STA237 - Activity 7

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1. We have $f(y_1, y_2) = 2(1 - y_1), 0 \leq y_1, y_2 \leq 1$. Then,

$$U_1 = U = Y_1\ Y_2 \quad U_2 = Y_1$$

Then,

$$\begin{aligned} u_1 &= y_1 y_2 u_2 = y_1 \\ y_1 &= u_2 y_2 = \frac{u_1}{y_1} = \frac{u_1}{u_2} \\ y_1 &= u_2 y_2 = \frac{u_1}{u_2} \end{aligned}$$

Now, we can calculate the partials:

$$\frac{\partial y_1}{u_1} = 0 \qquad \frac{\partial y_1}{\partial u_2} = 1$$

$$\frac{\partial y_2}{\partial u_1} = \frac{1}{u_2} \qquad \frac{\partial y_2}{\partial u_2} = u_1(-1)u_2^{-2}$$

Now, we calculate the Jacobian:

$$\begin{aligned} J &= \begin{bmatrix} 0 & 1 \\ \frac{1}{u_2} & -\frac{u_1}{u_2^2} \end{bmatrix} \\ &= -\frac{1}{u_2} \end{aligned}$$

Hence:

$$\begin{aligned} f(u_1, u_2) &= f(y_1, y_2)|J| \\ &= 2(1 - y_1) \mid -\frac{1}{u_2} \mid \\ &= \frac{2(1 - u_2)}{u_2} \end{aligned}$$

Consider the domain: We are given $0 < y_1 \leq 1$ and $0 < y_2 \leq 1$, implying that $0 < u_2 \leq 1$ and $0 < \frac{u_1}{u_2} \leq 1$. Then, $0 < u_1 \leq u_2$.
Hence, $0 < u_1 \leq u_2 \leq 1$.
Therefore:

$$f(u_1, u_2) = \frac{2(1 - u_2)}{u_2},\ 0 < u_1 \leq u_2 \leq 1$$

Now we find $f(u_1)$:

$$\begin{aligned}
f(u_1) &= \int_{u_1}^{1} \frac{2(1-u_2)}{u_2} du_2 \\
&= 2\int_{u_1}^{1} \left[\frac{1}{u_2} - 1\right] du_2 \\
&= 2\left[\ln(u_2) - u_2\right]_{u_1}^{1} \\
&= 2(\ln(1) - 1 - \ln(u_1) - u_1)) \\
&= 2(u_1 - \ln(u_1) - 1),\ \ 0 < u_1 \leq 1, \quad \text{and 0 otherwise.}
\end{aligned}$$