Simplex method (Chap. 16)

- The FTLP allows us to transform the LP problem (infinite number of feasible points) to a problem over a finite number of points (BFSs).
- Simplex method: an organized way of going from one BFS to another to search for the global minimizer.
- The method uses numerical linear algebraic techniques.

Pivoting

- Given a matrix $\boldsymbol{A} = [a_{ij}]$, to *pivot* about the (p, q)th element means:
  1. Divide the pth row by the (p, q)th element a_{pq};
  2. Then, for each row $i \neq p$, subtract the ith row by a_{iq} times the pth row.
- In the resulting matrix, the qth column is the vector $\boldsymbol{e}_p$ (all 0s, except 1 in the pth component).
- Note that if a column of $\boldsymbol{A}$ is $\boldsymbol{e}_i$, $i \neq p$, then pivoting about the (p, q) element does not change the column (it remains $\boldsymbol{e}_i$).

Pivot equations

- Denote the resulting matrix by $\boldsymbol{A}' = [a'_{ij}]$.
- The entries of $\boldsymbol{A}'$ are related to those of $\boldsymbol{A}$ via the equations

$$\begin{aligned} a'_{pj} &= \frac{a_{pj}}{a_{pq}}, \\ a'_{ij} &= a_{ij} - \frac{a_{pj}}{a_{pq}} a_{iq}, \quad i \neq p, \end{aligned}$$

  where j is the column index.
- The above are called the *pivot equations*.
- The element a_{pq} is called the *pivot element*.

Moving from one basis to another (§16.3)

- Consider $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{A} = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n]$.
- Let $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ be a basis.

- Suppose we do row operations to the augmented matrix $[\boldsymbol{A}, \boldsymbol{b}]$ to obtain the canonical augmented matrix.
- To get the canonical augmented matrix, we simply pivot about the element $(1, 1)$, then $(2, 2)$, $\ldots$, (m, m).
- The canonical augmented matrix has the form

$$[\boldsymbol{I}_m, \boldsymbol{Y}_{m,n-m}, \boldsymbol{y}_0] =$$

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & y_{1\,m+1} & \cdots & y_{1n} & y_{10} \\ 0 & 1 & \cdots & 0 & y_{2\,m+1} & \cdots & y_{2n} & y_{20} \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & y_{m\,m+1} & \cdots & y_{mn} & y_{m0} \end{bmatrix}.$$

- Suppose we now want to consider a different basis, say

$$\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{p-1}, \boldsymbol{a}_q, \boldsymbol{a}_{p+1}, \ldots, \boldsymbol{a}_m\},$$

  where $q > m$.
- In other words, we want to replace the vector $\boldsymbol{a}_p$ by $\boldsymbol{a}_q$ in the basis.
- The canonical augmented matrix for the new basis will have vectors of the form $\boldsymbol{e}_i$ in the qth column and in the first m columns except the pth.
- How do we obtain the above canonical augmented matrix?
- We can start from $[\boldsymbol{A}, \boldsymbol{b}]$ and pivot about the appropriate elements (in the qth column and in the first m columns except the pth).
- Or, we can start with the previous canonical augmented matrix $[\boldsymbol{I}_m, \boldsymbol{Y}_{m,n-m}, \boldsymbol{y}_0]$ and pivot about the (p, q)th element.
- Both give the same result. (Why?)
- If we already have the canonical augmented matrix $[\boldsymbol{I}_m, \boldsymbol{Y}_{m,n-m}, \boldsymbol{y}_0]$, then the latter is preferable because it involves only one additional pivoting operation.

Adjacent basic feasible solutions

- Consider two BFSs, where the two associated bases differ by only one vector.
- Example: $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ and $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{p-1}, \boldsymbol{a}_q, \boldsymbol{a}_{p+1}, \ldots, \boldsymbol{a}_m\}$.

- Geometrically, the relationship between the above two BFSs is that they are *adjacent* vertices in the constraint set.
- See previous geometric example.

Summary: updating the canonical augmented matrix

- Given: a canonical augmented matrix corresponding to a basis.
- Consider a new basis that differs from the previous one only in that the vector $\boldsymbol{a}_p$ has been replaced by $\boldsymbol{a}_q$.
- Then, to obtain the new canonical augmented matrix, we simply pivot about the (p, q)th element.
- p = index of basis vector that leaves basis;
  q = index of column that enters basis.

Example:

- Consider
$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}.$$
- Consider basis $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$.
- To get canonical augmented matrix, pivot about $(1, 1)$ and $(2, 2)$:
$$\begin{bmatrix} 1 & 0 & -1 & -3.2 \\ 0 & 1 & 2 & 4.6 \end{bmatrix}.$$
- Basic solution: $[-3.2, 4.6, 0]^T$ (not feasible).
- Consider new basis $\{\boldsymbol{a}_1, \boldsymbol{a}_3\}$.
- To get canonical augmented matrix, pivot about $(2, 3)$:
$$\begin{bmatrix} 1 & 0.5 & 0 & -0.9 \\ 0 & 0.5 & 1 & 2.3 \end{bmatrix}.$$
- Basic solution: $[-0.9, 0, 2.3]^T$ (not feasible).
- Consider new basis $\{\boldsymbol{a}_2, \boldsymbol{a}_3\}$.

- To get canonical augmented matrix, pivot about $(1,2)$:

$$\begin{bmatrix} 2 & 1 & 0 & -1.8 \\ -1 & 0 & 1 & 3.2 \end{bmatrix}.$$

- Basic solution: $[0,-1.8,3.2]^T$ (not feasible).

- What if we had taken the canonical augmented matrix for $\{\boldsymbol{a}_1,\boldsymbol{a}_2\}$ and pivoted about $(1,3)$?
- We would obtained:

$$\begin{bmatrix} -1 & 0 & 1 & 3.2 \\ 2 & 1 & 0 & -1.8 \end{bmatrix}.$$

- Note that that basic solution is the same as what we got before: $[0,-1.8,3.2]^T$ (not feasible).

Simplex method: basic idea

- Start with an initial basis corresponding to a BFS.
- We then move to an adjacent BFS in such a way that the objective function decreases.
- If the stopping criterion is satisfied, we stop; otherwise, we repeat the process.

- We already know how to move from one basic solution to an adjacent one: if $\boldsymbol{a}_p$ leaves the basis and $\boldsymbol{a}_q$ enters, we pivot about the (p,q)th element of the canonical augmented matrix.
- How do we ensure that the adjacent basic solution we move to is feasible?
- In other words if we are given a q, how do we choose a p such that if we pivot about the (p,q)th element, the resulting rightmost column of the canonical augmented matrix will have all nonnegative elements?

Moving from one BFS to an adjacent BFS (§16.4)

- Given: $\boldsymbol{A}$ and $\boldsymbol{b}$.
- Suppose we have a basis corresponding to a BFS.
- For simplicity, assume the basis is $\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m\}$.
- Suppose we now want to put $\boldsymbol{a}_q$ into the basis, where $q>m$.
- One of the vectors $\boldsymbol{a}_p$, $1\le p\le m$, must leave the basis.
- Which p should leave? We want to choose p in such a way that the basic solution for the new basis is feasible (i.e., we have a new BFS).

- Suppose the canonical augmented matrix for the original basis is:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & y_{1\,m+1} & \cdots & y_{1n} & y_{10} \\ 0 & 1 & \cdots & 0 & y_{2\,m+1} & \cdots & y_{2n} & y_{20} \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & y_{m\,m+1} & \cdots & y_{mn} & y_{m0} \end{bmatrix}.$$

- Note that $y_{i0} \geq 0$, $i = 1, \ldots, m$.
- If $\boldsymbol{a}_q$ enters the basis and $\boldsymbol{a}_p$ leaves the basis, the new canonical augmented matrix is obtained by pivoting about the (p, q) element.
- The new elements are given by the pivot equations:

$$\begin{aligned} y'_{pj} &= \frac{y_{pj}}{y_{pq}}, \\ y'_{ij} &= y_{ij} - \frac{y_{pj}}{y_{pq}} y_{iq}, \quad i \neq p. \end{aligned}$$

- For last column, we have

$$\begin{aligned} y'_{p0} &= \frac{y_{p0}}{y_{pq}}, \\ y'_{i0} &= y_{i0} - \frac{y_{p0}}{y_{pq}} y_{iq}, \quad i \neq p \\ &= y_{iq} \left(\frac{y_{i0}}{y_{iq}} - \frac{y_{p0}}{y_{pq}} \right). \end{aligned}$$

- We want the elements y'_{i0} to be ≥ 0 (because these are the values of the basic variables in the new basic solution).
- If $y_{pq} < 0$, then $y'_{p0} < 0$. So, we need to consider only those p for which $y_{pq} > 0$.
- It is evident that we need p to be

$$p = \arg\min_i \left\{ \frac{y_{i0}}{y_{iq}} : y_{iq} > 0 \right\}.$$

- Any other p will lead to some y'_{i0} being < 0.
- Note: if $y_{iq} \leq 0$ for all i, then there are feasible points with arbitrarily large components (unbounded feasible set).
- To see this, first write

$$y_{10}\boldsymbol{a}_1 + \cdots + y_{m0}\boldsymbol{a}_m = \boldsymbol{b}.$$

- Similarly,

$$y_{1q}\boldsymbol{a}_1 + \cdots + y_{mq}\boldsymbol{a}_m = \boldsymbol{a}_q.$$

- Multiplying the second equation by $\varepsilon > 0$ and subtracting from the first, we obtain

$$(y_{10} - \varepsilon y_{1q})\boldsymbol{a}_1 + \cdots + (y_{m0} - \varepsilon y_{mq})\boldsymbol{a}_m + \varepsilon\boldsymbol{a}_q = \boldsymbol{b}.$$

- The above equation tells us that the following is a solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$:

$$[y_{10} - \varepsilon y_{1q}, \ldots, y_{m0} - \varepsilon y_{mq}, 0, \ldots, \varepsilon, \ldots, 0]^T,$$

  where the ε occurs at the qth component.

- If $y_{iq} \leq 0$ for all $i = 1, \ldots, m$, then we can see that the above solution is feasible for *any* $\varepsilon > 0$.

- In particular, the qth component can be made arbitrarily large.

Summary: finding an adjacent BFS

- Given a BFS corresponding to a basis $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$.

- We want to find an adjacent BFS corresponding to $\boldsymbol{a}_q$ entering the basis.

- The column $\boldsymbol{a}_p$ to leave the basis is the one with $p = \arg\min_i \{y_{i0}/y_{iq} : y_{iq} > 0\}$.

- If more than one p satisfies the above argmin, then we can choose either one. (Usual convention: let p be the smallest such index.) The new BFS will be degenerate in this case.

Finding a "better" adjacent BFS

- If we knew which column $\boldsymbol{a}_q$ to enter the basis, we know how to move to an adjacent BFS; i.e., given q, we can find out which p should leave.

- So, which column $\boldsymbol{a}_q$ should we have enter the basis?

- The goal is that the new (adjacent) BFS should have lower objective function value.

- Which q should we choose to achieve this goal?

Relative cost coefficients

- Consider the given basis $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$, with canonical augmented matrix

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & y_{1\,m+1} & \cdots & y_{1n} & y_{10} \\ 0 & 1 & \cdots & 0 & y_{2\,m+1} & \cdots & y_{2n} & y_{20} \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & y_{m\,m+1} & \cdots & y_{mn} & y_{m0} \end{bmatrix}.$$

- The objective function for the BFS is

$$z_0 = \sum_{i=1}^{m} c_i y_{i0}.$$

- Consider a new basis where $\boldsymbol{a}_q$ enters the basis and $\boldsymbol{a}_p$ leaves the basis.

- The objective function for the new BFS is

$$\begin{aligned} z &= \sum_{i=1, i \neq p}^{m} c_i y'_{i0} + c_q y'_{p0} \\ &= \sum_{i=1}^{m} c_i \left(y_{i0} - \frac{y_{p0}}{y_{pq}} y_{iq} \right) + c_q \frac{y_{p0}}{y_{pq}} \\ &= z_0 + \left[c_q - \sum_{i=1}^{m} c_i y_{iq} \right] \frac{y_{p0}}{y_{pq}}. \end{aligned}$$

- Let $z_q = \sum_{i=1}^{m} c_i y_{iq}$.

- We have

$$z = z_0 + (c_q - z_q) \frac{y_{p0}}{y_{pq}}.$$

- Hence, if $c_q - z_q < 0$, we conclude that $z < z_0$; i.e., the objective function for the new BFS is lower.

- Define

$$r_i = \begin{cases} 0 & \text{if } i = 1, \ldots, m \text{ (basic)} \\ c_i - z_i & \text{if } i = m+1, \ldots, n \text{ (nonbasic)} \end{cases}$$

  where z_i is as defined before.

- Then, we may write

$$z = z_0 + r_q \frac{y_{p0}}{y_{pq}}.$$

- We call r_i the ith *relative cost coefficient* (RCC).

- Note: the RCC for a basic variable is always 0.
- If $r_q < 0$, then the new BFS is better.
- In general, if $\boldsymbol{x}$ is any feasible solution, we can derive the equation

$$\boldsymbol{c}^T\boldsymbol{x} = z_0 + \sum_{i=m+1}^{n} r_i x_i.$$

- Therefore, we deduce the following result.
- Theorem (16.2): A BFS is optimal if and only if the corresponding RCC values are all ≥ 0.
- Recall that if $y_{iq} \leq 0$ for all $i = 1, \ldots, m$, then we can have feasible solutions with arbitrarily large qth component.
- If this is the case and $r_q < 0$, then we deduce that there exist feasible solutions with arbitrarily negative values.
- In this case we say that the problem is unbounded (i.e., the LP problem has no solution).

The Simplex Algorithm

1. Initialization: Given an initial BFS and its canonical augmented matrix.
2. Calculate the RCCs corresponding to the nonbasic variables.
3. If $r_j \geq 0$ for all j, then STOP—the current BFS is optimal.
4. Select a q such that $r_q < 0$.
5. If no $y_{iq} > 0$, then STOP—the problem is unbounded;
   else, calculate $p = \arg\min_i \{y_{i0}/y_{iq} : y_{iq} > 0\}$.
6. Update canonical augmented matrix by pivoting about the (p, q)th element.
7. Go to step 2.

Example (16.3)

- Consider the LP problem

$$\begin{array}{ll} \text{maximize} & [2 \;\; 5] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \text{subject to} & x_1 \leq 4 \\ & x_2 \leq 6 \\ & x_1 + x_2 \leq 8 \\ & x_1, x_2 \geq 0. \end{array}$$

- Must first convert to standard form:

$$
\begin{array}{ll}
\text{minimize} & -2x_1 - 5x_2 - 0x_3 - 0x_4 - 0x_5 \\
 & \begin{array}{cccccl} x_1 & & +x_3 & & & = 4 \\ & x_2 & & +x_4 & & = 6 \\ x_1 & +x_2 & & & +x_5 & = 8 \end{array} \\
\text{subject to} & \\
 & x_1, \ldots, x_5 \geq 0.
\end{array}
$$

- Note that the last three columns can act as a basis; the augmented matrix is automatically in canonical form.

- We use initial canonical augmented matrix

$$
\begin{array}{cccccc}
\boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 & \boldsymbol{b} \\
1 & 0 & 1 & 0 & 0 & 4 \\
0 & 1 & 0 & 1 & 0 & 6 \\
1 & 1 & 0 & 0 & 1 & 8
\end{array}
$$

  where the BFS is $\boldsymbol{x} = [0, 0, 4, 6, 8]^T$.

- The basic columns are $\{\boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5\}$.

- Note: the basic columns are not the first three. (But does not matter from computation point of view.)

- To begin, we compute the RCCs:

$$
\begin{aligned}
r_1 &= c_1 - z_1 = c_1 - (c_3 y_{11} + c_4 y_{21} + c_5 y_{31}) = -2, \\
r_2 &= c_2 - z_2 = c_2 - (c_3 y_{12} + c_4 y_{22} + c_5 y_{32}) = -5.
\end{aligned}
$$

Iteration 1:

- Both RCCs for nonbasic variables are < 0, so we can choose either one to enter the basis.

- Common practice: select the most negative RCC; i.e., $q = 2$.

- Compute $p = \arg\min\{y_{i0}/y_{i2} : y_{i2} > 0\} = 2$.

- We now update the canonical augmented matrix by pivoting about the $(2, 2)$th element:

$$
\begin{aligned}
y'_{2j} &= \frac{y_{2j}}{y_{22}} \\
y'_{ij} &= y_{ij} - \frac{y_{2j}}{y_{22}} y_{i2}, \ i \neq 2.
\end{aligned}
$$

- Resulting updated canonical augmented matrix:

$$\begin{array}{cccccc} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 & \boldsymbol{b} \\ 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 0 & 6 \\ 1 & 0 & 0 & -1 & 1 & 2 \end{array}$$

  with corresponding BFS $\boldsymbol{x} = [0, 6, 4, 0, 2]^T$.

- The nonbasic RCCs are

$$\begin{aligned} r_1 &= c_1 - z_1 = -2 \\ r_4 &= c_4 - z_4 = 5. \end{aligned}$$

Iteration 2:

- Choose $q = 1$.
- We find that $p = 3$.
- We pivot about the $(3, 1)$th element to obtain

$$\begin{array}{cccccc} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 & \boldsymbol{b} \\ 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 1 & 0 & 1 & 0 & 6 \\ 1 & 0 & 0 & -1 & 1 & 2 \end{array}$$

  with corresponding BFS $\boldsymbol{x} = [2, 6, 2, 0, 0]^T$.

- The nonbasic RCCs are:

$$r_4 = c_4 - z_4 = 3$$
$$r_5 = c_5 - z_5 = 2.$$

- Since the RCCs are all ≥ 0, we STOP.
- Current BFS is optimal: $\boldsymbol{x} = [2, 6, 2, 0, 0]^T$.
- Solution to original problem is $x_1 = 2$, $x_2 = 6$.

Matrix form of simplex algorithm (§16.5)

- To make the calculations simpler and more organized, we formulate the simplex method using matrix notation.
- To begin, consider

$$\begin{aligned} \text{minimize } \boldsymbol{c}^T\boldsymbol{x} & \\ \text{subject to } \boldsymbol{A}\boldsymbol{x} &= \boldsymbol{b} \\ \boldsymbol{x} &\geq \boldsymbol{0}. \end{aligned}$$

- Let the first m columns of $\boldsymbol{A}$ be the basic columns. Write $\boldsymbol{A} = [\boldsymbol{B}, \boldsymbol{D}]$ where $\boldsymbol{B}$ is the basis matrix.
- Similarly partition $\boldsymbol{c}^T = [\boldsymbol{c}_B^T, \boldsymbol{c}_D^T]$.
- Can represent original LP problem as

$$\begin{array}{ll} \text{minimize} & (\boldsymbol{c}_B^T\boldsymbol{x}_B + \boldsymbol{c}_D^T\boldsymbol{x}_D) \\ \text{subject to} & [\boldsymbol{B}\ \ \boldsymbol{D}]\begin{bmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_D \end{bmatrix} = \boldsymbol{B}\boldsymbol{x}_B + \boldsymbol{D}\boldsymbol{x}_D = \boldsymbol{b}, \\ & \boldsymbol{x}_B \geq \boldsymbol{0},\ \boldsymbol{x}_D \geq \boldsymbol{0}. \end{array}$$

- BFS corresponding to basis $\boldsymbol{B}$ is:

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_D \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix},$$

  with corresponding objective function value

$$z_0 = \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b}.$$

- Any feasible solution $\boldsymbol{x} = [\boldsymbol{x}_B^T, \boldsymbol{x}_D^T]^T$ satisfies

$$\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{D}\boldsymbol{x}_D,$$

- Therefore, we can express the objective function value for any $\boldsymbol{x}$ as

$$\begin{aligned} \boldsymbol{c}^T\boldsymbol{x} &= \boldsymbol{c}_B^T\boldsymbol{x}_B + \boldsymbol{c}_D^T\boldsymbol{x}_D \\ &= \boldsymbol{c}_B^T(\boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{D}\boldsymbol{x}_D) + \boldsymbol{c}_D^T\boldsymbol{x}_D \\ &= \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b} + (\boldsymbol{c}_D^T - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{D})\boldsymbol{x}_D. \end{aligned}$$

- Define $\boldsymbol{r}_D^T = \boldsymbol{c}_D^T - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{D}$.

- The elements of $\boldsymbol{r}_D$ are the RCCs corresponding to nonbasic variables.

- We have
$$\boldsymbol{c}^T\boldsymbol{x} = z_0 + \boldsymbol{r}_D^T\boldsymbol{x}_D.$$

- If $\boldsymbol{r}_D \geq \boldsymbol{0}$, then the BFS corresponding to $\boldsymbol{B}$ is optimal.

- Otherwise, we can find another BFS with lower objective function value, by bringing into the basis a column with negative RCC.

LP problem tableau

- Append to the bottom of the augmented matrix $[\boldsymbol{A}, \boldsymbol{b}]$ an extra row $[\boldsymbol{c}^T, 0]$:
$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{c}^T & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{D} & \boldsymbol{b} \\ \boldsymbol{c}_B^T & \boldsymbol{c}_D^T & 0 \end{bmatrix}.$$

- Name: *tableau* of the LP problem.

- All relevant information about the LP is contained in the problem tableau.

Canonical tableau

- We now apply elementary row operations to the problem tableau such that the top part of the tableau corresponding to the augmented matrix $[\boldsymbol{A}, \boldsymbol{b}]$ is transformed into canonical form.

- In other words, we want the resulting tableau to have $\boldsymbol{e}_i$ as the ith columns, $i = 1, \ldots, m$, where $\boldsymbol{e}_i$ is the vector with 1 in the ith position and 0 elsewhere.

- This operation corresponds to premultiplying the tableau by the matrix
$$\begin{bmatrix} \boldsymbol{B}^{-1} & \boldsymbol{0} \\ -\boldsymbol{c}_B^T\boldsymbol{B}^{-1} & 1 \end{bmatrix}.$$

- The result of this operation is
$$\begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{B}^{-1}\boldsymbol{D} & \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0}^T & \boldsymbol{c}_D^T - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{D} & -\boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b} \end{bmatrix}.$$

- Name: *canonical tableau* corresponding to the basis $\boldsymbol{B}$.

- Note that the last row of the canonical tableau contains the RCCs.

- Also, the last element in the last row is the negative of the value of the objective function corresponding to the BFS.

- Using the tableau is a convenient way of implementing the simplex algorithm.
- Updating the tableau immediately gives us both the values of the basic variables and the RCCs.
- In addition, the (negative of the) value of the objective function can be found in the lower right-hand corner of the tableau.
- Exercise: try doing it with the previous example.

Example (16.3)

- Consider the following LP problem:

$$
\begin{array}{ll}
\text{maximize} & 7x_1 + 6x_2 \\
\text{subject to} & 2x_1 + x_2 \leq 3 \\
& x_1 + 4x_2 \leq 4 \\
& x_1, x_2 \geq 0.
\end{array}
$$

- The above is not in standard form.
- Before applying the simplex algorithm, we must first transform the problem into standard form.
- We transform the problem into standard form by introducing slack variables x_3, x_4.
- The (standard form) problem tableau is:

$$
\begin{array}{cccccc}
 & \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{b} \\
 & 2 & 1 & 1 & 0 & 3 \\
 & 1 & 4 & 0 & 1 & 4 \\
\boldsymbol{c}^T & -7 & -6 & 0 & 0 & 0
\end{array}
$$

- Notice that the above tableau is already in canonical form with respect to the basis $[\boldsymbol{a}_3, \boldsymbol{a}_4]$.
- Hence, the last row contains the reduced cost coefficients, and the rightmost column contains the values of the basic variables.
- Since $r_1 = -7$ is the most negative RCC, we set $q = 1$.
- We have $p = \arg\min_i \{y_{i0}/y_{i1} : y_{i1} > 0\} = 1$.

Iteration 1:

- We pivot about the $(1,1)$th element of the tableau to obtain

$$\begin{array}{ccccc} 1 & 1/2 & 1/2 & 0 & 3/2 \\ 0 & 7/2 & -1/2 & 1 & 5/2 \\ 0 & -5/2 & 7/2 & 0 & 21/2 \end{array}$$

- In the above, only r_2 is negative. Hence, $q = 2$.

- We compute $p = 2$.

Iteration 2:

- We pivot about the $(2,2)$th element to obtain:

$$\begin{array}{ccccc} 1 & 0 & 4/7 & -1/7 & 8/7 \\ 0 & 1 & -1/7 & 2/7 & 5/7 \\ 0 & 0 & 22/7 & 5/7 & 86/7 \end{array}$$

- All the RCCs (last row) are ≥ 0.

- Hence, the corresponding BFS is optimal: $\boldsymbol{x} = [8/7, 5/7, 0, 0,]^T$, with objective function value $-86/7$ (lower right corner).

- Solution to original problem is simply $x_1 = 8/7$, $x_2 = 5/7$, with objective function value $86/7$.

Two-phase method

- The simplex method requires an initial basis.

- How to choose an initial basis?

- Brute force: arbitrarily choose m basic columns and transform the augmented matrix for the problem into canonical form. If rightmost column is positive, then we have a legitimate (initial) BFS. Otherwise, try again.

- Potentially requires $\binom{n}{m}$ tries, and is therefore not practical.

- Certain LP problems have obvious initial BFSs.

- Example: if we have constraints of the form $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$, and we add m slack variables $z_1, \ldots, z_m$, then the constraints in standard form become

$$[\boldsymbol{A}, \boldsymbol{I}_m] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{z} \end{bmatrix} = \boldsymbol{b}, \qquad \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{z} \end{bmatrix} \geq \boldsymbol{0},$$

where $\boldsymbol{z} = [z_1, \ldots, z_m]^T$.

The obvious initial basic feasible solution is

$$\begin{bmatrix} \mathbf{0} \\ \boldsymbol{b} \end{bmatrix},$$

and the basic variables are the slack variables.

- In general, an initial BFS is not always apparent (indeed, one may not even exist).
- We need a systematic way of finding an initial basis for the simplex method.
- Two-phase simplex method: a systematic way of initializing the simplex method.

- Consider a general LP in standard form:

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \mathbf{0}. \end{aligned}$$

- Define the following associated *artificial problem*:

$$\begin{aligned} \text{minimize} \quad & y_1 + y_2 + \cdots + y_m \\ \text{subject to} \quad & [\,\boldsymbol{A} \quad \boldsymbol{I}_m\,] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \boldsymbol{b} \\ & \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} \geq \mathbf{0}, \end{aligned}$$

where $\boldsymbol{y} = [y_1, \ldots, y_m]^T$.

- We call $\boldsymbol{y}$ the vector of *artificial variables*.
- Note that the artificial problem has an obvious initial BFS:

$$\begin{bmatrix} \mathbf{0} \\ \boldsymbol{b} \end{bmatrix}.$$

We can therefore solve this problem by the simplex method.

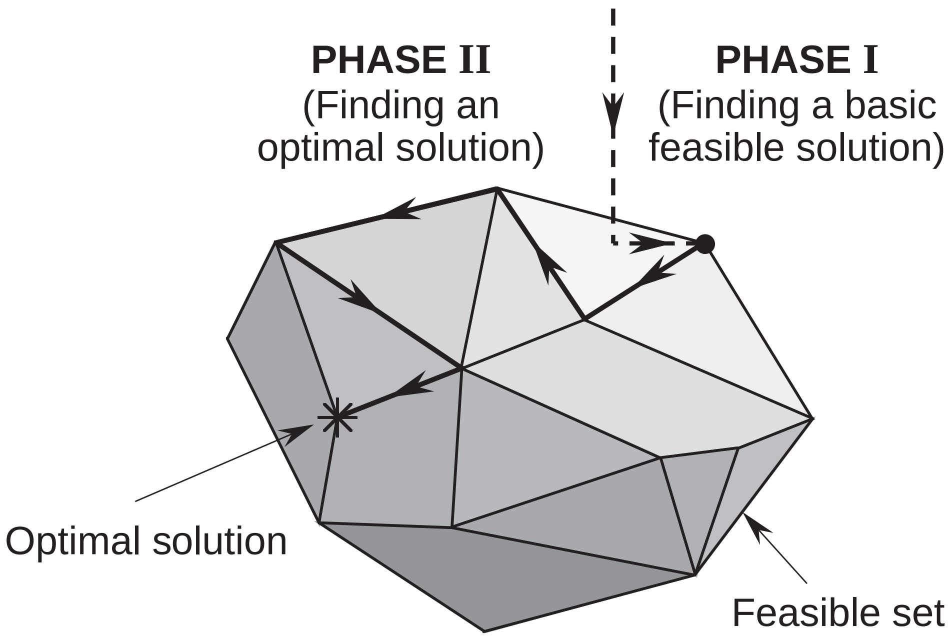
- Prop. (16.1): The original LP problem has a BFS if and only if the associated artificial problem has an optimal feasible solution with objective function value zero.

- Based on the previous proposition, we always have a systematic way of finding an initial basis.

- If we apply the simplex method to the artificial problem, and it terminates with objective function value zero, then we know that the artificial variables are not basic (assuming non-degeneracy).
- Hence, the final BFS for the artificial problem can be used as an initial BFS for the original problem.

Two-phase simplex method:

- Phase I: solve the artificial problem using simplex method.
- Phase II: use the BFS resulting from phase I to initialize the simplex algorithm to solve the original LP problem.



Example (16.4)

- Consider

$$\begin{aligned} \text{minimize} \quad & 2x_1 + 3x_2 \\ \text{subject to} \quad & 4x_1 + 2x_2 - x_3 = 12 \\ & x_1 + 4x_2 - x_4 = 6 \\ & x_1, \dots, x_4 \geq 0. \end{aligned}$$

- There is no obvious BFS that we can use to initialize the simplex method.
- Therefore, we use the two-phase method.

Phase I:

- Introduce artificial variables $x_5, x_6 \geq 0$, and an artificial objective function $x_5 + x_6$.
- Problem tableau:

$$\begin{array}{cccccccc} & \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 & \boldsymbol{a}_6 & \boldsymbol{b} \\ & 4 & 2 & -1 & 0 & 1 & 0 & 12 \\ & 1 & 4 & 0 & -1 & 0 & 1 & 6 \\ \boldsymbol{c}^T & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$

- We update the last row to transform it into canonical form.
- Initial canonical tableau:

$$\begin{array}{ccccccc} 4 & 2 & -1 & 0 & 1 & 0 & 12 \\ 1 & 4 & 0 & -1 & 0 & 1 & 6 \\ -5 & -6 & 1 & 1 & 0 & 0 & -18 \end{array}$$

- We now proceed to use the simplex algorithm for the above (artificial) LP.
- After two iterations, we get the canonical tableau

$$\begin{array}{ccccccc} 1 & 0 & -2/7 & 1/7 & 2/7 & -1/7 & 18/7 \\ 0 & 1 & 1/14 & -2/7 & -1/14 & 2/7 & 6/7 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$

- Both of the artificial variables have been driven out of the basis, and the current BFS is optimal.
- We now proceed to phase II.

Phase II:

- We start by deleting the columns corresponding to the artificial variables in the last tableau in phase I, and revert back to the original objective function.
- We obtain

$$\begin{array}{cccccc} & \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{b} \\ & 1 & 0 & -2/7 & 1/7 & 18/7 \\ & 0 & 1 & 1/14 & -2/7 & 6/7 \\ \boldsymbol{c}^T & 2 & 3 & 0 & 0 & 0 \end{array}$$

- We transform the last row so that the zeros appear in the basis columns (to make it into canonical form).

- We obtain

$$
\begin{array}{ccccc}
1 & 0 & -2/7 & 1/7 & 18/7 \\
0 & 1 & 1/14 & -2/7 & 6/7 \\
0 & 0 & 5/14 & 4/7 & -54/7
\end{array}
$$

- All the RCCs are ≥ 0.

- Hence, the optimal solution is

$$\boldsymbol{x} = [18/7, 6/7, 0, 0]^T$$

  and the optimal cost is $54/7$.

