The transition matrix is

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{pmatrix}$$

$\boxed{\#5 \quad 2/5}$

Example 2.1.8

For the random walk in Ex. 2.1.2, the state space

$$S = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

The transition probabilities are

$$P_{ij} = \begin{cases} p & , \ j = i+1, \\ 1-p, & , \ j = i-1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$P = \begin{pmatrix} \ddots & & \ddots & & & & & & 0 \\ & & 0 & 1-p & 0 & p & 0 & & \\ & & & 0 & 1-p & 0 & p & 0 & \\ & 0 & & & 0 & 1-p & 0 & p & 0 & \cdots \\ & & & & & \ddots & & & \ddots \end{pmatrix}$$

Example 2.1.9

We generalize the random walk to allow the possibility that the particle can remain at is current location.

We let $\{p_i\}, \{q_i\}, \{r_i\}$ be numbers with

$$\begin{aligned} &0 < p_i < 1, \\ &0 < q_i < 1, \qquad \text{all } i \\ &0 \le r_i < 1, \end{aligned}$$

and

$$p_i + q_i + r_i = 1, \quad \text{all } i$$

We set

$$\begin{aligned} P(X_{n+1} = i+1 \mid X_n = i) &= p_i \\ P(X_{n+1} = i-1 \mid X_n = i) &= q_i \\ P(X_{n+1} = i \mid X_n = i) &= r_i \end{aligned}$$

We have

$$P = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ \cdots & 0 & q_{-1} & r_{-1} & p_{-1} & 0 & \cdots & & 0 \\ & \cdots & 0 & q_0 & r_0 & p_0 & 0 & \cdots \\ 0 & & \cdots & 0 & q_1 & r_1 & p_1 & 0 & \cdots \\ & & & & & \ddots & \ddots & \ddots \end{pmatrix}$$

Example 2.1.10 Discrete Queueing

Customers arrive for a service and take their place in a waiting line. During each period of time, a single customer is served, provided there is a customer in line, otherwise no service is provided. During each service period, new customers may arrive.

We assume that the number of customers that arrive during the n^{th} period is a random variable C_n whose distribution is independent of time, and

$$P(C_n = k) = a_k, \quad k = 0, 1, 2, \ldots,$$

where $a_k \geq 0$ and $\sum_{k=0}^{\infty} a_k = 1$. We assume that the $\{C_n\}$ are independent.

The state of the system is the number X_n of customers waiting in line at time n, $n = 0, 1, 2, \ldots$. If the present state is i, then after one time period, the state is

$$j = \begin{cases} i - 1 + C_n & i \geq 1, \\ C_n, & i = 0, \end{cases}$$

where C_n is the number of new customers arriving during the period. We have

$$X_{n+1} = \max\{X_n - 1, 0\} + C_n$$

and

$$P = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & & \\ a_0 & a_1 & a_2 & \cdots & & \\ 0 & a_0 & a_1 & a_2 & \cdots & \\ 0 & 0 & a_0 & a_1 & a_2 & \cdots \\ \vdots & \ddots & & & \ddots & \end{pmatrix}$$

We can define a number of Markov chains using a discrete-valued random variable Y, with range $\{0,1,2,\ldots\}$ and $P(Y = i) = a_i \geq 0$, with $\sum_{i=0}^{\infty} a_i = 1$. We let $Y_1, Y_2, \ldots$ be independent observations of Y.

<u>Example 2.1.11</u> <u>Independent Random Variables</u>

Consider the process $X_n = Y_n$, $n = 0,1,2,\ldots$ Then

$$P = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_0 & a_1 & a_2 & a_3 & \cdots \\ a_0 & a_1 & a_2 & a_3 & \cdots \\ \vdots & \ddots & & & \end{pmatrix}$$

The identical rows express the fact that X_{n+1} is independent of X_n.

Example 2.1.12

Let

$$X_n = \max\{Y_0, Y_1, \dots, Y_n\}$$

$n = 1, 2, 3, \dots$, be the partial maxima of $\{Y_i\}$ and set $X_0 = Y_0$. Exercise: X_n is a Markov chain and since

$$X_{n+1} = \max\{X_n, Y_{n+1}\}$$

$$P = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & a_0+a_1 & a_2 & a_3 & \cdots \\ 0 & 0 & a_0+a_1+a_2 & a_3 & \cdots \\ 0 & 0 & 0 & a_0+\dots+a_3 & \cdots \\ \vdots & & \ddots & & \end{pmatrix}$$

Example 2.1.13 Partial Sums

We set

$$X_0 = 0$$
$$X_n = Y_1 + \dots + Y_n, \quad n \geq 1$$

Since $X_n = X_{n-1} + Y_n$ for $n \geq 1$,

$$P(X_{n+1} = j \mid X_1 = i_1, \ldots, X_{n-1} = i_{n-1}, X_n = i)$$

$$= P(Y_{n+1} = j - i \mid Y_1 = i_1,\ Y_2 = i_2 - i_1, \ldots, Y_n = i - i_{n-1})$$

$$= P(Y_{n+1} = j - i) \quad \text{(by independence)}$$

$$= P(X_{n+1} = j \mid X_n = i)$$

Also,

$$P(X_{n+1} = j \mid X_n = i)$$

$$= P(Y_1 + \cdots + Y_{n+1} = j \mid Y_1 + \cdots + Y_n = i)$$

$$= P(Y_{n+1} = j - i)$$

$$= \begin{cases} a_{j-i}, & j \geq i \\ 0, & j < i \end{cases}$$

Using independence again. This gives

$$P = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & & \\ 0 & a_0 & a_1 & a_2 & \cdots & \\ 0 & 0 & a_0 & a_1 & a_2 & \cdots \\ \vdots & & \ddots & & \ddots & \end{pmatrix}$$

The examples so far suggest

Theorem 2.1.2

The transition probability matrix satisfies

(1) $P_{ij} \geq 0$ all i, j

(2) $\sum_{j=0}^{\infty} P_{ij} = 1$ all i

Proof

Exercise

Definition 2.1.5

A matrix with these properties is called a stochastic Matrix.

§2.2 First analysis of dynamic behavior

We now set out to analyze the behavior of a Markov chain over time. There are three time scales

- one step
- a finite number of steps
- asymptotic behavior as time $\to \infty$.

The transition matrix describes what happens over one step.

Suppose we want the distribution of values of a chain two steps ahead. Let S denote the state space.

Set

$$P_{ij}^{(2)} = P(X_2 = j \mid X_0 = i)$$

and condition on the intermediate step X_1. The law of total probability gives

$$P_{ij}^{(2)} = \sum_{k \in S} P(X_2 = j \mid X_0 = i, X_1 = k) P(X_1 = k \mid X_0 = i)$$

$$\big\downarrow \text{Markov property}$$

$$= \sum_{k \in S} P(X_2 = j \mid X_1 = k) P(X_1 = k \mid X_0 = i)$$

$$= \sum_{k \in S} P_{ik} P_{kj}$$

This argument makes good sense as we have to visit some state k to go from state i to state j in two steps.

Matrix multiplication leads to the conclusion that $P_{ij}^{(2)}$ is the i^{th}, j^{th} entry in the (formal) matrix P^2.

Definition 2.2.1

The n-step transition probability matrix $P^{m,m+n}$ is the matrix of n-step transition probabilities

$$P^{m,m+n} = \left(P_{ij}^{m,m+n} \right) = \left(P\left(X_{m+n} = j \,/\, X_m = i \right) \right)$$

So we are looking at the probability of moving from state i to state j in n steps, where we begin at state i in the m^{th} step.

The assumption of homogeneity means that $P^{m,m+1} = P$, the transition matrix, for all m.
It turns out that $P^{m,m+n}$ is always independent of m, one consequence of the following result.

Theorem 2.2.1 Chapman - Kolmogorov Equations

$$(2.2.1) \quad P_{ij}^{m,m+n+r} = \sum_{k=0}^{\infty} P_{ik}^{m,m+n} \, P_{kj}^{m+n,m+n+r} \qquad m,n,r \geq 0$$

(k in state space)

In particular, formally,

$$P^{m,m+n+r} = P^{m,m+n}\, P^{m+n,m+n+r}$$

$$P^{m,m+n} = P^{n}$$

<u>Proof</u>

$$P_{ij}^{m,m+n+r} = P(X_{m+n+r} = j \mid X_m = i)$$

$$= \underbrace{\sum_k P(X_{m+n+r} = j,\ X_{m+n} = k \mid X_m = i)}_{\text{The chain must pass through some intermediate step } k.}$$

Now for events A, B, C

$$P(A \cap B \mid C) = P(A \mid B \cap C)\, P(B \mid C)$$

$A:\ X_{m+n+r} = j$

$B:\ X_{m+n} = k$

$C:\ X_m = i$

$$P_{ij}^{m,m+n+r} = \sum_{k \in S} P(X_{m+n+r} = j \mid X_{m+n} = k,\ X_m = i)\, P(X_{m+n} = k \mid X_m =$$

Markov property

$$= \sum_k P(X_{m+n+r} = j \mid X_{m+n} = k)\, P(X_{m+n} = k \mid X_m = i)$$

$\boxed{\#6 \quad 2/7}$

As a consequence