For the initial investigation, we assume the Markov chain is irreducible and aperiodic. The main result is

Theorem 3.4.1

For an irreducible, aperiodic Markov chain,

$$(3.4.1) \qquad P_{ij}^n \longrightarrow \frac{1}{\mu_j} \quad \text{as } n \to \infty \quad \text{for all } i, j$$

(μ_j is the mean recurrence time).

Note that this limiting value is the same for every state i. In words, P^n converges to a matrix in which the rows have equal entries

$$P^n \to \begin{pmatrix} \frac{1}{\mu_0} & \frac{1}{\mu_1} & \frac{1}{\mu_2} & \cdots \\ \frac{1}{\mu_0} & \frac{1}{\mu_1} & \frac{1}{\mu_2} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

Definition 3.4.1

If there exists a probability distribution q on the state space S such that

$$P_{ij}^n \xrightarrow[n\to\infty]{} q_j \quad \text{for all } i,j\in S$$

then q is called the limit distribution of the chain.

The intuition behind a limit distribution is that q_j describes the probability that the chain is in state j at some late time, and by that time, the chain has "forgotten" where it started. More concretely,

$$P(X_n=j) = \sum_i P(X_0=i)P_{ij}^n \xrightarrow[n\to\infty]{} \frac{1}{\mu_j},$$

regardless of the initial distribution of X_0.

There are some immediate consequences

Theorem 3.4.2

(a) If the chain is transient or null recurrent then $P_{ij}^n \to 0$ for all i, j.

(b) If the chain is positive recurrent, then $P_{ij}^n \to \pi_j = \mu_j^{-1}$, where π is the unique stationary distribution.

Theorem 3.4.3

If X_n is an irreducible chain with period d, then Y_n with $Y_n = X_{nd}$, $n \geq 0$, is an aperiodic chain and so

$$P_{jj}^{nd} = P(Y_n = j \mid Y_0 = j) \to \frac{d}{\mu_j} \text{ as } n \to \infty.$$

We can also prove

Proof of Theorem 3.1.6

Let $C(i)$ be the irreducible closed set of states that contains the recurrent state i. If $C(i)$ is aperiodic, the result follows immediately. The periodic case is treated as in Thm 3.4.3.

Before proving Thm 3.4.1, we discuss

stationary and limiting distributions, which are not the same idea. Consider a Markov chain at a late time n. The stationary distribution gives the long time proportions of time spent in the different states up to time n. The limit distribution gives the proportions of time spent in the states at time n, so we think of considering many realizations of the chain at time n.

<u>Example 3.4.2</u>

Consider Ex. 3.4.1 again. The stationary distribution $(\frac{1}{2}, \frac{1}{2})$ says that equal amounts of time have been spent in both states up to some large time, say $n = 1000$, regardless of the initial state.

If we look precisely at time $n = 1000$, however, the chain must be in the same state it started in. If we run the chain again from the same initial state, it ends up in the same state again, and multiple realizations produce 0 proportion of time at $n = 1000$ in the other state.

We summarize the most favorable case

Theorem 3.4.4

An ergodic Markov chain has the property that it has both stationary and limiting distributions and these are equal.

In words, an ergodic Markov chain has the property that the proportion of times spent in its states up to a long time n is equal to the proportion of times spent in different states at time n for many realizations.

Proof of Theorem 3.4.1

We treat different cases.

The simplest case is a transient chain. In this case, Thm 3.1.2 (3) implies $P_{ij}^n \to 0$ as $n \to \infty$ for all i, j.

The recurrent cases are treated with an important technique called coupling.

Definition 3.4.2

Let X_n, Y_n be independent Markov chains with common state space S and transition probability matrix P. The <u>coupled chain</u>

$Z_n = (X_n, Y_n)$ takes values in $S \times S$.

<u>Theorem 3.4.5</u>

Z_n is a Markov chain with probabilities

$$P_{ij,kl} = P_{ik} P_{jl}$$

If X_n, Y_n are irreducible and aperiodic then Z_n is irreducible.

<u>Proof</u>

$$P_{ij,kl} = P(Z_{n+1} = (k,l) \mid Z_n = (i,j))$$

$$= P(X_{n+1} = k \mid X_n = i)\, P(Y_{n+1} = l \mid Y_n = j)$$

If X, Y are irreducible and aperiodic, for any states i, j, k, l, there exists $N = N(i,j,k,l)$ such that $P_{ik}^n P_{jl}^n > 0$ for all $n \geq N$, which implies Z_n is irreducible (Exercise).

This is the only place that we use the assumption that X is aperiodic.

We now assume that X_n is positive recurrent, which implies that it has a unique stationary distribution π.
Consider $Y = X$ in constructing Z.
As an exercise, it follows that the coupled chain Z has stationary distribution $\nu = (\nu_{ij} : i, j \in S)$, $\nu_{ij} = \pi_i \pi_j$, and hence Z is also positive recurrent.

Choose $X_0 = i$, $Y_0 = j$, so $Z_0 = (i, j)$. Choose any state $s \in S$ and let

$$T = \min\{n \geq 1 : Z_n = (s, s)\}$$

denote the time of the first passage of Z_n to (s, s). The recurrence of Z_n implies that $P(T < \infty) = 1$ (Exercise).

$\boxed{\#19 \quad 4/3}$

Observation:
Now suppose $m \leq n$ and $X_m = Y_m$. Then X_n and Y_n are identically distributed since the distributions of X_n and Y_n depend only on P and the common value at m. Thus, conditional on $\{T \leq n\}$, X_n and Y_n have the same distribution.

We use this observation and the finiteness of T to show the ultimate distributions of X_n and Y_n are independent of their starting points.