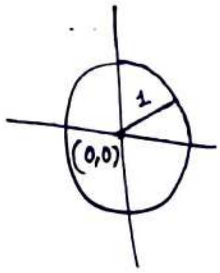
21-7-22

Circles equation in complex plane:

1) Unit circle: $x^2+y^2=1$

$$\Rightarrow \sqrt{x^2+y^2}=1$$

$$\Rightarrow \boxed{|z|=1} \quad \left(\because |z|=\sqrt{x^2+y^2}\right).$$



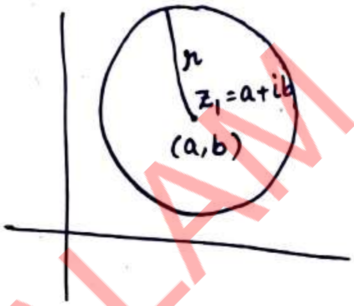
2) Circle in $\mathbb{C}$ plane:

$$(x-a)^2+(y-b)^2=r^2$$

$$\Rightarrow \sqrt{(x-a)^2+(y-b)^2}=r$$

$$\Rightarrow |z-(a+ib)|=r$$

$$\Rightarrow \boxed{|z-z_1|=r}$$

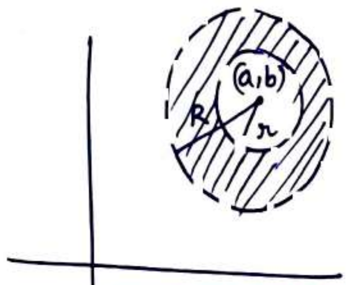


3) ~~Annulus~~ region in $\mathbb{C}$:

Let us have two circles with radius r & R having common centre z_1 such that $r<R$.

So, the region $r<|z-z_1|<R$

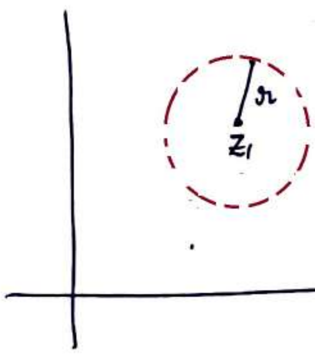
is called annulus region R in $\mathbb{C}$



Boundary points of both circles are not included in the annulus region

Neighbourhood (nbd) of a point:

An open circular disc $|z-z_1|<r$ is called nbd of a point z_1 (centre).



$\rightarrow$ boundary points are not included, so open.

Function of Complex variables (complex function)

A function from a complex set to a complex set is called function of complex variables.

$$\boxed{f(z) = w} \quad \text{where, } \begin{array}{l} z = x+iy \\ w = u+iv \end{array}$$

Ex-1: Let $w = f(z) = z^2 + 3z$ is a complex function. Find u & v if $w = u+iv$. **Also find f(2+3i).**

Sol: $w = f(z) = z^2+3z$

$= (x+iy)^2 + 3(x+iy) \quad (\because z = x+iy)$

$= x^2 - y^2 + 2ixy + 3x + 3iy$

$\therefore \; w = \underbrace{(x^2-y^2+3x)}_{u} + i\underbrace{(2xy+3y)}_{v}$

$\therefore$ Required $\begin{array}{l} u = x^2-y^2+3x \\ v = 2xy+3y \end{array} \Rightarrow f(z) = u+iv$

Now, let us find $f(2+3i)$

Then $x = 2$ and $y = 3$

$\therefore \; f(x+iy) = (x^2-y^2+3x) + i(2xy+3y)$

$\Rightarrow (4-9+6) + i(12+9)$

$\therefore \boxed{f(2+3i) = 1+21i}$

Ex-2: Let $w = f(z) = 2iz + 6\bar{z}$ is a complex function. Find u, v if $w = u + iv$ and $f(2+3i)$

Sol: $w = f(z) = 2i(x+iy) + 6(x-iy)$

$$= \underbrace{(6x-2y)}_{u} + i\underbrace{(2x-6y)}_{v}$$

u= $(6x-2y)$

v= $(2x-6y)$

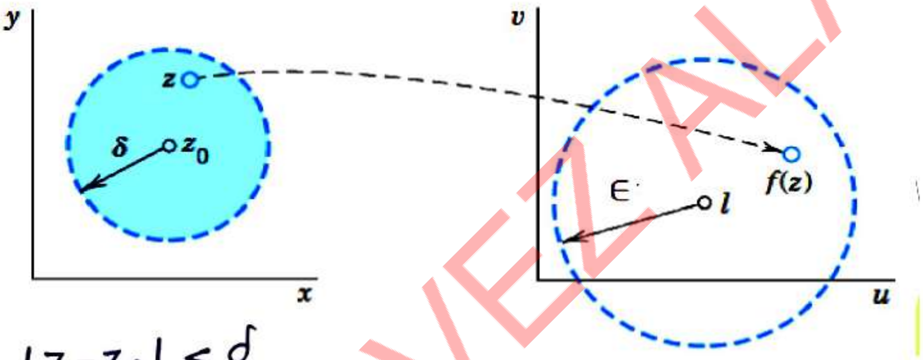
putting $x = \frac{1}{2}$ and $y = 2$, we get

$$w = f(z) = \left[6\left(\tfrac{1}{2}\right) - 2(2)\right] + i\left[2\left(\tfrac{1}{2}\right) - 6(2)\right]$$

$$\therefore \quad w = f\left(\tfrac{1}{2} + 2i\right) = \underline{-1 - 11i}.$$

Limit of a complex function:

The limit of a function $f(z)$ at z_0 is a number l such that $\boxed{\lim_{z \to z_0} f(z) = l}$ (exist)



$|z - z_0| < \delta$

$\boxed{|f(z) - l| < \varepsilon}$

If $\forall$ positive real ε, we can find a positive real δ such that for all $z \neq z_0$, in disc $|z - z_0| < \delta$ we have

$\therefore \boxed{|f(z) - l| < \varepsilon}$

Note: If limit exists, it should be unique.

Ex-1: Find limit of $f(z) = \frac{\bar{z}}{z}$ at $z = 0$.

Sol) $$\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{(x,y)\to(0,0)} \frac{x-iy}{x+iy}$$

$\therefore$ Along x-axis, $\lim_{(x,y)\to(x,0)} f(z) = \frac{x-i0}{x+i0} = \frac{\cancel{x}}{\cancel{x}} = 1$

$\therefore$ Along y-axis, $\lim_{(x,y)\to(0,y)} f(z) = \frac{0-iy}{0+iy} = \frac{-\cancel{iy}}{\cancel{iy}} = -1$

$\therefore$ Along $y = mx$, $\lim_{(x,y)\to(x,mx)} f(z) = \frac{x-imx}{x+imx} = \frac{1-im}{1+im}$

$$\lim_{z\to 0} f(z) = \begin{cases} 1, & \text{along } x\text{-axis} \\ -1, & \text{along } y\text{-axis} \\ \frac{1-im}{1+im}, & \text{along } y = mx \end{cases}$$

Hence, limit doesn't exist.

Since, depending on m, it is not unique.

Note: For the limit in real valued functions $f(x)$, we are approaching along real line (left / right); but for complex form $f(x)$ we are approaching from any direction.

Continuity:

A function is said to be continous at $z = z_0$, if following statements are true.

(i) $f(z_0)$ exists

(ii) $\lim\limits_{z \to z_0} f(z) = f(z_0)$

Ex-①: Check the continuity of $f(z) = \dfrac{z^2+4}{z^2+9}$ at $z = i$ & $z = 3i$

Sol) at $z = i$, $f(i) = \dfrac{-1+4}{-1+9} = \dfrac{3}{8}$

$$\lim_{z \to i} \frac{z^2+4}{z^2+9} = \frac{3}{8}$$

Here, $\lim\limits_{z \to i} f(z) = f(i)$. So $f(z)$ is said to be a continous function at $z = i$.

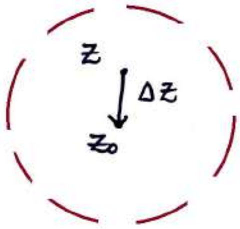
at $z = 3i$, $f(3i)$ = undefined (not existing)

So, at $z = 3i$, $f(z)$ is not continous.

Differentiability:

A function $f(z)$ is said to be differentiable at $z = z_0$ if $\underset{z\to z_0}{lt} \dfrac{f(z)-f(z_0)}{z-z_0}$ exists.

we denote it by $f'(z_0)$ or $\left.\dfrac{df}{dz}\right|_{z=z_0}$



So, $$\boxed{f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}}$$

Here, $z = z_0 + \Delta z$

where, $\Delta z = \Delta x + i\Delta y$

and $z_0 = x_0 + iy_0$

Also, we can write above formula as,

$$f'(x_0) = \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\cancel{z_0}+\Delta z-\cancel{z_0}}$$

$$\therefore \boxed{f'(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}}$$

Note: If $f(x)$ is diff. at each point of domain D then we say that $f(x)$ is diff. in D.

Ex-①: show that $f(z) = |z|$ is diff. at $z = 0$.

Sol) At $z=0$, $f'(z) = f'(0) = \lim_{z\to 0} \dfrac{f(z)-f(0)}{z-0} = \lim_{z\to 0} \dfrac{|z|-0}{z-0}$

$$\therefore f'(0) = \lim_{z\to 0} \frac{|z|}{z} = \lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2+y^2}}{x+iy}$$

Along x-axis, $\lim_{(x,y)\to(x,0)} \dfrac{\sqrt{x^2+0}}{x+i0} = \pm\dfrac{\cancel{x}}{\cancel{x}} = \pm 1$

Along y-axis, $\lim_{(x,y)\to(0,y)} \dfrac{\sqrt{0+y^2}}{0+iy} = \pm\dfrac{\cancel{y}}{i\cancel{y}} = \pm\dfrac{1}{i}$

$\Big\}$ not unique

Here limit doesn't exist, so $f(z)$ is not diff. at $z=0$.

Ex-2: Show that,

$f(z) = z^2$ is differentiable for all $z_0 \in \mathbb{C}$ and has derivative $2z_0$.

Sol)

$f(z) = z^2$

$f(z_0+\Delta z) = (z_0+\Delta z)^2$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0+\Delta z)^2 - z_0^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{\Delta z}(\overset{0}{\cancel{\Delta z}} + 2z_0)}{\cancel{\Delta z}}$$

$$\therefore \quad f'(z_0) = 2z_0 \quad \text{(exists)}.$$

Ex-3: Check the differentiability of $f(z) = \bar{z}$.

Sol) Taking any arbitary point $z_0 \in \mathbb{C}$

$f(z) = \bar{z}$

$f(z_0+\Delta z) = \overline{z_0+\Delta z}$

$= \overline{z_0} + \overline{\Delta z}$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{\overline{z_0}} + \overline{\Delta z} - \cancel{\overline{z_0}}}{\Delta z}$$

$$\Rightarrow \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \begin{cases} 1, & \text{along } x\text{-axis} \\ -1, & \text{along } y\text{-axis} \\ \dfrac{1-im}{1+im}, & \text{along } y = mx \end{cases}$$

Here, limit is not unique, so not diff. in $\mathbb{C}$.