STA237 Notes

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1 Introduction

1.1 Basic Definitions

1. Scientific Question - A question created by an experimenter.
2. Experiment - A task to collect information in order to answer a scientific question.
3. Sample Space (Ω) - The set of all possible outcomes or results of an experiment.
   For example, $\Omega = \{H, T\}$ is the sample space of tossing a coin.
4. Subsets of the sample space are called events.
   Events all use typical set operations (complements, union, intersection, etc.).

1.2 Properties of Events

1. We call events A, B mutually exclusive if A, B have no outcomes in common. That is, $A \cap B = \emptyset$
2. **Demorgan's Law** - For any two events A, B, we have $(A \cup B)^c = A^c \cap B^c$, and $(A \cap B)^c = A^c \cup B^c$.
3. A **Probability Function** (P) on a finite sample space Ω assigns to each event in A in Ω a number $P(A)$ in $[0, 1]$ such that:

   (a) $P(\Omega) = 1$, and

   (b) $P(A \cup B) = P(A) + P(B)$, if A, B are disjoint.
   The number $P(A)$ is the probability for which A occurs.

   Suppose we had two events A, B, and $P(A) \cap P(B) \neq \emptyset$. We have:

   (a) Elements of ONLY A: $A \cap B^c$

   (b) Elements of A AND B: $A \cap B$

   (c) Elements of ONLY B: $B \cap A^c$

   Then:

   (a) $P(A) = P(A \cap B^c) + P(A \cap B)$

   (b) $P(B) = P(B \cap A^C) + P(A \cap B)$

   (c) $P(A \cup B) = P(A \cap B^c) + P(A \cap B) + P(B \cap A^c)$
   Then: $P(A \cup B) = P(A) - P(A \cap B) + P(A \cap B) + P(B) - P(A \cap B)$
   $= P(A) + P(B) - P(A \cap B)$

   Therefore, we have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

   We know that $P(A) \subseteq P(\Omega)$, and the complement A^c is mutually exclusive. $P(\Omega) = 1$, and thus:

   $$P(\Omega) = 1 = P(A^c) + P(A)$$

   Therefore: $P(A^c) = 1 - P(A)$.

4. A and B are **independent** if $P(A \cap B) = P(A) \cdot P(B)$.

1.2.1 Axioms

Suppose Ω is a sample space associated with an experiment. To every event A in Ω, we assign a number $P(A)$ (called the probability of A), so that the following axioms hold:

1. Axiom 1: $P(A) \geq 0$
2. Axiom 2: $P(S) = 1$
3. Axiom 3: If $A_1, A_2, ..., A_n$ form a sequence of pairwise mutually exclusive events in Ω (that is, $A_i \cap A_j = \emptyset$ if $i \neq j$), then
$$P(A_1 \cup A_2 \cup ... \cup A_n) = \sum_{i=1}^{n} P(A_i)$$

1.3 Tools for Counting Sample Points

With m elements $a_1, a_2, ..., a_m$, and $b_1, b_2, ..., b_n$, it is possible to form $mn = m \times n$ pairs containing one element from each group.

An ordered arrangement of r distinct objects is called a **permutation**. The number of ways of ordering n distinct objects taken r at a time will be designated by the symbol P_r^n. That is:

$$P_r^n = n(n-1)(n-2)...(n-(r+1)) = \frac{n!}{(n-r)!}$$

The number of unordered subsets of size r chosen (without replacement from n available objects is:

$$\binom{n}{r} = \frac{P_r^n}{r!} = \frac{n!}{r!(n-r)!}$$

Sometimes it is denoted as C_r^n.

2 Conditional Probability

Conditional probability is the likelihood of an event occurring based on the occurrence of a previous event. That is, for two events R, L, the conditional probability of R given L is $P(R|L)$.
It is denoted by:

$$P(A|C) = \frac{P(A \cap C)}{P(C)},$$

provided $P(C) > 0$.

Note that $P(R|L) + P(R^c|L) = 1$:

$$\begin{aligned} P(R|L) + P(R^c|L) &= \frac{P(A \cap C}{P(C)} + \frac{P(A^c \cap C}{P(C)} \\ &= \frac{P(C)}{P(C)} && \text{Since } P(A), P(A^c) \text{ are mutually exclusive, the union of the intersections is } P(\\ &= 1 \end{aligned}$$

For example, suppose we had the following events:

1. L: Born in a long month (31 days)
$L = \{Jan, Mar, May, Jul, Aug, Oct, Dec\}$;

2. R: Born in a month with letter r
   $R = \{Jan, Feb, Mar, Apr, Sep, Oct, Nov, Dec\}$

This means that the conditional probability of R given L is:

$$\begin{aligned} P(R|L) &= \frac{1/3}{7/12} \\ &= \frac{4}{7} \end{aligned}$$

2.0.1 Multiplication Rule

For any events A, C:

$$\begin{aligned} P(A|C) &= \frac{P(A \cap C)}{P(C)} \\ P(A \cap C) &= P(A|C) \cdot P(C) \end{aligned}$$

2.1 Independent Events

Events A, C are **independent** if and only if the probability of A is the same when we know that C has occurred. That is:

$$P(A|C) = P(A)$$

Then:

$$\begin{aligned} \frac{P(A \cap C)}{P(C)} &= P(A) \\ P(A \cap C) &= P(A) \cdot P(C) \end{aligned}$$

2.2 Partitions

For some positive integer k, let the sets $B_1, B_2, ..., B_k$ be such that:

1. $\Omega = B_1 \cup B_2 \cup ... \cup B_k$
2. $B_i \cap B_j = \emptyset$, for $\imath \neq j$.

Then, the collection of sets $\{B_1, B_2, ..., B_k\}$ is said to be a partition of Ω.

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STA237 Exercises

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3 Exercises

3.1 Probability

1. Suppose $P(A) = 0.5, P(A \cap B) = 0.2$, and $P(A \cup B) = 0.7$. Find:

   (a) $P(B)$

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ 0.7 &= 0.5 + P(B) - 0.2 \\ 0.7 &= 0.3 + P(B) \\ P(B) &= 0.4 \end{aligned}$$

   (b) $P(\text{exactly one of two events occurs})$
   This means the probability of no elements from the intersection. Let this area be $P(O)$.

$$\begin{aligned} P(A \cup B) &= P(O) + P(A \cap B) \\ 0.7 &= P(O) + 0.2 \\ P(O) &= 0.5 \end{aligned}$$

   (c) $P(\text{neither event occurs})$
   This means the probability of no elements in A, B. Call this probability $P(X)$.

$$\begin{aligned} P(X) &= P(\Omega) - P(A \cup B) \\ &= 1 - 0.7 \\ &= 0.3 \end{aligned}$$

2. Suppose we throw a coin two times. The sample space for this experiment is:

$$\Omega = \{H, T\} \times \{H, T\} = \{(H, H), (H, T), (T, H), (T, T)\}$$

   If it is a fair coin, then all four outcomes have equal possibilities:

$$P((H, H)) = P((H, T)) = P((T, H)) = P((T, T)) = \frac{1}{4}$$

   What is the probability of getting at least one tail?
   Looking at the sample space, the probability is $\frac{3}{4}$.
   Alternatively, we can solve as follows, where RP represents the required probability:

$$\begin{aligned} P(\Omega) &= RP + P(HH) \\ 1 &= RP + P(HH) \\ &= RP + P(H)P(H) && \text{Since it is independent.} \\ &= RP + \frac{1}{2} \cdot \frac{1}{2} \\ 1 &= RP + \frac{1}{4} \\ RP &= \frac{3}{4} \end{aligned}$$

   Suppose we wanted to throw three coins, what would happen to the sample space?

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Then, there are $2 \cdot 2 \cdot 2 = 2^3$ possibilities (to keep on with this pattern, flipping a coin 6 times would have 2^6 possibilities, and so on).
What is the probability of at least one head appearing in a 3-coin toss?

$$\begin{aligned}
P(X) &= 1 - P(TTT) \\
&= 1 - P(T)P(T)P(T) \\
&= 1 - \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \\
&= 1 - \frac{1}{8} \\
&= \frac{7}{8}
\end{aligned}$$

3. A survey of students in a class was asked about their birth order (1 = oldest/only child), and which college of the university they were enrolled in.

Birth Order	1 or Only	2 or More	Total
Arts	Science	34	23
57			
Agriculture	52	41	93
Human Ecology	15	28	43
Other	12	17	30
Total	113	110	223

Suppose we selected a random student from this class.

(a) What is the probability we select a Human Ecology student?
P(Human Ecology) = $\frac{43}{223}$

(b) What is the probability we select a first-born student?
P(First Born) = $\frac{113}{223}$

(c) What is the probability that the student is a first-born and a Human Ecology student?
P(C) = $P(A \cap B) = \frac{15}{223}$

(d) What is the probability that the student is a first born or a Human Ecology student?

$$\begin{aligned}
P(D) &= P(A \cup B) \\
&= P(A) + P(B) - P(\cap B) \\
&= \frac{43}{223} + \frac{113}{223} - \frac{15}{223} \\
&= \frac{141}{223}
\end{aligned}$$

4. A sample space consists of 5 simple events, E_1, E_2, E_3, E_4, E_5.

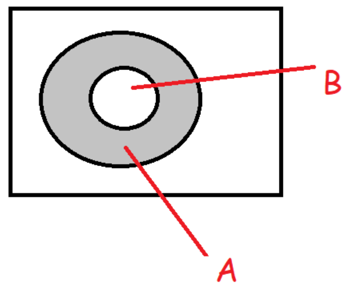
(a) If $P(E_1) = P(E_2) = 0.15$, $P(E_3) = 0.4$, and $P(E_4) = 2P(E_5)$, find the probabilities of E_4 and E_5.

$$\begin{aligned}
1 &= 0.15 + 0.15 + 0.4 + 2P(E_5) + 2P(E_5) \\
0.3 &= 3P(E_5) \\
P(E_5) &= 0.1
\end{aligned}$$

Then:

$$\begin{aligned}
P(E_4) &= 2(0.1) \\
&= 0.2
\end{aligned}$$

5. If A, B are events, $B \subset A$, $P(A) = 0.6$ and $P(B) = 0.2$, then find $P(A \cap B^c)$.
   To visualize:



$$P(A) = P(A \cap B^c) + P(B)0.6 \qquad = P(A \cap B^c) + 0.2$$
$$P(AcapB^c) = 0.4$$

6. An experiment involves tossing a pair of dice and observing the numbers on the upper faces. Find the number of sample points in Ω and the sample space for the experiment The sample points are $\Omega_{\{}m, n\} = \{1, 2, 3, 4, 5, 6\}$.
   The sample space is taking the cross product of the sample points (i.e., $\Omega_1 \times \Omega_2$). Then:

$$\Omega\{(1, 1), (1, 2), ..., (6, 6)\}$$

   There is a $m \cdot n = 6 \cdot 6 = 36$ chance for a single probability.

7. The names of three employees are randomly drawn without replacement from a bowl containing the names of 30 employees of a small company. The person whose names is drawn first receives \$100, and the individuals whose names are drawn second and third receive \$50 and \$25, respectively. How many samples are associated with this experiment? There are 30 employees and 3 people are picked. So, $r = 3, n = 30$. Hence:

$$\begin{aligned}\Omega &= P_3^{30} \\ &= \frac{30!}{(30-3)!} \\ &= \frac{30!}{27!}\end{aligned}$$

8. Find the number of ways of selecting two applicants out of five.
   Here, $n = 5, r = 2$.

$$\begin{aligned}\binom{5}{2} &= \frac{5!}{2!(5-2)!} \\ &= \frac{5!}{2!3!} \\ &= \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} \\ &= 10\end{aligned}$$

3.2 Conditional Probability

1. Let $N = R^c$ be the event ‘born in a month without r’, and L is the event ‘born in a long month’. What is the conditional probability $P(N|L)$?
   $P(R^c|L) = \frac{3}{7}$
   Note that $P(R|L) + P(R^c|L) = \frac{4}{7} + \frac{3}{7} = 1$.

2. A survey asked: 'Are you currently in a relationship?', and 'Are you involved in club sports?'. The survey found that 33% were in a relationship, and 25% were involved in sport. 11% said yes to both. Suppose you meet a student who is on a sports team. What is the probability they are also in a relationship?
   Let A be students in a relationship, and B be students in a club sport.
   $P(A) = 0.33$, $P(B) = 0.25$, $P(A \cap B) = 0.11$. We have:

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{0.11}{0.25} \\ &= 0.44 \end{aligned}$$

   We know that $P(A|B) \neq P(A)$, A and B must be dependent events.

3. A bowl contains 6 red and 4 blue balls, A child selects two balls at random. What is the probability at least one of them is red?
   We have:
   (a) $\{R, R\} = \frac{6}{10}, \frac{5}{9}$
   (b) $\{R, B\} = \frac{6}{10}, \frac{4}{9}$
   (c) $\{B, R\} = \frac{4}{10}, \frac{6}{9}$
   (d) $\{B, B\} = \frac{4}{10}, \frac{3}{9}$

   Note that P(at least one red) $+P$(no red) $= 1$, so:

$$\begin{aligned} P(one\_R) &= 1 - P(no\_R) \\ &= 1 - P(BB) \\ &= 1 - P(B_1 \cap B_2) \\ &= 1 - P(B_2|B_1)P(B_1) \\ &= 1 - \frac{3}{9} \cdot \frac{4}{10} \\ &= 1 - \frac{2}{15} \\ &= \frac{13}{15} \end{aligned}$$