STA237 Exercises

Contents

1 Exercises

1.1 Probability

1. Suppose $P(A) = 0.5, P(A \cap B) = 0.2$, and $P(A \cup B) = 0.7$. Find:

   (a) $P(B)$

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ 0.7 &= 0.5 + P(B) - 0.2 \\ 0.7 &= 0.3 + P(B) \\ P(B) &= 0.4 \end{aligned}$$

   (b) $P(\text{exactly one of two events occurs})$
   This means the probability of no elements from the intersection. Let this area be $P(O)$.

$$\begin{aligned} P(A \cup B) &= P(O) + P(A \cap B) \\ 0.7 &= P(O) + 0.2 \\ P(O) &= 0.5 \end{aligned}$$

   (c) $P(\text{neither event occurs})$
   This means the probability of no elements in A, B. Call this probability $P(X)$.

$$\begin{aligned} P(X) &= P(\Omega) - P(A \cup B) \\ &= 1 - 0.7 \\ &= 0.3 \end{aligned}$$

2. Suppose we throw a coin two times. The sample space for this experiment is:

$$\Omega = \{H, T\} \times \{H, T\} = \{(H, H), (H, T), (T, H), (T, T)\}$$

   If it is a fair coin, then all four outcomes have equal possibilities:

$$P((H, H)) = P((H, T)) = P((T, H)) = P((T, T)) = \frac{1}{4}$$

   What is the probability of getting at least one tail?
   Looking at the sample space, the probability is $\frac{3}{4}$.
   Alternatively, we can solve as follows, where RP represents the required probability:

$$\begin{aligned} P(\Omega) &= RP + P(HH) \\ 1 &= RP + P(HH) \\ &= RP + P(H)P(H) && \text{Since it is independent.} \\ &= RP + \frac{1}{2} \cdot \frac{1}{2} \\ 1 &= RP + \frac{1}{4} \\ RP &= \frac{3}{4} \end{aligned}$$

   Suppose we wanted to throw three coins, what would happen to the sample space?

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Then, there are $2 \cdot 2 \cdot 2 = 2^3$ possibilities (to keep on with this pattern, flipping a coin 6 times would have 2^6 possibilities, and so on).
What is the probability of at least one head appearing in a 3-coin toss?

$$\begin{aligned} P(X) &= 1 - P(TTT) \\ &= 1 - P(T)P(T)P(T) \\ &= 1 - \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \\ &= 1 - \frac{1}{8} \\ &= \frac{7}{8} \end{aligned}$$

3. A survey of students in a class was asked about their birth order (1 = oldest/only child), and which college of the university they were enrolled in.

Birth Order	1 or Only	2 or More	Total
Arts & Science	34	23	57
Agriculture	52	41	93
Human Ecology	15	28	43
Other	12	17	30
Total	113	110	223

Suppose we selected a random student from this class.

(a) What is the probability we select a Human Ecology student?
P(Human Ecology) = $\frac{43}{223}$

(b) What is the probability we select a first-born student?
P(First Born) = $\frac{113}{223}$

(c) What is the probability that the student is a first-born and a Human Ecology student?
P(C) = $P(A \cap B) = \frac{15}{223}$

(d) What is the probability that the student is a first born or a Human Ecology student?

$$\begin{aligned} P(D) &= P(A \cup B) \\ &= P(A) + P(B) - P(\cap B) \\ &= \frac{43}{223} + \frac{113}{223} - \frac{15}{223} \\ &= \frac{141}{223} \end{aligned}$$

4. A sample space consists of 5 simple events, E_1, E_2, E_3, E_4, E_5.

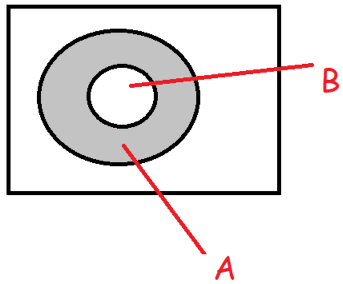
(a) If $P(E_1) = P(E_2) = 0.15$, $P(E_3) = 0.4$, and $P(E_4) = 2P(E_5)$, find the probabilities of E_4 and E_5.

$$\begin{aligned} 1 &= 0.15 + 0.15 + 0.4 + 2P(E_5) + 2P(E_5) \\ 0.3 &= 3P(E_5) \\ P(E_5) &= 0.1 \end{aligned}$$

Then:

$$\begin{aligned} P(E_4) &= 2(0.1) \\ &= 0.2 \end{aligned}$$

5. If A, B are events, $B \subset A$, $P(A) = 0.6$ and $P(B) = 0.2$, then find $P(A \cap B^c)$.
   To visualize:



$$P(A) = P(A \cap B^c) + P(B)0.6 \qquad = P(A \cap B^c) + 0.2$$
$$P(AcapB^c) = 0.4$$

6. An experiment involves tossing a pair of dice and observing the numbers on the upper faces. Find the number of sample points in Ω and the sample space for the experiment The sample points are $\Omega_{\{}m, n\} = \{1, 2, 3, 4, 5, 6\}$.
   The sample space is taking the cross product of the sample points (i.e., $\Omega_1 \times \Omega_2$). Then:

$$\Omega\{(1,1),(1,2),...,(6,6)\}$$

   There is a $m \cdot n = 6 \cdot 6 = 36$ chance for a single probability.

7. The names of three employees are randomly drawn without replacement from a bowl containing the names of 30 employees of a small company. The person whose names is drawn first receives \$100, and the individuals whose names are drawn second and third receive \$50 and \$25, respectively. How many samples are associated with this experiment? There are 30 employees and 3 people are picked. So, $r = 3, n = 30$. Hence:

$$\begin{aligned}\Omega &= P_3^{30} \\ &= \frac{30!}{(30-3)!} \\ &= \frac{30!}{27!}\end{aligned}$$

8. Find the number of ways of selecting two applicants out of five.
   Here, $n = 5, r = 2$.

$$\begin{aligned}\binom{5}{2} &= \frac{5!}{2!(5-2)!} \\ &= \frac{5!}{2!3!} \\ &= \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} \\ &= 10\end{aligned}$$

1.2 Conditional Probability

1. Let $N = R^c$ be the event 'born in a month without r', and L is the event 'born in a long month'. What is the conditional probability $P(N|L)$?
   $P(R^c|L) = \frac{3}{7}$
   Note that $P(R|L) + P(R^c|L) = \frac{4}{7} + \frac{3}{7} = 1$.

2. A survey asked: 'Are you currently in a relationship?', and 'Are you involved in club sports?'. The survey found that 33% were in a relationship, and 25% were involved in sport. 11% said yes to both. Suppose you meet a student who is on a sports team. What is the probability they are also in a relationship?
   Let A be students in a relationship, and B be students in a club sport.
   $P(A) = 0.33$, $P(B) = 0.25$, $P(A \cap B) = 0.11$. We have:

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{0.11}{0.25} \\ &= 0.44 \end{aligned}$$

   We know that $P(A|B) \neq P(A)$, A and B must be dependent events.

3. A bowl contains 6 red and 4 blue balls, A child selects two balls at random. What is the probability at least one of them is red?
   We have:
   (a) $\{R, R\} = \frac{6}{10}, \frac{5}{9}$
   (b) $\{R, B\} = \frac{6}{10}, \frac{4}{9}$
   (c) $\{B, R\} = \frac{4}{10}, \frac{6}{9}$
   (d) $\{B, B\} = \frac{4}{10}, \frac{3}{9}$

   Note that $P(\text{at least one red}) + P(\text{no red}) = 1$, so:

$$\begin{aligned} P(one\_R) &= 1 - P(no\_R) \\ &= 1 - P(BB) \\ &= 1 - P(B_1 \cap B_2) \\ &= 1 - P(B_2|B_1)P(B_1) \\ &= 1 - \frac{3}{9} \cdot \frac{4}{10} \\ &= 1 - \frac{2}{15} \\ &= \frac{13}{15} \end{aligned}$$

4. A recent highway safety study found that in 77% of all accidents, the driver was wearing a seatbelt. Accident reports indicated that 92% of those drivers did not experience serious injury (defined as hospitalization or death), but only 63% of the non-belted were okay.

   (a) What is the probability that a driver was seriously injured?
   (b) What is the probability that a driver who was seriously injured was not wearing a seatbelt?

   Let B is those that are wearing a seat bet, and NB is for those who did not wear a seat belt. Then, let I represent those who experienced a serious injury.
   We have $P(B) = 0.77$, $P(NB) = 1 - 0.77 = 0.23$, and $P(I^c|B) = 0.92$, $P(I^c|NB) = 0.63$.
   Then, $P(I|B) = 0.08$, $P(I|NB) = 0.37$.
   [a)]Want to find $P(I)$:

$$\begin{aligned} P(I) &= P(B \cap I) + P(NB \cap I) \\ &= P(I|B)P(B) + P(I|NB)P(NB) \\ &= 0.08 \cdot 0.77 + 0.37 \cdot 0.23 \\ &= 0.0616 + 0.0851 \\ &= 0.1467 \end{aligned}$$

Want to find $P(NB|I)$.

$$\begin{aligned}
P(I|NB) &= \frac{P(NB \cap I)}{P(I)} \\
&= \frac{P(I|NB)P(I)}{0.1467} \\
&= \frac{0.23 \cdot 0.37}{0.1467} \\
&= \frac{0.0851}{0.1467} \\
&= 0.58
\end{aligned}$$

(5) A student figures he has a 30% chance of being let out of class late. If he leaves class late, there is a 45% chance he will miss the bus. What is the probability that he is let out of class late and misses the bus?
Let L be the chance he leaves class late. Let M be the chance he misses the bus.
$P(L) = 0.3$, $P(M|L) = 0.45$. We want to find $P(L \cap M)$:

$$\begin{aligned}
P(L \cap M) &= P(M|L)P(L) \\
&= 0.45 \cdot 0.3 \\
&= 0.135
\end{aligned}$$

6. Leah flies from Moncton to Vancouver with a connection in Montreal. The probability her flight leaves on time is 15%. If the flight is on time, the chance her luggage makes the connecting flight in Montreal is 95%. But, if the flight is delayed, the probability the luggage will make it is only 65%.

   (a) Are the first flight leaving on time and the luggage making the connection independent events? Explain.

   (b) What is the probability that Leah's luggage arrives in Vancouver with her?

   Let F be the chance her flight leaves on time. Let L be the luggage being in the connecting flight.
   $P(F) = 0.15$, $P(L|F) = 0.95$, $P(L|F^C) = 0.65$.

   (a) We have $P(L|F) = 0.95$ and $P(L) = 0.695$, so they are not equal. Therefore, they are dependent events.

   (b)

$$\begin{aligned}
P(L) &= P(L|F)P(F) + P(L|F^c)P(F^c) \\
&= 0.95 \cdot 0.15 + 0.65 \cdot 0.85 \\
&= 0.1425 + 0.5525 \\
&= 0.695
\end{aligned}$$

7. The Ontario Lottery association claims your odds of winning a price on an Instant Win Crossword game are 1 in 3.09. This means that any ticket has 32.4% chance of winning a prize. Every Friday, you buy one Crossword game.

   (a) What is the probability you don't win a prize next Friday?

   (b) What is the probability you don't win a prize 6 Fridays in a row?

   (c) If you haven't won a prize for the past 6 Fridays, what is the probability you win a prize on your next game?

   (d) What is the probability you win on two of your next three games?

   Let W be the chance you win a prize. $P(W) = 0.324$.

(a) Want to find $P(W^c)$. $P(W^c) = 1 - 0.324 = 0.676$

(b) Want to find $P(W_1^c...W_6^c)$:

$$\begin{aligned} P(W_1^c...W_6^c) &= P(W_1^c) \cdot ... \cdot P(W_2^c) \\ &= P(W^c)^6 \\ &= 0.676^6 \\ &= 0.0954 \end{aligned}$$

(c) Want to find $P((W_1^c...W_6^c)^c)$. So: $P((W_1^c...W_6^c)^c) = 1 - 0.0954 = 0.9046$.

(d) Want to find $P(W_1W_2W_3^c)$:

$$\begin{aligned} P(W_1W_2W_3^c) &= 0.324 \cdot 0.3424 \cdot 0.676 \\ &= 0.071 \end{aligned}$$

1.3 Random Variables

1. Suppose we roll two dice. List the outcomes:

   (a) $\{X = 8\} = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$.
   $P(X = 8) = \frac{5}{36}$

   (b) $\{X = 3\} = \{(1,2),(2,1)\}$.
   $P(X = 3) = \frac{2}{36} = \frac{1}{18}$

   (c) $\{X = 13\} = \{\}$
   $P(X = 13) = 0$

2. Suppose the death rate in a year is 1 out of 1000 people, and that 2 out of 1000 people suffer some kind of disability. Then, we display the probability model for X = company payout in a table:

Policyholder Outcome	Payout x	P(X = x)
Death	10000	1/1000
Disability	5000	2/1000
Neither	0	998/1000

   What is $P(X = 0)$?
   We have $X = 0, 5000, 10000$, and $P(X = 10000) = \frac{1}{1000}$, $P(X = 5000) = \frac{2}{1000}$. We calculate:

$$\begin{aligned} P(X = 0) &= 1 - \frac{1}{1000} - \frac{2}{1000} \\ &= \frac{997}{1000} \end{aligned}$$

3. Take the previous example, and determine $E(X)$, where X is the company payout.
   Recall: $X = 0, 5000, 10000$, and $P(X = 10000) = \frac{1}{1000}$, $P(X = 5000) = \frac{2}{1000}$, and $P(X = 0) = \frac{997}{1000}$.
   To calculate $E(X)$:

$$\begin{aligned} E(X) &= \left(10000 \cdot \frac{1}{1000}\right) + \left(5000 \cdot \frac{2}{1000}\right) + \left(0 \cdot \frac{997}{1000}\right) \\ &= 10 + 10 \\ &= 20 \end{aligned}$$

   Next, suppose we wanted to calculate the variance of X.
   We have:

(a) $10000 - 20 = 9980$

(b) $5000 - 20 = 4980$

(c) $0 - 20 = -20$

So, we have:

$$\begin{aligned}\sigma^2 &= \sum_x (x-\mu)^2 P(x) \\ &= \left(9980^2 \cdot \frac{1}{1000}\right) + \left(4980^2 \cdot \frac{2}{1000}\right) + \left((-20)^2 \cdot \frac{997}{1000}\right) \\ &= 149600\end{aligned}$$

Then:

$$\begin{aligned}\sigma = SD(X) &= \sqrt{V(X)} \\ &= \sqrt{149600} \\ &= \$386.78\end{aligned}$$

4. If $E(X) = 10$ and $V(X) = 3$, then:

(a) Calculate $E(9x + 11)$.

$$\begin{aligned}E[9x+11] &= 9E(x) + 11 \\ &= (9 \cdot 10) + 11 \\ &= 90 + 11 = 101\end{aligned}$$

(b) Calculate $V(9x + 11)$.

$$\begin{aligned}V(9x+11) &= 9^2 V(x) \\ &= 9^2 \cdot 3 \\ &= 243\end{aligned}$$

(c)

$$\begin{aligned}\sigma^2 &= E(X-\mu)^2 \\ &= E(x^2 - 2\mu x + \mu^2) \\ &= E(x^2) - 2\mu E(X) + \mu^2 \\ &= E(x^2) - 2\mu^2 + \mu^2 \\ &= E(x^2) - \mu^2\end{aligned}$$

(d) If $E(X) = 10$ and $E(X^2) = 160$:

$$\sigma^2 = 160 - 10^2 = 60$$

5. The probability distribution for a random variable Y is given by the following table. Find the mean, variance, and standard deviation of Y.

y	p(y)
0	1/8
1	1/4
2	3/8
3	1/4

Then:

$$\begin{aligned} F(0) &= P(Y \leq 0) \\ &= P(Y = 0) = \frac{1}{8} \end{aligned}$$

$$\begin{aligned} F(1) &= P(Y \leq 1) \\ &= P(Y = 0) + P(Y = 1) \\ &= \frac{1}{8} + \frac{1}{4} = \frac{3}{8} \end{aligned}$$

$$\begin{aligned} F(2) &= P(Y \leq 2) \\ &= \frac{3}{8} + \frac{1}{4} + \frac{1}{8} = \frac{6}{8} \end{aligned}$$

$$\begin{aligned} F(3) &= P(Y \leq 3) \\ &= \frac{6}{8} + \frac{1}{4} = 1 \end{aligned}$$

6. Suppose we toss a coin n times. Let $n = 5$, and X be the number of heads in n trials. Suppose we wanted to find the probability of $P(SFFSS)$; we have:

$$\begin{aligned} P(SFFSS) &= P(S)P(F)P(F)P(S)P(S) \\ &= p \cdot (1-p) \cdot (1-p) \cdot p \cdot p \\ &= p^3(1-p)^{5-3} \end{aligned}$$

Then, since we know it is independent, we can write this as $P(X = 3)$. Thus, we have:

$$P(X = 3) = \binom{5}{3} p^3 (1-p)^{5-3}$$

7. A tennis player makes successful first serves 70% of the time. Assume each serve is independent of the others. If she serves, what is the probability she gets:

   (a) All six serves in?

   (b) Four serves in?

   (c) At least four serves in?

   (d) No more than four serves in?

   We have $P(S) = 0.7$ and $P(F) = 0.3$. Let X be the number of successful serves. Let $n = 6$, then $X \sim B(6, 0.7)$. We have:

$$P(X = x) = \binom{6}{k} 0.7^x 0.3^{6-x}$$

   To solve each part, we can just plug in values:

   (a)

$$\begin{aligned} P(X = 6) &= \binom{6}{6} 0.7^6 0.3^{6-6} \\ &= \frac{6!}{6!0!} \cdot 0.7^6 \\ &= 0.7^6 = 0.118 \end{aligned}$$

(b)

$$
\begin{aligned}
P(X=4) &= \binom{6}{4} 0.7^4 0.3^{6-4} \\
&= \frac{6!}{4!2!} 0.7^4 \cdot 0.3^2 \\
&= 0.324
\end{aligned}
$$

(c)

$$
\begin{aligned}
P(X \geq 4) &= P(X=4) + P(X=5) + P(X=6) \\
&= 0.744
\end{aligned}
$$

(d)

$$
\begin{aligned}
P(X \leq 4) &= P(X=4) + P(X=3) + ... + P(X=0) \\
&= 0.580
\end{aligned}
$$

8. Suppose that a lot of 5000 electrical fuses contain 5% defectives.

   (a) If a sample of 5 fuses is tested, find the probability of observing at least one defective.

   (b) If $n = 20$ fuses are randomly sampled from this lot, find the probability that at most 6 defectives are observed.

   (c) If $n = 20$ fuses are randomly sampled, find the probability that at least four defectives will be observed.

   We know this event is independent; we have $P(F) = 0.05$ and $P(S) = 0.95$ for 5000 fuses.
   Let X be the number of defectives in n fuses.

   (a) $n = 5$, we want to find $P(X \geq 1)$.
   Using the binomial distribution table, we have:

$$
\begin{aligned}
P(X \geq 1) &= 1 - P(X \leq 0) \\
&= 1 - 0.774 \\
&= 0.226
\end{aligned}
$$

   (b)

9. Suppose the probability of an engine malfunction during a one-hour period is $p = 0.02$. Find the probability that the engine will survive for two hours.
   We have $p = P(S) = 0.02$, so $q = P(F) = 0.98$. Let Y be the number of one-hour intervals until a malfunction occurs. We have:

$$
\begin{aligned}
P(y) &= 0.98^{y-1} \cdot 0.02 \qquad \text{For } y = 1, 2, ... \\
&= P(Y \geq 3) \\
&= 1 - P(Y \leq 2) \\
&= 1 - P(Y=1) - P(Y=2) \\
&= 1 - (0.98^1 \cdot 0.02) - (0.98^0 \cdot 0.02) \\
&= 0.9604
\end{aligned}
$$

10. If the probability of an engine malfunction is $p = 0.02$ and Y is the number of one-hour intervals until the first malfunction, find the mean and standard deviation of Y.

$$\begin{aligned} E(Y) &= \frac{1}{p} \\ &= \frac{1}{0.02} \\ &= 50 V(Y) = \frac{1-0.02}{0.02^2} \\ &= 2450 \end{aligned}$$

11. Suppose a professor randomly selects 3 new teaching assistants from a total of 10 applicants - 6 male, 4 female. Let X be the number of females who are hired.

    (a) Find the probability that no females are hired.

    (b) Find the probability that at least 2 females are hired.

    Here, $r = 4$, then $N - r = 6$. Set $n = 3$; we use hypergeometric distribution. First, calculate x:

$$\begin{aligned} \max\{0, n-(N-r)\} &: n-(N-r) \\ &: 3-6=-3 \\ &= 0 \\ \min\{r,n\} &: \min 4, 3 \\ &= 3 \end{aligned}$$

    Hence, $x = 0, 1, 2, 3$.

$$P(X = x) = \frac{\binom{4}{x}\binom{6}{3-x}}{\binom{10}{3}}$$

    (a)

$$\begin{aligned} P(X=0) &= \frac{\binom{4}{0}\binom{6}{3}}{\binom{10}{3}} \\ &= \frac{6!}{3!3!} \cdot \frac{7!3!}{10!} \\ &= \frac{1}{6} \end{aligned}$$

    (b) We want to find $P(X \geq 2)$, i.e., $1 - P(X = 0) - P(x = 1)$.

$$\begin{aligned} P(X \geq 2 &= 1 - \frac{1}{6} - \frac{\binom{4}{1}\binom{6}{2}}{\binom{10}{3}} \\ &= 1 - \frac{1}{6} - \frac{1}{2} \end{aligned}$$

12. Suppose there are X number of blue whale sightings per week. Assume X follows a Poisson distribution and the number of weekly sightings is 2.5.

    (a) Find the mean and standard deviation of X.

$$\begin{aligned} E(X) &= 2.5 \\ \sigma^2 = V(X) &= 2.5 \\ \sigma &\approx 1.58 \end{aligned}$$

(b) Find the probability that 5 sightings are made in one week.

$$P(X = 5) = \frac{2.5^5 e^{-2.5}}{5!}$$

We can use a Poisson Table:

$$\begin{aligned} P(X = 5 &= P(X \leq 5) - P(X \leq 4) \\ &= 0.958 - 0.891 \\ &= 0.067 \end{aligned}$$

(c) Find the probability that there are fewer than 2 sightings made per week. Use a Poisson Table:

$$\begin{aligned} P(X < 2) &= P(\leq 1) \\ &= 0.287 \end{aligned}$$

1.4 Continuous Random Variables

1. Suppose that:

$$F(y) = \begin{cases} 0 & y < 0 \\ y & 0 \leq y \leq 1 \\ 0 & y > 1 \end{cases}$$

Find the probability density function for Y.

Take the derivative, and get:

$$f(y) = \begin{cases} 0 & y < 0 \\ 1 & 0 \leq y \leq 1 \\ 0 & y > 1 \end{cases}$$

We can also write this as $f(y) = 1$ if $0 \leq y \leq 1$, and 0 otherwise.

2. Let Y be a continuous random variable with probability density function given by:

$$f(y) = \begin{cases} 3y^2 & 0 \leq y \leq 10 \\ \text{otherwise} & \end{cases}$$

Find $F(y)$.

To find $F(y)$ we take the anti-derivative.
For $y < 0$, we have:

$$\begin{aligned} F(y) &= P(Y \leq y) \\ &= \int_{-\infty}^{y} 0 dt \\ &= 0 \end{aligned}$$

For $0 \leq y \leq 1$, we have:

$$\begin{aligned} F9Y0 &= \int_{-\infty}^{y} f(t) dt \\ &= \int_{-\infty}^{0} 0 dt + \int_{0}^{y} 3t^2 dt \\ &= t^3 |_0^y \\ &= y^3 \end{aligned}$$

For $y > 1$:

$$\begin{aligned} F(y) &= \int_{-\infty}^{0} 0dt + \int_{0}^{1} 3t^2 dt + \int_{1}^{\infty} 0dt \\ &= 1 \end{aligned}$$

Hence, we have:

$$F(y) = \begin{cases} 0 & y < 0 \\ y^3 & 0 \le y \le 1 \\ 1 & y > 1 \end{cases}$$

3. Given $f(y) = cy^2, 0 \le y \le 2, f(y) = 0$ elsewhere, find the value of c for which $f(y)$ is a valid density function.

   For a probability density function $\int_{-\infty}^{\infty} f(y)dy = 1$, and have $\int_0^2 cy^2 dy = 1$. Then:

$$\begin{aligned} c\frac{y^3}{3}|_0^2 &= 1 \\ c\frac{8}{3} &= 1 \\ c &= \frac{3}{8} \end{aligned}$$

4. Find $P(1 \le Y \le 2)$ from the previous example.
   We have:

$$f(y) = \frac{3}{8}y^2 \text{ for } 0 \le y \le 2$$

$$\begin{aligned} P(1 \le Y \le 2) &= \int_1^2 \frac{3}{8}y^2 dy \\ &= \frac{1}{8}[y^3]_1^2 \\ &= \frac{1}{8}(8-1) \\ &= \frac{7}{8} \end{aligned}$$

   How about for $P(1 \le Y \le 5)$? We know that it is valid only for $0 \le y \le 2$, so we split it up:

$$P(1 \le Y \le 5) = P(1 \le Y \le 2) + P(2 \le Y \le 5)$$

5. From the previous example, find $\mu = E(Y)$ and $\sigma^2 = V(Y)$.
   We have $f(y) = \frac{3}{8}y^2$ for $0 \le y \le 2$, and 0 otherwise. To find the expected value:

$$\begin{aligned} \mu = E(Y) &= \int_0^2 y \cdot \frac{3}{8}y^2 dy \\ &= \frac{3}{8}\int_0^2 y^3 dy \\ &= \frac{3}{8}\left[\frac{y^4}{4}\right]_0^2 \\ &= \frac{3}{8}(4-0) \\ &= \frac{3}{2} \end{aligned}$$

Thus, $\mu = E(Y) = 1.5$.
Then, find the variance. Find $E(Y^2)$:

$$
\begin{aligned}
E(Y^2) &= \int_0^2 y^2 \cdot \frac{3}{8} y^2 dy \\
&= \frac{3}{8} \int_0^2 y^4 dy \\
&= \frac{3}{8} \left[\frac{y^5}{5} \right]_0^2 \\
&= \frac{3}{8} (\frac{32}{5} - 0) \\
&= \frac{12}{5}
\end{aligned}
$$

Hence, $E(Y^2) = 2.4$. Now, we can calculate $V(Y)$:

$$
\begin{aligned}
V(Y) &= E(Y^2) - \mu^2 \\
&= 2.4 - 1.5^2 \\
&= 0.15
\end{aligned}
$$

6. Esme is taking a train from Minneapolis to Boston, a distance of roughly 1400 miles.Her position is uniformly distributed between the two cities. What is the probability that she is past Chicago, which is 400 miles from Minneapolis?
   Let X be Esme's location, where it follows the uniform probability distribution on the interval $(0, 1400)$. We have $f(x) = \frac{1}{1400}$ if $0 \le x \le 1400$, and 0 otherwise.
   We want to find $P(X > 400)$, so:

$$
\begin{aligned}
\int_{400}^{1} 400 \frac{1}{1400} dx &= \frac{1}{1400} [x]_{400}^{1400} \\
&= \frac{1400 - 400}{1400} \\
&= \frac{10}{14}
\end{aligned}
$$

7. Let Z denote a normal random variable with mean 0 and standard deviation 1 (the standard normal random variable).
   That is, $Z \sim N(0, 1)$.

   (a) Find $P(Z > 2)$:
   Using the Normal curve table, we have:

$$
\begin{aligned}
P(Z > 2) &= 1 - P(Z \le 2) \\
&= 1 - 0.9772 \\
&= 0.0228
\end{aligned}
$$

   (b) Find $P(-2 \le Z \le 2)$:

$$
\begin{aligned}
P(-2 \le Z \le 2) &= P(Z \le 2) - P(Z < -2) \\
&= 0.9772 - 0.0228 \\
&= 0.9544
\end{aligned}
$$

   (c) Find $P(0 \le Z \le 1.73$:

$$
\begin{aligned}
P(0 \le Z \le 0.173) &= P(Z \le 1.73) - P(Z < 0) \\
&= 0.9582 - 0.5 \\
&= 0.4582
\end{aligned}
$$

8. The achievement scores for a college entrance exam are normally distributed with mean 75 and standard deviation 10. What fraction of the scores lie between 80 and 90?
   Let $Y \to$Score. We have $Y \sim N(75, 10)$. We want to find $P(80 \leq Y \leq 90)$.
   We want to standardize this to use the Normal curve table. So:

$$
\begin{aligned}
P(80 \leq Y \leq 90) &= P\left(\frac{80-75}{10} \leq \frac{Y-\mu}{\sigma} \leq \frac{90-75}{10}\right) \\
&= P(0.5 \leq Z \leq 1.5) \\
&= P(Z \leq 1.5) - P(Z < 0.5) \\
&= 0.9332 - 0.6915 \\
&= 0.2417
\end{aligned}
$$

9. If Z is the standard normal random variable, find the value z_0 such that:

   (a) $P(Z > z_0) = 0.5$:
   We know the mean is 0, so $z_0 = 0$.

   (b) $P(Z < z_0) = 0.8643$:
   Using the standard normal curve table, $z_0 = 1.10$.

   (c) $P(-z_0 < Z < z_0) = 0.9$:
   We can use the symmetric property, we calculate the unused area:

$$
\begin{aligned}
1 - 0.9 &= 0.1 \\
\frac{0.1}{2} &= 0.05
\end{aligned}
$$

   So, each side of the unused area has size 0.05. Then,

$$
P(Z \leq z_0) = 0.95 P(Z < -z_0) = 0.05
$$

   Hence, $z_0 = 1.645$.

   (d) $P(-z_0 < Z < z_0) = 0.99$:
   We have $1 - 0.99 = 0.01$, and so each side of the unused area has size 0.005. Then:

$$
P(Z \leq z_0) = 0.995 P(Z \leq -z_0) = 0.005
$$

   Then, $z_0 = 2.575$.

10. The width of bolts of fabric is normally distributed with mean $950mm$ and standard deviation $10mm$.

    (a) What is the probability that a randomly chosen bolt has a width of between 947 and 958 mm?
    Let $Y \to$ width, $Y \sim N(950, 10)$.
    We want to find $P(947 \leq Y \leq 958)$.

$$
\begin{aligned}
P\left(\frac{947-950}{10} \leq Z \leq \frac{958-950}{10}\right) &= P(-0.3 \leq Z \leq 0.8) \\
&= P(Z \leq 0.8) - P(Z < -0.3) \\
&= 0.7881 - 0.3821 \\
&= 0.406
\end{aligned}
$$

    (b) What is the appropriate value for C such that a randomly chosen bolt has width less than C with probability 0.8531?

We want to find $P(Z < C) = 0.8531$.

$$
\begin{aligned}
P\left(Z < -\frac{C-950}{10}\right) &= 0.8531 \\
P(Z < z_0) &= 0.8531,
\end{aligned}
$$

where $z_0 = \frac{C-950}{10}$. We find $z_0 = 1.05$ using the table then find C:

$$
\begin{aligned}
\frac{C-950}{10} &= 1.05 \\
C &= 1.05 \cdot 10 + 950 \\
&= 10.5 + 950 \\
&= 960.5
\end{aligned}
$$

11. If a die is rolled 600 times, what is the probability of rolling between 90 and 110 fours?
$\Omega = \{1, 2, 3, 4, 5, 6\}$, and $p = P(4) = \frac{1}{6}$.
Let $n = 600$, and $X \to$ number of fours in 600 rolls, where $X \sim B(600, \frac{1}{6})$. We also have $np = 600 \cdot \frac{1}{6} = 100$, and $n(1-p) = 600 \cdot \frac{1}{6} = 100$.
Hence, $\mu = np = 100$, and $\sigma^2 = np(1-p) = \frac{500}{6}$.

Now, we want to find $X \sim N(\mu, \sigma)$ - Note that because we are using continuous random variables, include decimals up to the tenth digit.

$$
\begin{aligned}
P(90 \leq Y \leq 110) &= P(89.5 \leq Y \leq 110.5) \\
&= P\left(\frac{89.5-100}{\sqrt{\frac{250}{3}}} \leq Z \leq \frac{110.5-100}{\sqrt{\frac{250}{3}}}\right) \\
&= P(-1.15 \leq Z \leq 1.15) \\
&= P(Z \leq 1.15) - P(Z < -1.15) \\
&= 0.8749 - 0.1251 \\
&= 0.7498
\end{aligned}
$$

Note, because it is a discrete random variable, using a strict relation ($<$), change it to $\leq$ and then add the decimal. For example, $P(X < 420)$ becomes $P(X \leq 419.5)$.

12. Suppose the length of time (in hours) between emergency arrivals at a hospital is modeled as an exponential distribution with $\beta = 2$. What is the probability that more than 5 hours pass without an emergency arrival?

Let $Y \to$ length of time (in hours) between emergency arrivals. We have $\beta = 2$. Use cumulative distribution:
We want to find $P(Y > 5)$.

$$
\begin{aligned}
P(Y > 5) &= e^{-y/\beta} \\
&= e^{-5/2} \\
&\approx 0.08208
\end{aligned}
$$

13. A manufacturer of microwave ovens is trying to determine the length of the warranty period it should attach to its magnetron tube. Preliminary testing shows the length of life X a magnetron tube has is the exponential probability distribution $\beta = 6.25$.

(a) Find the mean and standard deviation of X.

(b) Suppose a warranty period of five years is attached to the magnetron tube. What fraction of tubes must the manufacturer plan to replace, assuming the exponential model $\beta = 6.25$?

(c) What should the warranty period be if the manufacturer wants to replace 27.5% of the tubes?

(d) Find the probability that the lifespan of a magnetron tube is within the interval $[\mu - 2\sigma, \mu + 2\sigma]$.

Let $X \to$ the length of life in years, and $X \sim \exp(6.25)$, where $\beta = 6.25$.

(a) $\mu = E(X) = \beta = 6.25$
$\sigma = \beta = 6.25$.

(b)

$$\begin{aligned}P(X \leq 5) &= 1 - e^{-5/6.25}\\ &= 1 - 0.449329\\ &= 0.550671\end{aligned}$$

(c) Let $W \to$ required warranty period. We have $P(X \leq W) = 0.275$. Then:

$$\begin{aligned}1 - e^{-W/6.25} &= 0.275\\ e^{-W/6.25} &= 1 - 0.275\\ &= 0.725\\ -\frac{W}{6.25} &= \ln 0.725\\ &= -0.3216\\ W &= 6.25 \cdot 0.3216\\ &= 2.01\end{aligned}$$

(d) We have $\mu = 6.25 = \sigma$. We know $E(Y) = \frac{1}{\lambda}$ and $V(Y) = \frac{1}{\lambda^2}$.

$$\begin{aligned}\mu - 2\sigma &= 6.25 - 2 \cdot 6.25\\ &= -6.25\\ \mu + 2\sigma &= 6.25 + 2 \cdot 6.25\\ &= 18.75\end{aligned}$$

Then:

$$\begin{aligned}P(-6.25 \leq X \leq 18.75) &= P(0 \leq X \leq 18.75)\\ &= P(X \leq 18.75)\\ P(X \leq 18.75) &= 1 - e^{-18.75/6.25}\\ &= 1 - e^{-3}\\ &= 0.9502\end{aligned}$$

1.5 Multivariate

1. A local supermarket has 3 checkout counters. Two customers arrive at the counters at different times where there are no other customers. Each customer chooses a counter at random, independently of the other. Let Y_1 denote the number of customers who choose counter 1, and Y_2 the number who select counter 2. Find the joint probability of Y_1, Y_2.

   We let our sample space to be $\Omega = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$.

Then, for Y_1, we have $y_1 = 0, 1, 2$ as the number of people who chose counter 1, and Y_2 has $y_2 = 0, 1, 2$, for the number of people who chose counter 2. We have:

$$
\begin{aligned}
P(Y_1 = 0, Y_2 = 0) &= \frac{1}{9} \\
P(Y_1 = 0, Y_2 = 1) &= \frac{2}{9} \\
P(Y_1 = 0, Y_2 = 2) &= \frac{1}{9} \\
P(Y_1 = 1, Y_2 = 0) &= \frac{2}{9} \\
P(Y_1 = 2, Y_2 = 0) &= \frac{1}{9} \\
P(Y_1 = 1, Y_2 = 1) &= \frac{2}{9} \\
P(Y_1 = 2, Y_2 = 1) &= 0 \\
P(Y_1 = 2, Y_2 = 2) &= 0
\end{aligned}
$$

We want to find the join probability mass function of Y_1, Y_2.

	0	1	2
0	1/9	2/9	1/9
1	2/9	2/9	0
2	1/9	0	0

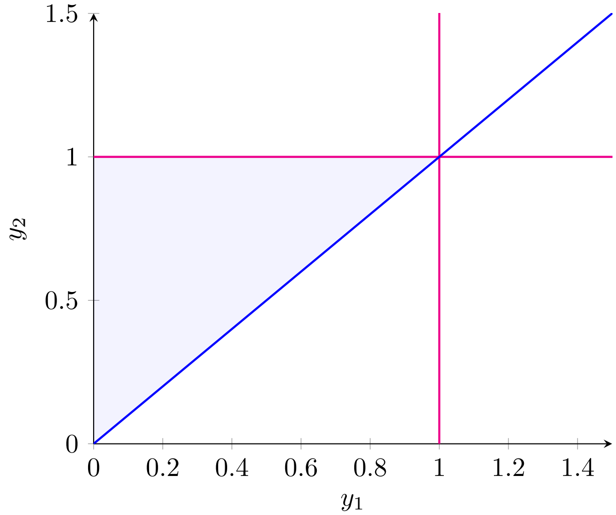
2. Consider the previous example. Find $F(-1, 2), F(1.5, 2)$, and $F(5, 7)$.

$$
\begin{aligned}
F(-1, 2) &= P(Y_1 \le -1, Y_2 \le 2) \\
&= 0 F(1.5, 2) \qquad\qquad = P(Y_1 \le 1.5, Y_2 \le 2) \\
&= \frac{8}{9} \\
F(5, 7) &= P(Y_1 \le 5, Y_2 \le 7) \\
&= 1
\end{aligned}
$$

3. Suppose a radioactive particle is randomly located in a square with sides of unit length. That is, if two regions within the unit square and of equal area are considered, then the particle is equally likely to be in either region.
   Let Y_1, Y_2 denote the coordinates of the particle's location. A reasonable model for the relative frequency histogram for Y_1, Y_2 is the bivariate analogue of the univariate uniform density function:

$$
f(y_1, y_2) = \begin{cases} 1, & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1 \\ 0, & \text{otherwise} \end{cases}
$$

   (a) Sketch the probability density surface.



(b) Find $F(0.2, 0.4)$.

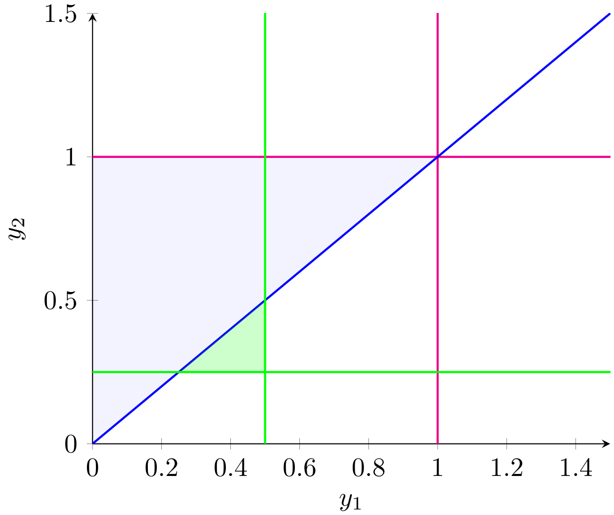
$$\begin{aligned} F(0.2, 0.4) &= P(Y_1 \leq 0.2, Y_2 \leq 0.4) \\ &= 0.2 \cdot 0.4 \cdot 1 \\ &= 0.08 \end{aligned}$$

(c) Find $P(0.1 \leq Y_1 \leq 0.3, 0 \leq Y - 2 \leq 0.5)$.

$$\begin{aligned} P(0.1 \leq Y_1 \leq 0.3, 0 \leq Y_2 \leq 0.5) &= (0.3 - 0.1) \cdot 0.5 \cdot 1 \\ &= 0.1 \end{aligned}$$

4. Gasoline is to be stocked in a bulk tank at the beginning of each week and then sold to individual customers. Let $Y_!$ denote the proportion of the capacity of the bulk tank that is available after the tank is stocked at the beginning of the week. Because of the limited supplies, Y_1 varies from week to week.
Let Y_2 denote the proportion of the capacity of the bulk tank sold during the week. Because Y_1 and Y_2 are proportions, the variables take on values $[0, 1]$. The amount sold, y_2 cannot exceed the amount available, y_1.
The join density function for Y_1, Y_2 is given by:

$$f(y_1, y_2) = \begin{cases} 3y_1 & 0 \leq y_2 \leq y_1 \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$



The blue line represents $y_1 = y_2$. Then, we have:

$$
\begin{aligned}
P(Y_1 \leq \frac{1}{2}, Y_2 \leq \frac{1}{4}) &= \int_{1/4}^{1/2} \left[\int_{1/4}^{y_1} 3y_1 \ dy_2 \right] dy_1 \\
&= \int_{1/4}^{1/2} 3y_1 \ [y_2]_{1/4}^{y_1} dy_1 \\
&= \int_{1/4}^{1/2} 3y_1(y_1 - \frac{1}{4}) dy_1 \\
&= \int_{1/4}^{1/2} \left(3y_1^2 - \frac{3y_1}{4} \right) dy_1 \\
&= \left[y_1^3 - \frac{3y_1^2}{8} \right]_{1/4}^{1/2} \\
&= \left[\frac{1}{2}^3 - \frac{3}{8} \left(\frac{1}{2} \right)^2 \right] - \left[\frac{1}{4}^3 - \frac{3}{8} \left(\frac{1}{4} \right)^2 \right] \\
&= \frac{1}{8} - \frac{3}{32} - \frac{1}{64} + \frac{3}{128} \\
&= \frac{16 - 12 - 2 + 3}{128} \\
&= \frac{5}{128}
\end{aligned}
$$

5. Let $f(y_1, y_2) = 3y_1$ for $0 \leq y_2 \leq y_1 \leq 1$. Calculate the marginal density function of Y_1.

$$
\begin{aligned}
f(y_1) &= \int_{-\infty}^{\infty} f(y_1, y_2) dy_2 \\
&= \int_{0}^{y_1} 3y_1 \ dy_2 \\
&= 3y_1 [y_2]_0^{y_1} \\
&= 3y_1^2
\end{aligned}
$$

Then, we have $f(y_1) = 3y_1^2$ if $0 \leq y_1 \leq 1$, and 0 otherwise.

Next, calculate:

$$\begin{aligned}
f(y_2) &= \int_{-\infty}^{\infty} f(y_1, y_2) dy_1 \\
&= \int_{y_2}^{1} 3y_1 \ dy_1 \\
&= \left[\frac{3y_1^2}{2}\right]_{y_2}^{1} \\
&= \frac{3}{2}(1 - y_2^2) \text{ if } 0 \leq y_2 \leq 1
\end{aligned}$$

To verify:

$$\begin{aligned}
\frac{3}{2}\int_0^1 (1 - y_2^2) dy_2 &= \frac{3}{2}\left[y_2 - \frac{y_2^3}{3}\right]_0^1 \\
&= \frac{3}{2}\left[1 - \frac{1}{3}\right] \\
&= 1
\end{aligned}$$

6. The conditional density function of Y_1 given $Y_2 = y_2$. We have:

$$\begin{aligned}
f(y_1|y_2) &= \frac{f(y_1, y_2)}{f(y_2)} \\
&= \frac{3y_1}{\frac{3}{2}(1 - y_2^2)} \\
&= \frac{2y_1}{1 - y_2^2}
\end{aligned}$$

Then, we find the range.
Since y_2 is fixed, consider the function y_2. We have $f(y_1, y_2) = 3y_1 : 0 \leq y_2 \leq y_1 \leq 1$, so:

$$f(y_1|y_2) = \frac{2y_1}{1 - y_2^2} \text{ if } y_2 \leq y_1 \leq 1$$

Then, if we wanted to get the density function in regards to y_1, we would get 1 if we integrated from $[0, 1]$.

Suppose we wanted to find $P(Y_1 \geq 0.7|Y_2 = 0.5)$. We use the continuous density function formula:

$$\begin{aligned}
f(y_1|Y_2 = 0.5) &= \frac{2y_1}{1 - 0.5^2} \text{ if } 0.5 \leq y_1 \leq 1 \\
&= \frac{2y_1}{0.75} \\
&= \frac{8}{3}y_1
\end{aligned}$$

We can check the density function by integration, and ensure it is equal to 1.

$$\begin{aligned}
\int_{0.5}^{1} \frac{8}{3}y_1 dy_1 &= \left[\frac{4}{3}y_1^2\right]_{1/2}^{1} \\
&= \frac{4}{3} - \frac{4}{3} \cdot \frac{1}{4} \\
&= 1
\end{aligned}$$

Hence:

$$\begin{aligned} P(Y_1 \geq 0.7 | Y_2 = 0.5) &= \int_{0.7}^{1} \frac{8}{3} y_1 \; dy_1 \\ &= \frac{4}{3} \left[y_1^2 \right]_{0.7}^{1} \\ &= \frac{4}{3}(1 - 0.7^2) \end{aligned}$$

7. From a group of 3 Republicans, 2 Democrats, and 1 independent, a committee of 2 people is randomly decided.
   Let Y_1 denote the number of Republicans, and Y_2 denote the number of Democrats on the committee. Find the joint probability function of Y_1, Y_2 and then find the marginal probability function of Y_1.

   We have two groups, and are given $n = 2$, and thus the probability follows a Hypergeometric Distribution.
   Then:

$$P(Y_1 = y_1, Y_2 = y_2) = \frac{\binom{3}{y_1} \cdot \binom{2}{y_2} \cdot \binom{1}{2-y_1-y_2}}{\binom{6}{2}}$$

   We can draw a probability table to find each probability, and plug in values we need.

$$\begin{aligned} P(Y_1 = 1, Y_2 = 2) &= \frac{\binom{3}{1}\binom{2}{1}\binom{1}{0}}{62} \\ &= \frac{6}{15} \\ P(Y_1 = 1, Y_2 = 0) &= \frac{3}{15} \\ P(Y_1 = 2, Y_2 = 0) &= \frac{3}{15} \\ P(Y_1 = 0, Y_2 = 2) &= \frac{1}{15} \\ P(Y_1 = 0, Y_2 = 1) &= \frac{2}{15} \end{aligned}$$

   Hence, we have:

y_1/y_2	0	1	2	$P(Y_2)$
0	0	3/15	3/15	6/15
1	2/15	6/15	0	8/15
2	1/15	0	0	1/15

   If we wanted to find the conditional discrete probability function, we get:

$$\begin{aligned} P(Y_1 = 0 | Y_2 = 1) &= \frac{P(Y_1 = 0, Y_2 = 1)}{P(Y_2 = 1)} \\ &= \frac{\frac{2}{15}}{\frac{8}{15}} \\ &= \frac{1}{4} \end{aligned}$$

8. Let $f(y_1, y_2) = \begin{cases} 2y_1 & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$

We have:

$$
\begin{aligned}
f(y_1) &= \int_{-\infty}^{\infty} f(y_1, y_2)\ dy_2 \\
f(y_1) &= \int_{-\infty}^{\infty} 0\ dy_2 = 0 \quad \text{if } 0 < y_1 \\
f(y_1) &= \int_0^1 2y_1 dy_2 \quad \text{if } 0 \le y_1 \le 1 \\
&= 2y_1 \left[y_2\right]_0^1 \\
&= 2y_1 \\
f(y_1) &= 2y_1
\end{aligned}
$$

Therefore:

$$
f(y_1) = \begin{cases} 2y_1 & \text{if } 0 \le y_1 \le 1 \\ 0 & \text{otherwise} \end{cases}
$$

Now, consider $f(y_2)$.
If $0 \le y_2 \le 1$:

$$
\begin{aligned}
f(y_2) &= \int_0^1 2y_1\ dy_1 \\
&= \left[y_1^2\right]_0^1 \\
&= 1
\end{aligned}
$$

So:

$$
f(y_2) = \begin{cases} 1 & \text{if } 0 \le y_2 \le 1 \\ 0 & \text{otherwise} \end{cases}
$$

9. Using the previous example about the Democrats and Republicans, take $Y_2 = 1$; find the probabilities for Y_1 (where $Y_1 = 0, 1, 2$.
   That is, we want to find $P(Y_1 = y_1 \mid Y_2 = 1)$.
   First, let $Y_1 = 0$:

$$
\begin{aligned}
P(Y_1 = y_1 \mid Y_2 = 1) &= \frac{P(Y_1 = y_1, Y_2 = 1)}{P(Y_2 = 1)} \\
(Y_1 = 0 \mid Y_2 = 1) &= \frac{P(Y_1 = 0, Y_2 = 1)}{P(Y_2 = 1)}
\end{aligned}
$$

   Using the table, we have:

$$
\begin{aligned}
\frac{P(Y_1 = 0, Y_2 = 1)}{P(Y_2 = 1)} &= \frac{2/15}{8/15} \\
&= \frac{1}{4}
\end{aligned}
$$

   Then, $Y_1 = 1$:

$$
\begin{aligned}
P(Y_1 = 1 \mid Y_2 = 1) &= \frac{P(Y_1 = 1, Y_2 = 1)}{P(Y_2 = 1)} \\
&= \frac{6/15}{8/15} \\
&= \frac{3}{4}
\end{aligned}
$$

Finally, do $Y_1 = 2$:

$$P(Y_1 \geq 2 \mid Y_2 = 1) = 0$$

We can check if these are independent:

$$\begin{aligned}
P(Y_1 = 1, Y_2 = 1) &= \frac{6}{15} \\
P(Y_1 = 1) \cdot P(Y_1 = 1) &= \frac{9}{15} \cdot \frac{8}{15} \\
&= \frac{72}{225}
\end{aligned}$$

Therefore, Y_1, Y_2 are dependent.

10. Let $f(y_1, y_2) = \begin{cases} 2 & 0 \leq y_2 \leq y_1 \leq 1 \\ 0 & \text{otherwise} \end{cases}$. We want to show they are independent.
We find the marginal density function of y_1:

$$\begin{aligned}
f(y_1) &= \int_0^y {}_1 2 \; dy_2 \\
&= 2 \left[y_2\right]_0^{y_1} \\
f(y_1) &= \begin{cases} 2y_1 & 0 \leq y_1 \leq 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Now, we want to find the marginal density function of y_2:

$$\begin{aligned}
f(y_2) &= \int_{y_2}^1 2 \; dy_1 \\
&= 2 \left[\right]_{y_2}^1 \\
&= 2(1 - y_2) \\
f(y_2) &= \begin{cases} 2(1 - y_2) & 0 \leq y_2 \leq 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Now, consider the conditional density function of y_1 given y_2):

$$\begin{aligned}
f(y_1 | y_2) &= \frac{f(y_1, y_2)}{f(y_2)} \\
&= \frac{2}{2(1 - y_2)} \\
&= \frac{1}{1 - y_2} \quad \text{if } y_2 \leq y_1 \leq 1 \\
f(y_1 \mid Y_2) &= \frac{1}{1 - 0.6} \\
&= 2.5 \quad \text{if } 0.6 \leq y_1 \leq 1
\end{aligned}$$

Thus, we have:

$$\begin{aligned}
P(Y_1 \geq 0.8 \mid Y_2 = 0.6) &= \int_{0.8}^1 2.5 \; dy_1 \\
&= 2.5 \left[y_1\right]_{0.8}^1 \\
&= 2.5 \cdot 0.2 \\
&= 0.5
\end{aligned}$$

1.6 Functions of Random Variables

1. Gasoline is stocked in a bulk tank once at the beginning of each week and then sold to customers. Let Y_1 denote the proportion of the capacity of the bulk tank that is available after the tank is stocked at the beginning of the week. Because of the limited supplies, Y_1 varies from week to week.
   Let Y_2 denote the proportion of capacity of the bulk tanks sold during the week. Because Y_1, Y_2 are proportions, both variables take values between 0 and 1.
   Further, the amount sold, y_2, cannot exceed the amount available (y_1).
   Suppose the joint density function for Y_1, Y_2 is given by:

$$f(y_1, y_2) = \begin{cases} 3y_1, & 0 \leq y_2 \leq y_1 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   Find the probability density function for $U = Y_1 - Y_2$, the proprtion amount of gasoline remaining at the end of the week. Use the density function of U to find $E(U)$.

   We have $F_u(u) = P(Y_1 - Y_2 \leq u)$, and so $y_2 = y_1 - u$. Then, there are 3 cases:

$$\begin{aligned}
u < 0 \implies F_u(u) &= P(Y_1 - Y_2 \leq u) \\
&= 0 \\
u > 1 \implies F_u(u) = 1 & \\
0 \leq u \leq 1 \implies F_u(u) &= 1 - \int_u^1 \left[\int_0^{y_1 - u} 3y_1 \; dy_2\right] dy_1 \\
&= 1 - \int_u^1 3y_1 \, [y_2]_0^{y_1 - u} \; dy_1 \\
&= 1 - \int_u^1 3y_1(y_1 - u) dy_1 \\
&= 1 - \int_u^1 [3y_1^2 - 3uy_1] dy_1 \\
&= 1 - \left[y_1^3 - \frac{3uy_1^2}{2}\right]_u^1 \\
&= 1 - (1 - \frac{3u}{2}) - (u^3 - \frac{3u^3}{2})) \\
&= \frac{3u}{2} - \frac{u^3}{2} \\
&= \frac{3}{2}(1 - u^2) \text{ if } 0 \leq u \leq 1
\end{aligned}$$

   Then, to find the expected value:

$$\begin{aligned}
E(u) &= \int_0^1 u \cdot \frac{3}{2}(1 - u^2) du \\
&= \frac{3}{2} \int_0^1 (u - u^3) du \\
&= \frac{3}{2} \left[\frac{u^2}{2} - \frac{u^4}{4}\right]_0^1 \\
&= \frac{3}{2} \left(\frac{1}{2} - \frac{1}{4}\right) \\
&= \frac{3}{2} \cdot \frac{1}{4} \\
&= \frac{3}{8}
\end{aligned}$$

2. Let (Y_1, Y_2) denote a random sample of size $n = 2$ from the uniform distribution for the interval $(0, 1)$. Find the probability density function for $U = Y_1 + Y_2$. Let $f(y) = \begin{cases} 1, & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$.

   Then, $f(y_1) = 1$. We have:

$$\begin{aligned} f(y_1, y_2) &= f(y_1)f(y_2) \\ &= 1 \text{ if } 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1. \end{aligned}$$

$$\begin{aligned} U &= Y_1 + Y_2 \\ F_u(u) &= P(U \leq u) \\ &= P(Y_1 + Y_2 \leq u) \end{aligned}$$

   We have four cases:

   (a) $F_u(u) = 0$ if $u < 0$

   (b) $F_u(u) = 0$ if $u > 2$

   (c) $F_u(u) = \frac{1}{2}u^2$ if $0 \leq u \leq 1$.
   We have:

$$\begin{aligned} F_u(u) &= \frac{1}{2}u \cdot u \cdot 1 \\ &= \frac{1}{2}u^2 \end{aligned}$$

   (d) $F_u(u) = 2u - 1 - \frac{u^2}{2}$ if $1 \leq u \leq 2$.

   Then:

$$f_u(u) = \begin{cases} u, & 0 \leq y \leq 1 \\ 2 - u, & 1 \leq u \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

3. Let Y have the probability density function given by:

$$f_y(y) = \begin{cases} \frac{y+1}{2}, & -1 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

   Find the density function for $U = Y^2$.

We have:

$$
\begin{aligned}
F_u(u) &= P(U \leq u) \\
&= P(Y^2 \leq u) \\
&= P(-\sqrt{u} \leq Y \leq \sqrt{u}) \\
&= P(Y \leq \sqrt{u}) - P(Y \leq -\sqrt{u}) \\
F_u(u) &= F_Y(\sqrt{u}) - F_Y(-\sqrt{u}) \\
f_u(u) &= f_Y(\sqrt{u})\frac{1}{2\sqrt{u}} + f_Y(-\sqrt{u})\frac{1}{2\sqrt{u}} \\
&= \frac{1}{2\sqrt{u}}(f_Y(\sqrt{u}) + f_Y(-\sqrt{u})) \\
&= \frac{1}{2\sqrt{u}}(\frac{\sqrt{u}+1}{2} + \frac{(-\sqrt{u}+1)}{2}) \\
&= \frac{1}{2\sqrt{u}}
\end{aligned}
$$

So:

$$
f_u(u) = \begin{cases} \frac{1}{2\sqrt{u}}, & 0 < u \leq 1 \\ 0, & \text{otherwise} \end{cases}
$$

Transformation Method:

1. Let Y have the probability density function given by:

$$
f_Y(y) = \begin{cases} 2y, & 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}
$$

Find the density function of random variable $U = -4Y + 3$.

We have:

$$
\begin{aligned}
u &= -4y + 3 \\
y &= \frac{3-u}{4}
\end{aligned}
$$

Then, $\frac{dy}{du} = \frac{-1}{4}$. So:

$$
\begin{aligned}
f_U(u) &= f_Y(y)\left|\frac{dy}{du}\right| \\
&= f_Y(\frac{3-u}{4})\left|\frac{-1}{4}\right| \\
&= 2 \cdot \frac{3-u}{4} \cdot \frac{1}{4} \\
&= \frac{3-u}{8}, \ 0 \leq y \leq 1 \\
&= \frac{3-u}{8}, \ 0 \leq \frac{3-u}{4} \leq 1 \\
&= \frac{3-u}{8}, -1 \leq u \leq 3
\end{aligned}
$$

2. Let Y_1, Y_2 be independent standard normal variables. If $U_1 = Y_1 + Y_2$ and $U_2 = Y_1 - Y_2$, what is the joint density of U_1, U_2?

We use $f(y_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_1^2}, -\infty < y_1, \infty$, and $f(y_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_2^2}, -\infty < y_2 < \infty$.
Then:

$$\begin{aligned} f(y_1, y_2) &= f(y_1)f(y_2) \\ &= \frac{1}{2\pi} e^{-\frac{1}{2}(y_1^2+y_2^2)}, \ -\infty < y_1, y_2 < \infty \end{aligned}$$

Then:

$$U_1 = Y_1 + Y_2 U_2 = Y_1 - Y_2$$
$$(1) u_1 = y_1 + y_2 (2) u_2 = y_1 - y_2$$

$$\begin{aligned} (1) + (2) &\implies u_1 + u_2 = 2y_1 \\ &\implies y_1 = \frac{1}{2}(u_1 + u_2) \\ (1) - (2) &\implies u_1 - u_2 = 2y_2 \\ &\implies y_2 = \frac{1}{2}(u_1 - u_2) \end{aligned}$$

So, we calculate the partial derivatives:

$$\begin{aligned} \frac{\partial y_1}{\partial u_1} &= \frac{1}{2} \frac{\partial y_1}{\partial u_2} = \frac{1}{2} \\ \frac{\partial y_2}{\partial u_1} &= \frac{1}{2} \frac{\partial y_2}{\partial u_2} = -\frac{1}{2} \end{aligned}$$

Calculate the Jacobian:

$$J = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

Hence:

$$\begin{aligned} y_1^2 + y_2^2 &= \left(\frac{u_1 + u_2}{2}\right)^2 + \left(\frac{u_1 - u_2}{2}\right)^2 \\ &= \frac{1}{4}\left(2u_1^2 + 2u_2^2\right) \\ &= \frac{u_1^2 + u_2^2}{2} \\ f(u_1, u_2) &= f(y_1, y_2)|J| \\ &= \frac{1}{2\pi} e^{-\frac{1}{2}(y_1^2+y_2^2)} \cdot \frac{1}{2} \\ &= \frac{1}{2 \cdot 2\pi} e^{-\frac{1}{4}(u-1^2+u_2^2)}, \ -\infty < u_1, u_2 < \infty \end{aligned}$$

3. Let Y_1, Y_2 be an independent exponential random variable, both with mean $\beta > 0$. Find the density function of $U = \frac{Y_1}{Y_1+Y_2}$.

1.7 Module 7

(a) Suppose we rolled a dice: $\Omega = \{1, 2, 3, 4, 5, 6\}$. The probability of rolling each side is $\frac{1}{6}$.
We want to find the theoretical mean of the population:

$$\begin{aligned} \mu = E(Y) &= \sum_y yP(y) \\ &= \frac{21}{6} = \frac{7}{2]} \end{aligned}$$

Then, we can find the variance:

$$\begin{aligned}
E(Y^2) &= \sum y^2 P(y) = \frac{91}{6} \\
\sigma^2 &= E(Y^2) - E(Y)^2 \\
&= \frac{91}{6} - \frac{21^2}{36} \\
&= \frac{105}{36}
\end{aligned}$$

Now, suppose we took a sample size $n = 2$. So, we have y_1, y_2. Then, we calculate the sample mean:

$$\overline{y} = \frac{y_1 + y_2}{2}$$

There is a $\frac{1}{36}$ chance of getting any 2 rolls of a die. Now, we want to calculate the sampling distribution of $\overline{y}, p(\overline{y})$.

4. Let $y \to$ Amount, with $\sigma = 1$ and $n = 9$. We have:

$$\begin{aligned}
P(|\overline{y} - \mu| < 0.3) &= P(-0.3 < \overline{y} - \mu < 0.3) \\
\frac{\sigma}{\sqrt{n}} &= \frac{1}{3}
\end{aligned}$$

So, $\overline{y} \sim N(\mu, \frac{1}{3})$. Then, $\frac{\overline{y}-\mu}{1/3} \sim N(0,1)$. Therefore:

$$\begin{aligned}
P(\frac{-0.3}{1/3} < Z < \frac{0.3}{1/3}) &= P(Z < 0.9) - P(Z < -0.9) \\
&= 0.8159 - 0.1841 \\
&= 0.6318
\end{aligned}$$

5. Suppose that 60% of all the city's voters favour the candidate. In a random sample of 100 voters, what is the probability that fewer than half are in favour of the candidate?
We have $p = P(\text{Favour the Candidate})$. Then, $p = 0.6$ and $n = 100$. We calculate:

$$\begin{aligned}
np &= 100 \cdot 0.6 \\
&= 60 > 10 \\
n(1-p) &= 100 - 60 \\
&= 40 > 10
\end{aligned}$$

Then:

$$\begin{aligned}
\sigma_{\hat{p}} &= \sqrt{\frac{p(1-p)}{n}} \\
&= \sqrt{\frac{0.6 \cdot 0.4}{100}} \\
&= 0.049
\end{aligned}$$

By Central Limit Theorem, we have $\hat{p} \sim N(0.6, 0.049)$. Hence:

$$\begin{aligned}
p(\hat{p} < 0.5) &= P(\frac{\hat{p} - p}{\sigma\hat{p}} < \frac{0.5 - 0.6}{0.049}) \\
&= P(Z < -2.04) \\
&= 0.0207
\end{aligned}$$

6. We have two populations:

	Population 1	Population 2
μ	μ_1	μ_2
σ	σ_1	σ^2
n	$n_1 = 6$	$n_2 = 10$

Note that the two are independent from one another, and we denote Population 1 as:

$$y_{11}, y_{12}, ..., y_{1n_1},$$

and Population 2 as:

$$y_{21}, y_{22}, ..., y_{2n_2}$$

We have $\sigma_1^2 = \sigma_2^2$, and so:

$$\frac{(n_1-1)s_1^2}{\sigma_1^2} \sim^2_{n_1-1},$$

and:

$$\frac{(n_2-1)s_2^2}{\sigma_2^2} \sim^2_{n_2-1}$$

Then, we have:

$$\frac{s_1^2}{\sigma_1^2} \cdot \frac{\sigma_2^2}{s_2^2} \sim F^{n_1-1}_{n_2-1}$$
$$\frac{s_1^2}{s_2^2} \sim F^{n_1-1}_{n-2-1}$$

Then:

$$n_1 = 6 \implies n_1 - 1 = 5$$
$$n_2 = 10 \implies n_2 - 1 = 9$$

Hence, $\frac{s_1^2}{s_2^2} \sim F_9^5$.

$$P(\frac{s_1^2}{s_2^2} \leq b) = 0.95$$

$$1 - P(\frac{s_1^2}{s_2^2} > b) = 0.95$$
$$P(\frac{s_1^2}{s_2^2} > b) = 0.05$$

Using the F Distribution table, we have $b = 3.48$.