Image and Kernel of a Linear Transformation

Let V and W be vector spaces, and let **T : V →W** be a linear transformation. The image of T, denoted by **Im(T) or Range(T)**, is the set

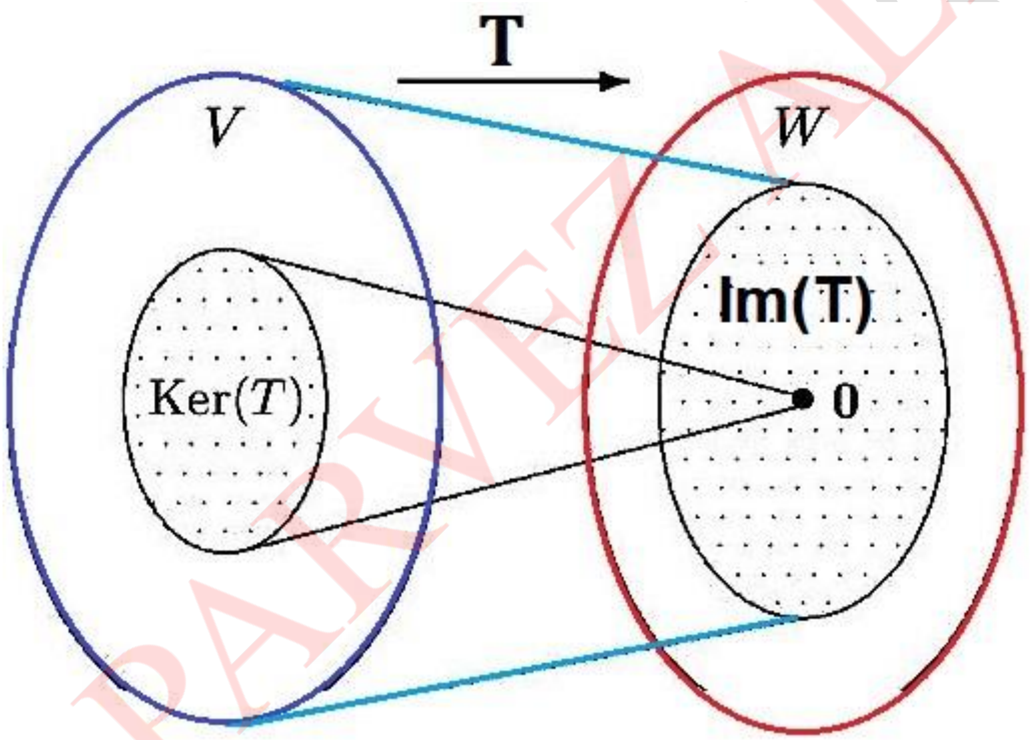
$$\boxed{\mathrm{Im}(T) \text{ or } \mathrm{R}(T) = \{T(v) \in \mathrm{W};\ \forall v \in V\} \subseteq W}$$

In other words, the image of T consists of individual images of all vectors of V.

Let V and W be vector spaces, and let **T : V →W** be a linear transformation. The kernel of T, denoted by **Ker(T)**, is the set

$$\boxed{\ker(T) = \{v \in V;\ T(v) = 0\} \subseteq V}$$

In other words, the kernel of T consists of all vectors of V that map to 0 in W.



$\boxed{\text{Ex}}$: (The kernel of the zero and identity transformations)

(a) $T(\mathbf{v})=\mathbf{0}$ (the zero transformation $T: V \to W$)

$$\boxed{\ker(T) = V}$$

(b) $T(\mathbf{v})=\mathbf{v}$ (the identity transformation $T: V \to V$)

$$\boxed{\ker(T) = \{\mathbf{0}\}}$$

Ques: Let $F:\mathbf{R}^3 \to \mathbf{R}^3$ be the projection mapping into the xy plane, i.e., defined by $F(x, y, z) = (x, y, 0)$. Find the kernel of F.

Solution: The points on the z axis, and only these points, map into the zero vector $0 = (0, 0, 0)$. Thus

$$\operatorname{Ker} F = \{(0, 0, c) : c \in \mathbf{R}\}.$$

$T(x, y, z) = (0, 0, 0)$

$\Rightarrow (x, y, 0) = (0, 0, 0)$

On solving we get: $x = 0$, $y = 0$ and z we cant find so taking it free variable so $z = c$.

Que: Find the image of the projection mapping $F(x, y, z) = (x, y, 0)$ in Problem 10.52.

Ans: The image of F consists precisely of those points in the xy plane: $\operatorname{Im} F = \{(a, b, 0) : a, b \in \mathbf{R}\}$.

$F(x, y, z) = (x, y, 0)$

$\Rightarrow$ *So* $(x, y, 0)$ is image.

We can take $x = a$ and $y = b$. So image is $(a, b, 0)$

Important Transform is given in the following Question:

Ques: Let $F:\mathbf{R}^3 \to \mathbf{R}^3$ be the linear mapping which rotates a vector about the z axis through an angle θ:

Find the kernel of F. $\boxed{F(x, y, z) = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta,\ z)}$

Let $(x, y, z) \in KerF$

So, $F(x, y, z) = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta,\ z) = (0,0,0)$

$$\left.\begin{aligned} x\cos\theta - y\sin\theta &= 0 \\ x\sin\theta + y\cos\theta &= 0 \\ z &= 0 \end{aligned}\right\} \Rightarrow z = 0,\ x = 0,\ y = 0$$

So, $KerF = \{(0,0,0)\} = \{\mathbf{0}\}$

Ex 1: (Finding the kernel of a linear transformation)

$$T(A) = A^T \quad (T : M_{3\times 2} \to M_{2\times 3})$$

Sol:

$$\ker(T) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

Ex 5: (Finding the kernel of a linear transformation)

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (T : R^3 \to R^2)$$

$\ker(T) = ?$

Remark:

Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear Transform defined by the (m x n) order matrix A. The kernel of A consists of all the solutions of system AX=0. Therefore, **Kernel** of A **[i.e., Ker(T)]** is nothing but a null space **N(A)** of matrix A.
The **Im(A) [i.e., Im(T)]** is just Column space **C(A)** of matrix A.

- Therefore, finding Kernel of T (if you know Matrix A corresponding to T) is same as finding Null space N(A) of matrices studied in **Module 4.**
- And finding basis of Im(T) (if you know Matrix A corresponding to T) is same as finding Basis of column space C(A) of matrices studied in **Module 4.**

Question 1: Let T:$R^4 \to R^5$ be a linear transformation with standard matrix A, such that T(X)=AX

$$A=\begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & 2 \end{bmatrix}_{5\text{x}4}$$

Then find Basis of **Im(T)** & Dim[**Im(T)**]

Solution: First we find Reduced Row Echelon form of A.

$$A=\begin{array}{c} \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & 2 \end{bmatrix}_{5\text{x}4} \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \bar{v}_1 & \bar{v}_2 & \bar{v}_3 & \bar{v}_4 \end{array} \end{array} \xrightarrow{\text{Reduced Row Echelon Form}} B=\begin{array}{c} \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ v_1 & v_2 & v_3 & v_4 \end{array} \end{array}$$

Basis for C(A)=Basis of Im(T)= $\{\bar{v}_1, \bar{v}_2, \bar{v}_4\} = \{(1,0,-3,3,2),(3,\ 1,0,4,0),(3,\ 0,-1,1,2)\}$

$$\text{or}\left\{\begin{bmatrix}1\\0\\-3\\3\\2\end{bmatrix},\begin{bmatrix}3\\1\\0\\4\\0\end{bmatrix},\begin{bmatrix}3\\0\\-1\\1\\2\end{bmatrix}\right\},$$

Dim C(A)=Dim Im(T)=3.

Question 2: Let T:R⁴→R³ be a linear transformation with standard matrix A, such that T(X)=AX

$$A=\begin{bmatrix}1 & 2 & -2 & 1\\ 3 & 6 & -5 & 4\\ 1 & 2 & 0 & 3\end{bmatrix}$$

Then find **Ker(T),** Basis of **Ker(T)** & Dim[**Ker(T)**].

Solution: The null space of A is the solution space of AX **=** 0 for X**=** $\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$

$$[A\,|\,0]=\left[\begin{array}{cccc|c}1 & 2 & -2 & 1 & 0\\ 3 & 6 & -5 & 4 & 0\\ 1 & 2 & 0 & 3 & 0\end{array}\right]\xrightarrow{\text{Row Echelon Form}}\left[\begin{array}{cccc|c}1 & 2 & -2 & 1 & 0\\ 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}\right]\begin{array}{l}\Rightarrow x_1+2x_2-2x_3+x_4=0\\ \Rightarrow \qquad\qquad x_3+x_4=0\end{array}$$

$\rho=2,\ n=4,\ n-\rho=4-2=2,$ two free variables. Let $x_4=t\Rightarrow x_3=-t$ & $x_2=s\Rightarrow x_1=-2s-3t$

Then the solution Space=Ker(T)= $\{(-2s-3t\,,s,-t,t)\,|\; s,t\in\mathbb{R}\}$**.**

$$\begin{aligned}(-2s-3t\,,s,-t,t)&=(-2s\,,s,0,0)+(-3t\,,0,-t,t)\\ &=s(-2,1,0,0)+t(-3,0,-1,1)\end{aligned},$$

So, Basis of Ker(T)= $B_{N(A)}=\{(-2,1,0,0),(-3,0,-1,1)\}$

Nullity=Dim[N(A)]=Dim[Ker(T)]=2. $\quad(\because n-\rho=2)$

- Rank of a linear transformation $T: V \to W$:

$$\boxed{rank(T) = \text{the dimension of } \mathrm{Im(T)}}$$

- Nullity of a linear transformation $T: V \to W$:

$$\boxed{nullity(T) = \text{the dimension of the kernel of } T}$$

- Note: Let $\boxed{T: R^n \to R^m}$ be the L.T. given by $T(\mathbf{x}) = A\mathbf{x}$, then

$$\boxed{rank(T) = rank(A)}$$

$$\boxed{nullity(T) = nullity(A)}$$

Rank-Nullity Theorem for Linear Transform

Let $T: V \to W$ be a L.T. form an $\underline{\text{n - dimensional}}$ vector space V into a vector space W. then

Dim of domain

$$\boxed{rank(T) + nullity(T) = n}$$

or

$$\boxed{\mathrm{Rank(T) + Ker(T) = n}}$$

We already studied this theorem in Module 3 in terms of matrices.

In Question 1, without finding Ker(T) we can find nullity that is DIm of Ker(T) using Rank nullity Theorem:
We have calculated Dim of Im(T)=3 that is Rank(T)=3.
So we have Rank(T) + Ker(T)=n
3 + Ker(T)=4
Ker(T)=4-3=1 #

In Question 2, without finding Im(T) we can find Rank(T) that is DIm of Im(T) using Rank nullity Theorem:
We have calculated Dim of Ker(T)=3 that is Nullity=2.
So we have Rank(T) + Ker(T)=n
Rank(T) + 2=4
Ker(T)=4-2=2 #

Que: For the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$, where $T(x, y, z) = (x - 2y + z, 2x + y + z)$:

(a) Find the rank of T.

(b) Without finding the kernel of T, use the rank-nullity theorem to find the nullity of T

Matrix representation of T is as $\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

It's a 3x2 matrix which means it represents a transformation from R^3 to R^2

(a) rank of T.

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & -1 \end{bmatrix}_{2 \times 3} \qquad m = 2, n = 3$$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & \frac{-1}{5} \end{bmatrix}$$

RRE form $$\sim \begin{bmatrix} 1 & 0 & \frac{3}{5} \\ 0 & 1 & \frac{-1}{5} \end{bmatrix}$$

Basis for Im(T)=$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \end{bmatrix} \right\}$

OR $\{(1, 0), (-2, 5)\}$

Dim Im(T)= Rank T= 2

Range of T, i.e., Im(T) = Spanned by $\{(1, 0), (-2, 5)\}$

By rank Nullity theorem: Rank(T) + Ker(T) = n

2 + Ker(T) = 3

Ker(T) = 3 - 1

Ker(T) = 1

Question for HW

Define the map $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \\ x_2 \end{bmatrix}$.

(a) Show that T is a linear transformation.

(b) Find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for each $\mathbf{x} \in \mathbb{R}^2$.

(c) Describe the null space (kernel) and the range of T and give the rank and the nullity of T.

Question: (Finding the rank and nullity of a linear transformation)

Let $T : R^5 \to R^7$ be a linear transformation.

(a) Find the dimension of the kernel of T if the dimension of the range is 2

(b) Find the rank of T if the nullity of T is 4

(c) Find the rank of T if $Ker(T) = \{0\}$

Sol:

$(a)\ \dim(\text{domain of } T) = 5$

$\dim(\text{kernel of } T) = n - \dim(\text{range of } T) = 5 - 2 = 3$

$(b) rank(T) = n - nullity(T) = 5 - 4 = 1$

$(c) rank(T) = n - nullity(T) = 5 - 0 = 5$

Theorem 4.1: *Let $T : V \to W$ be a linear transformation from a vector space V to a vector space W. Then the kernel* $\mathrm{Ker}(T)$ *and the image* $\mathrm{Im}(T)$ *are subspaces of V and W, respectively.*

Theorem 4.2: *Let V and W be vector spaces. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V and let $\mathbf{w}_1, \ldots, \mathbf{w}_n$ be any vectors (possibly repeated) in W. Then there exists a unique linear transformation $T : V \to W$ such that $T(\mathbf{v}_i) = \mathbf{w}_i$ for $i = 1, \ldots, n$.*

Corollary 4.3 *Let V and W be vector spaces, and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V. If S, $T : V \to W$ are linear transformations and $S(\mathbf{v}_i) = T(\mathbf{v}_i)$ for $i = 1, \ldots, n$, then $S = T$, i.e., $S(\mathbf{x}) = T(\mathbf{x})$ for all $\mathbf{x} \in V$.*