Notes - 1 April

Review from a question in class: why did we need a PGF last time? 1) Basic problem: finding $\Pi = \Pi P, \Pi$ pmf. Π exists sometimes. Relation to being positive recurrent. 2) Basic problem: find a bounded solution of $Y = PY$ with a row/column "missing", related to transient. Then we got to ex 3.3.8 - classic example: queuing theory. This example had several points: 1) Consider $\Pi = \Pi P$. Used generating functions. Came to the conclusion that Π exists $\iff \gamma = \Sigma_{k=0}^{\infty} k a_k < 1$, but only if chain is positive recurrent. So we needed to proved that the chain is positive recurrent so we can solve the problem. 2) We consider solution of $Y = PY, |Y_i| \leq 1$, with missing row/column (not full P). It this case using the fixed point theory we prove Y exists, but this holds iff chain is transient, which holds iff $\gamma > 1$. So, theorem is: positive recurrent if $\gamma < 1$, transient if $\gamma > 1$.

§3.4 - Limit Theorems

We explore the link between a stationary distribution and the behavior of P_{ij}^n as $n \to \infty$.

Example 3.4.1 - Consider ON/OFF system in ex 2.2.3 with P = (1-p p & q 1-q) $0 \leq p \leq 1, 0 \leq q \leq 1$. When $0 < p, q < 1, P^n \to \frac{1}{p+q}$(q p & q p) and we know there is a stationary distribution. Now suppose $p = q = 1$, the system changes states at every step. The stationary distribution satisfies $(\Pi_0, \Pi_1) = (\Pi_0, \Pi_1)$(0 1 & 1 0) $\Rightarrow \Pi_0 = \Pi_1 = 1/2$. We can compute, e.g. $P_{00}^n = \{$ 0 if n even, 1 if n odd. There is no limiting behavior in this case. However the states are periodic with period 2.

Theorem 3.4.1 - For an irreducible, aperiodic Markov chain, (3.4.1) $P_{ij}^n \to \frac{1}{\mu_j}$ as $n \to \infty$ for all i, j (μ_j is the mean recurrence time). The limiting value is the same for all states i. $P^n \to$ (1/mu0 1/mu0 $\ldots$ & 1/mu1 1/mu1 $\ldots$ & $\ldots)^T =$ (1/mu0 1/mu1 $\ldots$ & 1/mu0 1/mu1 $\ldots$ & $\ldots$).

Definition 3.4.1 - If there is a probability distribution q on the state space S such that $P_{ij}^n \to q_j$ for all $i, j \in S$ then q is a limit distribution of the chain.

Intuition: q_j describes the probability that the chain is in state j at some "late" time and by this time the chain has "forgotten" where it started. $P(X_n = j) = \Sigma_i P(X_0 = i) P_{ij}^n \to q_j$ regardless of the initial distribution of X_0. Consequences:

Theorem 3.4.2 - (a) If the chain is transient or null recurrent, then $P_{ij}^n \to 0$ for all i, j. (b) If the chain is positive recurrent, then $P_{ij}^n \to \Pi_j = \mu_j^{-1}$, where Π is the unique stationary distribution.

Theorem 3.4.3 - If X_n is an irreducible chain with period d, then $Y_n = X_{nd}, n \geq 0$, is an aperiodic, irreducible chain, $P_{jj}^{nd} = P(Y_n = j | Y_0 = j) \to \frac{d}{\mu_j}$ as $n \to \infty$. Immediately from this follows the proof of theorem 3.1.6 (see notes).

Connection between limiting and stationary distributions: consider a Markov chain at some "late" time n. The stationary distribution gives the proportion of time spent in the different states up to time n.

The limit distribution gives the proportion of "time" spent in the various states at the large time, where we count by considering many realizations.

Example 3.4.2 - Consider ON/OFF system, ex 3.4.1 again. The stationary distribution $(1/2, 1/2)$ says that equal amounts of time are spent in each state up to some large time, say $n = 1000$. If we look precisely at $n = 1000$, the chain must return to its initial state. Multiple realizations give probability 1 to be in the initial state and 0 to be in the other.

Theorem 3.4.4 - An ergodic Markov chain has the property that it has both stationary and limiting distributions and these are equal.

Proof of theorem 3.4.1 - We treat different cases. The simplest case is a transient chain, because theorem 3.1.2 (3) implies $P_{ij}^n \to 0$ as $n \to \infty$ for all i, j. The recurrent cases are treated with "coupling".

Definition - 3.4.2 - Let X_n, Y_n be independent Markov chains with common state space S and common probability transition matrix P. The coupled chain $Z_n = (X_n, Y_n)$ taking values in $S \times S$.

Theorem 3.4.5 - Z_n is a Markov chain with $P_{ij,kl} = P_{ik} P_{jl}$. If X_n, Y_n are irreducible and aperiodic, then Z_n is irreducible. Proof: $P_{ij,kl} = P(Z_{n+1} = (k, l) | Z_n = (i, j)) = P(X_{n+1} = k | X_n = i) \times P(X_{n+1} = l | Y_n = j). X_n, Y_n$ aperiodic, irreducible $\Rightarrow$ for any i, j, k, l there is an $N = N(i, j, k, l)$ such that $P_{ik}^n P_{jl}^n > 0, n \leq N$. Exercise: this implies Z_n is irreducible.

Comment: this is the only place we use the assumption X_n is aperiodic.

We assume X_n (in thm 3.4.1) is positive recurrent, so it has unique stationary distribution Π. (Consider $Y = X$ in the construction of Z.) Exercise: $Z_n = (X_n, Y_n)$ has a stationary distribution $\nu = (\nu_{ij}, i, j \in S), \nu_{ij} = \Pi_i \Pi_j$. This implies Z_n is also positive recurrent (due to stationary distribution). Choose $X_0 = i, Y_0 = j, Z_0 = (i, j)$. Choose $s \in S$. Set $T = \min(n \geq 1 : Z_n = (s, s))$. The recurrence of Z_n implies that

$P(T < \infty) = 1$ (exercise).