Module 5: Inner Product Spaces

Dot Product:

Let Vectors $x = (x_1, x_2, \dots x_n) \in R^n$
$y = (y_1, y_2, \dots y_n) \in R^n$

Then dot product of Vectors x and y defined as

$$\boxed{x \cdot y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n}$$

$$= [x_1, x_2, \dots x_n] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} ; y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\boxed{x \cdot y = x^T y}$$

Lenght/Mgnitude of a vector:

Let $x = (x_1, x_2, \dots x_n)$ be a Vector in R^n, then its magnitude is given by

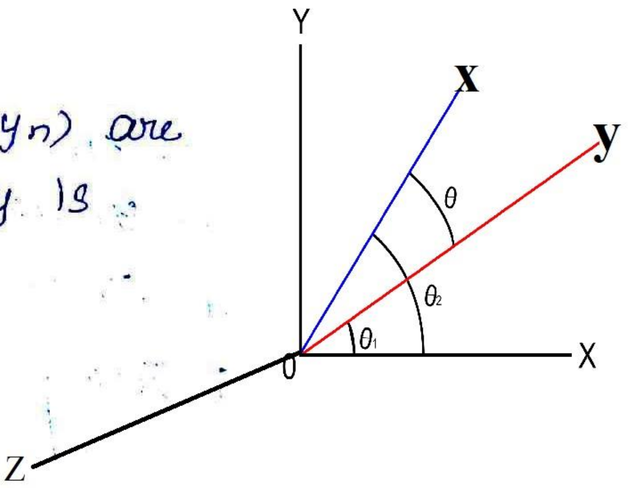
$$\boxed{\|x\| = (x \cdot x)^{1/2} = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}}$$

norm of x

Angle b/w two vectors:

Let $x = (x_1, x_2, \dots x_n)$ & $y = (y_1, y_2, \dots y_n)$ are Vectors in R^n, the angle b/w x and y is

$$\boxed{\cos\theta = \frac{x \cdot y}{\|x\| \|y\|}} \quad 0 \leq \theta \leq \pi$$



Hence $\boxed{x \cdot y = \|x\| \|y\| \cos\theta}$

Question: Find the angle b/w (4,7,9,1,3) and (2,1,1,6,8) in R^5. Also find their magnitudes of x, y and dot products of x & y.

Solution: Angle b/w x and y is :- $x = (4,7,9,1,3)$; $y = (2,1,1,6,8)$

$$\boxed{\cos\theta = \frac{x \cdot y}{\|x\|\|y\|}}$$

$x \cdot y = (4,7,9,1,3) \cdot (2,1,1,6,8) = (8+7+9+6+24) = 54$

$\|x\| = \sqrt{16+49+81+1+9} = \sqrt{156} = 12.48$

$\|y\| = \sqrt{4+1+1+36+64} = \sqrt{106} = 10.29$

$$\cos\theta = \frac{(8,7,9,6,24)}{\sqrt{156}\sqrt{106}} = \frac{(8+7+9+6+24)}{(12.48 \times 10.29)} = \frac{8+7+9+6+24}{128.42}$$

$$\cos\theta = \frac{54}{128.42} = 0.42 \quad \Rightarrow \quad \boxed{\theta = \cos^{-1}(0.42) = 65.16^\circ}$$

Magnitude of x and y

$\|x\| = (x \cdot x)^{1/2} = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} \Rightarrow \|x\| = 12.48$

$\|y\| = (y \cdot y)^{1/2} = \sqrt{y_1^2 + y_2^2 + \dots + y_n^2} \Rightarrow \|y\| = 10.29$

dot products of x and y :-

$x \cdot y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$

$= 8 + 7 + 9 + 6 + 24$

$\boxed{x \cdot y = 54}$

Inner Product:

Let V be a Vector Space over the field $\mathbb{R}$. And Inner Product over V denoted by $\langle , \rangle$ is a map from $V\times V$ to $\mathbb{R}$ satisfying following properties for vectors $x, y, z \in V$ and scalar α in $\mathbb{R}$.

$$\boxed{\langle , \rangle : V\times V \to \mathbb{R}}$$

property

Axiom (1): $\langle x, y\rangle = \langle y, x\rangle$ (Symmetry)

Axiom (2): $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ (additivity)

Axiom (3): $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$ (Homogeneity)

Axiom (4): $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ Iff $x=0$ (positive differentness)

Inner Product Space:

Let V be a Vector Space with an Inner product $\langle , \rangle$. Then the pair $(V, \langle , \rangle)$ is called an inner product space

Properties:

① If we take $V=\mathbb{R}^n$, then I.P.S $(\mathbb{R}^n, \langle , \rangle)$ is called Euclidian n-space.

② $\boxed{\|x\|^2 = \langle x, x\rangle}$ or $\boxed{\|x\| = \sqrt{\langle x, x\rangle}}$

③ $\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{x_1^2+x_2^2+\cdots x_n^2}$ by dot product/Eucledian IP/standard IP

④ Distance of two Vectors $x=(x_1, x_2, \ldots x_n)$ & $y=(y_1, y_2, \cdots y_n)$ is given by

$$\boxed{\begin{aligned} d(x,y) = \|x-y\| &= \sqrt{\langle x-y, y-x\rangle} \\ &= \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\cdots (x_n-y_n)^2} \end{aligned}}$$

⑤ $d(x,y) = d(y,x)$

Ques 1: Show that the following function defines as Inner product over $V=\mathbb{R}^2$.

$\boxed{\langle U,V\rangle = U_1V_1+2U_2V_2}$ where $U=(U_1,U_2)$; $V=(V_1,V_2)$.

Proof:- Axiom (1): $\langle x,y\rangle = \langle y,x\rangle$

$$\langle U,V\rangle = \langle V,U\rangle$$
$$\Downarrow \qquad \Downarrow$$
$$U_1V_1+2U_2V_2 = V_1U_1+2V_2U_2$$
$$U_1V_1+2U_2V_2 = U_1V_1+2U_2V_2$$
$$LHS = RHS$$

Axiom (2):- (additivity)

$$\langle x+y,z\rangle = \langle x,z\rangle+\langle y,z\rangle$$
$$\langle U+V,W\rangle = \langle U,W\rangle+\langle V,W\rangle.$$

$U=(U_1,U_2)$
$V=(V_1,V_2)$
$U+V=(U_1+V_1,U_2+V_2)$

$$\langle (U_1+V_1,U_2+V_2),(W_1,W_2)\rangle = \langle (U_1,U_2),(W_1,W_2)\rangle + \langle (V_1,V_2),(W_1,W_2)\rangle.$$

$$(U_1+V_1)W_1+2(U_2+V_2)W_2 = (U_1W_1+2U_2W_2)+(V_1W_1+2V_2W_2).$$

$$U_1W_1+V_1W_1+2U_2W_2+2V_2W_2 = U_1W_1+V_1W_1+2U_2W_2+2V_2W_2$$
$$LHS = RHS$$

Axiom (3):- Let $\alpha\in\mathbb{R}$,

$$\langle \alpha U,V\rangle = \langle \alpha(U_1,U_2),(V_1,V_2)\rangle$$
$$= \langle(\alpha U_1,\alpha U_2),(V_1,V_2)\rangle$$
$$= \alpha U_1V_1+2\alpha U_2V_2$$
$$= \alpha(U_1V_1+2U_2V_2)$$
$$\langle \alpha U,V\rangle = \alpha\langle U,V\rangle.$$ (Homogeneity satisfied).

Axiom (4):-

$$\langle U,U\rangle = \langle (U_1,U_2)(U_1,U_2)\rangle$$
$$= U_1^2+2U_2^2 \geq 0.$$
$$\langle U,U\rangle \geq 0.$$

and Iff $\langle U,U\rangle = 0$.
$$U_1^2+2U_2^2=0$$
$$\therefore U_1=0,\ U_2=0$$
$$\therefore U=(0,0)=0$$

So $\langle\ ,\ \rangle$ is inner product and $(\mathbb{R}^2,\langle\ ,\ \rangle)$ is I.P.S.

Ques 2: Show that the following function is not Inner product on R^3.

$$\langle u, v \rangle = u_1 v_1 - 2u_2 v_2 + u_3 v_3 \qquad u = (u_1, u_2, u_3) \in R^3$$

Soln:- Axiom (4):- $\langle u, u \rangle = \langle (u_1, u_2, u_3), (u_1, u_2, u_3) \rangle$

$= u_1^2 - 2u_2^2 + u_3^2$ It will be always -ve as all terms are square

Let $u = (1, 2, 1)$ So $\langle u, u \rangle = -6 < 0$ not satisfying condition.
$\quad u_1 \; u_2 \; u_3$

So function $\langle , \rangle$ is not I.P. and $(R^3, \langle , \rangle)$ is not IPS.

Euclidean Inner Product or Standard Inner Product:

An inner product with dot product is called inner product on vector space R^n.

$$\boxed{\langle x, y \rangle = x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n}$$

where vectors $x = (x_1, x_2, \ldots x_n)$; $y = (y_1, y_2, \ldots y_n)$.

Exercise:

Show (H.W) that R^n is Inner product space with dot products.

Hint:- For $x = (x_1, x_2, \ldots x_n)$, $y = (y_1, y_2, \ldots y_n)$
$\langle x, y \rangle = x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$ (show all axioms of IPS)

Q 3: Consider the Polynomial Vector space $P_n(\mathbb{R})$. For $f, g \in P_n(\mathbb{R})$. defined by the product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx.$$

Show that $\langle , \rangle$ is an inner product on $P_n(\mathbb{R})$.

Soln:-

Axiom (1)

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx = \int_0^1 g(x)f(x)\,dx = \langle g, f\rangle$$

LHS = RHS.

It satisfied Symmetry Condition.

Axiom ②:-

$$\langle f+g, h\rangle = \int_0^1 (f+g)(x).h(x)\,dx = \int_0^1 [f(x)+g(x)]\,h(x)\,dx$$

$$= \int_0^1 f(x)h(x)\,dx + \int_0^1 g(x)h(x)\,dx$$

$$= \langle f, h\rangle + \langle g, h\rangle$$

LHS = RHS

$\therefore$ It Satisfied additivity Condition.

Axiom ③:

$$\langle \alpha f, g\rangle = \int_0^1 \alpha f(x)g(x)\,dx = \alpha \int_0^1 f(x)g(x)\,dx = \alpha\langle f, g\rangle$$

LHS = RHS

$\therefore$ It satisfied homogeneity Condition.

Axiom ④:-

$$\langle f, f\rangle = \int_0^1 [f(x)]^2\,dx \geq 0$$

and $\langle f, f\rangle = 0$.

$$\int_0^1 [f(x)]^2\,dx = 0$$

If and only if $f(x) = 0$.

So, $\langle , \rangle$ is IP. and $(P_n(\mathbb{R}), \langle , \rangle)$ is IPS.

Ques 4: Consider the inner product space $P_2(R)$ with the inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx.$$

H.W. (A). Show that $(P_2(R), \langle , \rangle)$ is an IPS. Same as last Ques

(B). Find angle between $f(x) = x+2$ and $g(x) = 2x-3$.

Soln (B).

$$\boxed{\cos\theta = \frac{\langle f, g\rangle}{\|f\|\,\|g\|}} \qquad \boxed{\|f\| = \sqrt{\langle f, f\rangle}} \qquad \boxed{\|g\| = \sqrt{\langle g, g\rangle}}$$

$$\langle f, f\rangle = \int_0^1 f(x)f(x)\,dx = \int_0^1 (x+2)^2\,dx = \frac{19}{3}$$

$$\boxed{\|g\| = \sqrt{\langle g, g\rangle} = \sqrt{13/3}}$$

$$\langle g, g\rangle = \int_0^1 g(x)g(x)\,dx = \int_0^1 (2x-3)^2\,dx = \frac{13}{3}$$

$$\boxed{\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{19/3}}$$

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx = \int_0^1 (x+2)(2x-3)\,dx = \frac{-29}{6}$$

$$\cos\theta = \left(\frac{\frac{-29}{6}}{\sqrt{\frac{19\times 13}{9}}}\right)$$

$$\boxed{\theta = \cos^{-1}\left(\frac{-29}{2\sqrt{247}}\right)}$$

Ques 5: If $\langle p, q\rangle = a_0b_0 + a_1b_1 + a_2b_2$ is an inner product. Let $p(x) = 1-2x^2$ and $q(x) = 4-2x+x^2$ be polynomial in $P_2(R)$.

(a) Find $\langle p, q\rangle$

(b) Find $\|q\|$

(c) Find $d(p, q)$

Polynomial in $P_2(R)$

$$a_0x^0 + a_1x^1 + a_2x^2$$
$$b_0x^0 + b_1x^1 + b_2x^2$$

By Comparing $p(x) = 1-2x^2 \Rightarrow a_0 = 1, a_1 = 0, a_2 = -2$

$q(x) = 4-2x+x^2 \Rightarrow b_0 = 4, b_1 = -2, b_2 = 1.$

(a) $\langle p, q\rangle = a_0b_0 + a_1b_1 + a_2b_2 = 2$

(b) $\|q\| = \sqrt{\langle q, q\rangle} = \sqrt{b_0^2 + b_1^2 + b_2^2} = \sqrt{21}$

(c) $d(p, q) = \|p-q\| = \sqrt{c_0^2 + c_1^2 + c_2^2} = \sqrt{22}$

$$p - q = (1-2x^2) - (4-2x+x^2)$$
$$= \underset{c_0}{-3} + \underset{c_1}{2x} \underset{c_2}{-3x^2}$$

Exercise:

(6) H.W Let $V = R_n(R)$ be a V.S of real functions whose domain is $[a,b]$ with inner product

$$\langle f, g\rangle = \int_a^b f g \, dx.$$

Then prove that $(V, \langle , \rangle)$ is Inner product Space.

(7) Show that which of the following functions on R^2 are inner product for, where $x = (x_1, x_2)$, $y = (y_1, y_2)$.

(a) $\langle x, y\rangle = x_1 y_1 + x_2 y_2$

(b) $\langle U, V\rangle = U_1 V_1 + 3 U_2 V_2$ where $U = (U_1, U_2)$, $V = (V_1, V_2)$

(c) $\langle s, t\rangle = 4 s_1 t_1 + 4 s_2 t_2 - s_1 t_2 - s_2 t_1$ $s = (s_1, s_2)$, $t = (t_1, t_2)$

Theorem: (Cauchy - Schwarz inequality)

If x and y are vectors in an inner product space, then

$$\langle x, y\rangle^2 \leq \langle x, x\rangle \langle y, y\rangle$$

$$\text{or} \quad |\langle x, y\rangle| \leq \|x\| \cdot \|y\|$$

Problems:-

Q ① Let V be any inner product space $(V, \langle , \rangle)$, then show that

ⓐ $\|x\| \geq 0$

ⓑ $\|x\| = 0$ iff ($\Leftrightarrow$) $x = 0$

ⓒ $\|kx\| = |k| \, \|x\|$ — $k \in \mathbb{R}$ is any scalar

ⓓ $\|x+y\| \leq \|x\| + \|y\|$ (Triangle inequality)

Q ② Let $(V, \langle , \rangle)$ be any IPS, the show that

ⓐ $d(x,y) \geq 0$

ⓑ $d(x,y) = 0$ iff ($\Leftrightarrow$) $x = y$

ⓒ $d(x,y) = d(y,x)$

ⓓ $d(x,y) \leq d(x,z) + d(z,y)$ (Triangle inequality)

Q 3 Let $(V, \langle , \rangle)$ be an IPS. then

ⓐ $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$ — $k \in \mathbb{R}$

ⓑ $\langle x, ky\rangle = k\langle x, y\rangle$

ⓒ $\langle 0, x\rangle = 0 = \langle x, 0\rangle$

Hint: $\|x\| = \langle x, x\rangle$, $d(x,y) = \|x-y\| = \sqrt{\langle x-y, x-y\rangle}$

Orthogonal vectors :- Two vectors x and y in an IPS are said to be orthogonal (or $\perp$) if $\langle x, y\rangle = 0$

Note: $\boxed{\langle x, y\rangle = 0 \iff \theta = \pi/2}$

Lemma ① Let V be an IPS and let $x \in V$. Then the vector x is orthogonal to every $y \in V$ iff ($\Leftrightarrow$) $x = 0$.

i.e. $\langle x, y\rangle = 0 \quad \forall\ y \in V$ iff $x = 0$

Lemma ② Let V be an IPS and $\alpha = \{v_1, v_2, \cdots v_n\}$ be a basis for V. Then a vector x in V is orthogonal to every basis vector $v_i \in \alpha$ iff $x = 0$

Lemma ③ ~~If~~ vectors u and v in an IPS are orthogonal iff ($\Leftrightarrow$) u and v are L.I.

Theorem :- If $x_1, x_2, \cdots x_n$ are non-zero mutually orthogonal vectors in an IPS V (i.e. each vector is orthogonal to every other), then they are Linearly Independent.

proof: Suppose $c_1x_1 + c_2x_2 + \cdots c_nx_n = 0$ for scalars c_i

Then for ~~any~~ vectors x_i $\quad i = 1, 2, 3 \cdots n$

$$\langle 0, x_i\rangle = 0$$

$$\Rightarrow \langle c_1x_1 + c_2x_2 + \cdots c_nx_n, x_i\rangle = 0$$

$$\Rightarrow c_1\cancel{\langle x_1, x_i\rangle}^{0} + c_2\cancel{\langle x_2, x_i\rangle}^{0} + \cdots + c_i\langle x_i, x_i\rangle + \cdots + c_n\cancel{\langle x_n, x_i\rangle}^{0} = 0$$

$$\Rightarrow c_i\langle x_i, x_i\rangle = 0 \qquad \text{as all are } \perp$$

$$\Rightarrow c_i = 0 \qquad \because \langle x_i, x_i\rangle = \|x_i\|^2 \neq 0$$

$\therefore x_1, x_2, \cdots x_n$ are L.I. #