STA237 Exercises

Contents

1 Exercises

1.1 Probability

1. Suppose $P(A) = 0.5, P(A \cap B) = 0.2$, and $P(A \cup B) = 0.7$. Find:

   (a) $P(B)$

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ 0.7 &= 0.5 + P(B) - 0.2 \\ 0.7 &= 0.3 + P(B) \\ P(B) &= 0.4 \end{aligned}$$

   (b) $P(\text{exactly one of two events occurs})$
   This means the probability of no elements from the intersection. Let this area be $P(O)$.

$$\begin{aligned} P(A \cup B) &= P(O) + P(A \cap B) \\ 0.7 &= P(O) + 0.2 \\ P(O) &= 0.5 \end{aligned}$$

   (c) $P(\text{neither event occurs})$
   This means the probability of no elements in A, B. Call this probability $P(X)$.

$$\begin{aligned} P(X) &= P(\Omega) - P(A \cup B) \\ &= 1 - 0.7 \\ &= 0.3 \end{aligned}$$

2. Suppose we throw a coin two times. The sample space for this experiment is:

$$\Omega = \{H, T\} \times \{H, T\} = \{(H, H), (H, T), (T, H), (T, T)\}$$

   If it is a fair coin, then all four outcomes have equal possibilities:

$$P((H, H)) = P((H, T)) = P((T, H)) = P((T, T)) = \frac{1}{4}$$

   What is the probability of getting at least one tail?
   Looking at the sample space, the probability is $\frac{3}{4}$.
   Alternatively, we can solve as follows, where RP represents the required probability:

$$\begin{aligned} P(\Omega) &= RP + P(HH) \\ 1 &= RP + P(HH) \\ &= RP + P(H)P(H) && \text{Since it is independent.} \\ &= RP + \frac{1}{2} \cdot \frac{1}{2} \\ 1 &= RP + \frac{1}{4} \\ RP &= \frac{3}{4} \end{aligned}$$

   Suppose we wanted to throw three coins, what would happen to the sample space?

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Then, there are $2 \cdot 2 \cdot 2 = 2^3$ possibilities (to keep on with this pattern, flipping a coin 6 times would have 2^6 possibilities, and so on).
What is the probability of at least one head appearing in a 3-coin toss?

$$\begin{aligned} P(X) &= 1 - P(TTT) \\ &= 1 - P(T)P(T)P(T) \\ &= 1 - \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \\ &= 1 - \frac{1}{8} \\ &= \frac{7}{8} \end{aligned}$$

3. A survey of students in a class was asked about their birth order (1 = oldest/only child), and which college of the university they were enrolled in.

Birth Order	1 or Only	2 or More	Total
Arts	Science	34	23
57			
Agriculture	52	41	93
Human Ecology	15	28	43
Other	12	17	30
Total	113	110	223

Suppose we selected a random student from this class.

(a) What is the probability we select a Human Ecology student?
P(Human Ecology) = $\frac{43}{223}$

(b) What is the probability we select a first-born student?
P(First Born) = $\frac{113}{223}$

(c) What is the probability that the student is a first-born and a Human Ecology student?
P(C) = $P(A \cap B) = \frac{15}{223}$

(d) What is the probability that the student is a first born or a Human Ecology student?

$$\begin{aligned} P(D) &= P(A \cup B) \\ &= P(A) + P(B) - P(\cap B) \\ &= \frac{43}{223} + \frac{113}{223} - \frac{15}{223} \\ &= \frac{141}{223} \end{aligned}$$

4. A sample space consists of 5 simple events, E_1, E_2, E_3, E_4, E_5.

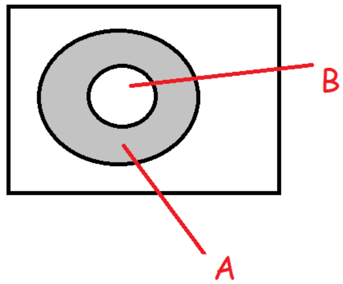
(a) If $P(E_1) = P(E_2) = 0.15$, $P(E_3) = 0.4$, and $P(E_4) = 2P(E_5)$, find the probabilities of E_4 and E_5.

$$\begin{aligned} 1 &= 0.15 + 0.15 + 0.4 + 2P(E_5) + 2P(E_5) \\ 0.3 &= 3P(E_5) \\ P(E_5) &= 0.1 \end{aligned}$$

Then:

$$\begin{aligned} P(E_4) &= 2(0.1) \\ &= 0.2 \end{aligned}$$

5. If A, B are events, $B \subset A$, $P(A) = 0.6$ and $P(B) = 0.2$, then find $P(A \cap B^c)$.
   To visualize:



$$P(A) = P(A \cap B^c) + P(B)0.6 \qquad = P(A \cap B^c) + 0.2$$
$$P(AcapB^c) = 0.4$$

6. An experiment involves tossing a pair of dice and observing the numbers on the upper faces. Find the number of sample points in Ω and the sample space for the experiment The sample points are $\Omega_{\{}m, n\} = \{1, 2, 3, 4, 5, 6\}$.
   The sample space is taking the cross product of the sample points (i.e., $\Omega_1 \times \Omega_2$). Then:
$$\Omega\{(1,1),(1,2),...,(6,6)\}$$
   There is a $m \cdot n = 6 \cdot 6 = 36$ chance for a single probability.

7. The names of three employees are randomly drawn without replacement from a bowl containing the names of 30 employees of a small company. The person whose names is drawn first receives \$100, and the individuals whose names are drawn second and third receive \$50 and \$25, respectively. How many samples are associated with this experiment? There are 30 employees and 3 people are picked. So, $r = 3, n = 30$. Hence:
$$\begin{aligned}\Omega &= P_3^{30} \\ &= \frac{30!}{(30-3)!} \\ &= \frac{30!}{27!}\end{aligned}$$

8. Find the number of ways of selecting two applicants out of five.
   Here, $n = 5, r = 2$.
$$\begin{aligned}\binom{5}{2} &= \frac{5!}{2!(5-2)!} \\ &= \frac{5!}{2!3!} \\ &= \frac{5\cdot4\cdot3\cdot2\cdot1}{2\cdot1\cdot3\cdot2\cdot1} \\ &= 10\end{aligned}$$

1.2 Conditional Probability

1. Let $N = R^c$ be the event 'born in a month without r', and L is the event 'born in a long month'. What is the conditional probability $P(N|L)$?
   $P(R^c|L) = \frac{3}{7}$
   Note that $P(R|L) + P(R^c|L) = \frac{4}{7} + \frac{3}{7} = 1$.

2. A survey asked: 'Are you currently in a relationship?', and 'Are you involved in club sports?'. The survey found that 33% were in a relationship, and 25% were involved in sport. 11% said yes to both. Suppose you meet a student who is on a sports team. What is the probability they are also in a relationship?
Let A be students in a relationship, and B be students in a club sport.
$P(A) = 0.33$, $P(B) = 0.25$, $P(A \cap B) = 0.11$. We have:

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{0.11}{0.25} \\ &= 0.44 \end{aligned}$$

We know that $P(A|B) \neq P(A)$, A and B must be dependent events.

3. A bowl contains 6 red and 4 blue balls, A child selects two balls at random. What is the probability at least one of them is red?
We have:

   (a) $\{R, R\} = \frac{6}{10}, \frac{5}{9}$
   (b) $\{R, B\} = \frac{6}{10}, \frac{4}{9}$
   (c) $\{B, R\} = \frac{4}{10}, \frac{6}{9}$
   (d) $\{B, B\} = \frac{4}{10}, \frac{3}{9}$

Note that $P(\text{at least one red}) + P(\text{no red}) = 1$, so:

$$\begin{aligned} P(one\_R) &= 1 - P(no\_R) \\ &= 1 - P(BB) \\ &= 1 - P(B_1 \cap B_2) \\ &= 1 - P(B_2|B_1)P(B_1) \\ &= 1 - \frac{3}{9} \cdot \frac{4}{10} \\ &= 1 - \frac{2}{15} \\ &= \frac{13}{15} \end{aligned}$$

4. A recent highway safety study found that in 77% of all accidents, the driver was wearing a seatbelt. Accident reports indicated that 92% of those drivers did not experience serious injury (defined as hospitalization or death), but only 63% of the non-belted were okay.

   (a) What is the probability that a driver was seriously injured?
   (b) What is the probability that a driver who was seriously injured was not wearing a seatbelt?

Let B is those that are wearing a seat bet, and NB is for those who did not wear a seat belt. Then, let I represent those who experienced a serious injury.
We have $P(B) = 0.77$, $P(NB) = 1 - 0.77 = 0.23$, and $P(I^c|B) = 0.92$, $P(I^c|NB) = 0.63$.
Then, $P(I|B) = 0.08$, $P(I|NB) = 0.37$.
[a)]Want to find $P(I)$:

$$\begin{aligned} P(I) &= P(B \cap I) + P(NB \cap I) \\ &= P(I|B)P(B) + P(I|NB)P(NB) \\ &= 0.08 \cdot 0.77 + 0.37 \cdot 0.23 \\ &= 0.0616 + 0.0851 \\ &= 0.1467 \end{aligned}$$

Want to find $P(NB|I)$.

$$
\begin{aligned}
P(I|NB) &= \frac{P(NB \cap I}{P(I)} \\
&= \frac{P(I|NB)P(I)}{0.1467} \\
&= \frac{0.23 \cdot 0.37}{0.1467} \\
&= \frac{0.0851}{0.1467} \\
&= 0.58
\end{aligned}
$$

5. A student figures he has a 30% chance of being let out of class late. If he leaves class late, there is a 45% chance he will miss the bus. What is the probability that he is let out of class late and misses the bus?
   Let L be the chance he leaves class late. Let M be the chance he misses the bus.
   $P(L) = 0.3$, $P(M|L) = 0.45$. We want to find $P(L \cap M)$:

$$
\begin{aligned}
P(L \cap M) &= P(M|L)P(L) \\
&= 0.45 \cdot 0.3 \\
&= 0.135
\end{aligned}
$$

6. Leah flies from Moncton to Vancouver with a connection in Montreal. The probability her flight leaves on time is 15%. If the flight is on time, the chance her luggage makes the connecting flight in Montreal is 95%. But, if the flight is delayed, the probability the luggage will make it is only 65%.

   (a) Are the first flight leaving on time and the luggage making the connection independent events? Explain.

   (b) What is the probability that Leah's luggage arrives in Vancouver with her?

   Let F be the chance her flight leaves on time. Let L be the luggage being in the connecting flight.
   $P(F) = 0.15$, $P(L|F) = 0.95$, $P(L|F^C) = 0.65$.

   [a)]We have $P(L|F) = 0.95$ and $P(L) = 0.695$, so they are not equal. Therefore, they are dependent events.

$$
\begin{aligned}
P(L) &= P(L|F)P(F) + P(L|F^c)P(F^c) \\
&= 0.95 \cdot 0.15 + 0.65 \cdot 0.85 \\
&= 0.1425 + 0.5525 \\
&= 0.695
\end{aligned}
$$

7. The Ontario Lottery association claims your odds of winning a price on an Instant Win Crossword game are 1 in 3.09. This means that any ticket has 32.4% chance of winning a prize. Every Friday, you buy one Crossword game.

   (a) What is the probability you don't win a prize next Friday?

   (b) What is the probability you don't win a prize 6 Fridays in a row?

   (c) If you haven't won a prize for the past 6 Fridays, what is the probability you win a prize on your next game?

   (d) What is the probability you win on two of your next three games?

   Let W be the chance you win a prize. $P(W) = 0.324$.

   (a) Want to find $P(W^c)$. $P(W^c) = 1 - 0.324 = 0.676$

(b) Want to find $P(W_1^c...W_6^c)$:

$$\begin{aligned} P(W_1^c...W_6^c) &= P(W_1^c) \cdot ... \cdot P(W_2^c) \\ &= P(W^c)^6 \\ &= 0.676^6 \\ &= 0.0954 \end{aligned}$$

(c) Want to find $P((W_1^c...W_6^c)^c)$. So: $P((W_1^c...W_6^c)^c) = 1 - 0.0954 = 0.9046$.

(d) Want to find $P(W_1W_2W_3^c)$:

$$\begin{aligned} P(W_1W_2W_3^c) &= 0.324 \cdot 0.3424 \cdot 0.676 \\ &= 0.071 \end{aligned}$$

1.3 Random Variables

1. Suppose we roll two dice. List the outcomes:

   (a) $\{X = 8\} = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$.
   $P(X = 8) = \frac{5}{36}$

   (b) $\{X = 3\} = \{(1,2),(2,1)\}$.
   $P(X = 3) = \frac{2}{36} = \frac{1}{18}$

   (c) $\{X = 13\} = \{\}$
   $P(X = 13) = 0$

2. Suppose the death rate in a year is 1 out of 1000 people, and that 2 out of 1000 people suffer some kind of disability. Then, we display the probability model for X = company payout in a table:

Policyholder Outcome	Payout x	P(X = x)
Death	10000	1/1000
Disability	5000	2/1000
Neither	0	

   What is $P(X = 0)$?
   We have $X = 0, 5000, 10000$, and $P(X = 10000) = \frac{1}{1000}$, $P(X = 5000) = \frac{2}{1000}$. We calculate:

$$\begin{aligned} P(X = 0) &= 1 - \frac{1}{1000} - \frac{2}{1000} \\ &= \frac{997}{1000} \end{aligned}$$

3. Take the previous example, and determine $E(X)$, where X is the company payout.
   Recall: $X = 0, 5000, 10000$, and $P(X = 10000) = \frac{1}{1000}$, $P(X = 5000) = \frac{2}{1000}$, and $P(X = 0) = \frac{997}{1000}$.
   To calculate $E(X)$:

$$\begin{aligned} E(X) &= (10000 \cdot \frac{1}{1000}) + (5000 \cdot \frac{2}{1000}) + (0 \cdot \frac{997}{1000}) \\ &= 10 + 10 \\ &= 20 \end{aligned}$$

   Next, suppose we wanted to calculate the variance of X.
   We have:

   (a) $10000 - 20 = 9980$

(b) $5000 - 20 = 4980$

(c) $0 - 20 = -20$

So, we have:

$$\begin{aligned}\sigma^2 &= \sum_x (x-\mu)^2 P(x) \\ &= (9980^2 \cdot \frac{1}{1000}) + (4980^2 \cdot \frac{2}{1000}) + ((-20)^2 \cdot \frac{997}{1000}) \\ &= 149600\end{aligned}$$

Then:

$$\begin{aligned}\sigma = SD(X) &= \sqrt{V(X)} \\ &= \sqrt{149600} \\ &= \$386.78\end{aligned}$$

4. If $E(X) = 10$ and $V(X) = 3$, then:

(a) Calculate $E(9x + 11)$.

$$\begin{aligned}E[9x + 11] &= 9E(x) + 11 \\ &= (9 \cdot 10) + 11 \\ &= 90 + 11 = 101\end{aligned}$$

(b) Calculate $V(9x + 11)$.

$$\begin{aligned}V(9x + 11) &= 9^2 V(x) \\ &= 9^2 \cdot 3 \\ &= 243\end{aligned}$$

(c)

$$\begin{aligned}\sigma^2 &= E(X-\mu)^2 \\ &= E(x^2 - 2\mu x + \mu^2) \\ &= E(x^2) - 2\mu E(X) + \mu^2 \\ &= E(x^2) - 2\mu^2 + \mu^2 \\ &= E(x^2) - \mu^2\end{aligned}$$

(d) If $E(X) = 10$ and $E(X^2) = 160$:

$$\sigma^2 = 160 - 10^2 = 60$$

5. The probability distribution for a random variable Y is given by the following table. Find the mean, variance, and standard deviation of Y.

y	p(y)
0	1/8
1	1/4
2	3/8
3	1/4

Then:

$$\begin{aligned}F(0) &= P(Y \leq 0)\\ &= P(Y = 0) = \frac{1}{8}\end{aligned}$$

$$\begin{aligned}F(1) &= P(Y \leq 1)\\ &= P(Y = 0) + P(Y = 1)\\ &= \frac{1}{8} + \frac{1}{4} = \frac{3}{8}\end{aligned}$$

$$\begin{aligned}F(2) &= P(Y \leq 2)\\ &= \frac{3}{8} + \frac{1}{4} + \frac{1}{8} = \frac{6}{8}\end{aligned}$$

$$\begin{aligned}F(3) &= P(Y \leq 3)\\ &= \frac{6}{8} + \frac{1}{4} = 1\end{aligned}$$

6. Suppose we toss a coin n times. Let $n = 5$, and X be the number of heads in n trials. Suppose we wanted to find the probability of $P(SFFSS)$; we have:

$$\begin{aligned}P(SFFSS) &= P(S)P(F)P(F)P(S)P(S)\\ &= p \cdot (1-p) \cdot (1-p) \cdot p \cdot p\\ &= p^3(1-p)^{5-3}\end{aligned}$$

Then, since we know it is independent, we can write this as $P(X = 3)$. Thus, we have:

$$P(X = 3) = \binom{5}{3} p^3(1-p)^{5-3}$$

7. A tennis player makes successful first serves 70% of the time. Assume each serve is independent of the others. If she serves, what is the probability she gets:

   (a) All six serves in?
   (b) Four serves in?
   (c) At least four serves in?
   (d) No more than four serves in?

   We have $P(S) = 0.7$ and $P(F) = 0.3$. Let X be the number of successful serves. Let $n = 6$, then $X \sim B(6, 0.7)$. We have:

$$P(X = x) = \binom{6}{k} 0.7^x 0.3^{6-x}$$

   To solve each part, we can just plug in values:

   (a)

$$\begin{aligned}P(X = 6) &= \binom{6}{6} 0.7^6 0.3^{6-6}\\ &= \frac{6!}{6!0!} \cdot 0.7^6\\ &= 0.7^6 = 0.118\end{aligned}$$

(b)

$$\begin{aligned}P(X=4) &= \binom{6}{4} 0.7^4 0.3^{6-4}\\ &= \frac{6!}{4!2!} 0.7^4 \cdot 0.3^2\\ &= 0.324\end{aligned}$$

(c)

$$\begin{aligned}P(X \geq 4) &= P(X=4)+P(X=5)+P(X=6)\\ &= 0.744\end{aligned}$$

(d)

$$\begin{aligned}P(X \leq 4) &= P(X=4)+P(X=3)+...+P(X=0)\\ &= 0.580\end{aligned}$$

8. Suppose that a lot of 5000 electrical fuses contain 5% defectives.
   (a) If a sample of 5 fuses is tested, find the probability of observing at least one defective.
   (b) If $n = 20$ fuses are randomly sampled from this lot, find the probability that at most 6 defectives are observed.
   (c) If $n = 20$ fuses are randomly sampled, find the probability that at least four defectives will be observed.

   We know this event is independent; we have $P(F) = 0.05$ and $P(S) = 0.95$ for 5000 fuses.
   Let X be the number of defectives in n fuses.

   (a) $n = 5$, we want to find $P(X \geq 1)$.
   Using the binomial distribution table, we have:

$$\begin{aligned}P(X \geq 1) &= 1 - P(X \leq 0)\\ &= 1 - 0.774\\ &= 0.226\end{aligned}$$

   (b)

9. Suppose the probability of an engine malfunction during a one-hour period is $p = 0.02$. Find the probability that the engine will survive for two hours.
   We have $p = P(S) = 0.02$, so $q = P(F) = 0.98$. Let Y be the number of one-hour intervals until a malfunction occurs. We have:

$$\begin{aligned}P(y) &= 0.98^{y-1} \cdot 0.02 && \text{For } y = 1, 2, ...\\ &= P(Y \geq 3)\\ &= 1 - P(Y \leq 2)\\ &= 1 - P(Y=1) - P(Y=2)\\ &= 1 - (0.98^1 \cdot 0.02) - (0.98^0 \cdot 0.02)\\ &= 0.9604\end{aligned}$$

10. If the probability of an engine malfunction is $p = 0.02$ and Y is the number of one-hour intervals until the first malfunction, find the mean and standard deviation of Y.

$$\begin{aligned} E(Y) &= \frac{1}{p} \\ &= \frac{1}{0.02} \\ &= 50 V(Y) = \frac{1 - 0.02}{0.02^2} \\ &= 2450 \end{aligned}$$

11. Suppose a professor randomly selects 3 new teaching assistants from a total of 10 applicants - 6 male, 4 female. Let X be the number of females who are hired.

   (a) Find the probability that no females are hired.

   (b) Find the probability that at least 2 females are hired.

   Here, $r = 4$, then $N - r = 6$. Set $n = 3$; we use hypergeometric distribution. First, calculate x:

$$\begin{aligned} \max\{0, n - (N - r)\} &: n - (N - r) \\ &: 3 - 6 = -3 \\ &= 0 \\ \min\{r, n\} &: \min 4, 3 \\ &= 3 \end{aligned}$$

   Hence, $x = 0, 1, 2, 3$.

$$P(X = x) = \frac{\binom{4}{x}\binom{6}{3-x}}{\binom{10}{3}}$$

   (a)

$$\begin{aligned} P(X = 0) &= \frac{\binom{4}{0}\binom{6}{3}}{\binom{10}{3}} \\ &= \frac{6!}{3!3!} \cdot \frac{7!3!}{10!} \\ &= \frac{1}{6} \end{aligned}$$

   (b) We want to find $P(X \geq 2)$, i.e., $1 - P(X = 0) - P(x = 1)$.

$$\begin{aligned} P(X \geq 2 &= 1 - \frac{1}{6} - \frac{\binom{4}{1}\binom{6}{2}}{\binom{10}{3}} \\ &= 1 - \frac{1}{6} - \frac{1}{2} \end{aligned}$$

12. Suppose there are X number of blue whale sightings per week. Assume X follows a Poisson distribution and the number of weekly sightings is 2.5