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EG 520
HW 6

17.3 a. Problem:

$$\begin{array}{rl} \text{maximize} & 2x_1 + 3x_2 \\ \text{subject to} & x_1 + 2x_2 \le 4 \\ & 2x_1 + x_2 \le 5 \\ & x_1, x_2 \ge 0 \end{array}$$

Solution: $x_1 = 2, x_2 = 1$ with objective value $= 7$.

b. Dual is:

$$\begin{array}{rl} \text{minimize} & 4y_1 + 5x_2 \\ \text{subject to} & y_1 + 2y_2 \le 2 \\ & 2y_1 + y_2 \le 3 \\ & y_1, y_2 \ge 0 \end{array}$$

Solution: $y_1 = 1\frac{1}{3}, x_2 = \frac{1}{3}$ with objective value $= 7$.

17.6 a. Dual is:

$$\begin{array}{rl} \text{maximize} & \lambda_1 + \cdots + \lambda_n \\ \text{subject to} & a_1\lambda_1 \le 1 \\ & \vdots \\ & a_n\lambda_n \le 1 \\ \text{where} & 0 < a_1 < \cdots < a_n \end{array}$$

b. Duality theorem: If the primal problem has an optimal solution, then so does the dual, and the optimal values of their respective objective functions are equal.

17.9 If $\boldsymbol{\mu}^T \boldsymbol{x} = 0$ then atleast one element of $\boldsymbol{\mu}$ or $\boldsymbol{x}$ per pair is 0.

19.6a Problem:

$$\begin{array}{rl} \text{minimize} & 2x_1 + 3x_2 - 4 \\ \text{subject to} & x_1x_2 = 6 \end{array}$$

Thus $\nabla f(\boldsymbol{x}) = [2,3]^T, \nabla h(\boldsymbol{x}) = [x_2, x_1]^T$. So the minimizers satisfy:

$$\begin{aligned} 2 + \lambda x_2 &= 0 \\ 3 + \lambda x_1 &= 0 \\ x_1 x_2 &= 6 \end{aligned}$$

Clearly λ, x_1, x_2 must all be nonzero. If any are then the equations would be $2 = 0$ or $3 = 0$. Solving the first equation for x_2 yields $x_2 = \frac{-2}{\lambda}$. Solving the second equation for x_1 yields $x_1 = \frac{-3}{\lambda}$. Substituting these into the third equation yields $\frac{-2}{\lambda}\frac{-3}{\lambda} = 6 \Rightarrow \frac{6}{\lambda^2} = 6 \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = 1, -1$. For $\lambda = 1 : x_2 = -2, x_1 = -3$. For $\lambda = -1 : x_2 = 2, x_1 = 3$. Analysis is the same if this problem is changed to a maximization problem.

19.10 Problem:

$$\begin{array}{rl} \text{maximize} & ax_1 + bx_2 \\ \text{subject to} & x_1^2 + x_2^2 = 2 \end{array}$$

Thus $\nabla f(\boldsymbol{x}) = [a, b]^T, \nabla h(\boldsymbol{x}) = [2x_1, 2x_2]^T$. So the maximizers satisfy:

$$\begin{aligned} a + 2\lambda x_1 &= 0 \\ b + 2\lambda x_2 &= 0 \\ x_1^2 + x_2^2 &= 2 \end{aligned}$$

Given the solution of $[1,1]^T$, $a + 2\lambda(1) = 0, b + 2\lambda(1) = 0 \Rightarrow a + 2\lambda = 0, b + 2\lambda = 0 \Rightarrow a + 2\lambda = b + 2\lambda \Rightarrow a = b$.

19.11a Problem:

$$\begin{array}{rl} \text{maximize} & x_1x_2 - 2x_1 \\ \text{subject to} & x_1^2 - x_2^2 = 0 \end{array}$$

Thus $\nabla f(\boldsymbol{x}) = [x_2 - 2, x_1]^T, \nabla h(\boldsymbol{x}) = [2x_1, -2x_2]^T$. So the minimizers satisfy:

$$\begin{aligned} x_2 - 2 + 2\lambda x_1 &= 0 \\ x_1 - 2\lambda x_2 &= 0 \\ x_1^2 - x_2^2 &= 0 \end{aligned}$$

From the third equation, $x_1^2 = x_2^2$, which is satisfied by both $[1,1]^T$ and $[-1,1]^T$.

Solving the first equation for x_2 yields $x_2 = 2 - 2\lambda x_1$. Solving the second equation for x_1 yields $x_1 = 2\lambda x_2$. Substituting yields $x_2 = 2 - 2\lambda(2\lambda x_2) = 2 - 4\lambda^2 x_2 \Rightarrow 2 = (1 + 4\lambda^2)x_2 \Rightarrow x_2 = \frac{2}{1+4\lambda^2} \Rightarrow 2 - 2\lambda x_1 = \frac{2}{1+4\lambda^2}$.

19.15a