(4.5.16) $\quad M(t) = at + i, \quad \lambda = \mu,$

(4.5.17) $\quad M(t) = \frac{a}{\lambda-\mu}\left(e^{(\lambda-\mu)t} - 1\right) + i e^{(\lambda-\mu)t}, \lambda \neq \mu$

We can derive an equation for the variance similarly.

It is interesting to note that

$$M(t) \longrightarrow \begin{cases} \infty & , \lambda \geq \mu \\ \frac{a}{\mu-\lambda} & , \lambda < \mu \end{cases}$$

This indicates there is a limiting distribution if $\lambda < \mu$.

#28 5/8

Example 4.5.2

Consider a Markov process on $S = \{0,1\}$ where the infinitesimal matrix is

$$A = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}.$$

- The process alternates between 0 and 1 with the sojourn times in 0 i.i.d. with exponential distribution with parameter α

$$\lim_{t \to \infty} P_{ij}(t) = \pi_j \geq 0$$

exist for all i. It is possible that $\pi_j = 0$, even for all j.

<u>Definition 4.6.1</u>

If the $\{\pi_j\}$ in Thm 4.6.1 are strictly positive and satisfy

$$\sum_j \pi_j = 1$$

they are called the limiting probability distribution for the process.

<u>Theorem 4.6.2</u>

If $X(t)$ is a birth-death process with no absorbing states and with limiting distribution $\{\pi_j\}$, then $\{\pi_j\}$ is also a stationary distribution, i.e.

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}(t), \qquad t \geq 0.$$

<u>Proof</u>

We have

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s)$$

If $\sum_j \theta_j = \infty$, then $\pi_j = 0$ for all j and there is no limiting probability distribution ($\lim_{t\to\infty} P_{ij}(t) = 0$ all j).

Proof

Starting with the Kolmogorov forward equations

$$(4.6.3) \quad \begin{cases} P_{i0}'(t) = -\lambda_0 P_{i0}(t) + \mu_1 P_{i1}(t) \\ P_{ij}'(t) = \lambda_{j-1} P_{ij-1}(t) - (\lambda_j + \mu_j) P_{ij}(t) + \mu_{j+1} P_{ij+1}(t), \; j \geq 1, \\ P_{ij}(0) = \delta_{ij} \end{cases}$$

Passing to the limit as $t \to \infty$, observing that the right-hand limit exists, means that the limits of the derivatives also exist. Since $P_{ij}(t)$ is converging to a constant,

$$\lim_{t\to\infty} P_{ij}'(t) = 0$$

We obtain

$$0 = -\lambda_0 \pi_0 + \mu_1 \pi_1$$

(4.6.4)

$$0 = \lambda_{j-1}\pi_{j-1} - (\lambda_j + \mu_j)\pi_j + \mu_{j+1}\pi_{j+1}, \quad j \geq 1$$

We find the solution by induction. We have

$$\pi_1 = \theta_1 \pi_0$$

Assuming $\pi_k = \theta_k \pi_0$, $k = 1, 2, \ldots, j$, we find $\pi_{j+1} = \theta_{j+1}\pi_0$. If $\sum_j \pi_j = 1$ and $\sum_j \theta_j < \infty$, then we can sum

$$\pi_0 = \theta_0 \pi_0$$
$$\pi_1 = \theta_1 \pi_0$$
$$\vdots$$
$$+ \overline{\qquad\qquad}$$
$$\sum_j \pi_j = 1 = \left(\sum_k \theta_k\right)\pi$$

to find $\pi_0 = \left(\sum_k \theta_k\right)^{-1}$, proving the result. If $\sum_j \theta_j = \infty$, then $\pi_0 = 0$.

Example 4.6.1

Consider linear growth with immigration, birth parameters $\lambda_n = a + n\lambda$, and death parameters $\mu_n = n\mu$. We saw that if $\lambda < \mu$, the population mean converges,

$$m(t) \to \frac{a}{\mu - \lambda}, \quad t \to \infty.$$

We compute the limiting distribution when $\lambda < \mu$. We have

$$\theta_0 = 1$$

$$\theta_1 = a/\mu$$

$$\theta_2 = \frac{a(a+\lambda)}{\mu \cdot 2\mu}$$

$$\theta_3 = \frac{a(a+\lambda)(a+2\lambda)}{\mu \cdot 2\mu \cdot 3\mu}$$

$$\vdots$$

$$\theta_k = \binom{(a/\lambda)+k-1}{k} \left(\frac{\lambda}{\mu}\right)^k$$

Using $$(1-x)^{-N} = \sum_{k=0}^{\infty} \binom{N+k-1}{k} x^k, \quad |x| < 1,$$

$$\sum_{k=0}^{\infty} \Theta_k = \sum_{k=0}^{\infty} \binom{(a/\lambda)+k-1}{k} \left(\frac{\lambda}{\mu}\right)^k = \left(1-\frac{\lambda}{\mu}\right)^{-(a/\lambda)}, \quad \lambda < \mu,$$

This means

$$\pi_0 = (1-\lambda/\mu)^{a/\lambda}$$

and

$$\pi_k = \left(\frac{\lambda}{\mu}\right)^k \frac{(a/\lambda)((a/\lambda)+1)\cdots(a/\lambda+k-1)}{k!} \left(1-\frac{\lambda}{\mu}\right)^{a/k},$$

$k > 1$

<u>Example 4.6.2</u> <u>Logistic Process</u>

Suppose a process ranges between two fixed integers $N < M$ for $t \geq 0$.

We assume the birth and death rates per individual at time t are

$$\lambda = \alpha(M - X(t))$$

$$\mu = \beta(X(t) - N)$$

and that individuals in the population act independently.

The birth and death rates for the population are

$$\lambda_n = \alpha n (M-n)$$

$$\mu_n = \beta n (n-N)$$

We might expect $X(t)$ to fluctuate between N and M because of the rates.

We can show that

$$\theta_{N+m} = \frac{\lambda_N \lambda_{N+1} \cdots \lambda_{N+m-1}}{\mu_{N+1}\mu_{N+2}\cdots \mu_{N+m}} = \frac{N}{N+m}\binom{M-N}{m}\left(\frac{\alpha}{\beta}\right)^m$$

and the stationary distribution is

$$\pi_{N+m} = \frac{c}{N+m}\binom{M-N}{m}\left(\frac{\alpha}{\beta}\right)^m$$

for $m = 0, 1, 2, \ldots, M-N$.

Chapter 5 Markov Chain Monte Carlo Methods

In general, it is very difficult to compute realizations of a random vector X whose component random variables are dependent. We describe a way to do this.

§5.1 Background

We consider a finite state Markov chain $\{X_n\}$, with state space $S = \{1, 2 \ldots N\}$. We let P_{ij} be the transition probability matrix and X_0 the initial condition.

We assume X_n is irreducible and aperiodic. We let $\{\pi_j\}$ be the stationary distribution of X_n, so

$$(5.1.1) \quad \begin{cases} \pi = \pi P \\ \sum_{j=1}^{N} \pi_j = 1 \end{cases}$$

$\{\pi_j\}$ is also the limiting distribution so

$$(5.1.2) \qquad \pi_j = \lim_{n \to \infty} P(X_n = j) \quad , j = 1 \ldots, N.$$

Sometimes, the following method can be used to solve (5.1.1). Suppose there are positive numbers θ_j, $j=1\ldots N$, such that

$$(5.1.2)\quad \begin{cases} \theta_i P_{ij} = \theta_j P_{ji}, & i \neq j \\ \sum_{j=1}^{N} \theta_j = 1 \end{cases}$$

Summing yields

$$\sum_{i=1}^{N} \theta_i P_{ij} = \theta_j \sum_{i=1}^{N} P_{ij} = \theta_j$$

so $\theta_j = \pi_j$, $j=1\ldots, N$.

We also use

<u>Theorem 5.1.1</u>

Let h be a function on S, then with probability 1,

$$(5.1.3)\quad \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} h(X_i) = \sum_{j=1}^{N} \pi_j h(j)$$

<u>Proof</u>

$\frac{1}{n}\sum_{i=1}^{n} h(X_i)$ is the expected value of $\{h(X_i)\}$. If $P_j(n)$ denotes the proportion of time that the chain is in state j during times $1, 2, \ldots, n$, then

$$\frac{1}{n}\sum_{i=1}^{n} h(X_i) = \sum_{j=1}^{N} h(j) P_j(n).$$

But. $P_j(n) \longrightarrow \pi_j$ as $n \to \infty$.

Recall that we say a Markov chain is time reversible if

$$\pi_i P_{ij} = \pi_j P_{ji}, \quad j \neq i.$$

If the initial state is chosen according to $\{\pi_j\}$, then the sequence of states considered backwards is a Markov chain with transition probability matrix P_{ij}.

Suppose we want to generate the value of a random variable X having probability mass function

$$P(X = j) = P_j, \quad j = 1, \dots, N$$

If we can generate an irreducible, aperiodic Markov chain with limiting distribution $\{P_j\}$, then we could approximately generate such a random variable by running the chain for n steps to obtain the value of X_n for n large.

If the goal is to generate many random variables with distribution $\{p_j\}$ so as to estimate

$$(5.1.4) \quad E(h(X)) = \sum_{j=1}^{N} h(j)\, p_j$$

then we can estimate $E(h(X))$ using

$$(5.1.5) \quad \frac{1}{n} \sum_{i=1}^{n} h(X_i)$$

Since the early states can be strongly affected by the initial condition, we may disregard the first k states and use

$$(5.1.6) \quad \hat{\theta} = \frac{1}{n-k} \sum_{i=k+1}^{n} h(X_i)$$

for k sufficiently large.

$\boxed{\#29 \quad 8/14}$

§5.2 The Hastings-Metropolis Algorithm

Let $\{b(j)\}_{j=1}^{m}$ be positive numbers and $B = \sum_{j=1}^{m} b(j)$. We assume m is large and B is difficult to compute. We want to simulate a