STA237 - Activity 6

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November 2, 2022

1. (a) We want to find the marginal density functions for Y_1, Y_2.

   The marginal density function for Y_1:

$$
\begin{aligned}
f(y_1) &= \int_0^1 4y_1 \; y_2 \; dy_2 \\
&= 2y_1 \int_0^1 2y_2 \; dy_2 \\
&= 2y_1 \left[y_2^2\right]_0^1 \\
&= 2y_1
\end{aligned}
$$

   Hence:

$$
f(y_1) = \begin{cases} 2y_1 & 0 \le y_1 \le 1 \\ 0 & \text{otherwise} \end{cases}
$$

   We can do this similarly for the marginal density of Y_2:

$$
f(y_2) = \begin{cases} 2y_2 & 0 \le y_2 \le 1 \\ 0 & \text{otherwise} \end{cases}
$$

   (b)

$$
P(Y_1 \le \frac{1}{2} \mid Y_2 \ge \frac{3}{4}) = \frac{P(Y_1 \le \frac{1}{2}, Y_2 \ge \frac{3}{4})}{P(Y_2 \ge \frac{3}{4})}
$$

   First, solve the numerator:
   We consider the range of $P(Y_1 \le \frac{1}{2}, Y_2 \ge \frac{3}{4})$, and get our bounds $y_1 \in [0, \frac{1}{2}]$, and $y_2 \in [\frac{3}{4}, 1]$.

$$
\begin{aligned}
\int_0^{1/2} \left[\int_{3/4}^1 4y_1 y_2 \; dy_2 \right] dy_1 &= \int_0^{1/2} 2y_1 \left[\int_{3/4}^1 2y_2 \; dy_2 \right] dy_1 \\
&= \int_0^{1/2} 2y_1 \left[y_2^2\right]_{3/4}^1 dy_1 \\
&= \int_0^{1/2} 2y_1 \left(1 - \frac{9}{16}\right) dy_1 \\
&= \frac{7}{16} \int_0^{1/2} 2y_1 \; dy_1 \\
&= \frac{7}{16} \left[y_1^2\right]_0^{1/2} \\
&= \frac{7}{16} \cdot \frac{1}{4} \\
&= \frac{7}{64}
\end{aligned}
$$

Next, we can consider the denominator $P(Y_2 \geq \frac{3}{4})$. We have:

$$
\begin{aligned}
1 - P(Y_2 < \frac{3}{4}) &= 1 - \int_0^{3/4} 2y_2 \; dy_2 \\
&= 1 - \left[y_2^2\right]_0^{3/4} \\
&= 1 - \frac{9}{16} \\
&= \frac{7}{16}
\end{aligned}
$$

Therefore, we compute:

$$
\begin{aligned}
P(Y_1 \leq \frac{1}{2} \mid Y_2 \geq \frac{3}{4}) &= \frac{\frac{7}{64}}{\frac{7}{16}} \\
&= \frac{1}{4}
\end{aligned}
$$

(c) We want to find the conditional density function of Y_1:

$$
\begin{aligned}
f(y_1|y_2) &= \frac{f(y_1, y_2)}{f(y_2)} \\
&= \frac{4y_1 \; y_2}{2y_2} \\
&= 2y_1
\end{aligned}
$$

Hence,

$$
f(y_1|y_2) = \begin{cases} 2y_1 & 0 \leq y_1 \leq 1 \\ 0 & \text{otherwise} \end{cases}
$$

(d) We want to find the conditional density function of Y_2:

$$
\begin{aligned}
f(y_2|y_1) &= \frac{f(y_1, y_2)}{f(y_1)} \\
&= 2y_2
\end{aligned}
$$

Therefore:

$$
f(y_2|y_1) = \begin{cases} 2y_2 & 0 \leq y_2 \leq 1 \\ 0 & \text{otherwise} \end{cases}
$$

(e) We want to find $P(Y_1 \leq \frac{3}{4} \mid Y_2 = \frac{1}{2})$. Use the conditional density function:

$$
\begin{aligned}
f(y_1|Y_2 = \frac{1}{2}) &= 2y_1 \text{ if } 0 \leq y_1 \leq 1 \\
P(Y_1 \leq \frac{3}{4} \mid Y_2 = \frac{1}{2}) &= \int_0^{3/4} [2y_1 \; dy_1] \\
&= \left[y_1^2\right]_0^{3/4} \\
&= \frac{9}{16}
\end{aligned}
$$