Contents

Chapter 1

Rules for weak Martin-Löf type theory

1.1 Dependent type theory

1.1.1 Judgments and contexts

Dependent type theory consists of type judgments, term judgments, and context judgments:

$$A \text{ type} \qquad a : A \qquad \Gamma \text{ ctx}$$

Rules to form the empty context and to extend the context by a term judgment:

$$\frac{}{() \text{ ctx}} \qquad \frac{\Gamma \text{ ctx} \quad \Gamma \vdash A \text{ type}}{(\Gamma, a : A) \text{ ctx}}$$

1.1.2 Variable rule

Variable rule:

$$\frac{\Gamma, a : A, \Delta \text{ ctx}}{\Gamma, a : A, \Delta \vdash a : A}$$

1.1.3 Admissible structural rules

Let $\mathcal{J}$ be any arbitrary judgment.

Weakening rule:

$$\frac{\Gamma, \Delta \vdash \mathcal{J} \quad \Gamma \vdash A \text{ type}}{\Gamma, a : A, \Delta \vdash \mathcal{J}}$$

Substitution rule:

$$\frac{\Gamma \vdash a : A \quad \Gamma, b : A, \Delta(b) \vdash \mathcal{J}(b)}{\Gamma, \Delta(a) \vdash \mathcal{J}(a)}$$

1.2 Basic type formers

The basic type formers of dependent type theory are dependent function types, dependent pair types, and identity types.

1.2.1 Formation rules

Formation rules for dependent function types:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B(x) \text{ type}}{\Gamma \vdash \prod_{x:A} .B(x) \text{ type}}$$

Formation rules for dependent pair types:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B(x) \text{ type}}{\Gamma \vdash \sum_{x:A} B(x) \text{ type}}$$

Formation rules for identity types:

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma, a : A, b : A \vdash \mathrm{Id}_A(a, b) \text{ type}}$$

1.2.2 Introduction rules

Introduction rules for dependent function types:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B(x) \text{ type} \quad \Gamma, x : A \vdash b(x) : B(x)}{\Gamma \vdash \lambda^{x:A.b(x)}_{\prod_{x:A} B(x)} : \prod_{x:A} .B(x)}$$

Introduction rules for dependent pair types:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B(x) \text{ type}}{\Gamma, x : A, y : B(x) \vdash \mathrm{pair}^{A,B}_{\sum}(x, y) : \sum_{x:A} B(x)}$$

Introduction rules for identity types:

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma, a : A \vdash \mathrm{refl}_A(a) : \mathrm{Id}_A(a, a)}$$

1.2.3 Elimination rules

Elimination rules for dependent function types:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B(x) \text{ type} \quad \Gamma \vdash f : \prod_{x:A} B(x)}{\Gamma, x : A \vdash f(x) : B(x)}$$

Elimination rules for dependent pair types:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B(x) \text{ type} \quad \Gamma, z : \sum_{x:A} B(x) \vdash C(z) \text{ type}}{\Gamma \vdash \mathrm{ind}^{A,B,C}_{\sum} : \prod_{g:\prod_{x:A} \prod_{y:B(x)} C(\mathrm{pair}^{A,B}_{\sum}(x,y))} \prod_{z:\sum_{x:A} B(x)} C(z)}$$

Dependent elimination rule for identity types:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A, y : A, p : x =_A y \vdash C(x, y, p) \text{ type}}{\Gamma \vdash \mathrm{ind}^{A,C}_{=} : \prod_{t:\prod_{x:A} C(x,x,\mathrm{refl}_A(x))} \prod_{x:A} \prod_{y:A} \prod_{p:x=_A y} C(x, y, p)}$$

1.2.4 Computation rules

Computation rules for dependent function types

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B(x) \text{ type} \quad \Gamma, x : A \vdash b(x) : B(x)}{\Gamma \vdash \beta^{x:A.b(x)}_{\prod_{x:A} B(x)} : \prod_{x:A} \lambda^{x:A.b(x)}_{\prod_{x:A} B(x)}(x) =_{B(x)} b(x)}$$

Computation rules for dependent pair types:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B(x) \text{ type} \quad \Gamma, z : \sum_{x:A} B(x) \vdash C(z) \text{ type}}{\Gamma \vdash \beta_{\sum}^{A,B,C} : \prod_{g:\prod_{x:A}\prod_{y:B(x)} C(\mathrm{pair}_{\sum}^{A,B}(x,y))} \prod_{x:A}\prod_{y:B(x)} \mathrm{ind}_{\sum}^{A,B,C}(g, \mathrm{pair}_{\sum}^{A,B}(x,y)) =_{C(\mathrm{pair}_{\sum}^{A,B}(x,y))} g(x,y)}$$

Computation rules for identity types:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A, y : A, p : x =_A y \vdash C(x,y,p) \text{ type}}{\Gamma \vdash \beta_{\mathrm{ind}_=}^{A,C} : \prod_{t:\prod_{x:A} C(x,x,\mathrm{refl}_A(x))} \prod_{x:A} \mathrm{ind}_{\mathrm{Id}}^{A,C}(t,x,x,\mathrm{refl}_A(x)) =_{C(x,x,\mathrm{refl}_A(x))} t(x)}$$

1.2.5 Uniqueness rules

Uniqueness rules for dependent function types:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B(x) \text{ type}}{\Gamma \vdash \eta_{\prod_{x:A} B(x)} : \prod_{f:\prod_{x:A} B(x)} f =_{\prod_{x:A} B(x)} \lambda_{\prod_{x:A} B(x)}^{x:A.f(x)}}$$

1.2.6 Extensionality rules

Extensionality rules for dependent function types:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B(x) \text{ type}}{\Gamma \vdash \mathrm{homToId}_{\prod_{x:A} B(x)} : \prod_{f:\prod_{x:A} B(x)} \prod_{g:\prod_{x:A} B(x)} \prod_{H:\prod_{x:A} f(x) =_{B(x)} g(x)} f =_{\prod_{x:A} B(x)} g}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B(x) \text{ type}}{\Gamma \vdash \mathrm{ext}_{\prod_{x:A} B(x)} : \begin{array}{l} \prod_{f;\prod_{x:A} B(x)} \prod_{g:\prod_{x:A} B(x)} \prod_{H:\prod_{x:A} f(x)=_{B(x)} g(x)} \\ \mathrm{ind}_=(\lambda f.\lambda x.\mathrm{refl}_{B(x)}(f(x)), f, g, \mathrm{homToId}_{\prod_{x:A} B(x)}(f,g,H)) =_{\prod_{x:A} f(x) =_{B(x)} g(x)} H \end{array}}$$

1.3 Judgmental equality

We add to the type theory two additional judgments called judgmental equality of types and judgmental equality of terms:

$$\Gamma \vdash A \equiv A' \text{ type} \quad \Gamma \vdash a \equiv a' : A$$

1.3.1 Structural rules for judgmental equality

Reflexivity of judgmental equality

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash A \equiv A \text{ type}}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash a : A}{\Gamma \vdash a \equiv a : A}$$

Symmetry of judgmental equality

$$\frac{\Gamma \vdash A \equiv B \text{ type}}{\Gamma \vdash B \equiv A \text{ type}}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash a \equiv b : A}{\Gamma \vdash b \equiv a : A}$$

Transitivity of judgmental equality

$$\frac{\Gamma \vdash A \equiv B \text{ type} \quad \Gamma \vdash B \equiv C \text{ type}}{\Gamma \vdash A \equiv C \text{ type}}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash a \equiv b : A \quad b \equiv c : A}{\Gamma \vdash a \equiv c : A}$$

Substitution of judgmentally equal terms:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash a \equiv b : A \quad \Gamma, x : A, \Delta(x) \vdash B(x) \text{ type}}{\Gamma, \Delta(a) \vdash B(a) \equiv B(b) \text{ type}}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash a \equiv b : A \quad \Gamma, x : A, \Delta(x) \vdash B(x) \text{ type}}{\Gamma, \Delta(b) \vdash B(a) \equiv B(b) \text{ type}}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash a \equiv b : A \quad \Gamma, x : A, \Delta(x) \vdash c(x) : B(x)}{\Gamma, \Delta(a) \vdash c(a) \equiv c(b) : B(a)}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash a \equiv b : A \quad \Gamma, x : A, \Delta(x) \vdash c(x) : B(x)}{\Gamma, \Delta(b) \vdash c(a) \equiv c(b) : B(b)}$$

Judgmental variable conversion rule:

$$\frac{\Gamma \vdash A \equiv B \text{ type} \quad \Gamma, x : A, \Delta \vdash \mathcal{J}}{\Gamma, x : B, \Delta \vdash \mathcal{J}}$$

1.3.2 Judgmental congruence rules for dependent function types

$$\frac{\begin{array}{c}\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B(x) \text{ type} \quad \Gamma \vdash A' \text{ type} \quad \Gamma, x : A' \vdash B'(x) \text{ type} \\ \Gamma \vdash A \equiv A' \text{ type} \quad \Gamma, x : A \vdash B(x) \equiv B'(x) \text{ type}\end{array}}{\Gamma \vdash \prod_{x:A} B(x) \equiv \prod_{x:A'} B'(x) \text{ type}}$$

$$\frac{\begin{array}{c}\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B(x) \text{ type} \quad \Gamma, x : A \vdash b(x) : B(x) \quad \Gamma, x : A \vdash b'(x) : B(x) \\ \Gamma, x : A \vdash b(x) \equiv b'(x) : B(x)\end{array}}{\Gamma \vdash \lambda^{x:A.b(x)}_{\prod_{x:A} B(x)} \equiv \lambda^{x:A.b'(x)}_{\prod_{x:A} B(x)} : \prod_{x:A} .B(x)}$$

$$\frac{\begin{array}{c}\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B(x) \text{ type} \quad \Gamma \vdash f : \prod_{x:A} B(x) \quad f' : \prod_{x:A} B(x) \\ \Gamma \vdash f \equiv f' : \prod_{x:A} B(x)\end{array}}{\Gamma, x : A \vdash f(x) \equiv f'(x) : B(x)}$$

$$\frac{\begin{array}{c}\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B(x) \text{ type} \quad \Gamma, x : A \vdash b(x) : B(x) \quad \Gamma, x : A \vdash b'(x) : B(x) \\ \Gamma, x : A \vdash b(x) \equiv b'(x) : B(x)\end{array}}{\Gamma \vdash \beta^{x:A.b(x)}_{\prod_{x:A} B(x)} \equiv \beta^{x:A.b'(x)}_{\prod_{x:A} B(x)} : \prod_{x:A} b(x) =_{B(x)} \lambda^{x:A.b(x)}_{\prod_{x:A} B(x)}(x)}$$

$$\frac{\begin{array}{c}\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B(x) \text{ type} \quad \Gamma, x : A \vdash B'(x) \text{ type} \\ \Gamma, x : A \vdash B(x) \equiv B'(x) \text{ type}\end{array}}{\Gamma \vdash \eta_{\prod_{x:A} B(x)} \equiv \eta_{\prod_{x:A} B'(x)} : \prod_{f:\prod_{x:A} B(x)} f =_{\prod_{x:A} B(x)} \lambda^{x:A.f(x)}_{\prod_{x:A} B(x)}}$$

$$\frac{\begin{array}{c}\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B(x) \text{ type} \quad \Gamma, x : A \vdash B'(x) \text{ type} \\ \Gamma, x : A \vdash B(x) \equiv B'(x) \text{ type}\end{array}}{\Gamma \vdash \mathrm{homToId}_{\prod_{x:A} B(x)} \equiv \mathrm{homToId}_{\prod_{x:A} B'(x)} : \prod_{f;\prod_{x:A} B(x)} \prod_{g:\prod_{x:A} B(x)} \left(\prod_{x:A} f(x) =_{B(x)} g(x)\right) \to (f =_{\prod_{x:A} B(x)} g)}$$

$$\frac{\begin{array}{c}\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B(x) \text{ type} \quad \Gamma, x : A \vdash B'(x) \text{ type} \\ \Gamma, x : A \vdash B(x) \equiv B'(x) \text{ type}\end{array}}{\Gamma \vdash \mathrm{ext}_{\prod_{x:A} B(x)} \equiv \mathrm{ext}_{\prod_{x:A} B'(x)} : \begin{array}{l}\prod_{f:\prod_{x:A} B(x)} \prod_{g:\prod_{x:A} B(x)} \prod_{H:\prod_{x:A} f(x) =_{B(x)} g(x)} \\ \mathrm{rec}_{=}(\lambda f.\lambda x.\mathrm{refl}_{B(x)}(f(x)), f, g, \mathrm{homToId}_{\prod_{x:A} B(x)}(f, g, H)) =_{\prod_{x:A} f(x) =_{B(x)} g(x)} H\end{array}}$$

1.3.3 Judgmental congruence rules for dependent pair types

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B(x) \text{ type} \quad \Gamma \vdash A' \text{ type} \quad \Gamma, x : A' \vdash B'(x) \text{ type} \atop \Gamma \vdash A \equiv A' \text{ type} \quad \Gamma, x : A \vdash B(x) \equiv B'(x) \text{ type}}{\Gamma \vdash \sum_{x:A} B(x) \equiv \sum_{x:A'} B'(x) \text{ type}}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B(x) \text{ type} \quad \Gamma \vdash A' \text{ type} \quad \Gamma, x : A' \vdash B'(x) \text{ type} \atop \Gamma \vdash A \equiv A' \text{ type} \quad \Gamma, x : A \vdash B(x) \equiv B'(x) \text{ type}}{\Gamma, x : A, y : B(x) \vdash \mathrm{pair}_{\Sigma}^{A,B} \equiv \mathrm{pair}_{\Sigma}^{A',B'} : \sum_{x:A} B(x)}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B(x) \text{ type} \quad \Gamma, z : \sum_{x:A} B(x) \vdash C(z) \text{ type} \atop \Gamma, z : \sum_{x:A} B(x) \vdash C'(z) \text{ type} \quad \Gamma, z : \sum_{x:A} B(x) \vdash C(z) \equiv C'(z) \text{ type}}{\Gamma \vdash \mathrm{ind}_{\Sigma}^{A,B,C} \equiv \mathrm{ind}_{\Sigma}^{A,B,C'} : \prod_{g:\prod_{x:A}\prod_{y:B(x)} C(\mathrm{pair}_{\Sigma}^{A,B}(x,y))} \prod_{z:\sum_{x:A} B(x)} C(z)}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B(x) \text{ type} \quad \Gamma, z : \sum_{x:A} B(x) \vdash C(z) \text{ type} \atop \Gamma, z : \sum_{x:A} B(x) \vdash C'(z) \text{ type} \quad \Gamma, z : \sum_{x:A} B(x) \vdash C(z) \equiv C'(z) \text{ type}}{\Gamma \vdash \beta_{\Sigma}^{A,B,C} \equiv \beta_{\Sigma}^{A,B,C'} : \prod_{g:\prod_{x:A}\prod_{y:B(x)} C(\mathrm{pair}_{\Sigma}^{A,B}(x,y))} \prod_{x:A}\prod_{y:B(x)} \mathrm{ind}_{\Sigma}^{A,B,C}(g, \mathrm{pair}_{\Sigma}^{A,B}(x,y)) =_{C(\mathrm{pair}_{\Sigma}^{A,B}(x,y))} g(x,y)}$$

1.3.4 Judgmental congruence rules for identity types

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash A' \text{ type} \quad \Gamma \vdash A \equiv A' \text{ type}}{\Gamma, x : A, y : A \vdash x =_A y \equiv x ='_A y}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash A' \text{ type} \quad \Gamma \vdash A \equiv A' \text{ type}}{\Gamma \vdash \mathrm{refl}_A \equiv \mathrm{refl}_{A'} : \prod_{x:A} x =_A x}$$

$$\frac{\Gamma \vdash A \equiv A' \text{ type} \quad \Gamma, x : A, y : A, p : x =_A y \vdash C(x,y,p) \text{ type} \quad \Gamma, x : A', y : A', p : x =_{A'} y \vdash C'(x,y,p) \text{ type} \atop \Gamma, x : A, y : A, p : x =_A y \vdash C(x,y,p) \equiv C'(x,y,p) \text{ type}}{\Gamma \vdash \mathrm{ind}_{=}^{A,C} \equiv \mathrm{ind}_{=}^{A',C'} : \prod_{t:\prod_{x:A} C(x,x,\mathrm{refl}_A(x))} \prod_{x:A}\prod_{y:A}\prod_{p:x=_A y} C(x,y,p)}$$

$$\frac{\Gamma \vdash A \equiv A' \text{ type} \quad \Gamma, x : A, y : A, p : \mathrm{Id}_A(x,y) \vdash C(x,y,p) \text{ type} \quad \Gamma, x : A', y : A', p : x =_{A'} y \vdash C'(x,y,p) \text{ type} \atop \Gamma, x : A, y : A, p : x =_A y \vdash C(x,y,p) \equiv C'(x,y,p) \text{ type}}{\Gamma \vdash \beta_{\mathrm{ind}_=}^{A,C} \equiv \beta_{\mathrm{ind}_=}^{A',C'} : \prod_{t:\prod_{x:A} C(x,x,\mathrm{refl}_A(x))} \prod_{x:A} \mathrm{ind}_{=}^{A,C}(t,x,x,\mathrm{refl}_A(x)) =_{C(x,x,\mathrm{refl}_A(x))} t(x)}$$

1.4 Equivalence types

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash \mathrm{isContr}(A) \text{ type}} \qquad \frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash \mathrm{isContr}(A) \equiv \sum_{x:A}\prod_{y:A} x =_A y \text{ type}}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B(x) \text{ type}}{\Gamma \vdash \exists! x : A.B(x) \text{ type}} \qquad \frac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B(x) \text{ type}}{\Gamma \vdash \exists! x : A.B(x) \equiv \mathrm{isContr}\left(\sum_{x:A} B(x)\right) \text{ type}}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \simeq B \text{ type}} \qquad \frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \simeq B \equiv \sum_{f:\prod_{x:A} B}\prod_{y:B} \exists! x : A.f(x) =_B y \text{ type}}$$

1.5 Empty type

Formation rule for the empty type

$$\frac{\Gamma \text{ ctx}}{\Gamma \vdash \emptyset \text{ type}}$$

Elimination rule for the empty type:

$$\frac{\Gamma, x : \emptyset \vdash C(x) \text{ type}}{\Gamma \vdash \mathrm{ind}_{\emptyset}^{C} : \prod_{x:\emptyset} C(x) \text{ type}}$$

Judgmental congruence rules for the empty type:

$$\frac{\Gamma, x : \emptyset \vdash C(x) \equiv C'(x) \text{ type}}{\Gamma \vdash \mathrm{ind}_{\emptyset}^{C} \equiv \mathrm{ind}_{\emptyset}^{C'} : \prod_{x:\emptyset} C(x) \text{ type}}$$

1.6 Unit type

$$\frac{\Gamma \text{ ctx}}{\Gamma \vdash \mathbb{1} \text{ type}} \qquad \frac{\Gamma \text{ ctx}}{\Gamma \vdash \mathbb{1} \equiv \prod_{x:\emptyset} \emptyset \text{ type}}$$

1.7 Type of booleans

Formation rule for the type of booleans

$$\frac{\Gamma \text{ ctx}}{\Gamma \vdash \mathbb{2} \text{ type}}$$

Introduction rule for the type of booleans

$$\frac{\Gamma \text{ ctx}}{\Gamma \vdash 0 : \mathbb{2}} \qquad \frac{\Gamma \text{ ctx}}{\Gamma \vdash 1 : \mathbb{2}}$$

Elimination rules for the type of booleans:

$$\frac{\Gamma, x : \mathbb{2} \vdash C(x) \text{ type}}{\Gamma \vdash \mathrm{ind}_{\mathbb{2}}^{C} : \prod_{a:C(0)} \prod_{b:C(1)} \prod_{x:\mathbb{2}} C(x)}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma, x : \mathbb{2} \vdash \mathrm{typerec}_{\mathbb{2}}^{A,B}(x) \text{ type}}$$

Computation rules for the type of booleans:

$$\frac{\Gamma, x : \mathbb{2} \vdash C(x) \text{ type}}{\Gamma \vdash \beta_{\mathbb{2}}^{0,C} : \prod_{a:C(0)} \prod_{b:C(1)} \mathrm{ind}_{\mathbb{2}}^{C}(a, b, 0) =_{C(0)} a}$$

$$\frac{\Gamma, x : \mathbb{2} \vdash C(x) \text{ type}}{\Gamma \vdash \beta_{\mathbb{2}}^{1,C} : \prod_{a:C(0)} \prod_{b:C(1)} \mathrm{ind}_{\mathbb{2}}^{C}(a, b, 1) =_{C(1)} b}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash \beta_{\mathbb{2}}^{0,A,B} : \mathrm{typerec}_{\mathbb{2}}^{A,B}(0) \simeq A}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash \beta_{\mathbb{2}}^{1,A,B} : \mathrm{typerec}_{\mathbb{2}}^{A,B}(1) \simeq B}$$

Extensionality rule for the type of booleans:

$$\frac{\Gamma \text{ ctx}}{\Gamma \vdash \mathrm{ext}_{\mathbb{2}} : \prod_{x:\mathbb{2}} \prod_{y:\mathbb{2}} (x =_{\mathbb{2}} y) \simeq (\mathrm{typerec}_{\mathbb{2}}^{\emptyset,\mathbb{1}}(x) \simeq \mathrm{typerec}_{\mathbb{2}}^{\emptyset,\mathbb{1}}(y))}$$

Judgmental congruence rules for the type of booleans:

$$\frac{\Gamma, x : 2 \vdash C(x) \equiv C'(x) \text{ type}}{\Gamma \vdash \mathrm{ind}_2^C \equiv \mathrm{ind}_2^{C'} : \prod_{a:C(0)} \prod_{b:C(1)} \prod_{x:2} C(x)}$$

$$\frac{\Gamma \vdash A \equiv A' \text{ type} \quad \Gamma \vdash B \equiv B' \text{ type}}{\Gamma, x : 2 \vdash \mathrm{typerec}_2^{A,B}(x) \equiv \mathrm{typerec}_2^{A',B'}(x) \text{ type}}$$

$$\frac{\Gamma, x : 2 \vdash C(x) \equiv C'(x) \text{ type}}{\Gamma \vdash \beta_2^{0,C} \equiv \beta_2^{0,C'} : \prod_{a:C(0)} \prod_{b:C(1)} \mathrm{ind}_2^C(a, b, 0) =_{C(0)} a}$$

$$\frac{\Gamma, x : 2 \vdash C(x) \equiv C'(x) \text{ type}}{\Gamma \vdash \beta_2^{1,C} \equiv \beta_2^{1,C'} : \prod_{a:C(0)} \prod_{b:C(1)} \mathrm{ind}_2^C(a, b, 1) =_{C(1)} b}$$

$$\frac{\Gamma \vdash A \equiv A' \text{ type} \quad \Gamma \vdash B \equiv B' \text{ type}}{\Gamma \vdash \beta_2^{0,A,B} \equiv \beta_2^{0,A',B'} : \mathrm{typerec}_2^{A,B}(0) \simeq A}$$

$$\frac{\Gamma \vdash A \equiv A' \text{ type} \quad \Gamma \vdash B \equiv B' \text{ type}}{\Gamma \vdash \beta_2^{1,A,B} \equiv \beta_2^{1,A',B'} : \mathrm{typerec}_2^{A,B}(1) \simeq B}$$

1.8 Sum types

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A + B \text{ type}}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A + B \equiv \sum_{x:2} \mathrm{typerec}_2^{A,B}(x) \text{ type}}$$

1.9 Product types

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \times B \text{ type}}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \times B \equiv \prod_{x:2} \mathrm{typerec}_2^{A,B}(x) \text{ type}}$$

1.10 Function types

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \to B \text{ type}}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \to B \equiv \prod_{x:A} B \text{ type}}$$

1.11 Type of propositions

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash \mathrm{isProp}(A) \text{ type}} \qquad \frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash \mathrm{isProp}(A) \equiv \prod_{x:A} \prod_{y:A} x =_A y \text{ type}}$$

Formation rules for the type of propositions:

$$\frac{\Gamma \text{ ctx}}{\Gamma \vdash \mathrm{Prop} \text{ type}}$$

Introduction rule for the type of propositions:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash \mathrm{proptrunc}_A : \mathrm{isProp}(A)}{\Gamma \vdash A : \mathrm{Prop}}$$

Elimination rules for the type of propositions:

$$\frac{\Gamma \vdash A : \mathrm{Prop}}{\Gamma \vdash A \text{ type}} \qquad \frac{\Gamma \vdash A \equiv A' : \mathrm{Prop}}{\Gamma \vdash A \equiv A' \text{ type}}$$

$$\frac{\Gamma \text{ ctx}}{\Gamma \vdash \mathrm{proptrunc} : \prod_{A:\mathrm{Prop}} \mathrm{isProp}(A)}$$

Extensionality rules for the type of propositions:

$$\frac{\Gamma \text{ ctx}}{\Gamma \vdash \mathrm{ext}_{\mathrm{Prop}} : \prod_{A:\mathrm{Prop}} \prod_{B:\mathrm{Prop}} (A =_{\mathrm{Prop}} B) \simeq (A \simeq B)}$$

1.12 Predicate logic

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash [A] \text{ type}} \qquad \frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash [A] \equiv \prod_{P:\mathrm{Prop}} (A \to P) \to P \text{ type}}$$

$$\frac{\Gamma \text{ ctx}}{\Gamma \vdash \bot \text{ type}} \qquad \frac{\Gamma \text{ ctx}}{\Gamma \vdash \bot \equiv [\emptyset] \text{ type}}$$

$$\frac{\Gamma \text{ ctx}}{\Gamma \vdash \top \text{ type}} \qquad \frac{\Gamma \text{ ctx}}{\Gamma \vdash \top \equiv [\mathbb{1}] \text{ type}}$$

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash \neg A \text{ type}} \qquad \frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash \neg A \equiv [A \to \emptyset] \text{ type}}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \vee B \text{ type}} \qquad \frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \vee B \equiv [A + B] \text{ type}}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \wedge B \text{ type}} \qquad \frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \wedge B \equiv [A \times B] \text{ type}}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \implies B \text{ type}} \qquad \frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \implies B \equiv [A \to B] \text{ type}}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \iff B \text{ type}} \qquad \frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \iff B \equiv [A \simeq B] \text{ type}}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B(x) \text{ type}}{\Gamma \vdash \exists x : A.B(x) \text{ type}} \qquad \frac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B(x) \text{ type}}{\Gamma \vdash \exists x : A.B(x) \equiv [\sum_{x:A} B(x)] \text{ type}}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B(x) \text{ type}}{\Gamma \vdash \forall x : A.B(x) \text{ type}} \qquad \frac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B(x) \text{ type}}{\Gamma \vdash \forall x : A.B(x) \equiv [\prod_{x:A} B(x)] \text{ type}}$$

1.13 Quotient sets

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash \mathrm{EquivRel}(A) \text{ type}}$$

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash \mathrm{EquivRel}(A) \equiv \sum_{R:A \times A \to \mathrm{Prop}} \prod_{x:A} R(x,x) \times \prod_{y:A} (R(x,y) \to R(y,x)) \times \prod_{z:A} (R(x,y) \times R(y,z)) \to R(x,z) \text{ type}}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash R : \mathrm{EquivRel}(A)}{\Gamma \vdash A/R \text{ type}}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash R : \mathrm{EquivRel}(A)}{\Gamma \vdash A/R \equiv \sum_{P:A \to \mathrm{Prop}} \exists x : A.\forall y : A.P(x) =_{\mathrm{Prop}} \pi_1(R)(x,y) \text{ type}}$$

1.14 Excluded middle and axiom of choice

$$\frac{\Gamma\ \mathrm{ctx}}{\Gamma \vdash \mathrm{lem} : \prod_{P:\mathrm{Prop}} P + \neg P}$$

$$\frac{\Gamma \vdash A\ \mathrm{type}}{\Gamma \vdash \mathrm{isSet}(A)\ \mathrm{type}} \qquad \frac{\Gamma \vdash A\ \mathrm{type}}{\Gamma \vdash \mathrm{isSet}(A) \equiv \prod_{x:A}\prod_{y;A} \mathrm{isProp}(x =_A y)\ \mathrm{type}}$$

$$\frac{\Gamma \vdash A\ \mathrm{type} \quad \Gamma, x : A \vdash B(x)\ \mathrm{type} \quad \Gamma, x : A, y : B(x) \vdash C(x,y)\ \mathrm{type}}{\Gamma \vdash \mathrm{choice}_{A,B,C} : (\mathrm{isSet}(A) \times \prod_{x:A} \mathrm{isSet}(B(x))) \to \forall x : A.\exists y : B(x).C(x,y) \to \exists g : \prod_{x:A} B(x).\forall x : A.C(x, g(x))}$$

$$\frac{\Gamma \vdash A \equiv A'\ \mathrm{type} \quad \Gamma, x : A \vdash B(x) \equiv B'(x)\ \mathrm{type} \quad \Gamma, x : A, y : B(x) \vdash C(x,y) \equiv C'(x,y)\ \mathrm{type}}{\Gamma \vdash \mathrm{choice}_{A,B,C} \equiv \mathrm{choice}_{A',B',C'} : \begin{array}{c}(\mathrm{isSet}(A) \times \prod_{x:A} \mathrm{isSet}(B(x))) \to \\ \forall x : A.\exists y : B(x).C(x,y) \to \exists g : \prod_{x:A} B(x).\forall x : A.C(x, g(x))\end{array}}$$

1.15 Natural numbers type

Formation rules for the natural numbers type:

$$\frac{\Gamma\ \mathrm{ctx}}{\Gamma \vdash \mathbb{N}\ \mathrm{type}}$$

Introduction rules for the natural numbers type:

$$\frac{\Gamma\ \mathrm{ctx}}{\Gamma \vdash 0 : \mathbb{N}} \qquad \frac{\Gamma\ \mathrm{ctx}}{\Gamma \vdash s : \mathbb{N} \to \mathbb{N}}$$

Elimination rules for the natural numbers type:

$$\frac{\Gamma, x : \mathbb{N} \vdash C(x)\ \mathrm{type}}{\Gamma \vdash \mathrm{ind}_{\mathbb{N}}^{C} : \prod_{c_0 : C(0)} \prod_{c_s : \prod_{x:\mathbb{N}} C(x) \to C(s(x))} \prod_{x:\mathbb{N} C(x)}}$$

Computation rules for the natural numbers type:

$$\frac{\Gamma, x : \mathbb{N} \vdash C(x)\ \mathrm{type}}{\Gamma \vdash \beta_{\mathbb{N}}^{0,C} : \prod_{c_0 : C(0)} \prod_{c_s : \prod_{x:\mathbb{N}} C(x) \to C(s(x))} \mathrm{ind}_{\mathbb{N}}^{C}(c_0, c_s, 0) =_{C(0)} c_0}$$

$$\frac{\Gamma, x : \mathbb{N} \vdash C(x)\ \mathrm{type}}{\Gamma \vdash \beta_{\mathbb{N}}^{s,C} : \prod_{c_0 : C(0)} \prod_{c_s : \prod_{x:\mathbb{N}} C(x) \to C(s(x))} \prod_{x:\mathbb{N}} \mathrm{ind}_{\mathbb{N}}^{C}(c_0, c_s, s(x)) =_{C(s(x))} c_s(x)(\mathrm{ind}_{\mathbb{N}}^{C}(c_0, c_s, x))}$$

Judgmental congruence rules for the natural numbers type:

$$\frac{\Gamma, x : \mathbb{N} \vdash C(x) \equiv C'(x)\ \mathrm{type}}{\Gamma \vdash \mathrm{ind}_{\mathbb{N}}^{C} \equiv \mathrm{ind}_{\mathbb{N}}^{C'} : \prod_{c_0 : C(0)} \prod_{c_s : \prod_{x:\mathbb{N}} C(x) \to C(s(x))} \prod_{x:\mathbb{N} C(x)}}$$

$$\frac{\Gamma, x : \mathbb{N} \vdash C(x) \equiv C'(x)\ \mathrm{type}}{\Gamma \vdash \beta_{\mathbb{N}}^{0,C} \equiv \beta_{\mathbb{N}}^{0,C'} : \prod_{c_0 : C(0)} \prod_{c_s : \prod_{x:\mathbb{N}} C(x) \to C(s(x))} \mathrm{ind}_{\mathbb{N}}^{C}(c_0, c_s, 0) =_{C(0)} c_0}$$

$$\frac{\Gamma, x : \mathbb{N} \vdash C(x) \equiv C'(x)\ \mathrm{type}}{\Gamma \vdash \beta_{\mathbb{N}}^{s,C} \equiv \beta_{\mathbb{N}}^{s,C'} : \prod_{c_0 : C(0)} \prod_{c_s : \prod_{x:\mathbb{N}} C(x) \to C(s(x))} \prod_{x:\mathbb{N}} \mathrm{ind}_{\mathbb{N}}^{C}(c_0, c_s, s(x)) =_{C(s(x))} c_s(x)(\mathrm{ind}_{\mathbb{N}}^{C}(c_0, c_s, x))}$$

Extensionality principle of the natural numbers type:

$$\frac{\Gamma \text{ ctx}}{\Gamma \vdash \mathrm{Eq}_{\mathbb{N}} : \mathbb{N} \times \mathbb{N} \to \mathbb{2}}$$

$$\frac{\Gamma \text{ ctx}}{\Gamma \vdash \beta_{\mathbb{N}}^{0,0,\mathbb{2}} : \mathrm{Eq}_{\mathbb{N}}(0,0) =_{\mathbb{2}} 1}$$

$$\frac{\Gamma \text{ ctx}}{\Gamma \vdash \beta_{\mathbb{N}}^{0,s,\mathbb{2}} : \prod_{x:\mathbb{N}} \mathrm{Eq}_{\mathbb{N}}(0, s(x)) =_{\mathbb{2}} 0}$$

$$\frac{\Gamma \text{ ctx}}{\Gamma \vdash \beta_{\mathbb{N}}^{s,0,\mathbb{2}} : \prod_{x:\mathbb{N}} \mathrm{Eq}_{\mathbb{N}}(s(x), 0) =_{\mathbb{2}} 0}$$

$$\frac{\Gamma \text{ ctx}}{\Gamma \vdash \beta_{\mathbb{N}}^{s,s,\mathbb{2}} : \prod_{x:\mathbb{N}} \prod_{y:\mathbb{N}} \mathrm{Eq}_{\mathbb{N}}(s(x), s(y)) =_{\mathbb{2}} \mathrm{Eq}_{\mathbb{N}}(x, y)}$$

$$\frac{\Gamma \text{ ctx}}{\Gamma \vdash \mathrm{ext}_{\mathbb{N}} : \prod_{x:\mathbb{N}} \prod_{y:\mathbb{N}} (x =_{\mathbb{N}} y) \simeq \mathrm{typerec}_{\mathbb{2}}^{\emptyset,\mathbb{1}}(\mathrm{Eq}_{\mathbb{N}}(x, y))}$$