Notes - 19 feb

On homework, should be x_m, m is index in Markov chain.

Continuing on Branching Processes.

Definition 2.5.2 - Branching Process Let $\{Z_{nj} n \geq 1, j \geq 1\}$ be iid non-negative integer-value random variables with pmf p_k. Below, if a random sum has zero summands we give it the value zero. The branching process X_n is defined (2.5.1) $X_0 = 1, X_1 = Z_{1,X_0}, X_2 = Z_{21} + Z_{22} + \cdots + Z_{2X_1}, \ldots, X_n = Z_{n1} + Z_{n2} + \cdots + Z_{nX_{n-1}}$. Z_{nj} = number of members in nth generation who are offspring of the jth member of the n-1st generation. Note: if $X_n = 0$, then $X_{n+1} = 0$, so 0 is absorbing.

Theorem 2.5.2 - X_n is a Markov chain. Proof: excercise.

The branching process is defined in terms of random sums. We develop a few more useful facts.

Definition 2.5.3 - Random Sums - Let $Y_1, Y_2, \ldots$ be a sequence of iid. r.v. Let N be a non-negative, integer-valued r.v. independent of $\{Y_j\}$. Let N have pmf $p_N(n) = p(N = n), n = 0, 1, 2, \ldots$. Set $X = 0 for N = 0, Y_1 + \cdots + Y_N for N > 0$. X is a random sum.

Theorem 2.5.3 - Assume that $\{Y_i\}$ and N have finite moments. $E(Y_i) = \mu, Var(Y_i) = \sigma^2, E(N) = \nu, Var(N) = \tau^2$. Then (2.5.2) $E(X) = \mu\nu$. (2.5.3) $Var(X) = \nu\sigma^2 + \mu^2\tau^2$. Proof: $E(X) = \Sigma_{n=0}^{\infty} E(X|N = n)p_N(n) = \Sigma_{n=0}^{\infty} E(Y_1 + \cdots + Y_N|N = n)p_N(n) = \Sigma_{n=0}^{\infty} E(Y_1 + \cdots + Y_n)p_N(n) = \mu\Sigma_{n=0}^{\infty} np_N(n)$. (2.5.3) is excercise.

Little notation: if Y is a random variable with pmf $\{p_k\}$, we define $\mu = E(Y), \sigma^2 = Var(Y)$.

Theorem 2.5.4 - Let X_n be a branching process with pmf $\{P_k\}$ and assume μ, σ^2 are finitie. Let $M_{(n)}, V_{(n)}$ be the mean and variance of X_n conditioned on $X_0 = 1$. (2.5.4) $M(n) = \mu^n$. (2.5.5) $V(n) = \sigma^2\mu^{n-1} * \{n for \mu = 1, \frac{1-\mu^n}{1-\mu} for \mu \neq 1$. Proof: using (2.5.2) in (2.5.1) gives (2.5.6) $\{M(n+1) = \mu M(n), V(n+1) = \sigma^2 M(n) + \mu^2 V(n). X_0 = 1, M(0) = 1, V(0) = 0$. Now iterate. $M(n+1) = \mu M(n) = \mu\mu M(n-1) = \mu^3 M(n-2) = \ldots$. This gives the result.

The mean population increases geometrically if $\mu > 1$, decreases geometrically if $\mu < 1$, and is fixed if $\mu = 1$. The variance increases or decreases geometrically if $\mu > 1$ or $\mu < 1$. It increases linearly if $\mu = 1$.

The probability of extinction:

Definition 2.5.4 - The random time of extinction N is the first time for which $X_N = 0$. This is an absorption time. We let (2.5.7) $U_n = P(N \leq n|X_0 = 1) = P(X_n = 0|X_0 = 1)$ be the probability of extinction at or prior to the nth generation, conditioned on $X_0 = 1$.

Theorem 2.5.5 - We have (2.5.8) $U_0 = 0, U_1 = P_0, U_n = \Sigma_{k=0}^{\infty} P_k(U_{n-1})^k, n \geq 2$. Proof: the single parent X_0 has $Z_{1,X_0} = k$ offspring. These offspring in turn have more offspring. If the original population dies out in n generations, then each of these k offspring lines die out in $n-1$ generations, or less. The k offspring lines are independent of each other and have the same statistics as the original generation. The probability that one of the k offspring lines dies out is U_{n-1}. So the probability that they all die out is $(U_{n-1})^k$. The total law of probability gives (2.5.8).

Example 2.5.3 - $P_0(1/4), P_1(1/8), P_2(1/2), P_3(1/8), P_4(0), \ldots$.

Recall §1.5. Recall Thm 1.5.3: if $Y_1, \ldots, Y_n$ are indep r.v. having prob. generating functions $P_{Y_1}, \ldots, P_{Y_n}$, then the generating function for $X = Y_1 + \cdots + Y_n$ is (2.5.9) $P_X(s) = P_{Y_1}(s) \cdots P_{Y_n}(s)$.

Thm 1.5.1: if a r.v. Y has pmf $\{P_k\}$ and prob. generating function P_Y, (2.5.10) $\frac{dP_Y(1)}{ds} = P_1 + 2P_2 + 3P_3 + \cdots = E(Y)$. (2.5.11) $Var(Y) = \frac{d^2P_Y(s)}{ds^2}|_{s=1} + \frac{dP_Y(s)}{ds}|_{s=1} - (\frac{dP_Y(s)}{ds}|_{s=1})^2$. See § 9.2 in text.

Theorem 2.5.6 - If $Z_1, Z_2, \ldots$ is a sequence of iid r.v. with common generating function P_Z and if $N > 0$ is an integer-valued non-negative r.v. indep of $\{Z_i\}$ with prob generating function P_N, then $X = Z_1 + \cdots + Z_N$ has prob. generating function (2.5.12) $P_X(s) = P_N(P_Z(s))$. Proof: $P_X(s) = E(s^X) = E(E(s^X|N)) = \Sigma_n E(s^X|N = n)P(N = n) = \Sigma_n E(s^{Z_1 + \cdots + Z_n})P(N = n) = \Sigma_n E(S^{Z_1}) \cdots E(s^{Z_n})P(N = n) = \Sigma_n (P_Z(s))^n P(N = n) = P_N(P_Z(s))$.

Returning to a branching process X_n, we assume the offspring pmf $\{P_k\}$ has generating function $\phi(s) = E(s^{X_1}) = \Sigma_k P_k s^k$. We want the prob. generating function ϕ_n for X_n.

Theorem 2.5.7 - We have (2.5.13) $\phi_{m+n}(s) = \phi_m(\phi_n(s)) = \phi_n(\phi_m(s))$. (2.5.14) $\phi_n(s) = \phi(\phi(\phi(\ldots\phi(s)))\ldots)$ (n compositions). Proof: Every member of the $(m+n)$th generation has a unique ancestor in the mth generation. So $X_{m+n} = Z_1 + \cdots + Z_{X_m}, Z_i$ = number of members of the (m+n)th generation that descend from the ith member of the mth generation. The Z_i are iid r.v. with the same distribution as X_n, by the Markov property. By theorem 2.5.6, $\phi_{m+n}(s) = \phi_m(\phi_{Z_1}(s)), \phi_{Z_1}(s) = \phi_n(s)$.