

STA237 Exercises

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1 Exercises

1.1 Probability

1. Suppose $P(A) = 0.5$, $P(A \cap B) = 0.2$, and $P(A \cup B) = 0.7$. Find:

(a) $P(B)$

$$\begin{aligned}P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\0.7 &= 0.5 + P(B) - 0.2 \\0.7 &= 0.3 + P(B) \\P(B) &= 0.4\end{aligned}$$

(b) $P(\text{exactly one of two events occurs})$

This means the probability of no elements from the intersection. Let this area be $P(O)$.

$$\begin{aligned}P(A \cup B) &= P(O) + P(A \cap B) \\0.7 &= P(O) + 0.2 \\P(O) &= 0.5\end{aligned}$$

(c) $P(\text{neither event occurs})$

This means the probability of no elements in A, B . Call this probability $P(X)$.

$$\begin{aligned}P(X) &= P(\Omega) - P(A \cup B) \\&= 1 - 0.7 \\&= 0.3\end{aligned}$$

2. Suppose we throw a coin two times. The sample space for this experiment is:

$$\Omega = \{H, T\} \times \{H, T\} = \{(H, H), (H, T), (T, H), (T, T)\}$$

If it is a fair coin, then all four outcomes have equal possibilities:

$$P((H, H)) = P((H, T)) = P((T, H)) = P((T, T)) = \frac{1}{4}$$

What is the probability of getting at least one tail?

Looking at the sample space, the probability is $\frac{3}{4}$.

Alternatively, we can solve as follows, where RP represents the required probability:

$$\begin{aligned}P(\Omega) &= RP + P(HH) \\1 &= RP + P(HH) \\&= RP + P(H)P(H) && \text{Since it is independent.} \\&= RP + \frac{1}{2} \cdot \frac{1}{2} \\1 &= RP + \frac{1}{4} \\RP &= \frac{3}{4}\end{aligned}$$

Suppose we wanted to throw three coins, what would happen to the sample space?

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Then, there are $2 \cdot 2 \cdot 2 = 2^3$ possibilities (to keep on with this pattern, flipping a coin 6 times would have 2^6 possibilities, and so on).

What is the probability of at least one head appearing in a 3-coin toss?

$$\begin{aligned}
 P(X) &= 1 - P(TTT) \\
 &= 1 - P(T)P(T)P(T) \\
 &= 1 - \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \\
 &= 1 - \frac{1}{8} \\
 &= \frac{7}{8}
 \end{aligned}$$

3. A survey of students in a class was asked about their birth order (1 = oldest/only child), and which college of the university they were enrolled in.

| Birth Order | 1 or Only | 2 or More | Total |
|----------------|-----------|-----------|-------|
| Arts & Science | 34 | 23 | 57 |
| Agriculture | 52 | 41 | 93 |
| Human Ecology | 15 | 28 | 43 |
| Other | 12 | 17 | 30 |
| Total | 113 | 110 | 223 |

Suppose we selected a random student from this class.

- (a) What is the probability we select a Human Ecology student?
 $P(\text{Human Ecology}) = \frac{43}{223}$
- (b) What is the probability we select a first-born student?
 $P(\text{First Born}) = \frac{113}{223}$
- (c) What is the probability that the student is a first-born and a Human Ecology student?
 $P(C) = P(A \cap B) = \frac{15}{223}$
- (d) What is the probability that the student is a first born or a Human Ecology student?

$$\begin{aligned}
 P(D) &= P(A \cup B) \\
 &= P(A) + P(B) - P(\cap B) \\
 &= \frac{43}{223} + \frac{113}{223} - \frac{15}{223} \\
 &= \frac{141}{223}
 \end{aligned}$$

4. A sample space consists of 5 simple events, E_1, E_2, E_3, E_4, E_5 .

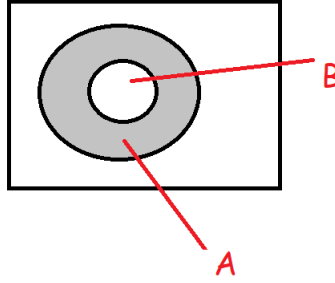
- (a) If $P(E_1) = P(E_2) = 0.15$, $P(E_3) = 0.4$, and $P(E_4) = 2P(E_5)$, find the probabilities of E_4 and E_5 .

$$\begin{aligned}
 1 &= 0.15 + 0.15 + 0.4 + 2P(E_5) + 2P(E_5) \\
 0.3 &= 3P(E_5) \\
 P(E_5) &= 0.1
 \end{aligned}$$

Then:

$$\begin{aligned}
 P(E_4) &= 2(0.1) \\
 &= 0.2
 \end{aligned}$$

5. If A, B are events, $B \subset A$, $P(A) = 0.6$ and $P(B) = 0.2$, then find $P(A \cap B^c)$.
To visualize:



$$\begin{aligned} P(A) &= P(A \cap B^c) + P(B)0.6 &= P(A \cap B^c) + 0.2 \\ P(A \cap B^c) &= 0.4 \end{aligned}$$

6. An experiment involves tossing a pair of dice and observing the numbers on the upper faces. Find the number of sample points in Ω and the sample space for the experiment. The sample points are $\Omega_{\{m, n\}} = \{1, 2, 3, 4, 5, 6\}$.
The sample space is taking the cross product of the sample points (i.e., $\Omega_1 \times \Omega_2$). Then:

$$\Omega\{(1, 1), (1, 2), \dots, (6, 6)\}$$

There is a $m \cdot n = 6 \cdot 6 = 36$ chance for a single probability.

7. The names of three employees are randomly drawn without replacement from a bowl containing the names of 30 employees of a small company. The person whose name is drawn first receives \$100, and the individuals whose names are drawn second and third receive \$50 and \$25, respectively. How many samples are associated with this experiment? There are 30 employees and 3 people are picked. So, $r = 3, n = 30$. Hence:

$$\begin{aligned} \Omega &= P_3^{30} \\ &= \frac{30!}{(30-3)!} \\ &= \frac{30!}{27!} \end{aligned}$$

8. Find the number of ways of selecting two applicants out of five.
Here, $n = 5, r = 2$.

$$\begin{aligned} \binom{5}{2} &= \frac{5!}{2!(5-2)!} \\ &= \frac{5!}{2!3!} \\ &= \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} \\ &= 10 \end{aligned}$$

1.2 Conditional Probability

1. Let $N = R^c$ be the event 'born in a month without r', and L is the event 'born in a long month'. What is the conditional probability $P(N|L)$?

$$P(R^c|L) = \frac{3}{7}$$

Note that $P(R|L) + P(R^c|L) = \frac{4}{7} + \frac{3}{7} = 1$.

2. A survey asked: ‘Are you currently in a relationship?’, and ‘Are you involved in club sports?’. The survey found that 33% were in a relationship, and 25% were involved in sport. 11% said yes to both. Suppose you meet a student who is on a sports team. What is the probability they are also in a relationship?

Let A be students in a relationship, and B be students in a club sport.

$P(A) = 0.33$, $P(B) = 0.25$, $P(A \cap B) = 0.11$. We have:

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{0.11}{0.25} \\ &= 0.44 \end{aligned}$$

We know that $P(A|B) \neq P(A)$, A and B must be dependent events.

3. A bowl contains 6 red and 4 blue balls, A child selects two balls at random. What is the probability at least one of them is red?

We have:

- (a) $\{R, R\} = \frac{6}{10}, \frac{5}{9}$
- (b) $\{R, B\} = \frac{6}{10}, \frac{4}{9}$
- (c) $\{B, R\} = \frac{4}{10}, \frac{6}{9}$
- (d) $\{B, B\} = \frac{4}{10}, \frac{3}{9}$

Note that $P(\text{at least one red}) + P(\text{no red}) = 1$, so:

$$\begin{aligned} P(\text{one_R}) &= 1 - P(\text{no_R}) \\ &= 1 - P(BB) \\ &= 1 - P(B_1 \cap B_2) \\ &= 1 - P(B_2|B_1)P(B_1) \\ &= 1 - \frac{3}{9} \cdot \frac{4}{10} \\ &= 1 - \frac{2}{15} \\ &= \frac{13}{15} \end{aligned}$$

4. A recent highway safety study found that in 77% of all accidents, the driver was wearing a seatbelt. Accident reports indicated that 92% of those drivers did not experience serious injury (defined as hospitalization or death), but only 63% of the non-belted were okay.

- (a) What is the probability that a driver was seriously injured?
- (b) What is the probability that a driver who was seriously injured was not wearing a seatbelt?

Let B is those that are wearing a seat belt, and NB is for those who did not wear a seat belt. Then, let I represent those who experienced a serious injury.

We have $P(B) = 0.77$, $P(NB) = 1 - 0.77 = 0.23$, and $P(I^c|B) = 0.92$, $P(I^c|NB) = 0.63$.

Then, $P(I|B) = 0.08$, $P(I|NB) = 0.37$.

[a)] Want to find $P(I)$:

$$\begin{aligned} P(I) &= P(B \cap I) + P(NB \cap I) \\ &= P(I|B)P(B) + P(I|NB)P(NB) \\ &= 0.08 \cdot 0.77 + 0.37 \cdot 0.23 \\ &= 0.0616 + 0.0851 \\ &= 0.1467 \end{aligned}$$

Want to find $P(NB|I)$.

$$\begin{aligned}
 P(I|NB) &= \frac{P(NB \cap I)}{P(I)} \\
 &= \frac{P(I|NB)P(I)}{0.1467} \\
 &= \frac{0.23 \cdot 0.37}{0.1467} \\
 &= \frac{0.0851}{0.1467} \\
 &= 0.58
 \end{aligned}$$

- (b) A student figures he has a 30% chance of being let out of class late. If he leaves class late, there is a 45% chance he will miss the bus. What is the probability that he is let out of class late and misses the bus?

Let L be the chance he leaves class late. Let M be the chance he misses the bus.

$P(L) = 0.3$, $P(M|L) = 0.45$. We want to find $P(L \cap M)$:

$$\begin{aligned}
 P(L \cap M) &= P(M|L)P(L) \\
 &= 0.45 \cdot 0.3 \\
 &= 0.135
 \end{aligned}$$

6. Leah flies from Moncton to Vancouver with a connection in Montreal. The probability her flight leaves on time is 15%. If the flight is on time, the chance her luggage makes the connecting flight in Montreal is 95%. But, if the flight is delayed, the probability the luggage will make it is only 65%.

- (a) Are the first flight leaving on time and the luggage making the connection independent events? Explain.
- (b) What is the probability that Leah's luggage arrives in Vancouver with her?

Let F be the chance her flight leaves on time. Let L be the luggage being in the connecting flight.

$P(F) = 0.15$, $P(L|F) = 0.95$, $P(L|F^c) = 0.65$.

[a)] We have $P(L|F) = 0.95$ and $P(L) = 0.695$, so they are not equal. Therefore, they are dependent events.

$$\begin{aligned}
 P(L) &= P(L|F)P(F) + P(L|F^c)P(F^c) \\
 &= 0.95 \cdot 0.15 + 0.65 \cdot 0.85 \\
 &= 0.1425 + 0.5525 \\
 &= 0.695
 \end{aligned}$$

- (b) The Ontario Lottery association claims your odds of winning a prize on an Instant Win Crossword game are 1 in 3.09. This means that any ticket has 32.4% chance of winning a prize. Every Friday, you buy one Crossword game.

- (a) What is the probability you don't win a prize next Friday?
- (b) What is the probability you don't win a prize 6 Fridays in a row?
- (c) If you haven't won a prize for the past 6 Fridays, what is the probability you win a prize on your next game?
- (d) What is the probability you win on two of your next three games?

Let W be the chance you win a prize. $P(W) = 0.324$.

- (a) Want to find $P(W^c)$. $P(W^c) = 1 - 0.324 = 0.676$

(b) Want to find $P(W_1^c \dots W_6^c)$:

$$\begin{aligned} P(W_1^c \dots W_6^c) &= P(W_1^c) \cdot \dots \cdot P(W_6^c) \\ &= P(W^c)^6 \\ &= 0.676^6 \\ &= 0.0954 \end{aligned}$$

(c) Want to find $P((W_1^c \dots W_6^c)^c)$. So: $P((W_1^c \dots W_6^c)^c) = 1 - 0.0954 = 0.9046$.

(d) Want to find $P(W_1 W_2 W_3^c)$:

$$\begin{aligned} P(W_1 W_2 W_3^c) &= 0.324 \cdot 0.3424 \cdot 0.676 \\ &= 0.071 \end{aligned}$$

1.3 Random Variables

1. Suppose we roll two dice. List the outcomes:

(a) $\{X = 8\} = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$.
 $P(X = 8) = \frac{5}{36}$

(b) $\{X = 3\} = \{(1, 2), (2, 1)\}$.
 $P(X = 3) = \frac{2}{36} = \frac{1}{18}$

(c) $\{X = 13\} = \{\}$
 $P(X = 13) = 0$

2. Suppose the death rate in a year is 1 out of 1000 people, and that 2 out of 1000 people suffer some kind of disability. Then, we display the probability model for X = company payout in a table:

| Policyholder Outcome | Payout x | P(X = x) |
|----------------------|----------|----------|
| Death | 10000 | 1/1000 |
| Disability | 5000 | 2/1000 |
| Neither | 0 | 998/1000 |

What is $P(X = 0)$?

We have $X = 0, 5000, 10000$, and $P(X = 10000) = \frac{1}{1000}$, $P(X = 5000) = \frac{2}{1000}$. We calculate:

$$\begin{aligned} P(X = 0) &= 1 - \frac{1}{1000} - \frac{2}{1000} \\ &= \frac{997}{1000} \end{aligned}$$

3. Take the previous example, and determine $E(X)$, where X is the company payout.

Recall: $X = 0, 5000, 10000$, and $P(X = 10000) = \frac{1}{1000}$, $P(X = 5000) = \frac{2}{1000}$, and $P(X = 0) = \frac{997}{1000}$.

To calculate $E(X)$:

$$\begin{aligned} E(X) &= \left(10000 \cdot \frac{1}{1000}\right) + \left(5000 \cdot \frac{2}{1000}\right) + \left(0 \cdot \frac{997}{1000}\right) \\ &= 10 + 10 \\ &= 20 \end{aligned}$$

Next, suppose we wanted to calculate the variance of X .

We have:

(a) $10000 - 20 = 9980$

(b) $5000 - 20 = 4980$

(c) $0 - 20 = -20$

So, we have:

$$\begin{aligned}\sigma^2 &= \sum_x (x - \mu)^2 P(x) \\ &= \left(9980^2 \cdot \frac{1}{1000}\right) + \left(4980^2 \cdot \frac{2}{1000}\right) + \left((-20)^2 \cdot \frac{997}{1000}\right) \\ &= 149600\end{aligned}$$

Then:

$$\begin{aligned}\sigma &= SD(X) = \sqrt{V(X)} \\ &= \sqrt{149600} \\ &= \$386.78\end{aligned}$$

4. If $E(X) = 10$ and $V(X) = 3$, then:

(a) Calculate $E(9x + 11)$.

$$\begin{aligned}E[9x + 11] &= 9E(x) + 11 \\ &= (9 \cdot 10) + 11 \\ &= 90 + 11 = 101\end{aligned}$$

(b) Calculate $V(9x + 11)$.

$$\begin{aligned}V(9x + 11) &= 9^2 V(x) \\ &= 9^2 \cdot 3 \\ &= 243\end{aligned}$$

(c)

$$\begin{aligned}\sigma^2 &= E(X - \mu)^2 \\ &= E(x^2 - 2\mu x + \mu^2) \\ &= E(x^2) - 2\mu E(X) + \mu^2 \\ &= E(x^2) - 2\mu^2 + \mu^2 \\ &= E(x^2) - \mu^2\end{aligned}$$

(d) If $E(X) = 10$ and $E(X^2) = 160$:

$$\sigma^2 = 160 - 10^2 = 60$$

5. The probability distribution for a random variable Y is given by the following table. Find the mean, variance, and standard deviation of Y .

| y | $p(y)$ |
|-----|--------|
| 0 | 1/8 |
| 1 | 1/4 |
| 2 | 3/8 |
| 3 | 1/4 |

Then:

$$\begin{aligned}F(0) &= P(Y \leq 0) \\&= P(Y = 0) = \frac{1}{8}\end{aligned}$$

$$\begin{aligned}F(1) &= P(Y \leq 1) \\&= P(Y = 0) + P(Y = 1) \\&= \frac{1}{8} + \frac{1}{4} = \frac{3}{8}\end{aligned}$$

$$\begin{aligned}F(2) &= P(Y \leq 2) \\&= \frac{3}{8} + \frac{1}{4} + \frac{1}{8} = \frac{6}{8}\end{aligned}$$

$$\begin{aligned}F(3) &= P(Y \leq 3) \\&= \frac{6}{8} + \frac{1}{4} = 1\end{aligned}$$

6. Suppose we toss a coin n times. Let $n = 5$, and X be the number of heads in n trials. Suppose we wanted to find the probability of $P(SFFSS)$; we have:

$$\begin{aligned}P(SFFSS) &= P(S)P(F)P(F)P(S)P(S) \\&= p \cdot (1-p) \cdot (1-p) \cdot p \cdot p \\&= p^3(1-p)^{5-3}\end{aligned}$$

Then, since we know it is independent, we can write this as $P(X = 3)$. Thus, we have:

$$P(X = 3) = \binom{5}{3} p^3 (1-p)^{5-3}$$

7. A tennis player makes successful first serves 70% of the time. Assume each serve is independent of the others. If she serves, what is the probability she gets:
- (a) All six serves in?
 - (b) Four serves in?
 - (c) At least four serves in?
 - (d) No more than four serves in?

We have $P(S) = 0.7$ and $P(F) = 0.3$. Let X be the number of successful serves. Let $n = 6$, then $X \sim B(6, 0.7)$. We have:

$$P(X = x) = \binom{6}{k} 0.7^x 0.3^{6-x}$$

To solve each part, we can just plug in values:

(a)

$$\begin{aligned}P(X = 6) &= \binom{6}{6} 0.7^6 0.3^{6-6} \\&= \frac{6!}{6!0!} \cdot 0.7^6 \\&= 0.7^6 = 0.118\end{aligned}$$

(b)

$$\begin{aligned}P(X = 4) &= \binom{6}{4} 0.7^4 0.3^{6-4} \\&= \frac{6!}{4!2!} 0.7^4 \cdot 0.3^2 \\&= 0.324\end{aligned}$$

(c)

$$\begin{aligned}P(X \geq 4) &= P(X = 4) + P(X = 5) + P(X = 6) \\&= 0.744\end{aligned}$$

(d)

$$\begin{aligned}P(X \leq 4) &= P(X = 4) + P(X = 3) + \dots + P(X = 0) \\&= 0.580\end{aligned}$$

8. Suppose that a lot of 5000 electrical fuses contain 5% defectives.

- (a) If a sample of 5 fuses is tested, find the probability of observing at least one defective.
- (b) If $n = 20$ fuses are randomly sampled from this lot, find the probability that at most 6 defectives are observed.
- (c) If $n = 20$ fuses are randomly sampled, find the probability that at least four defectives will be observed.

We know this event is independent; we have $P(F) = 0.05$ and $P(S) = 0.95$ for 5000 fuses. Let X be the number of defectives in n fuses.

- (a) $n = 5$, we want to find $P(X \geq 1)$.

Using the binomial distribution table, we have:

$$\begin{aligned}P(X \geq 1) &= 1 - P(X \leq 0) \\&= 1 - 0.774 \\&= 0.226\end{aligned}$$

(b)

9. Suppose the probability of an engine malfunction during a one-hour period is $p = 0.02$. Find the probability that the engine will survive for two hours.

We have $p = P(S) = 0.02$, so $q = P(F) = 0.98$. Let Y be the number of one-hour intervals until a malfunction occurs. We have:

$$\begin{aligned}P(y) &= 0.98^{y-1} \cdot 0.02 && \text{For } y = 1, 2, \dots \\&= P(Y \geq 3) \\&= 1 - P(Y \leq 2) \\&= 1 - P(Y = 1) - P(Y = 2) \\&= 1 - (0.98^1 \cdot 0.02) - (0.98^0 \cdot 0.02) \\&= 0.9604\end{aligned}$$

10. If the probability of an engine malfunction is $p = 0.02$ and Y is the number of one-hour intervals until the first malfunction, find the mean and standard deviation of Y .

$$\begin{aligned} E(Y) &= \frac{1}{p} \\ &= \frac{1}{0.02} \\ &= 50 \\ V(Y) &= \frac{1 - 0.02}{0.02^2} \\ &= 2450 \end{aligned}$$

11. Suppose a professor randomly selects 3 new teaching assistants from a total of 10 applicants - 6 male, 4 female. Let X be the number of females who are hired.

- (a) Find the probability that no females are hired.
 (b) Find the probability that at least 2 females are hired.

Here, $r = 4$, then $N - r = 6$. Set $n = 3$; we use hypergeometric distribution. First, calculate x :

$$\begin{aligned} \max\{0, n - (N - r)\} &: n - (N - r) \\ &: 3 - 6 = -3 \\ &= 0 \\ \min\{r, n\} &: \min 4, 3 \\ &= 3 \end{aligned}$$

Hence, $x = 0, 1, 2, 3$.

$$P(X = x) = \frac{\binom{4}{x} \binom{6}{3-x}}{\binom{10}{3}}$$

(a)

$$\begin{aligned} P(X = 0) &= \frac{\binom{4}{0} \binom{6}{3}}{\binom{10}{3}} \\ &= \frac{6!}{3!3!} \cdot \frac{7!3!}{10!} \\ &= \frac{1}{6} \end{aligned}$$

(b) We want to find $P(X \geq 2)$, i.e., $1 - P(X = 0) - P(x = 1)$.

$$\begin{aligned} P(X \geq 2) &= 1 - \frac{1}{6} - \frac{\binom{4}{1} \binom{6}{2}}{\binom{10}{3}} \\ &= 1 - \frac{1}{6} - \frac{1}{2} \end{aligned}$$

12. Suppose there are X number of blue whale sightings per week. Assume X follows a Poisson distribution and the number of weekly sightings is 2.5.

- (a) Find the mean and standard deviation of X .

$$\begin{aligned} E(X) &= 2.5 \\ \sigma^2 &= V(X) = 2.5 \\ \sigma &\approx 1.58 \end{aligned}$$

- (b) Find the probability that 5 sightings are made in one week.

$$P(X = 5) = \frac{2.5^5 e^{-2.5}}{5!}$$

We can use a Poisson Table:

$$\begin{aligned} P(X = 5) &= P(X \leq 5) - P(X \leq 4) \\ &= 0.958 - 0.891 \\ &= 0.067 \end{aligned}$$

- (c) Find the probability that there are fewer than 2 sightings made per week. Use a Poisson Table:

$$\begin{aligned} P(X < 2) &= P(\leq 1) \\ &= 0.287 \end{aligned}$$

1.4 Continuous Random Variables

1. Suppose that:

$$F(y) = \begin{cases} 0 & y < 0 \\ y & 0 \leq y \leq 1 \\ 0 & y > 1 \end{cases}$$

Find the probability density function for Y .

Take the derivative, and get:

$$f(y) = \begin{cases} 0 & y < 0 \\ 1 & 0 \leq y \leq 1 \\ 0 & y > 1 \end{cases}$$

We can also write this as $f(y) = 1$ if $0 \leq y \leq 1$, and 0 otherwise.

2. Let Y be a continuous random variable with probability density function given by:

$$f(y) = \begin{cases} 3y^2 & 0 \leq y \leq 10 \\ \text{otherwise} \end{cases}$$

Find $F(y)$.

To find $F(y)$ we take the anti-derivative.

For $y < 0$, we have:

$$\begin{aligned} F(y) &= P(Y \leq y) \\ &= \int_{-\infty}^y 0 dt \\ &= 0 \end{aligned}$$

For $0 \leq y \leq 10$, we have:

$$\begin{aligned} F(y) &= \int_{-\infty}^y f(t) dt \\ &= \int_{-\infty}^0 0 dt + \int_0^y 3t^2 dt \\ &= t^3 \Big|_0^y \\ &= y^3 \end{aligned}$$

For $y > 1$:

$$\begin{aligned} F(y) &= \int_{-\infty}^0 0dt + \int_0^1 3t^2 dt + \int_1^{\infty} 0dt \\ &= 1 \end{aligned}$$

Hence, we have:

$$F(y) = \begin{cases} 0 & y < 0 \\ y^3 & 0 \leq y \leq 1 \\ 1 & y > 1 \end{cases}$$

3. Given $f(y) = cy^2, 0 \leq y \leq 2, f(y) = 0$ elsewhere, find the value of c for which $f(y)$ is a valid density function.

For a probability density function $\int_{-\infty}^{\infty} f(y)dy = 1$, and have $\int_0^2 cy^2 dy = 1$. Then:

$$\begin{aligned} c \frac{y^3}{3} \Big|_0^2 &= 1 \\ c \frac{8}{3} &= 1 \\ c &= \frac{3}{8} \end{aligned}$$

4. Find $P(1 \leq Y \leq 2)$ from the previous example.

We have:

$$f(y) = \frac{3}{8}y^2 \text{ for } 0 \leq y \leq 2$$

$$\begin{aligned} P(1 \leq Y \leq 2) &= \int_1^2 \frac{3}{8}y^2 dy \\ &= \frac{1}{8}[y^3]_1^2 \\ &= \frac{1}{8}(8 - 1) \\ &= \frac{7}{8} \end{aligned}$$

How about for $P(1 \leq Y \leq 5)$? We know that it is valid only for $0 \leq y \leq 2$, so we split it up:

$$P(1 \leq Y \leq 5) = P(1 \leq Y \leq 2) + P(2 \leq Y \leq 5)$$

5. From the previous example, find $\mu = E(Y)$ and $\sigma^2 = V(Y)$.

We have $f(y) = \frac{3}{8}y^2$ for $0 \leq y \leq 2$, and 0 otherwise. To find the expected value:

$$\begin{aligned} \mu = E(Y) &= \int_0^2 y \cdot \frac{3}{8}y^2 dy \\ &= \frac{3}{8} \int_0^2 y^3 dy \\ &= \frac{3}{8} \left[\frac{y^4}{4} \right]_0^2 \\ &= \frac{3}{8}(4 - 0) \\ &= \frac{3}{2} \end{aligned}$$

Thus, $\mu = E(Y) = 1.5$.

Then, find the variance. Find $E(Y^2)$:

$$\begin{aligned} E(Y^2) &= \int_0^2 y^2 \cdot \frac{3}{8} y^2 dy \\ &= \frac{3}{8} \int_0^2 y^4 dy \\ &= \frac{3}{8} \left[\frac{y^5}{5} \right]_0^2 \\ &= \frac{3}{8} \left(\frac{32}{5} - 0 \right) \\ &= \frac{12}{5} \end{aligned}$$

Hence, $E(Y^2) = 2.4$. Now, we can calculate $V(Y)$:

$$\begin{aligned} V(Y) &= E(Y^2) - \mu^2 \\ &= 2.4 - 1.5^2 \\ &= 0.15 \end{aligned}$$

6. Esme is taking a train from Minneapolis to Boston, a distance of roughly 1400 miles. Her position is uniformly distributed between the two cities. What is the probability that she is past Chicago, which is 400 miles from Minneapolis?

Let X be Esme's location, where it follows the uniform probability distribution on the interval $(0, 1400)$. We have $f(x) = \frac{1}{1400}$ if $0 \leq x \leq 1400$, and 0 otherwise.

We want to find $P(X > 400)$, so:

$$\begin{aligned} \int_{400}^1 400 \frac{1}{1400} dx &= \frac{1}{1400} [x]_{400}^{1400} \\ &= \frac{1400 - 400}{1400} \\ &= \frac{10}{14} \end{aligned}$$

7. Let Z denote a normal random variable with mean 0 and standard deviation 1 (the standard normal random variable).

That is, $Z \sim N(0, 1)$.

- (a) Find $P(Z > 2)$:

Using the Normal curve table, we have:

$$\begin{aligned} P(Z > 2) &= 1 - P(Z \leq 2) \\ &= 1 - 0.9772 \\ &= 0.0228 \end{aligned}$$

- (b) Find $P(-2 \leq Z \leq 2)$:

$$\begin{aligned} P(-2 \leq Z \leq 2) &= P(Z \leq 2) - P(Z < -2) \\ &= 0.9772 - 0.0228 \\ &= 0.9544 \end{aligned}$$

- (c) Find $P(0 \leq Z \leq 1.73)$:

$$\begin{aligned} P(0 \leq Z \leq 1.73) &= P(Z \leq 1.73) - P(Z < 0) \\ &= 0.9582 - 0.5 \\ &= 0.4582 \end{aligned}$$

8. The achievement scores for a college entrance exam are normally distributed with mean 75 and standard deviation 10. What fraction of the scores lie between 80 and 90?

Let $Y \rightarrow \text{Score}$. We have $Y \sim N(75, 10)$. We want to find $P(80 \leq Y \leq 90)$.

We want to standardize this to use the Normal curve table. So:

$$\begin{aligned} P(80 \leq Y \leq 90) &= P\left(\frac{80 - 75}{10} \leq \frac{Y - \mu}{\sigma} \leq \frac{90 - 75}{10}\right) \\ &= P(0.5 \leq Z \leq 1.5) \\ &= P(Z \leq 1.5) - P(Z < 0.5) \\ &= 0.9332 - 0.6915 \\ &= 0.2417 \end{aligned}$$

9. If Z is the standard normal random variable, find the value z_0 such that:

(a) $P(Z > z_0) = 0.5$:

We know the mean is 0, so $z_0 = 0$.

(b) $P(Z < z_0) = 0.8643$:

Using the standard normal curve table, $z_0 = 1.10$.

(c) $P(-z_0 < Z < z_0) = 0.9$:

We can use the symmetric property, we calculate the unused area:

$$\begin{aligned} 1 - 0.9 &= 0.1 \\ \frac{0.1}{2} &= 0.05 \end{aligned}$$

So, each side of the unused area has size 0.05. Then,

$$P(Z \leq z_0) = 0.95P(Z < -z_0) = 0.05$$

Hence, $z_0 = 1.645$.

(d) $P(-z_0 < Z < z_0) = 0.99$:

We have $1 - 0.99 = 0.01$, and so each side of the unused area has size 0.005. Then:

$$P(Z \leq z_0) = 0.995P(Z \leq -z_0) = 0.005$$

Then, $z_0 = 2.575$.

10. The width of bolts of fabric is normally distributed with mean 950mm and standard deviation 10mm.

(a) What is the probability that a randomly chosen bolt has a width of between 947 and 958 mm?

Let $Y \rightarrow \text{width}$, $Y \sim N(950, 10)$.

We want to find $P(947 \leq Y \leq 958)$.

$$\begin{aligned} P\left(\frac{947 - 950}{10} \leq Z \leq \frac{958 - 950}{10}\right) &= P(-0.3 \leq Z \leq 0.8) \\ &= P(Z \leq 0.8) - P(Z < -0.3) \\ &= 0.7881 - 0.3821 \\ &= 0.406 \end{aligned}$$

(b) What is the appropriate value for C such that a randomly chosen bolt has width less than C with probability 0.8531?

We want to find $P(Z < C) = 0.8531$.

$$P\left(Z < -\frac{C - 950}{10}\right) = 0.8531$$

$$P(Z < z_0) = 0.8531,$$

where $z_0 = \frac{C-950}{10}$. We find $z_0 = 1.05$ using the table then find C :

$$\frac{C - 950}{10} = 1.05$$

$$C = 1.05 \cdot 10 + 950$$

$$= 10.5 + 950$$

$$= 960.5$$

11. If a die is rolled 600 times, what is the probability of rolling between 90 and 110 fours?

$\Omega = \{1, 2, 3, 4, 5, 6\}$, and $p = P(4) = \frac{1}{6}$.

Let $n = 600$, and $X \rightarrow$ number of fours in 600 rolls, where $X \sim B(600, \frac{1}{6})$. We also have $np = 600 \cdot \frac{1}{6} = 100$, and $n(1 - p) = 600 \cdot \frac{1}{6} = 100$.

Hence, $\mu = np = 100$, and $\sigma^2 = np(1 - p) = \frac{500}{6}$.

Now, we want to find $X \sim N(\mu, \sigma)$ - Note that because we are using continuous random variables, include decimals up to the tenth digit.

$$P(90 \leq Y \leq 110) = P(89.5 \leq Y \leq 110.5)$$

$$= P\left(\frac{89.5 - 100}{\sqrt{\frac{250}{3}}} \leq Z \leq \frac{110.5 - 100}{\sqrt{\frac{250}{3}}}\right)$$

$$= P(-1.15 \leq Z \leq 1.15)$$

$$= P(Z \leq 1.15) - P(Z < -1.15)$$

$$= 0.8749 - 0.1251$$

$$= 0.7498$$

Note, because it is a discrete random variable, using a strict relation ($<$), change it to \leq and then add the decimal. For example, $P(X < 420)$ becomes $P(X \leq 419.5)$.