# ECE/MATH 520, Spring 2008

## Exam 3: Due 5/8/08, 9:30am, at the ECE front office

**Name:**________________________ 75 mins.; Total 50 pts.

**1.** (15 pts.) Let $\boldsymbol{P} \in \mathbb{R}^{n \times n}$ be a matrix with the property that each element is in the real interval $[0, 1]$, and the sum of the elements of each row is equal to $1$; call such a matrix a *stochastic matrix*. Now consider a vector $\boldsymbol{x} \geq \boldsymbol{0}$ such that $\boldsymbol{x}^\top \boldsymbol{e} = 1$, where $\boldsymbol{e} = [1, \ldots, 1]^\top$; call such a vector $\boldsymbol{x}$ a *probability vector*.

We wish to prove the following result: For any stochastic matrix $\boldsymbol{P}$, there exists a probability vector $\boldsymbol{x}$ such that $\boldsymbol{x}^\top \boldsymbol{P} = \boldsymbol{x}^\top$. Although this is a key result in probability theory (under the topic of *Markov chains*), our argument is based on duality theory (for linear programming), consisting of the following parts.

a. Consider the primal linear program:

$$\begin{array}{ll} \text{maximize} & \boldsymbol{x}^\top \boldsymbol{e} \\ \text{subject to} & \boldsymbol{x}^\top \boldsymbol{P} = \boldsymbol{x}^\top \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{array}$$

Write down the dual of this problem.

b. Show that the dual is not feasible (i.e., there does not exist a feasible solution to the dual). *Hint:* Derive a contradiction based on $\boldsymbol{P}\boldsymbol{y} > \boldsymbol{y}$; think about the largest element of $\boldsymbol{y}$ (call it $y_i$).

c. Is the primal feasible? What can you deduce about whether or not the primal is unbounded?

d. Use part c to deduce the desired result: that there exists a vector $\boldsymbol{x} \geq \boldsymbol{0}$ such that $\boldsymbol{x}^\top \boldsymbol{P} = \boldsymbol{x}^\top$ and $\boldsymbol{x}^\top \boldsymbol{e} = 1$.

**2.** (10 pts.) Consider the following problem:

$$\begin{array}{rl} \text{minimize} & f(x) \\ \text{subject to} & h(x) = 0 \end{array}$$

where $f : \mathbb{R} \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$, $f, h \in \mathcal{C}^1$, and there is a local minimizer $x^*$.

Is possible that $x^*$ fails to satisfy the Lagrange condition? Whatever your answer, give a complete explanation.

**3.** (10 pts.) Suppose we have a convex optimization problem.

a. Consider the following three feasible points: $[1, 0, 0]^\top$, $[0, 1, 0]^\top$, $[0, 0, 1]^\top$. Suppose that all three have objective function value $1$. What can you say about the objective function value of the point $(1/3)[1, 1, 1]^\top$? Explain fully.

b. Suppose we know that the three points in part a are global minimizers. What can you say about the point $(1/3)[1, 1, 1]^\top$? Explain fully.

**4.** (15 pts.) This question is on duality theory for general *nonlinear* convex programming problems.

Consider the following optimization problem:

$$\begin{array}{ll} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{g}(\boldsymbol{x}) \le \boldsymbol{0}, \end{array}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is convex, each component of $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^m$ is convex, and $f, \boldsymbol{g} \in \mathcal{C}^1$. Let us call the above problem the *primal* problem.

Define the *dual* of the above problem as:

$$\begin{array}{ll} \text{maximize} & q(\boldsymbol{\mu}) \\ \text{subject to} & \boldsymbol{\mu} \ge \boldsymbol{0}, \end{array}$$

where $q$ is defined by

$$q(\boldsymbol{\mu}) = \min_{\boldsymbol{x} \in \mathbb{R}^n} l(\boldsymbol{x}, \boldsymbol{\mu}),$$

with $l(\boldsymbol{x}, \boldsymbol{\mu})$ the Lagrangian at $\boldsymbol{x}, \boldsymbol{\mu}$.

Prove the following results:

a. If $\boldsymbol{x}_0$ and $\boldsymbol{\mu}_0$ are feasible points in the primal and dual, respectively, then $f(\boldsymbol{x}_0) \ge q(\boldsymbol{\mu}_0)$.

b. If $\boldsymbol{x}_0$ and $\boldsymbol{\mu}_0$ are feasible points in the primal and dual, respectively, and $f(\boldsymbol{x}_0) = q(\boldsymbol{\mu}_0)$, then $\boldsymbol{x}_0$ and $\boldsymbol{\mu}_0$ are optimal solutions to the primal and dual, respectively.

c. If the primal has an optimal (feasible) solution, then so does the dual, and their objective function values are equal. (You may assume regularity.)