

Notes - 8 April

X_n irreducible positive recurrent M.C. Stationary distribution Π . Time reversal $Y_n = X_{N-n}, 0 \leq n \leq N$. Y_n is a M.C. $P(Y_{n+1} = j | Y_n = i) = \frac{\Pi_j}{\Pi_i} P_{ji}$. X_n is reversible if $\Pi_i P_{ij} = \Pi_j P_{ji}$ for all i, j .

If there is a distribution Π such that (3.5.1) holds, then Π is the stationary distribution and X_n is reversible.

Example 3.5.1 - Consider ON/OFF system in ex 2.2.3, $P = (1-p, p \text{ \& } q, 1-q), 0 \leq p \leq 1, 0 \leq q \leq 1$. (3.5.1) $\Rightarrow \Pi_0 P_{00} = \Pi_0 P_{00}, \dots$ (see notes). $\Rightarrow \Pi_0 P = \Pi_1 q$. Recall the stationary distribution is $\Pi_0 = \frac{q}{p+q}, \Pi_1 = \frac{p}{p+q}$. These equations hold, so the chain is reversible.

Example 3.5.2 - Ehrenfest Model of Diffusion - (See notes for drawing.) Two containers (A, B), stopped at top, connected with a small tube at the neck. Filled with M total molecules of a gas. At each time, one molecule picked at random moves between the containers. X_n = number of molecules in A at time n . X_n is a M.C. $P = (0, 1, 0, \dots \text{ \& } 1/m, 0, 1-(1/m), 0, \dots \text{ \& } 0, 2/m, 0, 1-(2/m), 0, \dots \text{ \& } \dots)$. $P_{i,i+1} = 1 - \frac{i}{m}, P_{i,i-1} = \frac{i}{m}, 0 \leq i \leq m$. If we look for solutions of (3.5.1) we find $\Pi_i = \binom{m}{i} (\frac{1}{2})^m$, which is a stationary distribution and the chain is reversible.

§ 3.6 - Chains with finitely many states

As discussed before, the theory is much simpler when there is a finite state space. Theorem 3.2.6 implies If X_n is irreducible and has a finite state space, then it is positive recurrent.

Theorem 3.6.1 - An irreducible aperiodic chain with a finite state space has a stationary distribution which is also a limiting distribution.

Theorem 3.6.2 - Perron-Fröbenius - If P is the transition probability matrix of an irreducible chain with period d and a finite state space, (1) $\lambda_0 = 1$ is an eigenvalue of P . (2) the d complex roots: $\lambda_1 = e^{2\pi i/d}, \lambda_2 = e^{4\pi i/d}, \dots, \lambda_d = e^{2\pi i \frac{d-1}{d}}, i = \sqrt{-1}, (\lambda_m)^d = 1$ are eigenvalues of P (look in notes for cool drawing of the unit circle with d roots drawn as spokes). (3) the remaining eigenvalues $\lambda_{d+1}, \dots, \lambda_m$ satisfy $|\lambda_m| \leq 1$.

What this means: the power method. Let P be a matrix. Suppose $\{V_1, \dots, V_N\}$ is a basis of eigenvectors. $PV_i = \lambda_i V_i$. Let V be arbitrary, then $V = \sum_{j=1}^N \alpha_j V_j$ for some $\{\alpha_1, \dots, \alpha_N\}$. Think of V as being the initial condition of a markov process (X_0) . So [some stuff that I don't get and isn't in the notes.] $P^n V = \sum_j \alpha_j \lambda_j^n V_j$. Suppose λ_1 is the largest $|\lambda_1| > |\lambda_j|, j \neq 1$. More stuff I don't understand.

Example 3.6.1 - Suppose the chain is aperiodic and P has distinct eigenvalues. There is a B with $P = B^{-1} \Lambda B = B^{-1}$ (diagonal matrix with λ_i in diagonals, $0 \leq i \leq N, 0$ else) $B, P^n = B^{-1} \Lambda^n B$ and $|\lambda_i|^n \rightarrow 0$ for $i \geq 1, |\lambda_0| = 1$.

§ 3.7 - Branching Processes

Unfortunately the preceding theory is not directly useful. Consider the branching process: $\{Z_{nj}, n \geq 1, j \geq 1\}$ i.i.d. nonnegative i.v., pmf $\{P_k\}$. $X_0 = 1, X_1 = Z_{1,X_0}, X_2 = Z_{21} + Z_{22} + \dots + Z_{2,X_1}, \dots, X_n = Z_{n,1} + \dots + Z_{n,X_{n-1}}$. If there is a strictly positive probability that each family is empty, $P(X_n = 0) > 0$ then 0 is an absorbing state. Hence, 0 is a positive recurrent state and all other states are transient. The chain is not irreducible, but there is a unique stationary distribution: $\Pi_0 = 1, \Pi_i = 0$ for $i \geq 1$. This gives no interesting information. The difficulty is that the process can behave in a number of different ways depending on, for example, whether or not it becomes extinct. We study the behavior conditional on some event, e.g. extinction. Recall $\rho_k = P(X_1 = k), \phi(s) = E(s^{X_1}), N = \inf\{n : X_n = 0\} = \text{time of extinction} [\inf = \min]$. $N = \infty$ if $X_n \neq 0$ for all n . If $N = \infty$, the process grows without bound and if $N < \infty$, the process is limited in size and eventually becomes extinct. Recall the probability $\eta = P(N < \infty)$ of extinction is the smallest nonnegative root of $S = \phi(s)$. We let $E_n = \{n < N < \infty\}$ be the event that extinction occurs at some time larger than n . We studied the distribution of the X_n conditional on E_n . We let $P_{0,j}(n) = P(X_n = j | E_n)$ be the conditional probability that $X_n = j$ given the future extinction. We try to compute $\Pi_{0,j} = \lim_{n \rightarrow \infty} P_{0,j}(n)$ (if the limit exists). To avoid trivial cases, we assume $0 < \rho_0 + \rho_1 < 1, \rho_0 > 0$ (some chance we will have more than one offspring). This implies $0 < P(E_n) < 1$ and $0 < \eta \leq 1$.

Theorem 3.7.1 - If $E(X_1) < \infty$, then $\Pi_{0,j} = \lim_{n \rightarrow \infty} P_{0,j}(n)$ exists. The generating function $G(s) = \sum_{j=0}^{\infty} \Pi_{0,j} s^j$ satisfies (3.7.1) $G(\eta^{-1} \phi(s\eta)) = \alpha G(s) + 1 - \alpha, \alpha = \phi'(\eta)$. If $\mu = E(X_1) \leq 1$, then $\eta \equiv 1$ and $\alpha = \mu$, so (3.7.2) $G(\phi(s)) = \mu G(s) + 1 - \mu$ for all $\mu, G'(\eta) \leq 1$ and $G'(\eta) = 1 \iff \mu = 1$. See outline of proof in notes.

Theorem 3.7.2 - (1) If $E(X_1) \neq 1, \sum_j \Pi_{0,j} = 1$, (2) If $E(X_1) = 1, \Pi_{0,j} = 0$ for all j .

§ 3.8 Review (in notes)