

Quantifying gerrymandering using the vote distribution

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Abstract

To assess the presence of gerrymandering, one can consider the shapes of districts or the distribution of votes. The *efficiency gap*, which does the latter, plays a central role in a 2016 federal court case on the constitutionality of Wisconsin’s state legislative district plan. Unfortunately, however, the efficiency gap reduces to proportional representation, an expectation that is not a constitutional right. We present a new measure of partisan asymmetry that does not rely on the shapes of districts, is simple to compute, is provably related to the “packing and cracking” integral to gerrymandering, and that avoids the constitutionality issue presented by the efficiency gap. In addition, we introduce a generalization of the efficiency gap that also avoids the equivalency to proportional representation. We apply the first function to US congressional and state legislative plans from recent decades to identify candidate gerrymanders.

1 Introduction

To *gerrymander* is to intentionally choose voting districts so as to actively (dis)advantage one or more groups¹. While gerrymandering can focus on various types of groups, such as incumbents or those formed by shared race, our focus in this article is the case in which the disadvantaged group is a political party, that is, *partisan gerrymandering*.

¹There is no universally agreed upon definition. In particular, usage in the courts differs from colloquial usage.

Approaches towards preventing gerrymanders include vesting the responsibility of drawing districts in non-partisan commissions (see [Win08]); drawing districts by putatively neutral algorithms (such as the shortest splitline method [KYS07]); or moving away from the current district-based single member plurality system (see [Gro81] for various alternatives). Since efforts along these lines have not yet eradicated gerrymandering, there is a need for methods that can accurately identify any gerrymanders that do occur.

Historically, gerrymanders have been identified by unusual district boundaries [Gri]. In fact, it was the resemblance of a Massachusetts senatorial district to a salamander that gives gerrymandering its name (Elbridge Gerry signed off on the district in 1812 while governor of Massachusetts). To this day, unexpected shapes of districts lead to both zoomorphic comparisons (e.g., Pennsylvania’s 6th congressional “dragon” district) and allegations of gerrymandering.

In response, a number of researchers have created compactness metrics as a way to help identify gerrymandered districts (see [NGCH90] for an overview); a recent, related Markov-chain based approach is found in [CFP17]. While such approaches are important tools, the fact that partisan asymmetries arise naturally from “human geography” [CR13] strongly suggests gerrymanders can exist without contorted boundaries. By analogy, just as one can be ill and yet not have a fever, so can one have a gerrymander without violating compactness.

For more than four decades, vote-distribution asymmetries have been analyzed via the “seat-votes curve” (see, for example, [Tuf73, GK94, KGC08, Nag15]), the computation of which requires significant statistical assumptions. In 2014, McGhee [McG14] introduced an alternative, the *efficiency gap* (see also [MS15]). This function is easily computed from its elegant definition without any assumptions at all when all races are contested. A short derivation (see [McG14] or the derivation of equation (6)) shows that a fair district plan according to the efficiency gap is one in which the seats won and the vote won are proportional with constant 2: Winning 53% (that is, 50% + 3%) of the vote will earn 56% (that is, 50% + 2 · 3%) of the seats when the efficiency gap is zero.

As detailed in [MS15, III.C], several decades of court cases predicated on partisan gerrymandering have, until recently, been uniformly unsuccessful in the federal courts: On November 21, 2016, a three-member circuit court panel ruled in *Whitford v. Gill* [Wis] that the districts drawn for the Wisconsin lower house are an unconstitutional partisan gerrymander. This is the first time such a determination has been made regarding an alleged partisan gerrymander [Win16]. A significant part of the panel’s decision relies on the supporting evidence of asymmetry afforded by the efficiency gap. However, in his dissent in *Whitford v. Gill*, Chief Judge Griesbach objects to the reliance on the efficiency gap, in large part on the grounds that the US Supreme Court has ruled that *there is no constitutional right to proportional representation* [Vie]. A natural question is whether there are alternative functions that do not reduce to proportional representation.

Below we do define a family of functions, indexed by nonnegative numbers τ , that specializes to (twice) the efficiency gap when $\tau = 0$. As we’ll describe in Section 4, for $\tau > 0$ these functions do not reduce to proportionality. However, the focus in this article will be on a new function for identifying gerrymandering we term the *declination*. It is a measure of asymmetry

in the vote distribution that relies only on the fraction of seats each party wins in conjunction with the aggregate vote each party uses to win those seats. (The declination is essentially an angle associated to the vote distribution; its name is in analogy with the angular difference between true north and magnetic north.) The declination does not require the significant statistical assumptions required to compute the seats-votes curve nor the proportional-representation assumption embedded in the efficiency gap.

To be useful for identifying gerrymanders, our functions must measure asymmetry that arises from gerrymandering rather than from other sources. Asymmetries that are caused by partisan gerrymandering arise from two primary techniques. The first is to *pack* certain districts with members of party P so that party P wins those districts overwhelmingly, thereby wasting votes that could have been helpful in winning districts elsewhere. A second technique, naturally paired with the first, is to *crack* the party- P voters by distributing them among districts so as to prevent them from having sufficient power to win elections outside of the packed districts. Overall, party P wins a small number of overwhelming victories while suffering a large number of narrow losses. In Theorem 1 we rigorously prove that the declination increases in response to packing and cracking.

In Section 2 we define the declination and two variations. Analyses of elections in recent decades using the declination are presented in Section 3. The τ -gap family of functions that generalizes the efficiency gap is introduced in Section 4. In Section 5 we formally present the theorem showing that both the declination and the τ -gap increase in response to packing and cracking. In Section 6 we discuss the relative merits of the declination, the τ -gap, the efficiency gap and the mean-median. In Section 7 we outline the data sets used as well as the statistical model used to impute votes in uncontested elections. We conclude in Section 8. Additional figures and tables are included in the appendix.

2 The declination

Our definition of the declination stems from the observation that in a randomly chosen (i.e., ungerrymandered) district plan, the 0.5 threshold for the fraction of democratic votes in a district does not play a distinguished role. Quirks of geography and sociology will certainly shape the distribution in various ways that we will not attempt to model. But we expect a phase shift across the 0.5 threshold no more than we would expect one across 0.4 or 0.56. Partisan gerrymandering, almost by definition, modifies a natural distribution in a manner that treats the 0.5 threshold as special. Accordingly, one approach to recognizing gerrymanders is to contrast the set of values below 0.5 with the set of values above 0.5.

In order to compare these two parts of the vote distribution, we introduce some notation: Define an *election with N districts* to be a triple $\mathcal{E} = (P, Q, \mathbf{p})$ consisting of political parties P and Q along with a vector $\mathbf{p} = (p_1, p_2, \dots, p_N)$ that records the fraction of the two-party vote won by party P in each of the N districts. We assume, without loss of generality, that our

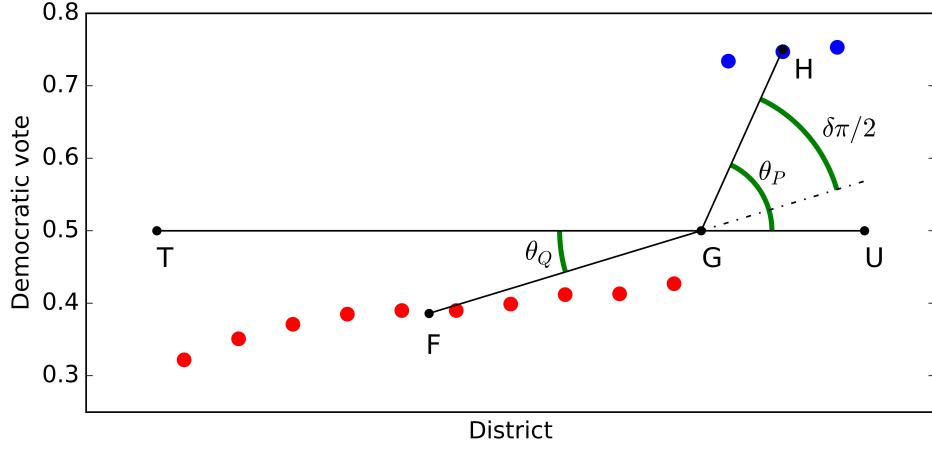


Figure 1: Illustration of the angles θ_P and θ_Q arising in the calculation of $\delta(\mathcal{E}) \approx 0.54$ for the 13-district 2014 North Carolina congressional election. Districts have been sorted in increasing order of democratic vote share.

districts are ordered in increasing order of party- P vote share:

$$0 \leq p_1 \leq p_2 \leq \dots \leq p_k \leq \frac{1}{2} < p_{k+1} \leq \dots \leq p_N \leq 1. \quad (1)$$

Set $k' = N - k$ and let $A = \{(i/N - 1/2N, p_i) : 1 \leq i \leq k\}$ and $B = \{(i/N - 1/2N, p_i) : k + 1 \leq i \leq N\}$. Let $F = (k/2N, \bar{y})$ and $H = (k/N + k'/2N, \bar{z})$ be the centers of mass of the points in A and B , respectively (we therefore assume that $k, k' \geq 1$). Set $G = (k/N, 1/2)$, $T = (0, 1/2)$ and $U = (1, 1/2)$; see Fig. 1. In a district plan in which neither side has an inherent advantage, we would expect the point G to lie on the line \overline{FH} . Deviation of G above \overline{FH} is indicative of an advantage for party P while deviation below is indicative of an advantage for party Q . This suggests, as a measure of partisan asymmetry, the *declination*, $\delta(\mathcal{E}) = 2(\theta_P - \theta_Q)/\pi$, where

$$\theta_P = \angle HGU = \arctan\left(\frac{2\bar{z} - 1}{k'/N}\right) \text{ and } \theta_Q = \angle FGT = \arctan\left(\frac{1 - 2\bar{y}}{k/N}\right). \quad (2)$$

Dividing by $\pi/2$ converts from radians to fractions of 90 degrees; possible values of the declination are between -1 and 1 .

To each party we are associating a line whose direction encodes the average vote the party gets in the districts it wins along with the total number of districts it wins. When a party uses lots of extra votes to win few districts, the slope of this line is high. The declination, up to the $\pi/2$ -scaling, is the angle between the lines associated to the two parties.

Fig. 2 illustrates how the declination and the efficiency gap evaluate different elections differently. In this figure and the remainder of the article, we identify Party P with the Democrats

and Party Q with the Republicans. As a result, positive values of the declination are favorable to Republicans.

As illustrated in Figure 3.A, packing or cracking a single district leads to a change of approximately $\pm 2/N$ in the declination. Hence, as a measure of the number of seats affected by the partisan advantage, we also consider $\delta_N(\mathcal{E}) = \delta(\mathcal{E})N/2$.

As a second variation, the value of $\tilde{\delta}(\mathcal{E}) = \delta(\mathcal{E}) \ln(N)/2$ has minimal correlation with N (see Fig. 3.B) and hence is useful when comparing the declinations of elections with different numbers of districts.

3 Analyses of elections

Fig. 4 displays the distribution of $\tilde{\delta}$ values over time for state legislative (lower house) and congressional races going back to 1972 (see Section 7 for a detailed description of the data used). We prove in Theorem 1 that the declination increases (resp., decreases) as a result of gerrymandering engendered by packing and cracking by party Q (resp., party P).

There is no gold standard for assessing gerrymandering. As a result, there is no direct way to validate the declination as a measure of gerrymandering. However, we can still review the extent to which the declination agrees with other measures of gerrymandering. Table 1 lists the values of $\tilde{\delta}$, δ and δ_N for congressional races in 2012. There is marked consistency between the declination and who controlled the redistricting process subsequent to the 2010 census. Also, while disproportionality may not be sufficient evidence to denounce a district plan as an unconstitutional gerrymander, a successful gerrymander by definition results in one party winning fewer seats than it would under a neutral plan. This is consistent with what happened in the first four states in Table 1 (Pennsylvania, Ohio, Virginia and North Carolina): As noted in [Wan16], the Democrats earned approximately half the statewide vote in 2012 in each of these states while garnering no more than 31% of the seats in any of the four.

There is also strong agreement between the declination and compactness metrics (we work here with $\tilde{\delta}$; δ and δ_N are similarly concurrent). In [Ing14], each 2012 congressional district is scored by the Polsby-Popper compactness metric (scaled to run from 0 to 100; higher scores indicate less compact). The nine states with at least eight districts and at least one district scoring greater than 90 are North Carolina, Ohio, Pennsylvania, Virginia (the first four listed in Table 1); California, Illinois and Maryland (three of the last four in Table 1); as well as Florida and Texas. Intriguingly, the declination identifies Indiana as a likely gerrymander, in contrast to its evaluation in [Ing14] as a state with very compact districts.

The matter of determining a standard for what qualifies as an unconstitutional partisan gerrymander is beyond the scope of this article. (See [MS15] for a comprehensive treatment utilizing the efficiency gap, [MSK15] for the seats-votes curve and [MB15, Wan16] for the mean-median difference.) In addition to the nontrivial task of addressing guidance from the courts, two significant issues that must be addressed are the reality that any measure of asymmetry in vote distributions will vary from election to election (see Fig. 5) and that there may be an inherent

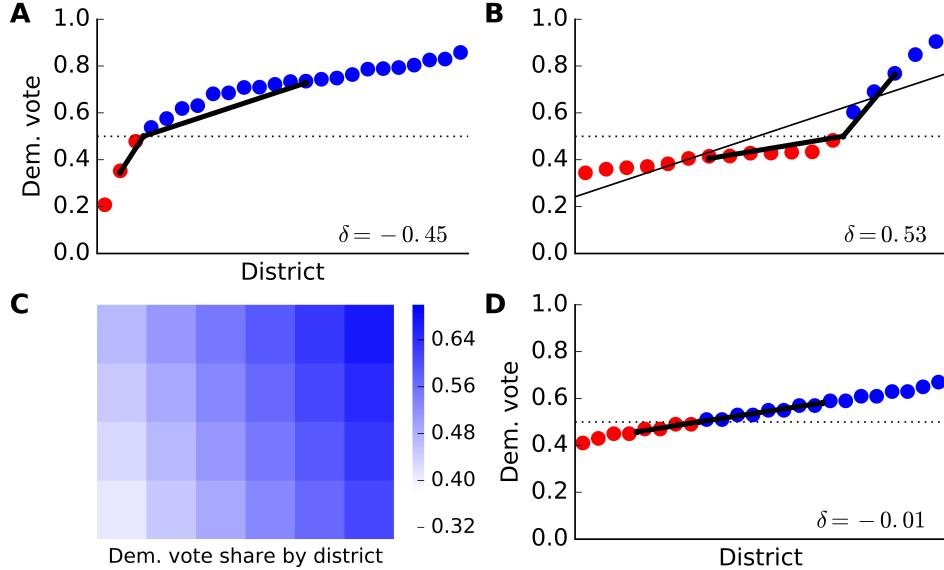


Figure 2: **(A)** In the 24-district 1974 Texas congressional elections, the Democrats won approximately 68% of the vote and 87.5% of the seats, yielding an efficiency gap of close to zero. The declination, however, is strongly negative as a consequence of marked change in slope of the distribution at 0.5. **(B)** The vote distribution for the 18-district 2012 Pennsylvania congressional election. The mean district vote fraction for the Democrats is 0.504. Also plotted is the regression line (slope 0.52, intercept 0.24; $r = 0.875, p < 0.001$) over all districts. The fact that it crosses the 0.5 vote threshold approximately five districts earlier than the actual results do is consistent with $\delta_N = 4.8$. **(C)** Hypothetical election results for a district plan with 24 districts whose democratic vote share ranges from 0.41 in the southwest to 0.67 in the northeast. The Democrats win 16 seats with a mean vote share of 0.54. **(D)** The vote distribution from the hypothetical election. The declination is near zero due to the regularity with which the democratic vote varies among districts. The efficiency gap is far from zero at -0.17 due to the fact that this election has a constant of proportionality close to one.

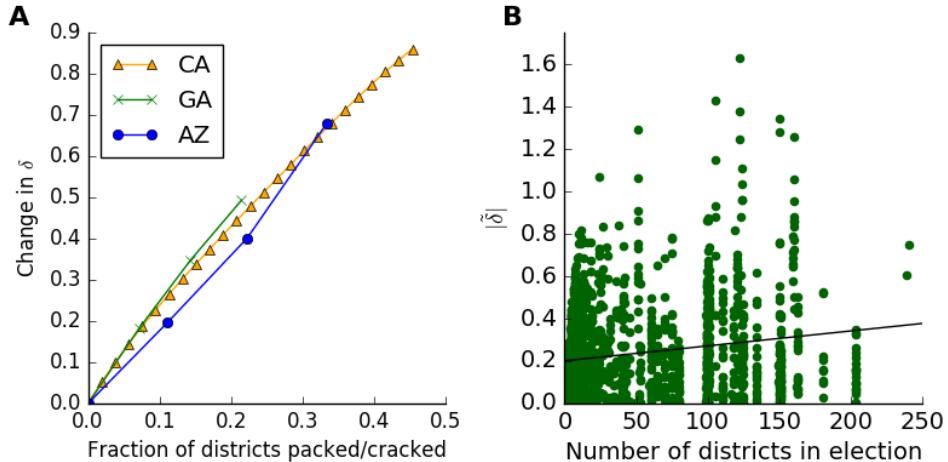


Figure 3: **(A)** Change in declination relative to fraction of seats packed (California 2012 congressional election) or cracked (Arizona and Georgia 2012 congressional elections). Note that the elections referenced are used to provide an initial distribution to which packing/cracking is artificially applied. **(B)** Plot of N versus $|\tilde{\delta}|$ for 1 472 legislative and congressional elections. The regression line is $0.0007N + 0.2011$ with an r -value of 0.1769 and $p < 0.0001$.

partisan asymmetry due to how voters are distributed geographically. We briefly address each issue.

We consider the first problem by determining how large the declination must be in absolute value before we are confident that it will not equal zero for a different election in the same ten-year districting cycle. Figs. 10 and 11 illustrate the total range of $\tilde{\delta}$ values for each state in each redistricting cycle. With respect to our historical data set of 1 195 elections included in these figures, for over eighty percent of the elections \mathcal{E} with $|\tilde{\delta}(\mathcal{E})| > 0.47$, the declination (i.e., the partisan advantage) persists in sign over the course of the entire ten-year redistricting cycle in which \mathcal{E} lies. With this confidence, therefore, we conclude from Table 1 (without examining data from 2014 or 2016) that the partisan advantage for the congressional district plans in Pennsylvania, Ohio, Virginia, North Carolina, Michigan, Indiana and Maryland are likely to persist through 2020. We note that the 2012 Wisconsin legislative election, with $\tilde{\delta} = 0.48$, just meets this threshold (also see Fig. 6). For reference, the most extreme values of the declination for congressional races since 1972 are sorted in Table 2.

The second major issue, inherent partisan advantages arising from how voters are distributed geographically, is an important one. As explored in [CR13], such realities may lead to one party naturally garnering more than 55% of the seats with only 50% of the vote. For example, if we take the republican advantage in Pennsylvania to be 8% (following [CR13]), this seat advantage translates into a value of $\tilde{\delta}$ of approximately 0.23. Since $\tilde{\delta} = 0.76 > 0.47 + 0.23$ for the 2012 Pennsylvania congressional election, we can still conclude with high confidence that the partisan advantage for the Republicans *beyond any geographic advantage* will persist through

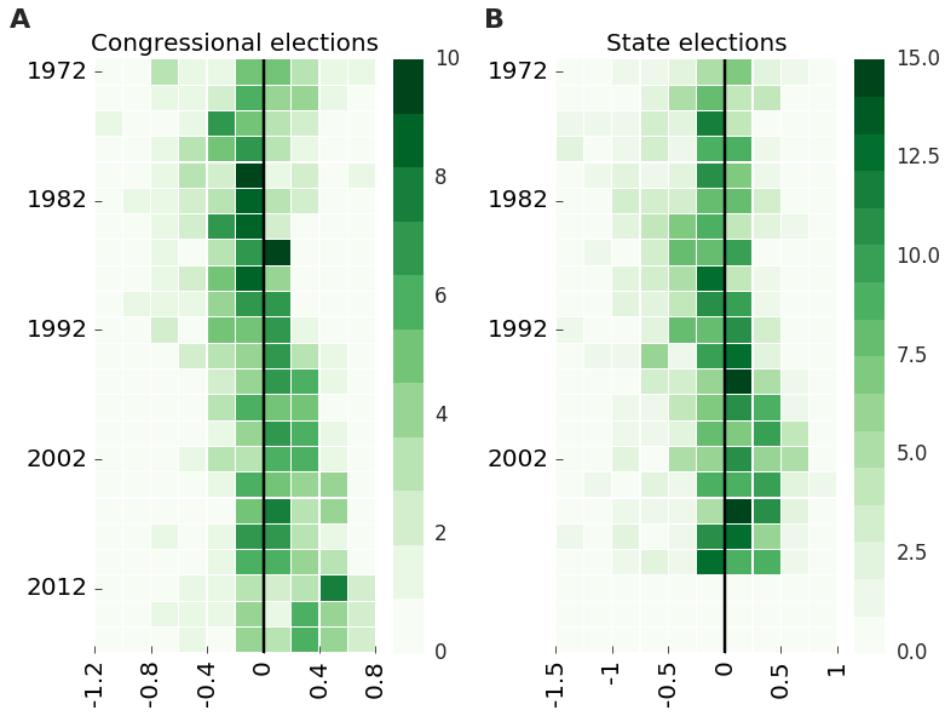


Figure 4: Values of $\tilde{\delta}$ stratified by year for 461 congressional elections (**A**) and 606 state elections (**B**) with at least eight districts (following [MS15]) and with each party winning at least one seat. Data for state elections after 2010 not collected. For each heat map, the number of included elections in a given year varies (for the state elections, states have gradually moved away from the multi-member districts (see Section 7); for congressional elections there are sporadic instances in which one party wins all seats).

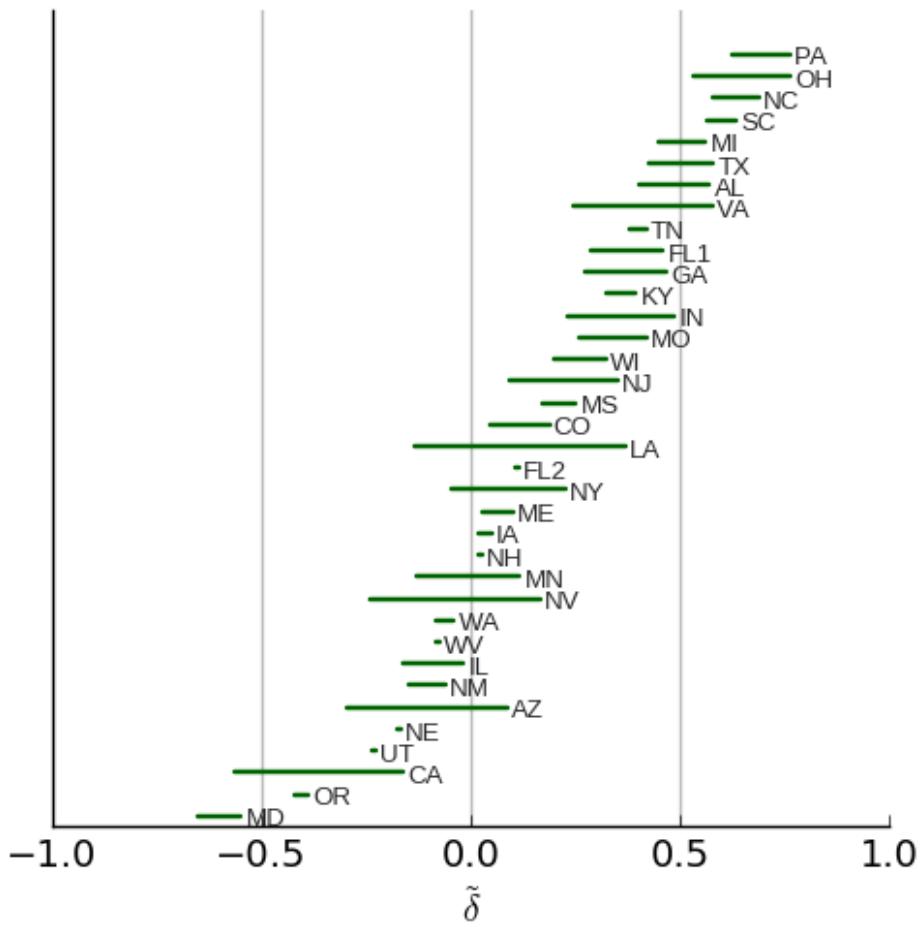


Figure 5: Range of $\tilde{\delta}$ between 2012 and 2016 for congressional elections. Since Florida's congressional districts were changed for the 2016 election, that plan is denoted by FL2 while the plan in place from 2012–2014 is denoted by FL1. States were omitted if one party won all seats every year of the cycle.

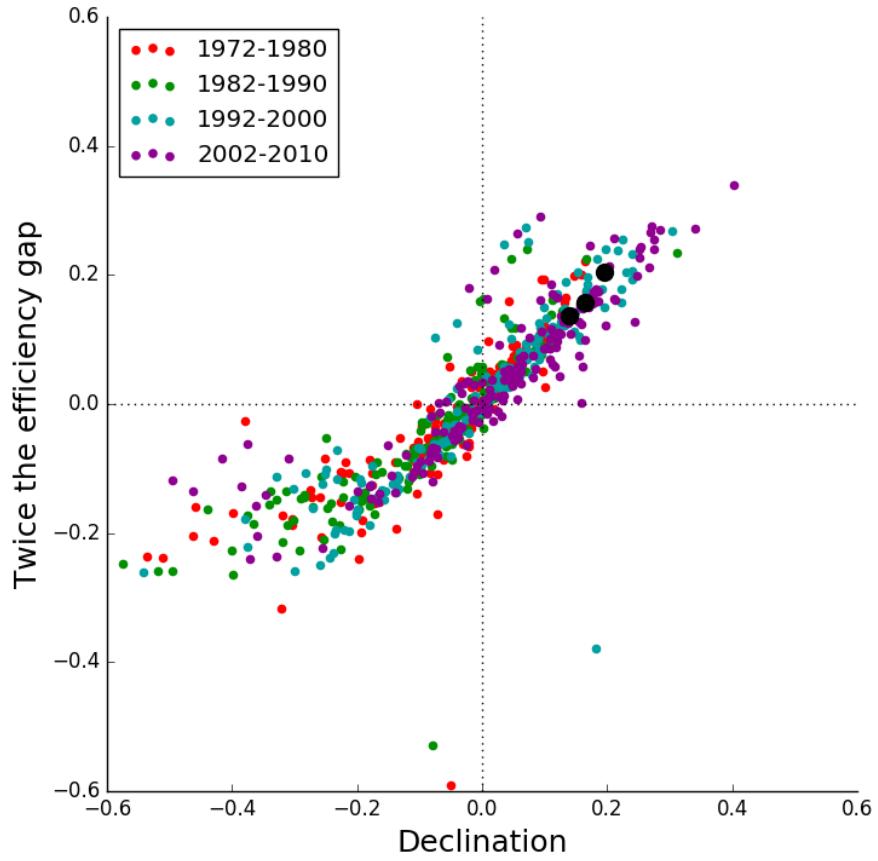


Figure 6: Comparison of the declination and (twice) the efficiency gap for 451 state legislative elections from 1972 to 2010. States with multi-member districts at any time since 1972 are omitted. Declination (rather than $\tilde{\delta}$) is used even though we are comparing districts with different sizes in order to be consistent with the efficiency gap. The correlation is $r = 0.76$ with $p < 0.001$. The three large black circles indicate the Wisconsin elections for 2012, 2014 and 2016.

Table 1: Values of the declination and its variants for the twenty 2012 US Congressional elections in states with at least eight districts for which each party won at least one seat. Political control of the redistricting process taken from [CC16]: R indicates republican control, D indicates democratic control, S indicates split control. Redistricting control in New Jersey and California was ostensibly non-partisan, but evidence suggests partisan control as indicated, see [Cai12, PL11] as noted in [CC16]. Voting Rights Act (pre-clearance) states are marked by an asterisk.

$\tilde{\delta}$	δ	δ_N	State	No. Seats	Control
0.76	0.53	4.8	PA	18	R
0.76	0.55	4.4	OH	16	R
0.58	0.48	2.6	VA	11	R*
0.58	0.45	2.9	NC	13	R
0.56	0.43	3.0	MI	14	R
0.49	0.44	2.0	IN	9	R
0.46	0.28	3.8	FL	27	R
0.44	0.24	4.4	TX	36	R*
0.42	0.40	1.6	MO	8	R
0.42	0.38	1.7	TN	9	R
0.35	0.28	1.7	NJ	12	R
0.32	0.31	1.2	WI	8	R
0.28	0.21	1.5	GA	14	R
0.16	0.09	1.3	NY	27	S
0.12	0.11	0.5	MN	8	S
-0.08	-0.07	-0.3	WA	10	S
-0.16	-0.11	-1.0	IL	18	D
-0.19	-0.10	-2.6	CA	53	D
-0.30	-0.27	-1.2	AZ	9	S*
-0.55	-0.53	-2.1	MD	8	D

2020. Ultimately, the proper way to account for any inherent geographic advantages requires answering the following: To what extent is it constitutional for a district plan to exacerbate (or mitigate) existing, natural advantages?

Additional summary tables and figures are included in the appendix. Specifically, Tables 3 and 4 list the declination for all 1142 congressional elections and 646 state legislative elections in the data set. Figs. 8 and 9 depict the actual vote distributions (utilizing imputed vote values for uncontested races (see Section 7) along with the declination for the 2016 congressional elections and the 2008 state legislative elections, respectively.

4 A generalization of the efficiency gap

In this section we introduce a variation on the efficiency gap function. We first review the definition of the efficiency gap before generalizing it in the next subsection.

McGhee defines a vote as *wasted* if either 1) it is a vote for the winning candidate in excess of the one-more-than-50% needed to win the election or 2) it is a vote for the losing candidate. The motivation for this definition is that packing and cracking produce more wasted votes in won and lost districts, respectively. In light of this, the difference in the total number of votes wasted by each party should reflect the extent to which the packing and cracking treats the parties asymmetrically.

Given an election $\mathcal{E} = (P, Q, \mathbf{p})$, party P 's *waste* in district i , $w_P(i)$, is $p_i - 1/2$ in those districts it wins and p_i in those districts it loses. Party Q 's waste is $w_Q(i) = 1/2 - w_P(i)$.

McGhee [McG14, Eq. (2)] (using different notation and terminology) made the following definition: The *efficiency gap* of \mathcal{E} is

$$\frac{\sum_{i=1}^N (w_P(i) - w_Q(i))}{N}. \quad (3)$$

A positive (resp., negative) efficiency gap indicates that party P (resp., Q) is wasting more votes. As such, “fair” district plans should have efficiency gaps close to zero.

4.1 The Gap_τ function

The efficiency gap takes a binary view of votes: Either a vote contributes to the first 50% required for a candidate's win, and is therefore not wasted at all, or it doesn't contribute, and is therefore completely wasted. We now explore a more general paradigm in which there is a continuum of waste that is shaped by a parameter τ .

As motivation for this generalization, consider the party- Q gerrymanderers. They are not aiming to have the party- P candidates win districts by exactly one vote above 50%. To do so would skirt disaster. If their projections are even slightly too optimistic, then the gerrymander will backfire spectacularly with each chosen candidate losing by a razor-thin margin. It is less reckless for party Q to aim for comfortable margins in each district they aim to win. Consequently, it is reasonable to consider the votes just above 50% as only partially wasted while treating any votes near 100% as almost fully wasted.

For simplicity, we view the votes won by a given candidate as ordered. For a winning candidate, the first half of the votes in the district contribute directly to the win. The subsequent votes for the winner, and all votes for the loser, are assigned waste values determined by the parameter τ . As will become apparent, setting $\tau = 0$ corresponds to weighting all wasted votes the same amount.

Given an N -district election \mathcal{E} , for each i , $1 \leq i \leq N$, set $a_i = 2p_i - 1$. When a_i is positive, party P wins district i and a_i is the fraction of the wasted votes that are wasted by P . When a_i is negative, party Q wins district i and $-a_i$ measures the fraction of wasted votes wasted by

party Q . For any nonnegative real number τ and any $i > k$, we define the τ -waste of party P in district i to be

$$w_{P,\tau}(i) = \int_0^{a_i} x^\tau dx = \frac{1}{\tau+1} a_i^{\tau+1}. \quad (4)$$

For $i \leq k$, similarly define $w_{Q,\tau}(i) = (-a_i)^{\tau+1}/(\tau+1)$. The remaining values are determined by requiring that the total waste in each district i , $w_{P,\tau}(i) + w_{Q,\tau}(i)$, is constant and equal to $\int_0^1 x^\tau dx = 1/(\tau+1)$. Set $w_{P,\tau} = \sum_{i=1}^N w_{P,\tau}(i)$ and $w_{Q,\tau} = \sum_{i=1}^N w_{Q,\tau}(i)$. Finally, define the τ -gap of election \mathcal{E} to be

$$\text{Gap}_\tau(\mathcal{E}) = \frac{w_{P,\tau} - w_{Q,\tau}}{w_{P,\tau} + w_{Q,\tau}}. \quad (5)$$

Note that for any election \mathcal{E} , $\text{Gap}_0(\mathcal{E})$ is twice the efficient gap.

We now derive a formula for the τ -gap that expresses it in terms of the a_i . First we note that

$$\begin{aligned} w_{P,\tau} &= \sum_{i=1}^k w_{P,\tau}(i) + \sum_{i=k+1}^N w_{P,\tau}(i) = \sum_{i=1}^k \left(\frac{1}{\tau+1} - \frac{(-a_i)^{\tau+1}}{\tau+1} \right) + \sum_{i=k+1}^N \frac{a_i^{\tau+1}}{\tau+1} \text{ and} \\ w_{Q,\tau} &= \sum_{i=1}^k w_{Q,\tau}(i) + \sum_{i=k+1}^N w_{Q,\tau}(i) = \sum_{i=1}^k \frac{(-a_i)^{\tau+1}}{\tau+1} + \sum_{i=k+1}^N \left(\frac{1}{\tau+1} - \frac{a_i^{\tau+1}}{\tau+1} \right). \end{aligned}$$

Second, $w_{P,\tau} + w_{Q,\tau} = N/(\tau+1)$. Plugging into equation (5), we find that

$$\begin{aligned} \text{Gap}_\tau(\mathcal{E}) &= \frac{\tau+1}{N} \left(- \sum_{i=1}^k 2 \frac{(-a_i)^{\tau+1}}{\tau+1} + \sum_{i=k+1}^N 2 \frac{a_i^{\tau+1}}{\tau+1} + \frac{k-k'}{\tau+1} \right) \\ &= 2 \left(\frac{- \sum_{i=1}^k (-a_i)^{\tau+1} + \sum_{i=k+1}^N a_i^{\tau+1}}{N} + \frac{1}{2} - \frac{k'}{N} \right). \end{aligned}$$

It follows that

$$\text{Gap}_\tau(\mathcal{E}) = 2 \left[\frac{\sum_{i=1}^N \epsilon_i (\epsilon_i a_i)^{\tau+1}}{N} + \frac{1}{2} - \frac{k'}{N} \right], \quad (6)$$

where $\epsilon_i = -1$ for $i \leq k$ and $\epsilon_i = 1$ for $i > k$.

We see that $\text{Gap}_0 = 0$ precisely when

$$\frac{\sum_{i=1}^N a_i}{N} = \frac{\sum_{i=1}^N 2p_i - 1}{N} = \frac{k'}{N} - \frac{1}{2}. \quad (7)$$

Let \bar{p} denote the overall vote fraction won by party P . Rewriting (7), we find that the efficiency gap being zero is equivalent to $2(\bar{p} - 1/2) = k'/N - 1/2$. We have recovered the observation of [McG14, MS15] that the efficiency gap is zero exactly when the excess in fraction of seats won above one half is twice the excess in vote fraction above one half.

For general τ , however, the relationship between seats won and votes earned in an election for which Gap_τ is zero is more complicated and depends on how the votes are distributed. When τ is a nonnegative even integer of the form $\tau = 2\ell$, we can express the condition succinctly since then

$$\text{Gap}_\tau(\mathcal{E}) = 2 \left[M_{2\ell+1}(\mathbf{a}) + \frac{1}{2} - \frac{k'}{N} \right], \quad (8)$$

where $M_{2\ell+1}(\mathbf{a})$ denotes the $(2\ell+1)^{\text{th}}$ moment of the a_i .

In the limit as $\tau \rightarrow \infty$, Gap_τ reduces (in the general case of $0 < |a_i| < 1$ for all i) to $1 - 2k'/N$. It follows that Gap_∞ is zero if and only if each party wins half of the seats, irrespective of overall vote share.

There are at least two natural ways in which the τ -gap could be further generalized. The first is by allowing the waste to be most pronounced close to the 50% threshold. This is conveniently attained by considering waste functions of the form $(1-a_i)^{-\tau}$ for $\tau < 0$. This choice has several appealing characteristics such as the fact that the winner and loser waste equal amounts more evenly than the 75%-25% split for the efficiency gap. In addition, the waste function is more sensitive to what is happening near the 50% threshold. Unfortunately, this latter property seems to reduce its utility greatly as there is much more noise due to random fluctuations in vote earned.

Second, while we have restricted our attention to power functions determined by a parameter τ , the concept of τ -waste could easily be extended to more sophisticated functions. One could also allow different functions for the votes wasted by the winner and the votes wasted by the loser.

5 Theorem on packing and cracking

We make precise the link between packing and cracking and the functions we introduce in the previous two sections. Define P -cracking to be the moving of party- P votes from district $k+1$ to districts $1, 2, \dots, k$ such that 1) the first k districts are still lost by party P after the redistribution and 2) district $k+1$ becomes a district that party P loses. Similarly, define P -packing to be the moving of party- P votes from district $k+1$ to districts $k+2, k+3, \dots, N$ such that district $k+1$ is now lost by party P . Using our new terminology, we state and prove the following

Theorem 1. Let $\mathcal{E} = (P, Q, \mathbf{p})$ be an election with \mathbf{p} in weakly increasing order. Let \bar{y} be the average party- P vote in the k districts lost by party P and \bar{z} be the corresponding average in the k' districts won by party P . Let p'_{k+1} be the party- P vote fraction remaining in district $k+1$ after P -cracking or P -packing and \mathcal{E}' the resulting election.

1. If $p'_{k+1} > \bar{y}$, then $\delta(\mathcal{E}') > \delta(\mathcal{E})$.
2. If $p'_{k+1} > p_k$, then $\text{Gap}_\tau(\mathcal{E}') > \text{Gap}_\tau(\mathcal{E})$.

Proof. Proof of (1). We refer the reader to Fig. 1 for the positions of points referenced. The P -packing or P -cracking of district $k + 1$ will move point G to the right by $1/N$ units. Each of points F and H will move to the right by only $1/2N$ units. Since $p'_{k+1} > \bar{y}$, point F will also move up. Since \mathbf{p} is arranged in increasing order, point H will also move up. Together, these shifts will decrease θ_Q and increase θ_P , thereby increasing δ . By symmetry, Q -packing and Q -cracking will decrease $\delta(\mathcal{E})$.

Proof of (2). We show only the details for P -cracking; the argument for P -packing is similar. Let $\mathbf{p}' = (p'_1, p'_2, \dots, p'_N)$ be the vote distribution for \mathcal{E}' . Since we are P -cracking the $(k + 1)^{\text{st}}$ district, we know that $p'_i = p_i$ for $i > k + 1$. Set $b_i = 2p'_i - 1$ for $i \leq k + 1$. Note in the below equations that all a_i and b_i for $i \leq k$ are *negative* (b_{k+1} is negative as well). Recall that

$$\begin{aligned}\text{Gap}_\tau(\mathcal{E}) &= 2 \left[\frac{-\sum_{i=1}^k (-a_i)^{\tau+1} + a_{k+1}^{\tau+1} + \sum_{i=k+2}^N a_i^{\tau+1}}{N} + \frac{1}{2} - \frac{k'}{N} \right] \text{ and} \\ \text{Gap}_\tau(\mathcal{E}') &= 2 \left[\frac{-\sum_{i=1}^k (-b_i)^{\tau+1} - (-b_{k+1})^{\tau+1} + \sum_{i=k+2}^N a_i^{\tau+1}}{N} + \frac{1}{2} - \frac{k'-1}{N} \right].\end{aligned}$$

We wish to show that the difference

$$\text{Gap}_\tau(\mathcal{E}') - \text{Gap}_\tau(\mathcal{E}) = \frac{\sum_{i=1}^k ((-a_i)^{\tau+1} - (-b_i)^{\tau+1}) - (-b_{k+1})^{\tau+1} - a_{k+1}^{\tau+1}}{N} + \frac{1}{N}$$

is positive. Multiplying by $N/(\tau + 1)$, we see that this is equivalent to showing that

$$\frac{\sum_{i=1}^k ((-a_i)^{\tau+1} - (-b_i)^{\tau+1}) + 1}{\tau + 1} \geq \frac{(-b_{k+1})^{\tau+1} + a_{k+1}^{\tau+1}}{\tau + 1}.$$

We make the following observations. First, since all votes being moved are going from district $k + 1$ to a district i with $i \leq k$, we must have $\sum_{i=1}^k ((-a_i) + b_i) = a_{k+1} - b_{k+1}$. Second, since a_{k+1} is positive $\sum_{i=1}^k ((-a_i) + b_i) \geq -b_{k+1}$. So choose constants $0 = c_0 \leq c_1 \leq c_2 \leq \dots \leq c_k = -b_{k+1}$ such that $(-a_i) - (-b_i) \geq c_i - c_{i-1}$ for $1 \leq i \leq k$. Straightforward calculus shows that, since x^τ is an increasing function and $-b_{k+1} \leq -a_i$ for all $i \leq k$ by hypothesis, that $((-a_i)^{\tau+1} - (-b_i)^{\tau+1}) \geq c_i^{\tau+1} - c_{i-1}^{\tau+1}$ for all $1 \leq i \leq k$. It follows that

$$\sum_{i=1}^k \int_{-b_i}^{-a_i} x^\tau dx = \sum_{i=1}^k \frac{((-a_i)^{\tau+1} - (-b_i)^{\tau+1})}{\tau + 1} \geq \frac{(-b_{k+1})^{\tau+1}}{\tau + 1}.$$

As $1/(\tau + 1) \geq a_{k+1}^{\tau+1}/(\tau + 1)$, the result follows. \square

6 Strengths and weaknesses of different functions

Our situation in identifying gerrymanders is analogous to that of assessing intelligence. Emotional intelligence and logical intelligence are best measured by different types of questions; no

single question type can capture the spectrum of ways in which intelligence can manifest itself. Every test for gerrymandering will have its strengths and weaknesses. In this section we attempt to illustrate, through concrete examples, ways in which each of the efficiency gap, the τ -gap, the mean-median difference and the declination can give misleading answers in their attempts to identify gerrymanders.

The oldest tool for this purpose, the seats-votes curve, has been around for over 40 years [Tuf73]. Notwithstanding its theoretical interest and applications in such areas as predictions (for example, [KGC08]), as it has not led to a manageable judicial standard for gerrymandering in this span, we will not address it directly in the remainder of this article.

6.1 The efficiency gap

The efficiency gap, through its role in *Whitford v. Gill*, has already proven its utility. And while its simplicity is an asset, it also has several drawbacks. First, by requiring that there be proportional representation for an election to be considered fair, it conflicts with constitutional law [Vie]. Second, it is simple to construct hypothetical examples for which a natural district plan leads to a constant of proportionality different from 2 (see, for example, Figs. 2.C,D). While the proportionality asserted by the efficiency gap may hold in some overall sense, for the instances for which it doesn't we are left in the difficult position of determining whether any deviation from this average law is due to gerrymandering or natural deviation. And, as illustrated by our definition of the τ -gap function, the proportionality with a constant of 2 depends on a subjective decision on how to weight various wasted votes. Third, there are historical examples in which the proportionality holds, but the shape of the vote distribution suggests significant partisan asymmetry (see, in particular, the 1974 Texas congressional election shown in Fig. 2.A as well as the 2012–2016 Tennessee congressional elections, the last of which is displayed in Fig. 8).

6.2 The τ -gap

The τ -gap family of functions is appealing due to its close connection to the efficiency gap and the fact that it does not reduce to proportionality. Additionally, when $\tau = 2$, the τ -gap is closely related to skewness of the a_i -distribution (see (8)), a standard measure of symmetry in statistical distributions. We also note that when $\tau = 2/5$, the correlation between the declination and the τ -gap is very high ($r^2 = 0.870, p < 0.001$ among 461 congressional elections). Unfortunately, the τ -gap family has its own drawbacks.

First, there is no reason to think that one value of τ will prove superior to other values in all cases. For example, consider the 1974 Texas and North Carolina congressional elections (see Fig. 2.A and Fig. 7.A, respectively). Proportionality is essentially met in both cases as reflected by the values of Gap_0 being close to zero; each election is fair according to the efficiency gap. In contrast, Gap_1 is approximately -0.38 for each, indicating strong asymmetry. While we have no absolute way of determining whether one or both is a gerrymander, evidence suggests that

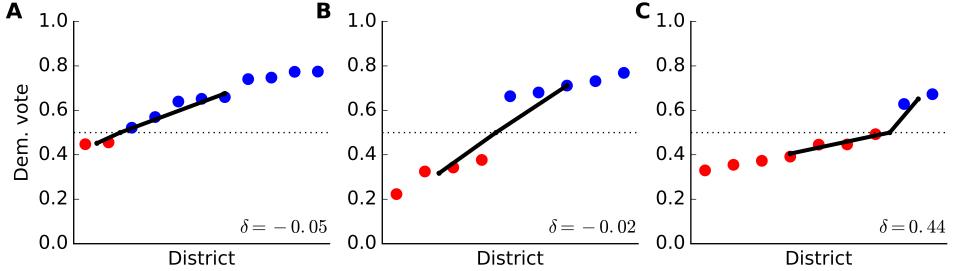


Figure 7: **(A)** Vote distribution for 1974 North Carolina congressional election with $\text{Gap}_0 = -0.1$ and $\text{Gap}_1 = -0.39$. **(B)** Vote distribution for 2006 Tennessee congressional election illustrating the mean-median difference of -0.13 . **(C)** Vote distribution for the 2012 Indiana congressional election showing a mean-median difference of 0.01 .

Gap_0 makes the correct assessment for the North Carolina election (little or no gerrymandering) while Gap_1 makes the correct assessment for the Texas election (gerrymandering)². We note that the declination yields what we believe to be the correct answer in both cases.

Second, even if there were one value of τ that was superior to all others, it is unclear what theoretical, rather empirical, justification could be used to support it. Third, as shown by (6), the τ -gap is essentially an interpolation, albeit a complicated one, between the proportionality with constant 2, when $\tau = 0$, and proportionality with constant 0, when $\tau = \infty$. (A fair election for the latter is one in which each party gets half the seats, regardless of the fraction of votes won by each side.)

6.3 The mean-median difference

The difference between the mean and the median of the democratic vote fraction among all districts has been suggested as yet another way of measuring partisan asymmetry (see, for example, Wang [Wan16] and [KMM⁺16, MB15]). A large difference between these values can indeed accompany an asymmetry in the vote distribution. For example, in the 2012 Pennsylvania congressional election (see Fig. 2.B), the mean is 0.50 while the median is 0.43, a difference of 0.07. However, the mean-median difference is very sensitive to the vote in a single district (practically speaking, this is less of an issue in the case of many districts). For example, in the 2006 Tennessee congressional election shown in Fig. 7.B, the difference is -0.13 . Had the democratic support been more unevenly distributed in the districts the Democrats won, the difference

²As described in [O'C90, pg. 35], there was not significant partisan rancor during the 1971 redistricting and the Democrats received approximately 63% of the congressional vote while winning 82% (9 of 11) congressional seats. Of course, these facts have little bearing on whether there may have been *racial* gerrymandering. For the Texas election, one author [May71] suggests that for the 1965 redistricting, protecting incumbents was a higher priority than advantaging the Democrats as a party. The constructions of a large number of safe seats for incumbents would certainly be consistent with the distribution seen.

would have been much smaller in absolute value. The declination, by relying on averages across districts rather than values within single districts is more robust in this respect.

Additionally, it is quite possible for packing and cracking to occur without having any effect on the mean-median difference. For example, consider the 2012 Indiana congressional election shown in Fig. 7.C. According to the mean-median difference, this is a fair election. However it is easy to see how less dominant wins in two districts could have translated into one or even two more seats for the Democrats: such changes need not affect the median.

It is also worth noting that the mean-median difference doesn't keep track of the number of seats won by each party. This is probably the main reason it is slightly more stable from election to election than the efficiency gap or the declination. On the other hand, this independence makes it harder to interpret the extent to which a particular value of the difference indicates actual impact on the electoral results.

6.4 The declination

The declination's primary weaknesses appear to occur when one party wins almost all of the seats. Fortunately, while gerrymandering may occur in such instances, these are typically not the cases of greatest interest. In most elections with a large number of seats, the vote fractions when sorted in increasing order increase essentially linearly except at the extremes. When one party dominates, this can lead to a declination that is large in absolute value even though the vote distribution as a whole is remarkably symmetric about the median democratic votes share. This is the case, for example in the 2008 Rhode Island state legislative election (shown in Fig. 9), in which the Democrats win two thirds of the vote. On the other hand, when there are few seats overall, it is not uncommon for one party to win all but one seat. In such a case, the declination is very sensitive to the exact vote fraction found in that one special district.

7 Data collection and Statistical methods

The US state legislature election data up through 2010 comes from [KBC⁺11]. We only included data on the lower house of the state legislature when two houses exist. The US congressional data through 2014 was provided by [Jac17]. Data on Wisconsin legislative elections from 2012 to 2016 were taken from Ballotpedia [Bal17]. Data for 2016 congressional races were taken from Wikipedia [Wik17] as they were not yet available from the Federal Election Commission.

The election data was analyzed using the python-based SageMath [S⁺16]. Python packages employed were pyStan [Tea16] for implementing a multilevel model to impute votes in uncontested races; Matplotlib [Hun07] and Seaborn [WBH⁺] for plotting and visualization; and SciPy [JOP⁺] for statistical methods.

For the state legislative elections, we restricted the analysis to elections that had no multi-member districts; vote fractions for uncontested races were imputed (i.e., we estimated what the values would have been had the races been contested).

Any election (year and state) in which there is at least one multi-member district (i.e., multiple winners in a single district) was excluded from our analysis. We ignored any third-party candidates and assumed there to be at most one Democrat and at most one Republican in each race. All vote percentages assigned to a candidate are therefore percentages of the two-party vote. We assume all districts have equal population and we do not take into account voter turnout in any way. Unopposed candidates are allowed. The declination is mildly sensitive to the vote distribution in each of these uncontested races (see below). Rather than assign 100% of the vote to the unopposed candidate, which is invariably not reflective of what would have happened had their been a two-candidate race, we imputed the vote fraction for such races.

Imputation of votes was done using a multilevel model of the form

$$y_i = \frac{1}{2} + \sigma_{j[i]} + \phi_{k[j[i]]} + \gamma_{\ell[i]} + \beta_1 W_i^D + \beta_2 W_i^R + \beta_3 I_i^D + \beta_4 I_i^R + \epsilon_i.$$

Here, y_i is the fraction of the vote garnered by the democratic candidate in district i ; $\sigma_{j[i]}$ is a random state effect; $\phi_{k[j[i]]}$ is a random district effect; $\gamma_{\ell[i]}$ is a random year effect; W_i^D and W_i^R are indicator variables for whether the democratic or republican candidate won, respectively; I_i^D and I_i^R are indicator variables for whether or not the democratic and republican candidates are incumbents, respectively; β_1 , β_2 , β_3 and β_4 are the corresponding random effects; and ϵ_i represents error due to individual characteristics of race i . The model was run separately for each redistricting cycle (typically of ten years), once for state legislatures and once for the US House of Representatives. As there are a number of cases in which states underwent major redistricting mid-decade, certain cycles were split. For example, Texas had one district plan for its congressional races during the years 1992–1996 and another for 1998–2000. These are represented in the Figure `reffig:congint.C` by TX1 and TX2, respectively.

There were 646 state elections in our data with a total of 68 955 races of which 25 371 of which were uncontested. There were 1 142 congressional elections with a total of 9 995 races of which 1 409 were uncontested.

When at least one race in a given district had been contested in a given cycle, imputation is straightforward using the values for the random and fixed effects provided by the model fit. We cross-validated the estimates by removing individual races, refitting and comparing the estimated to true values. We found a root mean square error of approximately 0.05 among 100 randomly chosen contested races. We note for comparison that if one uniformly assigns 65% of the vote to the winner, the error is approximately 0.09.

Of the 26 780 imputed values, there were 114 instances in which the data indicated a democratic winner, but the imputed value was less than or equal to 0.50. These imputed values were replaced with 0.505. Similarly, there were 40 instances in which a democratic loser was indicated, but the imputed value was greater than 0.50. These imputed values were replaced with 0.495.

There a number of instances in which a district was not contested at any time during a given reapportionment cycle. There were 11 770 state legislative district-cycle pairs. Of these, 1 281 were uncontested (i.e., the race in that district was not contested in any year of the cycle). The

corresponding numbers for congressional elections are 2030 and 41. For the vast majority of these cases, one of the two parties held the seat for the entire cycle. For these cases, we drew from the distribution of district effects stemming from districts that were consistently won by the same party throughout the cycle (with at least one contested race). There were a total of 19 state district-cycle pairs that were uncontested but held by both parties at some point during the cycle. The district effect in these cases was drawn at random from all district effects. This outcome did not occur for the congressional races.

We also performed a sensitivity analysis with respect to the imputed values by introducing a systemic bias of plus 3% to the democratic imputed votes for congressional elections. We then performed a linear regression of change in declination with respect to the fraction of races that had to be imputed. Elections in which one party won all of the seats were omitted. The slope and r^2 -value for the regression line was (0.09, 0.76). The corresponding regression line for state elections was similar, but with a higher r^2 -value.

For congressional elections with at least eight seats, 53% had at most 10% of the races uncontested while 90% had at most 40% of the seats uncontested. For state elections, only 31% had less than 10% of the seats uncontested while 92% had less than 65% of the races uncontested.

8 Conclusion

Among the tests discussed in the previous section, we believe that the declination, on the whole, possesses the most desirable combination of characteristics. First, it does not reduce to proportionality, and hence is not at odds with constitutional law. Second, it is readily visualized. Third, the declination is directly computed using fundamental aspects of the election, namely the number of seats each party wins along with the average vote fraction each earns in those wins. While the efficiency gap is perhaps even simpler, the general τ -gap is certainly much harder to visualize. Fourth, the declination is relatively robust with respect to the vote fraction in any individual district, unlike the mean-median difference. The declination also must change in the presence of packing and cracking, a feature that does not always hold for the mean-median difference.

The declination and compactness metrics are complementary, each captures particular characteristics of gerrymandering. These measures do not capture everything about an election and one must be careful to not ascribe more importance to a single number than is warranted (see [O’N16]). Nonetheless, they have utility through their ability to provide a consistent way to compare elections in different states and years. Assuming the courts ultimately accept one or more standards for ascertaining whether a district plan amounts to unconstitutional gerrymandering, it will be helpful if there is enough flexibility provided so that as our tools and understanding evolve, so do our classification standards.

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10 Appendix

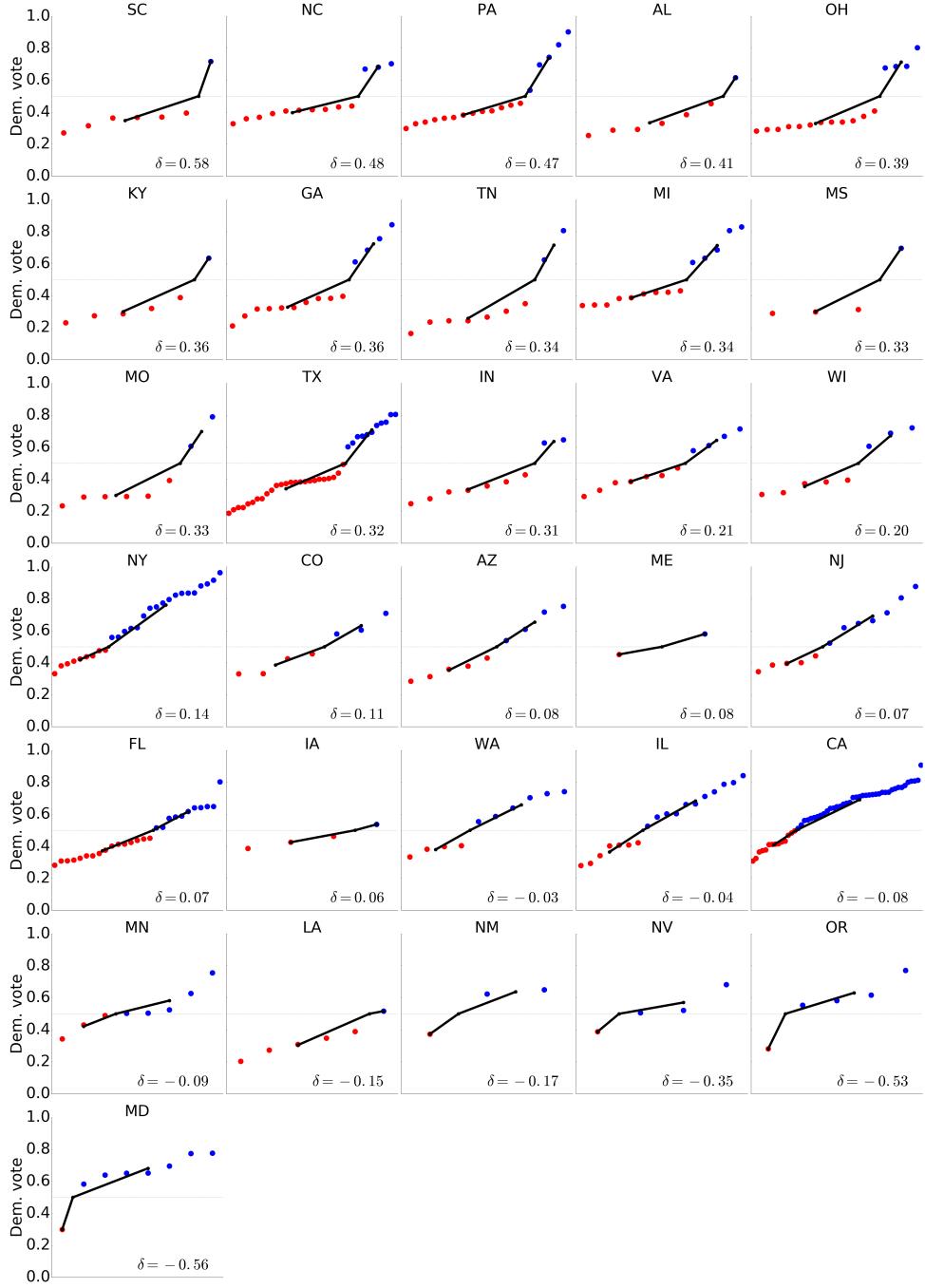


Figure 8: 2016 congressional races with at least one win by each party. Listed in decreasing order of declination.

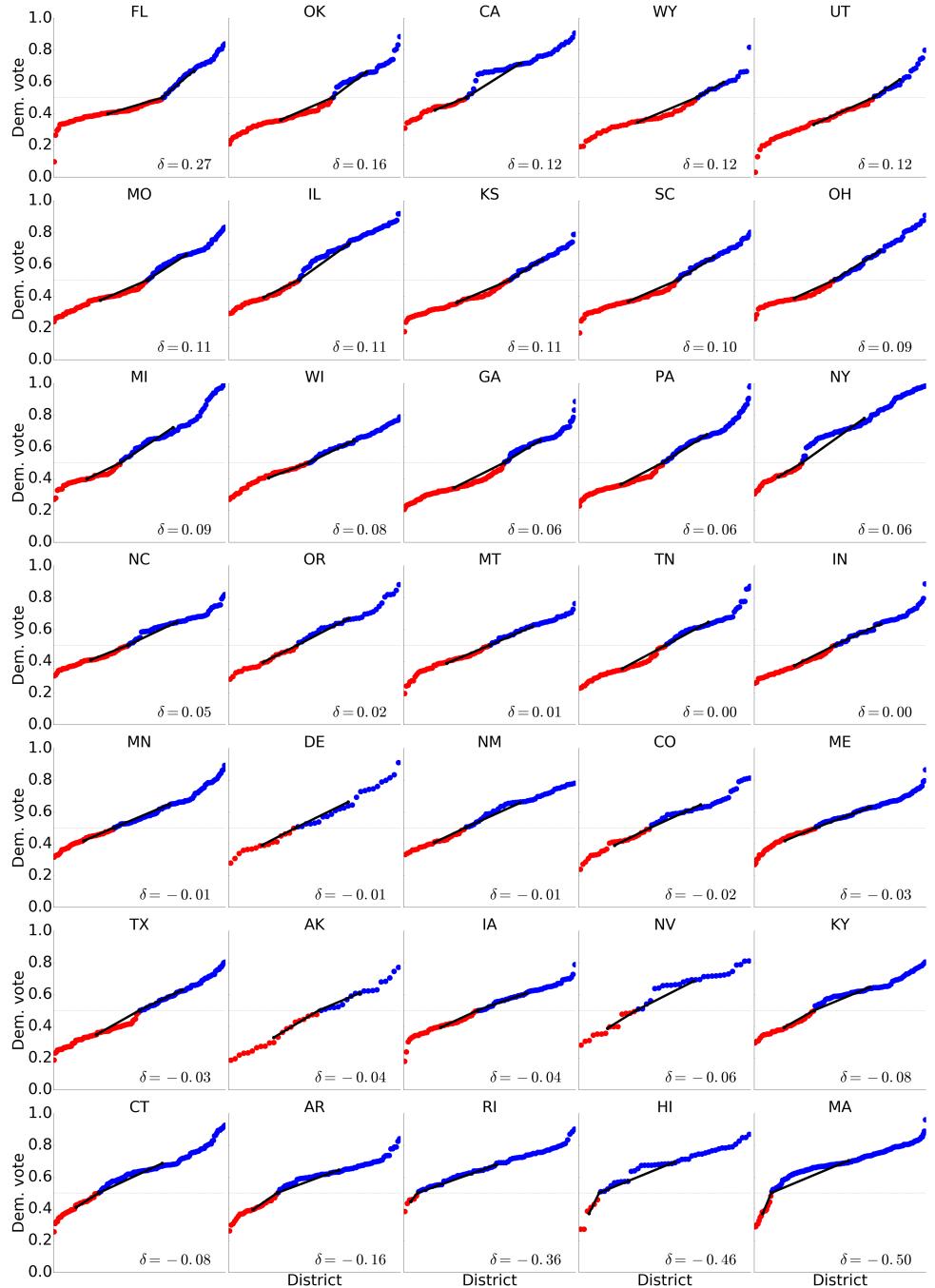


Figure 9: 2008 state legislative races with at least one win by each party. Listed in decreasing order of declination.

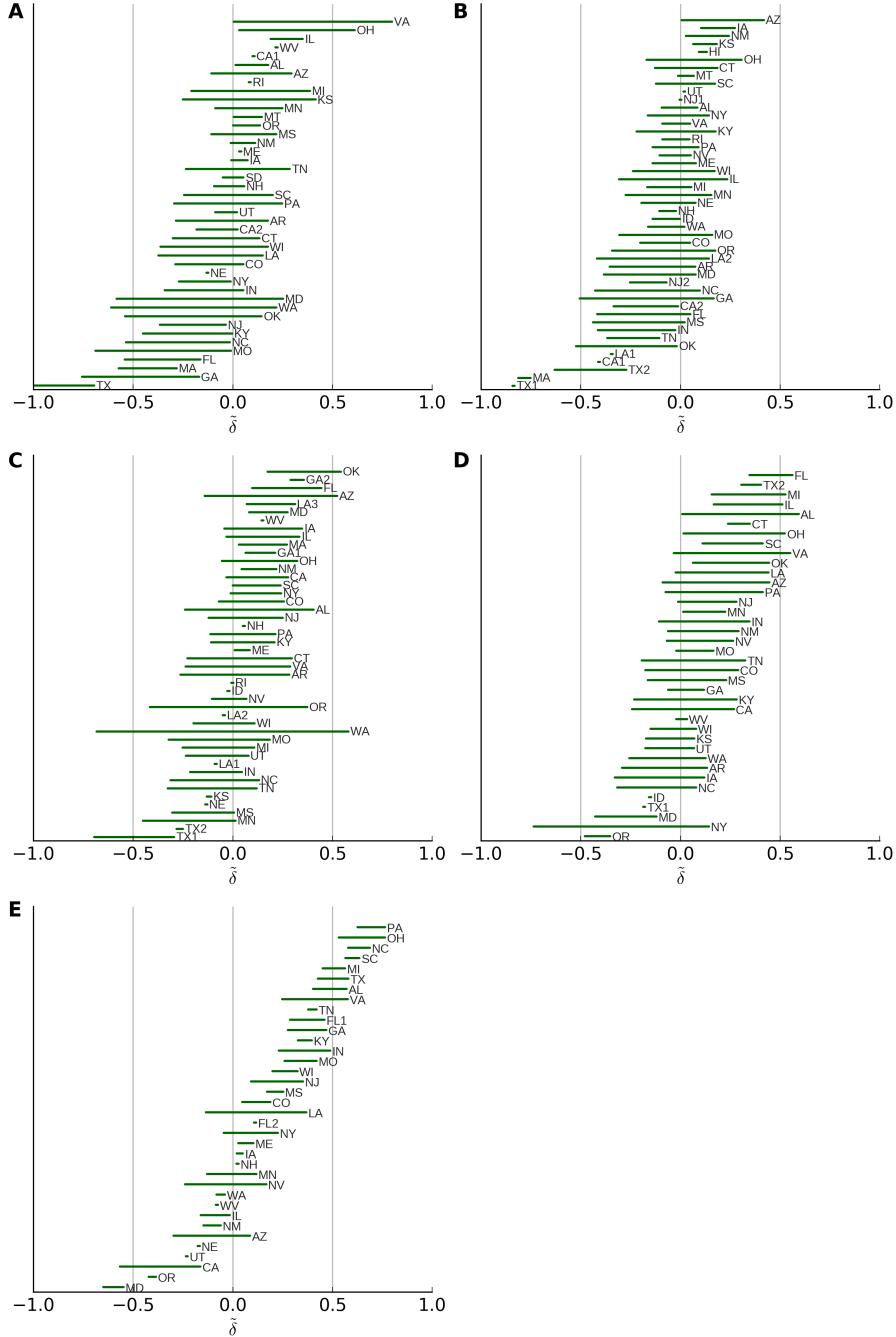


Figure 10: Range of $\tilde{\delta}$ over each districting cycle for congressional elections: **(A)** 1972–1980, **(B)** 1982–1990, **(C)** 1992–2000, **(D)** 2002–2010, **(E)** 2012–2016. Major redistrictings within each cycle are indicated by concatenation (e.g., TX1 refers to the district plan in effect at the beginning of the cycle and TX2 to the next plan). States were omitted if one party won all seats during the cycle. Subplot **(E)** is identical to Fig. 5 but is repeated here for comparison with earlier decades.

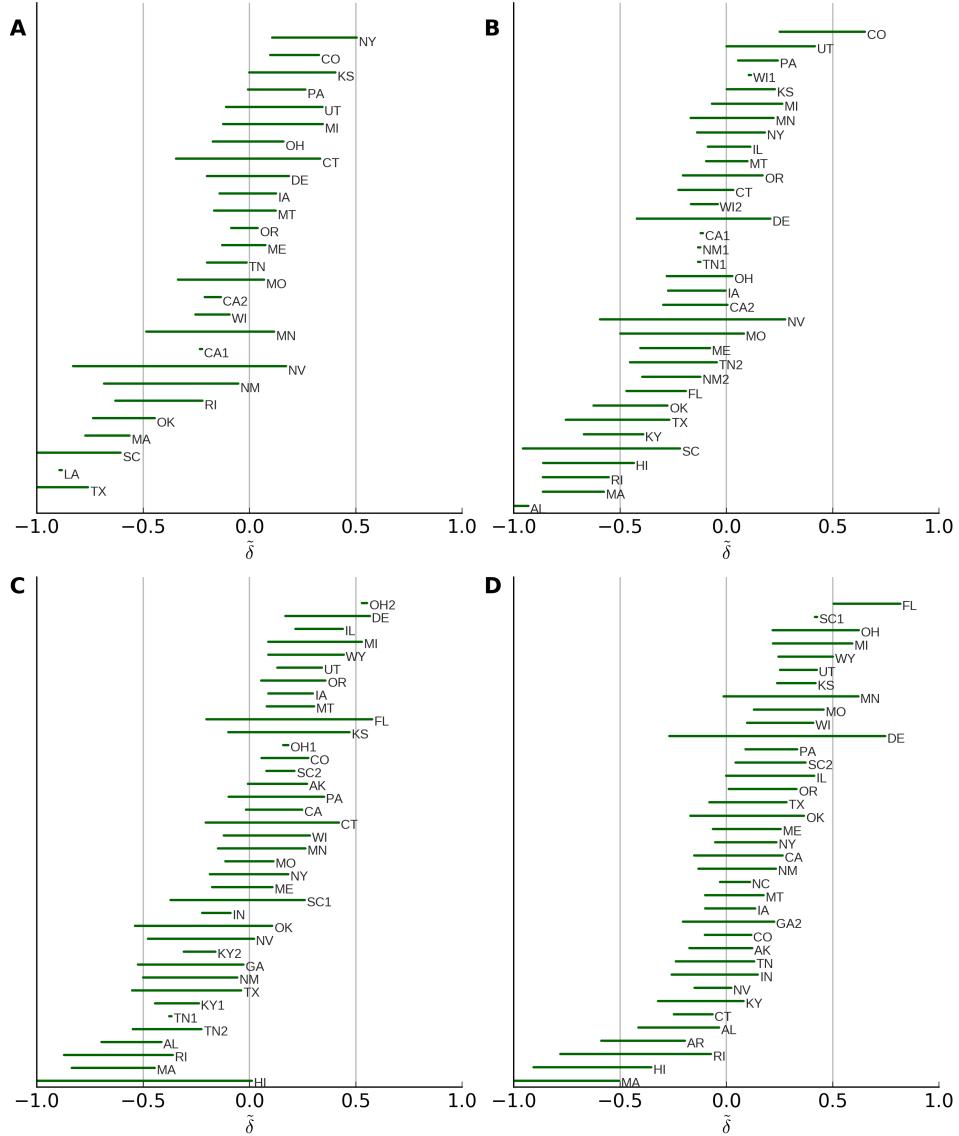


Figure 11: Range of $\tilde{\delta}$ over each districting cycle for state legislative elections: **(A)** 1972–1980, **(B)** 1982–1990, **(C)** 1992–2000, **(D)** 2002–2010. Major redistrictings within each cycle are indicated by concatenation (e.g., OH1 refers to the district plan in effect at the beginning of the cycle and OH2 to the next plan). States were omitted if one party won all seats during the cycle (or if the plan had multi-member districts).

Table 2: Most extreme values of $\tilde{\delta}$ for congressional elections since 1972.

Most positive $\tilde{\delta}$						Most negative $\tilde{\delta}$					
Year	State	Seats	$\tilde{\delta}$	δ	δ_N	Year	State	Seats	$\tilde{\delta}$	δ	δ_N
1980	VA	10	0.80	0.69	3.5	1976	TX	24	-1.07	-0.67	-8.1
2012	PA	18	0.76	0.53	4.7	1982	TX	27	-0.83	-0.51	-6.8
2012	OH	16	0.76	0.55	4.4	1990	MA	11	-0.82	-0.68	-3.7
2014	NC	13	0.69	0.54	3.5	1988	MA	11	-0.79	-0.66	-3.6
2016	PA	18	0.67	0.47	4.2	1984	MA	11	-0.77	-0.65	-3.5
2012	SC	7	0.63	0.65	2.3	1986	MA	11	-0.77	-0.64	-3.5
2014	PA	18	0.62	0.43	3.9	1972	GA	10	-0.76	-0.66	-3.3
2016	NC	13	0.61	0.48	3.1	1982	MA	11	-0.75	-0.63	-3.5
1972	OH	23	0.61	0.39	4.5	2008	NY	29	-0.74	-0.44	-6.3
2010	AL	7	0.59	0.61	2.1	1978	TX	24	-0.73	-0.46	-5.5
2014	SC	7	0.59	0.60	2.1	1980	TX	24	-0.72	-0.46	-5.5
1994	WA	9	0.58	0.53	2.4	1974	TX	24	-0.71	-0.45	-5.4
2016	TX	36	0.58	0.32	5.8	1972	TX	24	-0.70	-0.44	-5.3
2012	NC	13	0.58	0.45	2.9	1992	TX	30	-0.70	-0.41	-6.1
2012	VA	11	0.58	0.48	2.6	1972	MO	10	-0.69	-0.60	-3.0
1974	OH	23	0.57	0.37	4.2	1992	WA	9	-0.68	-0.62	-2.8
2012	AL	7	0.57	0.59	2.1	2014	MD	8	-0.65	-0.63	-2.5
2016	SC	7	0.56	0.58	2.0	1990	TX	27	-0.63	-0.38	-5.2
2012	MI	14	0.56	0.43	3.0	1978	WA	7	-0.61	-0.63	-2.2
2010	FL	25	0.56	0.35	4.4	1988	TX	27	-0.60	-0.36	-4.9
2002	FL	25	0.56	0.35	4.4	1994	TX	30	-0.59	-0.35	-5.2
2006	VA	11	0.55	0.46	2.5	1976	WA	7	-0.59	-0.60	-2.1
2016	OH	16	0.54	0.39	3.1	1980	MD	8	-0.58	-0.56	-2.2
1994	OK	6	0.54	0.60	1.8	2016	MD	8	-0.58	-0.56	-2.2
2014	OH	16	0.53	0.38	3.1	1980	MA	12	-0.57	-0.46	-2.8
2006	MI	15	0.53	0.39	2.9	2014	CA	53	-0.57	-0.29	-7.6
2004	FL	25	0.52	0.33	4.1	1974	WA	7	-0.56	-0.58	-2.0
2006	OH	18	0.52	0.36	3.3	2012	MD	8	-0.55	-0.53	-2.1
1998	AZ	6	0.52	0.58	1.8	1978	FL	15	-0.54	-0.40	-3.0
2000	AZ	6	0.51	0.57	1.7	1978	OK	6	-0.54	-0.60	-1.8
2010	IL	19	0.51	0.35	3.3	1978	NC	11	-0.54	-0.45	-2.5
2014	MI	14	0.50	0.38	2.7	1980	OK	6	-0.54	-0.60	-1.8

Table 3: Values of declination for congressional elections. Empty entries are due to one party winning all of the seats.

Table 4: Values of declination for state lower house elections. Empty entries are due to one party winning all of the seats or to the election containing multi-member districts.

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