Matrix Representation of an Inner Product $\langle\ \rangle$

Problem:

Show that for any Diagonal matrix $A = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$;

$d_i > 0$. $\langle x, y\rangle = x^T A y = d_1x_1y_1 + d_2x_2y_2 + d_3x_3y_3$ defines an IP.

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in R$ $\quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in R$ $\quad z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in R$.

(i) $\langle x, y\rangle = x^T A y = d_1x_1y_1 + d_2x_2y_2 + d_3x_3y_3$
$= d_1y_1x_1 + d_2y_2x_2 + d_3y_3x_3$
$= y^T A x$

$\langle x, y\rangle = \langle y, x\rangle$

(ii) $\langle x, y+z\rangle = x^T A (y+z)$
$= x^T A y + x^T A z$
$= \langle x, y\rangle + \langle x, z\rangle$

(iii) $\langle \alpha x, y\rangle = (\alpha x)^T A y$
$= \alpha x^T A y$

$\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$

(iv) $\langle x, x\rangle = d_1x_1^2 + d_2x_2^2 + d_3x_3^2 \geq 0$

and $\langle x, x\rangle = 0 \iff x_1 = x_2 = x_3 = 0 \iff x = 0$.

$\langle\ ,\ \rangle = x^T A y$ is an IP.

Matrix Representaion of an Inner Product $\langle \quad \rangle$:

We have already seen that $\langle x, y \rangle = x^T A y$ defines an IP on $\mathbb{R}^n$, provided matrix A is diagonal matrix with positive diagonal entries.

The converse is also true; every IP on a IPS can be expressed as in such a matrix product form.

Let $(V, \langle , \rangle)$ is an IPS and $B_V = \{v_1, v_2, \dots, v_n\}$ be the basis of V. Then for any x & $y \in V$ we have.

$$\boxed{\begin{aligned} x &= \sum_{i=1}^{n} \alpha_i v_i \\ y &= \sum_{j=1}^{n} \beta_j v_j \end{aligned}}$$

$$\therefore \langle x, y \rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \beta_j \underbrace{\langle v_i, v_j \rangle}_{a_{ij}} = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \beta_j (a_{ij}) = [x]_{B_1}^{T} A [y]_{B_2}$$

for $i, j = 1, 2, \dots n$.

We set $\boxed{a_{ij} = \langle v_i, v_j \rangle}$

So, matrix $A = [a_{ij}]$ is symmetric, because

$$\langle v_i, v_j \rangle = \langle v_j, v_i \rangle$$
$$a_{ij} = a_{ji}$$

$$\boxed{A = [a_{ij}]_{ij} = \begin{bmatrix} a_{11} & a_{12} & \dots \\ a_{21} & & \vdots \\ \vdots & & a_{nn} \end{bmatrix}}$$

Example 1:

$x \cdot y = \langle \ \rangle$

Let $V = R^n$ be an Euclidian space (IPS), then find its matrix representation with respect to Standard basis $\{e_1, e_2, \dots e_n\}$

We know that $A = [a_{ij}]$ $i, j = 1, 2, \dots n$

Standard Basis $= \{e_1, e_2, \dots e_n\}$. $a_{ij} = \langle e_i, e_j \rangle$

$e_1 = (1, 0, 0)$ 1st place

$e_2 = (0, 1, 0)$ 2nd place.

Case ①:

If $i = j$:-

$a_{ii} = \langle e_i, e_i \rangle$

$= 0^2 + 0^2 + \dots + 1^2 + \dots + 0^2$

$\boxed{a_{ij} = 1}$

$e_i = (0, 0, \dots 1, 0, \dots 0)$

ith place

$e_j = (0, 0, \dots 1, \dots 0)$

jth place.

Case ②:

If $i \neq j$:-

$a_{ij} = \langle e_i, e_j \rangle$

$= 0 \times 0 + \dots 1 \times 0 + \dots 0 \times 1 + \dots 0 \times 0$

ith place jth place.

$a_{ij} = 0$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ & & a_{33} & \\ \vdots & & & \\ a_{n1} & \dots & & a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & 1 & \\ 0 & 0 & 0 \dots & 1 \end{bmatrix}$$

Exmple 2: Let $V = P_2(\mathbb{R})$ be an IPS with Inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$ and basis $\{f_1(x) = 1,\ f_2(x) = x,\ f_3(x) = x^2\}$

find matrix representation of IP.

$$n = 2+1 = 3$$

$$A = [a_{ij}];\quad i,j = 1,2,3.$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$a_{11} = \langle f_1, f_1\rangle = \int_0^1 1\cdot 1\,dx = 1$$

$$a_{12} = \langle f_1, f_2\rangle = \int_0^1 1\cdot x\,dx = 1/2 = a_{21}$$

$$a_{23} = a_{32} = 1/4$$

$$a_{22} = 1/3$$

$$a_{33} = 1/5$$

$$A = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$