EE/M 520, Spring 2007
Exam 3: Due May 3 (9:30am at the ECE front office)

Solutions (version: May 4, 2007, 18:3) Total 50 pts.

1. (9 pts.) You are given the following linear programming problem:

$$\begin{array}{ll} \text{maximize} & c_1x_1 + \cdots + c_nx_n \\ \text{subject to} & x_1 + \cdots + x_n = 1 \\ & x_1, \ldots, x_n \geq 0, \end{array}$$

where $c_1, \ldots, c_n \in \mathbb{R}$ are constants.

a. Write down the dual linear program for the above given primal problem.

b. Suppose you know that $c_4 > c_i$ for all $i \neq 4$. Use this information to solve the dual.

c. Use part b to solve the given linear programming problem.

Ans.: a. By asymmetric duality, the dual is given by

$$\begin{array}{ll} \text{minimize} & \lambda \\ \text{subject to} & \lambda \geq c_i,\ i = 1, \ldots, n. \end{array}$$

b. The constraint in part a implies that λ is feasible if and only if $\lambda \geq c_4$. Hence, the solution is $\lambda^* = c_4$.

c. By the duality theorem, the optimal objective function value for the given problem is c_4. The only solution that achieves this value is $x_4^* = 1$ and $x_i^* = 0$ for all $i \neq 4$.

2. (10 pts.) Consider a primal linear programming problem and its dual (in either symmetric or asymmetric form). Let us view the $\boldsymbol{b}$ vector in the primal as a parameter that we can vary, and we wish to calculate the change in the optimal objective function value if we perturb $\boldsymbol{b}$ by a small perturbation $\Delta\boldsymbol{b}$ (i.e., replace $\boldsymbol{b}$ by $\boldsymbol{b} + \Delta\boldsymbol{b}$).

a. To make the problem precise, let $z(\boldsymbol{b})$ be the optimal value of the primal objective function. Let $\boldsymbol{\lambda}$ denote the corresponding optimal dual vector. Calculate the gradient of z at $\boldsymbol{b}$: $\nabla z(\boldsymbol{b})$. Write the answer in terms of $\boldsymbol{\lambda}$. You may assume that the optimal dual vector remains fixed in a neighborhood of $\boldsymbol{b}$; but if you do, you must explain why this assumption is reasonable. *Hint:* Use the duality theorem to see how $z(\boldsymbol{b})$ depends on $\boldsymbol{b}$.

b. Suppose that the first component of the optimal dual vector is $\lambda_1 = 3$. Now suppose we increase b_1 by a very small amount Δb_1. Determine the amount by which the optimal objective function value will change.

Ans.: a. Consider the dual; $\boldsymbol{b}$ does not appear in the constraint (but it does appear in the dual objective function). Thus, provided the level sets of the dual objective function do not exactly align with one of the faces of the constraint set (polyhedron), the optimal dual vector will not change if we perturb $\boldsymbol{b}$ very slightly. Now, by the duality theorem, $z(\boldsymbol{b}) = \boldsymbol{\lambda}^T \boldsymbol{b}$. Because $\boldsymbol{\lambda}$ is constant in a neighborhood of $\boldsymbol{b}$, we deduce that $\nabla z(\boldsymbol{b}) = \boldsymbol{\lambda}$.

b. By part a, we deduce that the optimal objective function value will change by $3\Delta b_1$.

3. (16 pts.) Consider the following problem:

$$\begin{array}{rl} \text{minimize} & x_1 x_2 \\ \text{subject to} & x_1 + x_2 \geq 2 \\ & x_2 \geq x_1. \end{array}$$

a. Write down the KKT condition for this problem.

b. Find all points (and KKT multipliers) satisfying the KKT condition. In each case, determine if the point is regular.

c. Find all points in part b that also satisfy the SONC.

d. Find all points in part c that also satisfy the SOSC.

e. Find all points in part c that are local minimizers.

Ans.: a. Write $f(\boldsymbol{x}) = x_1 x_2$, $g_1(\boldsymbol{x}) = 2 - x_1 - x_2$, and $g_2(\boldsymbol{x}) = x_1 - x_2$. The KKT condition is:

$$\begin{aligned} x_2 - \mu_1 + \mu_2 &= 0 \\ x_1 - \mu_1 - \mu_2 &= 0 \\ \mu_1(2 - x_1 - x_2) + \mu_2(x_1 - x_2) &= 0 \\ \mu_1, \mu_2 &\geq 0 \\ 2 - x_1 - x_2 &\leq 0 \\ x_1 - x_2 &\leq 0. \end{aligned}$$

b. It is easy to check that $\mu_1 \neq 0$ and $\mu_2 \not> 0$. This leaves us with only one solution to the KKT condition: $x_1^* = x_2^* = 1$, $\mu_1^* = 1$, $\mu_2^* = 0$. For this point, we have $Dg_1(\boldsymbol{x}^*) = [-1, -1]$ and $Dg_2(\boldsymbol{x}^*) = [1, -1]$. Hence, $\boldsymbol{x}^*$ is regular.

c. Both constraints are active. Hence, because $\boldsymbol{x}^*$ is regular, $T(\boldsymbol{x}^*) = \{\boldsymbol{0}\}$. This implies that the SONC is satisfied.

d. Now,

$$\boldsymbol{L}(\boldsymbol{x}^*, \boldsymbol{\mu}^*) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Moreover, $\tilde{T}(\boldsymbol{x}^*, \boldsymbol{\mu}^*) = \{\boldsymbol{y} : [-1, -1]\boldsymbol{y} = 0\} = \{\boldsymbol{y} : y_1 = -y_2\}$. Pick $\boldsymbol{y} = [1, -1]^T \in \tilde{T}(\boldsymbol{x}^*, \boldsymbol{\mu}^*)$. We have $\boldsymbol{y}^T \boldsymbol{L}(\boldsymbol{x}^*, \boldsymbol{\mu}^*)\boldsymbol{y} = -2 < 0$, which means that the SOSC fails.

e. In fact the point $\boldsymbol{x}^*$ is not a local minimizer. To see this, draw a picture of the constraint set and level sets of the objective function. Moving in the feasible direction $[1,1]^T$, the objective function increases; but moving in the feasible direction $[-1,1]^T$, the objective function decreases.

4. (15 pts.) Suppose you have a cake and you need to divide it among n different children. Suppose the ith child receives a fraction x_i of the cake. We will call the vector $\boldsymbol{x} = [x_1, \ldots, x_n]^T$ an *allocation*. We require that every child receives at least some share of the cake, and that the entire cake is completely used up in the allocation. We also impose the additional condition that the first child ($i = 1$) is allocated a share that is at least twice that of any other child. We say that the allocation is feasible if it meets all these requirements.

A feasible allocation $\boldsymbol{x}$ is said to be *proportionally fair* if for any other allocation $\boldsymbol{y}$,

$$\sum_{i=1}^{n} \frac{y_i - x_i}{x_i} \leq 0.$$

a. Let Ω be the set of all feasible allocations. Show that Ω is convex.

b. Show that a feasible allocation is proportionally fair if and only if it solves the following optimization problem:

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{i=1}^{n} \log(x_i) \\ \text{subject to} & \boldsymbol{x} \in \Omega. \end{array}$$

Ans.: a. We have

$$\Omega = \{\boldsymbol{x} : x_1 + \ldots + x_n = 1;\ x_1, \ldots, x_n > 0;\ x_1 \geq 2x_i,\ i = 2, \ldots, n\}.$$

So let $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $\alpha \in (0,1)$. Consider $\boldsymbol{z} = \alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}$. We have

$$\begin{aligned} z_1 + \cdots + z_n &= \alpha x_1 + (1-\alpha)y_1 + \cdots + \alpha x_n + (1-\alpha)y_n \\ &= \alpha(x_1 + \cdots + x_n) + (1-\alpha)(y_1 + \cdots + y_n) \\ &= \alpha + 1 - \alpha \\ &= 1. \end{aligned}$$

Moreover, for each i, because $x_i > 0$, $y_i > 0$, $\alpha > 0$ and $1 - \alpha > 0$, we have $z_i > 0$. Finally, for each i,

$$z_1 = \alpha x_1 + (1-\alpha)y_1 \geq \alpha 2x_i + (1-\alpha)2y_i = 2z_i.$$

Hence, $\boldsymbol{z} \in \Omega$, which implies that Ω is convex.

b. We first show that the negative of the objective function is convex. For this, we will compute its Hessian, which turns out to be a diagonal matrix with ith diagonal entry $1/x_i^2$, which is strictly positive. Hence, the Hessian is positive definite, which implies that the negative of the objective function is convex.

Combining the above with part a, we conclude that the problem is a convex optimization problem. Hence, the FONC (for set constraints) is necessary and sufficient. Let $\boldsymbol{x}$ be a given allocation. The FONC at $\boldsymbol{x}$ is $\boldsymbol{d}^T \nabla f(\boldsymbol{x}) \geq 0$ for all feasible directions $\boldsymbol{d}$ at $\boldsymbol{x}$. But because Ω is convex, the FONC can be written as $(\boldsymbol{y} - \boldsymbol{x})^T \nabla f(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{y} \in \Omega$. Computing $\nabla f(\boldsymbol{x})$ for $f(\boldsymbol{x}) = -\sum_{i=1}^{n} \log(x_i)$, we get the proportional fairness condition.