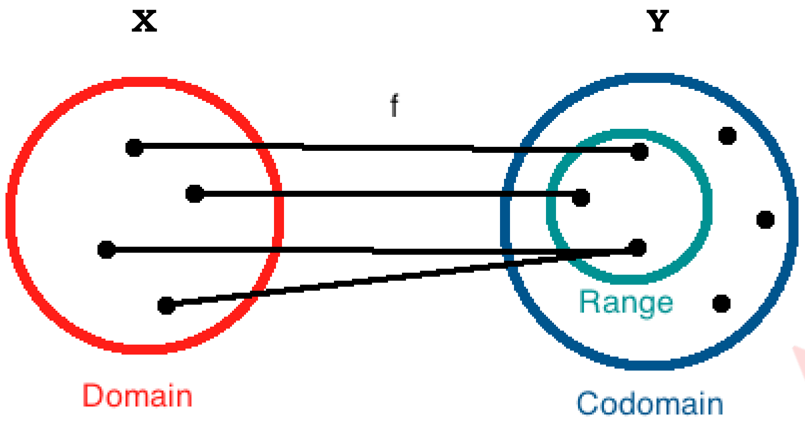
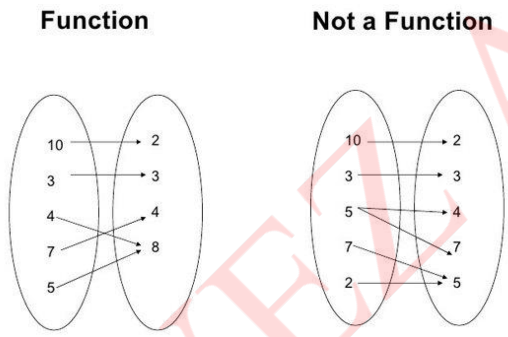
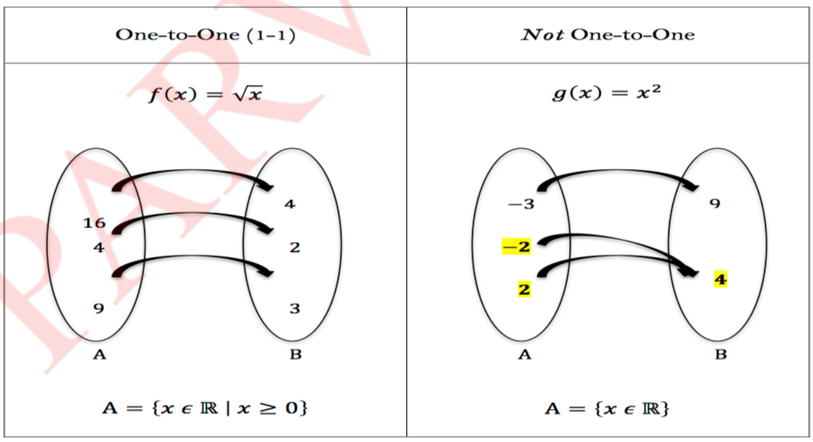
What is function?

A function from a set **X** to a set **Y** assigns to each element of **X** **exactly one element** of **Y**. The set **X** is called the **domain** of the function and the set **Y** is called the **codomain** of the function. Such function is denoted by $f : X \rightarrow Y$







Onto Function

Surjection

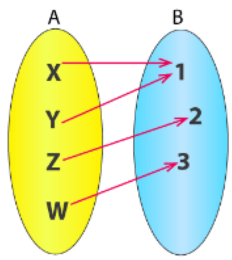


Fig. 1

Not a surjection

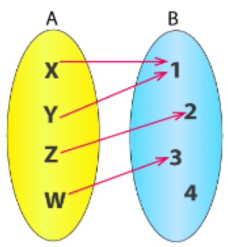


Fig. 2

Linear Transformations

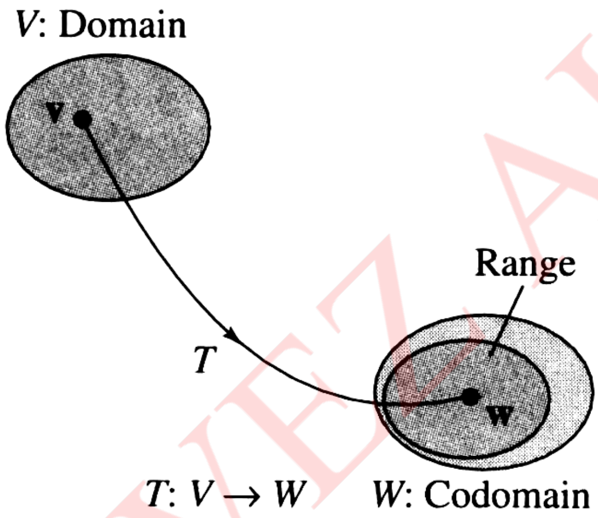
When we take given sets X and Y vectors spaces with some arithmetic rules the function is called **Linear Transform**. And we use T symbol instead of f.

Linear Transform: Let V and W are two vector spaces over R. A function that maps V into W, **T : V → W**, is called a linear transformation from V to W, if it hold following conditions

1. $T(v_1+v_2)=T(v_1)+T(v_2)$ for all $v_1,\ v_2 \in V$
2. $T(\alpha v)=\alpha T(v)$ for all $v \in V$ and for any scalar α.

These two conditions can be combined in a single one as

$T(v_1+\alpha v_2)=T(v_1)+\alpha T(v_2)$ for all $v_1,\ v_2 \in V$ and for any scalar α.



Linear operator: A linear transformation from a vector space V into itself **T : V → V** is called a **Linear operator.**

Example: Verify that $\mathbf{T : R^2 \to R^2}$ given by following is a linear transformation

$$\boxed{T(v_1,v_2)=(v_1-v_2,v_1+2v_2)}$$

Proof: Let $\mathbf{u}=(u_1,u_2),\ \mathbf{v}=(v_1,v_2)$: vector in R^2, c : any real number (scalar)

(1) Vector addition : $\mathbf{u}+\mathbf{v}=(u_1,u_2)+(v_1,v_2)=(u_1+v_1,u_2+v_2)$

$$\begin{aligned}T(\mathbf{u}+\mathbf{v})&=T(u_1+v_1,u_2+v_2)\\&=((u_1+v_1)-(u_2+v_2),(u_1+v_1)+2(u_2+v_2))\\&=((u_1-u_2)+(v_1-v_2),(u_1+2u_2)+(v_1+2v_2))\\&=(u_1-u_2,u_1+2u_2)+(v_1-v_2,v_1+2v_2)\end{aligned}$$

$$T(\mathbf{u}+\mathbf{v})=T(\mathbf{u})+T(\mathbf{v})$$

(2) Scalar multiplication $c\mathbf{u}=c(u_1,u_2)=(cu_1,cu_2)$

$$\begin{aligned}T(c\mathbf{u})&=T(cu_1,cu_2)=(cu_1-cu_2,cu_1+2cu_2)\\&=c(u_1-u_2,u_1+2u_2)\end{aligned}$$

$$T(c\mathbf{u})=cT(\mathbf{u})$$

Therefore, T is a linear transformation.

Ex. (Functions that are not linear transformations)

$(a)\ f(x)=\sin x$

$\sin(x_1+x_2)\neq\sin(x_1)+\sin(x_2) \Leftarrow f(x)=\sin x$ is not linear transformation

$\sin(\frac{\pi}{2}+\frac{\pi}{3})\neq\sin(\frac{\pi}{2})+\sin(\frac{\pi}{3})$

$(b)\ f(x)=x^2$

$(x_1+x_2)^2\neq x_1^2+x_2^2$ $\Leftarrow f(x)=x^2$ is not linear transformation

$(1+2)^2\neq 1^2+2^2$

$(c)\ f(x)=x+1$

$f(x_1+x_2)=x_1+x_2+1$

$f(x_1)+f(x_2)=(x_1+1)+(x_2+1)=x_1+x_2+2$

$f(x_1+x_2)\neq f(x_1)+f(x_2) \Leftarrow f(x)=x+1$ is not linear transformation

Example Consider the following functions:

(1) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$;

(2) $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2 - x$;

(3) $h : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $h(x, y) = (x - y, 2x)$;

(4) $k : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $k(x, y) = (xy, x^2 + 1)$. H.W.

One can easily see that g and k are not linear, while f and h are linear.

(I) $f : \mathbb{R} \to \mathbb{R}$

$f(x) = 2x$

$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

$2(\alpha x + \beta y) = \alpha 2x + \beta 2y$

$2\alpha x + 2\beta y = 2\alpha x + 2\beta y$

L.H.S = R.H.S

So f is L.T.

(II) $g : R \to R$.

$g(x) = x^2 - x$.

$g(\alpha x + \beta y) = \alpha g(x) + \beta g(y)$

$(\alpha x + \beta y)^2 - (\alpha x + \beta y) = \alpha(x^2 - x) + \beta(y^2 - y)$

$\alpha^2 x^2 + \beta^2 y^2 + 2\alpha\beta xy - \alpha x - \beta y = \alpha^2 x^2 - \alpha x + \beta y^2 - \beta y$

L.H.S $\neq$ R.H.S

g is not L.T.

(III) $h : \mathbb{R}^2 \to \mathbb{R}^2$

$h(x, y) = (x - y, 2x)$

$x = (x_1, x_2)$

$y = (y_1, y_2)$

$h(\alpha x + \beta y) = \alpha h(x) + \beta h(y)$

$h[\alpha(x_1, x_2) + \beta(y_1, y_2)] = \alpha h(x_1, x_2) + \beta h(y_1, y_2)$

$h[(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2)] = \alpha(x_1 - x_2, 2x_1) + \beta(y_1 - y_2, 2y_1)$

$[\alpha x_1 + \beta y_1 - \alpha x_2 - \beta y_2, 2(\alpha x_1 + \beta y_1)] = [\alpha(x_1 - x_2), 2x_1\alpha + \beta(y_1 - y_2), 2\beta y_1]$.

$[\alpha(x_1 - x_2) + \beta(y_1 - y_2), 2(\alpha x_1 + \beta y_1)] = [\alpha(x_1 - x_2) + \beta(y_1 - y_2), 2(\alpha x_1 + \beta y_1)]$

L.H.S = R.H.S.

$\therefore$ So h is linear transform.

Exercise: **Which of the following function T is linear transform.**

(1). $T:\mathbb{R}^3 \to \mathbb{R}^2;\quad T(x, y, z) = (|x|, 0)$

(2). $T:\mathbb{R}^2 \to \mathbb{R}^2;\quad T(x, y) = (\sin x, y)$

(3). $T:\mathbb{R}^3 \to \mathbb{R}^3;\quad T(x, y, z) = (x+y, 0, 2x+4z)$ HW

(4). $T:\mathbb{R}^2 \to \mathbb{R}^2;\quad T(x, y) = (x^2-y^2, x^2-y^2)$ HW

Soln:- (1) $T: R^3 \longrightarrow R^2$

$T(x, y, z) = (|x|, 0)$

Let $X = (x_1, y_1, z_1)$ & $Y = (x_2, y_2, z_2) \in R^3$

We need to show for L.T.

$T(\alpha X + \beta Y) \quad \alpha T(X) + \beta T(Y)$

$\Rightarrow T[\alpha(x_1, y_1, z_1) + \beta(x_2, y_2 z_2)] = \alpha T(x_1, y_1, z_1) + \beta T(x_2 y_2 z_2)$

$\Rightarrow T[\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2] = \alpha(|x_1|, 0) + \beta(|x_2|, 0)$

using given rule

$\Rightarrow (|\alpha x_1 + \beta x_2|, 0) \neq (\alpha|x_1| + \beta|x_2|, 0)$

$\because |\alpha x_1 + \beta x_2| \neq \alpha|x_1| + \beta|x_2|$ for all $\alpha, \beta \in R$

Hence T is not L.T. #

Soln (2):

$T: R^2 \longrightarrow R^2$

$T(x, y) = (\sin x, y)$

Let $X = (x_1, y_1)$ & $Y = (x_2 y_2) \in R^2$

Now we need to show that

$T(\alpha X + \beta Y) = \alpha T(X) + \beta T(Y)$

$\Rightarrow T[\alpha(x_1, y_1) + \beta(x_2 y_2)] = \alpha T(x_1 y_1) + \beta T(x_2 y_2)$

$\Rightarrow T[\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2] = \alpha(\sin x_1, y_1) + \beta(\sin x_2, y_2)$

$\Rightarrow [\sin(\alpha x_1 + \beta y_1), \alpha x_2 + \beta y_2] \neq (\alpha \sin x_1 + \beta \sin x_2, \alpha y_1 + \beta y_2)$

we can see $\sin(\alpha x_1 + \beta y_1) \neq \alpha \sin x_1 + \beta \sin x_2$

Hence T is not L.T. #

Example: Show that a mapping from derivative $\left(D \text{ or } \frac{d}{dt}\right)$ and integral $\left(J \text{ or } \int\right)$ mappings over the vector space of real polynomials are linear transform.

Proof:

Consider the vector space $V = \mathbf{P}(t)$ of polynomials over the real field $\mathbf{R}$.

Let $u(t)$ and $v(t)$ be any polynomials in V and let k be any scalar.

(a) Let $\mathbf{D}: V \to V$ be the derivative mapping. One proves in calculus that

$$\frac{d(u+v)}{dt} = \frac{du}{dt} + \frac{dv}{dt} \quad \text{and} \quad \frac{d(ku)}{dt} = k\frac{du}{dt}$$

That is, $\mathbf{D}(u+v) = \mathbf{D}(u) + \mathbf{D}(v)$ and $\mathbf{D}(ku) = k\mathbf{D}(u)$. Thus, the derivative mapping is linear.

(b) Let $\mathbf{J}: V \to \mathbf{R}$ be an integral mapping, say

$$\mathbf{J}(f(t)) = \int_0^1 f(t)\, dt$$

One also proves in calculus that,

$$\int_0^1 [u(t) + v(t)]dt = \int_0^1 u(t)\, dt + \int_0^1 v(t)\, dt$$

and

$$\int_0^1 ku(t)\, dt = k\int_0^1 u(t)\, dt$$

That is, $\mathbf{J}(u+v) = \mathbf{J}(u) + \mathbf{J}(v)$ and $\mathbf{J}(ku) = k\mathbf{J}(u)$. Thus, the integral mapping is linear.

Example Linear transformation, polynomials to polynomials

Define a function $S : P_4 \to P_5$ by

$$S(p(x)) = (x-2)p(x)$$

Then

$$S\left(p(x)+q(x)\right) = (x-2)(p(x)+q(x)) = (x-2)p(x) + (x-2)q(x) = S\left(p(x)\right) + S\left(q(x)\right)$$

$$S\left(\alpha p(x)\right) = (x-2)(\alpha p(x)) = (x-2)\alpha p(x) = \alpha(x-2)p(x) = \alpha S\left(p(x)\right)$$

So by Definition LT, S is a linear transformation. ⊠

Example: Show that a mapping from $M_{m\times n}$ to $M_{n\times m}$ is Linear transform.

$$T : M_{m\times n} \to M_{n\times m}$$

Show that T is a linear transformation.

Sol:

$$A, B \in M_{m\times n}$$

$$T(A+B) = (A+B)^T = A^T + B^T = T(A) + T(B)$$

$$T(cA) = (cA)^T = cA^T = cT(A)$$

Therefore, T is a linear transformation from $M_{m\times n}$ into $M_{n\times m}$.

Let T be a transformation defined by $T : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ x-z \end{bmatrix} \text{ for all } \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$$

Show that T is a linear transformation.

Solution

By Definition 5.1.1 we need to show that $T(k\vec{x}_1 + p\vec{x}_2) = kT(\vec{x}_1) + pT(\vec{x}_2)$ for all scalars k, p and vectors $\vec{x}_1, \vec{x}_2$. Let

$$\vec{x}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

Then

$$\begin{aligned}
T(k\vec{x}_1 + p\vec{x}_2) &= T\left(k\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + p\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) \\
&= T\left(\begin{bmatrix} kx_1 \\ ky_1 \\ kz_1 \end{bmatrix} + \begin{bmatrix} px_2 \\ py_2 \\ pz_2 \end{bmatrix}\right) \\
&= T\left(\begin{bmatrix} kx_1 + px_2 \\ ky_1 + py_2 \\ kz_1 + pz_2 \end{bmatrix}\right) \\
&= \begin{bmatrix} (kx_1 + px_2) + (ky_1 + py_2) \\ (kx_1 + px_2) - (kz_1 + pz_2) \end{bmatrix} \\
&= \begin{bmatrix} (kx_1 + ky_1) + (px_2 + py_2) \\ (kx_1 - kz_1) + (px_2 - pz_2) \end{bmatrix} \\
&= \begin{bmatrix} kx_1 + ky_1 \\ kx_1 - kz_1 \end{bmatrix} + \begin{bmatrix} px_2 + py_2 \\ px_2 - pz_2 \end{bmatrix} \\
&= k\begin{bmatrix} x_1 + y_1 \\ x_1 - z_1 \end{bmatrix} + p\begin{bmatrix} x_2 + y_2 \\ x_2 - z_2 \end{bmatrix} \\
&= kT(\vec{x}_1) + pT(\vec{x}_2)
\end{aligned}$$

- Zero transformation:

$$T : V \to W \qquad T(\mathbf{v}) = 0,\ \forall \mathbf{v} \in V$$

- Identity transformation:

$$T : V \to V \qquad T(\mathbf{v}) = \mathbf{v},\ \forall \mathbf{v} \in V$$

Both Zero trnasform and Identity transforms are Linear Transfrom. (Prove it HW)

Theorem *Let $T : V \to W$ be a linear transformation. Then*

(1) $T(\mathbf{0}) = \mathbf{0}$.

(2) *For any* $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_n \in V$ *and scalars* $k_1,\ k_2,\ \ldots,\ k_n$,

$$T(k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \cdots + k_n\mathbf{x}_n) = k_1T(\mathbf{x}_1) + k_2T(\mathbf{x}_2) + \cdots + k_nT(\mathbf{x}_n).$$

Sometimes it is given that to find LT of a vector, i.e. T(v)=? Then how we can find it? See the next example

Example: [Finding T(v) for given transform]

Let $T: R^3 \to R^3$ be a linear transformation such that

$$T(1,0,0) = (2,-1,4)$$
$$T(0,1,0) = (1,5,-2)$$
$$T(0,0,1) = (0,3,1)$$

Find $T(2, 3, -2)$.

Sol: We can write $(2,3,-2) = 2(1,0,0) + 3(0,1,0) - 2(0,0,1)$

Taking LT of both sides

$$T(2,3,-2) = 2T(1,0,0) + 3T(0,1,0) - 2T(0,0,1) \qquad (T \text{ is a L.T.})$$
$$= 2(2,-1,4) + 3(1,5,-2) - 2\,(0,3,1)$$
$$T(2,3,-2) = (7,7,0)$$

A linear transformation defined by a matrix:

We can represent a Linear Transform in Matrix multiplying from as **T(X)=AX**

Example:

The function $T : R^2 \to R^3$ is defined as $T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

(a) Show that T is a linear transformation form R^2 into R^3

(b) Find $T(\mathbf{v})$, where $\mathbf{v} = (2,-1)$

Sol: (a) $\boxed{T \text{ is a linear transformation form } R^2 \text{ into } R^3}$

(i) $T(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v})$ (vector addition)

(ii) $T(c\mathbf{u}) = A(c\mathbf{u}) = c(A\mathbf{u}) = cT(\mathbf{u})$ (scalar multiplication)

(b) $\boxed{\mathbf{v} = (2,-1)}$

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix}\underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{R^2 \text{ vector}} = \underbrace{\begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}}_{R^3 \text{ vector}}$$

$\therefore T(2,-1) = (6,3,0)$

Therefore in general a linear transformation can be represented as by a matrix transformation.

Matrix representation makes easier a Linear Transform Problems

Theorem: For an $m \times n$ matrix A, the transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by the matrix multiplication

$$\boxed{T(\mathbf{x}) = A\mathbf{x}}$$

is a linear transformation by the distributive law $A(\mathbf{x} + k\mathbf{y}) = A\mathbf{x} + kA\mathbf{y}$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and for any scalar $k \in \mathbb{R}$. Therefore, a matrix A, identified with T, may be considered to be a linear transformation of $\mathbb{R}^n$ to $\mathbb{R}^m$.

- Note:

R^n vector $\qquad$ R^m vector

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}$$

$$\boxed{T(\mathbf{v}) = A\mathbf{v}} \qquad \boxed{T : R^n \longrightarrow R^m}$$

Some standard LT and their Matrix form

We have some important examples of linear transformations like rotations, reflections, and projections in geometry defined in the following example.

Example for rotating a point (x, y) in the plane:

Ex. Show that the L.T. $T : R^2 \to R^2$ given by the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}_{2 \times 2}$$

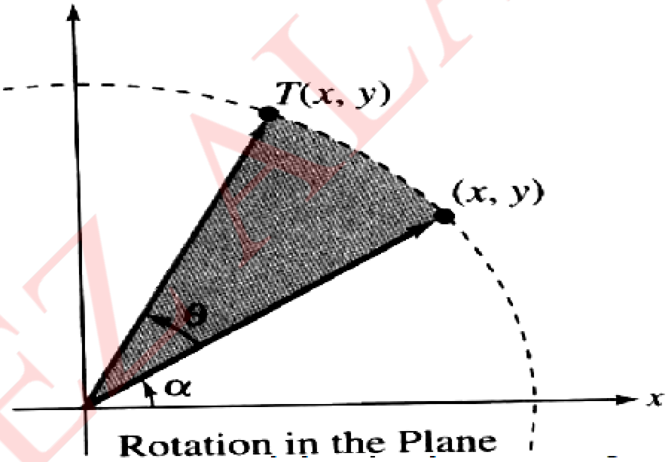
has the property that it rotates every vector in R^2 counterclockwise about the origin through the angle θ.

Sol:

$v = (x, y) = (r\cos\alpha, r\sin\alpha)$

r : the length of v

α : the angle from the positive x-axis counterclockwise to the vector v



Rotation in the Plane

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} r\cos\theta\cos\alpha - r\sin\theta\sin\alpha \\ r\sin\theta\cos\alpha + r\cos\theta\sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} r\cos(\theta+\alpha) \\ r\sin(\theta+\alpha) \end{bmatrix}$$

r : the length of $T(\mathbf{v})$ **[New vector got after rotating θ angle by old vector (x, y)]**

$\theta + \alpha$: the angle from the positive x-axis counterclockwise to the vector $T(\mathbf{v})$

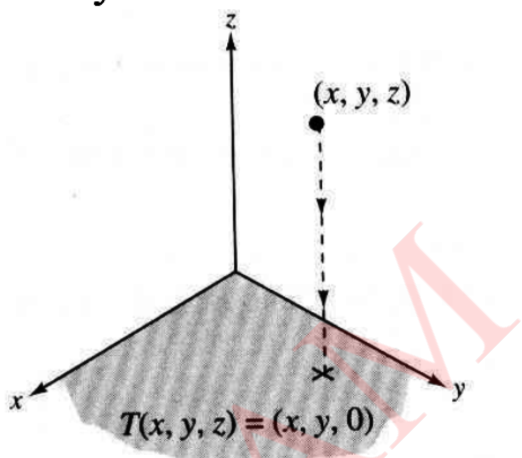
Thus, $T(\mathbf{v})$ is the vector that results from rotating the vector $\mathbf{v}$ counterclockwise through the angle θ.

Example for projection a point in the plane:

(a). Example in R^3

The linear transformation $T : R^3 \to R^3$ is given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Projection onto xy-plane

is called a projection in R^3.

by T(X)=AX= $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$

(b). Example in R^2

The projection on the x-axis is the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by, for $\mathbf{x} = (x, y) \in \mathbb{R}^2$,

A= $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

by T(X)=AX= $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$.

Example for reflection of a point about the axis:

The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by, for $\mathbf{x} = (x, y)$,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\text{by } T(X)=AX = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

This matrix A is reflecting (x, y) point in XY plane about the x-axis and giving point (x, -y)

• Reflection through the **y**-axis: $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$	• Reflection through the line **y=x** $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$
• Reflection through the line **y+x=0** $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$	• Reflection through the origin: $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$

Finding Rule for a Linear Transformations base problem

Ex. Show there is a unique linear map $T: \mathbb{R}^2 \to \mathbb{R}^2$ for which $T(3,1) = (2,-4)$ and $T(1,1) = (0,2)$. Also find the rule of the transform.

Solu. Since $(3,1)$ and $(1,1)$ are linearly independent, they form a basis for $\mathbb{R}^2$; hence such a linear map exists and is unique by Theorem "above theorem statement"

As $\{(3,1), (1,1)\}$ is forming basis ($\because$ L.I. & dim=2) for $T: R^2 \to R^2$

for any $X = (x,y) \in R^2$ domain,

$X = (x,y)$ can be generated by its basis

So,

$$(x,y) = \alpha_1 (3,1) + \alpha_2 (1,1) \quad — ①$$

$$(x,y) = (3\alpha_1 + \alpha_2, \alpha_1 + \alpha_2)$$

On comparing we get

$$3\alpha_1 + \alpha_2 = x$$

$$\alpha_1 + \alpha_2 = y$$

on solving for α_1 & α_2 we get

$$\alpha_1 = \frac{x}{2} - \frac{y}{2} \quad \& \quad \alpha_2 = -\frac{x}{2} + \frac{3}{2}y$$

Now taking L.T. of ①, we get

$$T(x,y) = \alpha_1 T(3,1) + \alpha_2 T(1,1)$$

$$= \left[\frac{x-y}{2}\right] T(3,1) + \left[\frac{-x+3y}{2}\right] T(1,1)$$

$$T(x,y) = \left[\frac{x-y}{2}\right] (2,-4) + \left[\frac{-x+3y}{2}\right] (0,2)$$

$$T(x,y) = (x-y, 5x-3y)$$

is required rule for given L.T.

We can also represent L.T. by means of matrix A as

$$T(X) = AX = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \#$$

Example Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(\mathbf{v}_1) = \mathbf{w}_1, \qquad T(\mathbf{v}_2) = \mathbf{w}_2, \qquad T(\mathbf{v}_3) = \mathbf{w}_3.$$

$$\mathbf{v}_1 = (1,\ 1,\ 1),\ \mathbf{v}_2 = (1,\ 1,\ 0),\ \mathbf{v}_3 = (1,\ 0,\ 0),\ \in \mathbb{R}^3$$

$$\mathbf{w}_1 = (1,\ 0),\ \mathbf{w}_2 = (2,\ -1),\ \mathbf{w}_3 = (4,\ 3) \quad \in \mathbb{R}^2$$

$$\beta = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\} \text{ be basis for } \mathbb{R}^3$$

Find a formula for $T(x_1, x_2, x_3)$, and then use it to compute $T(2,\ -3,\ 5)$.

Here we need to express x as L.C. of basis

Solution Let $X = (x_1, x_2 x_3) \in \mathbb{R}^3$

$X = (x_1\ x_2\ x_3) = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3$

$= \alpha_1 (1, 1, 1) + \alpha_2 (1, 1, 0) + \alpha_3 (1, 0, 0)$

$= (\alpha_1 + \alpha_2 + \alpha_3,\ \alpha_1 + \alpha_2,\ \alpha_1)$

On comparing, we get

$\alpha_1 + \alpha_2 + \alpha_3 = x_1$

$\alpha_1 + \alpha_2 = x_2$

$\alpha_3 = x_3$

we solve & get $\alpha_1 = x_3$

$\alpha_2 = x_2 - x_3$

$\alpha_3 = x_1 - x_2$

Now replacing

$T(x_1\ x_2\ x_3) = x_3 V_1 + (x_2 - x_3) V_2 + (x_1 - x_2) V_3$

Taking L.T. $T(x_1\ x_2 x_3) = x_3\, T(V_1) + (x_2 - x_3) T(V_2) + (x_1 - x_2)\, T(V_3)$

$= x_3 (1, 0) + (x_2 - x_3)(2, -1) + (x_1 - x_2)(4, 3)$

Rule $T(x_1, x_2 x_3) = (4x_1 - 2x_2 - x_3,\ 3x_1 - 4x_2 + x_3)$ #

$\therefore\ T(2, -3, 5) = (9, 23)$

In matrix form we can write as

$$T(X) = \begin{bmatrix} 4 & -2 & -1 \\ 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 - 2x_2 - x_3 \\ 3x_1 - 4x_2 + x_3 \end{bmatrix}$$

#

Que: Show there is a unique linear map $T: \mathbb{R}^2 \to \mathbb{R}$ for which $T(1,1) = 3$ and $T(0,1) = -2$. Find a formula for T. Find $T(8,2)$ and $T(-4,6)$.

Note: If slandered basis is given then finding rule is little easy. See next example

Example Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(\mathbf{e}_1) = \mathbf{w}_1, \quad T(\mathbf{e}_2) = \mathbf{w}_2, \quad T(\mathbf{e}_3) = \mathbf{w}_3.$$

$$\mathbf{w}_1 = (1,\ 0),\ \mathbf{w}_2 = (2,\ -1),\ \mathbf{w}_3 = (4,\ 3) \quad \in \mathbb{R}^2$$

$\alpha = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for the 3-space $\mathbb{R}^3$,

Find a formula for $T(x_1, x_2, x_3)$, and then use it to compute $T(2, -3, 5)$.

Solution:- $T: \mathbb{R}^3 \to \mathbb{R}^2$

Let $X = (x_1, x_2, x_3) \in \mathbb{R}^3$

$\therefore X = (x_1, x_2, x_3) = x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$

$X = (x_1, x_2, x_3) = x_1 e_1 + x_2 e_2 + x_3 e_3$

$\Rightarrow T(X) = x_1 T(e_1) + x_2 T(e_2) + x_3 T(e_3)$

$= x_1 w_1 + x_2 w_2 + x_3 w_3$

$= x_1(1,0) + x_2(2,-1) + x_3(4,3)$

Rule $T(x_1, x_2, x_3) = (x_1 + 2x_2 + 4x_3,\ -x_2 + 3x_3)$ #

$\therefore T(2,-3,5) = (16, 18)$ #

In matrix notation it can be written as

$$T(X) = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + 4x_3 \\ -x_2 + 3x_3 \end{bmatrix}$$