There is a definite connection to long time behavior. If $q > p$ and $\pi_0 > \pi_1$ the chain is more likely to be found in state 0 over a long time.

The assumption that the chain is irreducible is important.

Example 3.3.3

Consider the genotype example Ex 3.1.4, with

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{pmatrix}$$

This chain is not irreducible. We attempt to find a stationary distribution anyway. $\pi = \pi P$ becomes

$$(\pi_0 \ \pi_1 \ \pi_2) \begin{pmatrix} 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{pmatrix} = (\pi_0 \ \pi_1 \ \pi_2)$$

($AA = 0$, $Aa = 1$, $aa = 2$)
The first equation is

$$\pi_0 + \frac{1}{4}\pi_1 = \pi_0 \Rightarrow \pi_1 = 0.$$

The second equation therefore becomes $0 = 0$.

The third equation becomes $\pi_2 = \pi_2$.

Any distribution of the form

$$\pi = (\alpha, 0, 1-\alpha) \; , \; 0 \le \alpha \le 1,$$

qualifies as a stationary distribution.

The evolution is simple, we choose state 0 or 2 initially according to probabilities α, $1-\alpha$ respectively, and we remain in that initial state forever.

In the finite state space, we will prove later

<u>Theorem 3.3.2</u>

If the state space has r states, the equation $\pi P = \pi$ gives at most $r-1$ linearly independent equations. Including the equation $\sum_{j \in S} \pi_j = 1$,

we obtain at most r linearly independent equations.

There is always a solution. If the chain is not irreducible, there may be more than one solution.

Proof: later.

The situation with infinite state spaces is more complicated.

Example 3.3.4

Consider the gambler's ruin problem, Ex. 3.1.12 where A has a backer and B is infinitely rich. Assume $r_i = 0$ all i and $q_i = p_i = 1/2$ for all i, and that A starts with \$1. We have

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & \cdots & & \\ 1/2 & 0 & 1/2 & 0 & \cdots & \\ 0 & 1/2 & 0 & 1/2 & 0 & \cdots \\ \vdots & & \ddots & & & \ddots \end{pmatrix}$$

$\pi = \pi P$ gives

(eqn 1) $$\tfrac{1}{2}\pi_0 + \tfrac{1}{2}\pi_1 = \pi_0 \Rightarrow \pi_1 = \pi_0$$

(eqn 2) $$\tfrac{1}{2}\pi_0 + \tfrac{1}{2}\pi_2 = \pi_1 = \pi_0 \Rightarrow \pi_2 = \pi_0$$

Continuing, we find that if the distribution exists, then

$$\pi_n = \pi_0 \quad \text{for all } n,$$

and $\pi = (\pi_0, \pi_0, \ldots)$. But this is impossible because then $\sum_{i=1}^{\infty} \pi_0 \neq 1$. There are no stationary distributions.

In contrast, suppose $r_i = 0$ but $p_0 = p_1 = p_2 = \dots = p < 1/2$ for all i. Then

$$P = \begin{pmatrix} 1-p & p & 0 & \cdots & & \\ 1-p & 0 & p & 0 & \cdots & \\ 0 & 1-p & 0 & p & 0 & \cdots \\ \vdots & & \ddots & & \ddots & \end{pmatrix}$$

$\pi P = \pi$ gives

(1st eqn) $(1-p)\pi_0 + (1-p)\pi_1 = \pi_0 \Rightarrow \pi_1 = \frac{p}{1-p}\pi_0$

(2nd eqn) $p\pi_0 + (1-p)\pi_2 = \pi_1 \Rightarrow \pi_2 = \left(\frac{p}{1-p}\right)^2 \pi_0.$

In general,

$$\pi_n = \left(\frac{p}{1-p}\right)^n \pi_0, \quad n = 1, 2 \dots$$

Since $\sum \pi_j = 1$,

$$1 = \pi_0 \cdot \sum_{n=0}^{\infty} \left(\frac{p}{1-p}\right)^n = \pi_0 \frac{1-p}{1-2p}$$

and

$$\pi_n = \frac{1-2p}{1-p}\left(\frac{p}{1-p}\right)^n, \quad n \ge 0,$$

for the stationary distribution,

What is the difference in the last two examples?

Both chains are irreducible and recurrent. The first chain is null recurrent and the second is positive recurrent.

Theorem 3.3.3

An irreducible chain has a stationary distribution π if and only if all the states are positive recurrent.

In this case, there is a unique stationary distribution π, which satisfies $\pi = \pi P$. The solution is

$$\pi_i = \frac{1}{\mu_i}, \quad i \in S,$$

where μ_i is the mean recurrence time of i.

The proof is long. We prove some intermediate results first.

Definition 3.3.3

Fix state k and define

$\rho_i(k)$ = mean number of visits to state i between two successive visits to state k.

Let

$$T_k = \min\{n \geq 1 : X_n = k \mid X_0 = k\}$$

be the time of the first return to state k.

Define

$$N_i = \sum_{n=1}^{\infty} I_{\{X_n = i\} \cap \{T_k \geq n\}}$$

$\{X_n = i\} \cap \{T_k \geq n\}$ is the set where X_n is in state i and the time to return to state k is larger than n.

N_i counts the number of visits to state i between two successive visits to state k. Hence,

$$\rho_i(k) = E(N_i \mid X_0 = k)$$

Now $N_k = 1$, so $\rho_k(k) = 1$. We also have

$$(3.3.2) \qquad \rho_i(k) = \sum_{n=1}^{\infty} P(X_n = i, T_k \geq n \mid X_0 = k)$$

We let $\rho(k)$ be the vector with entries $(\rho_i(k))_{i \in S}$.

Theorem 3.3.4

The mean recurrence time μ_k satisfies

$$(3.3.3) \qquad \mu_k = \sum_{i \in S} p_i(k)$$

The vector $p(k)$ contains terms whose sum is the mean recurrence time.

Proof

Since the time between visits to state k must be spent in some state, we have

$$T_k = \sum_{i \in S} N_i$$

Taking expectations yields (3.3.3)

—

Theorem 3.3.5

For any state k of an irreducible, reccurent chain, the vector $p(k)$ satisfies $p_i(k) < \infty$ for all i and $p(k) = p(k)P$.

Proof

We first show $p_i(k) < \infty$. Set

$$l_{ki}(n) = P(X_n = i, T_k \geq n \mid X_0 = k)$$

which is the probability that the chain reaches state i in n steps, with no intermediate return to its starting point k.

The first return time to k equals $m+n$ if

- $X_m = i$
- there is no return to k before time m
- the next visit to k takes place after another n steps

This implies

$$f_{kk}(m+n) \geq l_{ki}(m) f_{ik}(n)$$

Since the chain is irreducible, there is an n with $f_{ik}(n) > 0$. Using this n,

$$l_{ki}(m) \leq \frac{f_{kk}(m+n)}{f_{ik}(n)} \qquad \text{(we know about } f_{kk}(n) \text{ by assumption)}$$

and so

$$\rho_i(k) = \sum_{m=1}^{\infty} l_{ki}(m) \leq \frac{1}{f_{ik}(n)} \sum_{m=1}^{\infty} f_{kk}(m+n)$$

$$\leq \frac{1}{f_{ik}(n)} < \infty.$$

To prove the second claim, we start with

$$\rho_i(k) = \sum_{n=1}^{\infty} l_{ki}(n)$$

Now, $l_{ki}(1) = P_{ki}$ and, for $n \geq 2$,

$$l_{ki}(n) = \sum_{\substack{j \in S \\ j \neq k}} P(X_n = i, X_{n-1} = j, T_k \geq n \mid X_0 = k)$$

step to i from state $j \neq k$

$$= \sum_{\substack{j \in S \\ j \neq k}} l_{kj}(n-1) P_{ji}.$$

Note we have conditioned on the intermediate state X_{n-1}. Summing for $n \geq 2$,

$$\rho_i(k) = P_{ki} + \sum_{\substack{j \in S \\ j \neq k}} \Big(\sum_{n \geq 2} l_{kj}(n-1) \Big) P_{ji}$$

$$= \rho_k(k) P_{ki} + \sum_{\substack{j \in S \\ j \neq k}} \rho_j(k) P_{ji} \qquad (\rho_k(k) = 1)$$

Theorem 3.3.6

Every positive recurrent irreducible Markov chain has a stationary distribution

Proof

We just proved that an irreducible, recurrent chain satisfies

$$\rho(k) = \rho(k) P$$

The components of $\rho(k)$ are nonnegative with sum μ_k. If $\mu_k < \infty$, the vector π with entries

$$\pi_i = \rho_i(k) / \mu_k$$

satisfies $\pi = \pi P$ and has nonnegative entries that sum to 1. π is a stationary distribution.

$\boxed{\#16 \quad 3/25}$

Summarizing so far, when the chain is recurrent and irreducible, there is a solution of $x = xP$ with nonnegative entries. It is an exercise to show that this solution can be taken to have strictly positive entries, and moreover the solution is unique up to a multiplicative factor.

We conclude

<u>Theorem 3.3.7</u>

If the chain is irreducible and recurrent, there is a solution x of $x = xP$ with strictly positive entries that is unique up to a multiplicative factor. The chain is positive if $\sum x_i < \infty$ and null if $\sum x_i = \infty$.