STA237 Tutorial 8

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Definition. If Y_1 and Y_2 are random variables with means μ_1, μ_2 respextively, the covariance of Y_1, Y_2 is:

$$Cov(Y_1, Y_2) = E\left[(Y_1 - \mu_1)(Y_2 - \mu_2)\right] = E(Y_1, Y_2) - E(Y_1)E(Y_2)$$

Note that if Y_1, Y_2 are independent random variables, then $Cov(Y_1, Y_2) = 0$.

Q1

We have $f(y_1, y_2) = \begin{cases} 3y_1, & 0 \le y_2 \le y_1 \le 1 \\ 0, & \text{otherwise} \end{cases}$. Then:

$$\begin{aligned} E(Y_1, Y_2) &= \int_0^1 \left[\int_0^{y_1} y_1 y_2 (3y_1) dy_2\right] \, dy_1 \\ &= \int_0^1 3y_1^2 \left[\int_0^{y_1} y_2 dy_2\right] dy_1 \\ &= \int_0^1 3y_1^2 \left[\frac{y_2^2}{2}\right]_0^{y_1} dy_1 \\ &= \int_0^1 3y_1^2 \left(\frac{y_1^2}{2}\right) dy_1 \\ &= \int_0^1 \frac{3}{2} y_1^4 \, dy_1 \\ &= \frac{3}{2}\left[\frac{y_1^5}{5}\right]_0^1 \\ &= \frac{3}{10} \end{aligned}$$

Then, to calculate the marginal density, we have:

$$\begin{aligned} f(y_1) &= \int_0^{y_1} 3y_1 \, dy_2 \\ &= 3y_1 \left[y_2\right]_0^{y_1} \\ &= 3y_1^2, \quad \text{if } 0 \le y_1 \le 1 \end{aligned}$$

Now, we can calculate the expected value of Y_1:

$$\begin{aligned}
E(Y_1) &= \int_0^1 y_1 \cdot 3y_1^2 \; dy_1 \\
&= \int_0^1 3y_1^3 \; dy_1 \\
&= 3\left[\frac{y_1^4}{4}\right]_0^1 \\
&= \frac{3}{4}
\end{aligned}$$

Finally, we calculate the marginal density of Y_2:

$$\begin{aligned}
f(y_2) &= \int_{y_2}^1 3y_1 \; dy_1 \\
&= 3\left[\frac{y_1^2}{2}\right]_{y_2}^1 \\
&= \frac{3}{2}(1 - y_2^2), \quad \text{if } 0 \leq y_2 \leq 1
\end{aligned}$$

And get the expected value for Y_2:

$$\begin{aligned}
E(Y_2) &= \int_0^1 y_2 \frac{3}{2}(1 - y_2^2) \; dy_2 \\
&= \frac{3}{2}\left[\frac{y_2^2}{2} - \frac{y_2^4}{4}\right]_0^1 \\
&= \frac{3}{2}\left(\frac{1}{2} - \frac{1}{4}\right) \\
&= \frac{3}{2} \cdot \frac{1}{4} \\
&= \frac{3}{8}
\end{aligned}$$

Therefore:

$$\begin{aligned}
Cov(Y_1, Y - 2) &= \frac{3}{10} - \frac{3}{4} \cdot \frac{3}{8} \\
&= \frac{3}{160} \\
&= 0.01875
\end{aligned}$$

Q2

Using the table, we have:

$$
\begin{aligned}
P(Y_1 = -1) &= \frac{5}{16} \\
P(Y_1 = 0) &= \frac{6}{16} \\
P(Y_1 = 1) &= \frac{5}{16}
\end{aligned}
$$

$$
\begin{aligned}
P(Y_2 = -1) &= \frac{5}{16} \\
P(Y_2 = 0) &= \frac{6}{16} \\
P(Y_2 = 1) &= \frac{5}{16}
\end{aligned}
$$

Take $P(Y_1 = 0, Y_2 = 0)$, which is equal to 0. However, if we multiply $P(Y_1 = 0) \cdot P(Y_2 = 0)$, we have $\frac{6}{16} \cdot \frac{6}{16}$, which is not equal to $P(Y_1 = 0, Y_2 = 0)$.

So now, we consider $E(Y_1, Y_2)$:

$$
\begin{aligned}
E(Y_1, Y_2) &= \sum_{y_1} \sum_{y_2} y_1 \ y_2 \ P(Y_1 = y_1, Y_2 = 2) \\
&= \left[-1 \cdot -1 \cdot \frac{1}{16}\right] + \left[0 \cdot -1 \cdot \frac{3}{16}\right] + \left[1 \cdot -1 \cdot \frac{1}{16}\right] + \left[-1 \cdot 0 \cdot \frac{3}{16}\right] \\
&= \frac{1}{16} - \frac{1}{16} - \frac{1}{16} + \frac{1}{16} \\
&= 0
\end{aligned}
$$

Then, consider $E(Y_1)$ and $E(Y_2)$:

$$
\begin{aligned}
E(Y_1) &= \sum_{y_1} y - 1 \ P(Y_1 = y_1) \\
&= -1 \cdot \frac{5}{16} + \left(0 \cdot \frac{6}{16}\right) = \left(1 \cdot \frac{5}{16}\right) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
E(Y_2) &= \sum_{y_2} y_2 \ P(Y_2 = y_2 \\
&= 0
\end{aligned}
$$

Therefore, independence implies that the covariance is 0, however, the covariance being equal to 0 does not imply independence.