ECE 514, Fall 2008
Exam 2: Due 12:30pm in class, October 30, 2008

Solutions (version: October 29, 2008, 20:13) 75 mins.; Total 50 pts.

1. (10 pts.) Let $X_1, X_2, \ldots$ be an i.i.d. sequence of random variables with Cauchy(1) distribution.

a. Define $M_n = (X_1 + \cdots + X_n)/n$, $n = 1, 2, \ldots$, as usual. Find the PDF of M_n.
Hint: Use characteristic functions.

b. Is it true that $\lim_{n\to\infty} \mathsf{P}\{|M_n| \geq \varepsilon\} = 0$ for every $\varepsilon > 0$? Justify your answer fully.

Ans.: a. Using $\varphi_{X_1}(\nu) = \mathsf{E}[e^{j\nu X_1}] = e^{-|\nu|}$,

$$\begin{aligned}
\varphi_{M_n}(\nu) &= \mathsf{E}[e^{j\nu M_n}] \\
&= \mathsf{E}[e^{j\nu(X_1+\cdots+X_n)/n}] \\
&= \mathsf{E}[e^{(j\nu X_1/n)+\cdots+(j\nu X_n/n)}] \\
&= \mathsf{E}[e^{(j\nu X_1/n)+\cdots+(j\nu X_n/n)}] \\
&= \mathsf{E}[e^{j\nu X_1/n}]\cdots\mathsf{E}[e^{j\nu X_n/n}] \quad \text{by independence} \\
&= (\varphi_{X_1}(\nu/n))^n \\
&= (e^{-|\nu/n|})^n \\
&= e^{-|\nu|}
\end{aligned}$$

Hence, M_n is also Cauchy(1), and so its PDF is $f(x) = (1/\pi)/(1+x^2)$.

b. For any real number $\varepsilon > 0$, we have $\mathsf{P}\{|M_n| \geq \varepsilon\} = \mathsf{P}\{|X_1| \geq \varepsilon\}$, which is a positive constant. Hence, it is not true that $\mathsf{P}\{|M_n| \geq \varepsilon\} \to 0$ for every $\varepsilon > 0$.

2. (13 pts.) In a recent article[1] published in *The Wall Street Journal*, the author displayed the following formulas for a quantity called the *Value-at-Risk*:

$$VaR_\alpha = \inf\{\ell \in \mathbb{R} : \mathsf{P}\{L > \ell\} \leq 1 - \alpha\} = \inf\{\ell \in \mathbb{R} : F_L(\ell) \geq \alpha\}.$$

This quantity is supposed to represent the amount of money for which the probability that we will lose this amount is no greater than some pre-specified threshold (but the author does not say exactly what the symbols mean).

a. Suppose F_L represents an exponential CDF with parameter $\lambda = 2$. Calculate $VaR_{0.5}$.

b. Suppose we are interested in calculating the amount of money such that the probability that we will lose this amount is no greater than 1%. In the formulas above, what do L, F_L, α, and VaR_α represent (for the quantity of interest to us here)?

c. Suppose we take α to be a random variable that has uniform distribution on $(0, 1)$. In this case, VaR_α is also a random variable. What is its distribution function?

[1] L. G. Crovitz, "The 1% Panic," *The Wall Street Journal*, October 13, 2008.

Ans.: a. We have $F_L(\ell) = 1 - e^{-2\ell}$. Hence,

$$\begin{aligned} VaR_{0.5} &= \inf\{\ell \in \mathbb{R} : F_L(\ell) \geq 0.5\} \\ &= \inf\{\ell \in \mathbb{R} : 1 - e^{-2\ell} \geq 0.5\} \\ &= \inf\{\ell \in \mathbb{R} : \ell \geq -\ln(0.5)/2\} \\ &= -\ln(0.5)/2 \\ &\approx 0.347. \end{aligned}$$

b. In this case, L is a random variable representing the amount of money we lose, F_L is the distribution function (CDF) of L, $\alpha = 0.99$, and $VaR_\alpha = VaR_{0.99}$ is the Value-at-Risk (i.e., the amount of money such that the probability that we will lose this amount is no greater than 1%)

c. We want to find $\mathsf{P}\{VaR_\alpha \leq x\}$ for any x. We will use the definition (formula) $VaR_\alpha = \inf\{\ell \in \mathbb{R} : F_L(\ell) \geq \alpha\}$.

We first show that $\mathsf{P}\{VaR_\alpha \leq x\} = \mathsf{P}\{\alpha \leq F_L(x)\}$. To see this, suppose $VaR_\alpha \leq x$. Then, by definition of VaR_α and by the monotonicity of F_L, we deduce that $F_L(x) \geq \alpha$. Hence, $\mathsf{P}\{VaR_\alpha \leq x\} \leq \mathsf{P}\{\alpha \leq F_L(x)\}$. Conversely, suppose $\alpha \leq F_L(x)$. Then, again by definition of VaR_α, $VaR_\alpha \leq x$. Hence, $\mathsf{P}\{VaR_\alpha \leq x\} \geq \mathsf{P}\{\alpha \leq F_L(x)\}$. This shows that $\mathsf{P}\{VaR_\alpha \leq x\} = \mathsf{P}\{\alpha \leq F_L(x)\}$.

Because α is uniform on $(0, 1)$, we deduce that $\mathsf{P}\{VaR_\alpha \leq x\} = F_L(x)$. In other words, the CDF of VaR_α is F_L.

3. (14 pts.) Consider the following terrorist detection system at an airport. There are two pieces of equipment used to check everyone: a metal detector and a baggage X-ray machine. When a terrorist walks through the metal detector, the detector rings an alarm with probability 0.95. When a regular person (non-terrorist) walks through the metal detector, the detector rings an alarm with probability 0.1. When a terrorist uses the baggage X-ray machine, the machine rings an alarm with probability 0.97. When a regular person (non-terrorist) uses the baggage X-ray machine, the machine rings an alarm with probability 0.02. Assume that 99% of people at the airport are not terrorists. Also assume that the two pieces of equipment have independent errors.

a. Suppose that a particular person at the airport uses the baggage X-ray machine without setting its alarm, but sets off the alarm at the metal detector. Use the MAP rule to determine if the person here is a terrorist.

b. Calculate the probability that the MAP rule will erroneous declare a regular person a terrorist.

Ans.: a. Let X be a random variable representing the classification of the person: T for terrorist and R for regular person. We have $\mathsf{P}\{X = T\} = 0.99$ and $\mathsf{P}\{X = R\} = 0.01$. Let Y be the output of the two checks: $Y = (i, j)$, where $i = 1$ means that the metal detector's alarm rings, $i = 0$ means it doesn't ring, $j = 1$ means the baggage alarm rings, and $j = 0$ means it doesn't.

The given observation is $Y = (1, 0)$. To apply the MAP rule, we need to calculate two quantities:

$$\mathsf{P}\{Y = (1,0)|X = T\}\mathsf{P}\{X = T\} = 0.95 \cdot 0.03 \cdot 0.01 = 0.000285,$$
$$\mathsf{P}\{Y = (1,0)|X = R\}\mathsf{P}\{X = R\} = 0.1 \cdot 0.98 \cdot 0.99 = 0.09702.$$

Hence, by the MAP rule, we pick R (i.e., the person is not a terrorist).

b. Let Ψ represent the MAP rule. The probability of interest here is

$$\mathsf{P}\{\Psi(Y) = T|X = R\} = \mathsf{P}\{Y \in D_T|X = R\}$$

where D_T is the set of inputs to Ψ that result in terrorist declaration (i.e., the inverse image of Ψ with respect to $\{T\}$). We know from part a that $(1,0) \notin D_T$ (because $\Psi((1,0)) = R$). Because $\mathsf{P}\{Y = (0,0)|X = T\} < \mathsf{P}\{Y = (1,0)|X = T\}$ and $\mathsf{P}\{Y = (0,0)|X = R\} > \mathsf{P}\{Y = (1,0)|X = R\}$, we conclude that $(0,0) \notin D_T$ also. Now, for $(0,1)$, we calculate

$$\begin{aligned}\mathsf{P}\{Y = (0,1)|X = T\}\mathsf{P}\{X = T\} &= 0.05 \cdot 0.97 \cdot 0.01 = 0.000485,\\ \mathsf{P}\{Y = (0,1)|X = R\}\mathsf{P}\{X = R\} &= 0.9 \cdot 0.02 \cdot 0.99 = 0.01782,\end{aligned}$$

which implies that $(0,1) \notin D_T$. Finally, for $(1,1)$,

$$\begin{aligned}\mathsf{P}\{Y = (1,1)|X = T\}\mathsf{P}\{X = T\} &= 0.95 \cdot 0.97 \cdot 0.01 = 0.009215,\\ \mathsf{P}\{Y = (1,1)|X = R\}\mathsf{P}\{X = R\} &= 0.1 \cdot 0.02 \cdot 0.99 = 0.00198,\end{aligned}$$

which implies that $(1,1) \in D_T$. Hence,

$$\mathsf{P}\{\Psi(Y) = T|X = R\} = \mathsf{P}\{Y = (1,1)|X = R\} = 0.1 \cdot 0.02 = 0.002.$$

4. (13 pts.) Consider a pair of independent random variables (X_1, X_2), where $X_i \sim \exp(1)$, $i = 1, 2$. Next, consider another pair of independent random variables (Y_1, Y_2), independent of the first pair, where $Y_i \sim \exp(2)$, $i = 1, 2$. Define a new pair of random variables (Z_1, Z_2) as follows: Toss a fair coin (independent of either given pairs); if the toss returns head, set $(Z_1, Z_2) = (X_1, X_2)$; otherwise, set $(Z_1, Z_2) = (Y_1, Y_2)$.

a. Find the marginal PDFs of Z_1 and Z_2. Are Z_1 and Z_2 identically distributed?

b. Find the joint PDF of (Z_1, Z_2). Are Z_1 and Z_2 independent?

Ans.: a. Let the Bernoulli random variable S represent the fair coin toss in the question. Now, it is clear that $f_{Z_i}(z_i) = 0$ for $z_i < 0$, $i = 1, 2$. We have, for $i = 1, 2$ and $z_i \geq 0$,

$$\begin{aligned}f_{Z_i}(z_i) &= f_{Z_i|S}(z_i|H)\mathsf{P}\{S = H\} + f_{Z_i|S}(z_i|T)\mathsf{P}\{S = T\}\\ &= f_{X_i}(z_i)(1/2) + f_{Y_i}(z_i)(1/2)\\ &= e^{-z_i}(1/2) + 2e^{-2z_i}(1/2)\\ &= e^{-z_i}/2 + e^{-2z_i}.\end{aligned}$$

Yes, Z_1 and Z_2 are identically distributed.

b. Use the notation f_Z for the joint density of (Z_1, Z_2); similarly, f_X and f_Y. As before, it is clear that $f_Z(z_1, z_2) = 0$ if either $z_1 < 0$ or $z_2 < 0$. For $z_1, z_2 \geq 0$, we have

$$\begin{aligned}f_Z(z_1, z_2) &= f_{Z|S}(z_1, z_2|H)\mathsf{P}\{S = H\} + f_{Z|S}(z_1, z_2|T)\mathsf{P}\{S = T\}\\ &= f_X(z_1, z_2)(1/2) + f_Y(z_1, z_2)(1/2)\\ &= e^{-z_1 - z_2}(1/2) + 4e^{-2z_1 - 2z_2}(1/2)\\ &= e^{-z_1 - z_2}/2 + 2e^{-2z_1 - 2z_2}.\end{aligned}$$

If Z_1 and Z_2 were independent, then $f_Z(z_1, z_2) = f_{Z_1}(z_1) f_{Z_2}(z_2)$ for all z_1, z_2. But this does not hold. To see this, take $z_1 = z_2 = 1$. We have

$$f_Z(1,1) = e^{-2}/2 + 2e^{-4} = 0.1043,$$

while

$$f_{Z_1}(1) f_{Z_2}(1) = (e^{-1}/2 + e^{-2})^2 = 0.1019.$$

Hence, Z_1 and Z_2 are not independent.