STA237 - Activity 4

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1. (a) We want to prove $\mu = E(Y) = \frac{\alpha}{\alpha+\beta}$.
   We have $f(y) = \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)}$, for $0 \leq y \leq 1$.

$$\begin{aligned} E(Y) &= \int_0^1 y \cdot \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)} dy \\ &= \frac{1}{B(\alpha,\beta)} \int_0^1 y^{(\alpha+1)-1}(1-y)^{\beta-1} dy \\ &= \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)} \\ &= \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \end{aligned}$$

   We know that $\Gamma(\alpha) = (\alpha-1)\cdot\Gamma(\alpha-1)$, so we have;

$$\begin{aligned} E(Y) &= \frac{\alpha\Gamma(\alpha)\Gamma(\alpha+\beta)}{(\alpha+\beta)\Gamma(\alpha+\beta)\Gamma(\alpha)} \\ &= \frac{\alpha}{\alpha+\beta} \end{aligned}$$

   (b) We want to prove $\sigma^2 = V(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

$$\begin{aligned} E(Y^2) &= \frac{1}{B(\alpha,\beta)} \int_0^1 y^2 y^{\alpha-1}(1-y)^{\beta-1} dy \\ &= \frac{1}{B(\alpha,\beta)} \int_0^1 y^{(\alpha+2)-1}(1-y)^{\beta-1} dy \\ &= \frac{B(\alpha+2,\beta)}{B(\alpha,\beta)} \\ &= \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)} \cdot \frac{\Gamma(\alpha+\beta)}{(\alpha)\Gamma(\beta)} \\ &= \frac{(\alpha+1)\alpha\Gamma(\alpha)\Gamma(\alpha+\beta)}{(\alpha+\beta+1)(\alpha+\beta)\Gamma(\alpha+\beta)\Gamma(\alpha)} && \text{By definition of } \Gamma. \\ &= \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)} \end{aligned}$$

Next, we use $\sigma^2 = V(Y) = E(Y^2) - \mu^2$:

$$\begin{aligned}
\sigma^2 &= E(Y^2) - \mu^2 \\
&= \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)} - \left(\frac{\alpha}{\alpha+\beta}\right)^2 \\
&= \frac{\alpha}{(\alpha+\beta+1)(\alpha+\beta)^2}\left[(\alpha+1)(\alpha+\beta) - \alpha(\alpha+\beta+1)\right] \\
&= \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}
\end{aligned}$$