EE 514, Fall 2005
Exam 3: November 17, 2005

Solutions (version: November 23, 2005, 12:18) 50 mins.; Total 50 pts.

1. (15 pts.) Let

$$A = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix},$$

where $\theta \in [0, 2\pi)$ represents some fixed angle. Consider a random vector $X = [X_1, X_2]'$ such that X_1 and X_2 are independent, with means 1 and 2, respectively, and variances σ_1^2 and σ_2^2, respectively. Let $Y = AX$.

a. Find the mean vector of Y.

b. Find the covariance matrix of Y.

c. For what set of values of θ, σ_1, and σ_2 will the components of Y be uncorrelated?

Ans.: a. The mean vector of Y is

$$\mathsf{E}[Y] = A\mathsf{E}[X] = \begin{bmatrix} \cos(\theta) + 2\sin(\theta) \\ -\sin(\theta) + 2\cos(\theta) \end{bmatrix}.$$

b. The covariance matrix of Y is

$$\begin{aligned} C_Y &= AC_XA' \\ &= \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \\ &= \begin{bmatrix} \sigma_1^2\cos^2(\theta) + \sigma_2^2\sin^2(\theta) & -\sigma_1^2\sin(\theta)\cos(\theta) + \sigma_2^2\sin(\theta)\cos(\theta) \\ -\sigma_1^2\sin(\theta)\cos(\theta) + \sigma_2^2\sin(\theta)\cos(\theta) & \sigma_1^2\sin^2(\theta) + \sigma_2^2\cos^2(\theta) \end{bmatrix} \end{aligned}$$

c. The components of Y are uncorrelated iff C_Y is diagonal, which is true iff

$$\sigma_1^2\sin(\theta)\cos(\theta) = \sigma_2^2\sin(\theta)\cos(\theta).$$

The set of values of σ_1, σ_2, and θ for which this holds is given by: $\theta = 0$ or $\theta = \pi/2$ or $\theta = \pi$ or $\theta = 3\pi/2$ or $\sigma_1 = \sigma_2$.

2. (15 pts.) Suppose a zero-mean random vector X has covariance matrix

$$C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

a. Find the eigenvalues of C.

b. Find P such that $P'P = PP' = I$ and $P'CP$ is diagonal.

c. Find a fixed vector v such that $X = Zv$ a.s. for some real-valued random variable Z.
Hint: Karhunen-Loéve expansion of X.

Ans.: a. The characteristic polynomial of C is

$$\det(\lambda I - C) = \lambda^2 - 2\lambda = \lambda(\lambda - 2).$$

Hence, the eigenvalues of C are 0 and 2.

b. Eigenvectors of C are in the nullspace of $-C$ and $2I - C$. By inspection, we see that $[1, -1]'$ and $[1, 1]'$ are eigenvectors of C, corresponding to eigenvalues 0 and 2. So, normalizing these eigenvectors, we can set

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

It is now easy to verify that

$$P'CP = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

c. The Karhunen-Loéve expansion of X is

$$X = PY = Y_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + Y_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

But $\mathsf{E}[Y_1^2] = 0$, which means that $Y_1 = 0$ a.s. Hence,

$$X = Y_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{a.s.}$$

So the answer is $v = [1, 1]'$.

3. (14 pts.) Suppose we wish to estimate a quantity, represented by a real-valued random variable X with density $f_X(x) = (1/2)e^{-|x-1|}$ (shifted Laplace). For this estimation, we take two measurements of X, represented by $Y_i = X + W_i$, $i = 1, 2$, where W_1 and W_2 represent i.i.d. noise, independent of X, with density $f_W(w) = (1/2)e^{-|w|}$.

a. Let $Y = [Y_1, Y_2]'$. Find m_X, m_Y, C_X, C_{XY}, and C_Y.

b. Find the Wiener filter for estimating X based on Y_1 and Y_2. (Write the filter as a simple expression involving Y_1 and Y_2.)

Ans.: a. We have $m_X = 1$ and $C_X = 2$, and hence $m_Y = [1, 1]'$. Now,

$$\begin{aligned} C_{XY} &= \mathsf{E}[(X-1)[Y_1 - 1, Y_2 - 1]] \\ &= \mathsf{E}[[(X-1)(Y_1 - 1), (X-1)(Y_2 - 1)]], \\ &= [\mathsf{E}[(X-1)(Y_1 - 1)], \mathsf{E}[(X-1)(Y_2 - 1)]], \end{aligned}$$

where, for $i = 1, 2$,

$$\begin{aligned}
\mathsf{E}[(X-1)(Y_i-1)] &= \mathsf{E}[(X-1)(X+W_i-1)] \\
&= \mathsf{E}[(X-1)(X-1)+(X-1)W_i] \\
&= \mathsf{E}[(X-1)^2]+\mathsf{E}[(X-1)W_i] \\
&= \mathsf{E}[(X-1)^2]+\mathsf{E}[X-1]\mathsf{E}[W_i] \quad \text{by independence} \\
&= 2.
\end{aligned}$$

Hence,

$$C_{XY} = [2, 2].$$

Also,

$$\begin{aligned}
C_Y &= \mathsf{E}\left[\begin{bmatrix} X-1+W_1 \\ X-1+W_2 \end{bmatrix}[X-1+W_1, X-1+W_2]\right] \\
&= \begin{bmatrix} \mathsf{E}[(X-1+W_1)^2 & \mathsf{E}[(X-1+W_1)(X-1+W_2)] \\ \mathsf{E}[(X-1+W_1)(X-1+W_2)] & \mathsf{E}[(X-1+W_2)^2] \end{bmatrix},
\end{aligned}$$

where

$$\begin{aligned}
\mathsf{E}[(X-1+W_i)^2] &= \mathsf{E}[(X-1)^2]+\mathsf{E}[W_i^2]+2\mathsf{E}[X-1]\mathsf{E}[W_i] \quad \text{by independence} \\
&= 4,
\end{aligned}$$

and, again by independence,

$$\begin{aligned}
\mathsf{E}[(X-1+W_1)(X-1+W_2)] &= \mathsf{E}[(X-1)^2]+\mathsf{E}[W_1]\mathsf{E}[W_2] \\
&\quad +\mathsf{E}[X-1]\mathsf{E}[W_1]+\mathsf{E}[X-1]\mathsf{E}[W_2] \\
&= 2.
\end{aligned}$$

Hence,

$$C_Y = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}.$$

The above calculations are simpler if we use vector/matrix notation. Specifically, first write $\bar{X} = [X, X]'$, so that $Y = \bar{X} + W$. Also note that the mean vector of $\bar{X}$ is equal to m_Y. Hence,

$$\begin{aligned}
C_{XY} &= \mathsf{E}[(X-m_X)(Y-m_Y)'] \\
&= \mathsf{E}[(X-m_X)(\bar{X}-m_Y+W)'] \\
&= \mathsf{E}[(X-m_X)(\bar{X}-m_Y)']+\mathsf{E}[X-m_X]\mathsf{E}[W'] \quad \text{by independence} \\
&= [2, 2].
\end{aligned}$$

Also, because $\bar{X}$ and W are indpedendent,

$$C_Y = C_{\bar{X}} + C_W = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}.$$

b. Hence, the Wiener filter is given by $A(Y - m_Y) + m_X$, where

$$A = C_{XY}C_Y^{-1} = [2, 2]\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}^{-1} = [2, 2]\frac{1}{12}\begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} = \frac{1}{3}[1, 1].$$

The Wiener filter is therefore given by

$$\frac{Y_1 + Y_2 + 1}{3}$$

(not quite the same as simply averaging the measurements).

4. (6 pts.) Suppose $[X, Y]'$ is a Gaussian random vector with mean $[1, 2]'$ and covariance matrix

$$C = \begin{bmatrix} 0.7 & 1 \\ 1 & 2 \end{bmatrix}.$$

Find the Wiener, MMSE (general), and MAP estimator of X given Y.

Ans.: We have $C_X = 0.7$, $C_{XY} = C_{YX} = 1$, and $C_Y = 2$. Compute A by

$$A = \frac{C_{XY}}{C_Y} = \frac{1}{2}.$$

Then, $C_{X|Y} = 0.7 - 0.5 = 0.2$. In this case, the *a posteriori* density $f_{X|Y}(x|y)$ is Gaussian (with variance 0.2). Hence, all three estimators are equal to

$$g(Y) = A(Y - 2) + 1 = \frac{Y}{2}.$$