ECON 5803 — Spring Semester 2018: Homework Assignment 3

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Due Date: 2/28, (in class)

Problem 1: Optimal consumption for 3 periods

Consider a household who lives for three periods and whose preferences are represented by the following utility function

$$U(c_0, c_1, c_2) = \frac{c_0^{1-\theta}}{1-\theta} + \beta \frac{c_1^{1-\theta}}{1-\theta} + \beta^2 \frac{c_2^{1-\theta}}{1-\theta} \quad , \tag{1}$$

where $0 < \beta < 1$, $\theta > 0$, $\theta \neq 1$, and $c_t \geq 0$ for $t = 0, 1, 2$. In the first and second period of life the household receives income $y_0 = y_1 = \bar{y} > 0$ and nothing in the final period of life, i.e., $y_2 = 0$. The household may save and borrow at the real interest rate r in order to transfer income across periods. Given this environment, do the following:

(a) Verify that the period utility function $u(\cdot)$ satisfies the assumptions made in class: $u'(c) > 0$, $u''(c) < 0$. Why are these assumptions important?

(b) Solve for the household's lifetime budget constraint.

(c) Express the household's maximization problem as a Lagrangian function. Derive the associated necessary conditions.

(d) Suppose $\beta(1 + r) = 1$: Find the level of consumption in each period. That is, find the exact optimal values for c_0, c_1, c_2, ..., as a function of exogenous variables only.

(e) **This is a bonus question and not required for full credit:** It turns out that for the special case in which $\theta \to 1$ the utility function considered in this problem is equivalent to $u(c) = \ln(c)$. To show this do the following steps. First, subtract 1 from the numerator, i.e., write the utility function as $\tilde{u}(c) = \frac{c^{1-\theta}-1}{1-\theta}$. The transformed utility function $\tilde{u}$ will yield the exact same optimal choices as $u(c)$, and thus, the two utility functions represent the same preferences regarding consumption. If this is not obvious to you convince yourself by solving the utility maximization problem with $\tilde{u}(c)$ instead of $u(c)$. Now show that $\lim_{\theta \to 1} \tilde{u}(c) = \ln(c)$. (Hint: You will have to use de l'Hôpital's rule to compute this limit. In fact, this is the only reason why we subtract 1 from the numerator. Otherwise we couldn't compute the limit.)

Problem 2: Interest rate vs. subjective discount factor

Now consider a household who lives forever (a dynasty if you will). The household's preferences are given by

$$U(\{c_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t \ln(c_t) \quad \text{with } 0 < \beta < 1, \text{ and } c_t \geq 0 \text{ for } t = 0, 1, 2, ... \tag{2}$$

The household receives income $y_t = \bar{y}$ in every period. The household may save and borrow at the real interest rate r in order to transfer income across periods. Given this environment, do the following:

(a) Verify that the period utility function $u(\cdot)$ satisfies the assumptions made in class: $u'(c) > 0$, $u''(c) < 0$.

(b) Assume that the household cannot run a Ponzi scheme. Solve for the household's lifetime budget constraint.

(c) Express the household's maximization problem as a Lagrangian function. Derive the associated associated necessary conditions.

(d) Suppose $\beta(1 + r) = 1$: Find the level of consumption in each period. That is, compute the exact optimal values for c_t for all $t = 0, 1, 2, 3, ...$

(e) Suppose $\beta(1 + r) > 1$: Describe (don't actually solve the model) the implied path of consumption. Explain your answer. (Hint: You may want to use the Euler equation for consumption to support your arguments. In fact, for this exercise it is instructive to plot c_{t+1}^* as a function of c_t^* in the (c_t, c_{t+1}) space.)

(f) Suppose $\beta(1+r) < 1$: Describe (don't actually solve the model) the implied path of consumption. Explain your answer.

(g) Does the time path (not the level) of consumption depend on the particular path of income, $\{y_t\}_{t=0}^{\infty}$? For example, would your answers to (d)-(f) be any different if the income stream was strictly increasing (instead of flat): $0 < y_0 < y_1 < y_2 < ...$? Would your answers change if you only made money in the first period and zero in all other periods? Explain your answer.

Problem 3: Temporary vs. Permanent shocks to income

Again, consider a household who lives forever (a dynasty if you will). The household's preferences are given by

$$U(\{c_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{with } c_t \geq 0 \text{ for } t = 0, 1, 2, ..., \tag{3}$$

where the period utility function is monotonically increasing and strictly concave in c. The household optimally chooses its sequence of consumption $\{c_t\}_{t=0}^{\infty}$, with $c_t \geq 0$ for all t, based on the sequence of earnings $\{y_t\}_{t=0}^{\infty}$, with $y_t > 0$ for all t. The household may borrow/lend at the real interest rate r. Assume that the household's patience is perfectly reflected by that of the market, and thus $\beta = (1+r)^{-1}$.

(a) Solve for the household's optimal sequence of consumption, $\{c_t^*\}_{t=0}^{\infty}$. (Feel free to refer to some results from the previous problems where appropriate. If you choose to do so, please clearly reference which results you are referring to.)

(b) Suppose the government gives the household a temporary transfer in period $t = 0$ of size $dy_0 = 1$. (Think of this payment as something like the G. W. Bush tax refund check.) By how much will optimal consumption change if the real interest rate is 1%?

(c) Suppose the household gets promoted at time $t = 0$ and receives a permanent pay-raise of size $dy_t = 1$ for all t. By how much will optimal consumption change if the real interest rate is 1%?

(d) Based on your results from (b) and (c), what do you conclude about temporary vs. permanent shocks to income?

Before attempting this problem set, please carefully read the reading labeled "Economic Growth with Endogenous Consumption (Obstfeld & Rogoff)". In part I of this problem set you still study precisely the model described in that reading. This is the same model as the one we solved in class but here we assume both population and technology growth.

Problem 4: Competitive Equilibrium in the Neoclassical Growth Model

In this problem we will verify that, within the environment of the Ramsey-Cass-Koopmans growth model, a competitive equilibrium (think "free markets") can indeed achieve the same equilibrium outcome as a centrally planned economy. This involves the following steps (here I just list the steps but I give more detailed instructions for how to do that further down):

1. Rewrite the model under the following two assumptions:
   (a) Households own capital and labor and their only income is from renting/selling these resources to firms at competitive prices
   (b) Firms hire labor and rent capital from the households at competitive prices in order to maximize their profits
2. Solve the model in this new environment. This entails the following steps:
   (a) Solve the firms' profit maximization problem to characterize the demand for labor and capital.
   (b) Solve the households' utility maximization problem to characterize the demand for consumption as well as the supply of capital and labor.
   (c) Impose that supply needs to equal demand in all markets
3. Check whether the equilibrium in the competitive environment actually yields the same solution as the central planner's problem that we solved in class (and Obstfeld & Rogoff solve in the reading).

Firms

We'll start with the firms' problem since this is the simpler one to solve. Essentially the problem is the same as our "basic model of production" from early on in the class. In every period firms maximize their period profits by choosing capital and labor:

$$\max_{K_t, L_t} F(K_t, A_t L_t) - w_t L_t - r_t K_t \tag{4}$$

Notice that I wrote the maximization problem in terms of the household capital stock K_t. Since all firms are assumed to be identical and the production function exhibits constant returns to scale, it doesn't matter what level of aggregation we choose to write down the firms' problem. Writing it at the household level is convenient, as it allows us to characterize equilibrium wages. If this is not obvious to you, convince yourself that your answer to question (a) below is independent of whether you write the problem in terms of capital per worker $k_t = \frac{K_t}{L_t}$, capital per effective worker $k_t^E = \frac{K_t}{A_t L_t}$, or at the aggregate level $K_t^A = H K_t$, where $H \geq 1$ is the number of households in the economy.

(a) Derive the first order condition implicitly characterizing the demand for capital.

(b) Derive the first order condition implicitly characterizing labor demand.

(c) Use Euler's formula to show that with a constant returns to scale production function profits need to be zero.

(d) Use the fact that profits are zero to show that, at the firms' optimal choice, wages are determined by

$$w_t = A_t \left[f(k_t^E) - f'(k_t^E) k_t^E \right],$$

where $k_t^E \equiv \frac{K_t}{A_t L_t}$ and $f(k_t^E) = \frac{F(K_t, A_t L_t)}{A_t L_t}$ is the production function in in units of effective labor.

Households

Like in reading, the household planner seeks to maximize the household's lifetime utility. Again, each household owns capital K_t and each of the L_t household members own one unit of time to perform productive labor services. What's new here is that there is no social planner who optimally allocates aggregate resources. Here the households need to sell their labor the firms (at the competitive wage rate w_t) and they may rent their capital to the firms (at competitive rental rate r_t) in order to finance their consumption. Thus, the household's budget constraint in every period is given by

$$K_{t+1} + C_t = w_t L_t + (1 + r_t) K_t, \tag{5}$$

where the left hand side is the household's total spending and the right hand side is the household's total income in period t.

Remember that the household seeks to maximize its lifetime utility

$$U = L_0 \sum_{t=0}^{\infty} \beta^t (1 + n)^t u(c_t), \tag{6}$$

where $c_t \equiv \frac{C_t}{L_t}$ represents the consumption of each individual within the household and the number of people within each household L_t grows at the constant rate $(1 + n) > 0$. L_0 is the exogenously given initial number of household members.

(e) Show that the household budget constraint in per worker terms is given by $(1 + n)k_{t+1} + c_t = w_t + (1 + r_t)k_t$, where $k_t \equiv \frac{K_t}{L_t}$ represents capital per household member.

(f) Write the household's optimization problem as a constrained optimization problem. Be precise about what the household takes as given each period and what it needs to choose.

(g) Transform the constrained optimization problem into an unconstrained optimization problem by writing a Lagrangian.

(h) Show that the inter-temporal Euler equation that characterizes the optimal path of consumption for the household is given by $u'(c_t) = \beta(1 + r_{t+1})u'(c_{t+1})$. Interpret this equation. How does this Euler equation differ from the one that the social planner follows (i.e. the one we derived in class)?

Market Clearing

In principle, the next step would be to set supply equal to demand for all goods. Since we are working with the two implicit functions $F(\cdot,\cdot)$ and $u(\cdot)$ we can't actually solve for the demand and supply explicitly. The only supply function we know explicitly is the one for labor, since it is exogenously given by L_t. However, despite the fact that we can't literally set supply equal to demand, we can enforce that the conditions that characterize supply and demand for both capital and labor need to be jointly satisfied.

(h) Show that the first order condition from part (a), characterizing the the demand for capital, combined with the Euler equation from part (h), characterizing the supply of capital, implies that the following Euler equation needs to hold in general equilibrium:

$$u'(c_t) = [1 + f'(k^E_{t+1})]\beta u'(c_{t+1}) \tag{7}$$

This is precisely the Euler equation that characterized the solution to the social planner's problem. Therefore, we confirmed that a competitive environment with optimizing households and firms leads to the same outcome as the one that the social planner would have chosen. This finding is generally true for competitive equilibiria and is known as the first fundamental theorem of welfare economics.

Problem 5: Optimal Choice of Labor Supply

Suppose there is no population growth and no technology growth. A representative individual chooses sequences of consumption and hours worked in order to maximize lifetime utility $U = \sum_{t=0}^{\infty} \beta^t u(c_t, 24 - h_t)$, where c_t is the individual's amount of consumption and h_t are the hours the individual is working at time t. The consumer's utility function is increasing in both arguments, i.e. the household likes consumption and leisure, $\ell_t = 24 - h_t$. The individual receives income from working at the hourly wage rate w_t and savings, which pay period interest r_t . Thus, the individual's budget constraint is given by $c_t + s_{t+1} = w_t h_t + (1 + r_t)s_t$.

(a) Write the consumer's optimization problem. Be careful to state explicitly what she is choosing and what she takes as given.

(b) Derive the first order condition for the optimal choice of hours/leisure. Interpret this condition.

(c) Derive the Euler equation characterizing the optimal consumption/savings decision. In what way does this Euler equation differ from the one you obtained in Part I? Explain.

(d) Redo parts (a) through (c) under the assumption that both prices, w_t and r_t, are $i.i.d$ random variables whose distribution (but not individual future values) are known to the consumer. Further assume that the household now maximizes the expected value of its lifetime utility, that is $U = E_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t, 24 - h_t)\right]$. Which of the results change, which don't? Explain why.