Finding A^n by using Caley Hamilton Theorem

First we find Ch. Eqn. thne replace lembda by A (C.H. Theorem)
Then in f(A) we multilply by A step by step,
and then we replacing last power value of A ....
See the following example.

Q: Find A^4 **for matrix A=** $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Soln: **Ch. Eqn.:** $|A - \lambda I| = \lambda^2 - 5\lambda - 2 = 0 \ldots(1)$

By Caley Hamilton Theorem, we can write

$$A^2 - 5A - 2I = 0$$

$$A^2 = 5A + 2I \quad \ldots\ldots\ldots\ldots\ldots(2)$$

Multiply both sides by A.

$$A^3 = 5A^2 + 2A$$

$$A^3 = 5(5A + 2I) + 2A \quad \text{(Replaced } A^2 \text{ by (2)}$$

$$A^3 = 27A + 10I$$

Since you are finding the fourth power, Multiply the expression by A again.

$$A^4 = 27A^2 + 10A$$

$$A^4 = 27(5A + 2I) + 10A \quad \text{(Replaced } A^2 \text{ by (2)}$$

$$A^4 = 145A + 54I$$

$$A^4 = 145\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 54\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 145 & 290 \\ 435 & 580 \end{bmatrix} + \begin{bmatrix} 54 & 0 \\ 0 & 54 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 190 & 290 \\ 435 & 634 \end{bmatrix} \quad \#$$

Q: For matrix $A = \begin{bmatrix} -2 & 4 \\ -1 & 3 \end{bmatrix}$ then A^5 equals to :

$$|A - \lambda I| = \begin{vmatrix} -2-\lambda & 4 \\ -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - \lambda - 2 = 0$$

By C.H. Theorem $A^2 - A - 2I = 0$ ....(1)

$\Rightarrow A^2 = A + 2I$ ....(2)

Multiplying A in (2) $\Rightarrow A^3 = A^2 + 2A$

$\Rightarrow A^3 = 3A + 2I$ ***(Putting*** A^2 ***by (2)]***

Multiplying A $\Rightarrow A^4 = 3A^2 + 2A$

$\Rightarrow A^4 = 3A + 6I + 2A$ ***(Putting*** A^2 ***by (2)]***

$= 5A + 6I$

Multiplying A $\Rightarrow A^5 = 5A^2 + 6A$

$= 5A + 10I + 6A$ ***(Putting*** A^2 ***by (2)]***

$\Rightarrow A^5 = 11A + 10I$

Computation of Matrix Exponantial e^{At}

Let A be the matrix of order n.

We can take $e^{At} = a_0 I + a_1 A + a_2 A^2 + \cdots + a_{n-1} A^{n-1}$ -- (1)

Let $\lambda = \lambda_1, \lambda_2, \cdots \lambda_n$ are Eigen values of A, then

Putting $A = \lambda_1, \lambda_2 \cdots \lambda_n$ by Caley Hamilton theorem

$$\left[\begin{array}{lll} A = \lambda_1 & \Rightarrow & e^{\lambda_1 t} = a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \cdots + a_{n-1}\lambda_1^{n-1} \\ A = \lambda_2 & \Rightarrow & e^{\lambda_2 t} = a_0 + a_1\lambda_2 + \cdots a_{n-1}\lambda_2^{n-1} \\ & & \text{—— —— ——} \\ A = \lambda_n & \Rightarrow & e^{\lambda_n t} = a_0 + a_1\lambda_n + \cdots a_{n-1}\lambda_n^{n-1} \end{array}\right]$$

We solve system & calculate constants $a_0, a_1, \cdots a_n$.

After putting a_is we get $e^{A\lambda}$ from (1).

Example: Find e^{At} for $A=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}$

Soln: ch. eqn. $|A-\lambda I| = \lambda^2+3\lambda+2=0$

$$\lambda = \underset{\lambda_1}{-1},\ \underset{\lambda_2}{-2}$$

as $n=2$
so upto $n-1$
$=2-1=1$ term

Assuming $e^{At} = a_0 I + a_1 A \quad -(1)$

Putting $A=\lambda_1=-1 \Rightarrow e^{-t} = a_0 - a_1 \quad -(2)$

Putting $A=\lambda_2=-2 \Rightarrow e^{-2t} = a_0 - 2a_1 \quad -(3)$

Solving (2) & (3), we get $a_0 = 2e^{-t} - e^{-2t}$
$a_1 = e^{-t} - e^{-2t}$

After putting a_0 & a_1 in (1) we get

$$e^{At} = (2e^{-t} - e^{-2t})I + (e^{-t} - e^{-2t})A \quad \#$$

or. $$e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

Q (2) Find e^{At} for $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Soln: Ch. eqn. $|A - \lambda I| = \lambda^2 + 1 = 0$

$\lambda_1 = i, \quad \lambda_2 = -i$

Assuming $e^{At} = a_0 I + a_1 A \quad \text{--- (1)}$

Putting $A = i \Rightarrow e^{+it} = a_0 + i a_1$

Putting $A = -i \Rightarrow e^{-it} = a_0 - i a_1$

On solving $a_0 = \frac{e^{it} + e^{-it}}{2} = \cos t$

$a_1 = \frac{e^{it} - e^{-it}}{2i} = \sin t$

So, by (1) $e^{At} = \cos t \, I + \sin t \, A$

$= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$ #