Towards analysis on the adeles

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This essay presents the rudiments of adelic analysis. Section 1 begins by introducing the adeles through example. Next, we define a suitable topology on the adeles, eventually discussing the group of additive characters over the adeles. Section 2 introduces the beginnings of adelic Fourier analysis, first introducing Schwartz functions and their various properties. I conclude with a proof of the adelic Poisson summation formula. Although of significant importance, we do not spend much time discussing the *ideles*, the multiplicative subgroup of the adeles. The exposition follows [2].

1 Constructing the adeles

In this section, I introduce the *adeles* over a number field $\mathbf{k}$, denoted $\mathbf{A_k}$ (or $\mathbf{A}$ when the underlying field is clear). I work over $\mathbf{k} = \mathbf{Q}$ rather than a general number field in order to simplify computations, though all of our results extend to arbitrary number fields, modulo some minor bookkeeping.

1.1 Example: the character group of $\mathbf{Q}$

To motivate the coming discussion, we consider the character group of the group $\mathbf{Q}^{(p)} = \{a/p^n : a, n \in \mathbf{Z},\ p \text{ prime}\}$. Obviously for $\chi \in \widehat{\mathbf{Q}^{(p)}}$,

$$\chi\left(\frac{a}{p^n}\right) = \chi\left(\frac{1}{p^n}\right)^a, \tag{1}$$

so we need only concern ourselves with $\chi(1/p^k)$, for $0 \leq n$. We can see that the characters satisfy the relationship

$$\chi\left(\frac{1}{p^{n+1}}\right)^p = \chi\left(\frac{1}{p^n}\right). \tag{2}$$

Conversely, any set of numbers $\{\chi(1/p^n) : n \geq 0\}$ satisfying relationship (2) along with the property $|\chi(1/p^n)| = 1$ yields a character on $\mathbf{Q}^{(p)}$ determined by relationship (1).

Since $|\chi(1/p^n)| = 1$, we know

$$\chi\left(\frac{1}{p^n}\right) = \exp\left(2\pi i \cdot \frac{\alpha_n}{p^n}\right),$$

where $\alpha_n \in \mathbf{R}$ is determined mod p^n. Because equation (2) gives us

$$\exp\left(2\pi i \cdot \frac{\alpha_{n+1}}{p^{n+1}}\right)^p = \exp\left(2\pi i \cdot \frac{\alpha_n}{p^n}\right),$$

we can write

$$\alpha_{n+1} \equiv \alpha_n \ (p^n).$$

We can split any α_n into two components,

$$\alpha_n = -\alpha + \beta_n,$$

where $\alpha = -\alpha_0$ is an element of $\mathbf{R}$ determined mod 1, and β_n is an integer determined mod p^n satisfying the relationship

$$\beta_{n+1} \equiv \beta_n \ (p^n).$$

In other words, the β_n are truncations of the p-adic series

$$\beta = a_0 + a_1 p + a_2 p^2 + \cdots, \qquad 0 \leq a_k \leq p.$$

Using the above data and equation (1), we rewrite the characters as

$$\chi\left(\frac{a}{p^n}\right) = \exp\left(2\pi i \cdot (-\alpha + \beta) \cdot \frac{a}{p^n}\right).$$

We conclude that *every character on $\mathbf{Q}^{(p)}$ is determined by the pair of numbers* (α, β). Observe that

$$\left|\chi\left(\frac{a}{p^n}\right)\right| = \left|\frac{\exp\left(\frac{2\pi i\cdot\beta a}{p^n}\right)}{\exp\left(\frac{2\pi i\cdot\alpha a}{p^n}\right)}\right|,$$

so $\chi(a/p^n) \equiv 1$ if and only if α is a rational number whose denominator is not divisible by p and $\beta = \alpha$ (as a p-adic representation). Thus the character group of $\mathbf{Q}^{(p)}$ is the additive group of pairs (α, β), quotiented out by the pairs (r, r), where r ranges over all rational numbers with denominators not divisible by p.

The character group of $\mathbf{Q}$ is considerably more complicated than that of $\mathbf{Q}^{(p)}$, but the general pattern is similar. Indeed, $\chi \in \widehat{\mathbf{Q}}$ is given by an *infinite* sequence,

$$a = (a_\infty, a_2, a_3, \ldots, a_p, \ldots),$$

where a_p is a p-adic number, and a_∞ is a real number. Moreover, for p sufficiently large, the a_p become p-adic *integers*. These sequences are *adeles*, and the character χ_a associated to such a sequence is given by

$$\chi_a(r) = \exp\left(2\pi i\cdot(-a_\infty r + a_2 r + \cdots + a_p r + \cdots)\right)$$

The sum on the right can be truncated by chopping off all of the p-adic integer elements, as these contribute nothing to the exponential. The sum in parentheses becomes a finite sum of rational numbers, so the definition is indeed grammatical. The set of adeles forms an additive group under componentwise addition. Note that $\chi_{a+a'}(r) = \chi_a(r)\chi_{a'}(r)$, so the map

$$\mathbf{A} \to \widehat{\mathbf{Q}}, \qquad a \mapsto \chi_a(r)$$

is a homomorphism for any $r \in \mathbf{Q}$. One can verify that the kernel of the homomorphism consists of the *principal adeles*

$$a = (\alpha, \alpha, \ldots, \alpha, \ldots),$$

where α is a rational number. There is an obvious isomorphism between the subgroup of principal adeles and the rational numbers. Occasionally $\mathbf{Q}^\Delta$ is used to differentiate the *diagonal embedding* of $\mathbf{Q}$ into $\mathbf{A}$ from $\mathbf{Q}$

itself, but from this point forward, we freely use $\mathbf{Q}$ to denote the principal adeles. We may now characterize the group $\widehat{\mathbf{Q}}$:

$$\widehat{\mathbf{Q}} \cong \mathbf{A}/\mathbf{Q}. \tag{3}$$

We will prove this assertion later. The adeles form a ring under compenentwise addition and multiplication. The set of invertible adeles are called the *ideles*, and they form a group under componentwise multiplication. We use $\mathbf{J}$ to denote the idele group.

1.2 Topology of the Adeles

Developing the adele topology from an intrinsic standpoint takes a considerable amount of time. The goal of this essay is to present the rudiments of analysis over the adeles, so this section is somewhat classic in its approach. That is, I make a number of statements regarding the topology of the adeles, and I do not provide justification for why they are true. For a good introduction to the adele topology, see [1].

Consider the subgroup $\mathbf{A}^o$ of the adeles consisting of sequences where each p-adic component is a p-adic integer. We can write $\mathbf{A}^o$ as

$$\mathbf{A}^o = \mathbf{R} \times \prod_{p \neq \infty} \mathbf{Z}_p,$$

where $\mathbf{Z}_p$ is the subgroup of p-adic integers. Obviously, $\mathbf{A}^o$ inherits the the product topology, and we declare $\mathbf{A}^o$ to be an open set in $\mathbf{A}$. A sequence of adeles (a_n) is said to converge to an adele a if it converges to a componentwise *and* if there is an N such that for $n \geq N$, the numbers $a_p - a_{n_p}$ are p-adic integers. Due to the compactness of the groups $\mathbf{Z}_p$, the adele topology is locally compact.

One might suspect that the idele group $\mathbf{J}$ is given the subset topology, but this is not the case. Indeed, one can verify that the inverse mapping is not continuous under this topology. Rather, let $H = \{(x, y) \in \mathbf{A} \times \mathbf{A} : xy = 1\}$. As explained in [3], the idele topology is identified with the subset topology of H.

Recall that $\mathbf{Q} = \{(r, r, \cdots, r, \cdots) : r \in \mathbf{Q}\} \subset \mathbf{A}$ is the set of principal adeles in $\mathbf{A}$. We will prove that $\mathbf{A}/\mathbf{Q}$ is compact, but first we need two lemmas.

Lemma 1.1. *The principal adeles* $\mathbf{Q}$ *are discrete in* $\mathbf{A}$.

Proof. Assume that $\mathbf{Q}$ is not discrete in $\mathbf{A}$. Then there exists a sequence of principal adeles (r_n) converging to zero. By the topology introduced above, this implies that for sufficiently large n, the number r_n is a p-adic integer. But this means that r_n is a non-zero integer, and therefore the sequence r_n does not converge to zero in the topology of $\mathbf{R}$. $\square$

Lemma 1.2. *The set* $F = \{a \in A : 0 \leq a_\infty \leq 1,\ |a_p|_p \leq 1\}$ *is a fundamental domain of the additive group* $\mathbf{A}$ *relative to* $\mathbf{Q}$.

Proof. Let $a = (a_2, a_3, \ldots, a_p, \ldots, a_\infty)$ be an arbitrary adele, and define α as the sum of the fractional parts of the p-adic components of a. Because the sequence a becomes p-adic integers for large enough p, the sum defined by α is a finite sum of rational numbers. Choose an integer n such that $0 \leq a_\infty - \alpha - n < 1$, and consider the principal adele $r = \alpha + n$. We can see that $a - r \in F$ by design. In other words, $\mathbf{A}$ *is a union of sets* $r + F$*, where* r *ranges over the principal adeles*. To complete the proof, we must show that these sets are pairwise disjoint

Consider the sets $r + F$ and F – if r is an integer and $a \in F$, then $r + a_\infty > 1$ does not belong to the interval $[0, 1)$, and it follows that $r + a \notin F$. If r is *not* an integer, then $|r|_p > 1$ for some p, and thus $|r + a_p|_p > 1$ for $a \in F$. Once again, it follows that $r + a \notin F$. $\square$

Using the above lemmas, proving that $\mathbf{A}/\mathbf{Q}$ is compact is immediate.

Proposition 1.1. *The quotient* $\mathbf{A}/\mathbf{Q}$ *is compact.*

Proof. Consider the quotient map,

$$q : \mathbf{A} \to \mathbf{A}/\mathbf{Q}, \qquad a \mapsto a + \mathbf{Q}.$$

By Lemma 1.2, we need only worry about the image of F. But F is a compact set and q is a continuous mapping, so $q(F)$ is compact, and the result follows. $\square$

1.3 The group of additive characters on A

To construct a suitable environment for Fourier analysis, we must consider the group of additive characters over $\mathbf{A}$. Consider the function

$$\sigma : \mathbf{A} \to \mathbf{R}, \qquad \sigma(a) \equiv -a_\infty + \sum_{p \neq \infty} a_p \pmod 1.$$

Because a_p is a p-adic integer for large enough p, the above sum is well-defined. The function σ has several important properties.

- For arbitrary $a, a' \in \mathbf{A}$, $\sigma(a + a') = \sigma(a) + \sigma(a')$.
- For each embedding $\mathbf{Q}_p$ in $\mathbf{A}$, we have that $\sigma(a_p) = 0$ if and only if a_p is a p-adic integer.
- If $a \in \mathbf{Q}$, then $\sigma(a) = 0$.

The first two properties are immediate. The third property takes a bit of work, but it is not too difficult to prove. We may define an associated additive character,

$$\psi(a) = \exp(2\pi i\sigma(a)).$$

Recall that a ring L is *self-dual* if every additive character χ is of the form

$$\chi(x) = \chi_0(ax), \qquad a \in L,$$

where χ_0 is some fixed character on L.

Theorem 1.1. *The ring of adeles is self-dual with generating character*

$$\psi(a) = \exp(2\pi i\sigma(a))$$

Proof. Given the results on σ, we know that $\psi(a_p) \equiv 1$ on every subring $\mathbf{Q}_p$ of $\mathbf{A}$ if and only if a_p is a p-adic integer, and we know that $\psi(a) \equiv 1$ for $a \in \mathbf{Q}$. We proceed by showing that every additive character $\chi(a)$ on $\mathbf{A}$ takes form $\psi(ba)$, for some $b \in \mathbf{A}$.

We begin by observing that any character χ on $\mathbf{A}$ can be decomposed multiplicatively:

$$\chi(a) = \prod_p \chi(a_p).$$

This is not totally trivial, but nonetheless standard. Likewise, for some $a \in \mathbf{A}$, we may write

$$a = \lim_{p \to \infty} a^{(p)},$$

where $a^{(p)} = (a_\infty, a_2, \ldots, a_p, 0, 0, \ldots)$. Because χ is a continuous function,

$$\chi(a) = \lim_{p \to \infty} \chi(a^{(p)}).$$

Furthermore, because $\psi(a_p) \equiv 1$ on $\mathbf{Q}_p$, the character $\chi(a_p)$ on $\mathbf{Q}_p$ can be represented as

$$\chi(a_p) = \psi(b_p a_p), \qquad b_p \in \mathbf{Q}_p.$$

So by the decomposition of χ given above,

$$\chi(a) = \psi(b_\infty a_\infty) \cdot \psi(b_2 a_2) \cdots \psi(b_p a_p), \qquad a \in \mathbf{A}, \tag{4}$$

truncating the product because $\psi(b_k a_k) \equiv 1$ for $k > p$.

If we can show that $b = (b_\infty, b_2, \ldots, b_p, \ldots) \in \mathbf{A}$, then it follows that $\mathbf{A}$ is self-dual. Suppose that $b \notin \mathbf{A}$. This implies that there are infinitely many b_{p_k} that are not p-adic integers. Because $\psi(a_p) \equiv 1$ on $\mathbf{Q}_p$ if and only if a_p is a p-adic integer, we may choose a_{p_k} such that $\psi(b_{p_k} a_{p_k}) \not\equiv 1$. Therefore we may fix a $\varepsilon > 0$ and choose a p_k-adic integer a_{p_k} such that

$$|\psi(b_{p_k} a_{p_k}) - 1| > \varepsilon.$$

We may do the same thing for any index p_k. But then the infinite product,

$$\prod_{k=1}^{\infty} \psi(b_{p_k} a_{p_k}),$$

does not converge, contradicting Equation (4). Thus $b \in \mathbf{A}$, and we conclude that $\mathbf{A}$ is self-dual with generating character

$$\psi(a) = \exp(2\pi i \sigma(a))$$

□

With the self-duality of $\mathbf{A}$ in hand, it is easy to show that the isomorphism on line (3) holds.

Corollary 1.1. *The character group of* $\mathbf{A}/\mathbf{Q}$ *is isomorphic to the additive group of rational numbers.*

Proof. The character group of $\mathbf{A}/\mathbf{Q}$ is isomorphic to the subgroup of all characters on $\mathbf{A}$ for which $\chi(a) \equiv 1$ for every $a \in \mathbf{Q}$. By Theorem 1.1, we know that χ takes the form

$$\chi(a) = \psi(ba)$$

for some $b \in \mathbf{A}$. Putting our conditions together, we want to find the set of $b \in \mathbf{A}$ such that

$$\psi(ba) \equiv 1, \qquad a \in \mathbf{Q}.$$

We already know that $\psi(a) \equiv 1$ on $\mathbf{Q}$, so if $b \in \mathbf{Q}$, we have $\psi(ba) \equiv 1$. Conversely, it is easy to see that $\psi(ba) \not\equiv 1$ for $b \notin \mathbf{Q}$. We conclude that the character group of $\mathbf{A}/\mathbf{Q}$ takes the form

$$\psi(ba),$$

where $b \in \mathbf{Q}$. The result follows. $\square$

2 Analysis over the adeles

Many interesting number-theoretic results can be discovered by extending our analysis environment to the adeles. Doing so has a considerable start-up cost, but the benefits far outweigh the initial difficulties. Before we proceed, we need to consider a special class of functions $S(\mathbf{A})$, the space of *Schwartz functions* over the adeles. These functions are frequently referred to as *Schwartz-Bruhat* functions.

Definition 2.1 (Schwartz-Bruhat function)**.** Let $\mathbf{A}$ be the ring of adeles over a number field $\mathbf{k}$. A *Schwartz-Bruhat* function $\varphi \in S(\mathbf{A})$ is a product,

$$\varphi(a) = \prod_p \varphi_p(a_p),$$

of Schwartz functions defined locally. For a finite place p, $\varphi_p(a)$ is a locally constant function of compact support, whereas $\varphi_\infty(a_\infty)$ is a *linear combination* of products of classical Schwartz functions.

We proceed with a brief discussion of measure on the adeles.

2.1 Invariant Measures on the Adeles

Recall that the group of adeles $\mathbf{A}$ is a locally compact topological group, and therefore has an invariant measure denoted $\mathrm{d}a$. We normalize the measure according to the following condition:

$$\int_F \mathrm{d}a = 1,$$

where $F = \{a \in \mathbf{A} : 0 \le a_\infty < 1, |a_p|_p \le 1\}$ is the fundamental domain discussed earlier.

Given a sufficiently "nice" function f over the adeles with decomposition

$$f(a) = \prod_p f_p(a_p),$$

we may think of the measure $\mathrm{d}a$ as a product of invariant measures,

$$\mathrm{d}a = \prod_p \mathrm{d}a_p.$$

Each $\mathrm{d}a_p$ is the usual Haar measure over the additive group $\mathbf{Q}_p$, with $\mathrm{d}a_\infty$ being the Haar measure over the additive group of real numbers. That is,

$$\int f(a)\ \mathrm{d}a = \prod_p \int f_p(a_p)\ \mathrm{d}a_p.$$

One may similarly define an invariant measure $\mathrm{d}\lambda$ over the multiplicative group of ideles, $\mathbf{J}$.

2.2 Fourier transform of Schwartz functions

Let $\psi(a) = \exp(2\pi i \sigma(a))$. For a Schwartz function $\varphi \in S(\mathbf{A})$, we define the *Fourier transform* as

$$\hat{\varphi}(\xi) = \int \varphi(x)\psi(\xi x)\ \mathrm{d}x$$

By the formula for the inverse Fourier transform, we have

$$\varphi(x) = c \cdot \int \hat{\varphi}(\xi)\psi(-\xi x)\ \mathrm{d}\xi,$$

meaning $\hat{\hat{\varphi}}(-a) = c^{-1}\varphi(a)$. Under the given normalization of the measure $\mathrm{d}x$, $c = 1$. We claim that $\hat{\varphi}$ is a Schwartz function as well. Before we prove this result, we prove a lemma. Let φ_p be the characteristic function of the integers $\mathbf{Z}_p$ over the local field $\mathbf{Q}_p$.

Lemma 2.1. *The Fourier transform takes φ_p to itself. That is,*

$$\hat{\varphi_p}(\xi) = \begin{cases} 1 & \text{for } |\xi|_p \le 1, \\ 0 & \text{for } |\xi|_p > 1 \end{cases}.$$

Proof. Let $\psi(x) = \exp(2\pi i x)$ denote the usual exponential character on $\mathbf{Q}_p$, and let φ denote φ_p for the remainder of the proof. Because φ is a characteristic function, we have

$$\hat{\varphi}(\xi) = \int_{\mathbf{Q}_p} \varphi(x)\psi(\xi x)\ \mathrm{d}x = \int_{\mathbf{Z}_p} \psi(\xi x)\ \mathrm{d}x.$$

Let $\xi = p^k v$, where $|v| = 1$. By a quick change of variable,

$$\hat{\varphi}(\xi) = \int_{\mathbf{Z}_p} \psi(p^k v x)\ \mathrm{d}x = |p|^{-k} \int_{p^k\mathbf{Z}_p} \psi(y)\ \mathrm{d}y.$$

Obviously $\psi(x) \equiv 1$ on $p^k\mathbf{Z}_p$ for $k \geq 0$, so

$$\hat{\varphi}(\xi) = |p|^{-k} \int_{p^k\mathbf{Z}_p} \mathrm{d}y = \int_{\mathbf{Z}_p} \mathrm{d}x = 1.$$

But if $k < 0$, then ψ is a nontrivial character on $p^k\mathbf{Z}_p$, forcing the integral to 0. We conclude that

$$\hat{\varphi_p}(\xi) = \begin{cases} 1 & \text{for } |\xi|_p \leq |p|^0 = 1, \\ 0 & \text{for } |\xi|_p > |p|^0 = 1 \end{cases}.$$

□

Proposition 2.1. *The Fourier transform maps $S(\mathbf{A})$ to itself.*

Proof. Recall that for $\varphi \in S(\mathbf{A})$, we have the decomposition,

$$\varphi(a) = \prod_v \varphi_v(a_v).$$

The proposition reduces to a series of lemmas regarding the Fourier transforms of the local Schwartz-functions, φ_v. Recall that $\varphi_\infty(a_\infty)$ is differentiable function decreasing faster than any power of $|a_\infty|$ as $|a_\infty| \to \infty$. The Fourier transform of such a function is a function of the same form, so the proposition holds at the infinite place.

Next, consider the Schwartz function φ_v for a finite place v. We know that φ_v takes the form

$$\varphi_v(a_v) = \begin{cases} 1 & \text{for } |a_v|_v \leq 1, \\ 0 & \text{for } |a_v|_v > 1 \end{cases}.$$

By Lemma 2.1, the Fourier transform carries a function of this form into itself. From the formula $\hat{\hat{\varphi}}(-a) = \varphi(a)$, it follows that the Fourier transform maps $S(\mathbf{A})$ into itself. □

2.3 The Poisson summation formula

Let $\varphi \in \mathbf{S}(\mathbf{A})$ be an adelic Schwartz function with Fourier transform $\hat{\varphi}$. We prove a generalized version of the Poisson summation formula.

Theorem 2.1 (The Poisson summation formula)**.** *Let $\lambda \in \mathbf{J}$. Then we have the following formula:*

$$\sum_{\alpha \in \mathbf{Q}} \varphi(\lambda \alpha) = \frac{1}{|\lambda|} \sum_{\alpha \in \mathbf{Q}} \varphi(\lambda^{-1} \alpha).$$

Proof. Consider the auxiliary function defined over $\mathbf{A}$,

$$\Phi(a) = \sum_{\alpha \in \mathbf{Q}} \varphi(\lambda(\alpha + a))$$

We know that Schwartz functions are summable, and it immediately follows that Φ is well defined and summable on $\mathbf{A}/\mathbf{Q}$. Observe that by some extension of Fubini's theorem,

$$\int_{\mathbf{A}} |\varphi(\lambda a)| \, \mathrm{d}a = \int_{\mathbf{A}/\mathbf{Q}} \left(\sum_{\alpha \in \mathbf{Q}} |\varphi(\lambda(\alpha + a))| \right) \mathrm{d}a.$$

We can see that $\Phi(a)$ is constant on the cosets of $\mathbf{Q}$ and summable on the compact group $\mathbf{A}/\mathbf{Q}$, so $\Phi(a)$ has a corresponding Fourier series with respect to the characters ψ. Indeed, we may write the series as

$$\Phi(a) = \sum_{\beta \in \mathbf{Q}} c_\beta(\Phi) \psi(\beta a),$$

with Fourier coefficients

$$c_\beta(\Phi) = \int_{\mathbf{A}/\mathbf{Q}} \Phi(a) \psi(-\beta a) \, \mathrm{d}a.$$

Substituting Φ for its expression, we get

$$\begin{aligned} c_\beta(\Phi) &= \int_{\mathbf{A}/\mathbf{Q}} \left(\sum_{\alpha \in \mathbf{Q}} \varphi(\lambda(\alpha + a)) \right) \psi(-\beta a)\, \mathrm{d}a \\ &= \int_{\mathbf{A}} \varphi(\lambda a) \psi(-\beta a)\, \mathrm{d}a \\ &= \frac{1}{|\lambda|} \int_{\mathbf{A}} \varphi(a) \psi(-\beta \lambda^{-1} a)\, \mathrm{d}a \\ &= \frac{1}{|\lambda|} \psi(-\beta \lambda^{-1}). \end{aligned}$$

This gives us

$$\sum_{a \in \mathbf{Q}} \varphi(\lambda(\alpha + a)) = \frac{1}{|\lambda|} \sum_{\beta \in \mathbf{Q}} \varphi(-\lambda^{-1}\beta) \psi(\beta a).$$

Set $a = 0$ to obtain the Poisson summation formula. $\square$

References

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