EE/M 520, Spring 2007

Exam 2: Due April 19 (9:30am at the ECE front office)

Solutions (version: April 19, 2007, 16:7) Total 50 pts.

1. (8 pts.) Suppose you are given two different types of concrete. The first type contains 30% cement, 40% gravel, and 30% sand (all percentages of weight). The second type contains 10% cement, 20% gravel, and 70% sand.

How many pounds of each type of concrete should you mix together so that you get a concrete mixture that has as close as possible to a total of 5 pounds of cement, 3 pounds of gravel, and 4 pounds of sand? Formulate and solve the problem using a linear least-squares method.

Ans.: The problem can be formulated as a least-squares problem with

$$\boldsymbol{A} = \begin{bmatrix} 0.3 & 0.1 \\ 0.4 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix},$$

where the decision variable is $\boldsymbol{x} = [x_1, x_2]^T$, and x_1 and x_2 are the amounts of concrete of the first and second types, respectively. After some algebra, we obtain the solution:

$$\boldsymbol{x}^* = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{b} = \frac{1}{(0.34)(0.54) - (0.32)^2} \begin{bmatrix} 0.54 & -0.32 \\ -0.32 & 0.34 \end{bmatrix} \begin{bmatrix} 3.9 \\ 3.9 \end{bmatrix} = \begin{bmatrix} 10.6 \\ 0.961 \end{bmatrix}.$$

2. (16 pts.) Consider the problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} \\ \text{subject to} & \|\boldsymbol{x}\|^2 = 1, \end{array}$$

where $\boldsymbol{Q} = \boldsymbol{Q}^T > 0$. Suppose we apply a fixed step size projected gradient algorithm to this problem.

a. Derive a formula for the update equation for the algorithm (i.e., write down an explicit formula for $\boldsymbol{x}^{(k+1)}$ as a function of $\boldsymbol{x}^{(k)}$, $\boldsymbol{Q}$, and the fixed step size α). You may assume that the argument in the projection operator to obtain $\boldsymbol{x}^{(k)}$ is never zero.

b. Is it possible for the algorithm not to converge to an optimal solution, even if the step size $\alpha > 0$ is taken to be sufficiently small? Explain fully.
Hint: Any solution to this optimization problem is an eigenvector of $\boldsymbol{Q}$ with smallest eigenvalue.

c. Show that for $0 < \alpha < 1/\lambda_{\max}$ (where $\lambda_{\max}$ is the largest eigenvalue of $\boldsymbol{Q}$), the fixed step size projected gradient algorithm (with step size α) converges to an optimal solution, provided $\boldsymbol{x}^{(0)}$ is not orthogonal to the eigenvectors of $\boldsymbol{Q}$ corresponding to the smallest eigenvalue.
Hint: Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ be orthonormal eigenvectors of $\boldsymbol{Q}$ ordered according to ascending eigenvalues (which you may assume to be distinct). Note that $\boldsymbol{v}_1$ is the optimal solution. Then write $\boldsymbol{x}^{(k)} = y_1^{(k)}\boldsymbol{v}_1 + \cdots + y_n^{(k)}\boldsymbol{v}_n$, and assume that $y_1^{(0)} \neq 0$.

Ans.: a. The projection operator in this case simply maps any vector to the closest point on the unit circle. Therefore, the projection operator is given by $\mathbf{\Pi}[\boldsymbol{x}] = \boldsymbol{x}/\|\boldsymbol{x}\|$, provided $\boldsymbol{x} \neq \mathbf{0}$. The update equation is

$$\boldsymbol{x}^{(k+1)} = \beta_k(\boldsymbol{x}^{(k)} - \alpha \boldsymbol{Q}\boldsymbol{x}^{(k)}) = \beta_k(\boldsymbol{I} - \alpha\boldsymbol{Q})\boldsymbol{x}^{(k)},$$

where $\beta_k = 1/\|(\boldsymbol{I} - \alpha\boldsymbol{Q})\boldsymbol{x}^{(k)}\|$ (i.e., it is whatever constant scaling is needed to make $\boldsymbol{x}^{(k+1)}$ have unit norm).

b. If we start with $\boldsymbol{x}^{(0)}$ being an eigenvector of $\boldsymbol{Q}$, then $\boldsymbol{x}^{(k)} = \boldsymbol{x}^{(0)}$ for all k. Therefore, if the corresponding eigenvalue is not the smallest, then clearly the algorithm is stuck at a point that is not optimal.

c. We have

$$\begin{aligned}
\boldsymbol{x}^{(k+1)} &= \beta_k(\boldsymbol{I} - \alpha\boldsymbol{Q})\boldsymbol{x}^{(k)} \\
&= \beta_k(\boldsymbol{I} - \alpha\boldsymbol{Q})(y_1^{(k)}\boldsymbol{v}_1 + \cdots + y_n^{(k)}\boldsymbol{v}_n) \\
&= \beta_k(y_1^{(k)}(\boldsymbol{I} - \alpha\boldsymbol{Q})\boldsymbol{v}_1 + \cdots + y_n^{(k)}(\boldsymbol{I} - \alpha\boldsymbol{Q})\boldsymbol{v}_n).
\end{aligned}$$

But $(\boldsymbol{I} - \alpha\boldsymbol{Q})\boldsymbol{v}_i = (1 - \alpha\lambda_i)\boldsymbol{v}_i$, where λ_i is the eigenvalue corresponding to $\boldsymbol{v}_i$. Hence,

$$\boldsymbol{x}^{(k+1)} = \beta_k(y_1^{(k)}(1 - \alpha\lambda_1)\boldsymbol{v}_1 + \cdots + y_n^{(k)}(1 - \alpha\lambda_n)\boldsymbol{v}_n),$$

which means that $y_i^{(k+1)} = \beta_k y_i^{(k)}(1 - \alpha\lambda_i)$. In other words, $y_i^{(k)} = \beta^{(k)} y_i^{(0)}(1 - \alpha\lambda_i)^k$, where $\beta^{(k)} = \Pi_{i=0}^{k-1}\beta_k$. Rewriting $\boldsymbol{x}^{(k)}$,

$$\begin{aligned}
\boldsymbol{x}^{(k)} &= \sum_{i=1}^{n} y_i^{(k)}\boldsymbol{v}_i \\
&= y_1^{(k)}\left(\boldsymbol{v}_1 + \sum_{i=2}^{n}\frac{y_i^{(k)}}{y_1^{(k)}}\boldsymbol{v}_i\right).
\end{aligned}$$

But, assuming $y_1^{(0)} \neq 0$,

$$\frac{y_i^{(k)}}{y_1^{(k)}} = \frac{y_i^{(0)}(1 - \alpha\lambda_i)^k}{y_1^{(0)}(1 - \alpha\lambda_1)^k} = \frac{y_i^{(0)}}{y_1^{(0)}}\left(\frac{1 - \alpha\lambda_i}{1 - \alpha\lambda_1}\right)^k.$$

But because $(1 - \alpha\lambda_i)/(1 - \alpha\lambda_1) < 1$ (because the $\lambda_i > \lambda_1$ for $i > 1$ and $\alpha < 1/\lambda_{\max}$), we deduce that

$$\frac{y_i^{(k)}}{y_1^{(k)}} \to 0,$$

which implies that $\boldsymbol{x}^{(k)} \to \boldsymbol{v}_1$ as required.

3. (10 pts.) Suppose you are given two different types of concrete. The first type contains 30% cement, 40% gravel, and 30% sand (all percentages of weight). The second type contains 10% cement, 20% gravel, and 70% sand. The first type of concrete costs \$5 per pound, while the second type costs \$1 per pound.

How many pounds of each type of concrete should you buy and mix together so that your cost is minimized but you get a concrete mixture that has at least a total of 5 pounds of cement, 3 pounds of gravel, and 4 pounds of sand? Formulate and solve the problem using a linear programming method. (No need to give details on how you solved it; just give the answer.)

Ans.: The problem can be represented as

$$\begin{array}{ll} \text{minimize} & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}, \end{array}$$

where

$$\boldsymbol{A} = \begin{bmatrix} 0.3 & 0.1 \\ 0.4 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}, \qquad \boldsymbol{c} = \begin{bmatrix} 5 \\ 1 \end{bmatrix},$$

Using a graphical method or converting to standard form using surplus variables and applying the simplex algorithm, we get a solution of $[0, 50]^T$, which means that we should purchase 50 pounds of the second type of concrete.

4. (16 pts.) You are given a linear programming problem in standard form. Suppose you use the two-phase simplex method and arrive at the following canonical tableau in phase I:

$$\begin{bmatrix} ? & 0 & 1 & 1 & ? & ? & 0 & 6 \\ ? & 0 & 0 & ? & ? & ? & 1 & \alpha \\ ? & 1 & 0 & ? & ? & ? & 0 & 5 \\ \gamma & 0 & 0 & \delta & ? & ? & \beta & 0 \end{bmatrix}.$$

The variables α, β, γ, and δ are unknowns to be determined. Those entries marked with "?" are unspecified. The only thing you are told is that the value of γ is either 2 or -1.

a. Determine the value of α. Explain fully.

b. Determine the value of β. Explain fully.

c. Determine the value of γ. Explain fully.

d. Determine the value of δ. Explain fully.

e. Does the given linear programming problem have a feasible solution? If yes, find it. If not, explain why.

Ans.: a. The value of α must be 0, because the objective function value is 0 (lower right corner), and α is the value of an artificial variable.

b. The value of β must be 0, because it is the RCC value corresponding to a basic column.

c. The value of γ must be 2, because it must be a positive value. Otherwise, there is a feasible solution to the artificial problem with objective function value smaller than 0, which is impossible.

d. The value of δ must be 0, because we must be able to bring the fourth column into the basis without changing the objective function value.

e. The given linear programming problem does indeed have a feasible solution: $[0, 5, 6, 0]^T$. We obtain this by noticing that the right-most column is a linear combination of the second and third columns, with coefficients 5 and 6.