STA237 Exercises

Contents

1 Exercises

1.1 Probability

1. Suppose $P(A) = 0.5, P(A \cap B) = 0.2$, and $P(A \cup B) = 0.7$. Find:

   (a) $P(B)$

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ 0.7 &= 0.5 + P(B) - 0.2 \\ 0.7 &= 0.3 + P(B) \\ P(B) &= 0.4 \end{aligned}$$

   (b) $P(\text{exactly one of two events occurs})$
   This means the probability of no elements from the intersection. Let this area be $P(O)$.

$$\begin{aligned} P(A \cup B) &= P(O) + P(A \cap B) \\ 0.7 &= P(O) + 0.2 \\ P(O) &= 0.5 \end{aligned}$$

   (c) $P(\text{neither event occurs})$
   This means the probability of no elements in A, B. Call this probability $P(X)$.

$$\begin{aligned} P(X) &= P(\Omega) - P(A \cup B) \\ &= 1 - 0.7 \\ &= 0.3 \end{aligned}$$

2. Suppose we throw a coin two times. The sample space for this experiment is:

$$\Omega = \{H, T\} \times \{H, T\} = \{(H, H), (H, T), (T, H), (T, T)\}$$

   If it is a fair coin, then all four outcomes have equal possibilities:

$$P((H, H)) = P((H, T)) = P((T, H)) = P((T, T)) = \frac{1}{4}$$

   What is the probability of getting at least one tail?
   Looking at the sample space, the probability is $\frac{3}{4}$.
   Alternatively, we can solve as follows, where RP represents the required probability:

$$\begin{aligned} P(\Omega) &= RP + P(HH) \\ 1 &= RP + P(HH) \\ &= RP + P(H)P(H) && \text{Since it is independent.} \\ &= RP + \frac{1}{2} \cdot \frac{1}{2} \\ 1 &= RP + \frac{1}{4} \\ RP &= \frac{3}{4} \end{aligned}$$

   Suppose we wanted to throw three coins, what would happen to the sample space?

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Then, there are $2 \cdot 2 \cdot 2 = 2^3$ possibilities (to keep on with this pattern, flipping a coin 6 times would have 2^6 possibilities, and so on).
What is the probability of at least one head appearing in a 3-coin toss?

$$\begin{aligned} P(X) &= 1 - P(TTT) \\ &= 1 - P(T)P(T)P(T) \\ &= 1 - \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \\ &= 1 - \frac{1}{8} \\ &= \frac{7}{8} \end{aligned}$$

3. A survey of students in a class was asked about their birth order (1 = oldest/only child), and which college of the university they were enrolled in.

Birth Order	1 or Only	2 or More	Total
Arts & Science	34	23	57
Agriculture	52	41	93
Human Ecology	15	28	43
Other	12	17	30
Total	113	110	223

Suppose we selected a random student from this class.

(a) What is the probability we select a Human Ecology student?
P(Human Ecology) = $\frac{43}{223}$

(b) What is the probability we select a first-born student?
P(First Born) = $\frac{113}{223}$

(c) What is the probability that the student is a first-born and a Human Ecology student?
P(C) = $P(A \cap B) = \frac{15}{223}$

(d) What is the probability that the student is a first born or a Human Ecology student?

$$\begin{aligned} P(D) &= P(A \cup B) \\ &= P(A) + P(B) - P(\cap B) \\ &= \frac{43}{223} + \frac{113}{223} - \frac{15}{223} \\ &= \frac{141}{223} \end{aligned}$$

4. A sample space consists of 5 simple events, E_1, E_2, E_3, E_4, E_5.

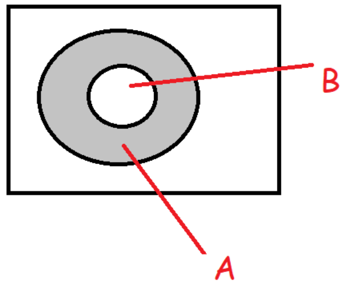
(a) If $P(E_1) = P(E_2) = 0.15$, $P(E_3) = 0.4$, and $P(E_4) = 2P(E_5)$, find the probabilities of E_4 and E_5.

$$\begin{aligned} 1 &= 0.15 + 0.15 + 0.4 + 2P(E_5) + 2P(E_5) \\ 0.3 &= 3P(E_5) \\ P(E_5) &= 0.1 \end{aligned}$$

Then:

$$\begin{aligned} P(E_4) &= 2(0.1) \\ &= 0.2 \end{aligned}$$

5. If A, B are events, $B \subset A$, $P(A) = 0.6$ and $P(B) = 0.2$, then find $P(A \cap B^c)$.
   To visualize:



$$P(A) = P(A \cap B^c) + P(B)0.6 \qquad = P(A \cap B^c) + 0.2$$
$$P(AcapB^c) = 0.4$$

6. An experiment involves tossing a pair of dice and observing the numbers on the upper faces. Find the number of sample points in Ω and the sample space for the experiment The sample points are $\Omega_{\{}m, n\} = \{1, 2, 3, 4, 5, 6\}$.
   The sample space is taking the cross product of the sample points (i.e., $\Omega_1 \times \Omega_2$). Then:

$$\Omega\{(1,1),(1,2),...,(6,6)\}$$

   There is a $m \cdot n = 6 \cdot 6 = 36$ chance for a single probability.

7. The names of three employees are randomly drawn without replacement from a bowl containing the names of 30 employees of a small company. The person whose names is drawn first receives \$100, and the individuals whose names are drawn second and third receive \$50 and \$25, respectively. How many samples are associated with this experiment? There are 30 employees and 3 people are picked. So, $r = 3, n = 30$. Hence:

$$\begin{aligned}\Omega &= P_3^{30} \\ &= \frac{30!}{(30-3)!} \\ &= \frac{30!}{27!}\end{aligned}$$

8. Find the number of ways of selecting two applicants out of five.
   Here, $n = 5, r = 2$.

$$\begin{aligned}\binom{5}{2} &= \frac{5!}{2!(5-2)!} \\ &= \frac{5!}{2!3!} \\ &= \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} \\ &= 10\end{aligned}$$

1.2 Conditional Probability

1. Let $N = R^c$ be the event 'born in a month without r', and L is the event 'born in a long month'. What is the conditional probability $P(N|L)$?
   $P(R^c|L) = \frac{3}{7}$
   Note that $P(R|L) + P(R^c|L) = \frac{4}{7} + \frac{3}{7} = 1$.

2. A survey asked: 'Are you currently in a relationship?', and 'Are you involved in club sports?'. The survey found that 33% were in a relationship, and 25% were involved in sport. 11% said yes to both. Suppose you meet a student who is on a sports team. What is the probability they are also in a relationship?
   Let A be students in a relationship, and B be students in a club sport.
   $P(A) = 0.33$, $P(B) = 0.25$, $P(A \cap B) = 0.11$. We have:

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{0.11}{0.25} \\ &= 0.44 \end{aligned}$$

   We know that $P(A|B) \neq P(A)$, A and B must be dependent events.

3. A bowl contains 6 red and 4 blue balls, A child selects two balls at random. What is the probability at least one of them is red?
   We have:
   (a) $\{R, R\} = \frac{6}{10}, \frac{5}{9}$
   (b) $\{R, B\} = \frac{6}{10}, \frac{4}{9}$
   (c) $\{B, R\} = \frac{4}{10}, \frac{6}{9}$
   (d) $\{B, B\} = \frac{4}{10}, \frac{3}{9}$

   Note that P(at least one red) $+P$(no red) $= 1$, so:

$$\begin{aligned} P(one\_R) &= 1 - P(no\_R) \\ &= 1 - P(BB) \\ &= 1 - P(B_1 \cap B_2) \\ &= 1 - P(B_2|B_1)P(B_1) \\ &= 1 - \frac{3}{9} \cdot \frac{4}{10} \\ &= 1 - \frac{2}{15} \\ &= \frac{13}{15} \end{aligned}$$

4. A recent highway safety study found that in 77% of all accidents, the driver was wearing a seatbelt. Accident reports indicated that 92% of those drivers did not experience serious injury (defined as hospitalization or death), but only 63% of the non-belted were okay.

   (a) What is the probability that a driver was seriously injured?
   (b) What is the probability that a driver who was seriously injured was not wearing a seatbelt?

   Let B is those that are wearing a seat bet, and NB is for those who did not wear a seat belt. Then, let I represent those who experienced a serious injury.
   We have $P(B) = 0.77$, $P(NB) = 1 - 0.77 = 0.23$, and $P(I^c|B) = 0.92$, $P(I^c|NB) = 0.63$.
   Then, $P(I|B) = 0.08$, $P(I|NB) = 0.37$.
   [a)]Want to find $P(I)$:

$$\begin{aligned} P(I) &= P(B \cap I) + P(NB \cap I) \\ &= P(I|B)P(B) + P(I|NB)P(NB) \\ &= 0.08 \cdot 0.77 + 0.37 \cdot 0.23 \\ &= 0.0616 + 0.0851 \\ &= 0.1467 \end{aligned}$$

Want to find $P(NB|I)$.

$$\begin{aligned} P(I|NB) &= \frac{P(NB \cap I)}{P(I)} \\ &= \frac{P(I|NB)P(I)}{0.1467} \\ &= \frac{0.23 \cdot 0.37}{0.1467} \\ &= \frac{0.0851}{0.1467} \\ &= 0.58 \end{aligned}$$

(5) A student figures he has a 30% chance of being let out of class late. If he leaves class late, there is a 45% chance he will miss the bus. What is the probability that he is let out of class late and misses the bus?
Let L be the chance he leaves class late. Let M be the chance he misses the bus.
$P(L) = 0.3$, $P(M|L) = 0.45$. We want to find $P(L \cap M)$:

$$\begin{aligned} P(L \cap M) &= P(M|L)P(L) \\ &= 0.45 \cdot 0.3 \\ &= 0.135 \end{aligned}$$

6. Leah flies from Moncton to Vancouver with a connection in Montreal. The probability her flight leaves on time is 15%. If the flight is on time, the chance her luggage makes the connecting flight in Montreal is 95%. But, if the flight is delayed, the probability the luggage will make it is only 65%.

   (a) Are the first flight leaving on time and the luggage making the connection independent events? Explain.

   (b) What is the probability that Leah's luggage arrives in Vancouver with her?

   Let F be the chance her flight leaves on time. Let L be the luggage being in the connecting flight.
   $P(F) = 0.15$, $P(L|F) = 0.95$, $P(L|F^C) = 0.65$.

   [a)]We have $P(L|F) = 0.95$ and $P(L) = 0.695$, so they are not equal. Therefore, they are dependent events.

$$\begin{aligned} P(L) &= P(L|F)P(F) + P(L|F^c)P(F^c) \\ &= 0.95 \cdot 0.15 + 0.65 \cdot 0.85 \\ &= 0.1425 + 0.5525 \\ &= 0.695 \end{aligned}$$

(7) The Ontario Lottery association claims your odds of winning a price on an Instant Win Crossword game are 1 in 3.09. This means that any ticket has 32.4% chance of winning a prize. Every Friday, you buy one Crossword game.

   (a) What is the probability you don't win a prize next Friday?

   (b) What is the probability you don't win a prize 6 Fridays in a row?

   (c) If you haven't won a prize for the past 6 Fridays, what is the probability you win a prize on your next game?

   (d) What is the probability you win on two of your next three games?

   Let W be the chance you win a prize. $P(W) = 0.324$.

   (a) Want to find $P(W^c)$. $P(W^c) = 1 - 0.324 = 0.676$

(b) Want to find $P(W_1^c ... W_6^c)$:

$$\begin{aligned} P(W_1^c ... W_6^c) &= P(W_1^c) \cdot ... \cdot P(W_2^c) \\ &= P(W^c)^6 \\ &= 0.676^6 \\ &= 0.0954 \end{aligned}$$

(c) Want to find $P((W_1^c ... W_6^c)^c)$. So: $P((W_1^c ... W_6^c)^c) = 1 - 0.0954 = 0.9046$.

(d) Want to find $P(W_1 W_2 W_3^c)$:

$$\begin{aligned} P(W_1 W_2 W_3^c) &= 0.324 \cdot 0.3424 \cdot 0.676 \\ &= 0.071 \end{aligned}$$

1.3 Random Variables

1. Suppose we roll two dice. List the outcomes:

   (a) $\{X = 8\} = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$.
   $P(X = 8) = \frac{5}{36}$

   (b) $\{X = 3\} = \{(1, 2), (2, 1)\}$.
   $P(X = 3) = \frac{2}{36} = \frac{1}{18}$

   (c) $\{X = 13\} = \{\}$
   $P(X = 13) = 0$

2. Suppose the death rate in a year is 1 out of 1000 people, and that 2 out of 1000 people suffer some kind of disability. Then, we display the probability model for X = company payout in a table:

Policyholder Outcome	Payout x	P(X = x)
Death	10000	1/1000
Disability	5000	2/1000
Neither	0	998/1000

   What is $P(X = 0)$?
   We have $X = 0, 5000, 10000$, and $P(X = 10000) = \frac{1}{1000}$, $P(X = 5000) = \frac{2}{1000}$. We calculate:

$$\begin{aligned} P(X = 0) &= 1 - \frac{1}{1000} - \frac{2}{1000} \\ &= \frac{997}{1000} \end{aligned}$$

3. Take the previous example, and determine $E(X)$, where X is the company payout.
   Recall: $X = 0, 5000, 10000$, and $P(X = 10000) = \frac{1}{1000}$, $P(X = 5000) = \frac{2}{1000}$, and $P(X = 0) = \frac{997}{1000}$.
   To calculate $E(X)$:

$$\begin{aligned} E(X) &= \left(10000 \cdot \frac{1}{1000}\right) + \left(5000 \cdot \frac{2}{1000}\right) + \left(0 \cdot \frac{997}{1000}\right) \\ &= 10 + 10 \\ &= 20 \end{aligned}$$

   Next, suppose we wanted to calculate the variance of X.
   We have:

   (a) $10000 - 20 = 9980$

(b) $5000 - 20 = 4980$

(c) $0 - 20 = -20$

So, we have:

$$\begin{aligned}
\sigma^2 &= \sum_x (x-\mu)^2 P(x) \\
&= \left(9980^2 \cdot \frac{1}{1000}\right) + \left(4980^2 \cdot \frac{2}{1000}\right) + \left((-20)^2 \cdot \frac{997}{1000}\right) \\
&= 149600
\end{aligned}$$

Then:

$$\begin{aligned}
\sigma &= SD(X) = \sqrt{V(X)} \\
&= \sqrt{149600} \\
&= \$386.78
\end{aligned}$$

4. If $E(X) = 10$ and $V(X) = 3$, then:

(a) Calculate $E(9x + 11)$.

$$\begin{aligned}
E[9x + 11] &= 9E(x) + 11 \\
&= (9 \cdot 10) + 11 \\
&= 90 + 11 = 101
\end{aligned}$$

(b) Calculate $V(9x + 11)$.

$$\begin{aligned}
V(9x + 11) &= 9^2 V(x) \\
&= 9^2 \cdot 3 \\
&= 243
\end{aligned}$$

(c)

$$\begin{aligned}
\sigma^2 &= E(X-\mu)^2 \\
&= E(x^2 - 2\mu x + \mu^2) \\
&= E(x^2) - 2\mu E(X) + \mu^2 \\
&= E(x^2) - 2\mu^2 + \mu^2 \\
&= E(x^2) - \mu^2
\end{aligned}$$

(d) If $E(X) = 10$ and $E(X^2) = 160$:

$$\sigma^2 = 160 - 10^2 = 60$$

5. The probability distribution for a random variable Y is given by the following table. Find the mean, variance, and standard deviation of Y.

y	p(y)
0	1/8
1	1/4
2	3/8
3	1/4

Then:

$$
\begin{aligned}
F(0) &= P(Y \leq 0) \\
&= P(Y = 0) = \frac{1}{8}
\end{aligned}
$$

$$
\begin{aligned}
F(1) &= P(Y \leq 1) \\
&= P(Y = 0) + P(Y = 1) \\
&= \frac{1}{8} + \frac{1}{4} = \frac{3}{8}
\end{aligned}
$$

$$
\begin{aligned}
F(2) &= P(Y \leq 2) \\
&= \frac{3}{8} + \frac{1}{4} + \frac{1}{8} = \frac{6}{8}
\end{aligned}
$$

$$
\begin{aligned}
F(3) &= P(Y \leq 3) \\
&= \frac{6}{8} + \frac{1}{4} = 1
\end{aligned}
$$

6. Suppose we toss a coin n times. Let $n = 5$, and X be the number of heads in n trials.
   Suppose we wanted to find the probability of $P(SFFSS)$; we have:

$$
\begin{aligned}
P(SFFSS) &= P(S)P(F)P(F)P(S)P(S) \\
&= p \cdot (1-p) \cdot (1-p) \cdot p \cdot p \\
&= p^3(1-p)^{5-3}
\end{aligned}
$$

   Then, since we know it is independent, we can write this as $P(X = 3)$. Thus, we have:

$$
P(X = 3) = \binom{5}{3} p^3 (1-p)^{5-3}
$$

7. A tennis player makes successful first serves 70% of the time. Assume each serve is independent of the others. If she serves, what is the probability she gets:

   (a) All six serves in?

   (b) Four serves in?

   (c) At least four serves in?

   (d) No more than four serves in?

   We have $P(S) = 0.7$ and $P(F) = 0.3$. Let X be the number of successful serves.
   Let $n = 6$, then $X \sim B(6, 0.7)$. We have:

$$
P(X = x) = \binom{6}{k} 0.7^x 0.3^{6-x}
$$

   To solve each part, we can just plug in values:

   (a)

$$
\begin{aligned}
P(X = 6) &= \binom{6}{6} 0.7^6 0.3^{6-6} \\
&= \frac{6!}{6!0!} \cdot 0.7^6 \\
&= 0.7^6 = 0.118
\end{aligned}
$$

(b)

$$\begin{aligned} P(X=4) &= \binom{6}{4} 0.7^4 0.3^{6-4} \\ &= \frac{6!}{4!2!} 0.7^4 \cdot 0.3^2 \\ &= 0.324 \end{aligned}$$

(c)

$$\begin{aligned} P(X \geq 4) &= P(X=4) + P(X=5) + P(X=6) \\ &= 0.744 \end{aligned}$$

(d)

$$\begin{aligned} P(X \leq 4) &= P(X=4) + P(X=3) + ... + P(X=0) \\ &= 0.580 \end{aligned}$$

8. Suppose that a lot of 5000 electrical fuses contain 5% defectives.

   (a) If a sample of 5 fuses is tested, find the probability of observing at least one defective.
   (b) If $n = 20$ fuses are randomly sampled from this lot, find the probability that at most 6 defectives are observed.
   (c) If $n = 20$ fuses are randomly sampled, find the probability that at least four defectives will be observed.

   We know this event is independent; we have $P(F) = 0.05$ and $P(S) = 0.95$ for 5000 fuses.
   Let X be the number of defectives in n fuses.

   (a) $n = 5$, we want to find $P(X \geq 1)$.
   Using the binomial distribution table, we have:

   $$\begin{aligned} P(X \geq 1) &= 1 - P(X \leq 0) \\ &= 1 - 0.774 \\ &= 0.226 \end{aligned}$$

   (b)

9. Suppose the probability of an engine malfunction during a one-hour period is $p = 0.02$. Find the probability that the engine will survive for two hours.
   We have $p = P(S) = 0.02$, so $q = P(F) = 0.98$. Let Y be the number of one-hour intervals until a malfunction occurs. We have:

   $$\begin{aligned} P(y) &= 0.98^{y-1} \cdot 0.02 && \text{For } y = 1, 2, ... \\ &= P(Y \geq 3) \\ &= 1 - P(Y \leq 2) \\ &= 1 - P(Y=1) - P(Y=2) \\ &= 1 - (0.98^1 \cdot 0.02) - (0.98^0 \cdot 0.02) \\ &= 0.9604 \end{aligned}$$

10. If the probability of an engine malfunction is $p = 0.02$ and Y is the number of one-hour intervals until the first malfunction, find the mean and standard deviation of Y.

$$
\begin{aligned}
E(Y) &= \frac{1}{p} \\
&= \frac{1}{0.02} \\
&= 50 V(Y) = \frac{1-0.02}{0.02^2} \\
&= 2450
\end{aligned}
$$

11. Suppose a professor randomly selects 3 new teaching assistants from a total of 10 applicants - 6 male, 4 female. Let X be the number of females who are hired.

    (a) Find the probability that no females are hired.

    (b) Find the probability that at least 2 females are hired.

    Here, $r = 4$, then $N - r = 6$. Set $n = 3$; we use hypergeometric distribution. First, calculate x:

$$
\begin{aligned}
\max\{0, n-(N-r)\} &: n-(N-r) \\
&: 3-6=-3 \\
&= 0 \\
\min\{r, n\} &: \min 4, 3 \\
&= 3
\end{aligned}
$$

Hence, $x = 0, 1, 2, 3$.

$$
P(X = x) = \frac{\binom{4}{x}\binom{6}{3-x}}{\binom{10}{3}}
$$

(a)

$$
\begin{aligned}
P(X=0) &= \frac{\binom{4}{0}\binom{6}{3}}{\binom{10}{3}} \\
&= \frac{6!}{3!3!} \cdot \frac{7!3!}{10!} \\
&= \frac{1}{6}
\end{aligned}
$$

(b) We want to find $P(X \geq 2)$, i.e., $1 - P(X = 0) - P(x = 1)$.

$$
\begin{aligned}
P(X \geq 2 &= 1 - \frac{1}{6} - \frac{\binom{4}{1}\binom{6}{2}}{\binom{10}{3}} \\
&= 1 - \frac{1}{6} - \frac{1}{2}
\end{aligned}
$$

12. Suppose there are X number of blue whale sightings per week. Assume X follows a Poisson distribution and the number of weekly sightings is 2.5.

    (a) Find the mean and standard deviation of X.

$$
\begin{aligned}
E(X) &= 2.5 \\
\sigma^2 = V(X) &= 2.5 \\
\sigma &\approx 1.58
\end{aligned}
$$

(b) Find the probability that 5 sightings are made in one week.

$$P(X = 5) = \frac{2.5^5 e^{-2.5}}{5!}$$

We can use a Poisson Table:

$$\begin{aligned} P(X = 5 &= P(X \leq 5) - P(X \leq 4) \\ &= 0.958 - 0.891 \\ &= 0.067 \end{aligned}$$

(c) Find the probability that there are fewer than 2 sightings made per week. Use a Poisson Table:

$$\begin{aligned} P(X < 2) &= P(\leq 1) \\ &= 0.287 \end{aligned}$$

1.4 Continuous Random Variables

1. Suppose that:

$$F(y) = \begin{cases} 0 & y < 0 \\ y & 0 \leq y \leq 1 \\ 0 & y > 1 \end{cases}$$

Find the probability density function for Y.

Take the derivative, and get:

$$f(y) = \begin{cases} 0 & y < 0 \\ 1 & 0 \leq y \leq 1 \\ 0 & y > 1 \end{cases}$$

We can also write this as $f(y) = 1$ if $0 \leq y \leq 1$, and 0 otherwise.

2. Let Y be a continuous random variable with probability density function given by:

$$f(y) = \begin{cases} 3y^2 & 0 \leq y \leq 10 \\ \text{otherwise} & \end{cases}$$

Find $F(y)$.

To find $F(y)$ we take the anti-derivative.
For $y < 0$, we have:

$$\begin{aligned} F(y) &= P(Y \leq y) \\ &= \int_{-\infty}^{y} 0 dt \\ &= 0 \end{aligned}$$

For $0 \leq y \leq 1$, we have:

$$\begin{aligned} F9Y0 &= \int_{-\infty}^{y} f(t) dt \\ &= \int_{-\infty}^{0} 0 dt + \int_{0}^{y} 3t^2 dt \\ &= t^3 |_0^y \\ &= y^3 \end{aligned}$$

For $y > 1$:

$$\begin{aligned} F(y) &= \int_{-\infty}^{0} 0dt + \int_{0}^{1} 3t^2 dt + \int_{1}^{\infty} 0dt \\ &= 1 \end{aligned}$$

Hence, we have:

$$F(y) = \begin{cases} 0 & y < 0 \\ y^3 & 0 \le y \le 1 \\ 1 & y > 1 \end{cases}$$

3. Given $f(y) = cy^2, 0 \le y \le 2, f(y) = 0$ elsewhere, find the value of c for which $f(y)$ is a valid density function.

   For a probability density function $\int_{-\infty}^{\infty} f(y)dy = 1$, and have $\int_0^2 cy^2 dy = 1$. Then:

$$\begin{aligned} c\frac{y^3}{3}|_0^2 &= 1 \\ c\frac{8}{3} &= 1 \\ c &= \frac{3}{8} \end{aligned}$$

4. Find $P(1 \le Y \le 2)$ from the previous example.
   We have:

$$f(y) = \frac{3}{8}y^2 \text{ for } 0 \le y \le 2$$

$$\begin{aligned} P(1 \le Y \le 2) &= \int_1^2 \frac{3}{8}y^2 dy \\ &= \frac{1}{8}[y^3]_1^2 \\ &= \frac{1}{8}(8-1) \\ &= \frac{7}{8} \end{aligned}$$

   How about for $P(1 \le Y \le 5)$? We know that it is valid only for $0 \le y \le 2$, so we split it up:

$$P(1 \le Y \le 5) = P(1 \le Y \le 2) + P(2 \le Y \le 5)$$

5. From the previous example, find $\mu = E(Y)$ and $\sigma^2 = V(Y)$.
   We have $f(y) = \frac{3}{8}y^2$ for $0 \le y \le 2$, and 0 otherwise. To find the expected value:

$$\begin{aligned} \mu = E(Y) &= \int_0^2 y \cdot \frac{3}{8}y^2 dy \\ &= \frac{3}{8}\int_0^2 y^3 dy \\ &= \frac{3}{8}\left[\frac{y^4}{4}\right]_0^2 \\ &= \frac{3}{8}(4-0) \\ &= \frac{3}{2} \end{aligned}$$

Thus, $\mu = E(Y) = 1.5$.
Then, find the variance. Find $E(Y^2)$:

$$\begin{aligned}
E(Y^2) &= \int_0^2 y^2 \cdot \frac{3}{8} y^2 dy \\
&= \frac{3}{8} \int_0^2 y^4 dy \\
&= \frac{3}{8} \left[\frac{y^5}{5} \right]_0^2 \\
&= \frac{3}{8} (\frac{32}{5} - 0) \\
&= \frac{12}{5}
\end{aligned}$$

Hence, $E(Y^2) = 2.4$. Now, we can calculate $V(Y)$:

$$\begin{aligned}
V(Y) &= E(Y^2) - \mu^2 \\
&= 2.4 - 1.5^2 \\
&= 0.15
\end{aligned}$$

6. Esme is taking a train from Minneapolis to Boston, a distance of roughly 1400 miles.Her position is uniformly distributed between the two cities. What is the probability that she is past Chicago, which is 400 miles from Minneapolis?
   Let X be Esme's location, where it follows the uniform probability distribution on the interval $(0, 1400)$. We have $f(x) = \frac{1}{1400}$ if $0 \leq x \leq 1400$, and 0 otherwise.
   We want to find $P(X > 400)$, so:

$$\begin{aligned}
\int_{400}^{1} 400 \frac{1}{1400} dx &= \frac{1}{1400} [x]_{400}^{1400} \\
&= \frac{1400 - 400}{1400} \\
&= \frac{10}{14}
\end{aligned}$$

7. Let Z denote a normal random variable with mean 0 and standard deviation 1 (the standard normal random variable).
   That is, $Z \sim N(0, 1)$.

   (a) Find $P(Z > 2)$:
   Using the Normal curve table, we have:

$$\begin{aligned}
P(Z > 2) &= 1 - P(Z \leq 2) \\
&= 1 - 0.9772 \\
&= 0.0228
\end{aligned}$$

   (b) Find $P(-2 \leq Z \leq 2)$:

$$\begin{aligned}
P(-2 \leq Z \leq 2) &= P(Z \leq 2) - P(Z < -2) \\
&= 0.9772 - 0.0228 \\
&= 0.9544
\end{aligned}$$

   (c) Find $P(0 \leq Z \leq 1.73$:

$$\begin{aligned}
P(0 \leq Z \leq 0.173) &= P(Z \leq 1.73) - P(Z < 0) \\
&= 0.9582 - 0.5 \\
&= 0.4582
\end{aligned}$$

8. The achievement scores for a college entrance exam are normally distributed with mean 75 and standard deviation 10. What fraction of the scores lie between 80 and 90?
Let $Y \rightarrow$Score. We have $Y \sim N(75, 10)$. We want to find $P(80 \leq Y \leq 90)$.
We want to standardize this to use the Normal curve table. So:

$$\begin{aligned}
P(80 \leq Y \leq 90) &= P\left(\frac{80-75}{10} \leq \frac{Y-\mu}{\sigma} \leq \frac{90-75}{10}\right) \\
&= P(0.5 \leq Z \leq 1.5) \\
&= P(Z \leq 1.5) - P(Z < 0.5) \\
&= 0.9332 - 0.6915 \\
&= 0.2417
\end{aligned}$$

9. If Z is the standard normal random variable, find the value z_0 such that:

   (a) $P(Z > z_0) = 0.5$:
   We know the mean is 0, so $z_0 = 0$.

   (b) $P(Z < z_0) = 0.8643$:
   Using the standard normal curve table, $z_0 = 1.10$.

   (c) $P(-z_0 < Z < z_0) = 0.9$:
   We can use the symmetric property, we calculate the unused area:

$$\begin{aligned}
1 - 0.9 &= 0.1 \\
\frac{0.1}{2} &= 0.05
\end{aligned}$$

   So, each side of the unused area has size 0.05. Then,

$$P(Z \leq z_0) = 0.95 P(Z < -z_0) = 0.05$$

   Hence, $z_0 = 1.645$.

   (d) $P(-z_0 < Z < z_0) = 0.99$:
   We have $1 - 0.99 = 0.01$, and so each side of the unused area has size 0.005. Then:

$$P(Z \leq z_0) = 0.995 P(Z \leq -z_0) = 0.005$$

   Then, $z_0 = 2.575$.

10. The width of bolts of fabric is normally distributed with mean $950mm$ and standard deviation $10mm$.

   (a) What is the probability that a randomly chosen bolt has a width of between 947 and 958 mm?
   Let $Y \rightarrow$ width, $Y \sim N(950, 10)$.
   We want to find $P(947 \leq Y \leq 958)$.

$$\begin{aligned}
P\left(\frac{947-950}{10} \leq Z \leq \frac{958-950}{10}\right) &= P(-0.3 \leq Z \leq 0.8) \\
&= P(Z \leq 0.8) - P(Z < -0.3) \\
&= 0.7881 - 0.3821 \\
&= 0.406
\end{aligned}$$

   (b) What is the appropriate value for C such that a randomly chosen bolt has width less than C with probability 0.8531?

We want to find $P(Z < C) = 0.8531$.

$$\begin{aligned} P\left(Z < -\frac{C-950}{10}\right) &= 0.8531 \\ P(Z < z_0) &= 0.8531, \end{aligned}$$

where $z_0 = \frac{C-950}{10}$. We find $z_0 = 1.05$ using the table then find C:

$$\begin{aligned} \frac{C-950}{10} &= 1.05 \\ C &= 1.05 \cdot 10 + 950 \\ &= 10.5 + 950 \\ &= 960.5 \end{aligned}$$

11. If a die is rolled 600 times, what is the probability of rolling between 90 and 110 fours?
$\Omega = \{1, 2, 3, 4, 5, 6\}$, and $p = P(4) = \frac{1}{6}$.
Let $n = 600$, and $X \to$ number of fours in 600 rolls, where $X \sim B(600, \frac{1}{6})$. We also have $np = 600 \cdot \frac{1}{6} = 100$, and $n(1-p) = 600 \cdot \frac{1}{6} = 100$.
Hence, $\mu = np = 100$, and $\sigma^2 = np(1-p) = \frac{500}{6}$.

Now, we want to find $X \sim N(\mu, \sigma)$ - Note that because we are using continuous random variables, include decimals up to the tenth digit.

$$\begin{aligned} P(90 \leq Y \leq 110) &= P(89.5 \leq Y \leq 110.5) \\ &= P\left(\frac{89.5-100}{\sqrt{\frac{250}{3}}} \leq Z \leq \frac{110.5-100}{\sqrt{\frac{250}{3}}}\right) \\ &= P(-1.15 \leq Z \leq 1.15) \\ &= P(Z \leq 1.15) - P(Z < -1.15) \\ &= 0.8749 - 0.1251 \\ &= 0.7498 \end{aligned}$$

Note, because it is a discrete random variable, using a strict relation ($<$), change it to $\leq$ and then add the decimal. For example, $P(X < 420)$ becomes $P(X \leq 419.5)$.

12. Suppose the length of time (in hours) between emergency arrivals at a hospital is modeled as an exponential distribution with $\beta = 2$. What is the probability that more than 5 hours pass without an emergency arrival?

Let $Y \to$ length of time (in hours) between emergency arrivals. We have $\beta = 2$. Use cumulative distribution:
We want to find $P(Y > 5)$.

$$\begin{aligned} P(Y > 5) &= e^{-y/\beta} \\ &= e^{-5/2} \\ &\approx 0.08208 \end{aligned}$$

13. A manufacturer of microwave ovens is trying to determine the length of the warranty period it should attach to its magnetron tube. Preliminary testing shows the length of life X a magnetron tube has is the exponential probability distribution $\beta = 6.25$.

(a) Find the mean and standard deviation of X.

(b) Suppose a warranty period of five years is attached to the magnetron tube. What fraction of tubes must the manufacturer plan to replace, assuming the exponential model $\beta = 6.25$?

(c) What should the warranty period be if the manufacturer wants to replace 27.5% of the tubes?

(d) Find the probability that the lifespan of a magnetron tube is within the interval $[\mu - 2\sigma, \mu + 2\sigma]$.

Let $X \to$ the length of life in years, and $X \sim \exp(6.25)$, where $\beta = 6.25$.

(a) $\mu = E(X) = \beta = 6.25$
$\sigma = \beta = 6.25$.

(b)

$$\begin{aligned} P(X \leq 5) &= 1 - e^{-5/6.25} \\ &= 1 - 0.449329 \\ &= 0.550671 \end{aligned}$$

(c) Let $W \to$ required warranty period. We have $P(X \leq W) = 0.275$. Then:

$$\begin{aligned} 1 - e^{-W/6.25} &= 0.275 \\ e^{-W/6.25} &= 1 - 0.275 \\ &= 0.725 \\ -\frac{W}{6.25} &= \ln 0.725 \\ &= -0.3216 \\ W &= 6.25 \cdot 0.3216 \\ &= 2.01 \end{aligned}$$

(d) We have $\mu = 6.25 = \sigma$. We know $E(Y) = \frac{1}{\lambda}$ and $V(Y) = \frac{1}{\lambda^2}$.

$$\begin{aligned} \mu - 2\sigma &= 6.25 - 2 \cdot 6.25 \\ &= -6.25 \\ \mu + 2\sigma &= 6.25 + 2 \cdot 6.25 \\ &= 18.75 \end{aligned}$$

Then:

$$\begin{aligned} P(-6.25 \leq X \leq 18.75) &= P(0 \leq X \leq 18.75) \\ &= P(X \leq 18.75) \\ P(X \leq 18.75) &= 1 - e^{-18.75/6.25} \\ &= 1 - e^{-3} \\ &= 0.9502 \end{aligned}$$

1.5 Multivariate

1. A local supermarket has 3 checkout counters. Two customers arrive at the counters at different times where there are no other customers. Each customer chooses a counter at random, independently of the other. Let Y_1 denote the number of customers who choose counter 1, and Y_2 the number who select counter 2. Find the joint probability of Y_1, Y_2.

   We let our sample space to be $\Omega = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$.

Then, for Y_1, we have $y_1 = 0, 1, 2$ as the number of people who chose counter 1, and Y_2 has $y_2 = 0, 1, 2$, for the number of people who chose counter 2. We have:

$$P(Y_1 = 0, Y_2 = 0) = \frac{1}{9}$$
$$P(Y_1 = 0, Y_2 = 1) = \frac{2}{9}$$
$$P(Y_1 = 0, Y_2 = 2) = \frac{1}{9}$$
$$P(Y_1 = 1, Y_2 = 0) = \frac{2}{9}$$
$$P(Y_1 = 2, Y_2 = 0) = \frac{1}{9}$$
$$P(Y_1 = 1, Y_2 = 1) = \frac{2}{9}$$
$$P(Y_1 = 2, Y_2 = 1) = 0$$
$$P(Y_1 = 2, Y_2 = 2) = 0$$

We want to find the join probability mass function of Y_1, Y_2.

	0	1	2
0	1/9	2/9	1/9
1	2/9	2/9	0
2	1/9	0	0

2. Consider the previous example. Find $F(-1, 2), F(1.5, 2)$, and $F(5, 7)$.

$$\begin{aligned} F(-1,2) &= P(Y_1 \leq -1, Y_2 \leq 2) \\ &= 0F(1.5,2) & &= P(Y_1 \leq 1.5, Y_2 \leq 2) \\ &= \frac{8}{9} \\ F(5,7) &= P(Y_1 \leq 5, Y_2 \leq 7) \\ &= 1 \end{aligned}$$