Diagonal Matrix: A square matrix (of n x n order) in which every element except the principal diagonal elements is zero is called a Diagonal Matrix.

$$A_{n \times n} = \begin{bmatrix} a_{11} & 0 & 0 & . & . & 0 \\ 0 & a_{22} & 0 & . & . & 0 \\ 0 & 0 & a_{33} & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & a_{nn} \end{bmatrix}_{n \times n}$$

Scalar Matrix: It is a special kind of diagonal matrix, in which the diagonal contains the same element.

$$\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

The identity matrix is also an example of a scalar matrix.

Identity Matrix: Identity Matrix is the matrix which is n × n square matrix where the diagonal consist of 1's and the other elements are all zeros. It is also called as a Unit Matrix. We represent it by I_n

$$I_1 = [1], \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Null Matrix: Null Matrix is a matrix having zero as each of its elements.

$$O_{2\,x\,2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad O_{3x3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$O_{1x4} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \quad O_{3x2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2 × 3 order null matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3 × 3 order null matrix

The Inverse of a Matrix

Inverse matrix:

A matrix **A** of dimension **n** x **n** (**should be square matrix only**) is called invertible **if and only if** there exists another matrix **B** of the same dimension, such that

$$AB = BA = I,$$

where **I** is the is the identity matrix of the same order.

Matrix **B** is known as the **inverse of matrix A** and symbolically we represents **B** by A^{-1}.

$$AA^{-1} = A^{-1}A = I.$$

Definitions:

- **Invertible Matrix:** An invertible matrix is a square matrix **that has an inverse.**
- **Non-Singular matrix:** A matrix is said to be non-singular, if its determinant value is not equal to zero, i.e. $|A| \neq 0$. Hence, inverse of any matrix is possible if and only if its determinant is non-zero. Therefore, **Non-Singular matrix** also known as **Invertible Matrix.**
- **Singular matrix:** A matrix is said to be singular, if its determinant is zero, i.e. $|A| = 0$. Hence, inverse of Singular matrices are not possible. Therefore, **Singular matrices** are known as **Non-Invertible.**

Properties of inverse matrices:

1) $(A^{-1})^{-1} = A$
2) $(A^T)^{-1} = (A^{-1})^T$
3) $(KA)^{-1} = \frac{1}{K} A^{-1}$, for any $k \neq 0$
4) $(AB)^{-1} = B^{-1}A^{-1}$.
5) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
6) $(A_1 A_2 \cdots A_n)^{-1} = A_n^{-1} A_{n-1}^{-1} \cdots A_2^{-1} A_1^{-1}$
7) $(A^n)^{-1} = (A^{-1})^n$, for any $n \in \mathbb{Z}^+$.

Finding inverse matrices by Gauss-Jordan Elimination method:

1. First we write the matrix A in the following form

$$[A \mid I]$$

2. Then we convert it into the **Reduced Row Echelon**, hence we get invers of A as

$$[A \mid I] \xrightarrow[\text{by finding Reduced Row Echelon form}]{\text{Gauss-Jordan Elimination}} [I \mid A^{-1}]$$

Q. Find the inverse of matrix A using Gauss-Jordan method:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix}$$

Soln:-

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$\frac{R_1}{2} \to R_1$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1/2 & -1/2 & 1/2 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$R_3 - R_1 \to R_3$ & $R_2 - R_1 \to R_2$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1/2 & -1/2 & 1/2 & 0 & 0 \\ 0 & -1/2 & -1/2 & -1/2 & 1 & 0 \\ 0 & 1/2 & 5/2 & -1/2 & 0 & 1 \end{array}\right]$$

$-2R_2 \to R_2$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1/2 & -1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1 & 1 & -2 & 0 \\ 0 & 1/2 & 5/2 & -1/2 & 0 & 1 \end{array}\right]$$

$R_1 - \frac{1}{2}R_2 \to R_1$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -2 & 0 \\ 0 & 1/2 & 5/2 & -1/2 & 0 & 1 \end{array}\right]$$

$R_3 - \frac{1}{2}R_2 \to R_3$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -2 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right]$$

$\frac{R_3}{2} \to R_3$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right]$$

$R_1 + R_3 \to R_1$ & $R_2 - R_3 \to R_2$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 3/2 & 1/2 \\ 0 & 1 & 0 & 3/2 & -5/2 & -1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right]$$

$$\Rightarrow \left[\, I \mid A^{-1} \,\right]$$

$$\therefore \quad A^{-1} = \begin{bmatrix} -1/2 & 3/2 & 1/2 \\ 3/2 & -5/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix} \quad \#$$

Exercise 1: Find the inverse of following matrices;

(1). $A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & 1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix}$ **(2).** $A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$ **(3).** $\begin{bmatrix} 2 & 4 & 6 \\ 2 & 0 & 2 \\ 6 & 8 & 14 \end{bmatrix}$

Note: If you can't get $\left[I \mid A^{-1} \right]$ form, then it means the matrix is not invertible (singular). See the Ex. (3), it's not possible to get $\left[I \mid A^{-1} \right]$ form for this.

Finding solution of a system of linear equations after finding inverse of matrix by using Gauss Jordan method

As we know that a system of 'm' **linear equations** of 'n' variables

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

can be written into the matrix from **AX=B** as

$$AX = B \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ & \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad \text{....................(1)}$$

where,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ & \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Therefore, equation (1) can be written left multiplying of A^{-1};

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ & \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}^{-1} \times \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad \text{....................(2)}$$

So, we can find solution of the system by finding A^{-1} and multiplying by B into A^{-1}

Note: If you can't get $\left[I \mid A^{-1} \right]$ form, then it means the coefficient matrix A is not invertible (singular), then the system could be have **no solution** or could be have **infinitely many solutions**.

Q. Find the solution of following system of Lin. eqn. by finding inverse of coeff. matrix using of Gauss-Jordan method.

$$2x_1 + x_2 - x_3 = 5$$
$$x_1 - x_3 = 1$$
$$x_1 + x_2 + 2x_3 = 3$$

Soln: We can write the given system as $AX = B$

$$\begin{bmatrix} 2 & +1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}$$

i.e. $X = A^{-1}B \quad - (*)$

So now we need to find A^{-1} by Gauss Jordan M.

$$A^{-1} = \begin{bmatrix} -1/2 & 3/2 & 1/2 \\ 3/2 & -5/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix}$$ (see last question already calculated)

Therefore putting A^{-1} in eqn $(*)$, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/2 & 3/2 & 1/2 \\ 3/2 & -5/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 7/2 \\ -1/2 \end{bmatrix}$$

$\Rightarrow x_1 = 1/2, \quad x_2 = 7/2 \quad \& \quad x_3 = -1/2$ #

Exercise: Write the following system of linear equations as AX=B and then solve it by computing A^{-1} using the Gauss Jordan Elimination method.

a) $2x - y + 3z = 2,\ y - 4z = 5,\ 2x + y - 2z = 7$
b) $x + 2y + 2z = 10,\ 2x - 2y + 3z = 1,\ 4x - 3y - +5z = 4$