Beyond the Circle: Measuring District Compactness using Graph Theory

Conference Paper · November 2016

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Dube, M.P., Clark, J.T. Beyond the Circle: Measuring District Compactness Using Graph Theory. Northeast Political Science Association Conference - Boston, MA (2016). November 10-12, 2016.

Beyond the Circle:
Measuring District Compactness Using Graph Theory

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Abstract

For decades, legislative districts have been drawn with two general mandates, contiguity and compactness. While contiguity is easily discernible, a single accepted measure of district compactness has not been found. Due to the various effects that gerrymandering and noncompact districts have on the aggregation of voter preferences, questions surrounding district compactness have direct implications of representation in democratic theory. Empirical political science has attempted on numerous occasions to formulate a single method by which to measure district compactness. These include various geometric measures such as circularity, convex hulls, and path connectedness, as well as methods that attempt to encompass voter dispersion. The real-world implementation of these compactness measures has been fraught with issues, however. Most apparent among these are natural and state boundaries, which greatly interfere with the implementation of geometric compactness measures. Examples of noncompactness induced by natural boundaries can be seen in the coast of Maine and the Chesapeake Bay in Maryland, while noncompactness induced by state boundaries can be seen in West Virginia and Louisiana. Further implementation issues may be found in methods that rely on voter dispersion. This is due to the boundaries that are used to collect apportionment data, namely census tracts, blocks, and precincts. U.S. Census Bureau delineations (tracts, blocks, and precincts) are the spatial granularity by which legislative districts are drawn, and have rigid boundaries. These boundaries not only proscribe the full use of most voter dispersion metrics of compactness, but also lead districts to be naturally conducive to irregular geometry. In this paper, we seek to overcome these geometric issues by using a different discipline of mathematics—graph theory—to formulate a new metric of district

compactness. This method utilizes graph partitioning with the established goal of balancing the population while minimizing the number of shapes that share an edge with one another. In doing so we formulate a new approach to compactness that is more reflective of the redistricting process, and overcomes traditional issues surrounding natural boundaries, disconnects, and population distribution.

Introduction

Since the original gerrymander was implemented by Elbridge Gerry in 1812, gerrymandering has been an issue of severe contention in the political arena. This is often done through the use of geometrically non-compact shapes as districts, by drawing districts around target demographics to achieve a desired result. With advances in mapmaking through the use of advanced proprietary geographic information system (GIS) software and subsequent advances in spatio-demographic modeling, redistricting and gerrymandering have become considerably more precise (Monmonier 2001). The use of these advanced techniques has at times run into conflict with the general mandate of redistricting- that districts be both compact and contiguous- spurring a voluminous literature on the redistricting process. While gerrymandering and redistricting as a whole have been examined from many different angles, we focus this paper around the issue of district shape and compactness. It has been shown that congressional districts have become considerably less compact in the past fifty years and longer (Hill 2011; Ansolabehere and Palmer 2015). It has also been shown that the shape of these districts effects voting knowledge, with voters in less compact districts showing reduced knowledge of voting and elected officials (Engstrom 2002; Hill 2011). Beyond the impact of compactness on voting behavior, issues of compactness in the redistricting process have often been raised in political discourse, and attempts have even been made to use such metrics as a way to measure gerrymandering (Graham 2014). This research is conflicting, as some researchers say that compactness inhibits gerrymandering (Hill 2011) while others say the use of compactness standard lowers minority representation (Barabas and Jerit 2004).

But what exactly is meant by "compactness?" While the individual metrics that have been used to measure compactness have varied, they all rely on geometric comparison. Reock (1961) synthesized a measure that compares the area of a district to the area of its minimum-bounding circle, essentially serving as a measure of circularity. Similarly, convex hulls have been used to compare the area of a district to the area of its minimum-bounding polygon (McGlone and Cheetham 2012.) Shwartzberg (1965) and Polsby-Popper (1991) give us methods to measure congressional district compactness by making comparisons between congressional districts and corresponding circles using area/perimeter ratios. While these are mathematically equivalent, the Schwartzberg measure compares the area/perimeter ratio of a congressional district to the perimeter of a circle with identical area, while the Polsby-Popper measure compares the perimeter/area ratio of a congressional district to a circle with the corresponding perimeter length.

The first major problem with these compactness metrics that we encounter is that they are not compatible, insomuch that districts that are compact to one measure are noncompact to another. For example, an elongated rectangle would be perfectly compact according to the convex hull measure, but would be noncompact compared to the Swartzberg/Polsby-Popper measure. This is an important distinction, as many states, such as Kansas, have sharp corners on their boundaries that could make the state problematic for examination using all of these compactness measures.

The second major problem that can be found with these measures of compactness is that they cannot be properly compared between states. While it my be demonstrated that one district in the interior of Illinois is more geometrically compact than a district on the coast of Maine, is this distinction actionable? We argue that it is not, for two reasons. First, Maine has natural boundaries and disconnects in the form of Islands and bays, giving it a naturally noncompact shape. As such, the floor to the optimal compactness of Maine is inherently much higher than the floor to the optimal compactness of a district in the interior of Illinois. Niemi et al (1990) seemingly provide us with an answer to this comparison problem, claiming that multiple measures of compactness are needed to fully analyze the compactness of congressional districts. This is where we encounter the second problem; none of the previously mentioned compactness measures takes into account how districts are actually drawn in practice. Unlike shapes, districts are not simply drawn onto a map. Instead, they are aggregations of units of space in the form of census tracts, blocks, and precincts. This is important because these borders are asymmetric and rigid; they are often oddly shaped, and cannot be easily changed. As a result, census tracts do not naturally form geometrically compact shapes very often, making geometric compactness measures tenuous at best. Instead, we argue that the only way to adequately measure compactness of congressional districts is by measuring the number of connections between adjacent census tracts on the border of districts. In order to do this, we must rely on a different field of mathematics, moving beyond geometry to graph theory. In doing this, we will create a compactness measure that emphasizes minimum connection between districts, thus providing a measure of compactness that is not only comparable between states but one that also reflects the manner in which districts are created in practice.

Edge Compactness

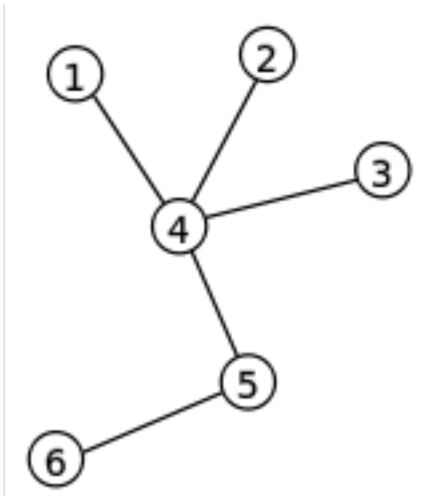
While every state has its own standard for drawing congressional districts, it is universal that every district in a state must be contiguous and have population within a specified percentage of equality in population size. This problem can be reduced to a problem in discrete mathematics and graph theory. The following definitions (Def. 1-7) provide the mathematical context for this work. This section provides both the definitions and the rationale for why they are necessary.

Given the necessity of contiguity, a structure that models the adjacency within a map is required. That structure is called a *graph* in discrete mathematics (Def. 1), distinct from graphs in other areas of mathematics. All graphs in discrete mathematics are constructed from a set of vertices and a set of edges. Depending upon the type of graph, the order in which vertices are connected is important, leading to the concepts of *origin* and *destination.* (Def. 2). Some graphs require directional information, while others do not. In the case of this study, directional information is not of particular usage; therefore, the domain of this project refers to *undirected graphs* (Def. 3). Figure 1 consolidates these definitions into a single graphic portraying the information contained.

Definition 1: Let V be a set of vertices $v_1,...,v_n$ and further let E be a set of edges $e_1,...,e_m$ of the form (a,b), where $a,b \in V$. The combination of the sets V and E form a **graph** $G_{V,E}$ (Hein, 1996).

Definition 2: Let e be an edge of the form (a,b). a is called the **origin** of the edge, and b is called the **destination** of the edge.

Definition 3: Let $G_{V,E}$ be a graph. If the origin and destination of each edge e_i has no functional consequence upon traversal, then $G_{V,E}$ is called **undirected**.



Vertices	Edges
1	(1,4)
2	(2,4)
3	(3,4)
4	(4,5)
5	(5,6)
6	

Figure 1: An undirected graph (Def. 3) on vertices {1, 2, 3, 4, 5, 6} and edges {(1,4), (2,4), (3,4), (4,5), (5,6)}.

Given that each district must belong to a single state and must be connected, a structure that can be extracted from a graph while maintaining all edge information is necessary to automatically construct a district. This structure is called an *induced subgraph* (Def. 4).

Definition 4: Let $G_{V,E}$ be a graph. Let $S_{N,C}$ also be a graph such that $N \subset V$ and $C \subset E$. $S_{N,C}$ is called a **subgraph** of $G_{V,E}$. If every $c_i \in C$ connects $a,b \in N$ and $E \backslash C$ does not contain any edges such that $a,b \in N$, $S_{N,C}$ is called an **induced subgraph**.

Districts, however, must represent a special case of induced subgraphs. These subgraphs must be *path-connected* (Def. 5), modeling the requirement of a contiguous district. For the purposes of redistricting, it is possible that two census tracts may form adjacency at a single point or at a finite number of points (Dube et al, 2015). This notion of contiguity is disallowed by redistricting laws. As such, each graph representing a state must remove all edges formed in this manner. Similarly, a state may have natural disconnections such as islands (Mount Desert Island in Maine) or peninsulas (Upper Peninsula of Michigan). Such natural disconnections must be appended as connections to create proper districts in the event that the area itself cannot solely comprise a set of complete districts.

Definition 5: Let $G_{V,E}$ be a graph. Consider $a,b \in V$. a and b are called **path-connected** if there exists a set of edges $C \in \mathrm{E}$ such that c_1 has origin a, c_n has destination b, and for every c_i and c_{i+1} the destination of c_i is the origin of c_{i+1} (Hein 1996).

Beyond the spatial problem, a district also has a population requirement. As such, each individual census tract must maintain knowledge of its population (and other demographics). This necessity forces the use of a system called *vertex weights* (Def. 6). A vertex can have numerous types of weights stored at the same time, such as total population, gender information, and other pertinent information.

Definition 6: Let $G_{V,E}$ be a graph. If each vertex v_i has an associated value to maintain for further mathematical analysis, then $G_{V,E}$ is called **vertex-weighted** (Hein 1996).

The last requirement of redistricting from a mathematical perspective is that each district in a state must have a population within a specified margin of error. Typically, that acceptable margin of error is on the order of +/- 3%. To subdivide a state into districts, the final mathematical piece necessary is that of the *k-way partition* (Def. 7). A k-way partition splits a graph into k parts. If those parts are of relatively equal weight, the k-way partition is referred to as *balanced*.

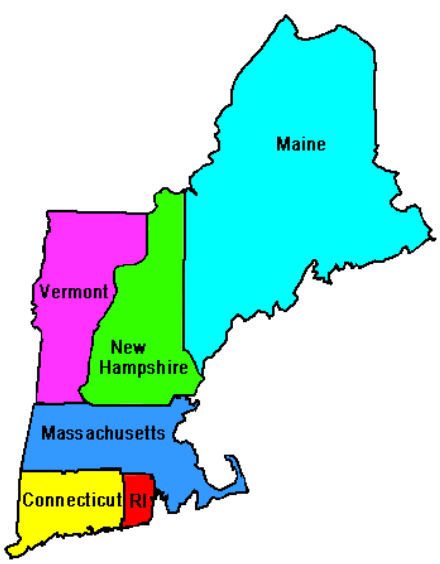
Definition 7: Let $G_{V,E}$ be a vertex-weighted, undirected graph, and further let D be a collection of k induced subgraphs of $G_{V,E}$ such that the vertex set of each D is mutually exclusive and together are jointly exhaustive of V. D is called a ***k*-way partition** of $G_{V,E}$. If each induced subgraph has total vertex weight within a specified tolerance t of all other induced subgraphs in D, then D is referred to as **balanced** (Hein 1996; Karypis and Kumar 1999).

Thus, the problem of constructing artificial districts is solved by constructing a balanced k-way partition such that each individual partition is a path-connected, induced subgraph of the state. This properly produces contiguous districts as well as districts of relatively equal populations. k-way partitioning algorithms such as *gpMetis* (Karypis and Kumar 1999) provide the option to reduce the number of edges that originate in one district and end in another, a concept we refer to as *edge-cut compactness*. This notion simulates the mathematical idea of compactness independent of the geometric lines that are often defined by physical environmental barriers or superimposed fiat objects (Edwards 1971; Taylor 1973; Young 1988; Smith, 2001; Lunday 2014).

Implementation

The data for this project came primarily from the United States Census Bureau, in two forms. The first data set is a map of all census tracts in the United States in shapefile format (.shp). The second data set consists of key demographic data collected by the American Community Survey (ACS) in 2013, in geodatabase format (.gdb) (U.S. Census Bureau 2016).

The census tract map was imported into ArcMap 10.3.1, a software suite to digitally process map information, to gather adjacency information between the census tracts using the *Polygon Neighbors* tool. Tracts adjacent at only a finite number of points were removed from the adjacency list (Dube et al. 2015). Additionally, all islands and otherwise isolated features are connected at the most logical point to the remainder of the map, such as by ferry terminals or bridges identified in Google Maps (Google 2016). The output of the Polygon Neighbors tool is an adjacency matrix that contains the connectivity information from the map (Fig. 2).



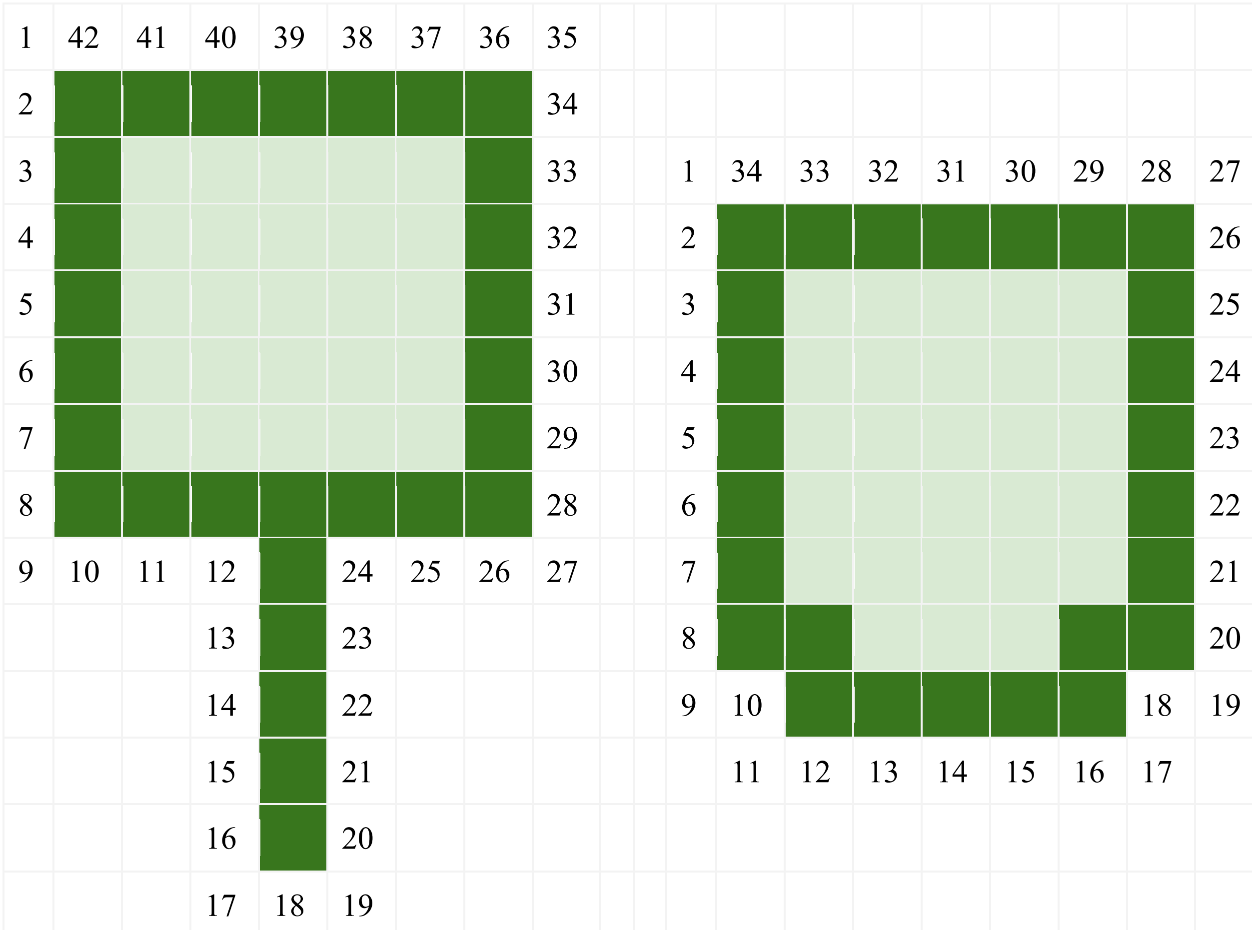
	ME	NH	VT	MA	CT	RI
ME	0	1	0	0	0	0
NH	1	0	1	1	0	0
VT	0	1	0	1	0	0
MA	0	1	1	0	1	1
CT	0	0	0	1	0	1
RI	0	0	0	1	1	0

Figure 2. New England and its corresponding adjacency matrix, the foundational tool for generating artificial districts upon census tracts using the Tiger Line shapefiles.

Using the adjacency matrix for each state and the corresponding population vector for those census tracts, *gpMetis* was used to generate a simulated redistricting map for each state. This algorithm is designed to make decisions based on the number of edge-cuts present. After concluding 5000 overall iterations (each with 1000 refinement iterations) from a random seed with a contiguity constraint, it returns the redistricting plan that it has encountered with the lowest edge-cut. While this may not be the minimum, the number of iterations (and refinement iterations) provides stable results, alerting to the approaching of a minimum edge-cut.

To demonstrate the effectiveness of edge-cut as a compactness measure, consider the following scenario. Consider the initial core of a district (in this case a 7 x 7 unit square). For the sake of presentation, we partition these shapes into rectangles, however, the same concept applies to all types of regular or irregular partitions. Fig. X shows that the choice of which "tracts" to append to the compact core has a dramatic impact on the edge-cut. The initial core in a regular rectangle partition has edge cut 32. By adding a 5-tract spike to the initial core (Fig. 3a), the edge-cut was increased by ten. By adding the 5-tract layer parallel to the same initial core, the edge-cut is only increased by two (Fig. 3b).

Census tracts, however, do not conform to these shape parameters or alignments. These divisions of space come in all shapes and sizes. It is thus possible to reduce the edge-cut simply by approaching a large census tract. It is also possible to reduce the edge-cut by avoiding areas where neighboring tracts exhibit a zipper-like arrangement, doubling the edge-cut based on a physical offset. Most units in the map of census tracts in the United States do not encounter these issues.



(a) (b)

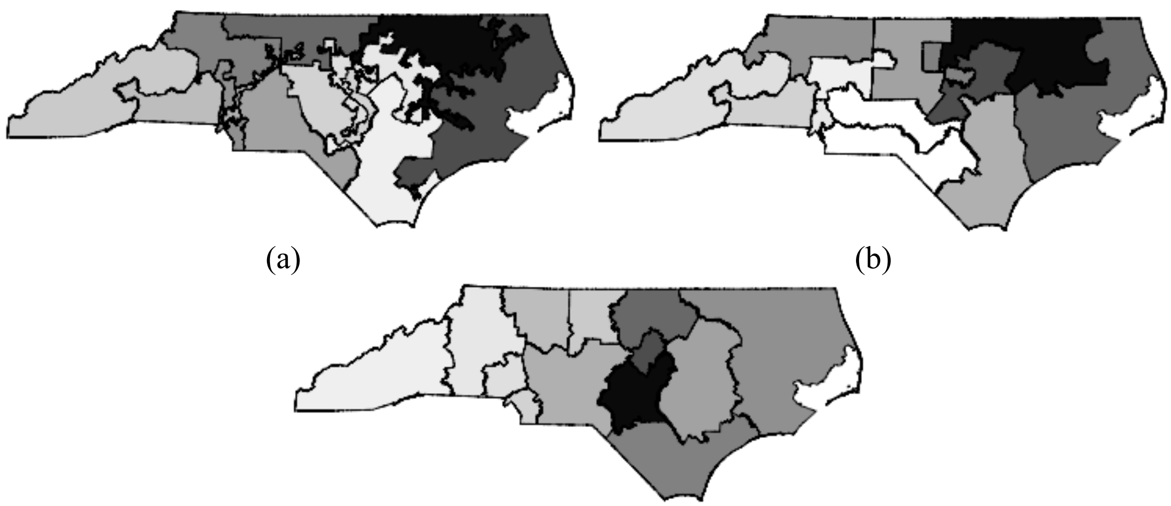
Figure 3: Extending the same district by five units of land, drastically impacting the number of neighboring cells.

Using the libraries *rgdal*, *maptools*, *plyr*, and *ggplot2* in R, each state's census tracts were joined to the corresponding assigned simulated district by *gpmetis* and extracted as a shapefile. These new shapefiles were passed to the Polygon Neighbors tool in ArcMap 10.3.1 to find the length of drawn borders within each state, absolving any concern about the coastline or the state border. The congressional districts for each state were also analyzed in this manner. Table 1 shows the results of the comparison for these two approaches, ranked by the deviance from the simulated district lines

Table 1. Comparison of State Internal Congressional Borders (2016 districts) to Simulation Internal Borders, Ranked by Deviance from Simulation

Rank	State	Redistricting Process	Actual Internal Border (miles)	Simulated Internal Border (miles)	Percentage
1	HI	I	64.686	16.106	24.9%
2	MD	D	1,112.262	389.586	35.0%
3	NH	R	343.824	146.479	42.6%
4	ME	I	346.690	184.706	53.3%
5	UT	R	1,394.698	800.298	57.4%
6	OH	R	3,270.952	1,920.264	58.7%
7	PA	R	3,617.939	2,178.259	60.2%
8	NJ	I	976.336	603.776	61.8%
9	LA	R	1,985.464	1,258.782	63.4%
10	MA	D	736.080	477.024	64.8%
11	KY	I	1,571.452	1,033.355	65.8%
12	VA	R	1,994.870	1,332.582	66.8%
13	SC	R	1,633.793	1,094.443	67.0%
14	IL	D	2,801.506	1,876.935	67.0%
15	AZ	I	2,300.829	1,564.003	68.0%
16	CT	I	399.025	272.419	68.3%
17	TN	R	1,909.620	1,353.775	70.9%
18	AL	R	2,129.918	1,513.517	71.1%
19	MS	I	999.935	741.091	74.1%
20	TX	R	8,763.370	6,662.505	76.0%
21	WV	I	774.700	592.126	76.4%
22	ID	I	655.006	507.338	77.5%
23	AR	D	1,249.114	974.280	78.0%
24	KS	R	854.644	677.666	79.3%
25	NC	R	2,399.474	1,921.100	80.1%
26	MI	R	1,886.647	1,526.514	80.9%
27	WI	R	1,752.059	1,444.021	82.4%
28	NE	I	514.708	435.448	84.6%
29	IA	I	926.837	785.060	84.7%
30	NM	I	960.014	825.699	86.0%
31	CO	I	1,631.681	1,412.781	86.6%
32	MO	I	1,710.322	1,482.320	86.7%
33	WA	D	2,117.531	1,856.975	87.7%
34	CA	I	7,038.968	6,219.263	88.4%
35	GA	R	2,435.832	2,162.662	88.8%
36	NY	I	1,908.839	1,715.244	89.9%
37	FL	R	2,884.507	2,620.531	90.8%
38	MN	I	1,414.705	1,296.339	91.6%
39	RI	D	46.418	43.086	92.8%
40	IN	R	1,148.228	1,129.503	98.4%
41	OR	I	1,178.763	1,169.373	99.2%
42	OK	R	1,080.085	1,077.647	99.8%
43	NV	I	652.677	874.997	134.1%

To assess a simple example of the impacts of this approach, consider the state of North Carolina, a state whose congressional districts were challenged after they were established in the year 2012. These congressional districts have since been redrawn for the current election cycle. Figure 4 looks at the districts of North Carolina in 2012, 2016, and via our simulated approach to minimize edge-cuts. While the 2016 redraw (2,399 miles) is far more efficient with its internal borders than its 2012 predecessor (4,083 miles), the proposed approach still shows a 20% improvement on this key metric (1,921 miles).



(a) (b) (c)

Figure 4: The state of North Carolina's district boundaries in (a) 2012, (b) 2016, and (c) an artificial simulation to minimize edge cuts. North Carolina's internal borders were reduced by 41% in the 2016 redraw, but the approach employed in this paper creates a reduction of 53%.

Conclusions

In this paper, we have identified an alternative approach to the study of compactness with regard to Congressional districts. Rather than focusing on geometric compactness, which is hindered by geographical and fiat boundaries, this paper suggests drawing attention to the census tract boundaries that form the edge of a district. This approach is generalizable to all states, independent of the shapes of both the state's border and the borders of all of its census tracts. In all but one state, this approach produces simulated districts that have lower internal boundary lengths than the currently established congressional districts. The exception is Nevada, a state with very large census tracts dispersed around the state, an ideal location to minimize edge cuts.

Edge-cut compactness represents an enormous opportunity for efficiently executing redistricting plans, independent of what a state border looks like. For states such as Maryland, Maine, Louisiana, West Virginia, and Mississippi, much of the border is geographically and geologically not conducive to producing geometrically compact districts. Furthermore, census tracts do not have a cookie-cutter shape, leading toward a lack of flexibility in attainable districts. Using edge-cut compactness allows for an immediate method to compare established redistricting plans on a level playing field, independent of the state's shape with each established compactness-inspired measurement.

One way to analyze these results is to consider the reduction at the federal level by employing such an approach. The current federal length of internal congressional boundaries (independent of state borders) is 75,575. This approach reduces that amount to 58,170 miles, a staggering difference of 17,405 miles, representing a 23% reduction.

Of further interest is the effect of the type of redistricting process within a state. In subdividing the states into separate types of redistricting processes, we have a platform with which to address the effectiveness of commissions at achieving boundaries comparable to this standard of compactness. The analysis of this data is shown in Table 2. Independent or mixed processes produce a higher rate of conformity to this standard than do partisan-based processes. This result points at the success of independent commissions in achieving more conscionable state congressional districts *vis a vis* compactness.

Table 2. Effect of Redistricting Process on Comparison between Actual Internal Borders and Simulated Internal Borders based on Internal Border Mileage.

Process Type	States	Actual Internal Borders (in miles)	Simulated Internal Borders (in miles)	Percentage
Independent	19	26,026.170	21,731.440	83.5%
Republican	18	41,485.930	30,720.550	74.3%
Democrat	6	8,062.911	5,617.887	69.7%

This comparison may be flawed in that states with smaller internal borders do not contribute as much to these total figures. As such, we have repeated the analysis using the proportions as the weights in Table 3. Independent commissions still produce a higher average of maintenance, though that percentage is not significant.

Table 3. Effect of Redistricting Process on Comparison between Actual Internal Borders and Simulated Internal Borders based on State Proportions.

Process Type	States	Maintenance Proportion (Avg.)
Independent	19	79.0%
Republican	18	74.1%
Democrat	6	70.9%

As we have shown, the use of graph theory and edge compactness provides us new methods by which to not only examine district compactness, but also to automatically create districts that are contiguous and ideally compact. It also introduces an entirely new logic to the way in which political scientists perceive how districts are drawn; instead of drawing lines on a map, districts are simply aggregations of previously created spatial delineations. This change in logic proscribes the use of geometric measures of district attributes, which may have deep impacts on how both academia and the judicial system analyze the impact of redistricting.

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