Coordinate of Vector

Let $B = \{u_1, u_2, \ldots u_n\}$ is the ordered basis for a vector space V (Dimension of $V = n$). Further $u \in V$ where $u = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n$ where $\alpha_i \in R$ (scalars) then these scalar co-ordinates $(\alpha_1, \alpha_2 \ldots \alpha_n)$ called co-ordinate of a vector V w.r.t. ordered basis 'B'.

Q.1 Find the co-ordinate of the vector (3, 4, 5) w.r.t. the ordered basis $B = \{ \overset{u_1}{(1,0,1)} \overset{u_2}{(1,1,0)} \overset{u_3}{(0,1,1)} \}$ of R^3.

$$U = \alpha_1 U_1 + \alpha_2 U_2 + \alpha_3 U_3 \quad \text{(By defination)}$$

Let $(3,4,5) = \alpha_1 (1,0,1) + \alpha_2 (1,1,0) + \alpha_3 (0,1,1)$

$\Rightarrow (3,4,5) = (\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_3)$

$\Rightarrow \alpha_1 + \alpha_2 = 3$ After solving we can get $\alpha_1 = 2$

$\alpha_2 + \alpha_3 = 4$ $\quad \alpha_2 = 1$

$\alpha_1 + \alpha_3 = 5$ $\quad \alpha_3 = 3$

So, co-ordinate vector is (2, 1, 3)

HW Q2. Find the co-ordinate vector of vector space (3, −2, 5, −4) w.r.t ordered basis $B = \{(0,1,1,1)\,(1,0,1,1)\,(1,1,0,1)\,(1,1,1,0)\}$

Q3. Find the co-ordinate vector of x^2+2x-1 w.r.t ordered basis $B = \{x+1,\ x^2+x-1,\ x^2-x+1\}$ of P_2

Let $U = x^2 + 2x - 1$

$$U = \alpha_1 U_1 + \alpha_2 U_2 + \alpha_3 U_3$$

$x^2+2x-1 = \alpha_1 (x+1) + \alpha_2 (x^2+x-1) + \alpha_3 (x^2-x+1)$

$x^2+2x-1 = (\alpha_2 + \alpha_3) x^2 + (\alpha_1 + \alpha_2 - \alpha_3) x + (\alpha_1 - \alpha_2 + \alpha_3)$

On comparing the coefficient of x^2, x & const.

$$\left\{\begin{array}{l} \alpha_2 + \alpha_3 = 1 \\ \alpha_1 + \alpha_2 - \alpha_3 = 2 \\ \alpha_1 - \alpha_2 + \alpha_3 = -1 \end{array}\right\} \quad \text{Ans} \rightarrow \left(\frac{1}{2}, \frac{5}{4}, \frac{-1}{4}\right)$$

Basis and dimension of Row and column spaces

Row and column spaces:

Let $A=\left[a_{ij}\right]_{m\times n}$ be an m x n matrix i.e.,

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ & \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Row Vectors:

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ & \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_m \end{bmatrix}$$

where, Row vectors of A are

$$\begin{aligned} r_1 &= [a_{11} \quad a_{12} \quad a_{13} \quad \cdots \quad a_{1n}] \\ r_2 &= [a_{21} \quad a_{22} \quad a_{23} \quad \cdots \quad a_{2n}] \\ r_3 &= [a_{31} \quad a_{32} \quad a_{33} \quad \cdots \quad a_{3n}] \\ \vdots & \qquad\quad \vdots \\ r_m &= [a_{m1} \quad a_{m2} \quad a_{m3} \quad \cdots \quad a_{mn}] \end{aligned}$$

Note: There are m Row vectors $\{r_1, r_2, r_3, ... r_m\}$ of A and each Row vector belongs to $\mathbb{R}^n$.

Column Vectors:

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ & \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} = [c_1 \quad c_2 \quad c_3 \quad \ldots \quad c_n]$$

where, column vectors of A are

$$\underbrace{\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \\ a_{m1} \end{bmatrix}}_{c_1} \quad \underbrace{\begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ \vdots \\ a_{m2} \end{bmatrix}}_{c_2} \quad \underbrace{\begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ \\ a_{m3} \end{bmatrix}}_{c_3} \quad \cdots \quad \underbrace{\begin{bmatrix} a_{1n} \\ a_{2n} \\ a_{3n} \\ \\ a_{mn} \end{bmatrix}}_{c_n}$$

Note: There are n Column vectors $\{c_1, c_2, c_3, ... c_n\}$ of A and each column vector belongs to $\mathbb{R}^m$.

Row space: Let $A=\left[a_{ij}\right]_{m\times n}$ be an m x n matrix, then the Row Space of A is the subspace in $\mathbb{R}^n$ is spanned by Row Vectors $\{r_1, r_2, r_3, ... r_m\}$ denoted by R(A).

Column space: Let $A=\left[a_{ij}\right]_{m\times n}$ be an m x n matrix, then the Column Space of A is the subspace in $\mathbb{R}^m$ is spanned by Row Vectors $\{c_1, c_2, c_3, ... c_n\}$ denoted by C(A).

Null space: Let $A=\left[a_{ij}\right]_{m\times n}$ be an m x n matrix, then the solution set of homogeneous equation $AX=0$ is called Null Space of A for any column vector $X=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$.

It is denoted by N(A); $\boxed{N(A)=\left\{X=\left(x_1, x_2, x_3, ... x_n\right)\in\mathbb{R}^n;\ \text{such that } AX=0\right\}}$.

Remarks:

1. Dimension of any space is equal to the cardinality of its basis. (see module 2 note)
2. **Dim**[**R(A)**] is called **Row rank** of matrix A.
3. **Dim**[**C(A)**] is called **Column rank** of matrix A.
4. **Dim**[**N(A)**] is called **Nullity** of matrix A.

   Nullity of A = **Dim**[N(A)] = **No. of free variables** in the solution of **AX=0**.
5. Since the Row vectors of A are just Column vectors of A^T and Column vectors of A are just Row vectors of A^T. Then Row space of A is Column space of A^T and Column space of A is Row space of A^T i.e.

   R(A)=C(A^T) and C(A)=R(A^T)
6. Row rank of A is always less or equal to n, i.e. $R(A)\le n$.
7. Column rank of A is always less or equal to m, i.e. $C(A)\le m$.

Theorem 4.1: (***Fundamental Theorem***)
If A is an m x n matrix, then the row space and the column space of A have the same dimension, i.e.,

Dim[**R(A)**] = **Dim**[**C(A)**]

Rank of Matrix A: The dimension of the row (or column) space of a matrix A is called the rank of A and is denoted by **Rank (A).** We also use **symbol** ρ for the Rank of A.

Dim[R(A)] = Dim[C(A)]= Rank (A) = ρ

Remark: **7.** By result 4 and Theorem 4.1; **Rank (A^T) = Rank (A) =** ρ

❖ Finding Basis for the Row Space of a matrix A:

- First convert the given matrix into **Row-Echelon** or **Reduced Row-Echelon** form.
- Collect the non-zero rows; it gives basis for the Row Space.
- Cardinality of the basis will be its dimension and called Row Rank.

Example 1: Find a basis of row space of $A=\begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & 2 \end{bmatrix}$

Solution: First we find Reduced Row Echelon form of A.

$$A=\begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & 2 \end{bmatrix}_{5x4} \xrightarrow{\text{Row Echelon Form}} B=\begin{array}{c} \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow u_1 \\ \leftarrow u_2 \\ \leftarrow u_3 \\ \leftarrow u_4 \\ \leftarrow u_5 \end{array}$$

Basis for R(A)= $\{u_1, u_2, u_3\} = \{(1,\ 3,\ 1,\ 3), (0,\ 1,\ 1,\ 0), (0,\ 0,\ 0,\ 1)\}$

or $\{[1\ \ 3\ \ 1\ \ 3], [0\ \ 1\ \ 1\ \ 0], [0\ \ 0\ \ 0\ \ 1]\}$.

Dim R(A)= Row Rank of A=3.

Observation: We can see that A is matrix of **5** x **4** order; $R(A) \subset \mathbb{R}^4$.

Example 2: Find a basis for the row space of matrix $A=\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$.

Solution: Given matrix $A=\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}_{3\times 5}$, we find Reduced Row Echelon form of A.

$$A=\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \xrightarrow{\text{Reduced Row Echelon Form}} B=\begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \leftarrow u_1 \\ \leftarrow u_2 \\ \leftarrow u_3 \end{array}$$

Basis for R(A)= $\{u_1, u_2\} = \{(1,\ -2,\ 0,\ -1,\ 3), (0,\ 0,\ 1,\ 2,\ -2)\}$

or $\{[1\ \ -2\ \ 0\ \ -1\ \ 3], [0\ \ 0\ \ 1\ \ 2\ \ -2]\}$.

Dim R(A)=Row Rank of A=2.

Observation: We can see that A is matrix of **3** x **5** order; $R(A) \subset \mathbb{R}^5$.

❖ Finding Basis for the Column Space of a matrix A:

- First convert the given matrix into **Reduced Row Echelon** form.
- Collect the corresponding column of '**Leading 1'** column; it gives basis for the Column Space.
- **Leading 1** means only single non-zero entry is '1' in column and rest entries are zero.
- Cardinality of the basis will be its dimension and called Column Rank.

Example 1: Find a basis for the column space of matrix $A=\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$.

Solution: Given matrix $A=\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}_{3\times 5}$, we find Reduced Row Echelon form of A.

$$A=\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \xrightarrow{\text{Reduced Row Echelon Form}} B=\begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \bar{v}_1 & \bar{v}_2 & \bar{v}_3 & \bar{v}_4 & \bar{v}_5 \end{array} \qquad \begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ v_1 & v_2 & v_3 & v_4 & v_5 \end{array}$$

Basis for C(A)= $\{\bar{v}_1, \bar{v}_3\}=\{(-3\ \ 1\ \ 2),(-1\ \ 2\ \ 5)\}$ or $\left\{\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}\right\}$

Dim C(A)=Column Rank of A=2.

Observation:

1. We can see that A is matrix of **3** x **5** order; $C(A)\subset\mathbb{R}^3$.

Example 2: Find a basis of **column** space of $A=\begin{bmatrix}1&3&1&3\\0&1&1&0\\-3&0&6&-1\\3&4&-2&1\\2&0&-4&2\end{bmatrix}$

Solution: First we find Reduced Row Echelon form of A.

$$A=\begin{array}{c}\begin{bmatrix}1&3&1&3\\0&1&1&0\\-3&0&6&-1\\3&4&-2&1\\2&0&-4&2\end{bmatrix}_{5\text{x}4}\\ \begin{matrix}\uparrow&\uparrow&\uparrow&\uparrow\\ \bar{v}_1&\bar{v}_2&\bar{v}_3&\bar{v}_4\end{matrix}\end{array} \xrightarrow{\text{Reduced Row Echelon Form}} B=\begin{array}{c}\begin{bmatrix}1&0&-2&0\\0&1&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}\\ \begin{matrix}\uparrow&\uparrow&\uparrow&\uparrow\\ v_1&v_2&v_3&v_4\end{matrix}\end{array}$$

Basis for C(A)= $\{\bar{v}_1, \bar{v}_2, \bar{v}_4\}=\{(1,0,-3,3,2),(3,1,0,4,0),(3,0,-1,1,2)\}$ or $\left\{\begin{bmatrix}1\\0\\-3\\3\\2\end{bmatrix},\begin{bmatrix}3\\1\\0\\4\\0\end{bmatrix},\begin{bmatrix}3\\0\\-1\\1\\2\end{bmatrix}\right\}$,

Dim C(A)=Column Rank of A=3.

Observation:

1. We can see that A is matrix of **5** x **4** order; $C(A)\subset\mathbb{R}^5$.

Alternate: **Here we have an alternate method to find Basis for Column space:**

- First convert the given matrix A into Transpose
- Then we find, **Row Echelon** or **Reduced Row Echelon** form of A^T.
- Collect the non-zero rows of A^T; it gives the basis for the Column Space.
- Cardinality of the basis will be its dimension and called Column Rank.

Note: We follow same rule as we used for finding basis for Row spaces, but here for A^T.

Example 1: Find a basis for the column space of matrix $A=\begin{bmatrix}-3&6&-1&1&-7\\1&-2&2&3&-1\\2&-4&5&8&-4\end{bmatrix}$

$$A^T = \begin{bmatrix} -3 & 1 & 2 \\ 6 & -2 & -4 \\ -1 & 2 & 5 \\ 1 & 3 & 8 \\ -7 & -1 & -4 \end{bmatrix} \xrightarrow{\text{Row Echelon Form}} B = \begin{bmatrix} 1 & -1/3 & -2/3 \\ 0 & 1 & 13/5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \leftarrow w_1 \\ \leftarrow w_2 \\ \\ \\ \\ \end{matrix}.$$

Basis for R(A^T)= $\{v_1, v_2\} = \{(1, \ -1/3, \ -2/3), (0, \ 1, \ 13/5)\}$

or $\{[1, \ -1/3, \ -2/3], [0, \ 1, \ 13/5]\}$,

So, **Basis for C(A)**= $\{v_1, v_2\} = \{(1, \ -1/3, \ -2/3), (0, \ 1, \ 13/5)\}$ or $\left\{ \begin{bmatrix} 1 \\ -1/3 \\ -2/3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 13/5 \end{bmatrix} \right\}$,

Dim C(A)=2.

Question: Find Basis and dimension of Row and Column spaces of matrix A= and hence find the Rank of the matrix.

$$A = \begin{bmatrix} 8 & -6 & 2 & 3 \\ -6 & 7 & -4 & 7 \\ 2 & -4 & 3 & 2 \end{bmatrix}$$

Solution

$$A = \begin{bmatrix} 8 & -6 & 2 & 3 \\ -6 & 7 & -4 & 7 \\ 2 & -4 & 3 & 2 \end{bmatrix} \xrightarrow{\textbf{Reduced Row Echelon form}} \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Basis for R(A)= {(1, 0, -1/2, 0), (0, 1, -1, 0), (0, 0, 0, 1)}

Bais for C(A) = {(8, -6, 2), (-6, 7, -4), (3, 7, 2)}

Rank A = Dim R(A) = Dim C(A) $=3$

Note: *In Exam you have to show steps of finding Row Echelon and Reduced Row Echelon forms. In my note, directly written, because already you are learned finding the Row Echelon and Reduced Row Echelon forms in module 1.*

❖ Finding Null Space & Basis for the Null Space of a matrix A:

Let A is an m x n matrix.

- First we solve $AX = 0$ for any $X = (x_1, x_2, x_3, ... x_n) \in \mathbb{R}^n$.
- The solution set of No-trivial solution gives Null Space.
- We can collect vectors after taking free variables common from the vectors. These vectors forms basis for the Null Space.
- Cardinality of the basis will be its dimension and called **Nullity**.

Quesion 1: Find the Null space of matrix A. And hence find its Basis and

$$A=\begin{pmatrix} 1 & -1 & -1 & 3 \\ 1 & 1 & -2 & 1 \\ 4 & -2 & 4 & 1 \end{pmatrix}$$

Soln: To find the null space of matrix A, we need to find solution of homogeneous linear system of eqn. AX=0 for $X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

The augmented matrix of this system is $\left(\begin{array}{cccc|c} 1 & -1 & -1 & 3 & 0 \\ 1 & 1 & -2 & 1 & 0 \\ 4 & -2 & 4 & 1 & 0 \end{array}\right)$

$$\begin{array}{c} R_2 - R_1 \to R_2 \\ \& \\ R_3 - 4R_1 \to R_3 \end{array} \left(\begin{array}{cccc|c} 1 & -1 & -1 & 3 & 0 \\ 0 & 2 & -1 & -2 & 0 \\ 0 & 2 & 8 & -11 & 0 \end{array}\right)$$

$$R_3 - R_2 \to R_3 \left(\begin{array}{cccc|c} 1 & -1 & -1 & 3 & 0 \\ 0 & 2 & -1 & -2 & 0 \\ 0 & 0 & 9 & -9 & 0 \end{array}\right)$$

$$\begin{array}{c} R_2/2 \to R_2 \\ R_3\ /9 \to R_3 \end{array} \left(\begin{array}{cccc|c} 1 & -1 & -1 & 3 & 0 \\ 0 & 1 & -1/2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \begin{array}{l} \Rightarrow x_1 - x_2 - x_3 + 3x_4 = 0 \\ \Rightarrow x_2 - 1/2x_3 - x_4 = 0 \\ \Rightarrow x_3 - x_4 = 0 \end{array}$$

Here, $n-\rho$=4-3=1, so assuming one free variable $x_3 = c$. Using the method of back substitution, we obtain $x_1 = -\frac{1}{2}c,\ x_2 = \frac{3}{2}c,\ x_3 = c,\ x_4 = c$.

Then the solution Space=N(A)= $\left\{\left(-\frac{1}{2}c, \frac{3}{2}c, c, c\right) \middle| c \in \mathbb{R}\right\}$.

$\left(-\frac{1}{2}c, \frac{3}{2}c, c, c\right) = c\left(-\frac{1}{2}, \frac{3}{2}, 1, 1\right) \Rightarrow$,

So, Basis of N(A)= $B_{N(A)} = \left\{\left(-\frac{1}{2}, \frac{3}{2}, 1, 1\right)\right\}$

Nullity=Dim[N(A)]=1. $(\because n-\rho=1)$

Question 2: Find the Null space of matrix A and hence find its Basis and nullity.

$$A=\begin{bmatrix} 4 & -8 & -2 \\ 3 & -5 & -2 \\ 2 & -8 & 1 \end{bmatrix}$$

Solution: To find Null Space N(A) of matrix A, we need to find solution of homogeneous system AX=0, for $X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Now, Solving the homogeneous system AX=0 by Gauss Elimination method –

Augmented Matrix $[A|B]=\left[\begin{array}{ccc|c} 4 & -8 & -2 & 0 \\ 3 & -5 & -2 & 0 \\ 2 & -8 & 1 & 0 \end{array}\right]$

$\frac{R_1}{4} \to R_1$ $\left[\begin{array}{ccc|c} 1 & -2 & -1/2 & 0 \\ 3 & -5 & -2 & 0 \\ 2 & -8 & 1 & 0 \end{array}\right]$

by Dr. Parvez Alam

$R_2-3R_1 \to R_2$ $\left[\begin{array}{ccc|c} 1 & -2 & -1/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 2 & -8 & 1 & 0 \end{array}\right]$

$R_3-2R_1 \to R_3$ $\left[\begin{array}{ccc|c} 1 & -2 & -1/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & -4 & 2 & 0 \end{array}\right]$

$R_3+4R_2 \to R_3$ $\left[\begin{array}{ccc|c} 1 & -2 & -1/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\Rightarrow x_1 - 2x_2 - \frac{1}{2}x_3 = 0$, $\Rightarrow x_2 - \frac{1}{2}t = 0$

by Dr. Parvez Alam

$R=2,\ \rho=2,\ n=3 \Rightarrow \rho = R < n$ (Infinitely many Soln.)

$n-\rho = 3-2 = 1$ free variable, so let $x_3 = t$

So, $x_2 = \frac{1}{2}t \Rightarrow x_1 = \frac{3}{2}t$

So, the solution Space = $N(A) = \left\{ \left(\frac{3}{2}t, \frac{1}{2}t, t\right) \mid t \in \mathbb{R} \right\}$ is Null space of A

$\left(\frac{3}{2}t, \frac{1}{2}t, t\right) = t\left(\frac{3}{2}, \frac{1}{2}, 1\right) \Rightarrow$ $\boxed{\text{Basis of } N(A) = B_{N(A)} = \left(\frac{3}{2}, \frac{1}{2}, 1\right)}$

$\boxed{\text{Nullity of } A = \text{Dim}[N(A)] = 1}$ $(\because n-\rho = 1)$

Ques 3: Find the nullspace of the matrix A. Also find its Basis and dimension

$$A=\begin{bmatrix}1 & 2 & -2 & 1\\ 3 & 6 & -5 & 4\\ 1 & 2 & 0 & 3\end{bmatrix}$$

Solution: The null space of A is the solution space of AX = 0 for X= $\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$

$$[A|0]=\left[\begin{array}{cccc|c}1 & 2 & -2 & 1 & 0\\ 3 & 6 & -5 & 4 & 0\\ 1 & 2 & 0 & 3 & 0\end{array}\right]\xrightarrow{\text{Row Echelon Form}}\left[\begin{array}{cccc|c}1 & 2 & -2 & 1 & 0\\ 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}\right]\begin{array}{l}\Rightarrow x_1+2x_2-2x_3+x_4=0\\ \Rightarrow \qquad\qquad x_3+x_4=0\end{array}$$

$\rho=2,\ n=4,\ n-\rho=4-2=2,$ two free variables. Let $x_4=t\Rightarrow x_3=-t$ & $x_2=s\Rightarrow x_1=-2s-3t$

Then the solution Space=N(A)= $\{(-2s-3t, s, -t, t)\mid s,t\in\mathbb{R}\}$.

$$\begin{aligned}(-2s-3t, s, -t, t)&=(-2s, s, 0, 0)+(-3t, 0, -t, t)\\ &=s(-2,1,0,0)+t(-3,0,-1,1)\end{aligned}$$

So, Basis of N(A)= $B_{N(A)}=\{(-2,1,0,0),(-3,0,-1,1)\}$

Nullity=Dim[N(A)]=2. $(\because n-\rho=2)$

Quesion 4: Find the Null space of matrix A. And hence find its Basis and dimension also.

$$A=\begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & -1 & 1 \end{bmatrix}$$

Soln: To find the null space of matrix A, we need to find solution of homogeneous linear system of eqn. AX=0 for $X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Augmented matrix for sytem AX=0 is

$$\left[\begin{array}{ccc|c} 2 & 1 & 4 & 0 \\ 3 & 2 & 5 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow{\text{Row-Echelon form}} \left[\begin{array}{ccc|c} 1 & 1/2 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\rho = 3$, $R=3$, $n=3$ $\Rightarrow$ $\rho = R = n$ (Unique soln.)

So, by back substitution we get

$x=0$, $y=0$, $z=0$

A. trivial solution.

by Dr. Parvez Alam

$$\boxed{N(A)=\{0\}, \text{ so Dim } N(A)=0}$$

Theorem: (Rank-Nullity Theorem)

Let $A=\left[a_{ij}\right]_{m\times n}$ be an m x n matrix then

1. **Dim R(A) + Dim N(A) = n (number of Column)**

   $\Rightarrow$ **Rank (A) + Nullity (A) = n**

2. **Dim R(A^T) + Dim N(A^T) = m (number of Rows)**

   $\Rightarrow$ **Rank (A^T) + Nullity (A^T) = m**

Note: If AX=0 has only trivial solution then **solution Space=N(A)=** $\{0\}$

Then **Dim N(A) = 0** $\Rightarrow$ **Rank (A)=n.**

Question: Find the Rank of the matrix A and using the Rank-nullity theorem find the dim of Null space of **A** and $\mathbf{A^T}$

$$\mathbf{A}=\begin{bmatrix} 8 & -6 & 2 & 3 \\ -6 & 7 & -4 & 7 \\ 2 & -4 & 3 & 2 \end{bmatrix}_{3\times 4}$$

Solution

$$\mathbf{A}=\begin{bmatrix} 8 & -6 & 2 & 3 \\ -6 & 7 & -4 & 7 \\ 2 & -4 & 3 & 2 \end{bmatrix} \xrightarrow{\text{Reduced Row Echelon form}} \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Basis for R(A)= $\{(1, 0, -1/2, 0), (0, 1, -1, 0), (0, 0, 0, 1)\}$

Rank A= Dim R(A) = 3

Here, m=3 & n=4, so by Rank- Nullity Theorem: **Rank (A) + Nullity (A) = n**

Nullity (A) = n -Rank (A) = 4 - 3 =1

By Rank (A) + Nullity (A^T) = m, we get

Nullity (A^T) = m - Rank (A) = 3 - 3 = 0