Date
Page

⊛ Eigen values and Eigen vectors of a square matrix :-

Let A be a square matrix of order n, then a non-zero column matrix X is said to be Eigen vector of matrix A, If it satisfy the equation

$$\boxed{\begin{array}{ll} & AX = \lambda X \\ \text{or} & (A - \lambda I) X = 0 \end{array}}$$; where λ is any real or complex no.

Here, λ is called Eigen value of the matrix A.

Note :- Eigen values are also called characteristic roots and latent values.

⊛ Characteristic Equation :-

The characteristic equation of matrix A of order n is defined by

$$|A - \lambda I| = 0$$

And this equation gives n^{th} order polynomial in λ

$$\boxed{a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0}$$

Note :- ① If A is ~~a~~ square matrix of order n, then it will have n eigen values and n eigen vectors.

② If A is square matrix of order 3, then its Characteristic eqn. is given by

$$\boxed{\lambda^3 - (\text{Trace of } A)\lambda^2 + (\text{sum of diagonal minors})\lambda - |A| = 0}$$

③ Trace of A = Sum of diagonal elements of A.

(4). For 2x2 matrix $\lambda^2 - (\text{Trace of A})\lambda + |A| = 0$

Note: 1. Sum of all Eigen Values=Trace=Sum of diagonal elements.
2. DetA=|A|=Multiplication of all Eigen Values.
You can verify above to points in your questions

Question ① Find the eigen values and eigen vector's of

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{bmatrix}$$

Soln. The characteristic eqn. of A is eqn. is given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & 4 \\ 0 & 2-\lambda & 0 \\ 3 & 1 & -3-\lambda \end{vmatrix} = 0$$

Rough

Trace $= 1+2-3=0$

$|A| = -30$

$D_1 = 2-0 = 2$

$D_2 = -6-0 = -6$

$D_3 = -3-12 = -15$

$D_1+D_2+D_3 = -19$

for D_3

$$D_3 = \begin{bmatrix} 1 & 4 \\ 3 & -3 \end{bmatrix}$$

By Solving determinant

$(1-\lambda)[(2-\lambda)(-3-\lambda)] + 4[0 - 3(2-\lambda)] = 0$

Ch. Eqn.

$\Rightarrow -\lambda^3 + 19\lambda - 30 = 0$

$\Rightarrow (2-\lambda)(\lambda+5)(\lambda-3) = 0$

$\Rightarrow \lambda = 2, 3, -5$

are eigen values of A.

OR

By formula

$\lambda^3 - (\text{Trace})\lambda^2 + (\text{Sum of diagonal minors})\lambda + |A| = 0$

$\Rightarrow \lambda^3 - 0 - 19\lambda + 30 = 0$

$\Rightarrow \lambda^3 - 19\lambda + 30 = 0$

$\Rightarrow (\lambda-3)$

$\Rightarrow$ by calc

$\lambda = 2, 3, -5$

are eigen values

The Eigen vectors for $\lambda = 2, 3, -5$ can be calculated by eqn.

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} 1-\lambda & 0 & 4 \\ 0 & 2-\lambda & 0 \\ 3 & 1 & -3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- (1)}$$

where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is eigen vector

(A) Eigen vector for $\lambda = 2$; Let $X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ is eigen vector for $\lambda = 2$

Then putting $\lambda = 2$ & $X = X_1$ in (1) we get

$$\begin{bmatrix} -1 & 0 & 4 \\ 0 & 0 & 0 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-1 \cdot x_1 + 0 \cdot y_1 + 4 \cdot z_1 = 0$$
$$\boxed{0 \cdot x_1 + 0 \cdot y_1 + 0 \cdot z_1 = 0} \quad \times$$
$$3 \cdot x_1 + 1 \cdot y_1 - 5 \cdot z_1 = 0$$

We take any two equation to find x1, y1, z1. But here already second equation is not usefule, so we take 1st and lastonly.

$$\frac{x_1}{\begin{vmatrix} 0 & 4 \\ 1 & -5 \end{vmatrix}} = \frac{-y_1}{\begin{vmatrix} -1 & 4 \\ 3 & -5 \end{vmatrix}} = \frac{z_1}{\begin{vmatrix} -1 & 0 \\ 3 & 1 \end{vmatrix}}$$

$$\frac{x_1}{-4} = \frac{+y_1}{+7} = \frac{z_1}{-1}$$

$$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} -4 \\ 7 \\ -1 \end{bmatrix} \quad \#$$

(B) Eigen vector for $\lambda = 3$; Let $X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ is eigen vector for $\lambda = 3$.

Putting $\lambda = 3$ & $X = X_2$ in (1)

$$\begin{bmatrix} -2 & 0 & 4 \\ 0 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2 \cdot x_2 + 0 \cdot y_2 + 4 \cdot z_2 = 0$$
$$0 \cdot x_2 - 1 \cdot y_2 + 0 \cdot z_2 = 0 \Rightarrow \frac{x_2}{\begin{vmatrix} 0 & 4 \\ -1 & 0 \end{vmatrix}} = \frac{-y_2}{\begin{vmatrix} -2 & 4 \\ 0 & 0 \end{vmatrix}} = \frac{z_2}{\begin{vmatrix} -2 & 0 \\ 0 & -1 \end{vmatrix}}$$
$$3 \cdot x_2 + 1 \cdot y_2 + 0 \cdot z_2 = 0$$

$$\frac{x_2}{4} = \frac{y_2}{0} = \frac{z_2}{2}$$

$$X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

(c) Eigen vector for $\lambda = -5$

Let $X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$ is eigen vector for $\lambda = -5$

Putting $\lambda = -5$ and $X = X_3$ in (1) we get

$$\begin{bmatrix} 6 & 0 & 4 \\ 0 & 7 & 0 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$6 \cdot x_3 + 0 \cdot y_3 + 4 z_3 = 0$$
$$0 \cdot x_3 + 7 \cdot y_3 + 0 \cdot z_3 = 0$$
$$3 \cdot x_3 + 1 \cdot y_3 + 2 \cdot z_3 = 0$$

We tcan ake any two equation to find x1, y1, z1. Here we are taking 1st and second eqns.

$$\frac{x_3}{\begin{vmatrix} 0 & 4 \\ 7 & 0 \end{vmatrix}} = \frac{-y_3}{\begin{vmatrix} 6 & 4 \\ 0 & 0 \end{vmatrix}} = \frac{z_3}{\begin{vmatrix} 6 & 0 \\ 0 & 7 \end{vmatrix}}$$

$$\frac{x_3}{-28} = \frac{-y_3}{0} = \frac{z_3}{42}$$

Multiplying by ~~4~~ 14 $\quad \frac{x_3}{-2} = \frac{y_3}{0} = \frac{z_3}{3}$

$$X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} \quad \#$$

Note:- ① We have to use any two equations to find x, y & z, ~~by~~ but we should select two eqns s.t. all values of x y z not be zero. Atleast one should be non-zero

② If all equations are the same or if we are getting x=0, y=0, z=0 for all pairs of equations, then we assume any two variables and then calculate third variable (for 3x3 matrix).

e.g. $x_1=k_1$ and $y_1=k_2$

V. Imp.

Q(2) Find eigen values and eigen vectors of

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Soln:- The ch. eqn. of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

Ch. Eqn. $\Rightarrow \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

$\Rightarrow \lambda = \underbrace{1, 1}_{\text{same}}, 5$ are eigen values of A.

The eigen vectors are given by
$(A - \lambda I) X = 0$

$$\begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad -(1)$$

(A) Eigen ~~value~~ vector for $\lambda = 1$

Let $X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ is eigen vector for A

Putting $\lambda = 1$ and $X = X_1$ in (1) we get

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$1 \cdot x_1 + 2 \cdot y_1 + 1 \cdot z_1 = 0$
$1 \cdot x_1 + 2 \cdot y_1 + 1 \cdot z_1 = 0$
$1 \cdot x_1 + 2 \cdot y_1 + 1 \cdot z_1 = 0$

Here all eqns are same, so values of x_1, y_1, z_1 will be 0. But we are looking for non-zero vector

$x_1 + 2y_1 + z_1 = 0$

we assum any value of y_1 & z_1

let $y_1 = K_1$ and $z_1 = K_2$

then $x_1 + 2K_1 + K_2 = 0$

$x_1 = -2K_1 - K_2$

Then $$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} -2K_1 - K_2 \\ K_1 \\ K_2 \end{bmatrix}$$

$$= \begin{bmatrix} -2K_1 \\ K_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -K_2 \\ 0 \\ K_2 \end{bmatrix}$$

$$= K_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + K_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Then $X_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are eigen vectors for $\lambda = 1$

(B) Eigen ~~values~~ vector for $\lambda = 5$:-

Let $X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$ is eigen vector for $\lambda = 5$

Putting $\lambda = 5$ and $X = X_3$, in (1)

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3 \cdot x_3 + 2 \cdot y_3 + 1 \cdot z_3 = 0$

$1 \cdot x_3 - 2 \cdot y_3 + 1 \cdot z_3 = 0$

$1 \cdot x_3 + 2\, y_3 - 3\, z_3 = 0$

We tcan ake any two equation to find x1, y1, z1. Here we are taking 1st and second eqns.

$$\frac{x_3}{\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{-y_3}{\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{z_3}{\begin{vmatrix} -3 & 2 \\ 1 & -2 \end{vmatrix}}$$

$$\frac{x_3}{4} = \frac{y_3}{4} = \frac{z_3}{4}$$

$$\frac{x_3}{1} = \frac{y_3}{1} = \frac{z_3}{1}$$

$$X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \#$$

Exercise:

Q1: Find the eigen values and eigen vectors of the following matrices

(i) $A = \begin{bmatrix} 7 & -2 & 2 \\ -2 & 1 & 4 \\ -2 & 4 & 1 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$ (iv) $A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

Q2: Find the Eigen values, corresponding Eigen vectors, and Eigen spaces of the following matrices.

1) $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

4) $A = \begin{bmatrix} 3 & -3 & 2 \\ -1 & 5 & -2 \\ -1 & 3 & 0 \end{bmatrix}$

2) $A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$

5) $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$

3) $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$