Properties of Eigen Values:-

1. The product of the eigen values of a matrix A is equal to its determinant.
2. If A is singular matrix then at least of the eigen values of A is zero and conversely.
3. If $\lambda_1, \lambda_2, \ldots \lambda_n$ are eigen values of a matrix A, then

   a. $\lambda_1^p, \lambda_2^p, \ldots \lambda_n^p$ are the eigen values of A^p, where p is any positive integer.

   b. $1/\lambda_1, 1/\lambda_2, \ldots 1/\lambda_n$ are the eigen values of A^{-1} .

   c. $k\lambda_1, k\lambda_2, \ldots k\lambda_n$ are the eigen values of kA, where k is a constant.

   d. $\lambda_1 + \mathbf{k}, \lambda_2 + \mathbf{k}, \ldots \lambda_n + \mathbf{k}$ are the eigen values of $\mathbf{A}+\mathbf{kI}$, where k is a constant.

4. Matrix A and its transpose A^T have same eigen values.

5. If A and B are similar matrices the A and B have same eigen values.

6. The diagonal entries are the eigen values of Diagonal matrices and lower/upper triangular matrices.

Diagonal matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Eigen Values: 1, 8, 4

$$\begin{bmatrix} 8 & 2 & 5 & -3 \\ 0 & 2 & 9 & 1 \\ 0 & 0 & -4 & 2 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

upper triangular matrix

Eigen Values: 8, 2, -4, 7

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 5 & 2 & 7 & 0 \\ 4 & 0 & -2 & 8 \end{bmatrix}$$

lower triangular matrix

Eigen Values: 2, -1, 7, 8

7. If a+ib is the Eigenvalue of any square matrix then a-ib will also be an Eigenvalue of the matrix.

CAYLEY HAMILTON THEOREM

Let A be the square matrix and $f(\lambda)=0$ is its characteristic equation, then

$$f(\mathrm{A})=0$$

where, $f(\lambda)=\left|\mathrm{A}-\lambda\mathrm{I}\right|=0$

OR

Every square matrix satisfies its own characteristics equation.

Example 1:-

Verify Cayley – Hamilton theorem for the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Hence compute A^{-1} .

Solution:- The characteristic equation of A is

$$|A-\lambda I| = 0 \quad \text{i.e.,} \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

or $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$ (on simplification)

So, $f(\lambda) = \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

To verify Cayley – Hamilton theorem, we have to show that f(A)= $A^3 - 6A^2 + 9A - 4I = 0$ … (1)

Now,

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -22 & -21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Now putting A^3, A^2, A and I in equation (1), we get

$$= \begin{bmatrix} 22 & -22 & -21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

This verifies Cayley – Hamilton theorem.

Now, pre – multiplying both sides of (1) by A^{-1} , we have

$$A^2 - 6A + 9I - 4A^{-1} = 0$$

$$\Rightarrow 4A^{-1} = A^2 - 6A + 9I$$

$$\Rightarrow 4A^{-1} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow \therefore A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Exercise

1. Verify Cayley-Hamilton theorem for $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$ and then find A^{-1}.

2. Verify Cayley-Hamilton theorem for $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix}$ and then find A^{-1}.

3. Q. 6 If the matrix $A = \begin{bmatrix} -2 & 4 \\ -1 & 3 \end{bmatrix}$, then A^5 equals to

(a) $6I + 5A$ (b) $22I + 21A$

(c) $10I + 11A$ (d) $10I + 21A$