$$= \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t}.$$

Theorem 4.3.4

For $0 < s < t$, $0 \le k \le n$,

$$(4.3.5) \quad P(N(s)=k \mid N(t)=n) = \frac{n!}{k!(n-k)!} \left(\frac{s}{t}\right)^k \left(1-\frac{s}{t}\right)^{n-k}$$

Proof

$$P(N(s)=k \mid N(t)=n) = \frac{P(N(s)=k,\ N(t)=n)}{P(N(t)=n)}$$

$$= \frac{P(N(s)=k,\ N(t)-N(s)=n-k)}{P(N(t)=n)}$$

homogeneity

$$= \frac{\left(e^{-\lambda s}\frac{(\lambda s)^k}{k!}\right)\left(e^{-\lambda(t-s)}\frac{(\lambda(t-s))^{n-k}}{(n-k)!}\right)}{\frac{e^{-\lambda t}(\lambda t)^n}{n!}}$$

$$= \frac{n!}{k!(n-k)!} \frac{s^k (t-s)^{n-k}}{t^n}$$

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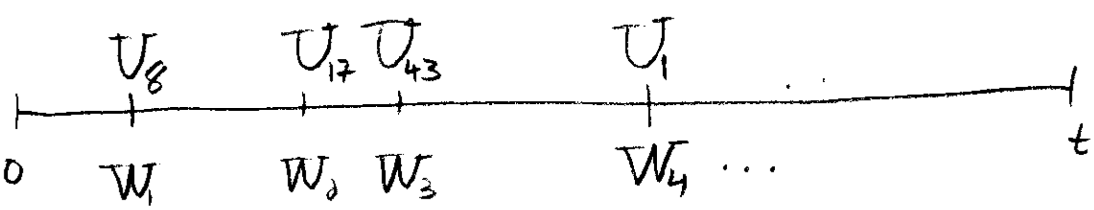
The next result says that conditioned on a fixed total number of events in an interval, the times of occurence of those events are uniformly distributed in a certain way.

We first compute a related probability density function. Consider an interval $(0,t]$ and choose a fixed number n of points uniformly on the interval, denoting the positions by $U_1, \ldots, U_n$. The p.d.f. is

$$f(s) = \begin{cases} \frac{1}{t}, & 0 \le s \le t, \\ 0, & \text{otherwise.} \end{cases}$$

Set $\{W_1, \ldots, W_n\}$ to be the positions $\{U_1, \ldots, U_n\}$ re-ordered from left to right.



The joint p.d.f. for $W_1, W_2, \ldots W_n$ is

(4.3.6) $$f_{W_1,\ldots,W_n}(w_1, w_2, \ldots, w_n) = n!\, t^{-n},$$

$$0 < w_1 < w_2 < \ldots < w_n \le t.$$

We can show this by induction.

$$f_{W_1 W_2}(w_1, w_2)\,\Delta w_1 \Delta w_2 = P(w_1 < W_1 \le w_1 + \Delta w_1,\ w_2 < W_2 \le w_2 + \Delta w_2)$$

$$= P(w_1 < U_1 \le w_1 + \Delta w_1,\ w_2 < U_2 \le w_2 + \Delta w_2)$$ 2 possible orders

$$+ P(w_1 < U_2 \le w_1 + \Delta w_1,\ w_2 < U_1 \le w_2 + \Delta w_2)$$

$$= 2\left(\frac{\Delta w_1}{t}\right)\left(\frac{\Delta w_2}{t}\right) = 2t^{-2}\Delta w_1 \Delta w_2$$

Dividing by $\Delta w_1 \Delta w_2$ and taking the limit gives (4.3.6). In general, there are $n!$ arrangements of $U_1, \ldots, U_n$ leading to the same ordered $W_1, \ldots, W_n$.

We now prove

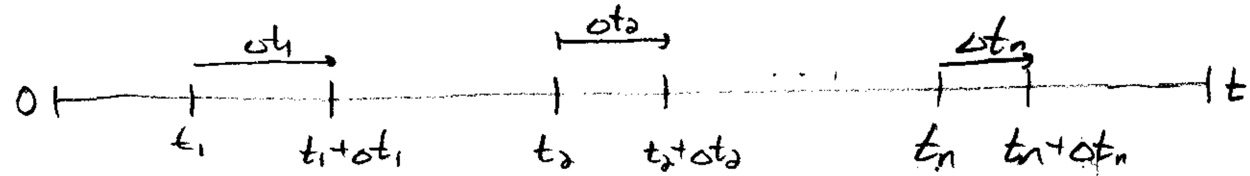
Theorem 4.3.5

Let $T_1, \ldots, T_n$ be arrival times in a Poisson process $N(t)$ with rate $\lambda > 0$. Conditioned on $N(t) = n$, the random variables $T_1, \ldots, T_n$ have joint p.d.f.

$$(4.3.7) \quad f_{T_1 \cdots T_n}(t_1, \ldots, t_n) = n!\, t^{-n}, \quad 0 < t_1 < \cdots < t_n \le t.$$

Proof

We create some subintervals from a set of times $t_1, \ldots, t_n$ and increments $\Delta t_1, \ldots, \Delta t_n$



The event

$$\{t_i < T_i < t_i + \Delta t_i,\ i = 1, \dots, n,\ N(t) = n\}$$

means no events occurred in $(0, t_1]$, $(t_1 + \Delta t_1, t_2]$, ..., $(t_n + \Delta t_n, t]$ and exactly one event occurred in each of $(t_1, t_1 + \Delta t_1]$, ..., $(t_n, t_n + \Delta t_n]$.

The intervals are disjoint and

$$P(N(t_1) = 0, \dots, N(t) - N(t_n + \Delta t_n) = 0)$$
$$= e^{-\lambda t_1} e^{-\lambda(t_2 - t_1 - \Delta t_1)} \cdots e^{-\lambda(t_n - t_{n-1} - \Delta t_{n-1})} e^{-\lambda(t - t_n - \Delta t_n)}$$
$$= e^{-\lambda t} e^{\lambda(\Delta t_1 + \dots + \Delta t_n)}$$
$$= e^{-\lambda t}\left(1 + o(\max_i \Delta t_i)\right)$$

while

$$P(N(t_1 + \Delta t_1) - N(t_1) = 1, \dots, N(t_n + \Delta t_n) - N(t_n) = 1)$$
$$= \lambda(\Delta t_1) \cdots \lambda(\Delta t_n)\left(1 + o(\max_i \Delta t_i)\right)$$

Hence,

$$f_{T_1 \cdots T_n | N(t)=n}(t_1, \ldots, t_n)\Delta t_1 \cdots \Delta t_n$$

$$= P(t_1 < T \le t_1 + \Delta t_1, \ldots, t_n < T_n \le t_n + \Delta t_n \mid N(t)=n)$$

$$+ O(\Delta t_1 \cdots \Delta t_n)$$

$$= \frac{P(t_1 < T_1 \le t_1 + \Delta t_1, \ldots, t_n < T_n \le t_n + \Delta t_n, N(t)=n)}{P(N(t)=n)} + O(\Delta t_1 \cdots \Delta t_n)$$

$$= \frac{e^{-\lambda t} \lambda \Delta t_1 \cdots \lambda \Delta t_n}{\frac{e^{-\lambda t}(\lambda t)^n}{n!}} \left(1 + o(\max_i \Delta t_i)\right)$$

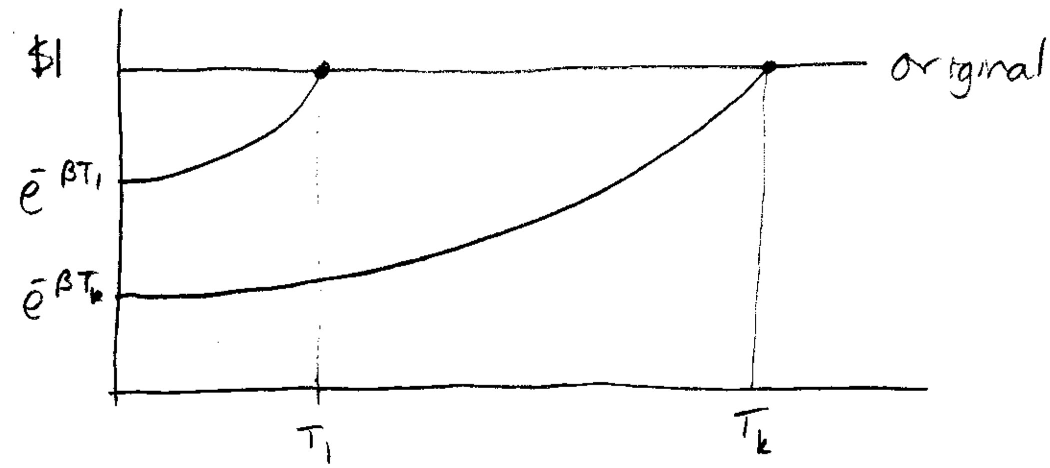
$$= n!\, t^{-n} \Delta t_1 \cdots \Delta t_n \left(1 + o(\max_i \Delta t_i)\right)$$

Dividing by $\Delta t_1 \cdots \Delta t_n$ and letting $\Delta t_1 \to 0, \ldots, \Delta t_n \to 0$ gives the result.

<u>Example 4.3.2</u>

Customers arrive at a facility according to a Poisson process of rate λ. Each customer pays \$1 on arrival. We want to evaluate the expected value of the total sum collected during the interval $(0,t]$, where the amounts are discounted back to time 0 according to a discount rate β. If a customer pays \$1 at event time T_k, then the discounted amount is

$$e^{-\beta T_k} \times 1.$$



We want to compute

$$M = E\left(\sum_{k=1}^{N(t)} e^{-\beta T_k} \right)$$

We condition on $N(t) = n$

$$M = \sum_{n=1}^{\infty} E\left(\sum_{k=1}^{n} e^{-\beta T_k} \,\middle|\, N(t) = n \right) P(N(t) = n)$$

Let $U_1, \ldots, U_n$ be iid uniformly distributed variables in $(0, t]$. Using Theorem 4.3.5,

each term is treated the same way

$$E\left(\sum_{k=1}^{n} e^{-\beta T_k} \,\middle|\, N(t) = n \right) = E\left(\sum_{k=1}^{n} e^{-\beta U_k} \right) = n\, E(e^{-\beta U_1})$$

$$= n t^{-1} \int_0^t e^{-\beta u}\, du = \frac{n}{\beta t}(1 - e^{-\beta t})$$

So

$$M = \frac{1}{\beta t}(1 - e^{-\beta t}) \underbrace{\sum_{n=1}^{\infty} n\, P(N(t) = n)}_{E(N(t))}$$

$$= \frac{\lambda}{\beta}(1 - e^{-\beta t})$$

The proof of the strong Markov property Thm 4.2.9 for birth processes uses

- weak Markov property
- temporal homogeneity

The strong Markov property is very important in the analysis of continuous time Markov processes.

When applied to a birth process, it implies that the new process $\tilde{N}(t)$ defined by

$$\tilde{N}(t) = N(t+T) - N(T), \quad t \geq 0,$$

conditional on $\{N(T) = i\}$ is also a birth process whenever T is a stopping time for N.

Exercise: show the intensities are $\lambda_i, \lambda_{i+1}, \ldots$

In the case of a Poisson process, $\tilde{N}(t) = N(T+t) - N(T)$ is also a Poisson process.

This leads to the observation of a fundamental characteristic.

Definition 4.3.1

A Poisson process N has stationary independent increments in the sense that

(1) the distribution of $N(t) - N(s)$ depends only on $t-s$

(2) any finite set of increments $\{N(t_i) - N(s_i),\ i = 1, 2, .., n\}$ are independent if $s_1 \le t_1 \le s_2 \le t_2 \le \dots \le t_n$.

It turns out that this property nearly characterizes a Poisson process.

Theorem 4.3.6

Suppose that $M(t)$, $t \ge 0$, is a non-decreasing, right-continuous, integer valued stochastic process with

(1) $M(0)=0$
(2) stationary independent increments
(3) M only has jump discontinuities of size 1 (implies the state space is $\{0,1,2,3,\ldots\}$)

Then M is a Poisson process.

<u>Outline of Proof</u>

For $u,v \geq 0$, we have

$$E(M(u+v)) = E(M(u)) + E(M(u+v)-M(u))$$

$$= E(M(u)) + E(M(v))$$

by the stationary increments assumption.

$E(M(u))$ is nondecreasing in u, so there is a λ such that

$$(4.3.8) \qquad E(M(u)) = \lambda u, \qquad u \geq 0$$

Let $T = \sup\{t : M(t)=0\}$ be the time of the first jump. Almost surely (right continuity), $M(T)=1$, and T is a stopping time for M. (Exercise: explain). Now

$$(4.3.9) \qquad E(M(s)) = E(E(M(s)\,|\,T)).$$

We have

$$E(M(s) \mid T) = 0 \qquad s < T,$$

while for $s \geq T$,

(4.3.10) $$E(M(s) \mid T=t) = E(M(t) \mid T=t) + E(M(s)-M(t) \mid T=t)$$

exercise.

$$= 1 + E(M(s-t))$$

by the stationary increments assumption. If $F(t)$ is the distribution function for T, then using (4.3.10) in (4.3.9) gives

$$E(M(s)) = \int_0^s (1 + E(M(s-t)))\, dF(t)$$

Since $E(M(s)) = \lambda s$,

(4.3.11) $$\lambda s = F(s) + \lambda \int_0^s (s-t)\, dF(t).$$

This is an integral equation for the unknown function F. It may be solved e.g. using Laplace transforms. We find

$$F(t) = 1 - e^{-\lambda t}, \quad t \geq 0$$

so T has the exponential distribution.

Now we may argue as for the sojourn time formulation to show the "inter-jump" times of M are independent and have the exponential distribution.

Hence, M is a Poisson process with intensity λ.

$\boxed{\#26 \ 4/31}$

§4.4 Death Processes

This is the complement of a birth process. It moves through states N, $N-1, \ldots, 0$, leading to absorption into 0, or extinction.

Definition 4.4.1

A death process $X(t)$ with death parameters $\mu_1, \ldots, \mu_N$, $\mu_i > 0$, is a process with state space $\{0, 1, 2, \ldots, N\}$ such that

(1) $X(0) = N$

(2) $s < t \Rightarrow X(s) \geq X(t)$

(3)

$$P\big(X(t+h) = k-m \mid X(t) = k\big) = \begin{cases} \mu_k h + o(h), & m = 1, \\ 1 - \mu_k h + o(h), & m = 0, \\ o(h), & m > 1. \end{cases}$$