MAT137 - Summer 2022 Notes

Contents

1 Sets, Quantifiers, and Conditionals

1.1 Symbols, Quantifiers, and Conditionals

Logical symbols
$\wedge$ = "and" — Exclusive
$\vee$ = "or" — Inclusive
$\neg$ = "not" — Negation

A	B	$A \wedge B$	A $\vee$ B
T	T	T	T
T	F	F	T
F	T	F	T
F	F	T	F

Quantifiers
$\forall$ = Universal quantifier (For all)
$\exists$ = Existential quantifier (There exists)
The negation of $\forall$ is $\exists$; the negation of $\exists$ is $\forall$.

Conditionals
$\implies$ = Implies — (If A, then B)
$\iff$ = Biconditional (If and only if) — (A $\implies$ B) $\wedge$ (B $\implies$ A)

A	B	$A \implies B$	A $\iff$ B
T	T	T	T
T	F	F	F
F	T	T	F
F	F	F	T

1.2 Logically Equivalent Statements

If two statements are **logically equivalent**, they will have the same true/false values. For example, the negation of A $\implies$ B is logically equivalent to A $\wedge$ ($\neg$ B).
For the implication A $\implies B$:
  $\neg$B $\implies$ $\neg$A — Contrapositive
  B $\implies$ A — Converse
  $\neg$A $\implies$ $\neg$B — Inverse

A	B	A $\implies$ B	$\neg$B $\implies$ $\neg$A	B $\implies$ A	$\neg$A $\implies$ $\neg$B
T	T	T	T	T	T
T	F	F	F	T	T
F	T	T	T	F	F
F	F	T	T	T	T

1.3 Examples of Logical Statements

1. **Students on fire**
   "No two students in this class are not on fire", written using logical symbols is:
   $\forall$x, y $\in S$, $(F(x) \wedge F(y))$

   To find the negation, we can write:
   $\exists$x, y $\in S$, $\neg(F(x) \wedge F(y))$
   In English, we would say, "There exists a pair of students in the class that not are on fire."

1.4 Sets

Sets are collections of elements: we use set builder notation to mathematically describe them.
For example, the **set of even numbers** is written as: $\{\forall n \in \mathbb{Z} : \exists a \in \mathbb{Z}, n = 2a\}$
In other words, it is the set of all integers n such that for some value a, n is a multiple of $2a$.

Notation
$\cup$ = Union — The set of items that are in the set A or B: $A \cup B = \{x : x \in A \lor x \in B\}$
$\cap$ = Intersection — The set of items that are in the set A and B: $A \cap B = \{x : x \in A \land x \in B\}$
$\emptyset$ = Empty Set — A set that contains no elements. If the set is empty, then it is implied that $\forall x \in \emptyset, x$ is true.

Symmetric Differences
Given two sets A and B, we can define:
$A \setminus B = \{x \in A : x \notin B\}$ — Set minus: The items in A that are not in B.
$A \triangle B = (A \setminus B) \cup (B \setminus A)$ — Symmetric Difference: The items that are not in both A and B (the union of two sets excluding the intersection).
Symmetric differences are also called disjunctive unions, and can be expressed using the XOR operation $\oplus$.

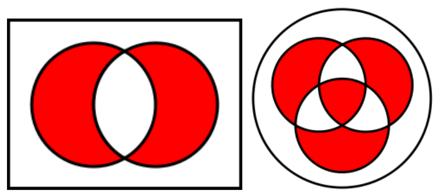


Figure 1: Visual representation of $A \triangle B$, and $(A \triangle B) \triangle C$.

Double Quantifiers
For multiple quantifiers, the order is very important. For example, let $S(x, y)$ = "x studies with y", where $x, y \in$ {Students}. Depending on how we quantify x and y, we get:

1. $\forall x, \forall y, S(x, y)$ — Every student studies with everyone.
2. $\forall x, \exists y, S(x, y)$ — Every student studies with at least one student.
3. $\exists x, \forall y, S(x, y)$ — There exists a student that studies with everyone.
4. $\exists x, \exists y, S(x, y)$ — There exists a student that studies with at least one student.

2 Functions

2.1 Injective Functions

Injective functions are "one-to-one", meaning that different x values produce different f(x) values.
In other words, a function $f : A \to B$ is injective $\iff \forall x_1, x_2 \in A, x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$
Equivalently, $\forall x_1, x_2 \in A, f(x_1) = f(x_2) \implies x_1 = x_2$.

2.1.1 Proving Injectivity

Determine whether $f(x) = 2x^3 + 7$ is injective or not.

Proof. We want to show that $\forall x, y \in \mathbb{R}, f(x_1) = f(y) \implies x = y$. Let $x, y \in \mathbb{R}$. Assume $f(x) = f(y)$. So, we must show that $x = y$.

$$
\begin{aligned}
& 2x^3 + 7 = 2y^3 + 7 && \text{by assumption} \\
\implies & 2x^3 = 2y^3 && \text{using arithmetic} \\
\implies & \frac{2x^3}{2} = \frac{2y^3}{2} \\
\implies & (x^3)^{\frac{1}{3}} = (y^3)^{\frac{1}{3}} \\
\implies & x = y
\end{aligned}
$$

So, $f(x)$ is injective.

$\square$

2.1.2 Proving Non-Injectivity

To prove a function $f : A \to B$ to not be injective, show that $\exists x, y \in A, f(x) = f(y) \land x \neq y$

For example, prove that $f(x) = x^2$ is not injective.

Proof. Let $f(x) = x^2$. We want to show that $\exists x, y \in \mathbb{R}, x^2 = y^2 \land x \neq y)$.
Let $x = 4$ and $y = -4$.
We can see that $(4)^2 = (-4)^2$, so $f(x) = f(y)$, but $x \neq y$.
Therefore, there exists two x values from the same y value, hence $y = x^2$ is not injective.

$\square$

2.1.3 Increasing Functions

A function $f : A \to B$ is said to be increasing if for $\forall a < b \in A, f(a) < f(b)$.
Similarly, a function is decreasing if for $\forall a < b \in A, f(a) > f(b)$.

2.2 Theorem: Increasing Functions Implies Injectivity

Claim. If f is increasing on A, then f is injective on A.

Proof. Let $a, b \in A$. Assume f is increasing. We want to show f is injective. Assume $a \neq b$ and show that $f(a) \neq f(b)$.
It was assumed that $a \neq b$ implies $a < b \implies f(a) < f(b)$ or $b < a \implies f(b) < f(a)$, since f is increasing.
Therefore, $f(a) \neq f(b)$. Thus, if f is increasing, then it is injective.

$\square$

2.3 Disproving a Theorem

Claim. We want to disprove that if f is injective on A, then f is increasing.

Proof. Let $f : A \to B$ be a function. We want to show that it is not the case that $\forall$ such f, injectivity implies increasing.
This means that there exists a function that it is injective and not increasing.
Let $f(x) = 2 - x$, with $x \in A$.
Injectivity: Let $a, b \in A$. If $2 - a = 2 - b$, it is implied that $a = b$.
Not increasing: We want to show that $\exists a, b \in A, a < b \land f(a) \geq f(b)$.
Let $a = 1, b = 3$, so $1 < b$ and $2 - 1 \geq 2 - 3$.

$$
\begin{aligned}
RHS : & 2 - 3 & \qquad LHS : & 2 - 1 \\
& -1 & & 1
\end{aligned}
$$

$1 \geq -1$, so $f(a) \geq f(b)$. Therefore, $\exists f : A \to B$ such that f is injective, and f is not increasing. $\square$

2.4 Floor Functions

Given an $x \in \mathbb{R}$, we define the **floor** of x, denoted $\lfloor x \rfloor$, as the largest integer $\leq x$. For example:

$$\lfloor \pi \rfloor = 3 \qquad \lfloor 7.9 \rfloor = 7 \qquad \lfloor -0.5 \rfloor = -1$$

3 Induction

Induction proofs are a type of mathematical proof, used to prove a statement $P(n)$ over a countable set (e.g., natural numbers $\mathbb{N}$).

3.1 Format of Induction Proofs

1. **Base Case:** Proving the statement for the first possible value for n.
2. **Hypothesis:** Assuming that statement $P(n)$ is possible for any given $n = k$.
3. **Induction Step:** Based on the hypothesis, prove the statement also holds for $n = k = 1$.

3.2 Example of Induction Proof

3.2.1 Example 1: Divisibility

Show that $n^3 + 2n$ is divisible by 3 for all positive integers n.

Proof. We WTS $\forall n \in \mathbb{N}, n^3 + 2n = 3x$ for some $x \in \mathbb{Z}$.

Base Case: Let $n = 1$. Then, $(1)^3 + 2(1) = 1 + 2 = 3$. Since $3 = 3x$ for $x = 1$, so our base case holds.

Induction Step: Assume that $n^3 + 2n = 3x$ be true for $n = k$. We want to show this holds for $n = k + 1$. So:

$$\begin{aligned}
(k+1)^3 + 2(k+1) &\implies (k^3 + 3x^2 + 3k + 1) + 2k + 2 && \text{By expanding.}\\
&\implies k^3 + 3x^2 + 5k + 3 && \text{Using arithmetic.}\\
&\implies (k^3 + 2k) + (3k^2 + 3k + 3) && \\
&\implies 3x + 3k^2 + 3k + 3 && \text{Since } k^3 + 2k = 3x\\
&\implies 3(x + 3k^2 + k + 1) && \text{By factoring 3.}
\end{aligned}$$

Hence, $(k + 1)^3 + 2(k + 1)$ must be divisible by 3.

Therefore, $n^3 + 2n$ is divisible by 3 for all positive integers n.

$\square$

4 Absolute Values

Absolute values, denoted $|n|$, is a value that is always considered 0 or positive.
It can be defined as a function $|x| : \mathbb{R} \to \mathbb{R}$ such that:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

4.1 Determining Absolute Values

See Tyler Holden's 137 Notes for examples (2.1.1).
Generally, break the absolute values into cases of when x will be positive, or when x is negative and we need to multiply the value by -1. By determining these cases, we can then either evaluate the absolute value, or determine its inequality to a polynomial.

4.2 Properties of Absolute Values

1. **Multiplicative:** If $x, y \in \mathbb{R}$, then $|xy| = |x||y|$.
2. **Non-degenerate:** $|x| = 0 \iff x = 0$.
3. **Triangle Inequality:** For any $x, y \in \mathbb{R}, |x + y| \leq |x| + |y|$.
   Similarly, the reverse triangle inequality: $||a| - |b|| \leq |a + b|$.

5 Limits

The limit $\lim_{x \to a} f(x) = L$ can be written as: $\forall \epsilon > 0, \exists \delta > 0 : 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$
By intuition, for any choice of ϵ, there is a δ value where the distance between x and a to be $\neq 0$ and arbitrarily small, it is implied that the distance between $f(x)$ and L is less than ϵ.

5.0.1 Some Properties of Limits

1. $\lim_{x \to a} f(x) + \lim_{x \to a} g(x) = \lim_{x \to a} (f(x) + g(x))$
   Let $\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M \implies \lim_{x \to a} f(x) + g(x) = L + M$.
   When we prove such limits, assume $0 < |x - a| < \delta$. Since $\delta = min\{\delta_1, \delta_2\}$, then we know that $|f(x) - L| < \epsilon_1$, and $|g(x) - M| < \epsilon_2$

   Using the Triangle Inequality, we can write:

$$\begin{aligned} |f(x) - L + g(x) - M| &\leq |f(x) - L| + |g(x) - M| && \text{By triangle inequality.} \\ &< \epsilon_1 + \epsilon_2 = \epsilon \end{aligned}$$

   Therefore, we would let $\epsilon_1 = \frac{\epsilon}{2}$, and let $\epsilon_2 = \frac{\epsilon}{2}$.

2. If f is continuous, $\lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)$.
3. $\lim_{x \to a} f(x)g(x) = (\lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$.

5.1 Examples of Limit Proofs

5.1.1 Example 1

We want to show that $\lim_{x \to 2}(4x + 1) = 9$.
So, $\forall \epsilon > 0, \exists \delta > 0 : 0 < |x - 2| < \delta \implies |(4x + 1) - 9| < \epsilon$
Rough Work:

$$\begin{aligned} |(4x + 1) - 9| &= |4x + 1 - 9| \\ &= |4x - 8| \\ &= |4(x - 2)| \\ &= 4|x - 2| \\ &\implies 4|x - 2| < 4\delta \end{aligned}$$

So, our choice of ϵ should be 4δ.

Proof. We WTS $\forall \epsilon > 0, \exists \delta > 0 : 0 < |x-2| < \delta \implies |(4x+1)-9| < \epsilon$.
Let $\epsilon \in \mathbb{R}$, with $\epsilon > 0$. Let $\delta = \frac{\epsilon}{4} > 0$.

$$\begin{aligned} 0 < |x-2| < \delta &\implies |x-2| < \delta \\ \implies |x-2| < \frac{\epsilon}{4} &\implies 4|x-2|\epsilon \\ &\implies |4x-8| = |(4x+1)-9| < \epsilon \end{aligned}$$

□

5.1.2 Example 2

We want to show that $\lim_{x\to -1}(x^2-3) = -2$
So, $\forall \epsilon > 0, \exists \delta > 0 : 0 < |x-(-1)| < \delta \implies |(x^2-3)-(-2)| < \epsilon$
Rough Work: We can assume that $|x+1| < \delta$.

$$\begin{aligned} |(x^2-3)-(-2)| &= |x^2-3+2| \\ &= |x^2-1| \\ &= |(x+1)(x-1)| \\ &= |x+1||x-1| < x\delta \end{aligned}$$

Assume $\delta < 1 \implies |x+1| < 1$

$$\begin{aligned} -1 < x+1 &< 1 \\ -2 < x &< 0 \\ -3 < x-1 &< -1 \\ |x-1| &< 3 \end{aligned}$$

Proof. WTS $\forall \epsilon > 0, \exists \delta > 0 : 0 < |x-(-1)| < \delta \implies |(x^2-3)-(-2)| < \epsilon$
Let $\epsilon \in \mathbb{R}, \epsilon > 0$.
Let $\delta = min\{\frac{\epsilon}{3}, 1\}$
Assume $0 < |x+1| < \delta \leq \frac{\epsilon}{3}$
Since $\delta \leq 1, |x+1| < \delta \implies -1 < (x+1) < 1$
$\implies -3 < (x-1) - 1$
$\implies |x-1| < 3 \implies |x-1||x+1| < 3\delta \leq 3\frac{\epsilon}{3}$
Therefore, $0 < |x+1| < \delta \implies |x-1||x+1| = |(x^2-3)-(-2)| < \epsilon$ □

5.1.3 Example 3

Prove that $\lim_{x\to 3}(x^2-2x+4) = 7$: $\forall \epsilon > 0, \exists \delta > 0 : 0 < |x-3| < \delta \implies |(x^2-2x+4)-7| < \epsilon$
Rough Work:

$$\begin{aligned} |x^2-2x+4-7| &= |x^2-2x-3| \\ &= |(x-3)(x+2)| \\ &= |x-3||x+1| \end{aligned}$$

If $\delta < 1$, then $|x-3| < 1 \implies -1 < (x-3) < 1$.
When we add 4, we get $3 < (x+1) < 5 \implies |x+1| < 5$

Proof. Let $\epsilon \in \mathbb{R}, \epsilon > 0$. Let $\delta = min\{\frac{\epsilon}{5}, 1\}$.
Assume $0 < |x-3| < \delta$, so $|x-3| < \delta \leq \frac{\epsilon}{5}$.
Since $\delta \leq 1 \implies |x-3| < 1$:

$$\begin{aligned} -1 < (x-3) < 1 &\implies 3 < (x+1) < 5 \\ &\implies |x+1| < 5 \end{aligned}$$

Therefore, $|(x^2 - 2x + 4) - 7| = |x^2 - 2x - 3|$
$= |x - 3||x + 1| < 5\delta \leq \epsilon$
Therefore, $0 < |x - 3| < \delta \implies |(x^2 - 2x + 4) - 7| < \epsilon$ □

5.1.4 Example 4

We want to show $\lim_{x \to 3} \frac{1}{x} = \frac{1}{3}$: $\forall \epsilon > 0, \exists \delta > 0, 0 < |x - 3| < \delta \implies |\frac{1}{x} - \frac{1}{3}| < \epsilon$
Rough Work:

$$\begin{aligned}
|\frac{1}{x} - \frac{1}{3}| &= |\frac{3}{3x} - \frac{x}{3x}| \\
&= |\frac{3 - x}{3x}| \\
&= \frac{|3 - x|}{3|x|} \\
&= \frac{|-(x - 3)|}{3|x|} \\
&= \frac{|x - 3|}{3|x|} \\
&= |x - 3|(\frac{1}{3})(\frac{1}{|x|})
\end{aligned}$$

Assume $|x - 3| < \delta$, and assume $\delta < 1$:

$$\begin{aligned}
\implies |x - 3| < 1 &\implies -1 < (x - 3) < 1 \\
&\implies -1 + 3 < x < 1 + 3 \\
&\implies 2 < x < 4 \\
&\implies |x| > 2 \implies \frac{1}{|x|} < \frac{1}{2}
\end{aligned}$$

Proof. Let $\epsilon \in \mathbb{R}$ and $\epsilon > 0$. Let $\delta = min\{1, 6\epsilon\}$.
Assume $0 < |x - 3| < \delta$.

$$\begin{aligned}
|\frac{1}{x} - \frac{1}{3}| &= |\frac{3 - x}{3x}| \\
&= \frac{|x - 3}{3|x|} < \delta(\frac{1}{3|x|}
\end{aligned}$$

and $|x - 3| \leq 1 \implies -1 < (x - 3) < 1 \implies 2 < x < 4$
So, $|x| > 2 \implies \frac{1}{|x|} < \frac{1}{2}$.
Therefore, $|\frac{1}{x} - \frac{1}{3}| < \delta(\frac{1}{6}) \leq \frac{6\epsilon}{6} = \epsilon$ □

5.2 Infinite Limits

Generally, we say if the limit exists at infinity (i.e., $\lim_{x \to \infty} = L$, then:

$$\forall \epsilon > 0, \exists M \in \mathbb{R} : x > M \implies |f(x) - L| < \epsilon$$

However, for a limit being equal to infinity (i.e., $\lim_x f(x) = \infty$, we say:

$$\forall M \in \mathbb{R}, \exists \delta > 0 : 0 < |x - c| < \delta \implies f(x) > M$$

5.2.1 Example

Prove $\lim_{x \to 3} \frac{1}{(x-3)^2} = \infty$

Rough Work. We want $\frac{1}{(x-3)^2} > M$. So, we have:

$$\begin{aligned} \frac{1}{|x-3||x_3|} &> \delta =?M \\ |x-3| &< \delta && \text{If } \delta \leq 1 \\ |x-3| &< 1 \\ \frac{1}{|x-3||x-3|} &> (1)(\delta) \end{aligned}$$

Proof. Let $M \in \mathbb{R}, M > 0$. Let $\delta = \min\{1, M\}$.
First, assume $|x-3| < \delta \leq 1$. Then, we get $|x-3| < 1$. This then implies:

$$\frac{1}{|x-3||x-3|} > \delta(1) \geq M(1) \implies \frac{1}{(x-3)^2} > M$$

□

5.3 One-Sided Limits

The limit that the function approaches as the x-values approach a limit from either the left or right side only. Denoted as $\lim_{x \to a^-} f(x)$ for the left-hand limit, and $\lim_{x \to a^+} f(x)$ for the right-hand limit.

Using the formal definition of limits, we write:

1. $\lim_{x \to a^+} f(x) = L \implies 0 < x - a < \delta$
2. $\lim_{x \to a^-} f(x) = L \implies -\delta < x - a < 0$

5.3.1 Example

We want to show $\lim_{x \to 0^+} \sqrt{x} = 0$: $\forall \epsilon > 0, \exists \delta > 0 : 0 < |x - 0| < \delta \implies |\sqrt{x}| < \epsilon$.
Rough Work: Let $\delta < 1$.

$$\begin{aligned} \sqrt{x} &< \epsilon \\ \sqrt{x}\sqrt{x} &< \epsilon\sqrt{x} \\ &< \delta \implies \sqrt{x} < \sqrt{\delta} \leq 1 \\ \sqrt{x} &< 1 \implies (\sqrt{x})^2 < \delta \end{aligned}$$

Proof. Let $\epsilon > 0, \epsilon \in \mathbb{R}$. Let $\delta = \min\{1, \epsilon^2\}$. Assume:

$$x < \delta \implies x < \epsilon \text{and} x < 1.$$
$$\implies \sqrt{x} < \sqrt{1} \implies \sqrt{x} < 1 \text{and} \sqrt{x} < \sqrt{\delta} \leq \sqrt{\epsilon^2}$$

Therefore, $\sqrt{x} < \epsilon$, as required. □

5.4 Exponential Limits

Exponential limits largely depend on the base (b) and the characteristics of its power $f(x)$ to determine the limit of $b^{f(x)}$.
To be precise, for a constant b, we have $\lim_{x \to a} b^{f(x)} \implies b^{\lim_{x \to a} f(x)}$.

In the case of $e^{f(x)}$, first sketch the graph of e^x, and use properties of $f(x)$ to evaluate the limit.

5.5 Rational Limits

The limits of rational functions that approach $\pm\infty$ can be estimated, and also help determine a function's asymptotes.

1. First, find all real values a for which the function is undefined.
2. For each value a, compute the limits $\lim_{x\to a^+}$ and $\lim_{x\to a^-}$.
   You can estimate such values by determining whether each factor of the function is positive or negative.
3. Using these answers, you can try to sketch the graph of the function.

5.6 Indeterminate Form

Indeterminate forms are improper limits that may not exist involving two functions.

1. Infinity over Infinity: $\frac{\infty}{\infty} \implies$ Indeterminate
2. Infinity minus Infinity: $\infty - \infty \implies$ Indeterminate
3. Zero over Zero: $\frac{0}{0} \implies$ Indeterminate
4. Zero times Infinity: $0 \cdot \infty \implies$ Indeterminate
5. Infinity to the Power of Zero: $\infty^0 \implies$ Indeterminate
6. One to the Power of Infinity: $1^\infty \implies$ Indeterminate

5.7 Squeeze Theorem

Let $a, L \in \mathbb{R}$. Let f, g, h be functions defined near a, except possibly at a.
If:

1. For x close to a but not a, $h(x) \le g(x) \le f(x)$
2. $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} h(x) = L$

Then:

1. $\lim_{x\to a} g(x) = L$

5.7.1 New Definition of Squeeze Theorem

Let $a \in \mathbb{R}$. Let $g(x)$ and $h(x)$ be functions defined near a, except possibly at a.
If:

1. For x close to a but not a, $h(x) \le g(x)$
2. $\lim_{x\to a} h(x) = \infty$

Then:

1. $\lim_{x\to a} g(x) = \infty$

5.7.2 Example Proof Using Squeeze Theorem

We want to show that $\lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0$.
We have to find $h(x) \le x^2 \sin(\frac{1}{x}$ and $g(x) \ge x^2 \sin(\frac{1}{x})$.

$$-1 \le \sin(\frac{1}{x}) \le 1 \implies -x^2 \le x^2 \sin(\frac{1}{x}) \le x^2$$

So, $\lim_{x\to 0} -x^2 = 0 = \lim_{x\to 0} x^2$.
Let $h(x) = -x^2, f(x) = x^2$. From above, we know $h(x) \le x^2 \sin(\frac{1}{x}) \le f(x)$.
Let $\lim_{x\to 0} h(x) = \lim_{x\to 0} f(x) = 0$. Then, by squeeze theorem, $\lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0$.

6 Continuity, IVT, EVT

6.1 Continuity of Functions

A function $f(x)$ is continuous at $x = a \iff \lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a^+} f(x)$.
Suppose f was discontinuous at $x = a$. We then have:

1. Jump Discontinuity: f from the left-side of a is a different value than from the right-side of a.
2. Removable Discontinuity: $\lim_{x \to a} f(x) \neq f(a)$.
3. Infinite Discontinuity: $\lim_{x \to a} f(x)$ DNE.

6.1.1 Min/Max and Continuity

Let $x, y \in \mathbb{R}$. We have:

$$f(x, y) = \frac{x + y + |x - y|}{2}$$

This equates to $\frac{x+y}{2} + \frac{|x+y|}{2}$, which means that $f(x, y) = \max\{x, y\}$.
Similarly, $g(x, y) = \frac{x+y-|x-y|}{2} \implies g(x, y) = \min\{x, y\}$

Theorem. If f, g are continuous functions, then $h(x) = \max\{f(x), g(x)\}$ is also a continuous functions.

6.2 Intermediate Value Theorem

Let f be a continuous function on $[a, b]$ and k is a value between $f(a)$ and $f(b)$, then $\exists c \in (a, b) s.t. f(x) = k$.

6.2.1 Example: Proving Multiple Solutions

Prove that $f(x) = x^4 - 2x - 100$ has at least two solutions.

Claim: We WTS $\exists r_1, r_2 \in \mathbb{R}, f(r_1) = f(r_2) = 0, r_1 \neq r_2$ and $f(x) = x^4 - 2x - 100$.

Proof. Let $k = 0$. We need $[a_1, b_1]$ such that 0 is in between $f(a_1)$ and $f(b_1)$.
Let $a_1 = 0$, $b_1 = 10 \implies f(a_1) < 0 \wedge f(b_1) > 0$.
Since f is a polynomial, we know it is continuous. Therefore, by IVT that $\exists c \in (0, 10)$ such that $f(c) = 0$.
Now let $k = 0, a_2 = -10, b_2 = 0 \implies f(a_2) > 0 and f(b_2) < 0$.
Since f is still a polynomial it is continuous, then we get from IVT that $\exists d \in (-10, 0)$ such that $f(d) = 0$.

Therefore, there exists two solutions $c \in (0, 10)$ and $d \in (-10, 0)$ such that $c^4 - 2c = 100$, and $d^4 - 2d = 100$ and $c \neq d$.

□

6.3 Extreme Value Theorem

Let f be a continuous function on $[a, b]$. Then f must have a minimum or maximum value on $[a, b]$.

7 Derivatives

A derivative is the instantaneous change of a function at certain $x = a$ value. Derivatives use properties of limits and can then help determine the function's characteristics.
We define for a function $f(x)$ its derivative:

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

7.1 Linear Approximation

Linear approximations can be used to easily approximate $f(x)$ near an $x = a$.
We take the formula for a slope:

$$m = \frac{y_1 - y_2}{x_1 - x_2}$$

and get out approximation for $f'(a)$:

$$f'(a) = \frac{y - f(a)}{x - a}$$

Using this f', we rearrange and get:

$$y = f(a) + f'(a)(x - a)$$

7.1.1 Tangent Lines and Differentiability

Does having a tangent line imply the function is differentiable?

Take the function:

$$f(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ x^2 & x \in \mathbb{Q} \end{cases}$$

First, we see if f is continuous at any point.
Take $x = 0$. Then $f(0) = 0^2 = 0$. Also, $\lim_{x \to 0^-} f(x) = 0$,and $\lim_{x \to 0^+} f(x) = 0$.
Hence, $y = f(x)$ is continuous at 0 and nowhere else.

Next, we determine if it is differentiable at $x = 0$. Take $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. We WTS $\forall \epsilon > 0, \exists \delta > 0 : 0 < |h| < \delta \implies \left| \frac{f(x+h)-f(x)}{h} - 0 \right| < \epsilon$ Consider

$$\begin{aligned} \frac{(x+h)^2 - x^2}{h} &= \frac{x^2 = 2xh + h^2}{h} \\ &= 2x + h \end{aligned}$$

Then, we have:

$$\begin{aligned} |(2x+h) - (2x)| &< \epsilon \text{Let } \delta = \epsilon \\ |0 + h - 0| &< \epsilon \text{Let } \delta = \epsilon \\ |f(x) - 2x| = |2x + h - 2x| &< \delta = \epsilon \end{aligned}$$

Hence, $f(x)$ is differentiable at $x = 0$ and nowhere else.

7.1.2 Absolute Values and Tangent Lines

Is $y = |x|$ differentiable at $x = a$?

Take the left and right side limits.

1. $\lim_{h \to 0^+} \frac{|a+h| - |a|}{h} = \lim_{h \to 0^+} \frac{a+h-a}{h} = 1$
2. $\lim_{h 0^-} \frac{-(a+h) - (-a)}{h} = -1$

Because the side limits are not equivalent, the limit does not exist. Hence, $|x|$ is not differentiable at $x = 0$.

7.2 Polynomial Derivatives

1. $(f \pm g)' = f' \pm g'$
2. $(cf)' = c(f')$ for $c \in \mathbb{R}$

Even if the function's exponent is not a polynomial, we can still use these basic derivative laws.
For example, $y = x^{\frac{1}{3}}$. Even though $\frac{1}{3}$ is not a polynomial, we can still say $y' = \frac{1}{3}x^{-\frac{2}{3}}$.

7.2.1 Example Proof

We want to find the derivative of $f(x) = \frac{2}{\sqrt{x}}$ at $x = 4$.

Using the definition of derivatives, find $\frac{2}{\sqrt{x}} = \frac{d}{dx}(2x^{-\frac{1}{2}})$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \qquad = \lim_{h\to 0} \frac{2(x+h)^{-\frac{1}{2}} - 2x^{-\frac{1}{2}}}{h}$$

$$= \lim_{h\to 0} \frac{2\sqrt{x} - 2\sqrt{x+h}}{\sqrt{x}\cdot\sqrt{x+h}} \cdot \frac{1}{h} \qquad \lim_{h\to 0} \frac{2\sqrt{x} - 2\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}$$

7.3 Differentiable Functions

Consider $f(x) = \begin{cases} x^2\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$
By Squeeze Theorem, $\lim_{x\to 0} x^2 \sin(\frac{1}{x}) = \lim_{x\to 0} \pm x^2 = 0$.
We know f is continuous and differentiable. So, we have:

$$f'(x) = \begin{cases} (2x)\sin(\frac{1}{x}) + x^2\cos(\frac{1}{x} \cdot (\frac{-1}{x^2}) \\ \text{no case, since } f' \text{does not have a limit at } x = 0 \end{cases}$$

Therefore, a function that is continuous/differentiable everywhere does not necessarily imply f' is continuous.

7.4 Chain Rule

Let f and g be differentiable functions and let $h = f \circ g$.
Then, the derivative $h' = f'(g(x)) \cdot g'(x)$.
Alternatively, we can write this as $\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$.

7.5 Trigonometric Derivatives

1. $\frac{d}{dx}\sin(x) = cos(x)$
2. $\frac{d}{dx}\cos(x) = -sin(x)$

Knowing these two, we can also find the derivatives of $\tan, \cot, \sec$, and $\csc$.
For example, take $\cot(x)$.

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\frac{\sin\theta}{\cos\theta}} = \frac{\cos\theta}{\sin\theta}$$

$$
\begin{aligned}
\frac{d}{d\theta}\cot\theta &= \frac{d}{d\theta}\frac{\cos\theta}{\sin\theta} \\
&= \frac{(-\sin\theta)(\sin\theta)-(\cos\theta)(\cos\theta)}{(\sin\theta)^2} \\
&= \frac{-\sin^2\theta -^2\theta}{\sin^2\theta} \\
&= \frac{-(\sin^2\theta+\cos^2\theta)}{\sin^2\theta} \\
&= \frac{-1}{\sin^2\theta} \\
&= -\csc^2\theta
\end{aligned}
$$

Similar for the other trigonometric functions.

7.6 Implicit Differentiation

Implicit derivatives simply use chain rule. We can write relationships of functions without solving for one variable as a function of the other. For example, $x^2 + y^2 = 25$.

7.6.1 Example of Solving Implicit Differentiation

Find $\frac{dy}{dx}$ from $x^5 + x^3y^4 = 3 - y^2$.

$$
\begin{aligned}
\frac{d}{dx} &: 5x^4 + (3x^2)(y^4) + (x^3)(4y^3\frac{dy}{dx}) \\
&= 0 - 2y\frac{dy}{dx} \\
x^3(4y^3)\frac{dy}{dx} + 2y\frac{dy}{dx} &= -5x^4 - 3x^2y^4 \\
\frac{dy}{dx}\frac{4x^3y^3+2y}{(4x^3y^3+2y)} &= \frac{-5x^4-3x^2y^4}{4x^3y^3+2y}
\end{aligned}
$$

This means that $\frac{dy}{dx} = \frac{-5x^4-3x^2y^4}{4x^3y^3+2y}$ when $4x^3y^3 + 2y \neq 0$

7.7 Inverse Functions

Let $f : (a, b)$. Then the inverse $f^{-1} : (b, a)$. We also have $f(a) = b \implies f^{-1}(b) = a$.
$y = f(x)$ is a function of x if it intersects each vertical line at most once. $y = f^{-1}(x)$ is a function of x is if intersects each vertical line at most once. The inverse relation of $y = f(x)$ is a function of $x \iff$ each horizontal line intersects the graph $y = f(x)$ at most once.

7.8 Composition of One-to-One Functions

7.8.1 Theorem 1

Theorem 1. Let f and g be functions with domain $\mathbb{R}$. If f, g are one-to-one, then $f \circ g$ is one-to-one.

Proof. Let f, g be arbitrary injective functions. We WTS $\forall x_1, x_2 \in \mathbb{R}, f(g(x_1)) = f(g(x_2)) \implies x_1 = x_2$. We know:

1. $\forall x_1, x_2 \in \mathbb{R}, f(x_1) = f(x_2) \implies x_1 = x_2$
2. $\forall y_1, y_2 \in \mathbb{R}, f(y_1) = f(y_2) \implies y_1 = y_2$
3. $\forall x_1, x_2 \in \mathbb{R}, g(x_1) = g(x_2) \implies x_1 = x_2$

Assume (2) and (3) and $f(g(x_1)) = f(g(x_2))$. By (2) and letting $y_1 = g(x_1)$ and $y_2 = g(x_2)$, we get that $y_1 = y_2 \implies g(x_1) = g(x_2)$. Then, by (3), $x_1 = x_2$.
Therefore, if f, g are injective, then $f \circ g$ is injective. ☐

7.8.2 Theorem 2

Theorem 2. Let f, g be functions with domain $\mathbb{R}$. If $f \circ g$ is injective, then g is injective.
By contrapositive, if g is not one to one, then $f(g(x))$ is not an injective function.

Proof. Assume f, g are functions, and g is not injective. So, $\exists x_1, x_2 \in \mathbb{R}, (g(x_1)) = g(x_2)) \wedge (x_1 \neq x_2)$ by assumption.
Consider $f \circ g$ at those two x-values.
Then, $f(g(x_1)) = f(g(x_2))$ since $g(x_1) = g(x_2)$. This is true, and therefore $x_1 \neq x_2$. Hence, $f \circ g$ is also not injective. ☐

7.8.3 Theorem 3

Claim 1. Let f, g be functions. If $f \circ g$ is injective, then f is injective.
This claim is false. We WTS there exists f, g such that $f \circ g$ is injective, but f is not injective.

Theorem 3. If f is increasing, then it is injective.

Proof. Let $f$be a function. Assume $f : A \to B$ is increasing: $\forall x_1, x_2 \in A, x_1 < x_2 \implies f(x_1) < f(x_2)$.
We WTS $\forall x_1, x_2 \in A, f(x_1) = f(x_2) \implies x_1 = x_2$, which is equivalent to:
$\forall x_1, x_2 \in A, x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$

Let $x_1, x_2 \in A$. If $x_1 < x_2 \implies x_1 \neq x_1$, but not the converse. We will assume $\forall((x_1, x_2, (x_1 < x_2 \implies f(x_1) < f(x_2)) \wedge (x_1 \neq x_2))$.
We have two cases:

1. $x_1 < x_2 \implies f(x_1) < f(x_2) \implies f(x_1) \neq f(x_2)$
2. $x_2 < x_2 \implies f(x_2) < f(x_1) \implies f(x_1) \neq f(x_2)$

☐

7.9 Derivatives of Exponentials and Logarithms

1. $\frac{d}{dx} e^x = e^x$
   Use chain rule for the exponent part of the function. For example:

$$\begin{aligned} \frac{d}{dx} e^{x^2} &= e^{x^2} \cdot \frac{d}{dx} x^2 \\ &= 2x \cdot e^{x^2} \end{aligned}$$

2. $\frac{d}{dx} a^x = a^x \ln a$
   Use chain rule for the exponent part of the function.

$$\frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}$$

3. $\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$
   If x is a function, we write:

$$\frac{d}{dx} log_a f(x) = \frac{f'(x)}{\ln a \cdot f(x)}$$

4. $\frac{d}{dx}\ln(x) = \frac{1}{x}$
Use chain rule if x is a function. For example:

$$\begin{aligned}\frac{d}{dx}\ln(2x+1) &= \ln'(2x+1)\cdot 2\\ &= \frac{2}{2x+1}\end{aligned}$$

7.9.1 More Complex Logarithmic Functions

Calculate the derivative of $f(x) = log_{x+1}(x^2+1)$.

$$\begin{aligned}y = log_{x+1}(x^2+1) &\implies (x+1)^y = x^2+1\\ &\implies \ln(x+1)^y = \ln(x^2+1)\\ &\implies y\ln(x+1) = \ln(x^2+1)\end{aligned}$$

Now, we take the derivative:

$$\begin{aligned}\text{Left Side: } &\frac{dy}{dx}\ln(x+1) + y\frac{1}{x+1} && = \frac{2x}{x+1}\\ \text{Right Side: } &\frac{dy}{dx} = \frac{\frac{2x}{x^2+1} - \frac{y}{x+1}}{\ln(x+1)}\\ &= \frac{d}{dx}\log_{x+1}(x^2+1)\\ &= \frac{\frac{dx}{x^2+1} - \frac{\log_{x+1}(x^2+1)}{x+1}}{\ln(x+1)}\end{aligned}$$

Therefore, the derivative of $(x+1)^y = x^2+1$ is $\frac{\frac{dx}{x^2+1} - \frac{\log_{x+1}(x^2+1)}{x+1}}{\ln(x+1)}$

7.10 Maxima/Minima

Local Extrema: For an interval arbitrarily small around the point of a function, if no value is greater/less than that point, then it is a local extremum.
Alternatively, we say that the slope of the tangent line at that point is 0.

Global Extrema: A local extrema, except it is the minimum/maximum of the entire function.

Note that if the function is on a closed interval, then we can only compare the interior points and not the endpoints (i.e., if f was on interval [0,1], then $0, 1$ would not have any extrema values.

7.10.1 Extrema and Differntiability

Claim. We WTS if f has domain $\mathbb{R}$, is continuous, has $f(0) = 0$, and $\forall x \in \mathbb{R}, f(x) \geq x$, then $f'(0) = 1$.

Proof. Let's assume f is differentiable on an open interval $(-\delta, \delta)$. Otherwise, this does not work.
Rough Work:

$$\begin{aligned}f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0}\frac{f(0+h)}{h}\\ &= 1\\ f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0}\frac{f(x)}{x}\end{aligned}$$

At $x = 0$, the graph has derivative equal to:

$$\lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{x}{x} = 1,$$

since we know f is differentiable near that point, we replace f with its tangent line (first order approximation) since as $x \to 0$, $y = x$ becomes a continually closer approximation. $\square$

7.10.2 Graphing Local, Global, Endpoint Extrema

Find the local, global, and endpoint extrema for $g(x) = x^{2/3}(x-1)^3$ on $[-1, 2]$.

$$\begin{aligned} g'(x) &= (\frac{2}{3}x^{-1/3})(x-1)^3 + (x^{2/3})(3(x-1)^2(1)) \\ &= \frac{2(x-1)^3}{3x^{1/3}} + 3x^{2/3}(x-1)^2 \\ &= \frac{2(x-1)^3 + 9x(x-1)^2}{3x^{1/3}} \end{aligned}$$

$$\begin{aligned} g''(x) &= \frac{[6(x-1)^2 + 9(x-1)^2 + 18x(x-1)][3x^{1/3}] - [2(x-1)^3 + 9x(x-1)^2]x^{-2/3}]}{9x^{2/3}} \\ &= 0 \text{ or DNE} \end{aligned}$$

To easily determine the minima and maxima, estimate the values for different intervals of $g(x), g'(x)$, and $g''(x)$.

Interval	$g(x)$	$g'(x)$	$g''(x)$
$[-1, -0.3]$	-	+	-
$[-0.3, 0]$	-	-	-
$[0, 0.5]$	-	+	+
$[0.5, 1]$	-	+	-
$[1, 2]$	+	+	+

From this, we can then easily a sketch of the graph $g(x)$.

Generally, to find locla extrema, look for $f' = 0$ and $f' = DNE$.
Then, find the $y-$values associated with each x bvalue (the critical points and endpoints).
Using the first derivative test and the second derivative test allows us to determine whether or not it is a local extremum.

7.11 Mean Value Theorem

7.11.1 Rolle's Theorem

Let $a < b$, $f : [a.b] \to \mathbb{R}$, with $f(a) = f(b)$. Assume f is continuous on $[a, b]$ and differentiable on (a, b)
Then, $\exists c \in (a, b)$ such that $f'(c) = 0$.

7.12 N-th Theorem of Rolle

Let N be a positive integer. Let $a < b$. Let f be a function defined on $[a, b]$. If:

1. Similar conditions of continuity and derivatives.
2. Similar conditions about a.
3. $f(b) = 0$

Then, $\exists c \in (a, b)$ such that $f^N(c) = 0$.

7.12.1 Theorem 1

Let f be a differentiable function on an interval $[a, b]$ with $a < b$. If $\forall x \in [a, b], f$ is not injective on $[a, b]$ then $\exists c \in (a, b)$ such that $f'(c) = 0$.

Proof. Rolle's Theorem states that between $f(a) = 0$ and $f(b) = 0$, there exists $c \in (a, b)$ such that $f'(c) = 0$ assuming f is continuous on $[a, b]$ and differentiable on (a, b).
For example, if we have a cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$, then $p'(x) = 3ax^2 + 2bx + c$, and $p''(x) = 6ax + 2b$.
$p'' = 0$ at $x = \frac{-2b}{6a}$ and nowhere else. $\square$

7.12.2 Theorem 2

Let $a < b$. Let f be a function defined on $[a, b]$. If:

1. f continuous on $[a, b]$
2. f differentiable on (a, b)
3. f non-injective on $[a, b]$

then, $\exists c \in (a, b)$ such that $f'(c) = 0$.

Note that we require this theorem in order to justify MVT.

7.12.3 Mean Value Theorem

Let f be a function defined on an interval $[a, b]$. If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. We define the equation of the secant line:

$$h(x) = f(a) + \left(\frac{f(b) - f(a)}{b - a}\right)(x - a)$$

Consider the difference $g(x) = f(x) - h(x)$, and note that $g(a) = g(b) = 0$ and g is differentiable.
By Rolle's Theorem, there exists $c \in (a, b)$ with $g'(c) = 0$. Hence:

$$f'(c) = h'(c) = \frac{f(b) - f(a)}{b - a}$$

$\square$

7.13 Antiderivatives

Antiderivatives, or the primitive of a function, F, is equal to an original function f.
That is, $F'(x) = f(x)$.
If we know the basic rules of differentiation for the function, it is easy to calculate the antiderivative.
For example:

1. **Polynomials:** For $f = ax^n$, where a, n are arbitrary values, $F = \frac{a}{n}x^{n+1}$
2. **Trigonometric Functions:** If $f = \sin(x)$, then $F = -\cos(x)$. If $f = \cos(x)$, then $F = \sin(x)$.

7.14 Monotonicity

A function f is said to be monotonically increasing on an interval $[a, b]$ where $\forall x_1, x_2 \in [a, b], x_1 < x_2$, $f(x_1) < f(x_2)$.
When we want to find the intervals for which a function is increasing/decreasing, find the first derivative of the function, find the derivative's roots, and determine which intervals of roots increase or decrease.

7.14.1 Finding When a Function Increases

Determine when the following function is increasing:

$$f(x) = \frac{1}{3}x^3 + x^2 - 3x - 2$$

First, find the first derivative: $f'(x) = x^2 + 2x - 3$
Now, find the roots:

$$\begin{aligned} 0 &= x^2 + 2x - 3 \\ &= (x+3)(x-1) \end{aligned}$$

So, we have $x = -3$ and $x = 1$.

Interval	f'(x)
$(-\infty, -3)$	+
$(-3, 1)$	-
$(1, \infty)$	+

Therefore, $f(x)$ increases for $x < -3$, decreases for $-3 < x < 1$, and increases for $x > 1$.

7.15 Related Rates

Related rates are an application of implicit differentiation, using the multiple variables to solve for a specific solution of a problem.

7.15.1 Example 1

We drop a pebble into a lake. It produces a circular ripple. When the radius is 2m and increasing at a rate of $10cm/s$, at what rate is the area increasing?

$$\begin{aligned} A &= \pi r^2 \\ \frac{dA}{dt} &= 2\pi r \frac{dr}{dx} \\ &= 2\pi 200 \cdot 10 \\ &= 4000\pi \end{aligned}$$

Therefore, it increases at a rate of $4000cm/s$.

7.15.2 Example 2

A 10m long ladder is leaning against a vertical wall and sliding. The top end of the ladder is 8 meters high and sliding down at a rate of 1m per second. At what rate is the bottom end sliding?

Let $a = 8m$, $c = 10m$, $\frac{da}{dt} = -1m/s$, $\frac{dc}{dt} = 0m/s$. To find b, use Pythagorean's Theorem.

$$\begin{aligned} b &= \sqrt{10^2 - 8^2} \\ &= \sqrt{100 - 64} \\ &= 6 \end{aligned}$$

Therefore, $b = 6m$.
Next, we want to calculate $\frac{db}{dt}$. We use Pythagrorean's Theorem and get:

$$2\frac{dc}{dt} = 2a(\frac{da}{dt}) + 2b(\frac{db}{dt})$$

So:

$$
\begin{aligned}
0 &= 2 \cdot 8 \cdot -1 + 2 \cdot 6 \cdot \frac{db}{dt} \\
&= -16 + 12\frac{db}{dt} \\
\frac{db}{dt} &= \frac{4}{3}
\end{aligned}
$$

Therefore, the ladder is sliding at $\frac{4}{3}$ m/s.

7.16 Optimization

When mazimizing or minimizing a function:

1. Determine the fixed values and the value we want to solve for.
2. Look at the minimums and maximums of the endpoints.
3. Solve for the optimized value.

7.16.1 Example 1

A farmer has $300m$ of fencing and wants to fence off a rectangular field and add an extra fence that divides the rectangular area in two equal parts down the middle. What is the largest area that the field can have?

Let l be the length of the field, and w be the width of the field.
Total Fencing $= 3l + 2w$
Area $= l \cdot w$
We can substitute $l = 100 - \frac{2}{3}w$. So:

$$
\begin{aligned}
A(w) &= (100 - \frac{2}{3}w)w \\
A &= 100w - \frac{2}{3}w^2 \\
A' &= 100 - \frac{4}{3}w \\
&\implies 100 - \frac{4}{3}w = 0 &&\implies w = 100 \cdot \frac{3}{4} = 75
\end{aligned}
$$

Therefore, $w = 75$. Then, $l = 100 - \frac{2}{3}75 = 50 = 50$.
So, the maximum area is $75 \cdot 50 = 3750m^2$.

7.16.2 Example 2

Find the point on the parabola $y^2 = 2x$ closest to the point $(1, 4)$.

We have $y^2 = 2x$, and $x = \frac{y^2}{2}$.
We want to find the point with the shortest distance to $(1, 4)$. We can use the distance formula:

$$D = \sqrt{(x-1)^2 + (y-4)^2}$$

Substituting $x = \frac{y^2}{2}$ we have:

$$\begin{aligned}
D &= \sqrt{(\frac{y^2}{2} - 1)^2 + (y-4)^2} \\
D' &= \frac{1}{2} \cdot ((\frac{y^2}{2} - 1)^2 + (y-4)^2)^{\frac{-1}{2}} \cdot (2(\frac{y^2}{2} - 1)(y) + 2(y-4)) \\
0 &= \frac{2y(\frac{y^2}{2} - 1) + 2y - 8)}{2\sqrt{((\frac{y^2}{2})^2 - 1)^2 + (y-4)^2}} \\
&\implies y^3 - 8 = 0 \\
&\implies y^3 = 8 \implies y = 2
\end{aligned}$$

Using the first derivative test, we know that $D'(1) < 0$, and $D'(3) > 0$. So, we know that $D'(2) = 0$. Therefore, $(2, 2)$ is the closest point to $(1, 4)$.

7.16.3 Example 3

The cost of fuel per hour for a certain airplane is proportional to the square of its speed and is 1200 per hour for a speed of $600 km/h$. After every 5000 hours flown, the aircraft must undergo an 8 million dollar safety inspection. What speed should the airplane fly in order to achieve the lowest cost per kilometre?

The cost of fuel per hour would be $k(v^2)$, since the cost is proportional (k is a constant we multiply by).
Let $c = cost, v = velocity$. We have $C_r = \$1200/hr$, $V = 600km/h$.
Let $x =$ number of 5000 hours. There are $5000x$ safety inspections, and the cost of one safety inspection is $\$8,000,000$.
We want to find the velocity v that would give the lowest cost of fuel per hour.

Let $C(v) = C_r(v) + C_s$, where:

$$C_s = \frac{80000000}{5000}$$

We want to minimize the $\$/km$. So, we have:

$$\begin{aligned}
f(v) &= \frac{C_r(v)}{v} + \frac{C_s}{v} \\
&= \frac{kv^2}{v^2} + \frac{C_s}{v} \\
f'(v) &= 0
\end{aligned}$$

So, we want to find the v.

7.17 L'Hopital's Rule

L'Hopital's rule is used for calculating limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
Note: Do NOT use this for $\frac{0}{\infty}$ or $\frac{\infty}{0}$!

If $\lim_{x \to a}$ or $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)}$ goes to $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then the limit $= \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

7.17.1 Indeterminate Forms

An indeterminate form is a form where substituting for x does not tell us the value of the limit.
Examples: $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, 1^\infty, 0^0$

1. Type 1: $\frac{0}{0} or \frac{\infty}{\infty}$

$$\begin{aligned}
\lim_{x\to 0}\frac{e^x-1-x}{x^2} &= \lim_{x\to 0}\frac{e^x-1}{2x} && \text{Using L'Hopital's}\\
&= \lim_{x\to 0}\frac{e^x}{2} && \text{Using L'Hopital's}\\
&= \frac{1}{2}
\end{aligned}$$

2. Type 2: $0\cdot\infty$

$$\begin{aligned}
lim_{x\to 0^+}x(\ln(x)) &= lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}\\
&= \lim_{x\to 0^+}\frac{\frac{1}{x}}{\frac{-1}{x^2}} && \text{Using L'Hopital's}\\
&= \lim_{x\to 0^+}\frac{1}{x}\cdot\frac{x^2}{-1}\\
&= \lim_{x\to 0^+} -x = 0
\end{aligned}$$

3. Type 3: $1^\infty, 0^0, \infty^0$

$$\lim_{x\to\infty}(1+\frac{1}{x})^x$$

Let $L=\lim_{x\to\infty}(1+\frac{1}{x})^x$, if it exists.
Then, $\ln\lim_{x\to\infty}(1+\frac{1}{x})^x = \ln L = LHS$

$$\begin{aligned}
LHS &= \lim_{x\to\infty} x\ln(1+\frac{1}{x}\\
&= \lim_{x\to\infty}\frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}\\
&= lim_{x\to\infty}\frac{\frac{1}{1+1/x}\cdot -1/x^2}{-1/x^2} && \text{Using L'hopital's}\\
&= \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}}\\
&= 1
\end{aligned}$$

7.17.2 L'Hopital's and Exponents

With polynomials, we can easily use L'Hopital's Rule multiple times at once. For example, when we compute:

$$\lim_{x\to\infty}\frac{x^7+5x^3+2}{e^x}$$

We know that the derivative of e^x is e^x, and it's limit as it goes to ∞ is ∞. Similarly, the above polynomial approaches ∞ as $x\to\infty$, so we need the polynomial to become a constant.
So, we can apply L'Hopital's 7 times to make it constant. Then we are left with $7\cdot 6\cdot ... = 7!$ for the numerator. Hence, we get:

$$\lim_{x\to\infty}\frac{7!}{e^x}=0$$

7.18 Concavity

Concavity is found by taking the second derivative of a function.
If $f''>0$, it concaves up. If $f''<0$, it concaves down.

7.18.1 Example: Graphing Functions

Sketch the graph of $y = x^x$ on $(0, \infty)$.
Take the first derivative:

$$\begin{aligned} y &= x^x \\ \ln y &= \ln x^x \\ \ln y &= x(\ln x) \\ \frac{1}{y}\frac{dy}{dx} &= (\ln x) + \frac{x}{x} \\ \frac{dy}{dx} &= (\ln x + 1)y \\ \frac{dy}{dx} &= (\ln x + 1)x^x \end{aligned}$$

So, $f'(x) = (\ln x + 1)x^x$
Now, take the second derivative:

$$\begin{aligned} \frac{dy}{dx} &= (\ln x + 1)x^x \\ \frac{d^2y}{dx} &= \frac{1}{x}x^x + (\ln x + 1)^2 x^x \end{aligned}$$

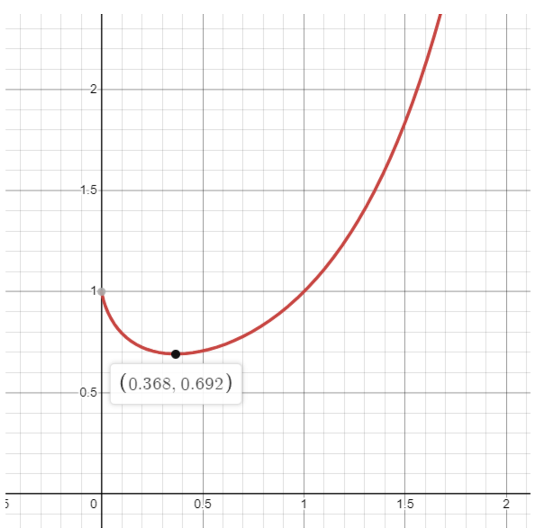
Now, we look at the values for which y, y', y'' are positive or negative.

1. y: $+$ for all values $(0, \infty)$
2. y': $-$ on $(0, \frac{1}{e}$, $+$ on $(\frac{1}{e}, \infty)$
3. y'': $+$ for all values $(0, \infty)$

Therefore, we know that y is concave up, decreases from 0 until $\frac{1}{e}$, and then increases from $\frac{1}{e}$ to ∞.
Now we observe the limits:

$$\begin{aligned} &\lim_{x \to 0^+} x^x = L \implies \lim_{x \to 0^+} x \ln x = \ln L \\ &\lim_{x \to \infty} x^x = \infty \\ &e^0 = e^{\ln L} \implies L = 1 \qquad\qquad \text{Since } \ln x + 1 = 0 \wedge e^{\ln x} = e^{-1} \end{aligned}$$

Now, we can sketch the graph:



7.19 Asymptotes

An **inflection point** is the point of the function where it switches concavity.
Asymptotes are imaginary lines that are approached by a curve of the graph but never *touch* the curve. To find the asymptotes of a function, we compare the numerators and denominators and figure out some information about the function.

7.19.1 Horizontal Asymptotes

If the numerator and denominator have the same degree, divide the coefficients by the largest term to get the horizontal asymptote.
If the numerator is less than the denominator, then the asymptote is $y = 0$.
If the numerator is greater than the denominator, then there is no horizontal asymptote.

For example, take $f(x) = \frac{3x^2+6x}{x^2+x}$. The numerator and denominator are both of degree 2, so we take the quotient of the leading coefficients.
$\frac{3}{1} = 3$, so there is a H.A. at $y = 3$.

7.19.2 Vertical Asymptotes

Simplify the function, and then find the zeroes of the denominator of the function to determine the vertical asymptotes of the function.

For example, for $f(x) = \frac{3x^2+6x}{x^2+x}$, we can simplify the denominator to be $x(x+1)$, and get a hole at $x = 0$, and a vertical asymptote at $x = 1$.

7.19.3 Oblique Asymptotes

Oblique asymptotes use a $y = mx + b$ formula. They are found by dividing its nnumerator by the denominator. The quotient of the division is the equation of the slant asymptote.

For example, the oblique asymptote of $y = \frac{3x^3-1}{x^2+2x}$, by dividing, is $y = 3x - 6$.

8 Introduction to Integrals

8.1 Summation

Sigma notation, or summations, take an index starting at i to n and calculate the sum of the function. It is denoted as:

$$\sum_{i=0}^{n}$$

For example: $\sum_{i=2}^{4}(2i+1) = (2 \cdot 2 + 1) + (2 \cdot 3 + 1) + (2 \cdot 4 + 1) = 21$

8.1.1 Simplifying Summations

We can rewrite sums and cancel certain terms out to simplify them. For example:

1. $\sum_{i=1}^{100} \tan i - \sum_{i=1}^{50} \tan i = \sum_{i=51}^{100} \tan i$
2. $\sum_{i=1}^{N}(2i-1)^5 = \sum_{i=0}^{N-1}(2i+1)^5 = 1^5 + 3^5 + ... + (2N-1)^5$
3. $\left[\sum_{k+1}^{N} x^k\right] + \left[\sum_{k=0}^{N} k \cdot x^{k+1}\right] = \left[\sum_{k=2}^{N} k \cdot x^k\right] + N^{n+1} + x$

8.1.2 Telescope Summation

Compute the exact value of:

$$\sum_{i=1}^{137}\left[\frac{1}{i}-\frac{1}{i+1}\right]$$

We can write the first few terms:

$$(\frac{1}{1}-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+...+(\frac{1}{137}-\frac{1}{138})$$

We see that all the values cancel out, so we're left with $\frac{1}{1}$ and $\frac{1}{138}$, so we have:

$$\sum_{i=1}^{137}\left[\frac{1}{i}-\frac{1}{i+1}\right]=1-\frac{1}{138}=\frac{137}{138}$$

This series is called a **telescoping series**, since part of each term gets cancelled by a later term, which then collapses to just 2 terms.

8.1.3 Partial Fractions and Summations

Compute the exact value of:

$$\sum_{i=1}^{10{,}000}\frac{1}{i(i+1)}$$

We note that:

$$\frac{1}{i(i+1)}=\frac{A}{i}+\frac{B}{i+1}=\frac{A(i+1)+Bi}{i(i+1)}$$

Knowing this, we have $i=-1,-B=1,i=0,A=1$.
Hence, we get:

$$\sum_{i=1}^{10{,}000}\frac{1}{i(i+1)}=\sum_{i=1}^{10{,}000}\left[\frac{1}{i}-\frac{1}{i+1}\right]$$

8.2 Suprema and Infima

Supremum - The lowest upper bound of a subset. It is always greater than or equal to the maximum of the subset.
Equivalently, for S being an upper bound of set A:

1. If R is an upper bound of A, then $S \leq R$.
2. $\forall R < S, \exists x \in A : R < x < S$.
3. $\forall \epsilon > 0, \exists x \in A : S - \epsilon < x \leq S$

Infimum - The greatest lower bound of a subset. It is always less than or equal to the minimum of the subset.
Equivalently, for L being a lower bound on set A:

1. If P is a lower bound of A, then $I \geq P$.
2. $\forall P > I, \exists x \in A : P > x > L$.
3. $\forall \epsilon > 0, \exists x \in A : I - \epsilon > x \geq P$

8.2.1 Consequences of Infima/Suprema

1. $\sup(f+g) < \sup(f) + \sup(g)$
2. $\sup(f)$ on $[a,c] \geq \max\{\sup(f)$ on $[a,b], \sup(f)$ on $[b,c]\}$
3. $\sup(cf) \neq c(\sup(f))$

8.3 Partitions

A set $P = \{x_1, x_2, ..., x_n\}$ is a partition of $[a,b]$ if:

1. P is finite
2. $x_1 = a, x_n = b$
3. $P \subset [a,b] \neq \emptyset$
   Note that we know $a < x_1 < ... < b$, so it is a strict relation.

8.3.1 Lower and Upper Sums

Suppose we had a function f. We can determine its area by taking the partition of its intervals.
We then take the lower sum ($L_P(f)$) and upper sums ($U_P(f)$) by looking at the minimums and maximums of each interval. Note that $L_P(f)) \leq$ actual area $\leq U_P(f))$
The shorter the intervals are between each point in a partition, the more accurate the upper and lower sums become to being the actual area.

8.3.2 Joining Partitions

Assume P, Q are two partitions of $[a,b]$.
We can take the union of these two partitions, and get a better refinement for the interval $[a,b]$ that can be used to take the integral.
In other words, the upper rectangles above $P \cup Q$ are contained in the bigger of the two rectangles from $U_P(f)$ and $U_Q(f)$ (similar for the lower rectangles).

8.3.3 Partitions at Different Intervals

Let $a < b < c$.

1. If P, Q partitions of $[a,b]$, then $P \cup Q$ is a partition of $[a,b]$.
2. If P, Q partitions of $[a,b]$, then $P \cap Q$ is a partition of $[a,b]$.
3. If P is a partition of $[a,b]$ and Q is a partition of $[b,c]$, then $P \cup Q$ is a partition of $[a,c]$.

8.4 Epsilon Definition of Integrals

Let f be a bounded function on $[a,b]$.
If f is integrable on $[a,b]$, then $\forall \epsilon > 0, \exists$ partition of P of $[a,b]$ such that $U_P(f) - L_P(f) < \epsilon$

From this, we also know that:

1. There exists a partition P_1 s.t. $U_{P_1}(f) < I + \frac{\epsilon}{2}$,
   since $I + \frac{\epsilon}{2} > I = \inf(U_P(f))$
2. There exists partition P_2 s.t. $L_{P_2}(f) > I - \frac{\epsilon}{2}$.
3. $U_{P_1}(f) - L_{P_2}(f) < \epsilon$,
   since $P = P_1 \cup P_2$

9 Calculating Integrals

9.1 Jump-Discontinued Functions

Let $f(x) = \begin{cases} 0 & x = 0 \\ 5 & 0 \leq x \leq 1 \end{cases}$ defined on $[0, 1]$.

1. Let $P = \{0, 0.2, 0.5, 0.9, 0.1\}$.
   Calculate $L_P(f)$ and $U_P(f)$ for this partition.
   Since only $f(0) = 0$, every value is the same except for one point where $L_P(f)$ has an area of $5 \cdot 0.2$ at one point.

2. When we fix an arbitrary partition $\{x_0, x_1, ..., x_N\}$ of $[0, 1]$. Then, $L_P(f)$ would be the same as $U_P(f)$ for all points except for the area for $5(x_1 - x_0)$.

3. Find a partition P with exactly 3 points (2 subintervals) such that $L_P(f) = 4.99$.
   Let $P = \{0, 1 - \frac{4.99}{5}, 1\}\}$
   $(1 - (1 - \frac{4.99}{5}) = \frac{4.99}{5}$, which we multiply by 5 to get 4.99.

4. The upper and lower integrals are both 5 (since for a single point, it still approaches the same area), therefore f is integrable.

Any function with a single discontinuity will still be integrable since the area can still be calculated.

9.2 Non-Continuous Functions

9.2.1 Theorem on Irrationals/Rationals

Theorem. Between every two irrational numbers is a rational number.
Similarly, between every two rational numbers is an irrational number.

9.2.2 Calculating the Integral

Let $f(x) = \begin{cases} 1/2 & 0 \leq x \leq 1/2 \\ 1 & 1/2 < x \leq 1 \wedge x \in \mathbb{Q} \\ 0 & 1/2 < x \leq 1 \wedge x \notin \mathbb{Q} \end{cases}$ be defined on $[0, 1]$.

1. Let $P = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$. Calculate $L_P(f), U_P(f)$.
   Then, we have $L = (0.2)(\frac{1}{2}) + (0.2)(\frac{1}{2}) + 0 + 0 + 0 = 0.2$, and
   $U = (0.2)(\frac{1}{2}) + (0.2)(\frac{1}{2}) + 0.2(1) + 0.2 + 0.2 = 0.8$.

2. We can construct a partition P such that $L_P(f) = \frac{1}{4}$ and $U_P(f) = \frac{3}{4}$, where the area at $[0, \frac{1}{2}]$ is $\frac{1}{4}$, and the area at $[\frac{1}{2}, 1]$ is $\frac{1}{2}$

3. The upper integral is equal to $\frac{3}{4}$, and the lower integral is $\frac{1}{4}$.
   Since the sup of the lower sums is not equal to the inf of upper sums, it is not integrable.

9.3 Sum of Non-Integrable Functions

We can find a bounded function f, g on $[0, 1]$ such that

1. f non-integrable on $[0, 1]$,
2. g non-integrable on $[0, 1]$,
3. $f + g$ integrable on $[0, 1]$.

Let $f = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$, and let $g = \begin{cases} 1 & x \notin \mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$
Then the integral of $f + g = 1$.

9.4 Properties of Integrals

Let
$$\int_0^2 f(x)dx = 3, \int_0^4 f(x)dx = 9, \int_0^4 g(x)dx = 2$$

We can compute:

1. $\int_0^2 f(t)dx = 2f(t)$ (since it is constant respect to x).
2. $\int_2^0 f(x)dx = -3$
3. $\int_2^4 f(x)dx = \int_0^4 f(x)dx - \int_0^2 f(x)dx = 9 - 3 = 6$
4. $\int_{-2}^0 f(x)dx$ cannot be computed.
5. $\int_0^4 [f(x) - 2g(x)]dx = 9 - 2(2) = 5$.

9.5 Integrals and Riemann Sums

The summation of an integral $\sum_{i=1}^n f(\epsilon_i)(\delta_x - \delta_{x-1})$ is considered the Riemann sum.
When we take the limit of the Riemann sum as the norm of the partition $||P||$ approaches zero, we can determine the actual area of the function:

$$A = \lim_{||P|| \to 0} \sum_{i=1}^n f(\epsilon_i)(\delta_x - \delta_{x-1})$$

which is also the definite integral.

9.5.1 Riemann Sums Backwards

Interpret the following limits as integrals.

$$1. \lim_{n\to\infty} \sum_{i=1}^n \frac{1}{n} \sin(\frac{i}{n}) \qquad 2. \lim_{n\to\infty} \sum_{i=1}^n \frac{n+1}{n^2}$$

We let f be a continuous function on $[0, 1]$. We can write a formula $\int_0^1 f(x)dx$ as a limit of Riemann sums, making the simplest choices you can.
Let n be the number of rectangles, and let i be the amount of the rectangle. So, we have:

$$\lim_{n\to} \sum_{i=1}^n (\frac{b-a}{n}) \cdot f(a + i(\frac{b-a}{n}))$$

where $\frac{b-a}{n}$ is the width of the rectangle (length of each subinterval), and $f(a + i(\frac{b-a}{n})$ is the length of each rectangle.

Now, we can compute the following limits:

1. $\lim_{n\to\infty} \sum_{i=1}^n \frac{1}{n} \sin(\frac{i}{n})$

$$\lim_{n\to\infty} \sum_{i=1}^n \frac{1}{n} \sin(\frac{i}{n})$$
$$= \int_a^{a+1} f(x)dx$$
$$= \int_0^1 \sin(x)dx$$

2.

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n+i}{n^2} &= \lim_{n\to\infty}\sum_{i=1}^{n}(\frac{1}{n})(\frac{n+1}{n}) \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}(\frac{1}{n})(1+\frac{i}{n}) \\
&= \int_1^2 x dx \\
&= 1.5
\end{aligned}$$

Note that:

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n}(\frac{b-a}{n})f(x_i)$$

and:

$$x_i = a + i(\frac{b-a}{n})$$

9.6 MVT and Integrals

Theorem. Let $a < b$. Let f be a continuous function on $[a, b]$. There exists $c \in [a, b]$ such that:

$$f(c) = \frac{1}{b-a}\int_a^b f(t)dt$$

Proof. We want to show there exists $c \in [a, b] s.t. f(c) = \frac{1}{b-a}\int_a^b f(x)dx$ on $[a, b]$ Since f is continuous on $[a, b]$, which is a closed bounded interval, EVT says there exists $c_1 \in [a, b]$ such that $f(c_1)$ is the minimum value of $[a, b]$, and there exists a maximum $f(c_2)$ where $c_2 \in [a, b]$.
The lower sum for f on $[a, b]$ using $P = \{a, b\}$ is $(b - a)(f(c_1))$, where $f(c_1)$ is the minimum value of f on $[a, b]$.
The upper sum for f on $[a, b]$ using P is $b - a(f(c_2))$.

We know continuous functions are integrable, so f is integrable. We then know by integrable that $L \leq \int_a^b f \leq U$.
We can divide everything by $(b - a)$ to get:

$$f(c_1) \leq \frac{1}{b-a}\int_a^b f \leq f(c_2)$$

Since f continuous, IVT says there exists a c such that $a < c < b$, where $f(c) = \frac{1}{b-a}\int_a^b f$. □

9.7 Initial Value Problem

Find a function f such that:

1. $\forall x \in \mathbb{R}, f''(x) = \sin(x) + x^2$,
2. $f'(0) = 5$,
3. $f(0) = 7$

Looking at f'', we know $f'(x) = -\cos x + \frac{x^3}{3} + 6$.
Then, $f(x) = -\sin(x) + \frac{x^4}{12} + 6x + 7$

9.8 Computing Antiderivatives

1. $\int x^5 dx = \frac{x^6}{6}$
2. $\int (3x^8 - 18x^5 + 1)dx = \frac{x^9}{3} - 3x^6 + x + C$
3. $\int \sqrt[3]{x}dx = \int x^{\frac{1}{3}}dx = \frac{3}{4}x^{\frac{4}{3}} + C$
4. $\int \frac{1}{x^9}dx = \int x^{-9}dx = \frac{x^{-8}}{-8} + C$

9.9 Completing the Squares

Sometimes, it is easier to use squares to compute values of functions.

Example: Complete the square for $2x^2 + 12x + 1$.

$$2(x^2 + 6x + A - A) + 1$$
$$2(x + A)(x + A) - B + 1$$

To get a square with a term 6x, we can let $A = 3$. Then:

$$2(x + 3)(x + 3) - 18 + 1 = 2(x + 3)^2 - 17$$

So, we have $2(x^2 + 6x + 9 - 9) + 1$.

10 Fundamental Theorem of Calculus

Let $F'(x) = f(x)$ (that is, $F(x)$ is the antiderivative of $f(x)$). Then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

We utilize the following terms:

1. **Antiderivatives** of $f(x)$ is a function with derivative equal to $f(x)$.
2. **Definite Integrals** are defined as $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$.
   It is the net area between f and the x-axis on $[a, b]$ AND the total change in the antiderivative (e.g., integral of velocity = total change in position/displacement).
3. **Indefinite Integrals** are defined as $\int f(x)dx = F(x) + C$.
   It is the integral defined when there is no upper and lower limit.

10.1 Graphing Functions Using FTC

Suppose we have a graph $F(x) = \int_0^x f(t)dt$. We want to graph $f(t)$.

1. Calculate some exact values of f using FTC, i.e., $f(1) - f(0) = \int_0^1 F(x)$.
2. Check when f is increasing, i.e., when $F(x) > 0$.
   Going from decreasing to increasing implies there is a minimum. Going from increasing to deccreasing implies there is a maximum.
   Note that integrating from right to left switches the sign of the width.

10.2 Some Statements on Integrals

1. Let $f(x) = \int_0^x g(t)dt$. Can an anti-derivative of $g(x)$ be written as $\int_0^x g(t)dt$?
   We have $\frac{d}{dx}\left[\int_0^x g(t)dt\right] = f'(x) = g(x)$. Since the derivatives are equal, the functions are equal except for maybe a constant.
   Note that $f(x) = \int_0^x g(t)dt$ is not true for all anti-derivatives of g; just one of them. We use the extra quantifier/constant but it does not always make the statement true.

2. If $f(0) = 0$, then $f(x) = \int_0^x g(t)dt$.
   We can interpret this as "If an anti-derivative of g is 0 at $x = 0$, then it can be written as $\int_0^x g(t)dt$"
   Note that there is no extra quantifier on f to make this true.

3. There exists a $C \in \mathbb{R}$ such that $f(x) = C + \int_0^x g(t)dt$. We can set C to just be equal to the integral to make the two values equal.

10.3 Calculating Derivatives Using FTC

Find $\frac{d}{dx}F(x)$ for the following integrals:

1. $F(x) = \int_0^1 e^{-x^2}dt$

$$\begin{aligned}\frac{d}{dx}F(x) &= \frac{d}{dx}[\text{constant}]\\ &= 0\end{aligned}$$

2. $F(x) = \int_0^x e^{-\sin(t)}dt$

$$\begin{aligned}\frac{d}{dx}F(x) &= \frac{d}{dx}[F(x) - F(0)]\\ &= F'(x) - 0 && \text{Where } F' = e^{-\sin(t)}\\ &= e^{-\sin(t)}\end{aligned}$$

3. $F(x) = \int_1^{x^2} \frac{\sin t}{t^2}dt$

$$\begin{aligned}\frac{d}{dx}F(x) &= \frac{d}{dx}G(x^2)\\ &= G'(x^2)\cdot(2x)\\ &= \frac{\sin(x^2)}{(x^2)^2}\cdot(2x) && \text{since } \int_1^x \frac{\sin t}{t^2}dt \implies G'(x) = \frac{\sin(x)}{x^2}\end{aligned}$$

4. $F(x) = \int_x^7 \sin^3(\sqrt{t})dt$

$$\begin{aligned}\int_x^7 \sin^3(\sqrt{t})dt &= -\int_7^x \sin^3(\sqrt{t})dt\\ \frac{d}{dx}\left(\int_x^7 \sin^3(\sqrt{t})dt\right) &= \frac{d}{dx}\left(-\int_7^x \sin^3(\sqrt{t})dt\right)\\ &= -\sin^3(\sqrt{x})\end{aligned}$$

5. $F(x) = \int_{2x}^{x^2} \frac{1}{1+t^3}dt$

$$\begin{aligned}\int_{2x}^{x^2} \frac{1}{1+t^3}dt &= \frac{d}{dx}\left[\int_0^{x^2} \frac{1}{1+t^3}dt - \int_0^{2x} \frac{1}{1+t^3}dt\right]\\ &= \frac{1}{1+(x^2)^3}\cdot(2x) - \frac{1}{1+(2x)^3}\cdot 2\end{aligned}$$

10.4 Generalizing FTC

Let f, u, v be differentiable functions with domain $\mathbb{R}$. We define:

$$F(x) = \int_{u(v)}^{v(x)} f(t)dt$$

Find a formula for $F'(x)$.

$$\begin{aligned} F(x) &= \int_0^{v(x)} f(t)dt - \int_0^{u(x)} f(t)dt \\ \frac{d}{dx}F(x) &= f(v(x)) \cdot v'(x) - f(u(x)) \cdot u'(x) \end{aligned}$$

10.5 Integrals With Substition

Suppose we have $[f(g(x))]'$. Using chain rule, we have it equal to $f'(g(x)) \cdot g'(x)$. So, when we take the integral, we can substitute $g(x)$ as a variable to simplify our calculations:

$$\begin{aligned} \int f'(g(x)) \cdot g'(x)dx &= \int f'(w)dw \qquad \text{Where } w = g(x), \quad \frac{dw}{dx} = g'(x), \quad dw = g'(x)dx \\ &= f(w) + c \\ &= f(g(x)) + c \end{aligned}$$

10.5.1 Examples Using Substitution

1. $\int (3x^2+4)\sqrt{x^3+4x-1}dx$
   Let $w = x^3+4x-1$, $\frac{dw}{dx} = (3x^2+4)$, $dw = (3x^2+4)dx$

$$\begin{aligned} \int (3x^2+4)\sqrt{x^3+4x-1}dx &= \int \sqrt{w}dw \\ &= \int w^{\frac{1}{2}}dw \\ &= \frac{2}{3}w^{\frac{3}{2}} + c \\ &= \frac{2}{3}(x^3+4x-1)^{\frac{3}{2}} + c \end{aligned}$$

2. $\int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \frac{4}{\sqrt{1-x^2}}dx$
   Let $x = \sin(w)$, and $= \cos(w)dw$. Note that $\arcsin(\sin w) = \arcsin x$, and $[\frac{1}{2}, \frac{1}{\sqrt{2}}]$ are in the domain

where $\arcsin(\sin(\theta)) = \theta$

$$\begin{aligned}
\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}}dx &= \int_{?}^{?} \frac{4\cos(w)}{\sqrt{1-\sin^2 w}}dw \\
&= \int_{?}^{?} \frac{4\cos w}{\cos^2 w}dw \\
&= \int_{?}^{?} 4dw \\
&= [4w]_{?}^{?} \\
&= [4\arcsin(x)]_{1/2}^{1/\sqrt{2}} && \text{Since } \frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}} \\
&= 4\arcsin(\frac{1}{\sqrt{2}}) \\
&= \frac{4\pi}{4} - \frac{4\pi}{6} \\
&= \frac{\pi}{3}
\end{aligned}$$

10.6 Computing Definite Integrals

1. $\int_1^2 x^3 dx$

$$\begin{aligned}
\int_1^2 x^3 dx &= \left[\frac{x^4}{4}\right]_1^2 \\
&= \frac{2^4}{4} - \frac{1^4}{4} = 4 - \frac{1}{4} = \frac{15}{4}
\end{aligned}$$

2. $\int_0^1 [e^x + e^{-x} - \cos(2x)]\, dx$

$$\begin{aligned}
\int_0^1 \left[e^x + e^{-x} - \cos(2x)\right] dx &= \left[e^x - e^{-x} - \frac{\sin(2x)}{2}\right]_0^1 \\
&= \left[e^1 - e^{-1} - \frac{\sin 2}{2}\right] - \left[e^0 - e^{-0} - \frac{\sin(0)}{2}\right] \\
&= (e - \frac{1}{e} - \frac{\sin 2}{2}) - (1 - 1 - 0) \\
&= e - \frac{1}{e} - \frac{\sin 2}{2}
\end{aligned}$$

3. $\int_{\pi/4}^{\pi/3} \sec^2 x dx$

$$\begin{aligned}
\int_{\pi/4}^{\pi/3} \sec^2 x dx &= [\tan(x)]_{\pi/4}^{\pi/3} \\
&= \tan\frac{\pi}{3} - \tan\frac{\pi}{4} \\
&= \frac{\sqrt{3}}{2} - 1
\end{aligned}$$

4. $\int_1^2 \left[\frac{d}{dx}(\frac{\sin^2 x}{1+arctan^2 x+e^{-x^2}})\right]$

$$\int_1^2 \left[\frac{d}{dx}(\frac{\sin^2 x}{1+arctan^2 x+e^{-x^2}})\right] = \frac{\sin^2(2)}{1+\arctan^2(2)+e^{-4}} - \frac{\sin^2(1)}{1+\arctan^2(1)+e^{-1}}$$

10.6.1 Aside About Circles

The collection of points (x, y) that are one unit away from $(0, 0)$ is:

$$1 = \sqrt{(x-0)^2 + (y-0)^2}$$

$$1 = x^2 + y^2 \implies y = \pm\sqrt{1-x^2}$$

10.7 Areas Using Integrals

When we calculate the areas under curves of a function, it can be zero or negative. However, we mostly deal with signed areas of the function which deal with negative and positive values of the graph.
If we want the total unsigned area, we can use absolute values, instead.

10.7.1 Examples

1. Write the integral between $y = \cos(x)$, the $x-$axis, from $[0, \pi]$.
   We know $\cos(x)$ is positive on $[0, \pi/2]$ and negative from $[\pi/2, \pi]$. So, our unsigned area is:

$$|\int_0^{\pi/2} \cos(x)dx| + \int_{\pi/2}^{\pi} \cos(x)dx$$

2. Write the integral between $y = x^2 + 3$ and $y = 3x + 1$.
   First, find the intersection points:

$$\begin{aligned} x^2 + 3 &= 3x + 1 \\ x^2 - 3x + 2 &= 0 \\ (x-1)(x-2) &= 0 \end{aligned}$$

   So, our intersection points are $x = 1, x = 2$. Next, we can calculate the integral.

$$\int_1^2 (3x+1) - (x^2+3)dx = \int_1^2 (3x - 2 - x^2)dx$$

10.7.2 Minimizing Area

For the function

$$f_a(x) = (1 + a) - ax^2$$

find the value of $a > 0$ that minimizes the area of the region bounded by the graph of f_a and the $x-$axis.
First, we set up the integral:

$$\int 1 + a - ax^2 dx$$

We want to find the endpoints of x, so we use the quadratic formula:

$$x_i = \frac{0 \pm \sqrt{-4(1+a)a}}{-2a}$$

Using this endpoint, we then can find this area (multiplying by 2 since it is symmetric above the $y-$axis:

$$2\int_0^{x_i} 1 + a - ax^2 dx$$

10.7.3 Finding Length From the Area

What is the length of the rectangle with a a base $[a, b]$, and the area $\int_a^b f(x)dx$?
Using the area of a rectangle we know that:

$$(b-a)(l) = \int_a^b f(x)dx$$

So, we can say that the length of the rectangle is:

$$l = \frac{1}{b-a}\int_a^b f(x)dx$$

This is equivalent to the **average value of a function**.

11 Integration Techniques

11.1 Simplification and Guess-and-Check

Simplification can be used to help make equations easier to compute and understand.

Example 1:

$$\begin{aligned}\int (x^2+6)^2 dx &= (x^2+6)(x^2+6)dx \\ &= (x^4+12x^2+36)dx\end{aligned}$$

Check: $\frac{d}{dx}x^5 = 5x^4$. So, $\frac{d}{dx}\frac{x^5}{5} = \frac{1}{\cancel{5}}\cancel{5}x^4$.
Then, $\frac{d}{dx}4x^3 = 12x^2$. Hence:

$$(x^4+12x^2+36)dx = \frac{x^5}{5} + 4x^3 + 36x + c$$

11.2 Integration By Substitution

For an integral $\int f'(g(x))g'(x)dx$, we can let $w = g(x)$, then $dw = g'(x)dx$.

$$\begin{aligned}\int \frac{5x+3}{\sqrt{\frac{5}{2}x^2-3x-1}}dx &= \int \frac{dw}{\sqrt{w}} && \text{Since } w = \frac{5}{2}x^2-3x-1, \quad dw = (5x+3)dx \\ &= \int w^{-1/2}dw \\ &= 2w^{1/2}+c \\ &= 2\sqrt{\frac{5}{2}x^2+3x-1}+c && \text{Since } \frac{d}{dw}w^{1/2} = \frac{1}{2}w^{-1/2}\end{aligned}$$

11.2.1 Examples

1. $\int \frac{\sin(x^{1/2}}{x^{1/2}}$
   Let $w = x^{1/2}dx$. Then $2dw = \frac{2}{2\sqrt{x}}dx$.

$$\begin{aligned}\int \frac{\sin(x^{1/2}}{x^{1/2}} &= \int 2\sin(w)dw \\ &= -2\cos(w)+c \\ &= -2\cos(\sqrt{x})+c\end{aligned}$$

2. $\int e^x \cos(e^x) dx$
Let $w = e^x, dw = e^x dx$.

$$\begin{aligned}\int e^x \cos(e^x) dx &= \int \cos(e^x) \cdot (e^x dx) \\ &= \int \cos(w) dw \\ &= \sin(w) + c \\ &= \sin(e^x) + c\end{aligned}$$

3. $\int \cot(x) dx = \int \frac{\cos(x)}{\sin(x)} dx$
Let $w = \sin(x), dw = \cos(x) dx$.

$$\begin{aligned}\int \frac{\cos(x)}{\sin(x)} dx &= \int \frac{dw}{w} \\ &= \int \frac{1}{w} \\ &= \ln|w| + c \\ &= \ln|\sin(x)| + c\end{aligned}$$

4. $\int x^2 \sqrt{x+1} dx$ Let $w = \sqrt{x+1} = (x+1)^{1/2}$. Then, $dw = \frac{1}{2}(x+1)^{-1/2} dx = \frac{1}{2\sqrt{x+1}} dx = \frac{1}{2w} dx$. Hence $2w dw = dx$.
Then, $w^2 = x + 1$, $x = w^2 - 1$, $x^2 = (w^2 - 1)^2$. So:

$$\begin{aligned}\int x^2 \sqrt{x+1} dx &= \int (w^2-1)^2 2w^2 \ dw \\ &= \int (w^4 - 2w^2 - 1) 2w^2 dw \\ &= \int 2w^6 - 4w^4 - 2w^2 dw \\ &= \frac{2}{7} w^7 - \frac{4}{5} w^5 - \frac{2}{3} w^3 + c \\ &= \frac{2}{7}(x+1)^{7/2} - \frac{4}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + c\end{aligned}$$

5. $\int \frac{e^2 x}{\sqrt{e^x+1}}$
Let $w = e^x, dw = e^x dx$.

$$\begin{aligned}\int \frac{e^2 x}{\sqrt{e^x+1}} &= \int \frac{e^x \cdot e^x}{\sqrt{e^x+1}} dx \\ &= \int \frac{w}{\sqrt{w+1}} dw \\ &= \int \frac{z-1}{\sqrt{z}} dw && \text{Let } z = w+1, \quad z-1 = w, \quad dz = dw \\ &= \int \frac{z}{z^{1/2}} - z^{-1/2} dz \\ &= \int z^{1/2} - z^{-1/2} dz \\ &= \frac{2}{3} z^{3/2} - 2z^{1/2} + c \\ &= \frac{2}{3}(e^x+1)^{3/2} - 2(e^x+1)^{1/2} + c\end{aligned}$$

6. $\int \frac{\ln(\ln(x))^2}{x \ln x} dx$
   Let $w = \ln(x), dw = \frac{1}{x}dx.$

$$
\begin{aligned}
\int \frac{\ln(\ln(x))^2}{x \ln x} dx &= \int \frac{(\ln(w))^2}{w} dw \\
&= z^2 dz && \text{Let } z = \ln(w), dz = \frac{1}{w}dw \\
&= \frac{z^3}{3} + c \\
&= \frac{(\ln(w))^3}{3} + c \\
&= \frac{[\ln(\ln(x)]^3}{3} + c
\end{aligned}
$$

7. $\int xe^{-x^2} dx$
   Let $w = -x^2$, and $\frac{dw}{-2} = \frac{\cancel{-2}x}{\cancel{-2}}dx.$

$$
\begin{aligned}
\int xe^{-x^2} dx &= e^w dw \\
&= e^w + c \\
&= e^{-x^2} + c
\end{aligned}
$$

8. $\int_0^2 \sqrt{x^3+1}x^2 dx$
   Let $w = x^3 + 1$, and $\frac{dw}{3} = \frac{\cancel{3}x^2}{\cancel{3}}dx$ Method 1:

$$
\begin{aligned}
\int_0^2 \sqrt{x^3+1}x^2 dx &= \int_?^? \frac{\sqrt{w}}{3} dw \\
&= \left[3\frac{2w^{3/2}}{3}\right]_?^? \\
&= \left[2(x^3+1)^{3/2}\right]_0^2 \\
&= 2(9^{3/2}) - 2
\end{aligned}
$$

   Method 2:
   If $x : 0 \to 2$, then $w : 0^3 + 1 \to 2^3 + 1$

$$
\begin{aligned}
\int_0^2 \sqrt{x^3+1}x^2 dx &= \int_1^9 \frac{\sqrt{3}}{3} dw \\
&= \left[2w^{3/2}\right]_1^9 \\
&= 2(9)^{3/2} - 2(1)^{3/2}
\end{aligned}
$$

11.3 Integration By Parts

For a derivative, we have $(f \cdot g)' = f'g + fg'$. Similarly for integrals:

$$
\begin{aligned}
\int (f(x)g(x))' dx &= \int f'(x)g(x)dx + \int f(x)g'(x)dx \\
f(x)g(x) &= \int f'(x)g(x)dx + \int f(x)g'(x)dx \\
\int f'(x)g(x)dx &= f(x)g(x) - \int f(x)g'(x)dx
\end{aligned}
$$

For example:

$$\int x\ln(x)dx$$

$$\int u\ dv = uv - \int v\ du$$

We don't know how to integrate $\ln(x)$, so let $u = \ln(x)$, with $dv = x\ dx$. Then, $du = \frac{1}{x}dx$, and $v = \frac{x^2}{2}$. Then:

$$\begin{aligned}\int x\ln(x)dx &= \frac{x^2}{2}\ln(x) - \int \frac{x^2}{2}\cdot\frac{1}{x}dx \\ &= \frac{x^2}{2}\ln(x) - \frac{1}{2}\int x\ dx \\ &= \frac{x^2}{2}\ln(x) - \frac{1}{2}\cdot\frac{x^2}{2} + c\end{aligned}$$

11.3.1 Examples

1. $\int xe^{-2x}dx$
   Let $u = x, dv = e^{-2x}dx$. Then, $du = dx, v = \frac{e^{-2x}}{-2}dx$.

$$\int xe^{-2x}dx = x\frac{e^{-2x}}{-2} - \int \frac{e^{-2x}}{-2}dx$$

   We have $\frac{d}{dx}\frac{e^{-2x}}{-2} = e^{-2x}$, and
   $\frac{d}{dx}\frac{e^{-2x}}{4} = \frac{1}{4}e^{-2x}(-2)$. So

$$x\frac{e^{-2x}}{-2} - \int \frac{e^{-2x}}{-2}dx = \frac{xe^{-2x}}{-2} - \frac{e^{-2x}}{4} + c$$

2. $\int x^2 \sin x\ dx$
   Let $u = x^2, dv = \sin x\ dx$. Then, $du = 2x\ dx, b = -\cos x$.

$$\int x^2\sin x\ dx = -x^2\cos(x) + 2\int x\cos(x)dx$$

   Then, let $u = x, dv = \cos x\ dx$. We also have $du = dx, v = \sin x$.

$$\begin{aligned}-x^2\cos(x) + 2\int x\cos(x)dx &= -x^2\cos x + 2[x\sin(x) - \int \sin x\ dx] \\ &= -x^2\cos x + 2x\sin x - 2[-\cos x] + c\end{aligned}$$

3. $\int \ln x\ dx$
   Let $u = \ln x, dv = dx$. Then, $du = \frac{1}{x}dx, v = x$.

$$\begin{aligned}\int \ln x\ dx &= x\ln x - \int x\frac{1}{x}dx \\ &= x\ln x - \int 1dx \\ &= x\ln x - x + c \\ &= x(\ln x - 1) + c\end{aligned}$$

4. $\int \sin\sqrt{x}dx$
   Let $u = \sin\sqrt{x}, dv = dx$. Then, $du = \frac{\cos\sqrt{x}}{2\sqrt{x}}, v = x$.

$$\int \sin\sqrt{x}dx = x\sin\sqrt{x} - \frac{1}{2}\int \frac{x}{\sqrt{x}}\cos\sqrt{x}\ d\sqrt{x}$$

   Then, let $u = \cos x, dv = x^{1/2}$, and $du = \frac{-\sin\sqrt{x}}{2\sqrt{x}}dx, v = \frac{2}{3}x^{3/2}$.

5. $\int_1^e (\ln(x))^3 dx$
   Let $u = (\ln x)^3, dv = dx$, then $du = \frac{3(\ln(x)^2}{x}dx, v = x$.

$$\int_1^e (\ln(x))^3 dx = \left[x(\ln x)^3\right]_1^3 - 3\int_1^e (\ln x)^2 dx$$

   Let $u = (\ln x)^2, dv = dx$, then $du = \frac{2\ln x}{x}dx, v = x$.

$$= \left[x(\ln x)^3\right]_1^e - 3\left[[x(\ln x)^2]_1^e - 2\int_1^e \ln x dx\right]$$

   Let $u = \ln x, dv = dx$, then $du = \frac{1}{x}dx, v = x$.

$$\begin{aligned}
&= \left[x(\ln x)^3\right]_1^e - 3\left[x(\ln x)^2\right]_1^e + 6\left[[x\ln x]_1^e - \int_1^e dx\right] \\
&= (e-0) - 3(e-0) + 6(e-0) - [x]_1^e \\
&= 3e - 1
\end{aligned}$$

11.4 Partial Fractions

The integrals of partial fractions are calculated similarly to rational fractions, however, partial fractions typically have a denominator with a larger degree than the numerator.
If the numerator is bigger than the denominator, use division or substitution.

1. $\int \frac{1}{(x+1)(x-3)}dx$
   Let this be equal to $\int \frac{A}{x+1} + \frac{B}{x-3}dx$. Then:

$$\frac{A(x-3) + B(x+1)}{(x+1)(x-3)} = \frac{1}{(x+1)(x-3)}$$

   So:

$$\begin{gathered}
Ax - 3A + Bx + B = 1 \\
(A+B)x + (B-3A) = 0x + 1
\end{gathered}$$

   Hence, we know $A + B = 0$, or $A = -B$, and for our constants we have $B - 3A = 1, B - 3(-B) = 1, 4B = 1$. Thus, $B = \frac{1}{4}$. Using this, we have:

$$\begin{aligned}
\int \frac{1}{(x+1)(x-3)}dx &= \int \frac{-1/4}{x+1} + \frac{1/4}{x-3}dx \\
&= \frac{-1}{4}\ln|x+1| + \frac{1}{4}\ln|x-3| + C
\end{aligned}$$

2. $\int \frac{1}{(x+1)^n} dx$
   We can use substitution. Let $w = x+1, dw = dx$.

$$\int \frac{1}{(x+1)^n} dx = \int \frac{1}{w^n} dw$$
$$= \int w^{-n} dw \qquad = \frac{w^{-n+1}}{-n+1} + C$$
$$= \frac{(x+1)^{1-n}}{1-n} + c$$

3. $\int \frac{(x+1)-1}{(x+1)^2} dx$ We can cancel out the $x+1$ term.

$$\int \frac{x+1}{(x+1)^2} - \frac{1}{(x+1)^2} dx$$
$$= \int \frac{1}{x+1} - (x+1)^{-2} dx$$
$$= \ln|x+1| + (x+1)^{-1} + c$$

   We can do a similar process with partial fractions.

$$\int \frac{A}{(x+1)^1} + \frac{B}{(x+1)^2} dx = \frac{A(x+1)+B}{(x+1)^2} = \frac{x}{(x+1)^2}$$

   So, we have:

$$Ax + A + B = x \iff Ax + (A+B) = 1x + 0$$

   So, $A = 1, A+B = 0$, meaning $A = -B, B = -1$. Then:

$$\int \frac{1}{x+1} - \frac{1}{(x+1)^2} dx,$$

   which then is solved the same way as above.

4. $\int \frac{2x+6}{(x+1)^2} dx$
   We have:

$$\int \frac{A}{x+1} + \frac{B}{(x+1)^2} dx = \int \frac{A(x+1)+B}{(x+1)^2} dx$$

   So, $2x+6 = Ax + A + B$. Then $2x = Ax \implies 2 = A$, and $6 = A+B \implies B = 4$. Thus:

$$\int \frac{2}{x+1} + \frac{4}{(x+1)^2} dx = 2\ln|x+1| - 4(x+1)^{-1} + c$$

5. $\int \frac{x^2}{(x+1)^3} dx$
   We have $\int \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} dx$
   Let $w = x+1 \implies x = w-1$. Then, $dw = dx$.

$$\int \frac{(w-1)^2}{w^3} dw = \int \frac{w^2}{w^3} - \frac{2w}{w^3} + \frac{1}{w^3} dw$$
$$= \int \frac{1}{w} - \frac{2}{w^2} + \frac{1}{w^3} dw$$
$$= \ln|w| + 2w^{-1} - \frac{w^{-2}}{2} + c$$
$$= \ln|x+1| + 2(x+1)^{-1} - \frac{(x+1)^{-2}}{2} + c$$

6. $\int \frac{p(x)}{x^4(x+1)^3(x+2)(x^2+1)(x^2+4)}$
   We can expand this (make sure to get all terms):

$$\int \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{E}{(x+1)} + \frac{F}{(x+1)^2} + \frac{G}{(x+1)^3} + \frac{H}{(x+2)} + \frac{I}{x^2+1} + \frac{J}{x^2+4} dx$$

12 Volumes

For any shape with the same cross-section (e.g., prisms), calculate the area of the face, then multiply it by the distance of the object.
More generally, we can say:

$$V = \int_a^b A(x)dx$$

1. **Prisms**
   Take a prism with a base defined as $x^2 + y^2 = 5$. with a height of 2.

   Generally, we solve volumes by multiplying 2 by the integral of half of the prism. In this case, it is $y = \sqrt{5 - x^2}$. Then, we know we go from a height of 0 to 2 for the distance of the prism, so we integrate from 0 to 2.
   Hence, our volume is $V = 2\int_0^2 \sqrt{5 - x^2}dx$

2. **Pyramids**
   Compute the volume of a pyramid with height H and the square base with side length L.

   Geometrically, we determine this as a square base that gets smaller and smaller as the height of the pyramid increases. If n represents the number of slices, as $n \to \infty$, the height approaches dy.
   We can then draw triangles as the cross section of the object, where the base of the triangle is $\frac{1}{2}L$ and the height is H.
   As the height gets smaller, we get smaller triangles, which are defined as $h = H - y$, and $2l = \frac{(H-y)L}{H}$.
   l describes half the distance of the cross section (the pyramid's base at height y).
   Now, we can calculate the volume: $\int_0^H (\frac{(H-y)L}{H})^2 dy$.

3. **Other Volumes**
   Find the volume of the shape with base bordered by the $x-$axis, $y-$axis, and $y = 6 - x$, and has a square cross-section, which is parallel to the y-axis.

   The resulting shape will look something like a wedge (a square base that forms a triangular point).
   The cross-section of the square base gets gradually smaller as x increases until $x = 6$ (where the function becomes a zero), so we know we integrate from 0 to 6.
   The height of the shape is dependent on the y value as $x \to 6$, so we know we take the value of y at each x value to determine the volume.
   Hence: $V = \int_0^6 = (y)^2 dx = \int_0^6 (6 - x)^2 dx$.

12.1 Volumes of Rotation

Using volumes of rotation is useful when we want to find the volume of something that started as a two-dimensional area. Usually, we rotate around a certain point (either in the y or x axis) in order to create circles in the third-dimension (or cylinders), of which the volume can then be calculated.
You can calculate these volumes using:

1. **Washers** - 2D circles that sometimes have holes in the centre of them.
   General Formula:

$$\int_a^b (\pi r^2)dx = \int_a^b \pi(f(x))^2 dx$$

If we rotate around the y-axis, we use $\int_a^b \pi(g(y))^2 dy$.

Example: Volume of a Sphere.
We know the equation of a circle with radius r centred around $(0,0)$ is $x^2 + y^2 = r^2$.
If we rotate the circle around the x-axis, we can produce a sphere. We integrate in terms of y, so we find the radius $y = \sqrt{r^2 - x^2}$. Hence, we get the integral $\int_{-r}^{r} \pi(\sqrt{r^2 - x^2})^2 dx$. Expanding this we have:

$$\begin{aligned}
\int_{-r}^{r} \pi(\sqrt{r^2 - x^2})^2 dx &= \int_{-r}^{r} \pi(r^2 - x^2) dx \\
&= \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r} \\
&= \pi \left[(r^3 - \frac{r^3}{3}) - (r^2(-r) - \frac{(-r)^3}{3}) \right] \\
&= \pi \left[\frac{2}{3} r^3 + \frac{2}{3} r^3 \right] \\
&= \frac{4}{3} \pi r^3
\end{aligned}$$

2. **Cylindrical Shells -** 3D cylinders that sometimes have holes in the centre of them).
General Formula: We calculate the area of the cylinder using the rectangular cross-section of a cylinder (defined as $2\pi r$), and then multiply this value by the height of the shape.
When we rotate around the $y-$axis, our x-values represent a different height of the cylindrical shells.
Hence:

$$\int_a^b (2\pi r) y dx = \int_a^b 2\pi f(x) \cdot x dx$$

Example: Determine the volume created by rotating Ω around the $y-$axis where Ω is the area bounded between $y = x^5 + x - 2$, $x = 2$, and the $x-$axis.

Here, we know we're integrating from 0 to 2 based on our bounds. When rotating around the $y-$axis, we integrate in terms of x, so we have the following formula:

$$\begin{aligned}
\int_0^2 2\pi x y dx &= \int_0^2 2\pi x(x^5 + x - 2) dx \\
&= 2\pi \int_0^2 (x^6 + x^2 - 2x) dx
\end{aligned}$$

12.1.1 More Examples

Let Ω be the finite area bounded between $y = x^2$, the $x-$axis, and $x = 1$.
Find the formulas for the following volumes of rotation:

1. Rotating around $y = 0$:
   (a) Washers: Our cross-section becomes circles that become bigger as they get closer to $x = 1$. Here, we use the area of a circle (πr^2) r is the perpendicular axis of rotation.
   Hence, our volume is $\int_0^1 \pi(x^2)^2 dx$.
   (b) Cylindrical Shells: We create cylinders that get bigger as they approach $x = 1$. We use the surface area of a cylinder, where r is the perpendicular axis of rotation, and h is parallel to the axis of rotation.
   Note that $y = x^2$, and so we get $x = 1 - \sqrt{y}$ as our radius.
   Hence, our volume is $\int_0^1 2\pi(y)(1 - \sqrt{y}) dy$

2. Rotating around $y = 2$:

   (a) Washers: Here, we have a "cut-out" for the volume, so we have the larger circle $\pi(2)^2(1)$, and our smaller cutout (which gets smaller as $x \to 1$).
   Hence, our volume is: $\int_0^1 \pi(2)^2 - \pi(2 - x^2)^2 dx$.

   (b) Cylindrical Shells: Here, we form a large bowl shape with an empty shape in the middle, so the radius of the cylinder decreases as it approaches 1, making it equal to $(2 - y)$. Then, similar to (1), our height is equal to $(1 - \sqrt{y})$.
   Thus, our volume is $\int_0^1 2\pi(2 - y)(1 - \sqrt{y})dy$.

3. Rotating around $y = -1$.

   (a) Washers: Here, because we rotate around a horizontal axis, we take the distance from $y = -1$ to $y = x^2$ for the outer circle, to the inner circle (with a radius of 1).
   Thus, our volume is $\int_0^1 \pi(x^2 - 1)dx$.

   (b) Cylindrical Shells: For the radius of the cylinder, we take the distance of $y = -1$ to $y = 0$, which is equal to $(y + 1)$. Next, we take the height of the cylinder, which is equal to $(1 - \sqrt{y})$.
   Hence, the volume is: $\int_0^1 2\pi(y + 1)()1 - \sqrt{y})dy$.

4. Rotating around $x = 0$.

   (a) Washers: Because we rotate around a vertical axis, the radius of our circles depend on y-values. Since our integral is from 0 to 1, we know the radius to calculate for the outer circle is 1.
   However, we form a hole in the middle, so we subtract the areas of the smaller circle in our integral to calculate the volume. This difference between the smaller and larger circle is $\sqrt{y}$ (since $y = x^2$).
   Hence, our volume is: $\int_0^1 \pi(1)^2 - \pi(\sqrt{y})^2 dy$.

   (b) Cylindrical Shells: Since we rotate around the vertical axis, the radius of our cylinder is dependent on the x-values, so the radius is always equal to x.
   Then, the height is just the y value, which we know is $y = x^2$ based on the given formula.
   Hence, our volume is: $\int_0^1 2\pi x(x^2)dx$

5. Rotating around $x = 4$.

   (a) Washers: Similar to (4), however, because of the large cylindrical distance to the vertical axis of rotation, the smaller circle has a constant size, while the outer circle gets larger/smaller as our y-value changes.
   The inner circle spans from $x = 1$ to $x = 4$, so the radius is equal to 3.
   The outer circle is the distance from our value of x to the axis of rotation ($x = 4$), and since we integrate in terms of y, the radius is equal to $(4 - \sqrt{y})$.
   Hence, our volume is equal to $\int_0^1 \pi(4 - \sqrt{y})^2 - \pi(3)^2 dy$.

   (b) Cylindrical Shells: Similar to (4); the radius of our cylinder is dependent on the values of x, so in other words, we want the distance from our vertical line at x and $x = 4$. Thus, the radius of the cylinder is $(4 - x)$.
   Then, the height is the y-coordinate for each value of x, and we know $y = x^2$, so the height is equal to x^2.
   Thus, our volume is: $\int_0^1 2\pi(4 - x)(x^2)dx$.

6. Rotating around $x = -3$.

   (a) Washers: Here, the radius of the outer circle is constant, with a radius equal to the distance of $x = 1$ to $x = -3$. Thus, the radius is 4.
   Our inner radius is not constant, and since our cross-section is dependent on the value of y, we take the distance from our value of x to the axis of rotation $x = 3$ in terms of y. So, our inner radius is $(3 + \sqrt{y})$.
   Thus, our volume is: $\int_0^1 \pi(4)^2 - \pi(3 + \sqrt{y})^2 dy$

(b) Cylindrical Shells: Since we rotate around a vertical axis, the radius of our cylinder is equal to x plus the distance to $x = 3$, so our radius is $(3 + x)$. Then, since the height is dependent on our value of y and we integrate in terms of x, the height is equal to x^2.
Hence, our volume is: $\int_0^1 2\pi(3 + x)(x^2)dx$

13 Improper Integrals

13.1 Big Theorem

$$\ln n < n^a < c^n < n! < n^n$$

That is , ln < polynomials < exponents < factorials < n^n,
where $a_n < b_n$ means $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.
We also write $\frac{b_n}{a_n} = \frac{1}{a_n/b_n} \to \infty$.

13.1.1 Calculations

1.

$$\begin{aligned}\lim_{n\to\infty} \frac{n! + 2e^n}{3n! + 4e^n} &= \lim_{n\to\infty} \frac{n! + 2e^n}{3n! + 4e^n} \cdot \frac{1/n!}{1/n!} \\ &= \lim_{n\to\infty} \frac{\frac{n!}{n!} + \frac{2e^n}{n!}}{\frac{3n!}{n!} + \frac{4e^n}{n!}} \\ &= \frac{1}{3}\end{aligned}$$

2.

$$\begin{aligned}\lim_{n\to\infty} \frac{2^n + (2n)^n}{2^{n+1} + n^2} &= \lim_{n\to\infty} \frac{2^n + 4n^2}{(2)2^n + n^2} \\ &= \lim_{n\to\infty} \frac{\frac{2^n}{2^n} + \frac{4n^2}{2^n}}{2(\frac{2^n}{2^n}) + \frac{n^2}{2^n}} \\ &= \frac{1}{2} && \text{Since } n^2 < c^n\end{aligned}$$

3.

$$\begin{aligned}\lim_{n\to\infty} \frac{5n^5 + 5^n + 5n!}{n^n} &= \lim_{n\to} \frac{5n^5}{n^n} + \frac{5^n}{n^n} + \frac{5n!}{n^n} \\ &= 0\end{aligned}$$

13.2 Improper Integrals

1. **Type-1 Improper Integrals** - Let f be a bounded, continuous function on $[c, \infty)$. How do we define the improper integral:

$$\int_c^\infty f(x)dx \ ?$$

We know this value may be finite, i.e., $\int_1^\infty \frac{1}{2^x} dx < \infty$, or potentially infinite, i.e., $\int_1^\infty \frac{1}{x} dx? = \infty$.

For example, take the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ... = 1$.
We can write this as $\int_1^\infty \frac{1}{2^x} dx$, which is finite and less than or equal to 1. We know that this overestimation $U_P(f) = 1$.

2. **Type-2 Improper Integrals** - Let f be a continuous function on $(a, b]$, possibly with $x = a$ as a vertical asymptote. How do we define the improper integral

$$\int_a^b f(x)dx$$

We have two options, where it **converges** (to a finite value $< \infty$), or **diverges** (to infinity).

13.2.1 Example of Improper Integrals

1.

$$\begin{aligned}
\int_1^\infty \frac{1}{x^2+x}dx &= \int_1^\infty \frac{1}{x} - \frac{1}{x+1}dx \\
&= \lim_{R\to\infty} \lim_1^R \frac{1}{x} - \frac{1}{x+1}dx \\
&= \lim_{R\to\infty} [\ln|x| - |n|]_1^R \\
&= \lim_{R\to\infty} \left[\ln|\frac{x}{x+1}|\right]_1^R && \text{By logarithm rules.} \\
&= \lim_{n\to\infty} \ln|\frac{R}{R+1}| - \ln\frac{1}{2} \\
&= 0 - \ln\frac{1}{2} \\
&= -\ln\frac{1}{2} \\
&= -[\ln 10 \ln 2] \\
&= \ln 2
\end{aligned}$$

Hence, the integral converges.

13.2.2 Important Improper Integrals

We can use the definition of the improper integral to determine for which values of $p \in \mathbb{R}$ each of the following improper integrals converge.

1. $\frac{1}{x^p}$, or the P-Test.

$$\begin{aligned}
\int_1^\infty \frac{1}{x^p}dx &= \lim_{R\to\infty} \int_1^R x^{-p}dx \\
&= \lim_{R\to\infty} \int_1^R x^{-p}dx \\
&= \lim_{R\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^R \\
&= \lim_{R\to\infty} \left[\frac{R^{-p+1}}{-p+1} - \frac{1}{-p+1}\right]
\end{aligned}$$

Hence, if $-p+1 > 0$, then $\int_1^\infty x^{-p}$ diverges.
So, $-p > -1 \implies p < 1$. Hence, $p < 1 \implies \int_1^\infty \frac{1}{x^p}dx$ diverges.

If $-p+1 < 0$, then we get

$$\lim_{R\to\infty} \left[\frac{1}{(-p+1)R^{p-1}} - \frac{1}{-p+1}\right],$$

which means that $-p < -1 \implies p > 1$, meaning that $\int_1^\infty \frac{1}{x^p} dx$ converges and equals $\frac{-1}{-p+1}$.

If $p = 0$, then:

$$\begin{aligned}
\lim_{1}^{\infty} \frac{1}{x} dx &= \lim_{R\to\infty} \int_1^R \frac{1}{x} dx \\
&= \lim_{R\to\infty} [\ln|x|]_1^R \\
&= \lim_{R\to\infty} \ln R - \ln 1 \\
&= \infty,
\end{aligned}$$

meaning that it diverges at $p = 0$.

Next, we determine the other endpoints.

$$\begin{aligned}
\int_0^1 \frac{1}{x^p} dx &= \lim_{t\to 0^+} \int_t^1 \frac{1}{x^p} dx \\
&= \lim_{t\to 0^+} \left[\frac{x^{-p+1}}{-p+1} \right]_t^1 \\
&= \lim_{t\to 0^+} \left[\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1} \right]
\end{aligned}$$

So, $p > 1 \implies -p + 1 < 0$, which diverges.
If $p < 1 \implies -p + 1 > 0$, which converges.
When $p = 1$,

$$\begin{aligned}
\int_0^1 \frac{1}{x} dx &= \lim_{t\to 0^+} \int_t^1 \frac{1}{x} dx \\
&= \lim_{t\to 0^+} [\ln|x|]_t^1 \\
&= \lim_{t\to 0^+} \ln t - \ln 1 \\
&= -\infty,
\end{aligned}$$

meaning it diverges.

So, when we calculate $\int_0^\infty \frac{1}{x^p} dx = \int_0^1 \frac{1}{x^p} dx + \int_1^\infty \frac{1}{x^p} dx$, either one of the parts of the integral diverge, meaning the integral diverges.

13.3 Probability

A non-negative function f defined on $(-\infty, \infty)$ is called a **probability density function** if

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

i.e., $\int_0^a f(x) dx$ is equal to the probability that some value is $\leq a$.

The mean of a probability density function is defined as

$$\mu = \int_{-\infty}^{\infty} x f(x) dx$$

Let $f = \begin{cases} Ce^{-kx} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

1. For $k > 0$, find a constant C such that the function f is a probability density function.

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} f(x)dx \\
&= \lim_{Q\to\infty} \int_{-Q}^{0} f(x)dx + \lim_{R\to\infty} \int_{0}^{R} f(x)dx \\
&= 0 + \lim_{R\to\infty} \int_{0}^{R} Ce^{-kx}dx \\
1 &= \lim_{R\to\infty} \left[\frac{Ce^{-kx}}{-k}\right]_0^R \\
&= \lim_{R\to\infty} \frac{C}{-k}e^{-kR} - \frac{C}{-k}e^{0} \\
&= \lim_{R\to\infty} \frac{C}{-ke^{kR}} + \frac{C}{k} \\
&= \frac{C}{k}
\end{aligned}$$

This means that $\frac{c}{k} = 1 \implies c = k$.

Hence, $f(x) = \begin{cases} \frac{1}{k}e^{-kx} & x \geq 0 \\ 0 & x < 0 \end{cases}$

2. Calculate the mean μ.

$$\begin{aligned}
\int_{-\infty}^{\infty} xf(x)dx &= \int_{-\infty}^{0} x(0)dx + \int_{0}^{\infty} x(k)e^{-kx}dx \\
&= \lim_{R\to\infty} \int_{0}^{R} kxe^{-kx}dx
\end{aligned}$$

Let $u = x, du = dx$, then $v = \frac{e^{-kx}}{-k}, dv = e^{-kx}dx$. Then:

$$k \int xe^{-x}dx = \frac{x}{-k}e^{-kx} + \frac{1}{k}\int e^{-kx}dx$$

Using this information, we can calculate the mean μ by completing the integral.

14 Sequences

Sequences are ordered collections of elements. That is, a sequence is a map $a : \mathbb{N} \to \mathbb{R}$, which is typically denoted as a_n or $(a_n)_{n=1}^{\infty}$.

Examples:

1. $\{a_n\}_{n=0}^{\infty} = \{1, 4, 9, 16, 25, ...\}$
   $a_n = (n+1)^2$
2. $\{b_n\}_{n=1}^{\infty} = \{1, -2, 4, -8, 16, -32, ...\}$
   $b_n = (-2)^{n-1}$, or $b_n = (-1)^{n+1} \cdot 2^{n-1}$
3. $\{c_n\}_{n=1}^{\infty} = \{\frac{2}{1!}, \frac{3}{2!}, \frac{4}{3!}, \frac{5}{4!}, ...\}$
   $c_n = \frac{n+1}{n!}$
4. $\{d_n\}_{n=1}^{\infty} = \{1, 4, 7, 10, 13, ...\}$
   $d_n = -2 + 3n$

14.0.1 Limits of a Sequence

Since we treat sequences similarly to functions, we can also treat them to have limits. Specifically, we can imagine sequences as having specific horizontal asymptotes.
A sequence a_n converges to a limit $L \in \mathbb{R}$ is $\forall \epsilon > 0, \exists M \in \mathbb{N} : k \geq M \implies |a_k - L| < \epsilon$. Then, we write:

$$\lim_{n \to \infty} a_n = L$$

Equivalently, we can write:

1. $\forall \epsilon > 0, \exists n_0 \in \mathbb{R}, \forall n \in \mathbb{N} : n \geq n_0 \implies |L - a_n| < \epsilon$
2. $\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N} : n \geq n_0 \implies |L - a_n| \leq \epsilon$
3. $\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N} : n \geq n_0 \implies |L - a_n| < \frac{1}{\epsilon}$
4. $k \in \mathbb{Z}^+, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N} : n \geq n_0 \implies |L - a_n| < \frac{1}{k}$

14.1 Sequences and Convergence

For any function f with domain $[0, \infty)$, we define a sequence as $a_n = f(n)$. Let $L \in \mathbb{R}$.
We have that $\lim_{x \to \infty} f(x) = L \implies \lim_{n \to} a_n = L$.
We also have $\lim_{n \to \infty} a_n = L \implies \lim_{n \to \infty} a_{n+1} = L$ (since this means $(a_n)_{n=1}^{\infty} \implies (a_n)_{n=2}^{\infty} = L$).

14.1.1 Convergence and Divergence

1. **Convergence**
   $\exists L \in \mathbb{R}, \forall N \in \mathbb{R}, \forall n \in \mathbb{N} : n > N \implies |a_n - L| < \epsilon$
2. **Divergence**
   $\forall L \in \mathbb{R}, \exists \epsilon > 0, \forall N \in \mathbb{R}, \exists n \in \mathbb{N} : n > N \land |a_n - L| \geq \epsilon$
3. **Divergent to Infinity**
   $\forall M \in \mathbb{R}, \exists n_0 \in \mathbb{R}, \forall n \in \mathbb{N} : n \geq n_0 \implies a_n > M$

14.2 Monotonicity and Sequences

For any function f with domain $[0, \infty)$, we define a sequence as $a_n = f(n)$. Then:

1. If f is increasing, then $\{a_n\}_{n=0}^{\infty}$ is increasing.
2. If f is bounded, then $\{a_n\}_{n=0}^{\infty}$ is bounded.

		Convergent	Divergent
Montonic	Bounded	Yes	No
	Unbounded	No	Yes
Not Monotonic	Bounded	Yes	Yes
	Unbounded	No	Yes

14.3 Recurrence

Consider the sequence $\{R_n\}_{n=0}^{\infty}$ defined by:

$$\begin{cases} & R_0 = 1 \\ \forall n \in \mathbb{N}, & R_{n+1} = \frac{R_n + 2}{R_n + 3} \end{cases}$$

Compute R_1, R_2, R_3.

We have:

$$\begin{aligned} R_1 &= \frac{R_0+2}{R_0+3} \\ &= \frac{1+2}{1+3} \\ &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned} R_2 &= \frac{R_1+2}{R_1+3} \\ &= \frac{3/4+2}{3/4+3} \\ &= \frac{11}{15} \end{aligned}$$

$$\begin{aligned} R_3 &= \frac{R_2+2}{R_2+3} \\ &= \frac{11/15+2}{11/15+3} \end{aligned}$$

Suppose we want to find the limit of this recurrence.
If $\lim_{n\to\infty} R_n$ exists, we call it L. We compute:

$$\begin{aligned} \lim_{n\to\infty} R_{n+1} &= \lim_{n\to\infty} \frac{R_n+2}{R_n+3} \\ L &= \frac{L+2}{L+3} \\ L(L+3) &= L+2 \\ L^2+3L &= L+2 \\ L^2+2L-2 &= 0 \end{aligned}$$

Then, by quadratic formula, $L = -1+\sqrt{3}$ or $L = -1-\sqrt{3}$.

14.4 Basic and Limit Comparison Test

1. Basic Comparison Test:
   For $0 \le a_n \le b_n$:
   (a) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
   (b) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.
2. Limit Comparison Test:
   For $a_n, b_n > 0$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = L (0 < L < \infty)$: $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both diverge or both converge.

14.4.1 Example of BCT

We want to determine whether $\int_1^{\infty} \frac{1}{x+e^x} dx$ is convergent or divergent.
We can try at least two comparisons:

1. Compare $\frac{1}{x}$ and $\frac{1}{x+e^x}$.
   Using BCT, we have:
   $$0 < \frac{1}{x+e^x} < \frac{1}{x}$$
   Then we write:
   $$\begin{aligned}\int_1^\infty \frac{1}{x}dx &= \lim_{R\to\infty}\int_1^R \frac{1}{x}dx \\ &= \lim_{R\to\infty} \ln R - \ln 1 \\ &= \infty\end{aligned}$$
   However, this doesn't give us any information about $\frac{1}{x+e^x}$, so this comparison does not work.

2. Compare $\frac{1}{e^x}$ and $\frac{1}{x+e^x}$
   We have:
   $$0 < \frac{1}{x+e^x} < \frac{1}{e^x}$$
   Then:
   $$\begin{aligned}\int_1^\infty \frac{1}{e^x}dx &= \lim_{R\to\infty}\int_1^R e^{-x}dx \\ &= \lim_{R\to\infty} [-e^{-x}]_1^R \\ &= \lim_{R\to\infty} \frac{-1}{e^R} - \frac{-1}{e} \\ &= \frac{1}{e}\end{aligned}$$
   So, we know that $\int_1^\infty \frac{1}{e^x}dx$ is convergent. Then, by BCT, $\int_1^\infty \frac{1}{x+e^x}dx$ also converges.

14.5 Absolute Convergence

The integral $\int_a^\infty f(x)dx$ is called **absolutely convergent** when $\int_a^\infty |f(x)|dx$ converges.

14.5.1 Absolute Convergent Implies Convergent

Theorem. If an integral is absolutely convergent, then it is regularly convergent.

Proof. Let $f_+(x) = \begin{cases} f(x), & f(x) \geq 0 \\ 0, & f(x) \leq 0 \end{cases}$ and $f_-(x) = \begin{cases} 0, & f(x) \geq 0 \\ |f(x)|, & f(x) \leq 0 \end{cases}$.
Then, we have;
$$\int_a^\infty f_+(x)dx + \int_a^\infty f_-(x)dx = \int_a^\infty |f(x)|dx$$
Since $\int_a^\infty |f(x)|dx$ converges, so does $\int_a^\infty f_+$ and $\int_a^\infty f_-$. Note that if either part was infinite or oscillating, we say the integral diverges.
Then,
$$\begin{aligned}\int_a^\infty f(x)dx &= \int_a^\infty f_+(x)dx - \int_a^\infty f_-(x)dx \\ f \leq f_+ &\implies \int f \leq f_+ \\ \int_a^\infty |f|dx &= \int_a^\infty f_+(x)dx + \int_a^\infty f_-(x)dx\end{aligned}$$

Knowing this, we have:

$$\begin{aligned} f_+(x) - f_-(x) = f(x) \\ &\leq f_+(x) + f_-(x) \leq \infty \end{aligned}$$

Since we know $f_+(x) - f_(x) = f(x)$ and $f_+(x) + f_-(x) = |f(x)|$, we know that $\int_a^\infty f(x)dx$ is convergent. $\square$

15 Different Methods of Convergence

Test	When to Use	Conclusions
Geometric Series	$\sum_{n=0}^\infty ar^n$	$\sum_{n=0}^\infty ar^n = \frac{a}{1-r}$ if $\lvert r\rvert < 1$; diverges if $\lvert r\rvert \geq 1$.
Necessary Condition	All series	If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges.
Integral Test	• $a_n = f(n)$ • f is continuous, positive, and decreasing. • $\int_1^\infty f(x)dx$ is easy to compute.	$\sum_{n=1}^\infty$ and $\int_1^\infty f(x)dx$ both converge or both diverge.
P-Series	$\sum_{n=1}^\infty \frac{1}{n^p}$	Converges for $p > 1$; diverges for $p \geq 1$.
Basic Comparison Test	$0 \leq a_n \leq b_n$	If $\sum_{n=1}^\infty b_n$ converges, then $\sum_{n=1}^\infty a_n$ converges. If $\sum_{n=1}^\infty a_n$ diverges, then $\sum_{n=1}^\infty b_n$ diverges.
Limit Comparison Test	$a_n, b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L(0 < L < \infty)$	Both series converge or both diverge.
Alternating Series Test	$\sum_{n=1}^\infty (-1)^n b_n,\ b_n \geq 0$	If: • $\forall n, b_n > 0$ • $\{b_n\}$ is decreasing • $\lim_{n\to\infty} b_n = 0$, then $\sum_{n=1}^\infty (-1)^n b_n$ is convergent.
Absolute Convergence	Series with positive terms and some negative terms	If $\sum_{n=1}^\infty \lvert a_n\rvert$ converges, then $\sum_{n=1}^\infty a_n$ absolutely converges.
Ratio Test	Any series (especially exponents/factorials)	For $\lim_{n\to\infty} \lvert \frac{a_{n+1}}{a_n} \rvert = L$: • If $L < 1$, then $\sum_{n=1}^\infty a_n$ absolutely converges. • If $L > 1$, then $\sum_{n=1}^\infty a_n$ diverges. • If $L = 1$, then it is inconclusive.

16 Series

A series is an operation of extending a finite sum to an infinite sum/series. We can divide them to be n'th partial sums for the sequence a_n.
Generally we write these as:

$$S_\infty = \lim_{n\to\infty} S_n$$

16.0.1 Telescoping Series

Consider the sequence

$$\sum_{n=1}^{\infty} \frac{1}{n^2+2n}$$

We expand the terms:

$$\begin{aligned} S_1 &= \frac{1}{3} \\ S_2 &= \frac{1}{3}+\frac{1}{8} && = \frac{11}{24} \end{aligned}$$

This does not give us nice numbers, so instead we can use the relation of partial fractions.

$$\begin{aligned} \frac{1}{n^2+2n} &= \frac{1}{n(n+2)} \\ &= \frac{A}{n}+\frac{B}{n+2} \\ &= \frac{A(n+2)+Bn}{n(n+2)} \\ &= \frac{An+2A+Bn}{n(n+2)} \end{aligned}$$

This implies:

$$\begin{aligned} 1 = (A+B)n+2A &\implies 1 = 2A \\ &\implies A = \frac{1}{2} \\ O_n = (A+B)_n &\implies B = \frac{-1}{2} \end{aligned}$$

Now we consider the summation:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^2+2n} &= \sum_{n=1}^{\infty} (\frac{1}{2n} - \frac{1}{2(n+2)}) \\ &= (\frac{1}{2}-\frac{1}{6}) + (\frac{1}{4}-\frac{1}{8}) + (\frac{1}{6}-\frac{1}{10}) + (\frac{1}{8}-\frac{1}{12}) + (\frac{1}{10}-\frac{1}{14}) + ... \end{aligned}$$

Hence, we have:

$$S_1 = \frac{1}{3}, \quad S_2 = \frac{1}{8}, S_n = \frac{1}{2}+\frac{1}{4} + \text{small value}$$

Then, we take the limit:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2}+\frac{1}{4}+(...) = \frac{3}{4}$$

And thus, we have:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+2n} = \frac{3}{4}$$

16.1 Series and Convergence

Assume $\forall n \in \mathbb{N}, a_n > 0$. Consider the series $\sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} S_n$.
Let $\{S_n\}_{n=0}^{\infty}$ be its sequence of partial sums.
In the following cases, we have:

1. $\exists M \in \mathbb{R} s.t. \forall n \in \mathbb{N}, S_n \leq M$
   Then, S_n is monotonic and bounded, and so S_n is convergent.

2. $\exists M > 0 s.t. \forall n \in \mathbb{N}, a_n \geq M$.
   Then, S_n is divergent.

Additionally, let $\sum_{n=0}^{\infty} a_n$ be a series. Let $\{S_n\}_{n=0}^{\infty}$ be its partial-sum sequence.

1. If $\sum_{n=0}^{\infty} a_n$ is convergent, then $\{S_n\}_{n=0}^{\infty}$ is bounded.
2. If $\{S_n\}_{n=0}^{\infty}$ is bounded and eventually monotonic, then $\sum_{n=0}^{\infty} a_n$ is convergent.
3. If $\forall n > 0, a_n > 0$, then $\{S_n\}_{n=0}^{\infty}$ is increasing.
4. If $\{S_n\}_{n=0}^{\infty}$ is increasing, then $\forall n > 0, a_n > 0$ (except maybe for a_1).
5. If $\forall n > 0, a_n \geq 0$, then $\{S_n\}_{n=0}^{\infty}$ is non-decreasing.
6. If $\{S_n\}_{n=0}^{\infty}$ is non-decreasing, then $\forall n > 0, a_n \geq 0$, except maybe a_1.

16.1.1 Odd/Even Partial Sums

Let $\sum_{n=0}^{\infty} a_n$ be a series. Let $\{S_n\}_{n=0}^{\infty}$ be its partial-sum sequence. we have:

1. If $\lim_{n\to\infty} S_{2n}$ exists and $\lim_{n\to\infty} a_n = 0$, we have that $\forall \epsilon > 0 \exists N \in \mathbb{N}, k \geq N \implies |\delta_{2k} - L| < \epsilon$.
   Then, we have $(S_{2k+1}) = (S_{2k}) + (a_{2k+1})$, and by triangle inequality, we have:

   $$|S_{2L} - L + a_{2k+1}| \leq |S_{2k} - L| + |a_{2k+1}| < \epsilon + |a_{2k+1}|$$

   We can then find an a_{2k+1} arbitrarily small if a_n can go to 0.
   Let $|a_{2k+1}| < \frac{\epsilon}{2}$. In other words, we take a k big enough such that:

   $$|S_{2k+1} - L| = |S_{2k} + a_{2k+1} - L| \leq |S_{2k} - L| + |a_{2k+1}| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

   $$S_{2k+1} = S_{2k} + a_{2k+1}$$

   Therefore, $|S_{2k+1} - L| < \epsilon$ for a big enough k. Then $\sum_{n=1}^{\infty} a_n$ converges (as long as a_n goes to 0), since $\lim_{k\to\infty} S_k = L$.
   $\sum_{n=0}^{\infty} a_n$ is convergent.

16.2 Harmonic Series

For each $n > 0$ we define:

$$r_n = \text{smallest power of 2 that is greater than or equal to } n$$

We consider the series $S = \sum_{n=1}^{\infty} \frac{1}{r_n}$.
We compute r_1 to r_8:

$$
\begin{aligned}
r_1 &= 0 & &\rightarrow \text{exponent in } 2^0 \geq 1 \\
r_2 &= 1 & &\rightarrow \text{exponent in } 2^1 \geq 2 \\
r_3 &= 2 & &\rightarrow 2^2 \geq 3 \\
r_4 &= 2 & &\rightarrow 2^2 \geq 4 \\
r_5 &= 3 & &\rightarrow 2^3 \geq 5 \\
r_6 &= 3 \\
r_7 &= 3 \\
r_8 &= 3
\end{aligned}
$$

Then, we calculate the series, and compare $\sum_{n=1}^{\infty} \frac{1}{r_n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$.
We have $\sum_{i=1}^{n} \frac{1}{r_n}$ will increase too quickly, since we keep adding one to it.
So, $\lim_{n\to\infty} S_n = \infty$ since $|S_n - L| < \epsilon$ will not hold for any $L \in \mathbb{R}, 0 < \epsilon < 1$.

Thus, $\sum_{n=1}^{\infty} \frac{1}{n} = \lim_{n\to\infty}(S_j - S_k)$, where $S_j - S_k$ from the partial sums of $\sum \frac{1}{r_n}$.

16.3 Decimal Expansions

Any real number can be defined as a number with a decimal expansion.
Infinite decimal expansions can be written as a series:

$$0.a_1a_2a_3a_4a_5 = \frac{a_1}{10} + \frac{a_2}{100} + \frac{a_3}{1000} + ...$$

These series $\sum_{n=1}^{\infty} \frac{a_n}{10^n}$ are always convergent!

Proof. We know that $\frac{a_n}{10^n} \leq \frac{9}{10^n}$, so BCT lets us compare x to $\sum_{n=1}^{\infty} \frac{9}{10^n}$:

$$\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + ... = 9(\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + ...)$$

Then, $\sum_{n=1}^{\infty} \frac{9}{10^n} = 9\sum_{n=1}^{\infty} \frac{1}{10^n}$ which is geometric with $r = \frac{1}{10}$, so $|r| < 1$. Hence. $\sum_{n=1}^{\infty} \frac{a}{10^n}$ is convergent.
Furthermore, from BCT, $x = \sum_{n=1}^{\infty} \frac{a_n}{10^n}$ converges.

□

16.3.1 Calculating Decimal Expansions

Can we say that $0.99999... = 1$?
We expand the term $0.999... = 0.9 + 0.09 + 0.009 + ...$, and get:

$$
\begin{aligned}
\sum_{n=1}^{\infty} &= \frac{9(1/10)}{1 - 1/10} / \frac{9}{10} \\
&\frac{9}{10} \cdot 109 \\
&= 1
\end{aligned}
$$

Alternatively, we could say that since $S_1 = 0.9.S_2 = 0.99, S_3 = 0.999$, then

$$\lim_{n\to\infty} S_n = 0.999... = 1$$

16.3.2 Other Expansions

We have 0.252525.... = 0.25 + 0.0025 + 0.000025 + ...:

$$\begin{aligned}\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{10^2}+\frac{1}{4}\cdot\frac{1}{10^4}+... &= \sum_{n=1}^{\infty}\frac{1}{4}(\frac{1}{100})^n \\ &= \frac{1/4}{1-1/100} \\ &= \frac{1}{4}\cdot\frac{100}{99} \\ &= \frac{25}{99}\end{aligned}$$

We have 0.3121212... = 0.3 + 0.012 + 0.00012 + 0.0000012 + ...

$$\begin{aligned}\sum_{n=1}^{\infty}(\frac{12}{10})(\frac{1}{100})^n+0.3 &= 0.3+\frac{0.012}{[1-1/100]} \\ &= 0.3+(0.012\cdot\frac{100}{99})\end{aligned}$$

16.4 Absolute Values and Series

Suppose we had a series $\{a_n\}_{n=1}^{\infty}$, which is convergent. Then, $\{a_n\}_{n=1}^{\infty}$ also converges.
Consider $\sum_{n=1}^{\infty} a_n$ where $\sum_{n=1}^{\infty} |a_n|$ converges.
Let $b_n = \begin{cases} a_n, & a_n \geq 0 \\ 0, & a_n < 0 \end{cases}$, which implies that $b_n \leq |a_n|$.
Then, $\sum b_n$ converges by the Basic Comparison Test with $\sum |a_n|$.
Then, the summation of the positive terms is equal to the summation of b_n, meaning that the summation of the positive terms is convergent.
Finally, let $c_n = \begin{cases} |a_n|, & a_n \leq 0 \\ 0, & a_n > 0 \end{cases}$. Then, $\sum_{n=1}^{\infty} c_n$ is equal to the summation of all the negative terms. We have $0 \leq c_n \leq |a_n|$, and then by the BCT, the summation of the negative terms converges since $\sum c_n$ is convergent.

16.5 Functions as Series

You know that when $|x| < 1$:

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

So, we can write the following function as a series. For example:

1.

$$\begin{aligned}g(x) &= \frac{1}{1+x} \\ &= \frac{1}{1-(-x)} \\ &= f(-x) \\ &= \sum_{n=0}^{\infty}(-x)^n\end{aligned}$$

2.

$$\begin{aligned} h(x) &= \frac{1}{1-x^2} \\ &= \frac{1}{1-(x^2)} \\ &= f(x^2) \\ &= \sum_{n=0}^{\infty}(x^2)^n \end{aligned}$$

3.

$$\begin{aligned} A(x) &= \frac{1}{2-x} \\ &= \frac{1}{2(1)-\frac{2x}{2}} \\ &= \frac{1}{2}(\frac{1}{1-\frac{x}{2}} \\ &= \frac{1}{2}f(\frac{x}{2}) \\ &= \frac{1}{2}\sum_{n=0}^{\infty}(\frac{x}{2})^n \end{aligned}$$

16.6 Tails of Series

1. If a series $\sum_{n=0}^{\infty} a_n$ converges, then the series $\sum_{n=7}^{\infty} a_n$ converges.
   (Take $\sum_{n=0}^{\infty} - \sum_{n=0}^{6}$).

2. If $\sum_{n=7}^{\infty} a_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges (as long as the terms exist.

16.7 Necessary Conditions of Series

1. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n}^{\infty} a_n$ is divergent.

2. If $\sum_{n}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

3. If $\sum_{n=0}^{\infty} a_n$ is convergent, then $\lim_{k\to\infty} [\sum_{n=k}^{\infty} a_n] = 0$.

4. If $\sum_{n=1}^{\infty} a_{2_n}$ and $\sum_{n=1}^{\infty} a_{2_{n+1}}$ are convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

16.8 Series are Linear

Let $\sum_{n=0}^{\infty} a_n$ be a series. Let $c \in \mathbb{R}$.
If $\sum_{n=0}^{\infty} a_n$ is convergent, then $\sum_{n=0}^{\infty}(ca_n)$ is convergent and $\sum_{n=0}^{L}(ca_n) = c\left[\sum_{n=0}^{L} a_n\right]$.

Proof. We have $S_k = \sum_{n=0}^{k} ca_n = c(\sum_{n=0}^{k} a_n)$. Then, $cL = \lim_{L\to\infty}\sum_{n=0}^{n=k} ca_n = \lim_{k\to\infty} S_k$.
We know $\lim_{k\to\infty}\sum_{n=0}^{k} a_n = L < \infty$ by $\sum a_n$ converging.
Then, $c(L) = c\lim_{k\to\infty}\sum_{n=0}^{k} a_n = \lim_{k\to\infty} c\sum_{k\to\infty}^{\infty} a_n$ by the property of limits.

Now, to show that $\sum ca_n$ converges, we need to show that the partial sum T_n has a limit.
Let $S_k = \sum_{k=1}^{k} a_n$. We know $\lim_{k\to\infty} S_k < \infty$ exists and is equal to L (since $\sum a_n$ convergent).

$$\begin{aligned}
\sum_{n=1}^{k} ca_n &= T_k \\
&= c(a_1 + a_2 + ... + a_k) \\
&= cS_k \\
\lim_{k\to\infty} T_k &= \lim_{k\to\infty} cS_k \\
&= c \lim_{k\to\infty} S_k \\
&= ck \\
&= c\sum_{n=0}^{\infty} a_n
\end{aligned}$$

$\square$

16.9 Intervals of Convergence

For each power series, determine the interval(s) of convergence:

1. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$
   We use the ratio test:

$$\begin{aligned}
\lim_{n\to\infty} |\frac{(x^{n+1})/(n+1)!)}{x^n/n!}| &= \lim_{n\to\infty} |\frac{x^{n+1}n!}{(n+1)!x^n}| \\
&= \lim_{n\to\infty} |\frac{x\cdot x^n n!}{(n+1)n!(x^n)}| \\
&= \lim_{n\to\infty} \frac{|x|}{n+1} \\
&= |x| \lim_{n\to\infty} \frac{1}{n+1} = 0
\end{aligned}$$

   So then, $L = 0$ for all real values of x. Hence, the series converges for all $x \in \mathbb{R}$.
   Then, the radius of convergence is ∞.

2. $\sum_{n=1}^{\infty} \frac{n^n}{42^n}(x^n) = \frac{1}{42}x + \frac{4}{42^2}x^2 + \frac{9}{42^3}x^3 + ...$
   We use the ratio test:

$$\begin{aligned}
\lim_{n\to\infty} |\frac{(n+1)^{n+1}x^{n+1}}{42^{n+1}} / \frac{n^n x^n}{42^n}| &= \lim_{n\to\infty} |\frac{(n+1)(n+1)^n x \cdot x^n(42^n)}{42(42^n)n^n x^a}| \\
&= \lim_{n\to\infty} (\frac{n+1}{42})(\frac{n+1}{n})^n |x| \\
&= |x| \cdot \text{something going to infinity} \\
&= \infty \text{for all } x \neq 0
\end{aligned}$$

   Then, when $x = 0, \sum_{i=1}^{\infty} 0$ converges.
   Thus, the center of interval of convergence is $x = 0$. Radius of convergence is $R = 0$.

3. $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2(2^{2n+1}}$

Use the ratio test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{(n+1)^2(2^2n+1)+1}\Big/\frac{(x-5)^n}{n^2(2^n+1)}\right| &= \lim_{n\to\infty}\left|\frac{(x-5)(x-5)^n n^2 2^{2n+1}}{(n+1)(n+1)2^{2n+3}(x-5)^n}\right| \\ &= \lim_{n\to\infty}\frac{|x-5|}{4}\frac{n^2}{n^2+2n+1} \\ &= |x-5|\frac{1}{4}\end{aligned}$$

So, this series converges when $\frac{|x-5|}{4} < 1$:

$$\begin{aligned}|x-5| &< 4 \\ -4 < x-5 &< 4 \\ -4+5 < x &< 4+5\end{aligned}$$

So, the interval of convergence is $[1,9]$ with a centre 5 and radius of convergence $= 4$.

16.10 Comparing Series

Consider the power series

$$\sum_n^\infty a_n x^n$$

$\sum_n^\infty a_n 3^n$	Absolutely Convergent	Conditionally Convergent	Divergent
$\sum_n^\infty a_n 2^n$	AC, D	AC, CC	Cannot be determined
$\sum_n^\infty a_n (-3)^n$	AC, D	CC, D	CC, D
$\sum_n^\infty a_n 4^n$	AC, CC, D	AC, CC, D	Cannot be determined

16.11 Computing Different Series

1. Compute $A = \sum_{n=1}^\infty \frac{n}{3^n}$.
   (a) Evaluate the sum of $\sum_{n=0}^\infty x^n$, which we know $\sum_{n=0}^\infty x^n = \frac{1}{1-x}$.
   (b) Use derivatives to relate $\sum_n^\infty x^n$ and $\sum_n^\infty nx^{n-1}$.
   $\frac{d}{dx}x^n = nx^{n-1}$.
   (c) Compute $\sum_{n=1}^\infty nx^{n-1}$ and $\sum_{n=1}^\infty nx^n$.
   (d) Compute A.

$$\begin{aligned}
A = \sum_{n=1}^{\infty} \frac{n}{3^n} &= \sum_{n=1}^{\infty} n(\frac{1}{3})^{n-1} \cdot (\frac{1}{3}) \\
&= \frac{1}{3} \sum_{n=1}^{\infty} n(\frac{1}{3})^{n-1} \\
&= \frac{1}{3} \left[\frac{d}{dx} (\sum_{n=1}^{\infty} (x)^n) \right]_{x=1/3} \\
&= \frac{1}{3} \left[\frac{d}{dx} \left(\frac{1}{1-x} \right) \right]_{x=1/3} \\
&= \frac{1}{3} \left[\left(\frac{1}{(1-x)^2} \right) \right]_{x=1/3} \\
&= \frac{1}{3} \left(\frac{1}{(1-1/3)^2} \right) \\
&= \frac{1}{3(4/9)} \\
&= \frac{1}{4/3} \\
&= \frac{3}{4}
\end{aligned}$$

2. Compute $B = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$.

   (a) Write $F(x) = \arctan(x)$ as a power series.
   The derivative of $\arctan(x)$ is $\frac{1}{1+x^2}$, which written as a power series, is:

$$1 - x^2 + x^4 - x^6 + x^8 - ...$$

   So, we have:

$$\begin{aligned}
\int \frac{1}{1+x^2} dx &= \int 1 dx - \int x^2 dx + \int x^4 dx - ... \\
\sqrt{3} arctan(x) &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... \\
&= \sqrt{3} \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)} \right)
\end{aligned}$$

   Then, let $x = \frac{1}{\sqrt{3}}$. Then:

$$\begin{aligned}
\sqrt{3} \arctan(\frac{1}{\sqrt{3}}) &= \sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\sqrt{3}^{2n+1}} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(\sqrt{3}^n)^n} \frac{\sqrt{3}}{\sqrt{3}}
\end{aligned}$$

   (b) Calculate B.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n} &= \sqrt{3} \arctan(\frac{1}{\sqrt{3}} \\
&= \frac{\sqrt{3}\pi}{2}
\end{aligned}$$

   Then, $\arctan(\frac{1}{\sqrt{3}}) = \frac{\pi}{2}$.

17 Taylor Polynomials

The Taylor Series is an infinite polynomial, or a series that can be evaluated at a value $x = a$ that gives the value $f(a)$. It is written as:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

The n-th Taylor polynomial for function f at point a is the approximation of f for that order near a. Additionally, we can also say $\lim_{x \to a} \frac{f(x)-P_n(x)}{(x-a)^n} = 0$, since P_n exists if f is differentiable. This then implies it is continuous.

Additionally, for calculating the tangent line, we have:

$$\begin{aligned} f'(a) \approx \frac{f(x)-f(a)}{x-a} &\implies f'(a)(x-a) \approx f(x) - f(a) \\ &\implies f(x) \approx f(a) + f'(a)(x-a) \end{aligned}$$

Where here, $f(a) = \frac{f^{(0)(a)}}{0!}x^0$, and $f'(a)$ is for $k = 1$.

The **Maclaurin Series** is the same formula as the Taylor series, however, we always take $a = 0$ in the formulas.

17.1 Evaluating Taylor Polynomials

1. Determine the Taylor Polynomial of degree 3 for $f(x) = \sin(x)$ at $a = 0$.

   Let $T_3 = a + bx + cx^2 + dx^3$.
   Take the derivative of $\sin(x)$ of different degrees to determine the polynomials (i.e., $T_3' = b+2cx+3dx^2$).

$$\begin{aligned} f(0) = T_3(0) &\implies a = 0 \\ f'(0) = T_3'(0) &\implies \cos(x) = b + 2cx + 3dx^2 \text{ at } x = 0 \\ &\implies 1 = b + 2c(0) + 3d(0)^2 \\ &\implies b = 1 \\ f''(0) = T_3''(0) &\implies -\sin(x) = 2x + 6dx \text{ at } x = 0 \\ &\implies 0 = 2x \\ &\implies c = 0 f'''(x) = T_3'''(0) && \implies -\cos(x) = 6d \\ &\implies -1 = 6d \\ &\implies d = \frac{-1}{6} = \frac{-1}{3!} \end{aligned}$$

   Then, we can calculate $T_3(x)$ as $1x - \frac{1}{3!}x^3$.

   Alternatively, we use the definition of the Taylor Polynomial:

$$\begin{aligned} \sum_{k=0}^{3} \frac{f^{(k)}(0)}{k!}(x-0)^k &= \frac{\sin(0)}{0!}x^0 + \frac{\cos(0)}{1!}x^1 + \frac{-\sin(0)}{2!}x^2 + \frac{-\cos(0)}{3!}x^3 \\ &= 0 + x + 0 - \frac{1}{3!}x^3 \end{aligned}$$

2. Determine the first derivative for $c(x) = \frac{-293}{8} + 29x + \frac{13}{4}x^2 - 3x^3 + \frac{3}{8}x^4$ at $x = 3$.

$$
\begin{aligned}
c'(x) &= 29 + \frac{13}{2}x - 9x^2 + \frac{3}{2}x^3 \\
c'(3) &= 29 + \frac{39}{2} - 9(3)^2 + \frac{3}{2}(3)^4 \\
&= 8
\end{aligned}
$$

3. Determine the first derivative for $d(x) = 29 + 8(x-3) - \frac{7}{2}(x-3)^2 + \frac{9}{6}(x-3)^3 + \frac{9}{24}(x-3)^4$.

$$
\begin{aligned}
d'(x) &= 8 - 7(x-3) + \frac{9}{2}(x-3)^2 + \frac{3}{2}(x-3)^3 \\
d'(3) &= 8
\end{aligned}
$$

   Surprisingly, (2) and (3) are the same polynomial and equate to the same value, but (2) has been algebraically modified using Taylor Polynomials!

17.1.1 Using Polynomials to find Derivatives

Consider the polynomial $P(x) = c_0 + c_1x + c_2x^2 + c_3x^3$. Find values of the coefficients that satisfy:

1. $P(0) = 1$
   $\implies c_0 = 1$
2. $P'(0) = 5$
   $\implies c_1 = 5$
3. $P''(0) = 3$
   $\implies 2c_2 = 3$
4. $P'''(0) = -7$
   $\implies 3(2)c_3 = -7$

Then, we can compute the Taylor polynomial as:

$$
P = 1 + 5x + \frac{3}{2}x^2 - \frac{7}{6}x^3
$$

17.1.2 Using Maclaurin Series

1. Determine the Maclaurin Series of $\cos(x)$: Using the derivatives, we have:

$$
\begin{aligned}
f(x) &= \cos(x) f(0) = 1 \\
f'(x) &= -\sin(x) f'(0) = 0 \\
f''(x) &= -\cos(x) f''(0) = -1 \\
f'''(x) &= \sin(x) f'''(0) = 0 \\
f^{(4)}(x) &= \cos(x) f^{(4)}(0) = 1
\end{aligned}
$$

   So, we know the series is periodic. Now, we write the series:

$$
\begin{aligned}
\cos(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x-0)^k \\
&= \frac{1}{0!}x^0 + \frac{0}{1!}x^1 + \frac{(-1)}{2!}x^2 \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + .. \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - ... ,
\end{aligned}
$$

which is true for at least $x = 0$.
Then:

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

2. Using (1), calculate the Maclaurin Series for $\sin(x)$.
Take the first derivative of $\cos(x)$:

$$\begin{aligned} -\sin(x) &= \frac{-2}{2}x + \frac{4}{4!}x^3 - \frac{6}{6!}x^5 + \frac{8}{8!}x^7 - ... \\ &= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - ... \\ \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \end{aligned}$$

3. Using (1), determine the Maclaurin Series for e^x.

$$\begin{aligned} f(x) &= e^x f(0) = 1 \\ f'(x) &= e^x f'(0) = 1 \end{aligned}$$

Then, we calculate:

$$\begin{aligned} e^x &= \frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ... \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \end{aligned}$$

17.2 Lagranges Remainder Theorem

Suppose we have $e^x \approx 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$
We know this is just an approximation (since it goes up to finitely many terms. We can add a remainder R_k to make it =, and get:

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + R_4$$

More generally, for any function f:

$$f(x) = T_n(x) + R_n(x),$$

where T_n is the Taylor polynomial of degree n, and R_n is the remainder when you subtract $f(x) - T_n(x)$.

Then, there exists a ζ in between a (where a is the point of tangency) and x such that $R_n(x) = \frac{f^{(n+1)}(\zeta)}{(n+1)!}(x - a)^{n+1}$.

So, if we know that $f^{(n+1)}(x) \leq M$ (the upper bound), then $R_n \leq \frac{M}{(n+1)!}(x-a)^{n+1}$.
As $n \to \infty$, $\frac{M}{(n+1)!}(x-a)^{n+1}$ goes to 0 (since M stays the same and the denominator gets bigger).
Assuming M is the same upper bound for all $f^{(k)}$, $\forall k$.
This means that the more terms we have of the Taylor Polynomial, then the smaller the remainder R_n is.

17.3 Boundedness and Taylor Series

Theorem. Let I be an open interval. Let $a \in I$. Let f be a C^∞ function on I. Let $S(x)$ be the Taylor series for f centred at a.
If $\exists A, B, \forall x \in I, \forall n \in \mathbb{N}, \forall \zeta \in J_{x,a}, A \leq f^{(n)}(\zeta) \leq B$,
then $\forall x \in I, f(x) = S(x)$.

17.4 Calculating Integrals Using Taylor Series

If an integral is too difficult to calculate, but we know the series of part of the integral, rather than doing integration by parts, we can substitute it with its series. TODO: w12p1

17.5

Take the function $e^{i\pi}$.
We have $e \approx 2.71...$ and $\pi \approx 3.14....$
Let $i = \sqrt{-1}$, which is an imaginary number. Then:

$$\begin{aligned}
i^2 &= (\sqrt{-1})^2 && = -1 \\
i^3 &= i(i^2) && = -i \\
i^4 &= (-1)(-1) && = 1 \\
i^5 &= 1(i) && = -i
\end{aligned}$$

Then, we have:

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... \\
e^{i\theta} &= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{1^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + ... \\
&= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + ... \\
&= (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ...) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ...)
\end{aligned}$$

We know the two series to be $\sin(x)$ and $\cos(x)$, so we can write:

$$\begin{aligned}
e^{i\theta} &= \cos\theta + i\sin\theta \\
&= \cos\pi + i(\sin\pi) \\
&= -1
\end{aligned}$$