rightarrow \infty} A_n$ .  $P(A) = \lim_{n \rightarrow \infty} P(A_n)$ .  $P(X_n = 0) \rightarrow_{n \rightarrow \infty} P(\text{ultimate extinction}) = \{1 \text{ for } p \leq q, q/p \text{ for } p > q$ .

Extinction occurs with probability 1 if  $\mu = E(X_1) = p/q = \frac{1}{(\frac{q}{p})}$  satisfies  $\mu < 1$ . Makes sense: if  $E(X_1) < 1$  then  $X_n = 0$  at some point.

Theorem 2.5.8 - let  $\phi$  be the prob. gen. function of the offspring. Then  $P(\text{ultimate extinction}) = \lim_{n \rightarrow \infty} P(X_n = 0) = \eta$ , where  $\eta$  is the smallest nonnegative fixed point of the equation (2.5.15)  $\phi(s) = s$ . (1) If  $|\phi'(1)| < 1, \eta > 1$ . (2) If  $|\phi'(1)| > 1, \eta < 1$ . (3) If  $|\phi'(1)| = 1, \eta = 1$  as long as the offspring distribution has positive variance.

Proof: Let  $\eta_n = P(X_n = 0)$ . Theorem 2.5.7 implies (2.5.16)  $\eta_n = \phi_n(0) = \phi(\phi_{n-1}(0)) = \phi(\eta_{n-1})$  (fixed point iteration). From the discussion above,  $\eta_n \uparrow \eta$ . Since  $\phi$  is continuous, take the limit in (2.5.16) gives  $\eta = \phi(\eta), \eta_n = \phi(\eta_{n-1}), \lim_{n \rightarrow \infty} \eta_n = \eta = \phi(\lim_{n \rightarrow \infty} \eta_{n-1}) = \phi(\eta)$ .

Suppose  $\psi$  is a nonnegative root of  $s = \phi(s)$ . We show  $\eta \leq \psi$ .  $\phi$  is nondecreasing on  $[0, 1]$ .  $\phi(s) = \sum_{k=0}^{\infty} P_k s^k$ .  $\eta_1 = \phi(0) \leq \phi(\psi) = \psi, \eta_2 = \phi(\eta_1) \leq \phi(\psi) = \psi, \dots, \eta \leq \psi$ .

To prove the rest, we use the fact that  $\phi$  is convex.  $\phi''(s) = E(X_1(X_1 - 1)s^{X_1-2}) \geq 0$  if  $s \geq 0$ .  $\phi$  is convex and nondecreasing on  $[0, 1]$  and  $\phi(1) = 1$ . The two curves  $Y = s$  and  $Y = \phi(s)$  can have two intersection points in  $[0, 1]$ ,  $\eta$  and 1. If  $|\phi'(1)| < 1$ ,  $\eta = 1$ . If  $|\phi'(1)| > 1$ ,  $\eta \neq 1$ . (Consult graphs in notes.)

Example 2.5.4 - An individual has a disease. Transmits the disease to some fraction of people that are contacted. Assume: a person is infectious and has the disease for a brief period. Each infected person contacts a number of people with a Poisson distribution with mean 10, and each contact has probability  $p$  of being infected. Model the infected population  $X_n$ . For  $p = .2$ , what is the prob. that the disease will die out. If  $p < .1$ , the disease will die out (ultimate extinction). prob gen func of a Poisson process with mean  $\mu$  is  $e^{\mu(s-1)}$ , so we solve  $s = e^{.2*10(s-1)} = e^{2(s-1)} \Rightarrow s \approx .2$ .

Example 2.5.5 - Suppose  $p_0 = 1/5, p_1 = 1/2, p_2 = 3/10, p_i = 0, i \geq 3$ .  $\mu = 1/2 + 2 * 3/10 = 11/10 > 1$ .  $\phi(s) = 1/5 + 1/2s + 3/10s^2$ . Verify  $\phi'(1) = \mu$ . Prob of ultimate extinction:  $1/5 + t/2 + \frac{3t^2}{10} = t \Rightarrow \eta = 2/3, \eta = 1$  (2/3 was circled, 1 not circled). See ex 2.5.6 in notes.

### 3 Long time behavior for Markov Chains

#### 3.1 Classification of States

Viewpoint here: think of the Markov chain as describing the motion of a particle in the state space. Does a particle return to its starting state within some time? It suffices to consider the distribution of the length of time until the particle returns the first time. Let  $X_n$  be a Markov chain with state space  $S$ .

Definition 3.1.1 - A state  $i$  is persistent or recurrent if  $P(X_n = i \text{ for some } n \geq 1 | X_0 = i) = 1$ . If  $P(X_n = i \text{ for some } n \geq 1 | X_0 = i) < 1$ ,  $i$  is transient.

Recall - Long time analysis - Classification of states. Recurrent state:  $P(X_n = i \text{ for some } n \geq 1 | X_0 = i) = 1$ . Transient state:  $P(X_n = i \text{ for some } n \geq 1 | X_0 = i) < 1$ . First passage time: the smallest time it takes to go from state  $i$  to state  $j$ . We are interested in mean first passage time.

Theorem 3.1.2 - (1)  $j$  is recurrent if  $\sum_n P_{jj}^n = \infty$ . (2)  $j$  is transient if  $\sum_n P_{jj}^n < \infty$ . (3) If  $j$  is transient, then  $P_{ij}^n \rightarrow 0$  as  $n \rightarrow \infty$  for all  $i$ .

Example 3.1.6 - Random Walk. We consider the simple random walk in ex 2.2.2.  $X_n = X_0 + \sum_{k=1}^n B_k$ ,  $\{B_k\}$  are i.i.d. Bernoulli variables.  $P(B_k = 1) = p, P(B_k = -1) = 1 - p = q$ . Consider state  $j$ .  $P_{jj}^{2n-1} = 0$  for  $n = 1, 2, 3, \dots$  ( $2n-1 = \text{odd numbers}, n = 1, 2, 3, \dots$ ) To return in  $2n$  steps: we must take  $n$  steps in one direction and then  $n$  in other direction. This has probability (3.1.3)  $P_{jj}^{2n} = \binom{2n}{n} p^n (1-p)^n = \frac{(2n)!}{n!n!} (p(1-p))^n$ . We approximate these terms by an expression valid for  $n$  large then consider the sum of the approximations. Deciding if a series converges or not is not affected if we drop a finite number of terms from the beginning of the series. We use an asymptotic expression for  $n!$  valid for  $n$  large.

Stirling formula - (3.1.4)  $n! \sim n^n \sqrt{n} e^{-n} \sqrt{2\pi}$   $n$  large, which means  $\lim_{n \rightarrow \infty} \frac{n!}{n^n \sqrt{n} e^{-n} \sqrt{2\pi}} = 1$ . We can substitute (3.1.4) into the series  $\sum_n P_{jj}^{2n}$  without affecting convergence/divergence.  $P_{jj}^{2n} \sim \frac{(4p(1-p))^n}{\sqrt{\pi n}}$ . When  $p = 1/2$ ,  $P_{jj}^{2n} \sim \frac{1}{\sqrt{\pi n}}$ ,  $\sum_n P_{jj}^n = \infty$ . Any state is recurrent when  $p = 1/2$ . If  $p \neq 1/2$ ,  $4p(1-p) < 1$ ,  $\sum_n P_{jj}^n < \infty$ . Any state is transient when  $p \neq 1/2$ . Note: theorem 3.1.3 implies that any state is either recurrent or transient.

Theorem 3.1.4 - the number of times  $N(i)$  that a Markov chain visits its starting point  $i$  satisfies  $P(N(i) = \infty) = \{1 \text{ if } i \text{ is recurrent, } 0 \text{ if } i \text{ is transient. Proof - After any return to } i, \text{ a subsequent return is guaranteed iff } f_{ii} = 1.$

Another classification -

Definition 3.1.5 - Let  $T_j = \min\{n \geq 1 : X_n = j\}$  be the time of the first visit to state  $j$  where  $T_j = \infty$  if  $X_n$  never visits  $j$ . ( $T_j$  depends on  $X_0$ .)

Theorem 3.1.5 -  $P(T_i = \infty | X_0 = i) > 0$  iff  $i$  is transient. When  $i$  is transient,  $E(T_i | X_0 = i) = \infty$ . What about recurrent states?

Definition 3.1.6 - The mean recurrence time  $\mu_i$  of a state  $i$  is:  $\mu_i = E(T_i | X_0 = i) = \{\sum_{n=1}^{\infty} f_{ii}(n)$  for  $i$  recurrent,  $\infty$  for  $i$  transient.  $\mu_i$  may be infinite when  $i$  is recurrent.

Definition 3.1.7 - A recurrent state  $i$  is null if  $\mu_i = \infty$  and positive if  $\mu_i < \infty$ .

Theorem 3.1.6 - A recurrent state is null iff  $P_{ii}^n \rightarrow 0$  as  $n \rightarrow \infty$  and if this holds,  $P_{ji}^n \rightarrow 0$  for all  $j$ . Proof later.

Example 3.1.7 - Consider the genotype example 3.1.4. AA and aa are recurrent ( $0 = \text{"aa"}$ ).  $f_{00}(1) = 1$ ,  $f_{00}(n) = 0$ ,  $n > 1$ ,  $\Rightarrow f_{00} = 1$ . These states are positive.

Example 3.1.8 - For simple random walk, ex 3.1.6, when  $p = 1/2$ ,  $P_{jj}^n \approx \frac{1}{\sqrt{\pi n}} \rightarrow 0$  as  $n \rightarrow \infty$ . So any state in a simple random walk with  $p = 1/2$  is null recurrent.

The last classification of states we discuss: recall in the simple random walk, the chain can return only with an even number of steps, 2, 4, 6, /dots all divisible by 2.

Definition 3.1.8 - The greatest common divisor of a set of integers  $\{n_1, n_2, \dots\}$  written  $\text{g.c.d.}(n_1, n_2, \dots)$  is the largest integer  $m$  such that  $m$  divides  $n_1, n_2, \dots$  all without remainder.

Example 3.1.9 -  $\text{gcd}(2, 4, 6, 8) = 2$ .  $\text{gcd}(2, 3, 5) = 1$ .

Definition 3.1.9 - The period  $d(i)$  of state  $i$  is  $d(i) = \text{gcd}\{n : P_{ii}^n > 0\}$ . If  $d(i) = 1$ ,  $i$  is called aperiodic. If  $d(i) > 1$ ,  $i$  is called periodic.

Ex 3.1.10 - Consider the OFF/ON system in ex 3.1.5. If  $0 < p < 1$ ,  $0 < q < 1$ , then  $P_{00}, P_{01}, P_{10}, P_{11}$  are all strictly between 0 and 1. Hence  $d(i) = 1$  for  $i = 0, 1$ . Suppose  $p = q = 1$ , then  $P_{00}^n > 0$  for  $n$  even,  $P_{00}^n = 0$  for  $n$  odd.  $d(0) = 2$ .

Example 3.1.11 - Simple random walk is periodic with  $d(i) = 2$  when  $p = 1/2$ .

Ex 3.1.12 - Consider gambler's ruin in §2.4, modified so A has \$1 initially, A has a backer that guarantees A's losses (ex 2.4.5), B is infinitely wealthy. We assume  $r_1 = r_2 = \dots = 0$ ,  $p_0 = p_1 = \dots = p$ ,  $q_1 = q_2 = \dots = q$ . Exercise:  $P = (q \text{ p } 0 \dots \& 0 \text{ q } 0 \text{ p } 0 \dots \& 0 \text{ 0 } q \text{ 0 } p \text{ 0 } \dots)$ .  $P_{11}^1 = 0$ ,  $P_{11}^2 > 0$ ,  $P_{11}^3 > 0$ ,  $d(i) = 1$ , single  $\text{gcd}(2, 3) = 1$ .

Definition 3.1.10 - If all the states of a Markov chain are aperiodic, we call the chain aperiodic.

Definition 3.1.11 - A state is ergodic if it is recurrent, positive, and aperiodic.

Ex 3.1.13 - Consider a branching process. 0 is absorbing and once there a chain never leaves. So  $P_{00}^n = 1$  for all  $n$ , and 0 is recurrent. Using the formulas for  $f_{ii}, \mu_0 = 1$ , 0 is positive, 0 is aperiodic, so 0 is ergodic. All other states are transient.

## 3.2 Classification of Chains

Definition 3.2.1 - State  $i$  communicates with state  $j$   $i \rightarrow j$  if the chain may visit  $j$  with positive probability having started in  $i$ .  $i \rightarrow j \Leftrightarrow P_{ij}^m > 0$  some  $m$ . If  $i \rightarrow j$  and  $j \rightarrow i$ , then  $i$  and  $j$  intercommunicate,  $i \leftrightarrow j$ .

Example 3.2.1 - In the roulette wheel in ex 2.1.6, all nonzero states intercommunicate, and 0 only communicates with itself.

Theorem 3.2.1 - If  $i \neq j$  then  $i \rightarrow j$  iff  $f_{ij} > 0$ . Proof: exercise. Note,  $i \leftrightarrow i$  since  $P_{ii}^0 = 1$ . If we fix  $i$ , we could search for all states  $j$  that intercommunicate with it.

Def 3.2.2 - All the states that intercommunicate with a given state form a communication class.  $i \leftrightarrow j, j \leftrightarrow k$ , so  $i \leftrightarrow k$ .

Definition 3.2.3 - An equivalence relation  $\sim$  on a set  $S$  is an operation on pairs of elements satisfying (1)  $a \sim a, a \in S$ , (2)  $a \sim b \Rightarrow b \sim a, a, b \in S$ , (3)  $a \sim b, b \sim c \Rightarrow a \sim c$ .

Theorem 3.2.2 -  $\leftrightarrow$  is an equivalence relation.

Example 3.2.2 - In the roulette wheel, ex 3.2.1, there are two communication classes  $\{0\}, \{1, 2, \dots, 38\}$ .

Example 3.2.3 - In the genotype example, ex 2.1.7, each state forms its own class,  $\{AA\}, \{aa\}, \{Aa\}$ .

Example 3.2.4 - In the ON/OFF system, ex 2.2.3, with  $0 < p < 1, 0 < q < 1$ , there is one class:  $\{ON, OFF\}$ .

Theorem 3.2.3 - If  $i \leftrightarrow j$ , (1)  $i$  is transient iff  $j$  is transient, (2)  $i$  and  $j$  have the same period, (3)  $i$  is null recurrent iff  $j$  is null recurrent.

Proof - (1) if  $i \leftrightarrow j$ , there are  $m, n \geq 0$  such that  $\alpha = P_{ij}^m P_{ji}^n > 0$ . By the Chapman-Kolmogorov equations (2.2.1)  $P_{ii}^{m+r+n} \geq P_{ij}^m P_{jj}^r P_{ji}^n$  for any integer  $r > 0$ . Summing over  $r$ ,  $\sum_{r=0}^{\infty} P_{ii}^r < \infty \Rightarrow \sum_{r=0}^{\infty} P_{jj}^r < \infty$ . The argument holds with  $j$  and  $i$  reversed. By thm 3.1.2, (1) holds. (2) exercise. (3) will be proved below.

Definition 3.2.4 - A set  $C$  of states in the state space  $S$  is closed if  $P_{ij} = 0$  for all  $i \in C$  and  $j \notin C$ . A closed set with one element is called absorbing. Once a Markov chain takes a value in a closed set, it never leaves.

Example 3.2.5 - Consider the chain with  $S = \{0, 1, 2\}$ .  $\{0, 2\}$  forms a closed set.

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1/4 & 1/2 & 1/4 \\ 1 & 0 & 0 \end{pmatrix}$$

Definition 3.2.5 - A set  $C$  of states in the state space  $S$  is irreducible if  $i \leftrightarrow j$  for all  $i, j \in C$ . The communication classes of a Markov chain are irreducible.

Because of theorem 3.2.3, Definition 3.2.6 - An irreducible set  $C$  is periodic, transient, or null recurrent if all or any of the states in  $C$  have these properties.

Definition 3.2.7 - If the entire state space is irreducible, we say the Markov chain is irreducible.

Theorem 3.2.4 - In an irreducible Markov chain, either all states are transient or all states are recurrent.

Theorem 3.2.5 - Decomposition Theorem - The state space  $S$  can be partitioned uniquely as  $S = T \cup C_1 \cup C_2 \cup \dots$ ,  $T = \{\text{transient states}\}$ ,  $\{C_i\} = \text{irreducible, closed sets of recurrent states}$ . Lots of words in notes.

Proof: Let  $\{C_j\}$  be the recurrent equivalence classes of intercommunication ( $\leftrightarrow$ ). We only need to show that each  $C_r$  is closed. Suppose on the contrary that  $i \in C_r$ ,  $j \notin C_r$ , and  $P_{ij} > 0$ ,  $j \not\rightarrow i$ , so  $P(X_n \text{ never returns to } i) \geq P(X_n \text{ reaches } j), P(X_n \neq i \text{ for } n \geq 1 | X_0 = i) \geq P(X_1 = j | X_0 = i) > 0$ . This contradicts the assumption that  $i$  is recurrent.

Markov chains with finite state spaces are special. For example, it is impossible to stay in transient states for all time.

Theorem 3.2.6 - If the state space is finite, then at least one state is recurrent and all recurrent states are positive.

Proof - assume all states are transient.  $1 = \sum_{j \in S} P_{ij}^n$  (finite sum). So,  $1 = \lim_{n \rightarrow \infty} \sum_{j \in S} P_{ij}^n = \sum_{j \in S} \lim_{n \rightarrow \infty} P_{ij}^n = 0$  (by thm 3.1.2(3)). (in general  $\lim_{n \rightarrow \infty} \sum_{i=0}^{\infty} a_i(n) \neq \sum_{i=0}^{\infty} \lim_{n \rightarrow \infty} a_i(n)$ ). That is a contradiction. The same argument works for the closed set of all null recurrent states (exercise).

Theorem 3.2.7 - Suppose the state space is finite.  $i$  is transient  $\iff$  there is a state  $j$  with  $i \rightarrow j$  but  $j \not\rightarrow i$ .

Example 3.2.6 - Consider a Markov chain with  $S = \{0, 1, 2, 3, 4\}$ .

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/2 & 1/8 & 3/8 \end{pmatrix}$$

$C_1 = \{0, 1\}$ ,  $C_2 = \{2, 3, 4\}$  are both closed. They are both irreducible. They must contain positive recurrent states.  $S = C_1 \cup C_2$ .

Example 3.2.7 - Consider random walk with  $(1 \ 0 \ 0 \dots \& \ q \ 0 \ p \ 0 \dots \& \ 0 \ q \ 0 \ p \ 0 \dots \& \ 0 \ 0 \ q \ 0 \ p \ 0 \dots \& \dots \& \dots \ 0 \ q \ 0 \ p \& \ 0 \dots \ 0 \ 0 \ 1)$ , states  $0, \dots, N$ .

Three classes  $C_1 = \{0\}$ ,  $T = \{1, 2, \dots, N-1\}$ ,  $C_2 = \{N\}$ .  $\{1, 2, \dots, N-1\} \rightarrow \{0\}$ , but  $\{0\} \not\rightarrow \{1, 2, \dots, N-1\}$ .  $\{1, 2, \dots, N-1\} \rightarrow \{N\}$ , but  $\{N\} \not\rightarrow \{1, 2, \dots, N-1\}$ .  $\{0\}$  and  $\{N\}$  are absorbing.

State  $i$  is recurrent:  $P(X_n = i \text{ some } n \geq 1 | X_0 = i) = 1$ .  $i$  is transient:  $P(X_n = i \text{ some } n \geq 1 | X_0 = i) < 1$ .  $f_{ij}(n) = P(X_1 \neq j, X_2 \neq j, \dots, X_n \neq j, X_{n+1} = j | X_0 = i)$ .  $f_{ij} = \sum_{n=1}^{\infty} f_{ij}(n)$ .  $j$  is recurrent  $\iff f_{jj} = 1$ .

Theorem -  $j$  is recurrent if  $\sum_n P_{jj}^n = \infty$ .  $j$  is transient if  $\sum_n P_{jj}^n < \infty$ .

$T_j = \min\{n \geq 1 : X_n = j\}$ . time of first visit to  $j(X_0 = i)$ .  $P(T_i = \infty | X_0 = i) > 0 \iff i$  is transient.  $\mu_i = E(T_i | X_0 = i) = \{\sum_{n=1}^{\infty} f_{ii}(n) \text{ for } i \text{ recurrent, } \infty \text{ for } i \text{ transient}\}$ . Recurrent state is null if  $\mu_i = \infty$ . Recurrent state is positive if  $\mu_i < \infty$ .

Theorem - Recurrent state is null iff  $P_{ii}^n \rightarrow 0$  as  $n \rightarrow \infty$ .

Example 3.2.8 -  $S = \{0, 1, 2, 3, 4, 5\}$ .

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

.  $\{0, 1\}, \{4, 5\}$  irreducible and closed. They can't(?) in positive recurrent states. 2, 3 are transient:  $2 \rightarrow 3 \rightarrow 5$ . But return to 2 or 3 from 5 is impossible.  $T = \{2, 3\}, C_1 = \{0, 1\}, C_2 = \{4, 5\}$ . All states have period 1 since  $P_{ii} > 0$  for all  $i$  (all entries on diagonal  $> 0$ ). 0, 1, 4, 5 are ergodic. We can compute  $f_0(1) = P_{00} = 1/2, f_{00}(n) = P_{01}(P_{11})^{n-2}P_{10} = 1/2(3/4)^{n-2}1/4, n \geq 2, \mu_0 = \sum_n f_{00}(n) * n = 3$ .

Example 3.2.9 - Success Runs -  $S = \{0, 1, \dots\}$ .  $P = (q_0 \ p_0 \ 0 \dots \ \& \ q_1 \ 0 \ p_1 \ 0 \dots \ \& \ q_2 \ 0 \ 0 \ p_2 \ 0 \dots \ \& \dots)$ .  $q_{ii}p_i \geq 0, q_i + p_i = 1$  for all  $i$ . This is a success run chain. Intuition: assume  $p_i = p$  for all  $i$ . We attempt independent Bernoulli trials with probability  $p$  of success. We count the number of successful trials in a row. If we had  $n$  successes in a row, we can extend the run to  $n+1$  if we have success on the next trial or we start over with a run of 0 if we fail in the next trial. This gives the row  $(q(0\text{th}) \ 0 \dots \ p((n+1)\text{st}) \ 0 \dots)$ . We assume  $0 < p_i < 1$  for all  $i$  so the chain is irreducible. This means state  $i$  is recurrent iff state 0 is recurrent. We have  $f_{00}(1) = q_0$ , and for  $n \geq 2, f_{00}(n) = P(X_1 = 1, X_2 = 2, \dots, X_{n-1} = n-1, X_n = 0 | X_0 = 0) = P_0 P_1 P_2 \dots P_{n-2} * q_{n-1}$ . Set  $U_n = \prod_{i=0}^n P_i, n \geq 0$  since  $q_{n-1} = 1 - P_{n-1}, f_{00}(n) = U_{n-2} - U_{n-1} = \prod_{i=0}^{n-2} P_i (1 - P_{n-1})$ . So  $\sum_{n=1}^{\infty} f_{00}(n) = q_0 + (U_0 - U_1) + \dots + (U_{N-1} - U_N) = q_0 + U_0 - U_N = 1 - U_N$ . 0 is recurrent iff  $U_N = \prod_{i=0}^N P_i \rightarrow 0$  as  $N \rightarrow \infty$ . L'Hopital's rule implies that if  $0 < P_i < 1$  for all  $i, U_N = \prod_{i=0}^N P_i \rightarrow 0 \iff \sum_{i=0}^{\infty} (1 - P_i) = \infty, \prod_{i=0}^{\infty} P_i > 0 \iff \sum_{i=0}^{\infty} (1 - P_i) < \infty$ . 0 is recurrent iff  $\sum_{i=0}^{\infty} (1 - P_i) = \infty$ , or the  $P_i$ 's cannot be too close to 1. If  $P_i = 1 - (1/2)^i$ , not recurrent.  $P_i$  constant then recurrent. (Chapter IV, section §3 in text.)

### 3.3 Stationary distributions and the limit theorem

We consider behavior as  $n \rightarrow \infty$ . Does the distribution of  $X_n$  converge to something?

Example 3.3.1 - ON/OFF system - ex 2.2.3 -  $P = (1 - p, p \ \& \ q, 1 - q)$ .  $P^n$  as before.  $0 < p < 1, 0 < q < 1, P^n \rightarrow \frac{1}{p+q} * (q \ p \ \& \ q \ p)$  [as before]. We choose the initial state  $X_0$  according to the probabilities  $P(X_0 = 0) = \nu_0, P(X_0 = 1) = \nu_1 = 1 - \nu_0$ .

Definition 3.3.1 - An initial distribution is a probability distribution for the initial state of a Markov chain. The probability distribution of  $X_1$ , conditioned on  $X_0$  is  $P(X_1 = j | X_0) = P_{0j}\nu_0 + P_{1j}\nu_1, j = 0, 1$ . Matrix notation  $(P(X_1 = 0 | X_0) P(X_1 = 1 | X_0)) = \nu p$ . Suppose we take  $\nu_0 = \frac{q}{q+p}, \nu_1 = \frac{p}{q+p}$ . If we compute,  $P(X_1 = 0) = (1 - p)\frac{q}{q+p} + q\frac{p}{q+p} = \frac{q}{p+q} = \nu_0$  and  $P(X_1 = 1) = \nu_1$ . In matrix notation  $\nu = \nu p$ . That particular initial distribution does not change over time.

Definition 3.3.2 - Let  $S$  = state space, The vector  $\Pi$  is a stationary distribution if  $\Pi = (\Pi_i)_{i \in S}$  satisfies (1)  $\Pi_i \geq 0$  all  $i, \sum_{i \in S} \Pi_i = 1$ , (2)  $\Pi = \Pi P (\Pi_j = \sum_{i \in S} \Pi_i P_{ij} \text{ all } j \in S)$ .  $P$  = probability transition matrix. These are also called invariant distributions and equilibrium distributions.

Theorem 3.3.1 - If  $\Pi$  is a stationary distribution, (3.3.1)  $\Pi P^n = \Pi$  for all  $n \geq 0$ . If  $X_0$  has distribution  $\Pi$ , then so does  $X_n$  for  $n \geq 0$ . Proof: exercise.

Aside: long time behavior of ODE's  $\dot{y} = f(y)$ . Stead-state/equilibrium solutions  $f(y_s) = 0 \Rightarrow y_s$  constant,  $\dot{y}_s = f(y_s) = 0$ .

We assume the chain is irreducible and explore the existence of stationary distributions.

Example 3.3.2 - Consider ex 3.3.1 (ON/OFF),  $\Pi = \Pi P = (\Pi_0 \Pi_1)(1 - p, p \ \& \ q, 1 - q) = (\Pi_0 \Pi_1) \Rightarrow \Pi_1 = p/q \Pi_0$  (1st equation),  $\Pi_1 = p/q \Pi_0$  (2nd equation),  $\Pi_0 + \Pi_1 = 1 \Rightarrow \Pi = (\frac{q}{p+q}, \frac{p}{p+q})$ .

The assumption that the chain is irreducible is important.

Example 3.3.3 - Consider the genotype example, ex 3.1.4:  $P = (1 \ 0 \ 0 \ \& \ 1/4 \ 1/2 \ 1/4 \ \& \ 0 \ 0 \ 1)$ . The chain is not irreducible. We try to find a stationary distribution anyway.  $\Pi = \Pi P$  becomes  $\Pi_0 \Pi_1 \Pi_2) (1 \ 0 \ 0 \ \& \ 1/4 \ 1/2 \ 1/4 \ \& \ 0 \ 0 \ 1) = (\Pi_0 \Pi_1 \Pi_2)$ . 1st:  $\Pi_0 + 1/4 \Pi_1 = \Pi_0 \Rightarrow \Pi_1 = 0$ . 2nd:  $0 = 0$ . 3rd:  $\Pi_2 = \Pi_2$ . Any distribution of the form  $\Pi = (\alpha, 0, 1 - \alpha, 0 \leq \alpha \leq 1)$  is a stationary distribution.

In the finite state space case, we later prove: Theorem 3.3.2 - If the state space has  $r$  states, then  $\Pi = \Pi P$  has at most  $r - 1$  linearly independent equations. Including the equation  $\sum_{j \in S} \Pi_j = 1$ , we obtain at most

$r$  linearly independent equations. There is always atleast one solution. If the chain is not irreducible there may be more than one solution. The situation with an infinite state space case is more complicated.

Example 3.3.4 - Consider the gambler's ruin model, ex 3.1.12, where A has a backer and B is infinitely rich. Assume  $r_i = 0$ , and  $q_i = p_i = 1/2$ , and A starts with \$1.  $P = (1/2 \ 1/2 \ 0 \dots \& \ 1/2 \ 0 \ 1/2 \ 0 \dots \& \ 0 \ 1/2 \ 0 \ 1/2 \ 0 \dots \& \dots)$ .  $\Pi = \Pi P$ . 1st:  $1/2\Pi_0 + 1/2\Pi_1 = \Pi_0 \Rightarrow \Pi_1 = \Pi_0$ . 2nd:  $1/2\Pi_0 + 1/2\Pi_2 = \Pi_1 = \Pi_0 \Rightarrow \Pi_2 = \Pi_0$ . Continuing, we find that IF the distribution exists, then  $\Pi_n = \Pi_0$  for all  $n$ .  $\Pi = (\Pi_0, \Pi_0, \dots) \Rightarrow \Sigma\Pi_0 = 1$ , which is impossible. So there are no such stationary distributions.

Suppose  $r_i = 0$ , but  $P_0 = P_1 = P_2 = \dots = p < 1/2$ .  $P = (1-p \ p \ 0 \dots \& \ 1-p \ 0 \ p \ 0 \dots \& \ 0 \ 1-p \ 0 \ p \ 0 \dots \& \dots)$ .  $\Pi P = \Pi$ . 1st:  $(1-p)\Pi_0 + (1-p)\Pi_1 = \Pi_0 \Rightarrow \Pi_1 = \frac{p}{1-p}\Pi_0$ . 2nd:  $\Pi_2 = (\frac{p}{1-p})^2\Pi_0, \dots, \Pi_n = (\frac{p}{1-p})^n\Pi_0, n \geq 1, \Sigma\Pi_i = 1 \Rightarrow 1 = \Pi_0 * \Sigma_{n=0}^{\infty} (\frac{p}{1-p})^n = \Pi_0 \frac{1-p}{1-2p}, \Pi_n = \frac{1-2p}{1-p} (\frac{p}{1-p})^n, n \geq 0$  for the stationary distribution.

In both cases, the chain is irreducible and recurrent. In the first example, it is null, and in the second it is positive.

Theorem 3.3.3 - An irreducible chain has the stationary distribution  $\Pi$  iff all the states are positive recurrent. In this case there is a unique stationary distribution. The solution is  $\Pi_i = \frac{1}{\mu_i}, i \in S$ , where  $\mu_i$  is the mean recurrence time. The proof has several parts.

Definition 3.3.3 - Fix state  $k$ . Define  $\rho_i(k)$  = mean number of visits to state  $i$  between two successive visits to state  $k$ .  $T_k = \min\{n \geq 1 : X_n = k | X_0 = k\}$  = time of first return to state  $k$ .  $N_i = \sum_{n=1}^{\infty} 1_{\{X_n=i\} \cap \{T_k \geq n\}}$  (count 1s).  $\{X_n = i\} \cap \{T_k \geq n\}$  is the event where  $X_n$  is in state  $i$  and the time to return to state  $k$  is larger than  $n$ .  $N_i$  counts the number of visits to state  $i$  between successive visits to state  $k$ . Hence,  $\rho_i(k) = E(N_i | X_0 = k)$ . Now  $N_k = 1$ , so  $\rho_k(k) = 1$ . We also have (3.3.2)  $\rho_i(k) = \sum_{n=1}^{\infty} P(X_n = i, T_k \geq n | X_0 = k)$ .  $\rho(k) = (\rho_1(k), \rho_2(k), \dots)$ .

Theorem 3.3.4 - The mean recurrence time  $\mu_k$  satisfies (3.3.3)  $\mu_k = \sum_{i \in S} \rho_i(k)$ . Proof: The time between visits to state  $k$  must be spent in some state.  $T_k = \sum_{i \in S} N_i$ . Take expectations.

Theorem 3.3.5 - For any state  $k$  of an irreducible, recurrent chain,  $\rho(k)$  satisfies  $\rho_i(k) < \infty$  for all  $i$  and  $\rho(k) = \rho(k)P$ . Proof: first,  $\rho_i(k) < \infty$ . Set  $l_{ki}(n) = P(X_n = i, T_k \geq n | X_0 = k)$  = probability that the chain reaches state  $i$  in  $n$  steps, with no intermediate returns to state  $k$ . Observation: the first return time to  $k$  is  $m+n$  if (1)  $X_m = i$ , (2) there is no return to  $k$  before  $m$  steps, and (3) the next visit to state  $k$  occurs after  $n$  more steps. This implies  $f_{kk}(m+n) \geq l_{ki}(m)f_{ik}(n)$ . Since the chain is irreducible, there is an  $n$  with  $f_{ik}(n) > 0$ . Using this  $n$ ,  $l_{ki}(m) \leq \frac{f_{kk}(m+n)}{f_{ik}(n)}$ . We have  $\rho_i(k) = \sum_{n=1}^{\infty} l_{ki}(n)$  (important, exercise: explain)  $\leq \frac{1}{f_{ik}(n)} \sum_{m=1}^{\infty} f_{kk}(m+n) \leq \frac{1}{f_{ik}(n)} < \infty$ . To prove the second claim, start with  $\rho_i(k) = \sum_{n=1}^{\infty} l_{ki}(n)$ . Now  $l_{ki}(n) = P_{ki}, n \geq 2, l_{ki}(n) = \sum_{j \in S, j \neq k} P(X_n = i | X_{n-1} = j, T_k \geq n | X_0 = k) = \sum_{j \in S, j \neq k} l_{kj}(n-1)P_{ji}$ . Summing in  $n$ ,  $\rho_j(k) = P_{ki} + \sum_{j \in S, j \neq k} (\sum_{n \geq 2} l_{kj}(n-1))P_{ji} = \rho_k(k)P_{ki} + \sum_{j \in S, j \neq k} \rho_j(k)P_{ji}$ .

Theorem 3.3.6 - Every positive recurrent irreducible chain has a stationary distribution. Proof: We just proved  $\rho(k) = \rho(k)P$ . The components of  $\rho(k)$  are nonnegative and sum to  $\mu_k$ . If  $\mu_k < \infty, \Pi : \Pi_i = \rho_i(k)/\mu_k \Rightarrow \Pi = \Pi P, \Pi$  is a p.m.f.

Decomposition Theorem -  $S$  = state space.  $\Pi$  is a stationary distribution. If (1)  $\Pi_j \geq 0, \sum_{j \in S} \Pi_j = 1$ , (!!!) (2)  $\Pi = \Pi P, \Pi$  is a LEFT eigenvector of the prob transition matrix. This implies  $\Pi = \Pi P^n, n \geq 1$ . Does  $\Pi$  exist? (Thm 3.3.2:) Finite dimension state space  $\Rightarrow$  theory of eigenvectors.

Theorem 3.3.3 (no assumption of finite state) - An irreducible chain has a stationary distribution  $\Pi \iff$  all the states are positive recurrent. In this case,  $\Pi$  is unique,  $\Pi_i = \frac{1}{\mu_i}, i \in S$ . Let  $\rho_i(k)$  = mean number of visits to state  $i$  between successive visits to state  $k$ .

Theorem (3.3.3) -  $\mu_k = \sum_{i \in S} \rho_i(k)$ .

Theorem - If  $\rho(k) = (\rho_i(k))$ , if the communication class of  $k$  is irreducible and recurrent, then  $\rho_i(k) < \infty$  for all  $i$  and  $\rho(k) = \rho(k)P$ .

Theorem - Every positive recurrent irreducible Markov chain has a stationary distribution.

Theorem 3.3.7 - If the chain is irreducible and recurrent there is a solution of  $x = xP$  with strictly positive entries that is unique up to a multiplicative factor. The chain is positive if  $\Sigma X_i < \infty$  and null if  $\Sigma X_i = \infty$ . Now we are ready to prove theorem 3.3.3.

Proof of Theorem 3.3.3 - Suppose  $\Pi$  is a stationary distribution. If all the states are transient then  $P_{ij}^n \rightarrow 0$  as  $n \rightarrow \infty$  by Thm 3.1.2. By (3.3.1), (3.3.4)  $\Pi_j = \sum_i \Pi_i P_{ij}^n \rightarrow 0, n \rightarrow \infty$  for all  $j$ . This contradicts defn 3.3.2(1), hence all states are recurrent. (More argument in notes.) We next show that all states are positive and  $\Pi_i = \mu_i^{-1}$  for all  $i$ . Suppose  $X_0$  has distribution  $\Pi$ . Exercise:  $\Pi_j \mu_j = \sum_{n=1}^{\infty} P(T_j \geq n | X_0 =$



$j)P(X_0 = j) = \sum_{n=1}^{\infty} P(T_j \geq n, X_0 = j)$ .  $P(T_j \geq 1, X_0 = j) = P(X_0 = j), n \geq 2, P(T_j \geq n | X_0 = j) = P(X_0 = j, X_m \neq j, 1 \leq m \leq n-1) = P(X_m \neq j, 1 \leq m \leq n-1) - P(X_m \neq j, 0 \leq m \leq n-1) = P(X_m \neq j, 0 \leq m \leq n-2) - P(X_m \neq j, 0 \leq m \leq n-1) = a_{n-2} - a_{n-1}, a_n = P(X_m \neq j, 0 \leq m \leq n)$ . Sum over  $n, \Pi_j \mu_j = P(X_0 = j) + P(X_0 \neq j) - \lim_{n \rightarrow \infty} a_n = 1 - \lim_{n \rightarrow \infty} a_n$ .  $a_n \rightarrow P(X_m \neq j \text{ for all } m) = 0$  since  $j$  is recurrent. We have shown that  $\mu_j \Pi_j = 1$  so  $\mu_j = \Pi_j^{-1} < \infty$  if  $\Pi_j < 0$ . To show  $\Pi_j > 0$  for all  $j$ , assume some  $\Pi_j = 0$ .  $0 = \Pi_j = \sum_{j(\text{might be instead of } i)} \Pi_i P_{ij}^n \geq \Pi_i P_{ij}^n$  for all  $i, n$ . This means  $\Pi_i = 0$  when  $j \rightarrow i$ . The chain is irreducible so  $\Pi_i = 0$  for all  $i$ . This contradicts  $\sum_i \Pi_i = 1$ . Hence  $\mu_j < \infty$  for all  $j$ , and all the states are positive. The other direction follows from the intermediate steps.

Proof of Thm 3.2.3 (3) is given on page 163.

Example 3.3.5 - OFF/ON system: ex 3.3.2 -  $P = (1/2 \ 1/2 \ \& \ 1/4 \ 3/4)$ . We can compute  $\Pi = (1/3, 2/3) \Rightarrow \mu_1 = 3, \mu_2 = 3/2$ . (What is new here is that we can now compute the mean recurrence times,  $\mu_i$ .)

Example 3.3.6 - Consider Gambler's Ruin in ex 3.3.4 where  $P < 1/2$ .  $P = (1-p \ p \ 0 \ \dots \ \& \ 1-p \ 0 \ p \ 0 \ \dots \ \& \ 0 \ 1-p \ 0 \ p \ 0 \ \dots \ \& \ \dots)$ . Using the formula for  $\Pi_n$  computed there,  $\mu_n = \Pi_n^{-1} = \frac{1-p}{1-2p} \left(\frac{1-p}{p}\right)^n, n \geq 0, p = 1/4 \Rightarrow \mu_n = \frac{3}{2} 3^n, n \geq 0$ .

Theorem 3.3.3 can be used to determine if an irreducible chain is positive recurrent.

Theorem 3.3.8 - Let  $s \in S$  be a state of an irreducible chain. The chain is transient iff there is a nonzero solution  $\{Y_i, i \in S\}$  of the equations (3.3.5)  $Y_i = \sum_{j \in S, j \neq s} P_{ij} Y_j, i \neq s$ , with  $|Y_j| \leq 1$  for all  $j$ .

Theorem 3.3.8 - Let  $s \in S$  be a state of an irreducible chain. The chain is transient  $\iff$  there is a nonzero solution  $\{Y_i, i \in S\}$  of (3.3.5)  $Y_i = \sum_{j \in S, j \neq s} P_{ij} Y_j, i \neq s$ , with  $|Y_i| \leq 1$  for all  $i$ .

Proof - The chain is transient iff  $s$  is transient. Suppose  $s$  is transient. Define (3.3.6)  $\tau_i(n) = P(\text{no visit to } s \text{ in the first } n \text{ steps} | X_0 = i) = P(X_m \neq s, 1 \leq m \leq n | X_0 = i)$ .  $\tau_i(1) = \sum_{j \neq s} P_{ij}$ , which is  $X_1 \neq s$ .  $\tau_i(n+1) = \sum_{j \neq s} P_{ij} \tau_j(n)$ . Furthermore,  $\tau_i(n) \geq \tau_i(n+1), \tau_i = \lim_{n \rightarrow \infty} \tau_i(n) = P(\text{no visit to } s | X_0 = i) = 1 - f_{is}$ . Exercise:  $\tau_i$  satisfies (3.3.5). Also  $\tau_i > 0$  for some  $i$ . Otherwise  $f_{is} = 1$  for all  $i \neq s$ . This implies (condition on  $X_1$ )  $f_{ss} = P_{ss}[X_1 = s] + \sum_{i \neq s} P_{si} f_{is}[X_1 \neq s] = \sum_i P_{si} = 1$  which contradicts the transiency of  $s$ . Let  $Y$  satisfy (3.3.5) with  $|Y_i| \leq 1$ .  $|Y_i| \leq \sum_{j \neq s} P_{ij} |Y_j| \leq \sum_{j \neq s} P_{ij} = \tau_i(1)$ . Going back to (3.3.5)  $|Y_i| \leq \sum_{j \neq s} P_{ij} \tau_j(1) = \tau_i(2) \dots$ . So,  $|Y_i| \leq \tau_i(n)$  for all  $n$ . Exercise: Let  $n \rightarrow \infty$  to show that  $\tau_i = \lim_{n \rightarrow \infty} \tau_i(n) > 0$  for some  $i$  ( $Y_i \neq 0$  for some  $i$  by assumption) which implies  $s$  is transient.

Theorem 3.3.9 - An irreducible chain is recurrent iff the only bounded solution of (3.3.5) is the zero solution.

Example 3.3.7 - Gambler's Ruin ex 3.1.2 -  $P = (q \ p \ 0 \ \& \ q \ 0 \ p \ 0 \ \& \ 0 \ q \ 0 \ p \ 0 \ \& \ \dots)$ ,  $q + p = 1$ . Set  $\gamma = \frac{p}{q}$ . (1) If  $q < p (\gamma > 1)$ , choose  $s = 0$  to test thm 3.3.8. (3.3.5) read:  $Y_0 = P_{01} Y_1 = p Y_1, Y_1 = P_{02} Y_2 = p Y_2, Y_2 = q Y_1 + p Y_3$  (this is what was written on the board, should the + be there?),  $\dots$ . Exercise: if  $Y_j = 1 - \gamma^{-j}$ , then  $Y$  solves the equations and the chain is transient. (2) We can solve  $\Pi = \Pi P$  to find a stationary solution with  $\Pi_j = \gamma^j (1 - \gamma) \iff q > p$ . The chain is positive recurrent  $\iff q > p$ .

Example 3.3.8 - Consider discrete queuing (ex 2.1.10). Customers arrive at a service place and take a place in a line (queue). In each period of time, 1 customer is served and a random number arrive.  $C_n$  = number of customers that arrive during  $n$ th period.  $P(C_n = k) = a_k$ , where  $a = \text{p.m.f.}$   $X_n$  = number of customers waiting in line at time  $n$ .  $X_{n+1} = \max\{X_n - 1, 0\} + C_n$ .  $P = (a_0 \ a_1 \ a_2 \ \dots \ \& \ a_0 \ a_1 \ a_2 \ \dots \ \& \ 0 \ a_0 \ a_1 \ a_2 \ \dots \ \& \ \dots)$ .  $\Pi = \Pi P, \Pi_0 = \Pi_0 a_0 + \Pi_1 a_0, \Pi_1 = \Pi_0 a_1 + \Pi_1 a_1 + \Pi_2 a_2, \Pi_2 = \Pi_0 a_2 + \Pi_1 a_2 + \Pi_2 a_1 + \Pi_3 a_0, \dots$  (3.3.7)  $\Pi_i = a_0 \Pi_{i+1} + \sum_{j=1}^{i+1} \Pi_j a_{i+1-j}$ . We use generating functions:  $\Pi(t) = \sum_{i=0}^{\infty} \Pi_i t^i$ . We multiply (3.3.7) by  $t^i$  and sum. (3.3.8)  $\Pi(t) = \Pi_0 * \sum_{i=0}^{\infty} a_i t^i + \sum_{i=0}^{\infty} \sum_{j=1}^{i+1} \Pi_j a_{i+1-j} t^i, 1 \leq j \leq i+1 \Rightarrow i \geq j-1, j \geq 1, (A(t) = \text{p.g.f. for } a) A(t) = \sum_{i=0}^{\infty} a_i t^i$ , the right-hand side of (3.3.8) is  $\Pi_0 A(t) + \sum_{j=1}^{\infty} \Pi_j t^{j-1} * \sum_{i=j-1}^{\infty} a_{i-j+1} t^{i-j+1} = \Pi_0 A(t) = t^{-1} (\sum_{j=1}^{\infty} \Pi_j t^j) A(t) = \Pi_0 A(t) + t^{-1} (\Pi(t) - \Pi_0 A(t))$  or  $\Pi(t) = \Pi_0 A(t) (1 - t^{-1} + t^{-1} \Pi(t) A(t)) \Rightarrow$  (3.3.9)  $\Pi(t) = \frac{\Pi_0 A(t)}{1 - \frac{1-A(t)}{1-t}}$ . The question is: when is it possible to specify  $\Pi_0$  so  $\Pi(1) = \sum_k \Pi_k = 1$ . This implies a stationary distribution exists. Since  $\{a_k\}$  is a pmf,  $A(1) = 1$ . We want to let  $t \uparrow 1$  in (3.3.9). We let  $\lim_{t \uparrow 1} \frac{1-A(t)}{1-t} = A(1) = \gamma = \sum_{k=0}^{\infty} k a_k, \gamma = \text{mean number of arrivals per service interval}$ . [Something illegible]  $t \uparrow 1$  in (3.3.9),  $\Pi(1) = \frac{\Pi(0)}{1-\gamma}$ . We can choose  $\Pi_0$  so  $\Pi(1) = 1 \iff 0 < \gamma < 1$  and then  $\Pi_0 = 1 - \gamma$ . The queuing chain is positive recurrent  $\iff \gamma < 1$ , which says that the mean number of arrivals does overwhelm the facility.

Next consider  $\gamma > 1$ . We show that (3.3.5) has a nonzero solution  $Y$  with  $0 \leq Y_i \leq 1$  for all  $i$ . We choose  $s = 0$ , use (3.3.5) to get (3.3.10)  $Y_1 = \sum_{i=1}^{\infty} a_i Y_i, \dots, Y_n = \sum_{i=0}^{\infty} a_i Y_{i+n-1}, n \geq 2$ . Guessing based on branching processes, we try  $Y_i = 1 - t^i, 0 < t < 1$ .  $Y_n : 1 - t^n = \sum_{i=0}^{\infty} a_i (1 - t^{i+n-1}) = 1 - (\sum_{i=0}^{\infty} a_i t^i) t^{n-1}$ , with  $A(t) =$

$\sum_{i=0}^{\infty} a_i t^i, t^n = A(t)t^{n-1} \Rightarrow t = A(t)$ . The branching process (fixed point) analysis  $\Rightarrow t = A(t)$  has a solution with  $0 < t < 1$  when  $\gamma > 1$ .  $\gamma > 1 \Rightarrow$  chain is transient. We argue that if the chain is transient, (3.3.10) has a nonzero solution. If the chain is transient, then for each  $j = 0, 1, 2, \dots$ , there is a last visit. There is therefore a last visit to any finite set  $\{0, 1, 2, \dots, M\}$ . So there is an  $n_0 = n_0(M)$  such that for  $n > n_0, X_n > M$ . Hence,  $X_n \rightarrow \infty$  as  $n \rightarrow \infty$ .  $A_{n+1}$  = number of arrivals in  $(n, n+1)$ .  $P(A_{n+1} = k) = a_k, E(A_{n+1}) = p, X_{n+1} = \max\{X_n - 1, 0\} + A_{n+1}, n \geq n_0, X_{n+1} = X_n + A_{n+1}$ . (This is a way to get out of a low customer state when we have a low number in the queue.) If  $N \geq n_0, \sum_{n=n_0}^N (X_{n+1} - X_n) = -(N - n_0) + \sum_{n=n_0}^N A_{n+1}, X_{N+1} - X_{n_0} = -(N - n_0) + \sum_{n=n_0+1}^{N+1} A_n, X_{N+1} - \sum_{n=1}^{N+1} (A_{n-1}) = X_{n_0} + n_0 - \sum_{n=1}^{n_0} A_n$  (entire last term up until  $=$  is constant, doesn't depend on  $N$ ). Therefore  $X_{N+1} \rightarrow \infty$  implies  $\sum_{n=1}^{N+1} (A_{n-1}) \rightarrow \infty$ . Exercise: a sum of iid rv with mean  $\mu$  converges to  $\infty \iff \mu > 0$  equivalently  $\rho > 1$ .

Review from a question in class: why did we need a PGF last time? 1) Basic problem: finding  $\Pi = \Pi P, \Pi$  pmf.  $\Pi$  exists sometimes. Relation to being positive recurrent. 2) Basic problem: find a bounded solution of  $Y = PY$  with a row/column “missing”, related to transient. Then we got to ex 3.3.8 - classic example: queuing theory. This example had several points: 1) Consider  $\Pi = \Pi P$ . Used generating functions. Came to the conclusion that  $\Pi$  exists  $\iff \gamma = \sum_{k=0}^{\infty} k a_k < 1$ , but only if chain is positive recurrent. So we needed to prove that the chain is positive recurrent so we can solve the problem. 2) We consider solution of  $Y = PY, |Y_i| \leq 1$ , with missing row/column (not full  $P$ ). In this case using the fixed point theory we prove  $Y$  exists, but this holds iff chain is transient, which holds iff  $\gamma > 1$ . So, theorem is: positive recurrent if  $\gamma < 1$ , transient if  $\gamma > 1$ .

### 3.4 Limit Theorems

We explore the link between a stationary distribution and the behavior of  $P_{ij}^n$  as  $n \rightarrow \infty$ .

Example 3.4.1 - Consider ON/OFF system in ex 2.2.3 with  $P = (1-p \text{ p } \& \text{ q } 1-q) \ 0 \leq p \leq 1, 0 \leq q \leq 1$ . When  $0 < p, q < 1, P^n \rightarrow \frac{1}{p+q} (q \text{ p } \& \text{ q } p)$  and we know there is a stationary distribution. Now suppose  $p = q = 1$ , the system changes states at every step. The stationary distribution satisfies  $(\Pi_0, \Pi_1) = (\Pi_0, \Pi_1) (0 \ 1 \ \& \ 1 \ 0) \Rightarrow \Pi_0 = \Pi_1 = 1/2$ . We can compute, e.g.  $P_{00}^n = \{ 0 \text{ if } n \text{ even, } 1 \text{ if } n \text{ odd. There is no limiting behavior in this case. However the states are periodic with period 2.}$

Theorem 3.4.1 - For an irreducible, aperiodic Markov chain, (3.4.1)  $P_{ij}^n \rightarrow \frac{1}{\mu_j}$  as  $n \rightarrow \infty$  for all  $i, j$  ( $\mu_j$  is the mean recurrence time). The limiting value is the same for all states  $i$ .  $P^n \rightarrow (1/\mu_0 \ 1/\mu_0 \dots \& \ 1/\mu_1 \ 1/\mu_1 \dots \& \dots)^T = (1/\mu_0 \ 1/\mu_1 \dots \& \ 1/\mu_0 \ 1/\mu_1 \dots \& \dots)$ .

Definition 3.4.1 - If there is a probability distribution  $q$  on the state space  $S$  such that  $P_{ij}^n \rightarrow q_j$  for all  $i, j \in S$  then  $q$  is a limit distribution of the chain.

Intuition:  $q_j$  describes the probability that the chain is in state  $j$  at some “late” time and by this time the chain has “forgotten” where it started.  $P(X_n = j) = \sum_i P(X_0 = i) P_{ij}^n \rightarrow q_j$  regardless of the initial distribution of  $X_0$ . Consequences:

Theorem 3.4.2 - (a) If the chain is transient or null recurrent, then  $P_{ij}^n \rightarrow 0$  for all  $i, j$ . (b) If the chain is positive recurrent, then  $P_{ij}^n \rightarrow \Pi_j = \mu_j^{-1}$ , where  $\Pi$  is the unique stationary distribution.

Theorem 3.4.3 - If  $X_n$  is an irreducible chain with period  $d$ , then  $Y_n = X_{nd}, n \geq 0$ , is an aperiodic, irreducible chain,  $P_{jj}^{nd} = P(Y_n = j | Y_0 = j) \rightarrow \frac{d}{\mu_j}$  as  $n \rightarrow \infty$ . Immediately from this follows the proof of theorem 3.1.6 (see notes).

Connection between limiting and stationary distributions: consider a Markov chain at some “late” time  $n$ . The stationary distribution gives the proportion of time spent in the different states up to time  $n$ .

The limit distribution gives the proportion of “time” spent in the various states at the large time, where we count by considering many realizations.

Example 3.4.2 - Consider ON/OFF system, ex 3.4.1 again. The stationary distribution  $(1/2, 1/2)$  says that equal amounts of time are spent in each state up to some large time, say  $n = 1000$ . If we look precisely at  $n = 1000$ , the chain must return to its initial state. Multiple realizations give probability 1 to be in the initial state and 0 to be in the other.

Theorem 3.4.4 - An ergodic Markov chain has the property that it has both stationary and limiting distributions and these are equal.

Proof of theorem 3.4.1 - We treat different cases. The simplest case is a transient chain, because theorem 3.1.2 (3) implies  $P_{ij}^n \rightarrow 0$  as  $n \rightarrow \infty$  for all  $i, j$ . The recurrent cases are treated with “coupling”.

Definition - 3.4.2 - Let  $X_n, Y_n$  be independent Markov chains with common state space  $S$  and common probability transition matrix  $P$ . The coupled chain  $Z_n = (X_n, Y_n)$  taking values in  $S \times S$ .

Theorem 3.4.5 -  $Z_n$  is a Markov chain with  $P_{ij,kl} = P_{ik}P_{jl}$ . If  $X_n, Y_n$  are irreducible and aperiodic, then  $Z_n$  is irreducible. Proof:  $P_{ij,kl} = P(Z_{n+1} = (k, l) | Z_n = (i, j)) = P(X_{n+1} = k | X_n = i) \times P(Y_{n+1} = l | Y_n = j)$ .  $X_n, Y_n$  aperiodic, irreducible  $\Rightarrow$  for any  $i, j, k, l$  there is an  $N = N(i, j, k, l)$  such that  $P_{ik}^n P_{jl}^n > 0, n \leq N$ . Exercise: this implies  $Z_n$  is irreducible.

Comment: this is the only place we use the assumption  $X_n$  is aperiodic.

We assume  $X_n$  (in thm 3.4.1) is positive recurrent, so it has unique stationary distribution  $\Pi$ . (Consider  $Y = X$  in the construction of  $Z$ .) Exercise:  $Z_n = (X_n, Y_n)$  has a stationary distribution  $\nu = (\nu_{ij}, i, j \in S), \nu_{ij} = \Pi_i \Pi_j$ . This implies  $Z_n$  is also positive recurrent (due to stationary distribution). Choose  $X_0 = i, Y_0 = j, Z_0 = (i, j)$ . Choose  $s \in S$ . Set  $T = \min(n \geq 1 : Z_n = (s, s))$ . The recurrence of  $Z_n$  implies that  $P(T < \infty) = 1$  (exercise).

(Review) Theorem 3.4.1 - irreducible, aperiodic MC.  $P_{ij}^n \rightarrow_{n \rightarrow \infty} \frac{1}{\mu_j}$  for all  $i, j$ .  $\mu_j$  is mean recurrent time.

Theorem 3.4.4 - An ergodic MC has both stationary and limiting distributions and these are equal.

Recurrent cases in proof:  $X_n, Y_n$  have common  $S, P$ .  $Z_n = (X_n, Y_n) \in S \times S, Z_n$  M.C.  $P_{ij,kl} = P_{ik}P_{jl}$ . Assume  $X_n$  positive recurrent.  $\Pi$  is unique stationary distribution. (With  $Y_n = X_n$ ),  $Z_n$  has stationary distribution  $\nu = (\nu_{ij}, i, j \in S), \nu_{ij} = \Pi_i \Pi_j$ .  $Z_n$  is positive recurrent. Choose  $X_0 = i, Y_0 = j, Z_0 = (i, j)$ . Choose  $s \in S$ .  $T = \min\{n \geq 1 : Z_n = (s, s)\}$ .  $Z_n$  recurrent  $\Rightarrow P(T < \infty) = 1$ .

Observation: Suppose  $m \leq n$  and  $X_m = Y_m$ . Then  $X_n$  and  $Y_n$  are identically distributed. Thus conditional on  $\{T \leq n\}$ ,  $X_n$  and  $Y_n$  have the same distribution. We use this observation and the fact that  $T$  is finite to prove that in large time, the distributions of  $X_n$  and  $Y_n$  are independent of the initial values.

Computation: Start from  $Z_0 = (i, j), P_{ik}^n = P(X_n = k) = P(X_n = k, T \leq n) + P(X_n = k, T > n) = P(Y_n = k, T \leq n) + P(Y_n = k, T > n) \leq P(Y_n = k) + P(T > n) = P_{jk}^n + P(T > n)$ . The symmetric argument implies  $P_{jk}^n \leq P_{ik}^n + P(T > n)$ . Hence,  $|P_{ik}^n - P_{jk}^n| \leq P(T > n) \rightarrow_{n \rightarrow \infty} 0$  ( $T$  finite!), for all  $i, j, k \in S$ . So (3.4.2)  $P_{ik}^n - P_{jk}^n \rightarrow 0$  as  $n \rightarrow \infty$  for all  $i, j, k \in S$ . If  $\lim_{n \rightarrow \infty} P_{ik}^n$  exists, then it is independent of  $i$ . We show the limit exists. We write  $\Pi_k - P_{jk}^n = \sum_i \Pi_i (P_{ik}^n - P_{jk}^n)$  (we can write this because  $\Pi$  is stationary ( $\Pi_k, P_{ik}^n$ ) and a pmf ( $P_{jk}^n$ )). For any finite set  $F \in S, \sum_{i \in S} \Pi_i |P_{ik}^n - P_{jk}^n| \leq \sum_{i \in F} \Pi_i |P_{ik}^n - P_{jk}^n| + 2 \sum_{i \notin F} \Pi_i$ . As  $n \rightarrow \infty$ , this converges to  $2 \sum_{i \notin F} \Pi_i$ . This converges to 0 as  $F \uparrow S$ . So (3.4.3)  $\Pi_k - P_{jk}^n = \sum_i \Pi_i (P_{ik}^n - P_{jk}^n) \rightarrow_{n \rightarrow \infty} 0$ . Read in notes about when  $X_n$  is null recurrent, 185-187.

More general version of theorem 3.4.1 (no proof given) that drops irreducibility. Theorem 3.4.6 - For any aperiodic state  $j$  of a Markov chain,  $P_{jj}^n \rightarrow \frac{1}{\mu_j}$  as  $n \rightarrow \infty$ . If  $i$  is any other state,  $P_{ij}^n \rightarrow \frac{f_{ij}}{\mu_j}$  as  $n \rightarrow \infty$ . More to the theorem in notes.

### 3.5 Reversibility

Some physical situations have the property that observations of the system taken at some times look the same if time runs forward or backward. Let  $X_n$  be a Markov chain,  $\{X_n, 0 \leq n \leq N\}$  irreducible, positive recurrent Markov chain, prob transition matrix  $P$  and stationary distribution  $\Pi$ .

Definition 3.5.1 - The reversed chain or time reversal  $Y_n$  is  $Y_n = X_{N-n}, 0 \leq n \leq N$ .

Theorem 3.5.1 -  $Y_n$  is a Markov chain with  $P(Y_{n+1} = j | Y_n = i) = \frac{\Pi_j}{\Pi_i} P_{ji}$ . Proof:  $P(Y_{n+1} = i_{n+1} | Y_n = i_n, \dots, Y_0 = i_0) = \frac{P(Y_k = i_k, 0 \leq k \leq n+1)}{P(Y_k = i_k, 0 \leq k \leq n)} = \frac{P(X_{N-n-1} = i_{n+1}, X_{N-n} = i_n, \dots, X_N = i_0)}{P(X_{N-n} = i_n, \dots, X_N = i_0)} = \frac{\Pi_{i_{n+1}} P_{i_{n+1}, i_n} P_{i_n, i_{n-1}}, \dots, P_{i_1, i_0}}{\Pi_{i_n} P_{i_n, i_{n-1}}, \dots, P_{i_1, i_0}} = \frac{\Pi_{i_{n+1}} P_{i_{n+1}, i_n}}{\Pi_{i_n}}$ .

Definition 3.5.2 - The chain is reversible if the probability transition matrices of  $X_n$  and its time reversal  $Y_n$  are the same, (3.5.1)  $\Pi_i P_{ij} = \Pi_j P_{ji}$  for all  $i, j$ . (3.5.1) are called the detailed balance equations. A transition matrix  $P$  and a probability distribution  $\lambda$  are in detailed balance if  $\lambda_i P_{ij} = \lambda_j P_{ji}$  for all  $i, j$ . An irreducible chain  $X_n$  with a stationary distribution  $\Pi$  is reversible in equilibrium if its probability transition matrix is in detailed balance with  $\Pi$ .

Theorem 3.5.1 - Let  $P$  be the probability transition matrix of an irreducible chain  $X_n$  and suppose there is a distribution  $\Pi$  with  $\Pi_i P_{ij} = \Pi_j P_{ji}$  for all  $i, j \in S$ . Then  $\Pi$  is the stationary distribution of  $X_n$  and  $X_n$  is reversible in equilibrium. Proof:  $\sum_i \Pi_i P_{ij} = \sum_i \Pi_j P_{ji} = \Pi_j \sum_i P_{ji} = \Pi_j$  or  $\Pi = \Pi P$ .

$X_n$  irreducible positive recurrent M.C. Stationary distribution  $\Pi$ . Time reversal  $Y_n = X_{N-n}, 0 \leq n \leq N$ .  $Y_n$  is a M.C.  $P(Y_{n+1} = j | Y_n = i) = \frac{\Pi_j}{\Pi_i} P_{ji}$ .  $X_n$  is reversible if  $\Pi_i P_{ij} = \Pi_j P_{ji}$  for all  $i, j$ .

If there is a distribution  $\Pi$  such that (3.5.1) holds, then  $\Pi$  is the stationary distribution and  $X_n$  is reversible.

Example 3.5.1 - Consider ON/OFF system in ex 2.2.3,  $P = (1-p, p \text{ \& } q, 1-q)$ ,  $0 \leq p \leq 1, 0 \leq q \leq 1$ . (3.5.1)  $\Rightarrow \Pi_0 P_{00} = \Pi_0 P_{00}, \dots$  (see notes).  $\Rightarrow \Pi_0 P = \Pi_1 q$ . Recall the stationary distribution is  $\Pi_0 = \frac{q}{p+q}, \Pi_1 = \frac{p}{p+q}$ . These equations hold, so the chain is reversible.

Example 3.5.2 - Ehrenfest Model of Diffusion - (See notes for drawing.) Two containers (A, B), stopped at top, connected with a small tube at the neck. Filled with  $M$  total molecules of a gas. At each time, one molecule picked at random moves between the containers.  $X_n$  = number of molecules in A at time  $n$ .  $X_n$  is a M.C.  $P = (0, 1, 0, \dots \& 1/m, 0, 1-(1/m), 0 \dots \& 0, 2/m, 0, 1-(2/m), 0, \dots \& \dots)$ .  $P_{i,i+1} = 1 - \frac{i}{m}, P_{i,i-1} = \frac{i}{m}, 0 \leq i \leq m$ . If we look for solutions of (3.5.1) we find  $\Pi_i = \binom{m}{i} (\frac{1}{2})^m$ , which is a stationary distribution and the chain is reversible.

### 3.6 Chains with finitely many states

As discussed before, the theory is much simpler when there is a finite state space. Theorem 3.2.6 implies If  $X_n$  is irreducible and has a finite state space, then it is positive recurrent.

Theorem 3.6.1 - An irreducible aperiodic chain with a finite state space has a stationary distribution which is also a limiting distribution.

Theorem 3.6.2 - Perron-Fröbenius - If  $P$  is the transition probability matrix of an irreducible chain with period  $d$  and a finite state space, (1)  $\lambda_0 = 1$  is an eigenvalue of  $P$ . (2) the  $d$  complex roots:  $\lambda_1 = e^{2\pi i/d}, \lambda_2 = e^{4\pi i/d}, \dots, \lambda_d = e^{2\pi i \frac{d-1}{d}}, i = \sqrt{-1}, (\lambda_m)^d = 1$  are eigenvalues of  $P$  (look in notes for cool drawing of the unit circle with  $d$  roots drawn as spokes). (3) the remaining eigenvalues  $\lambda_{d+1}, \dots, \lambda_m$  satisfy  $|\lambda_m| \leq 1$ .

What this means: the power method. Let  $P$  be a matrix. Suppose  $\{V_1, \dots, V_N\}$  is a basis of eigenvectors.  $PV_i = \lambda_i V_i$ . Let  $V$  be arbitrary, then  $V = \sum_{j=1}^N \alpha_j V_j$  for some  $\{\alpha_1, \dots, \alpha_N\}$ . Think of  $V$  as being the initial condition of a markov process ( $X_0$ ). So [some stuff that I don't get and isn't in the notes.]  $P^n V = \sum_j \alpha_j \lambda_j^n V_j$ . Suppose  $\lambda_1$  is the largest  $|\lambda_1| > |\lambda_j|, j \neq 1$ . More stuff I don't understand.

Example 3.6.1 - Suppose the chain is aperiodic and  $P$  has distinct eigenvalues. There is a  $B$  with  $P = B^{-1} \Lambda B = B^{-1}$  (diagonal matrix with  $\lambda_i$  in diagonals,  $0 \leq i \leq N$ , 0 else)  $B, P^n = B^{-1} \Lambda^n B$  and  $|\lambda_i|^n \rightarrow 0$  for  $i \geq 1$ .  $|\lambda_0| = 1$ .

### 3.7 Branching Processes

Unfortunately the preceding theory is not directly useful. Consider the branching process:  $\{Z_{nj}, n \geq 1, j \geq 1\}$  i.i.d. nonnegative i.v., pmf  $\{P_k\}$ .  $X_0 = 1, X_1 = Z_{1,X_0}, X_2 = Z_{21} + Z_{22} + \dots + Z_{2,X_1}, \dots, X_n = Z_{n,1} + \dots + Z_{n,X_{n-1}}$ . If there is a strictly positive probability that each family is empty,  $P(X_n = 0) > 0$  then 0 is an absorbing state. Hence, 0 is a positive recurrent state and all other states are transient. The chain is not irreducible, but there is a unique stationary distribution:  $\Pi_0 = 1, \Pi_i = 0$  for  $i \geq 1$ . This gives no interesting information. The difficulty is that the process can behave in a number of different ways depending on, for example, whether or not it becomes extinct. We study the behavior conditional on some event, e.g. extinction. Recall  $\rho_k = P(X_1 = k), \phi(s) = E(s^{X_1}), N = \inf\{n : X_n = 0\} = \text{time of extinction} [\inf = \min]$ .  $N = \infty$  if  $X_n \neq 0$  for all  $n$ . If  $N = \infty$ , the process grows without bound and if  $N < \infty$ , the process is limited in size and eventually becomes extinct. Recall the probability  $\eta = P(N < \infty)$  of extinction is the smallest nonnegative root of  $S = \phi(s)$ . We let  $E_n = \{n < N < \infty\}$  be the event that extinction occurs at some time larger than  $n$ . We studied the distribution of the  $X_n$  conditional on  $E_n$ . We let  $P_{0,j}(n) = P(X_n = j | E_n)$  be the conditional probability that  $X_n = j$  given the future extinction. We try to compute  $\Pi_{0,j} = \lim_{n \rightarrow \infty} P_{0,j}(n)$  (if the limit exists). To avoid trivial cases, we assume  $0 < \rho_0 + \rho_1 < 1, \rho_0 > 0$  (some chance we will have more than one offspring). This implies  $0 < P(E_n) < 1$  and  $0 < \eta \leq 1$ .

Theorem 3.7.1 - If  $E(X_1) < \infty$ , then  $\Pi_{0,j} = \lim_{n \rightarrow \infty} P_{0,j}(n)$  exists. The generating function  $G(s) = \sum_{j=0}^{\infty} \Pi_{0,j} s^j$  satisfies (3.7.1)  $G(\eta^{-1} \phi(s\eta)) = \alpha G(s) + 1 - \alpha, \alpha = \phi'(\eta)$ . If  $\mu = E(X_1) \leq 1$ , then  $\eta \equiv 1$  and  $\alpha = \mu$ , so (3.7.2)  $G(\phi(s)) = \mu G(s) + 1 - \mu$  for all  $\mu, G'(\eta) \leq 1$  and  $G'(\eta) = 1 \iff \mu = 1$ . See outline of proof in notes.

Theorem 3.7.2 - (1) If  $E(X_1) \neq 1, \sum_j \Pi_{0,j} = 1$ , (2) If  $E(X_1) = 1, \Pi_{0,j} = 0$  for all  $j$ .

### 3.8 Review (in notes)

## 4 Continuous Time Markov Chains

Continuous time chains stay in each state a random time that is a continuous random variable that may depend on the state.  $X(t)$  = state at time  $t, t \geq 0$ .  $X(t)$  may or may not be a Markov process.

### 4.1 The Poisson Process

Example 4.1.1 - We use a Geiger counter to observe the emission of particles from a radioactive source. If we switch on the counter at time zero, the count  $N(t)$  is the outcome of an apparently random process. Observations: (a)  $N(0) = 0, N(t) \in \{0, 1, 2, 3, \dots\}$ , (b) If  $s < t, N(s) \leq N(t)$ . We conjecture a continuity assumption: in a time period  $(t, t+h)$  the probability of an emission is proportional to  $h$  for small  $h$ .

Definition 4.1.1 - A Poisson process with intensity  $\lambda$  is a process  $N = \{N(t), t \geq 0\}$  taking values in  $S = \{0, 1, 2, 3, \dots\}$  such that (a)  $N(0) = 0$ , (b)  $S < t \Rightarrow N(s) \leq N(t)$ , (c)  $P(N(t+h) = n+m | N(t) = n) = \{\lambda h + O(h) \text{ for } m = 1, O(h) \text{ for } m > 1, 1 - \lambda h + O(h) \text{ for } m = 0\}$ .  $O(h)$  means an expression  $A(h)$  such that  $\lim_{h \rightarrow 0} \frac{|A(h)|}{h} \rightarrow 0$ , (d) If  $s < t$ , the number  $N(t) - N(s)$  of emissions in  $(s, t]$  is independent of the times of emissions in  $(0, s]$  (this is pretty much the Markov condition).

Definition 4.1.2 -  $N(t)$  = the number of arrivals or occurrences or events or emissions at time  $t$ .  $N$  is called a counting process.

Theorem 4.1.1 -  $N(t)$  has the Poisson distribution with parameter  $\lambda t$ , i.e., (4.1.1)  $P(N(t) = j) = \frac{(\lambda t)^j}{j!} e^{-\lambda t}, j = 0, 1, 2, \dots$

Proof: We condition  $N(t+h)$  on  $N(t)$ .  $P(N(t+h) = j) = \sum_i P(N(t) = i) P(N(t+h) = j | N(t) = i) = \sum_i P(N(t) = i) P((j-i) \text{ arrivals in } (t, t+h]) = P(N(t) = j-1) P(1 \text{ arrival in } (t, t+h]) + P(N(t) = j) P(\text{no arrivals in } (t, t+h]) + O(h)$ . Set  $P_j(t) = P(N(t) = j)$ ,  $P_j(t+h) = \lambda h P_{j-1}(t) + (1 - \lambda h) P_j(t) + O(h)$ ,  $P_0(t+h) = (1 - \lambda h) P_0(t) + O(h)$ ,  $j \neq 0$ . Assuming  $P_j(t)$  is a smooth function, we subtract  $P_j(t)$  from each side, divide by  $h$  and let  $h \downarrow 0$ .  $\lim_{h \rightarrow 0} \frac{P_j(t+h) - P_j(t)}{h}$ , etc. (4.1.2)  $P'_j(t) = \lambda P_{j-1}(t) - \lambda P_j(t)$ ,  $j \neq 0$ . (4.1.3)  $P'_0(t) = -\lambda P_0(t)$ . The initial condition is (4.1.4)  $P_j(0) = \delta_{j0} = \{1 \text{ for } j = 0, 0 \text{ for } j \neq 0\}$ . This is a big system of equations that we need to solve to go further.

Two approaches: (4.1.3) + (4.1.4) together yield  $P_0(t) = e^{-\lambda t}$ . We substitute into (4.1.2) with  $j = 1$ ,  $P'_1(t) = \lambda e^{-\lambda t} - \lambda P_1(t)$ .  $P'_1(t) + \lambda P_1(t) = \lambda e^{-\lambda t}$ ,  $e^{\lambda t} P'_1(t) + e^{\lambda t} \lambda P_1(t) = e^{\lambda t} \lambda e^{-\lambda t} \Rightarrow \frac{d}{dt}(e^{\lambda t} P_1(t)) = \lambda e^{\lambda t} P_1(t) = \lambda t + c \Rightarrow P_1(t) = \lambda t e^{-\lambda t}$ . Iteration yields  $P_j(t) = \frac{(\lambda t)^j}{j!} e^{-\lambda t}$ .

Second approach: Define a generating function  $G(s, t) = \sum_{j=0}^{\infty} P_j(t) s^j$ . Multiply (4.1.2) by  $s^j$  and sum (some details)  $\Rightarrow \frac{\nabla G}{\nabla t} = \lambda(s-1)G$ ,  $G(s, 0) = 1$ . The solution is (4.1.5)  $G(s, t) = e^{\lambda(s-1)t} = e^{-\lambda t} \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{j!} s^j$ . Reformulation of the process: important for computational purposes.

Definition 4.1.3 - Let  $T_0, T_1, T_2, \dots$  be given by (4.1.6)  $T_0 = 0, T_n = \inf_t \{N(t) = n\}$  ( $\inf = \min$ ).  $T_n$  is the arrival or waiting time for the  $n$ th event. The interarrival or sojourn times  $X_1, X_2, \dots$  are given by (4.1.7)  $X_n = T_n - T_{n-1}$ . If we know  $N$ , we can compute  $X_1, X_2, \dots$ . Vice versa if we know the entire collection of sojourn times  $\{X_n\}$  then (4.1.8)  $T_n = \sum_{i=1}^n X_i, N(t) = \max_{T_n \leq t} n$ . See picture in notes.

Theorem 4.1.2 - The random variables  $X_1, X_2, \dots$  are i.i.d. with exponential distribution with parameter  $\lambda$ .

### 4.2 Birth Processes

Definition 4.2.1 -  $N(t)$  (continuous-time birth process). Assume: (a)  $N(0) \geq 0$ , (b)  $s < t \Rightarrow N(s) \leq N(t)$ , (c)  $P(N(t+h) = n+m | N(t) = m) = \{\lambda_n h + O(h) \text{ for } m = 1, O(h) \text{ for } m > 1, 1 - \lambda_n h + O(h) \text{ for } m = 0\}$ , (d) If  $s < t$  then (conditional on the value of  $N(s)$ , the increment  $N(t) - N(s)$  is independent of the times of arrivals prior to  $s$ ).

Transition probabilities:  $P_{ij}(t) = P(N(t) = j | N(0) = i) = P(N(s+t) = j | N(s) = i), s \leq t$ .

Forward System - Assume  $\lambda_{-1} = 0, P_{ij}(0) = \delta_{ij}$ , then (4.2.1)  $P'_{ij}(t) = \lambda_{j-1} P_{ij-1}(t) - \lambda_j P_{ij}(t)$ .

Backward System - (4.2.2)  $P'_{ij}(t) = \lambda_i P_{i+1j}(t) - \lambda_i P_{ij}(t), j \geq i$ . (Yes, that is  $P_{i+1j}$ . I asked in class and verified it. I don't understand why...)

Theorem 4.2.3 - The forward system has a unique solution that also satisfies the backward system. Proof: First note that (4.2.3)  $P_{ij}(t) = 0$  if  $j < i$ . We solve the forward problem with  $j = i$ , so  $P'_{ii}(t) = \lambda_{i-1}P_{ii-1}(t) - \lambda_i P_{ii}(t)$  (first term after equals goes to zero), so (4.2.4)  $P_{ii}(t) = e^{-\lambda_i t}$ . We substitute into the forward equation with  $j = i + 1$  to find that  $P_{ii+1}(t)$  exists (using standard ODE theory). By induction, we conclude the solution of the forward system exists and is unique. We use the Laplace transform:  $\widehat{P_{ij}}(\theta) = \int_0^\infty e^{-\theta t} P_{ij}(t) dt$ ,  $P_{ij}(t) \xrightarrow{\widehat{}} \widehat{P_{ij}}(\theta)$ . This transform derivatives with respect to  $t$  to products in the  $\theta$  variable domain. If we transform both sides of the forward equation, we get  $(\theta + \lambda_j) \widehat{P_{ij}}(\theta) = \delta_{ij} + \lambda_{j-1} \widehat{P_{ij-1}}(\theta)$ . This difference equation can be solved: (4.2.5)  $\widehat{P_{ij}}(\theta) = \frac{1}{\lambda_j} \frac{\lambda_i}{\theta + \lambda_i} \frac{\lambda_{i+1}}{\theta + \lambda_{i+1}} \dots \frac{\lambda_j}{\theta + \lambda_j}$ ,  $j \geq i$ . Using the inverse Laplace transform gives  $P_{ij}(t)$ .

To show the claim about the backward equations, we also take the Laplace transform in the same way but now with the backward equation to find that any solution (call it  $\Pi$ —it may be  $P$ , may not be) with  $\widehat{\Pi_{ij}}(\theta) = \int_0^\infty e^{-\theta t} \Pi_{ij}(t) dt$  satisfies  $(\theta + \lambda_j) \widehat{\Pi_{ij}}(\theta) = \delta_{ij} + \lambda_i \widehat{\Pi_{i+1j}}(\theta)$ . Note that  $\widehat{P_{ij}}$  satisfies this equation, but so can other functions. (TeX note: the widehat change here is meaningless.)

Theorem 4.2.4 - If  $\{P_{ij}(t)\}$  is the unique solution of the forward system then any solution  $\{\Pi_{ij}\}$  of the backwards system satisfies  $P_{ij}(t) \leq \Pi_{ij}(t)$  for all  $t$ . Proof not given.

Observe: if (4.2.6)  $\sum_j P_{ij}(t) \equiv 1$  (for all  $t$ ), then Thm 4.2.4 would imply that  $\{P_{ij}\}$  is the unique solution of the backward system that is a probability distribution. However (4.2.6) may not hold.

Definition 4.2.5 - An explosion occurs if the birth rates  $\lambda_n$  increase sufficiently quickly that there is a positive probability that the process  $N$  can pass through all finite states in finite (bounded) time.

Definition 4.2.6 - Let  $T_\infty = \lim_{n \rightarrow \infty} T_n$  be the limit of the arrival times. We say  $N$  is honest if  $P(T_\infty = \infty) = 1$  and dishonest otherwise.

Theorem 4.2.5 - (4.2.6) holds (for each  $i$   $\{P_{ij}(t)\}$  is a probability distribution in  $j$ ) if and only if  $N$  is honest. Proof: (4.2.6) is equivalent to  $P(T_\infty > t) = 1$ , any  $t$ , why? Exercise.

Theorem 4.2.6 -  $N$  is honest  $\iff \sum_n \lambda_n^{-1} = \infty$ . This says that if the birth rates are sufficiently small (increase sufficiently slowly) then  $N$  is honest.  $\sum a_n = \infty$ ,  $a_n$  decreases sufficiently slowly, e.g.  $\sum_n \frac{1}{n} = \infty$ . If the  $\lambda_n$  increase sufficiently quickly that  $\sum_n \lambda_n^{-1}$  converges, then  $N$  is dishonest. We can think of the deficit as  $1 - \sum_j P_{ij}(t)$  as the probability  $P(T_\infty \leq t)$  of escaping to infinity at time  $t$  starting from state  $i$ .

Theorem 4.2.6 follows from:

Theorem 4.2.7 - Let  $X_1, X_2, \dots$  be independent random variables with  $X_n$  having the exponential distribution with parameter  $\lambda_{n-1}$  and let  $T_\infty = \sum_n X_n$ . Then  $P(T_\infty < \infty) = \{0 \text{ for } \sum_n \lambda_n^{-1} = \infty, 1 \text{ for } \sum_n \lambda_n^{-1} < \infty$ . Proof in notes. When the rates vary, the situation becomes more complicated.

Last topic in this section: what does the condition (d) mean? Recall that a sequence of random variables  $\{X_n, n \geq 0\}$  satisfies the Markov property if, conditional on the event  $\{X_n = i\}$  events related to the collection  $\{X_m, m > n\}$  are independent of events related to the collection  $\{X_m, m < n\}$ .

Theorem 4.2.8 - Weak Markov Property - Let  $N(t)$  be a birth process and  $T$  a fixed time. Conditional on the event  $\{N(T) = i\}$  the evolution of the process after  $T$  is independent of the evolution before  $T$ . Proof: definition 4.2.1(a).

Theorem 4.2.8 - Weak Markov Property -  $N(t)$  is a birth process.  $T$  is a fixed time. Conditional on the event  $\{N(T) = i\}$  the evolution of the process after time  $T$  is independent of the evolution prior to  $T$ . It is useful to let  $T$  be a random variable. This theorem as stated cannot hold for all possible random times  $T$ . For example, if  $T$  “looks into the future” the conclusion cannot hold.

Definition 4.2.7 - A random time  $T$  is a stopping time for the process  $N(t)$  if for all  $t \geq 0$ , the indicator function of the event  $\{T \leq t\}$  is a function of the values  $\{N(s), s \leq t\}$  of the process up to time  $t$ . We can decide if  $T$  has occurred by  $t$  by examining the values of  $N$  up to time  $t$ .

Example 4.2.5 - The arrival times  $T_1, \dots$ , are stopping times.  $I_{T_n=t} = \{1 \text{ for } N(s) \neq n, N(t) = n, s < t, 0 \text{ else. For fixed } k, \text{ the } k^{\text{th}} \text{ time a process } N \text{ visits a state } n \text{ is a stopping time. The last time a process visits a state } n \text{ is not a stopping time.}$

Example 4.2.6 -  $T_4 - 2$  and  $\frac{1}{2}(T_1 + T_2)$  are not stopping times (why?).

For stopping times we can prove:

Theorem 4.2.9 - Strong Markov Property - Let  $N(t)$  be a birth process and  $T$  a stopping time. Let  $A$  be an event that depends on  $\{N(s), s > t\}$  and  $B$  an event that depends on  $\{N(s), s < t\}$ . Then (4.2.9)  $P(A|N(T) = i, B) = P(A|N(T) = i)$ . Read outline of proof in notes.

### 4.3 More on Poisson Processes

While Poisson processes are special among continuous time processes, they also occur frequently in “nature”. Part of the reason is the Law of Rare Events.

Law of Rare Events - This says that in a situation in which a certain event can occur in any of a large number of possibilities but where the probability it occurs in any given possibility is small, then the total number of occurrences follows approximately a Poisson distribution.

Consider a large number  $N$  of Bernoulli trials. The probability of success in each trial is  $p$ . Let  $X_{N,p}$  = number of successes in the  $N$  trials. (4.3.1)  $P(X_{N,p} = k) = \frac{N!}{k!(N-k)!} p^k (1-p)^{N-k}$ ,  $k = 0, 1, \dots, N$ . We consider the limit as  $N \rightarrow \infty$  and  $p \rightarrow 0$  so that  $\mu = Np$  is fixed.  $P(X_{N,p} = k) = N(N-1) \cdots (N-k+1) \frac{p^k (1-p)^N}{k!}$ . Substitute  $p = \frac{\mu}{N} \Rightarrow P(X_{N,p} = k) = 1 * (1 - \frac{1}{N}) \cdots (1 - \frac{k-1}{N}) \frac{\mu^k (1 - \frac{\mu}{N})^N}{k!}$ . So  $N \rightarrow \infty (p \rightarrow 0)$ .  $1 * (1 - \frac{1}{N}) \cdots (1 - \frac{k-1}{N}) \rightarrow 1$ ,  $(1 - \frac{\mu}{N})^N \rightarrow e^{-\mu}$ ,  $(1 - \frac{\mu}{N})^k \rightarrow 1$ .

Theorem 4.3.1 - Law of Rare Events - If  $X_{N,p}$  is the number of successes in  $N$  Bernoulli trials with probability of success  $p$ , then (4.3.2)  $\lim_{N \rightarrow \infty, pN = \mu} P(X_{N,p} = k) = \frac{\mu^k e^{-\mu}}{k!}$ . (Poisson distribution with parameter  $\mu$ .) (4.3.2) can be used to compute Binomial probabilities.

Suppose the probability of success varies with each trial. Let  $Y_1, Y_2, \dots$ , be independent Bernoulli r.v. with  $P(Y_i = 1) = P_i$ ,  $P(Y_i = 0) = 1 - P_i$ .  $S_n = Y_1 + \cdots + Y_N$  = number of successes in  $N$  trials.  $P(S_N = k) = \sum_{(k)} \prod_{i=1}^N P_i^{Y_i} (1 - P_i)^{N-Y_i}$  where  $\sum_{(k)}$  is the sum over all 0, 1 valued  $Y_i$ s that sum  $Y_1 + \cdots + Y_N = k$ .

Theorem 4.3.2 - Law of Rare Events, v.2 - (4.3.3)  $|P(S_N = k) - \frac{\mu^k e^{-\mu}}{k!}| \leq \sum_{i=1}^N P_i^2$  with  $\mu = P_1 + \cdots + P_N$ . If the probabilities of success in the trials are small, we obtain a good approximation. Proof: see text, pg. 285.

Recall Theorem 4.1.1 - Let  $N(t)$  be a Poisson process. Then (4.1.1)  $P(N(t) = j) = \frac{(\lambda t)^j}{j!} e^{-\lambda t}$ ,  $j = 0, 1, \dots$  ( $\lambda$  = rate of the process).

Alternate Proof (using the Law of Rare Events. Divide  $[0, t]$  in to  $n$  subintervals of equal length  $h = \frac{t}{n}$ , setting  $Y_i = \{1 \text{ if there is an event in } ((i-1)\frac{t}{n}, i\frac{t}{n}], 0 \text{ otherwise. } S_n = Y_1 + \cdots + Y_n \text{ counts the number of subintervals that contain at least one event and } P_i = P(Y_i = 1) = \lambda \frac{t}{n} + o(\frac{t}{n}) \text{ by definition of a Poisson process. Using (4.3.) } |P(S_n = k) - \frac{\mu^k e^{-\mu}}{k!}| \leq n(\frac{\lambda t}{n} + o(\frac{t}{n}))^2 = \frac{(\lambda t)^2}{n} + 2\lambda + o(\frac{t}{n}) + no((\frac{t}{n})^2). \text{ Now } no(\frac{t}{n}) = t \frac{o(\frac{t}{n})}{\frac{t}{n}} = t \frac{o(h)}{h} \rightarrow_{n \rightarrow \infty, h \rightarrow 0} 0. S_n \text{ differs from } N(t) \text{ only if one of the subintervals has more than one event. } P(N(t) \neq S_n) = \sum_{i=1}^n P(\text{number of events in } ((i-1)\frac{t}{n}, i\frac{t}{n}] \geq 2) \leq n * o(\frac{t}{n}) \text{ by assumption. So } \lim_{n \rightarrow \infty} P(N(t) \neq S_n) = 0.$

Theorem 4.3.3 - The arrival time  $T_n$  has the gamma distribution with pdf (4.3.4)  $f_{T_n}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t}$ ,  $n = 1, 2, \dots, t \geq 0$ . Proof: the event  $\{T_n \leq t\}$  occurs iff there are atleast  $n$  events in  $(0, t]$ .  $N(t)$  is a Poisson distribution with parameter mean  $\lambda t$ . So the cdf of  $T_n$  is  $F_{T_n}(t) = P(T_n \leq t) = P(N(t) \geq n) = \sum_{k=n}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} = 1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k e^{-\lambda t}}{k!}$ . Differentiating, [some long messy computation, see notes]  $= \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}$ .

Theorem 4.3.4 - For  $0 < s < t$ ,  $0 \leq k \leq n$ , (4.3.5)  $P(N(s) = k | N(t) = n) = \frac{n!}{k!(n-k)!} (\frac{s}{t})^k (1 - \frac{s}{t})^{n-k}$ . Proof in notes.

The next result says that conditioned on a fixed total number of events in an interval, the times of occurrence of the events is uniformly distributed in a certain way.

Related computation: consider an interval  $(0, t]$ . Choose a fixed number  $n$  of points uniformly distributed on the interval. Positions =  $\{U_1, U_2, \dots, U_n\}$ . The pdf is  $f(s) = \frac{1}{t}$  for  $0 \leq s \leq t$ , 0 else. Set  $\{W_1, W_2, \dots, W_n\}$  be the positions  $\{U_1, U_2, \dots, U_n\}$  ordered from left to right. (See drawing in notes.) The joint pdf for  $W_1, \dots, W_n$  is (4.3.6)  $f_{W_1, W_2, \dots, W_n}(w_1, w_2, \dots, w_n) = n! t^{-n}$ ,  $0 < w_1 < \cdots < w_n \leq t$ . Arguing by induction,  $f_{W_1, W_2}(w_1, w_2) \Delta w_1 \Delta w_2 = P(w_1 < U_1 \leq w_1 + \Delta w_1, w_2 \leq U_2 \leq w_2 + \Delta w_2) + P(w_1 < U_2 \leq w_1 + \Delta w_1, w_2 \leq U_2 \leq w_2 + \Delta w_2) = 2(\frac{\Delta w_1}{t})(\frac{\Delta w_2}{t}) = 2t^{-2} \Delta w_1 \Delta w_2$ . Divide by  $\Delta w_1 \Delta w_2$ , take the limit as  $\Delta w_1 \Delta w_2 \rightarrow 0$ , then (4.3.6) is proved. In general there are  $n!$  arrangements of the  $U_i$ s.

Theorem 4.3.5 - Let  $T_1, \dots, T_n$  be arrival times for a Poisson process with rate  $\lambda > 0$ . Conditional on  $N(t) = n$ , the random variables  $T_1, \dots, T_n$  have joint pdf (4.3.7)  $f_{T_1, \dots, T_n}(t_1, \dots, t_n) = n! t^{-n}$ ,  $0 < t_1 < t_2 < \cdots < t_n \leq t$ . Proof: We create some subintervals from  $(0, t]$ , using a set of times  $t_1, \dots, t_n$  in increments  $\Delta t_1, \dots, \Delta t_n$ . (See drawing in notes.) The event  $\{t_i < T_i < t_i + \Delta t_i, i = 1, 2, \dots, n, N(t) = n\}$ . This means no events in  $(0, t_1), (t_1 + \Delta t_1, t_2), \dots, (t_n + \Delta t_n, t]$  and exactly one event in each interval  $(t_1, t_1 + \Delta t_1], (t_2, t_2 + \Delta t_2], \dots, (t_n, t_n + \Delta t_n]$ . The intervals are disjoint and  $P(N(t_1) = 0, \dots, N(t) - N(t_n + \Delta t_n) =$

$0) = e^{-\lambda t_1} e^{-\lambda(t_2 - t_1 - \Delta t_1)} \dots e^{-\lambda(t_n - t_{n-1} - \Delta t_{n-1})} \dots e^{-\lambda(t - t_n - \Delta t_n)} = e^{-\lambda t} (1 + o(\max_i \Delta t_i))$ ,  $P(N(t_1 + \Delta t_1) - N(t_1) = 1, \dots, N(t_n + \Delta t_n) - N(t_n) = 1) = \lambda(\Delta t_1) \lambda(\Delta t_2) \dots \lambda(\Delta t_n) (1 + o(\max_i \Delta t_i))$  [ $o(\max_i \Delta t_i)$  goes to zero faster than  $C \Delta t_i$ ]. Hence, more stuff in notes. Get to  $= n! t^{-n} \Delta t_1 \dots \Delta t_n (1 + o(\max_i \Delta t_i))$ . Divide by  $\Delta t_1 \dots \Delta t_n$ , take limit as  $\Delta t_1 \dots \Delta t_n \rightarrow 0$ .

Example 4.3.2 - Customers arrive at a facility according to a Poisson process with rate  $\lambda$ . Each customer pays \$1 when they arrive. We want to evaluate (compute) the expected value of the total sum collected during some interval  $(0, t]$  when the amounts are “discounted” back to time 0 at a rate  $\beta$ : If a customer pays \$1 at event time  $T_k$ , the discounted amount is  $e^{-\beta T_k} * \$1$ .  $M = E(\sum_{k=1}^{N(t)} e^{-\beta T_k})$ . We condition on  $N(t) = n!$ .  $M = \sum_{n=1}^{\infty} E(\sum_{k=1}^n e^{-\beta T_k} | N(t) = n) P(N(t) = n)$ . Let  $U_1, \dots, U_n$  be uniformly distributed iid variables in  $(0, t]$ . By theorem 4.3.5,  $E(\sum_{k=1}^n e^{-\beta T_k} | N(t) = n) = E(\sum_{k=1}^n e^{-\beta U_k})$ . Each term is treated in the same way:  $= n E(e^{-\beta U_1}) = n t^{-1} \int_0^t e^{-\beta u} du = \frac{n}{\beta t} (1 - e^{-\beta t})$ ,  $M = \frac{1}{\beta t} (1 - e^{-\beta t}) \sum_{n=1}^{\infty} n P(N(t) = n) = \frac{\lambda}{\beta} (1 - e^{-\beta t})$ .

The proof of the strong Markov property, theorem 4.2.9 for birth processes, uses the weak Markov property and temporal homogeneity. The strong Markov property is very important for the analysis of continuous processes. When applied to a birth process, it implies that the new process:  $\tilde{N}(t) = N(t + T) - N(t)$ ,  $t > 0$ , conditional on  $N(T) = i$ , is also a birth process when  $T$  is a stopping time. Exercise: show the rates are  $\lambda_i, \lambda_{i+1}, \dots$ .

Definition 4.3.1 - A Poisson process  $N$  has stationary, independent increments in the sense that (1) the distribution of  $N(t) - N(s)$  depends only on  $t - s$ , (2) any finite set of increments  $\{N(t_i) - N(s_i), i = 1, 2, \dots, n\}$  are independent if  $0 < s_1 \leq t_1 \leq s_2 \leq \dots \leq t_n$ . (Exercise: verify.)

Theorem 4.3.6 - Suppose that  $M(t)$ ,  $t \geq 0$ , is a nondecreasing, right-continuous, integer valued stochastic process with (1)  $M(0) = 0$ , (2) stationary independent increments, (3)  $M$  only has jump discontinuities of size 1. So  $M$  is a Poisson process. Outline of proof: for  $u, v \geq 0$ ,  $E(M(u+v)) = E(M(u)) + E(M(u+v) - M(u)) = E(M(u)) + E(M(v))$  [stationary increments].  $E(M(u))$  is nondecreasing in  $u$ , so there is a  $\lambda$  such that (4.3.8)  $E(M(u)) = \lambda u$ ,  $u \geq 0$ . Let  $T = \sup\{t : M(t) = 0\}$  be the time of the first jump (sup is like max). By right continuity almost surely,  $M(T) = 1$  and  $T$  is a stopping time for  $M$  (check!). Now (4.3.9)  $E(M(s)) = E(E(M(s)|T))$ ,  $E(M(s)|T) = 0$ ,  $s < T$ . For  $s \geq T$ , (4.3.10)  $E(M(s)|T = t) = E(M(t)|T = t) + E(M(s) - M(t)|T = t) = 1 + E(M(s-t))$ . If  $F(t)$  is the distribution function for  $T$ , then (4.3.10) into (4.3.9) yields  $E(M(s)) = \int_0^s (1 + E(M(s-t))) dF(t)$ .  $E(M(s)) = \lambda s \Rightarrow$  (4.3.11)  $\lambda s = F(s) + \lambda \int_0^s (s-t) dF(t)$ . This is called an integral equation for the unknown function  $F$ . This could be solved with Laplace transforms,  $F(t) = 1 - e^{-\lambda t}$ ,  $t \geq 0$ .  $T$  has the exponential distribution. We can make the same argument used for sojourn times to argue the inter-event times are independent and exponentially distributed. Conclude that  $M$  is a Poisson process with rate  $\lambda$ .