STA237 Notes

Contents

1 Introduction

1.1 Basic Definitions

1. Scientific Question - A question created by an experimenter.
2. Experiment - A task to collect information in order to answer a scientific question.
3. Sample Space (Ω) - The set of all possible outcomes or results of an experiment.
   For example, $\Omega = \{H, T\}$ is the sample space of tossing a coin.
4. Subsets of the sample space are called events.
   Events all use typical set operations (complements, union, intersection, etc.).

1.2 Properties of Events

1. We call events A, B mutually exclusive if A, B have no outcomes in common. That is, $A \cap B = \emptyset$
2. **Demorgan's Law** - For any two events A, B, we have $(A \cup B)^c = A^c \cap B^c$, and $(A \cap B)^c = A^c \cup B^c$.
3. A **Probability Function** (P) on a finite sample space Ω assigns to each event in A in Ω a number $P(A)$ in $[0, 1]$ such that:

   (a) $P(\Omega) = 1$, and
   (b) $P(A \cup B) = P(A) + P(B)$, if A, B are disjoint.
   The number $P(A)$ is the probability for which A occurs.

   Suppose we had two events A, B, and $P(A) \cap P(B) \neq \emptyset$. We have:

   (a) Elements of ONLY A: $A \cap B^c$
   (b) Elements of A AND B: $A \cap B$
   (c) Elements of ONLY B: $B \cap A^c$

   Then:

   (a) $P(A) = P(A \cap B^c) + P(A \cap B)$
   (b) $P(B) = P(B \cap A^C) + P(A \cap B)$
   (c) $P(A \cup B) = P(A \cap B^c) + P(A \cap B) + P(B \cap A^c)$
   Then: $P(A \cup B) = P(A) - P(A \cap B) + P(A \cap B) + P(B) - P(A \cap B)$
   $= P(A) + P(B) - P(A \cap B)$

   Therefore, we have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

   We know that $P(A) \subseteq P(\Omega)$, and the complement A^c is mutually exclusive. $P(\Omega) = 1$, and thus:

   $$P(\Omega) = 1 = P(A^c) + P(A)$$

   Therefore: $P(A^c) = 1 - P(A)$.

4. A and B are **independent** if $P(A \cap B) = P(A) \cdot P(B)$.

1.2.1 Axioms

Suppose Ω is a sample space associated with an experiment. To every event A in Ω, we assign a number $P(A)$ (called the probability of A), so that the following axioms hold:

1. Axiom 1: $P(A) \geq 0$
2. Axiom 2: $P(S) = 1$
3. Axiom 3: If $A_1, A_2, ..., A_n$ form a sequence of pairwise mutually exclusive events in Ω (that is, $A_i \cap A_j = \emptyset$ if $i \neq j$), then
$$P(A_1 \cup A_2 \cup ... \cup A_n) = \sum_{i=1}^{n} P(A_i)$$

1.3 Tools for Counting Sample Points

With m elements $a_1, a_2, ..., a_m$, and $b_1, b_2, ..., b_n$, it is possible to form $mn = m \times n$ pairs containing one element from each group.

An ordered arrangement of r distinct objects is called a **permutation**. The number of ways of ordering n distinct objects taken r at a time will be designated by the symbol P_r^n. That is:

$$P_r^n = n(n-1)(n-2)...(n-(r+1)) = \frac{n!}{(n-r)!}$$

The number of unordered subsets of size r chosen (without replacement from n available objects is:

$$\binom{n}{r} = \frac{P_r^n}{r!} = \frac{n!}{r!(n-r)!}$$

Sometimes it is denoted as C_r^n.

2 Conditional Probability

Conditional probability is the likelihood of an event occurring based on the occurrence of a previous event. That is, for two events R, L, the conditional probability of R given L is $P(R|L)$.
It is denoted by:

$$P(A|C) = \frac{P(A \cap C)}{P(C)},$$

provided $P(C) > 0$.

Note that $P(R|L) + P(R^c|L) = 1$:

$$\begin{aligned} P(R|L) + P(R^c|L) &= \frac{P(A \cap C)}{P(C)} + \frac{P(A^c \cap C)}{P(C)} \\ &= \frac{P(C)}{P(C)} && \text{Since they are mutually exclusive, the union of the intersections is } P(C). \\ &= 1 \end{aligned}$$

For example, suppose we had the following events:

1. L: Born in a long month (31 days)
$L = \{Jan, Mar, May, Jul, Aug, Oct, Dec\}$;

2. R: Born in a month with letter r
   $R = \{Jan, Feb, Mar, Apr, Sep, Oct, Nov, Dec\}$

This means that the conditional probability of R given L is:

$$\begin{aligned} P(R|L) &= \frac{1/3}{7/12} \\ &= \frac{4}{7} \end{aligned}$$

2.0.1 Multiplication Rule

For any events A, C:

$$\begin{aligned} P(A|C) &= \frac{P(A \cap C)}{P(C)} \\ P(A \cap C) &= P(A|C) \cdot P(C) \end{aligned}$$

2.1 Independent Events

Events A, C are **independent** if and only if the probability of A is the same when we know that C has occurred. That is:

$$P(A|C) = P(A)$$

Then:

$$\begin{aligned} \frac{P(A \cap C)}{P(C)} &= P(A) \\ P(A \cap C) &= P(A) \cdot P(C) \end{aligned}$$

2.2 Partitions

For some positive integer k, let the sets $B_1, B_2, ..., B_k$ be such that:

1. $\Omega = B_1 \cup B_2 \cup ... \cup B_k$
2. $B_i \cap B_j = \emptyset$, for $\imath \neq j$.

Then, the collection of sets $\{B_1, B_2, ..., B_k\}$ is said to be a partition of Ω.

2.2.1 The Law of Total Probability

Suppose that $\{B_1, B_2, ..., B_k\}$ is a partitions of Ω such that $P(B_i) > 0$ for $i = 1, 2, ..., k$. Then, for any event A:

$$\begin{aligned} P(A) &= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + ... + P(A|B_k)P(B_k) \\ &= \sum_{i=1}^{k} P(A|B_i)P(B_i) \end{aligned}$$

2.3 Bayes' Theorem

Suppose that $\{B_1, B_2, ..., B_k\}$ is a partition of Ω such that $P(B_i) > 0$, for $i = 1, 2, ..., k$. Then, for any event A:

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^{k} P(A|B_i)P(B_i)}$$

3 Random Variables

Discrete Variables are variables whose values can be measured by counting.
For example, a course mark: 0, 1, 2, ..., 100

Continuous Variables are impossible to count and can never properly be counted.
For example, time or weighs: 25 years, 10, months, ...

Categorical Variables take on a finite number of possible values, assigning units of observation to particular groups on the basis of qualitative properties.

For some event with sample space Ω taking multiple parameters ($e.g., \Omega = \{\sigma_1, \sigma_2\} : \sigma \in \{1, 2\}$), we can calculate the total outcome, i.e., the value of the function $X : \Omega \to \mathbb{R}$, given by:

$$X(\sigma_1, \sigma_2) = \sigma_1 + \sigma_2 \text{ for } (\sigma_1, \sigma_2) \in \Omega$$

We denote the event that the function S attains the value k by:

$$\{X = k\} = \{(\sigma_1, \sigma_2) \in \Omega : X(\sigma_1, \sigma_2) = k\}$$

We call X the **random variable**.
$X : \Omega \to \mathbb{R}$ is a **discrete random variable** if it takes on a finite number of values $a_1, a_2, ..., a_n$, **or** an infinite number of values $a_1, a_2, ...$
The probability that X takes on the value x, $P(X = x)$ is the sum of probabilities of all sample points in Ω that are assigned to the value x (i.e., $P(x) = P(X = x)$). We sometimes denote this as $p(x)$.
Then, the probability distribution of a discrete variable X can be represented by a formula, a table, or a graph that provides $P(X = x)$ for all x.

3.0.1 Result

For any discrete probability distribution, the following must be true:

1. $0 \leq p(x) \leq 1$ for all x
2. $\sum_x p(x) = 1$, where the summation is over all values of x with non-zero probability.

3.1 Expected Values of Random Variables

Let X be a discrete random variable with the probability function $p(x)$. Then, the expected value of X, $E(X)$, is defined as:

$$E(X) = \sum_x xP(x),$$

where $P(x) = P(X - x)$. Note that $E(x) = \mu = \sum_x xP(x)$.

3.1.1 Variance of Random Variables

If X is a random variable with **mean** $E(X) = \mu$, then the variance of a random variable X is the expected value of $(X - \mu)^2$. That is:

$$\sigma^2 = V(X) = E[(-\mu)^2]$$

The **standard deviation** of X is the positive square root of $V(X)$, or σ.

3.1.2 Results

1. Let X be a discrete random variable with probability function $p(x)$, and let c be a constant. Then,

$$\begin{aligned} E(c) &= \sum_x c \sum P(x) \\ &= c \cdot 1 \\ &= c \end{aligned}$$

   Therefore, $E(c) = c$.

2. Note that for the variance:

   (a)

$$\begin{aligned} V(c) &= E((c-\mu)^2) \\ &= E((c-c)^2) \\ &= 0 \end{aligned}$$

   (b)

$$\begin{aligned} V(cX) &= c^2 V(X) \\ V(aX+b) &= a^2 V(X) \end{aligned}$$

3. Let X be a discrete random variable with probability function $p(x)$, $g(x)$ be a function of X, and let c be a constant. Then:

$$\begin{aligned} E(cx) &= cE(x) \\ &= E[ax+b] \\ &= aE(x)+b \end{aligned}$$

   Therefore, $E[cg(X)] = cE(g(X))$.

4. Let X be a discrete random variable with probability function $p(x)$, and $g_1(X), g_2(X), ..., g_k(X)$ be k functions of X. Then:

$$E[g_1(X) + g_2(X) + ...g_k(X)] = E[g_1(X)] + E[g_2(X)] + ... + E[g_k(X)]$$

3.2 Distribution Function

The distribution function F of a random variable X is the function $F : \mathbb{R} \to [0, 1]$, defined by:

$$F(a) = P(X \leq a) \text{ for } -\infty < a < \infty$$

3.3 Bernoulli Distributions

The Bernoulli distribution is used to model an experiment with only two possible outcomes, referred to as a 'success' and 'failure', usually encoded as 1 and 0. A Bernoulli Trial is the term used to describe these experiments.
A discrete random variable X has a Bernoulli distribution with parameter p, where $0 \leq p \leq 1$, if its probability mass function is given by:

$$P(X = 1) = p \text{ and } P(X = 0) = 1 - p$$

We denote this distribution by $Ber(p)$.

3.3.1 Results

1. We calculate the mean μ:

$$\begin{aligned}\mu = E(x) &= \sum_x xP(x) \\ &= 0 \cdot (1-p) + 1 \cdot p \\ E(x) &= p\end{aligned}$$

   Similarly,

$$\begin{aligned}E(x^2) &= \sum_x x^2 P(x) \\ &= 0^2 \cdot (1-p) + 1^2 \cdot p \\ &= p\end{aligned}$$

2. We calculate the variance σ^2:

$$\begin{aligned}\sigma^2 = V(X) &= E(x^2) - \mu^2 \\ &= p - p^2 \\ V(x) &= p(1-p)\end{aligned}$$

For example: Suppose we flip a coin. Heads is a success (S), and Tail is a failure (F).
We have $P(S) = p$, and $P(F) = 1 - p$. We denote X as the number of heads (i.e., $X = 0, 1$).
Then, $P(X = 0) = 1 - p$, and $P(X = 1) = p$.

3.3.2 Probability Mass Functions

A probability mass function (pmf) is a function over the sample space of a discrete random variable X that shows $P(X)$ is equal to a specific value. That is:

$$P(X = x) = p^x(1-p)^{1-x}, \text{ where } x = 0, 1$$

3.4 Binomial Distributions

A discrete random variable X has a binomial distribution with parameters n, p, where $n = 1, 2, ...,$ and $0 \leq p \leq 1$, if its probability mass function is given by:

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \text{ for } x = 0, 1, 2, ..., n,$$

where $\binom{n}{k} = \frac{n!}{(n-x)!x!}$.
We denote this distribution by $B(n, p)$. We also have:

1. $E(X) = np$
2. $V(X) = np(1-p)$

Note that we have $X\tilde{B}(n, p)$

3.4.1 Properties of Binomial Distribution

1. The experiments consist of a fixed number, n, identical trials.
2. Each trial results in one of two outcomes (S, F).
3. $P(S) = p$ for every trial, and $P(F) = 1 - p$.
4. The trials are independent.

3.5 Geometric Distribution

A random variable Y is said to have a **geometric probability distribution** if and only if

$$p(y) = q^{y-1} \cdot p, \text{ where} y = 1, 2, 3, ...; 0 \leq p \leq 1$$

That is, $p(Y) = (1-p)^{y-1} \cdot p$.

This variable Y is the number of trials for which the first success occurs.

3.5.1 Properties of Geometric Distribution

1. The random variable with the geometric probability distribution is associated with an experiment that shares some of the characteristics of a binomial experiment.
2. Each trial has two outcomes, S, F.
3. $P(S) = p, P(F) = 1 - p$.
4. The trials are independent.
5. We are interested in the random variable Y, which is the number of trials on which the first success occurs.

3.5.2 Results of Geometric Probability Distribution

If Y is a random variable with a geometric distribution:

$$\mu = E(Y) = \frac{1}{p} \text{ and } \sigma^2 = V(Y) = \frac{1-p}{p^2}$$

3.6 Hypergeometric Random Variables

The hypergeometric probability distribution is a realistic model for some types of countable data. It has the following characteristics:

1. The experiment consists of randomly drawing n elements wihtout replacement from a set of N elements; r of which are S's, and $N - r$ are F's.
2. The hypergeometric random variable X is the number of S's in the draw of n elements.

Note that both the hypergeometric and binomial characteristics stipulate that each draw or trial results in one of two outcomes. The basic differences between these random variables is that **hypergeometric trials are dependent**, while binomial trials are independent.

3.6.1 Hypergeometric Probability Mass Function

We calculate the pmf of hypergeometric distributions as:

$$P(x) = \frac{\binom{r}{x} \cdot \binom{N-r}{n-x}}{\binom{N}{n}} : x = \max[0, n - (N - r)], ..., \min[r, n],$$

where N is the total number of elements, r is the number of S in N, n is the number of elements drawn, x is the number of S in n.

3.7 Poisson Probability Distribution

For a random variable X, it is said to have a Poisson probability distribution if and only if:

$$p(x) = \frac{\lambda^x e^{-\lambda}}{x!} \text{ for } x = 0, 1, 2, ..., \lambda > 0$$

We have $E(X) = \lambda$ and $V(X) = \lambda$.

4 Continuous Random Variables

A random variable that can take on any value in an interval is called **continuous**, and we can study probability distribution for continuous random variables.

4.1 Distribution Functions

Let Y denote any random variable. The **distribution function** of Y, denoted $F(y)$, is such that $F(y) = P(Y \leq y)$ for $-\infty < y\infty$.

A random variable Y with distribution function $F(y)$ is **continuous** if $F(y)$ is continuous, for $-\infty < y < \infty$.

4.1.1 Properties of Distribution Functions

If $F(y)$ is a distribution function, then:

1. $F(-\infty) = \lim_{y\to\infty} F(y) = 0$
2. $F(\infty) = \lim_{y\to\infty} F(y) = 1$
3. $F(y)$ is a non-decreasing function of y.
   If y_1, y_2 are any values such that $y_1 < y_2$, then $F(y_1) \leq F(y_2)$.

4.2 Probability Density Function

Let $F(y)$ be the distribution function for a continuous random variable Y. Then, $f(y)$, given by:

$$f(y) = \frac{dF(y)}{dy} = F'(y)$$

wherever the derivative exists, is called the **probability density function** for the random variable Y.

4.2.1 Properties of Density Functions

If $f(y)$ is a density function for a continuous random variable, then:

1. $f(y) \geq 0$ for all y, $-\infty < y < \infty$.
2. $\int_{-\infty}^{\infty} f(y)dy = 1$.

4.2.2 Results

If the random variable Y has a density function $f(y)$, and for $a < b$, the probability that Y falls into the interval $[a, b]$ is:

$$P(a \leq y) = \int_a^b f(y)dy$$

4.3 Expected Values for Continuous Random Variables

The expected value for a continuous random variable Y is:

$$E(Y) = \int_{-\infty}^{\infty} yf(y)dy$$

provided that the integral exists.

4.3.1 Results

Let $g(Y)$ be a function of Y. Then, the expected value fo $g(Y)$ is given by:

$$\mu = E[g(y)] = \int_{-\infty}^{\infty} g(y)f(y)dy,$$

provided that the integral exists.

Additionally, let c be a constant and let $g(Y), g_1(Y), g_2(Y), ..., g_k(Y)$ be functions of a continuous random variable Y. Then the following results hold:

1. $E(c) = c$
2. $E(c \cdot g(Y)) = cE(g(Y))$
3. $E(g_1(Y) + ... + g_k(Y)] = E[g_1(Y)] + ... + E[g_k(Y)]$

4.4 Variance in Continuous Random Variables

The variance of a random variable X is defined by:

$$\begin{aligned}\sigma &= V(X) \\ &= E(x-\mu)^2 \\ &= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx\end{aligned}$$

This process takes some time, so we can alternatively calculate this as:

$$V(X) = E(X)^2 - \mu^2$$

Knowing this, we then have $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$.

4.5 Uniform Probability Distribution

If $a < b$, a random variable Y is said to have a continuous **uniform probability distribution** on the interval (a, b) if and only if the density function of Y is:

$$f(y) = \begin{cases} \frac{1}{(b-a)} & a \leq y \leq b \\ 0 & \text{elsewhere} \end{cases}$$

4.5.1 Results

If $a < b$, and Y is a random variable uniformly distributed on the interval (a, b), then:

1. The mean:

$$\begin{aligned}\mu = E(Y) &= \int_{-\infty}^{\infty} yf(y)dy \\ &= \int_a^b y \cdot \frac{1}{(b-a)}dy \\ &= \frac{1}{b-a}\left[\frac{y^2}{2}\right]_a^b \\ &= \frac{b^2-a^2}{2(b-a} \\ &= \frac{a+b}{2}\end{aligned}$$

2. The variance:

$$\begin{aligned}
\mu^2 = E(Y^2) &= \int_a^b y^2 \cdot \frac{1}{b-a} dy \\
&= \frac{1}{b-a}\left[\frac{y^3}{3}\right]_a^b \\
&= \frac{b^3 - a^3}{3(b-a} \\
&= \frac{(b-a)(b^2+ab+a^2}{3(b-a)} \\
&= \frac{a^2+ab+b^2}{3}
\end{aligned}$$

Then:

$$\begin{aligned}
\sigma^2 = V(Y) &= E(Y^2) - \mu^2 \\
&= \frac{a^2+ab+b^2}{3} - \frac{a^2+2ab+b^2}{4} \\
&= \frac{4a^2+4ab+4b^2-3a^2+6ab+3b^2}{12} \\
&= \frac{a^2-2ab+b^2}{12} \\
&= \frac{(b-a)^2}{12}
\end{aligned}$$

4.6 Normal Probability Distribution

A random variable Y is said to have a **normal probability distribution** if and only if, for $\sigma > 0$ and $-\infty < \mu < \infty$, the density function of Y is:

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(\frac{1}{2}(\frac{y-\mu}{\sigma})^2} \ , \ -\infty < y < \infty$$

Then, $Y \sim N(\mu, \sigma)$.

4.6.1 Results

If Y is a normally distributed random variable with parameters μ and σ, then:

1. The mean:

$$E(Y) = \mu$$

2. The variance:

$$V(Y) = \sigma^2$$

However, calculating the integrals of these are extremely different to calculate, so we can standardize normal distributions in order to approximate them.

4.6.2 Standard Normal Distribution

For $Y \sim N(\mu, \sigma)$, we want to find the standard normal distribution Z:

$$Z = \frac{Y - \mu}{\sigma} \sim N(E(Z), V(Z))$$

We calculated the mean and variance:

$$\begin{aligned} E(Z) &= E(\frac{Y-\mu}{\sigma}) \\ &= \frac{1}{\sigma}E(Y-\mu) \\ &= \frac{1}{\sigma}(E(Y)-\mu)) \\ &= \frac{\mu-\mu}{\sigma} \\ &= 0, \end{aligned}$$

and also:

$$\begin{aligned} V(Z) &= V(\frac{V-\mu}{\sigma} \\ &= \frac{1}{\sigma^2}V(Y-\mu) \\ &= \frac{V(Y)}{\sigma^2} \\ &= \frac{\sigma^2}{\sigma^2} \\ &= 1 \end{aligned}$$

Therefore,

$$Z = \frac{Y-\mu}{\sigma} \sim N(0,1)$$

4.7 Normal Approximations of Binomial Distribution

Recall the formula for binomial distributions:

$$P(X = x) = \binom{1000}{x} 0.02^x \cdot 0.98^{n-x}$$

If $X = (n, p)$, and n is large, then the distribution of X is approximately normal with mean np and variance $np(1-p)$. Equivalently, the standardized random variable

$$\frac{X-\mu}{\sigma} \sim N(0,1) \implies \frac{X-np}{\sqrt{np(1-p)}}$$

has an approximate standard normal distribution.

4.8 Gamma Distribution

A random variable Y is said to have a **Gamma distribution** with parameters $\alpha > 0$ and $\beta > 0$ if and only if the density of Y is:

$$f(y) = \begin{cases} \frac{y^{\alpha-1}e^{-y/\beta}}{\beta^\alpha\Gamma(\alpha)} & y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

where:

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}dy$$

Note that the quantity $\Gamma(\alpha)$ is known as the Gamma function.
Using direct integration: $\Gamma(1) = 1$.
Using integration by parts: $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$, for any $\alpha > 1$ and $\Gamma(n) = (n-1)!$ for $n \in \mathbb{Z}$.

4.8.1 Results

1. We calculate the mean μ:

$$\begin{aligned}\mu = E(Y) &= \int_0^\infty \frac{y \cdot y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(x)} dy \\ &= \frac{1}{\beta^\alpha \Gamma(x)} \int_0^\infty y^\alpha e^{-y/\beta}\ dy\end{aligned}$$

   Let $z = \frac{y}{\beta}$. If $y = 0$, then $z = 0$. If $y \to \infty$, then $z \to \infty$.
   Then, we have $y = \beta \cdot z \implies dy = \beta\ dz$.
   Hence:

$$\begin{aligned}E(Y) &= \frac{1}{\beta^\alpha \Gamma(x)} \int_0^\infty (\beta z)^\alpha e^{-z} \beta\ dz \\ &= \frac{\beta}{\Gamma(\alpha)} \int_0^\infty z^{(\alpha+1)-1} e^{-z} dz \\ &= \frac{\beta}{\Gamma(\alpha)} \Gamma(\alpha + 1) \\ &= \frac{\beta\alpha}{\Gamma(\alpha)} \cdot \Gamma(\alpha) \\ \mu &= \alpha\beta\end{aligned}$$

2. We calculate the variance σ^2:

$$\begin{aligned}E(Y^2) &= \int_0^\infty \frac{y^2 y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)} dy \\ &= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty y^{\alpha+1} e^{-y/\beta} dy \\ &= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty (\beta z)^{\alpha+1} e^{-z} \beta\ dz \\ &= \frac{\beta^2}{\Gamma(\alpha)} \int_0^\infty z^{(\alpha+2)-1} e^{-z} dz \\ &= \frac{\beta^2}{\Gamma(\alpha)} \cdot \Gamma(\alpha + 2) \\ &= \frac{\beta^2(\alpha+1)(\Gamma(\alpha+1)}{\Gamma(\alpha)} \\ &= \frac{\beta^2(\alpha+1)(\alpha)\Gamma(\alpha)}{\Gamma(\alpha)} \\ &= \alpha^2\beta^2 + \alpha\beta^2 \\ \sigma^2 = V(Y) &= E(Y^2) - \mu^2 \\ &= \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2 \\ &= \alpha\beta^2\end{aligned}$$

4.9 Exponential Distribution

A random variable Y is said to have an **exponential distribution** with parameter:

$$\beta > 0 \text{ if and only if the density of } Y \text{ is } f(y) = \begin{cases} \frac{1}{\beta} e^{-y/\beta} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

We can then calculate the probabilities using:

$$\begin{aligned} F(y) = P(Y \leq y) &= \int_0^y \frac{1}{\beta} e^{-t/b} dt \\ &= \left[e^{-t/\beta} \right]_y^0 \\ &= 1 - e^{-y/\beta} \\ P(Y > y) &= e^{-y/\beta} \end{aligned}$$

We also have:

$$f(y) = \lambda e^{-\lambda y}$$

4.9.1 Results

1. We calculate the mean μ:

$$\begin{aligned} \sigma = E(Y) &= \int_0^\infty y \frac{1}{\beta} e^{-y/\beta} dy \\ &= \frac{1}{\beta} \int_0^\infty y \cdot e^{-y/\beta} dy \\ &= \frac{1}{\beta} \left(\left[\frac{ye^{-y\beta}}{1/\beta} \right]_\infty^0 - \int_0^\infty \frac{e^{-y/\beta}}{1/\beta} dy \right) \\ &= \left(\left[ye^{y/\beta} \right]_\infty^0 - \int_0^\infty e^{-y/\beta} dy \right) \\ &= 0 - 0 + \left[\frac{e^{y/-\beta}}{1/\beta} \right]_\infty^0 \\ &= \beta[1 - 0] && \text{Using l'Hopital's} \\ &= \beta \end{aligned}$$

2. We calculate the variance σ^2:

$$\begin{aligned} E(Y^2) &= \int_0^\infty y^2 \frac{1}{\beta} e^{-y/\beta} dy \\ &= \frac{1}{\beta} \left(\left[\frac{y^2 e^{-y/\beta}}{1/\beta} \right]_\infty^0 + \int_0^\infty \frac{e^{-y/\beta}}{1/\beta} 2y \; dy \right) \\ &= \left(\left[y^2 e^{-y/\beta} \right]_\infty^0 + \int_0^\infty e^{-y/\beta} 2y \; dy \right) \\ &= 0 + 2\beta \int_0^\infty \frac{ye^- y/\beta}{\beta} dy \\ &= 2\beta E(Y) \\ &= 2\beta^2 \\ \sigma^2 = V(Y) &= E(Y^2) - \mu^2 \\ &= 2\beta^2 - \beta^2 \\ &= \beta^2 \end{aligned}$$

5 Multivariate Probability Distributions

So far, we have only dealt with one-dimensional distributions of experiments and events. However, we can also consider the **weight** as well as the height of the events when conducting experiments.

5.1 Bivariate Probability Distributions

Let Y_1, Y_2 be discrete random variables. the **joint or bivariate probability function** for Y_1, Y_2, is given by:

$$p(y_1, y_2) = P(Y_1 = y_1, Y_2 = y_2), -\infty < y_1 < \infty,\ -\infty < y_2 < \infty$$

We also define the **joint distribution function**:

$$F(y_1, y_2) = P(Y_1 \leq y_1, Y_2 \leq y_2), -\infty < y_1 < \infty,\ -\infty < y_2 < \infty$$

5.1.1 Results

Let Y_1, Y_2 be discrete random variables with joint probability function $p(y_1, y_2)$. Then:

1. $p(y_1, y_2) \geq 0$ for all y_1, y_2.
2. $\sum_{y_1, y_2} p(y_1, y_2) = 1$, where the sum is over all values (y_1, y_2) that are assigned for non-zero probabilities.

5.1.2 Joint Probability Density

Let Y_1, Y_2 be continuous random variables with joint distribution function $F(y_1, y_2)$. If there is a non-negative function $f(y_1, y_2)$ such that:

$$F(y_1, y_2) = \int_{-\infty}^{y_1} \int_{-\infty}^{y_2} f(t_1, t_2) dt_2\ dt_1,$$

for all $-\infty < y_1 < \infty$, and $-\infty, < y_2 < \infty$; then Y_1, Y_2 are said to be **join continuous random variables**. The function $f(y_1, y_2)$ is called the **joint probability density function.**

In other words, we calculate the **volume** of the function in order to calculate the probability density.
It is often easier to draw a cubic surface of the probability function in order to calculate the probability density.

5.2 Marginal Probability Distributions

Let Y_1, Y_2 be jointly discrete random variables with probability function $p(y_1, y_2)$. Then, the **marginal probability functions** of Y_1, Y_2, respectively, are given by:

$$p_1(y_1) = \sum_{y_2} p(y_1, y_2) \text{ and } p_2(y_2) = \sum_{y_1} p(y_1, y_2)$$

We also have Y_1, Y_2 be jointly continuous random variables with joint density function $f(y_1, y_2)$. Then, the **marginal density functions** of Y_1, Y_2, respectively, are given by:

$$f_1(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2) dy_2 \text{ and } f_2(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2) dy_1$$

5.3 Conditional Probability Distribution

If Y_1, Y_2 are jointly continuous random variables with the joint density function $f(y_1, y_2)$, then the **conditional distribution function** of Y_1 given $Y_2 = y_2$ is:

$$F(y_1 | y_2) = P(Y_1 \leq y_2 | Y_2 = y_2)$$

Then, with Y_1, Y_2 as jointly continuous random variables with joint density $f(y_1, y_2)$ and marginal densities $f_1(y_1), f_2(y_2)$. For any y_2 such that $f_2(y_2) > 0$, the conditional density of Y_1 given $Y_2 = y_2$ is given by:

$$f(y_1|y_2) = \frac{f(y_1, y_2)}{f_2(y)2)}$$

Additionally, for any y_1 such that $f_2(y_2) > 0$, the conditional density of Y_2 given $Y_1 = y_1$ is given by:

$$f(y_2|y_1) = \frac{f(y_1, y_2)}{f_1(y_1)}$$

If Y_1, Y_2 are jointly discrete random variables with joint probability function $p(y_1, y_2)$ and marginal probability function $p_1(y_1)$ and $p_2(y_2)$, respectively, then the **conditional discrete probability function** of Y_1 given Y_2 is:

$$p(y_1|y_2) = P(Y_1 = y_1|Y_2 = y_2) = \frac{P(Y_1 = y_1, Y_2 = y_2)}{P(Y_2 = y_2)} = \frac{p_1(y_1, y_2)}{p_2(y_2)}$$

provided that $p_2(y_2) > 0$.

5.4 Independent Random Variables

Let Y_1 have distribution function $F_1(y_1)$, Y_2 have distribution function $F_2(y_2)$, and Y_1, Y_2 have joint distribution function $F(y_1, y_2)$. Then, Y_1, Y_2 are said to be **independent** if and only if:

$$F(y_1, y_2) = F_1(y_1) \cdot F_2(y_2)$$

for every pair of real numbers (y_1, y_2). If Y_1, Y_2 are not independent, they are dependent.

Additionally, if Y_1, Y_2 are discrete random variables with joint probability function $p(y_1, y_2)$, and we have the marginal probability functions $p_1(y_1), p_2(y_2)$, then Y_1, Y_2 are independent if and only if:

$$p(y_1, y_2) = p_1(y_1)p_2(y_2)$$

for all pairs of real numbers (y_1, y_2).

This is also true for continuous random variables iwth joint density function $f(y_1, y_2)$ with marginal density functions $f_1(y_1), f_2(y_2)$.

6 Functions of Random Variables

There are three methods for finding the probability distribution for a function of random variables.

6.1 Distribution Functions Method

Consider the single univariate example: If Y has probability density function $f(y)$ and if U is some function of Y, then we can find $F_u(u) = P(U \leq u)$ directly by integrating over the region for which $U \leq u$.
The probability density function for U is found by differentiating $F_u(u)$.

6.1.1 Example

For example:

A process for refining sugar yields up to 1 ton of pure sugar per day, but the actual amount produced, Y, is a random variable because of slowdowns. Suppose that Y has the density function given by:

$$f(y) = \begin{cases} 2y, & 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

The company is paid at the rate of \$300 per ton for the refined sugar, but also has a fixed overhead cost of \$100 per day. The daily profit in hundreds of dollars, is $U = 3Y - 1$. Find the probability density function for U.

We have:

$$\begin{aligned} F_u(u) &= P(U \leq u) \\ &= P(3Y - 1 \leq u) \\ &= P\left(Y \leq \frac{u+1}{3}\right) \\ &= F_Y\left(\frac{u+1}{3}\right) \end{aligned}$$

Then, $\frac{u+1}{3}$ could be less than 0, in between 0 and 1, or greater than 1.
We check each case:

$$\begin{aligned} F_Y(\frac{u+1}{3}) &= \int_{-\infty}^{\frac{u+1}{3}} f(y)\ dy \\ \frac{u+1}{3} < 0 &\iff u < -1 \\ &\implies F_u(u) = 0 \\ \frac{u+1}{3} > 1 &\iff u > 2 \\ &\implies F_u(u) = 1 \\ \frac{u+1}{3} &\iff -1 \leq u \leq 2 \\ &\implies F_u(u) = \int_0^{\frac{u+1}{3}} 2y\ dy \\ &= \left[y^2\right]_0^{\frac{u+1}{3}} \\ &= (\frac{u+1}{3})^2 \end{aligned}$$

Then, $F_u(u) = \frac{1}{9}(u+1)^2$ if $-1 \leq u \leq 2$. Hence:

$$f_u(u) = \begin{cases} \frac{1}{9}2(u+1), & -1 \leq u \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

6.2 Transformation Method

The transformation method is useful for finding probabilities of random variables using offshoots of the distribution function method.

That is, let Y have a probability density function $f_y(y)$. If $h(y)$ is either increasing or decreasing for all y such that $f_Y(y) > 0$, then $U = h(Y)$ has density function:

$$f_U(u) = f_Y(h^{-1}(u)) \left|\frac{dh^{-1}}{du}\right|,$$

where $\frac{dh^{-1}}{du} = \frac{d(h^{-1}(u))}{du}$.

6.2.1 Example

For example, let:

$$f_Y(y) = \begin{cases} 2y, & 0 \leq y \leq 1, \\ 0, & \text{otherwise} \end{cases}$$

Potentially, $U = 3Y - 1$ (?), so then:

$$\begin{aligned} h(y) &= 3y - 1 = u \\ u &= 3y - 1 \\ \frac{u+1}{3} &= y \end{aligned}$$

We have:

$$\begin{aligned} f_U(u) &= f_Y(y) \mid \frac{dy}{du} \mid \\ &= f_Y(\frac{u+1}{3})\frac{1}{3} \end{aligned}$$

Note that $f_Y(y) = 2y$ if $0 \leq y \leq 1$, so:

$$\begin{aligned} f_U(u) &= 2\frac{u+1}{3} \cdot \frac{1}{3} \\ &= \frac{2}{9}(u+1),\ 0 \leq \frac{u+1}{3} \leq 1 \\ &= \frac{2}{9}(u+1),\ -1 \leq u \leq 2 \end{aligned}$$

6.3 Bivariate Transformation Method

Suppose that Y_1, Y_2 are continuous random variables with joint density function $f_{Y_1,Y_2}(y_1, y_2)$, and that for all (y_1, y_2) such that $F_{Y_1,Y_2}(y_1, y_2) > 0$, then:

$$u_1 = h_1(y_1, y_2) \text{ and } u_2 = h_2(y_1, y_2)$$

Similarly:

$$U_1 = h_1(Y_1, Y_2) \text{ and } U_2 = h_2(Y_1, Y_2)$$

Then:

$$f_{u_1,u_2}(u_1, u_2) = f_{Y_1,Y_2}(y_1, y_2)|J|,$$

where $J \to$ Jacobian, which is equal to:

$$\begin{aligned} J &= \det \begin{bmatrix} \frac{\partial y_1}{\partial u_1} & \frac{\partial y_1}{\partial u_2} \\ \frac{\partial y_2}{\partial u_1} & \frac{\partial y_2}{\partial u_2} \end{bmatrix} \\ &= \frac{\partial y_1}{\partial u_1} \cdot \frac{\partial y_2}{\partial y_1} - \frac{\partial y_2}{\partial u_1} \cdot \frac{\partial y_1}{\partial u_2} \end{aligned}$$

6.3.1 Example

For example: Let Y_1, Y_2 have a joint density function given by:

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & 0 \leq y_1, 0 \leq y_2 \\ 0, & \text{otherwise} \end{cases}$$

Find the density function for $U = Y_1 + Y_2$.

We have $U = U_1 = Y_1 + Y_2$, and so $U_2 = Y_1$. Then:

$$u_1 = y_1 + y_2$$
$$u_2 = y_1$$

$$y_1 = u_2$$
$$y_2 = u_1 - u_2$$

We calculate the partial derivatives:

$$\frac{\partial y_1}{\partial u_1} = 0$$
$$\frac{\partial y_1}{\partial u_2} = 1$$
$$\frac{\partial y_2}{\partial u_1} = 1$$
$$\frac{\partial y_2}{\partial u_2} = -1$$

Then, calculate the Jacovian:

$$\begin{aligned} J &= \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \\ &= (0 \cdot -1) - (1 \cdot 1) \\ &= -1 \end{aligned}$$

Now, we can find $f(u_1, u_2)$:

$$\begin{aligned} f(u_1, u_2) &= f(y_1, y_2)|J| \\ &= e^{-(y_1+y_2)}|-1| \\ &= e^{-(y_1+y_2)} \\ &= e^{-u_1} \end{aligned}$$

To find the range, consider how $y_1 \geq 0$ and $y_2 \geq 0$. Then:

$$\begin{aligned} u_2 \geq 0 \text{ and } u_1 - u_1 \geq 0 &\implies u_2 \geq 0 \text{ and } u_1 \geq u_2 \\ &\implies u_1 \geq u_2 \geq 0 \end{aligned}$$

As an aside: Suppose we wanted to find the marginal density function of u_1. We have:

$$\begin{aligned} f(u_1) &= \int_{-\infty}^{\infty} f(u_1, u_2)\ du_2 \\ &= \int_0^{u_1} e^{-u_1} du_2 \\ &= e^{-u_1}\left[u_2\right]_0^{u_1} \\ &= u_1 e^{-u_1},\ u_1 \geq 0 \end{aligned}$$

7 Sampling Distributions and Central Limit Theorem

7.1 Introduction

Experimental units are objects (e.g., person, thing, transaction, event) about which we collect data.
Populations are sets of all units that we are interested in studying. For example, populations may include:

1. All employed workers in the US.
2. All registered voters in California.
3. Everyone who is afflicted with AIDS.
4. All the cars produced last year by a particular company.

For example, suppose we wanted to consider the population of a university. We have $N = 40000$, and we want to find $Y \rightarrow$ Age of a Student. We can then consider each student's age as $Y_1, Y_2, ..., Y_N$, where Y_N is the last (N'th) student.
To find the population mean μ, we would calculate:

$$\mu = \frac{1}{N}\sum_{i=1}^{N} Y_i$$

We also have the population variance σ^2:

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(Y_1 - \mu)^2$$

Suppose we wanted to take a random sample of the population, i.e., a sample size of $n = 100$. Then, we have the units $y_1, y_2, ..., y_n$.
From this, we can also calculate the sample mean:

$$\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

Similarly, we can calculate the sample variance:

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \overline{y})^2$$

Then, we can bridge the differences between the population and the sample using a **sampling distribution**.

7.2 Order Statistics

Let $Y_1, Y_2, ..., Y_n$ denote independent continuous random variables with distribution function $F(y)$ and density function $f(y)$.
We denote the **ordered random variables** Y_i by $Y_{(1)}, Y_{(2)}, ..., Y_{(n)}$, where $Y_{(1)} \leq Y_{(2)} \leq ... \leq Y_{(n)}$.
The maximum of the random variables is $Y_{(n)} = \max(Y_1, Y_2, ..., Y_n)$.
$Y_{(n)}$ is the maximum of $Y_1, ..., Y_n$, and $\{Y_{(n)} \leq y\}$ will occur if and only if $\{Y_i \leq y\}$ for every $i = 1, 2, ..., n$.
Then:

$$P(Y_{(n)} \leq y) = P(Y_1 \leq y_1, Y_2 \leq y_2, ..., Y_n \leq y)$$
$$P(Y_1 \leq y) \cdot P(Y_2 \leq y) ... P(Y_n \leq y) = P(Y_{(n)} \leq y) = F(Y_{(n)}(y)$$
$$P(Y_i \leq y) = F(y) \text{ for } i = 1, ..., n$$

Knowing these facts, we have the distribution function:

$$\begin{aligned}\prod_{i=1}^{n}(Y_i \leq y) &= (F(y))^n \\ &= F_{Y_{(n)}}(y)\end{aligned}$$

Then, we have the density function:

$$g_{(n)}(y) = n \cdot (F(y))^{n-1} \cdot f(y)$$

The minimum of the random variables $Y_{(1)} = \min(Y_1, Y_2, ..., Y_n)$ can be calculated as:

$$\begin{aligned} F_{Y_{(1)}}(y) &= P(Y_{(1)} \leq y) = 1 - P(Y_{(1)} > y \\ &= 1 - P(Y_1 > y, Y_2 > y, ..., Y_n > y) \\ &= 1 - \prod_{i=1}^{n} P(Y_i > y) \\ &= 1 - \prod_{i=1}^{n} (1 - F(y)) \\ &= 1 - [1 - F(y)]^n \end{aligned}$$

Next, we get that the density function as:

$$g_1(y) = n \cdot (1 - F(y))^{n-1} \cdot f(y)$$

7.3 Summary

For mean:

$$\mu = E(Y) = \frac{7}{2}$$

$$\mu_{\overline{y}} E(\overline{y}) = \frac{7}{2}$$

So, $E(\overline{y}) = \mu$.
For variance:

$$\sigma^2 = \frac{105}{36}$$

$$\sigma^2_{\overline{y}} V(\overline{y}) = \frac{105}{72} = \frac{105}{36 \cdot 2} = \frac{\sigma^2}{n}$$

Then, the sample mean is:

$$\overline{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$$

So:

$$\begin{aligned} \mu_{\overline{y}} &= E(\overline{y}) \\ &= E\left(\frac{1}{n} \sum_{i=1}^{n} y_i\right) \\ &= \frac{1}{n} \sum_{i=1}^{n} E(y_i) \\ &= \frac{1}{n} \sum_{i=1}^{n} \mu \\ &= \frac{n\mu}{n} = \mu \end{aligned}$$

Then:

$$\begin{aligned}
V(\overline{y}) &= V\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} V(y_i) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} \sigma^2 \\
&= \frac{n\sigma^2}{n^2} \\
&= \frac{\sigma^2}{n}
\end{aligned}$$

The standard deviation can be calculated as:

$$SD(\overline{y}) = SE(\overline{y}) = \frac{\sigma}{\sqrt{n}}$$

7.4 Central Limit Theorem

When we have a large number of samples, $\overline{y}$ follows a normal distribution (i.e., $\overline{y} \sim N(\mu, \frac{\sigma}{\sqrt{n}})$, but we want to know the distribution for Y such that $Y \sim (\mu, \sigma)$. This is Case 1, when we know the value for σ.
If σ is unknown, then:

$$\overline{y} \sim N(\mu, \frac{\sigma}{\sqrt{n}}) \implies \frac{\overline{y} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0, 1)$$

Then,

$$\frac{\overline{y} - \mu}{\frac{s}{\sqrt{n}}} \sim N(0, 1)$$

There is also a third case, where $Y \sim N(\mu, \sigma)$. Then,

$$\overline{y} \sim N(\mu, \frac{\sigma}{\sqrt{n}}$$

7.5 Sample Proportion

Suppose we had an event A with $p = P(A)$. Let y be the number of times A occurs when the experiment is repeated n independent times, and define $\hat{p} = \frac{y}{n}$. Then:

1. $E(\hat{p}) = p$
2. $V(\hat{p} = \frac{p(1-p)}{n}$, and $\sigma_{\hat{p}} = \sqrt{p(1-p)}n$.
3. As n increases, the distribution of $\hat{p}$ becomes a normal distribution. Note this is only true when:
   (a) It is a random sample.
   (b) It has the 10% condition.
   (c) $np \geq 10$ and $n(1-p) \geq 10$.

   Thus, $\hat{p} \sim N(p, \sqrt{\frac{p(1-p)}{n}}$.

7.6 t-Distributions

$Y \sim (\mu, \sigma)$. If we have an $n < 30$ and σ is unknown, use the t distribution.
Assume that the population is normal, then the random variable is:

$$T = \frac{\overline{y} - \mu}{s/\sqrt{n}} \sim t_{n-1},$$

which has $(n-1)$ degrees of freedom, t_{n-1}.
The t distribution is similar to a normal distribution in that it is symmetric, however, it relies on the degrees of freedom, df.
This t distribution is called **Gosset's Theorem**.

7.6.1 Using t Distribution Table

Get the degree of freedom $df = n - 1$, and find the value closest to the calculated region.

7.7 Chi-Squared Distribution

Let $y_1, ..., y_n \sim N(\mu, \sigma)$ be a normal distribution with mean μ and variance σ^2. Then:

$$\frac{(n-1)s^2}{\sigma^2} = \frac{1}{\sigma^2} \sum_{i=1}^{n} (y_i - \overline{y})^2,$$

which follows χ^2_{n-1}, a Chi-Squared Distribution with $(n-1)$ degrees of freedom.

7.8 F Distribution

Let W_1, W_2 be two independent Chi-Square distributed random variables with ν_1, ν_2 as their degrees of freedom, respectively. Then:

$$F = \frac{W_1/\nu_1}{W_2/\nu_2},$$

is said to have an F distribution with ν_1 numerator degrees of freedom, and ν_2 denominator degrees of freedom.