transition probability matrix between states $\{0,1,2\}$ is

$$P = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & \beta & \gamma \\ 0 & 0 & 1 \end{pmatrix}, \quad \alpha,\beta,\gamma > 0, \quad \alpha+\beta+\gamma = 1$$

If the Markov chain starts in state 0 or 2, it remains in 0 or 2 respectively (with probability 1). If it starts in state 1, it remains in state 1 for some (random) time then moves to state 0 or 2, where it is <u>trapped</u> or <u>absorbed</u>, i.e. remains for ever.

—start # 7 2/12

Questions:

1) In which state is the process absorbed into?

2) How long does it take to reach an absorbed state?

Let

$$T = \min\{n \geq 0, X_n = 0 \text{ or } X_n = 2\}$$

be the <u>time</u> <u>of</u> <u>absorption</u>.

The questions become

1) what is $P(X_T = 0 \mid X_0 = 1)$

(we can get $P(X_T = 2 \mid X_0 = 1) = 1 - P(X_T = 1 \mid X_0 = 1)$)

2) what is $E(T \mid X_0 = 1)$?

Consider

$$X_0 = 1 \begin{array}{l} \nearrow X_1 = 0 \quad \text{prob. } \alpha \\ \rightarrow X_1 = 1 \quad \text{prob. } \beta \\ \searrow X_2 = 2 \quad \text{prob. } \gamma \end{array}$$

Let $u = P(X_T = 0 \mid X_0 = 1)$

If $X_1 = 0$, $T = 1$, $X_T = 0$ prob. α

If $X_1 = 2$, $T = 1$, $X_T = 2$ prob. γ

If $X_1 = 1$, the computation repeats.

Now

$$P(X_T = 0 \mid X_1 = 0) = 1$$
$$P(X_T = 0 \mid X_1 = 2) = 0$$

$$P(X_T = 0 \mid X_1 = 1) = u$$

Using the total law of probability

$$u = P(X_T = 0 \mid X_0 = 1)$$

$$= \sum_{k=0}^{2} P(X_T = 0 \mid X_0 = 1, X_1 = k) P(X_1 = k \mid X_0 = 1)$$

visiting state k before going to 0

$$= \sum_{k=0}^{2} P(X_T = 0 \mid X_1 = k) P(X_1 = k \mid X_0 = 1)$$

(Markov property)

$$= 1 \cdot \alpha + u \cdot \beta + 0 \cdot \gamma$$

$$\Rightarrow \quad u = \alpha + \beta u$$

or

$$u = \frac{\alpha}{1-\beta} = \frac{\alpha}{\alpha+\gamma}$$

For the mean absorption time, we note $T \geq 1$.
If $X_1 = 0$, $X_1 = 2$, no further steps are needed

If $X_1 = 1$, the process has to continue

and an average of $E(T \mid X_0 = 1)$ additional steps are required for absorption. Note $X_1 = 1$ is like starting over by the Markov property.

Setting $\nu = E(T \mid X_0 = 1)$, we have

$$\nu = \underset{\text{1st step}}{1} + \underset{\substack{\text{additional}\\\text{steps}\\X_1=0}}{\alpha \cdot 0} + \overbrace{\underbrace{\beta \cdot \nu}_{\substack{\text{expected}\\\text{additional}\\\text{steps}\\X_1=1}} + \underset{\substack{\text{additional}\\\text{steps}\\X_1=2}}{\gamma \cdot 0}}^{\text{steps past first}} = 1 + \beta\nu$$

$$\Rightarrow \quad \nu = \frac{1}{1-\beta}.$$

This example generalizes to finite state Markov chains. Suppose X_n has state space $\{0, 1, 2, \ldots, N\}$.

<u>Definition 2.3.1</u>

States $0, 1, 2, \ldots, r-1$ are <u>transient</u> if $P_{ij}^n \to 0$ as $n \to \infty$ for $0 \le i, j \le r-1$

States $r, r+1, \ldots, N$ are <u>absorbing</u> if

- if $P_{ii} = 1$ for $r \le i \le N$.

We assume these conditions on X_n, which means the probability transition matrix has the form

(2.3.1) $$P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix}$$

where

0 is an $N-r+1 \times r$ zero matrix
I is an $(N-r+1)\times(N-r+1)$ identity matrix
$Q_{ij} = P_{ij}$, $0 \le i, j < r$

Starting in a transient state $X_0 = i$, $0 \le i \le r-1$, the process will remain transient for some random time, but will ultimately become trapped in some absorbing state.

We are interested in the mean time

until absorption and the probability distribution over the absorbing states. Fix k ; $r \le k \le N$. The probability of absorption into k depends on $X_0 = i$. Let $U_{ik} = U_i$ denote this probability. Starting from state i, with probability P_{ik}, the process goes to k, where it remains.

Alternatively, it could move to $j \ne k$, where $r \le j \le N$, and k would never be reached.

Finally, the chain could move to a transient state $0 \le j \le r-1$. By the Markov property, from state j, the probability of ultimate absorption into k is $U_j = U_{jk}$.

By the law of total probability,

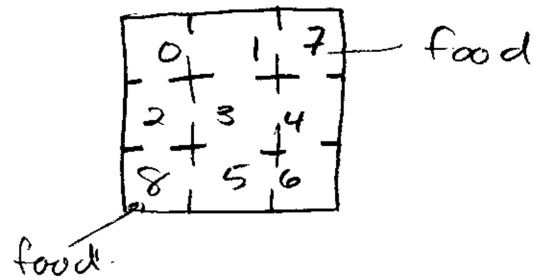
$$U_i = P(\text{Absorption into } k \mid X_0 = i)$$
$$= \sum_{j=0}^{N} P(\text{Absorption into } k \mid X_0 = i,\ X_1 = j)\ P_{ij}$$
$$= P_{ik} + \sum_{\substack{j=r \\ j \neq k}}^{N} P_{ij} \times 0 + \sum_{j=0}^{r-1} P_{ij} U_j$$

Thus, for a fixed absorbing state k, the probabilities $\{U_i\}$ satisfy the equations

$$(2.3.2)\quad U_i = P_{ik} + \sum_{j=0}^{r-1} P_{ij} U_j, \quad i = 0, \ldots, r-1.$$

<u>Example 2.3.2</u>

A white rat is put into a maze



We assume the rat moves at random and if there are k ways to leave a room, these are chosen at random with equal probability. If it reaches state 7 or 8, the rat remains there. We assume the rat changes rooms with each time and let X_n be the room occupied at time n.

We compute the transition matrix, and then compute the probabilities of absorption into state 7.

	0	1	2	3	4	5	6	7	8
0	0	1/2	1/2	0	0	0	0	0	0
1	1/3	0	0	1/3	0	0	0	1/3	0
2	1/3	0	0	1/3	0	0	0	0	1/3
3	0	1/4	1/4	0	1/4	1/4	0	0	0
4	0	0	0	1/3	0	0	1/3	1/3	0
5	0	0	0	1/3	0	0	1/3	0	1/3
6	0	0	0	0	1/2	1/2	0	0	0
7	0	0	0	0	0	0	0	1	0
8	0	0	0	0	0	0	0	0	1

The equations become

$$U_0 = \frac{1}{2}U_1 + \frac{1}{2}U_2$$

$$U_1 = \frac{1}{3} + \frac{1}{3}U_0 + \frac{1}{3}U_3$$

$$U_2 = \frac{1}{3}U_0 + \frac{1}{3}U_3$$

$$U_3 = \frac{1}{4}U_1 + \frac{1}{4}U_2 + \frac{1}{4}U_4 + \frac{1}{4}U_5$$

$$U_4 = \frac{1}{3} + \frac{1}{3}U_3 + \frac{1}{3}U_6$$

$$U_5 = \frac{1}{3}U_3 + \frac{1}{3}U_6$$

$$U_6 = \frac{1}{2}U_4 + \frac{1}{2}U_5$$

We conclude

$$U_0 = 1/2, \quad U_1 = 2/3, \quad U_2 = 1/3$$

$$U_3 = 1/2, \quad U_4 = 2/3, \quad U_5 = 1/3$$

$$U_6 = 1/2.$$

Definition 2.3.2

Let X_n be a Markov chain with state space $\{0,1,..,N\}$ and probability transition matrix in the form (2.3.1). The random absorption time T is

$$T = \min\{ n \geq 0 : X_n \geq r \}$$

We would like to compute statistics such as the mean time to absorption or the mean number of visits to a state before absorption

The argument used to derive (2.3.2) generaliz

To each transient state, we associate a "rate" $g(i)$, and we wish to compute the mean total rate accumulated up to absorption. We let W_i denote this mean corresponding to $\Xi_0 = i$,

$$W_i = E\left(\sum_{n=0}^{T-1} g(\Xi_n) \mid \Xi_0 = i \right)$$

Example 2.3.3

If we let $g(i) = 1$, then

$$\sum_{n=0}^{T-1} g(\Xi_n) = \sum_{n=0}^{T-1} 1 = T.$$

This provides motivation for the label "rate" - if we sum the rate, we get total time.

— start # 8 2/14 —

Example 2.3.4

For a transient state k, define

$$g(i) = \begin{cases} 1, & i = k, \\ 0, & i \neq k. \end{cases}$$

This gives $W_i = W_{ik}$, the mean number of visits to state k $(0 \le k \le r)$ before absorption.

Note: the sum $\sum_{n=0}^{T-1} g(\Xi_n)$ always includes $g(\Xi_0) = g(i)$.