# The Type of Language for Mathematical Programming

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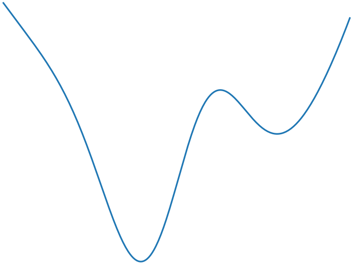
JuliaCon, June 23 2017

# What is an optimization problem?

the optimization contract:

- you give me a function, and I'll find you its minimum



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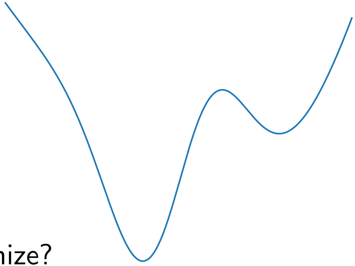
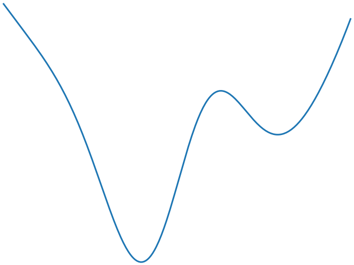


- why optimize?
  - fit a model to data (*e.g.*, understand customer preferences)
  - make predictions (*e.g.*, image recognition)
  - maximize revenue (*e.g.*, airline pricing)
  - maximize investment returns (*e.g.*, quant finance)
  - design a control system (*e.g.*, autopilot)
  - ...

## How to find the minimum



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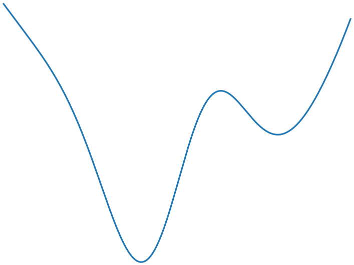

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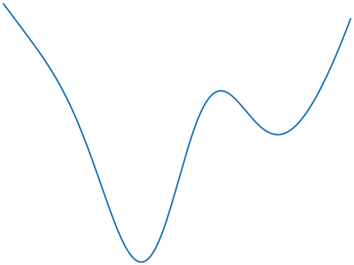

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- set derivative to zero

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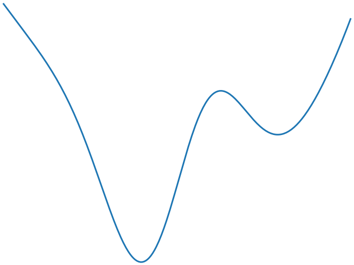

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- set derivative to zero ... if $\frac{d}{dx}f$ has an analytical formula

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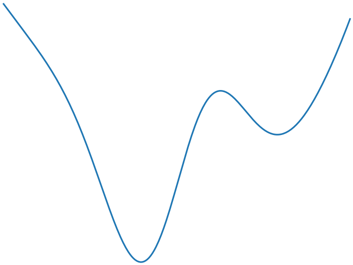

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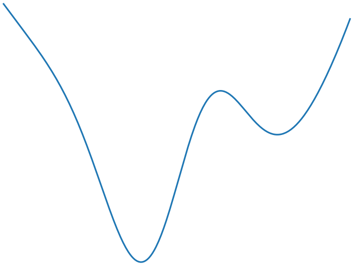

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- gradient descent / backprop ... if $f$ is differentiable

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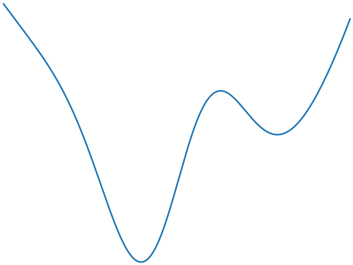

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- gradient descent / backprop ... if $f$ is differentiable
- other fancy optimization methods

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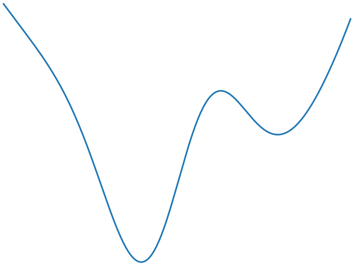

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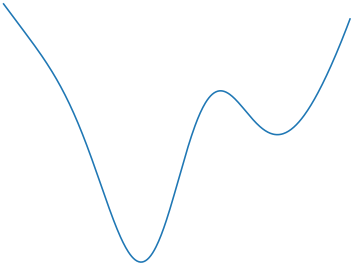

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- point

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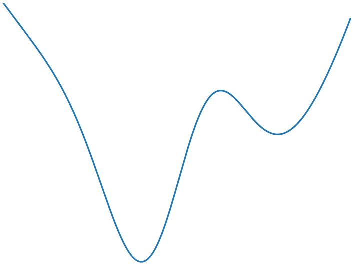

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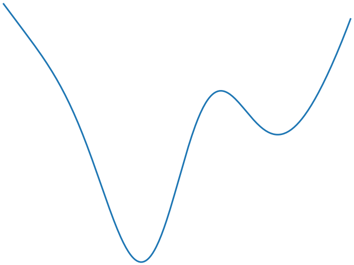

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**key question:** how will you give me the function?

## Example: what is $\frac{d}{dx}$?

what is

$$\frac{d}{dx}(x^2)|_{x=1}$$

?

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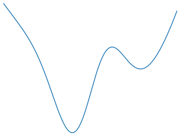
$$\frac{d}{dx}(x^2)|_{x=1}$$

? it's *bad notation!*

$x$ means three different things above:

- $x^2$ is the square function
- $\frac{d}{dx}$ is an operator that takes a function and returns its derivative (another function)
- $|_{x=1}$ evaluates the function at the argument 1

# How to give a function



- as a plot
- as an oracle

```
f(x) = x^2     › f
f(1)   1
f(2)   4
```

- as a type

```
type Square
end
f = Square()     Square()
evaluate(f::Square,x) = x^2
evaluate(f,1)   1
evaluate(f,2)   4
```

# How to give a function

**demo:**

`https://github.com/madeleineudell/JuliaCon17/types-for-opt.ipynb`

# How to give a function

moral:

- a function is a type
- on which various operations are defined
- which can be used to solve optimization problems

advantages:

- easy to understand
- easy to reuse code
- easy to extend by adding new methods

## What is an optimization problem?

optimization problem: nonlinear form

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m_1 \\
& h_i(x) = 0, \quad i = 1, \ldots, m_2 \\
& x \in \mathcal{C}
\end{array}
$$

- objective $f_0$
- inequality constraints $f_i$
- equality constraints $h_i$
- domain $\mathcal{C}$

advantages:

- easy to formulate

# Structure determines solvers

How should we solve this problem?

- LP solver?
- conic solver?
- nonlinear derivative based solver?
- operator splitting?

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**What do we know about this problem's structure?**

## Structure

useful kinds of structure:

- is the problem convex?
  - is the objective convex?
  - is the domain convex?
  - are the inequality constraints convex?
  - are the equality constraints affine?
- is the problem representable in some standard form?
  - convex: LP, QP, SOCP, SDP, ...
  - nonconvex: MILP, MISOCP ...
- is the problem smooth?
- ...

# Optimization in Julia

model specifies structure; solvers exploit structure

- model (*e.g.*, JuMP or Convex)
- glue (MathProgBase)
- solvers (*e.g.*, GLKP, Gurobi, Mosek, ECOS, ...)

JuliaOpt curates all these solvers: `http://www.juliaopt.org/`

# Two major approaches

- JuMP: user specifies structure
- Convex: solver detects structure

# JuMP vs Convex

JuMP

- lower level interface
- access to advanced solver features
- automatic differentiation
- support for conic and nonlinear programming

Convex

- automatic structure detection
- automatic convexity proof
- can only solve convex problems

# Convex in action

**demo:**

`https://github.com/madeleineudell/JuliaCon17/Convex-intro.ipynb`

## Convex: behind the scenes

Convex is a framework for detecting and exploiting structure in optimization problems.

what properties of functions does Convex use?

- evaluate
- verify convexity
- compute conic form

Induction detects; recursion exploits. Let's see how.

# Expressions: behind the scenes

(using prefix notation)

- $x + y \implies (+, (x, y))$
- $x[1] + x[2] \implies (+, ((\text{index}, (x, 1)), (\text{index}, (x, 2)))$
- $\log(x + 7y) \implies (\log, (+, (x, (*, (7, y)))))$

Every composite expression has

- a **head** (operation) and
- a (possibly empty) list of **children** (arguments).

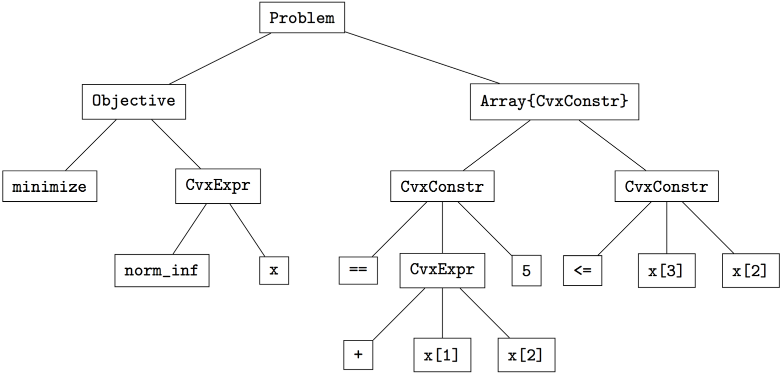
## Evaluate expressions recursively

to evaluate expression:

- apply top level function to value of argument
- *e.g.*, if top level function of expression is abs,

```
function evaluate(e::AbsAtom)
  return abs.(evaluate(e.children[1]))
end
```

# Abstract expression tree for an optimization problem



## Structure by induction

We use induction (and recursion) to move from properties of

- variables,
- constants, and
- functions

to properties of

- expressions,
- constraints, and
- problems.

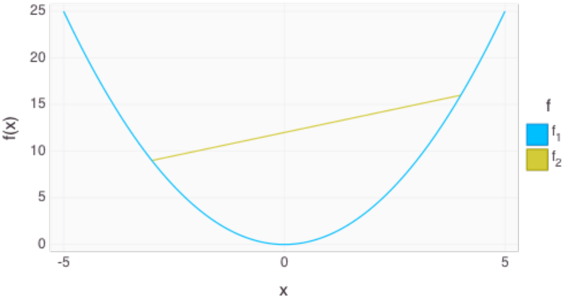
## Detecting structure: two case studies

- detect convexity
- transform to conic form

# Convexity

$f$ is *convex* if for all $\theta \in [0, 1]$

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$$




equivalently,

- $f$ has nonnegative (upward) curvature
- the graph of $f$ never lies above its chords

# Disciplined convex programming

**Disciplined convex programming** (DCP) [Grant, Boyd & Ye, 2006] provides a set of simple inductive rules to verify convexity:

- $f \circ g(x)$ is convex in $x$ if
  - $f$ is convex nondecreasing and $g$ is convex
  - $f$ is convex nonincreasing and $g$ is concave

  *cf.*, the chain rule:

$$(f \circ g)''(x) = f''(g(x))(g(x))^2 + f'(g(x))g''(x)$$

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A function is **DCP** if its convexity (or concavity) can be inferred from these composition rules.

(**N.B.** a function that is not DCP may still be convex)

# DCP: base case

A function `vexity` is defined on each data type (variable, constant, functions, constraints, problems) to return its **vexity**: constant, affine, convex, concave, or not DCP.

**base case:**

- *Constant.* Constants are **constant**.
- *Variable.* Variables are **affine**.

# DCP: inductive rule

**inductive rules:**

- *Expressions.* Functions each have known **curvature** (convex, concave, or affine) and **monotonicity** (increasing, decreasing, or none) in each of their arguments. Expressions check their convexity by examining convexity of arguments and following composition rules.
- *Constraints.* Constraints check their convexity by determining their left and right hand sides define convex sets.
- *Problems.* Problems check their convexity by verifying the objective and constraints are all convex.

# DCP: inductive rule

Composition rules are implemented as **arithmetic on vexities**:

$$\underbrace{\text{convex function}}_{\texttt{ConvexVexity}+}\ \underbrace{\text{nondecreasing}}_{\texttt{NonDecreasing}}\ \underbrace{\text{in}}_{*}\ \underbrace{\text{convex expression}}_{\texttt{ConvexVexity}}\ \underbrace{\text{is}}_{==}\ \underbrace{\text{convex}}_{\texttt{ConvexVexity}}$$

```
function vexity(x::AbstractExpr)
  monotonicities = monotonicity(x)
  vex = curvature(x)
  for i = 1:length(x.children)
    vex += monotonicities[i] * vexity(x.children[i])
  end
  return vex
end
```

# DCP in action

**demo:**

`https://github.com/madeleineudell/JuliaCon17/Convex-intro.ipynb`

# Conic form

Convex transforms optimization problems to conic form:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & b - Ax \in \mathcal{K}, \end{array}$$

where $\mathcal{K}$ is a **convex cone**:

$$x \in \mathcal{K} \iff rx \in \mathcal{K} \text{ for any } r > 0.$$

examples:

- zero cone $\mathcal{K} = \{0\}$
- positive orthant $\mathcal{K} = \{x : x_i >= 0, \ i = 1, \ldots, n\}$
- second order cone $\mathcal{K} = \{(x, t) : \|x\|_2 \leq t\}$
- positive semidefinite (PSD) cone
  $\mathcal{K} = \{X : X = X^T, \ v^T X v \geq 0, \ \forall v \in \mathbf{R}^n\}$
- products of cones

# Why conic form?

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & b - Ax \in \mathcal{K}, \end{array}$$

advantages:

- efficiently grok the structure of problem
- fast solvers

## Conic form for expressions

epigraph conic form for expressions:

$$f(x) = \begin{array}{ll} \text{min} & C[x,t] \\ \text{with variable} & t \\ \text{subject to} & A[x,t] \in \mathcal{K} \end{array}$$

(note: "objective" can be vector valued)

- function can be represented by tuple

$$(C, A, \mathcal{K})$$

## Conic form: base case

A function `conic_form` is defined on each data type to return the tuple $(C, A, \mathcal{K})$.

**base case:**

- *Constant.*

$$3 = \begin{array}{ll} \text{min} & 3 \\ \text{with variable} & \emptyset \\ \text{subject to} & \emptyset \end{array}$$

- *Variable.*

$$x = \begin{array}{ll} \text{min} & x \\ \text{with variable} & \emptyset \\ \text{subject to} & \emptyset \end{array}$$

## Conic form: inductive rule

**inductive rule:** if

$$f(y) = \begin{array}{ll} \text{min} & C^f[y, t^f] \\ \text{with variable} & t^f \\ \text{subject to} & A^f[y, t^f] \in \mathcal{K}^f, \end{array}$$

$$g(x) = \begin{array}{ll} \text{min} & C^g[x, t^g] \\ \text{with variable} & t^g \\ \text{subject to} & A^g[x, t^g] \in \mathcal{K}^g \end{array}$$

then

$$f(g(x)) = \begin{array}{ll} \text{min} & C^f[C^g \;\; I][x, t^g, t^f] \\ \text{with variable} & t^g, t^f \\ \text{subject to} & A^f[C^g I][x, t^g, t^f] \in \mathcal{K}^f \\ & A^g[x, t^g] \in \mathcal{K}^g \end{array}$$

## Conic form: inductive rule

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**proof:** $f$ is convex and increasing in its argument and $g$ is convex, so partial minimizations over $t^f$ and $t^g$ commute.

# Conic form: inductive rule

**in math:**

$$
f(g(x)) = \begin{array}{ll}
\text{min} & C^f[C^g \ \ I][x, t^g, t^f] \\
\text{with variable} & t^g, t^f \\
\text{subject to} & A^f[C^g I][x, t^g, t^f] \in \mathcal{K}^f \\
 & A^g[x, t^g] \in \mathcal{K}^g
\end{array}
$$

**in code:**

```
function conic_form(f::AbstractExpr)
  (Cg,Ag,Kg) = conic_form(f.children)
  (Cf,Af,Kf) = conic_form(f.head)
  return (Cf*[Cg I], [Af*[Cg I], [Ag 0]], [Kf, Kf])
end
```

# Coda: compilers

is Convex reproducing the compiler?

- yes: and we're not ashamed
  - type system + multiple dispatch makes it easy
  - so you can simulate a compiler
  - without understanding Julia's own compiler

other optimization software goes down the rabbit hole:

- Automatic Differentiation **demo:**

  `https://github.com/madeleineudell/JuliaCon17/types-for-opt.ipynb`

# The Type of Language for Mathematical Programming

Julia is the right language for mathematical programming:

- a function is a type
- on which various operations are defined
- which can be used to solve optimization problems

so use Julia for your mathematical programming:

- Julia has a tremendous ecosystem of optimization software
- use it!

more information (and code!)

- JuliaOpt: `http://www.juliaopt.org/`
- Convex: `http://www.github.com/JuliaOpt/Convex.jl`
- LowRankModels: `http://www.github.com/madeleineudell/LowRankModels.jl`

more optimization at JuliaCon:

- 3:52pm today: Mihir Paradkar on GraphGLRMs
- (yesterday) 5:09pm: Ayush Pandey on complex numbers in Convex