Eisenstein series: a modern view

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Consider the *classical Eisenstein series* of weight $\kappa = 2k \in \mathbf{Z}$ over the full modular group $\mathrm{SL}(2, \mathbf{Z})$:

$$E_{\mathcal{H}}(z) = \sum_{(c,d) \in \mathbf{Z}^2 \setminus (0,0)} (cz + d)^{-\kappa}, \quad z \in \mathcal{H}, .$$

where $\mathcal{H}$ is the usual upper half-plane. The modular form $E_{\mathcal{H}}$ is absolutely convergent and extends to a holomorphic function at cusps. Originally studied as a function of the domain $\mathcal{H}$, the Eisenstein series was soon found to have more general analogues. This chapter serves as a transition from the classical environment of modular forms defined on $\mathcal{H}$, to the modern environment of automorphic forms over a topological group quotient. We conclude with an examination of the *adelic Eisenstein series* attached to the kernel ε,

$$E_{\varepsilon}(g) = \sum_{P_{\mathbf{Q}} \setminus G_{\mathbf{Q}}} \varepsilon(\gamma g), \quad g \in G_{\mathbf{A}},$$

where P is the standard parabolic subgroup of $G = \mathrm{SL}(2, \cdot)$ (see Section 2). Transitioning from the classical viewpoint to a more modern perspective requires significant start-up cost, but this effort pays oy

This chapter works in some generality, defining modular forms over principal congruence subgroups of level L, Γ_L, of $\mathrm{SL}(2, \mathbf{Z})$. Working with levels provides motivation for the "lifting" of modular forms on the classical domain to automorphic forms on the adele group. The first two sections "lift" a modular form on the upper-half plane to an automorphic form over the adeles. The third section discusses the adelic Eisenstein series intrinsically. As it turns out, the kernel ε that is attached to an adelic Eisenstein series can be decomposed place-wise, where the corresponding ε_v is a *spherical* vector in a principal series representation of $\mathrm{SL}(2, \mathbf{Q}_v)$.

We closely follows an unpublished writeup by Jerry Shurman [5]. For a concise introduction to the classical theory of modular forms, see Chapter 7 of [3]. A more exhaustive introduction to both modular forms and the modularity theorem is offered by [2]. For an introduction to the theory of automorphic forms, see [1].

1 Modular forms on the upper half-plane

This section assumes some familiarity with classical modular forms, presenting the standard definitions at a rapid pace. Recall the classical domain

$$\mathcal{H} = \{z \in \mathbf{C} : \mathrm{im}(z) > 0\}$$

and the classical group

$$G_{\mathbf{Q}} = \mathrm{SL}(2, \mathbf{Q}) = \{\gamma \in M_{2\times 2}(\mathbf{Q}) : \det(\gamma) = 1\}.$$

We let $G_{\mathbf{Z}} = \mathrm{SL}(2, \mathbf{Z})$ denote the full *modular group*, and Γ_L a *principal congruence subgroup* of $G_{\mathbf{Z}}$,

$$\Gamma_L = \left\{\gamma \in G_{\mathbf{Z}} : \gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{L\mathbf{Z}}\right\}.$$

The group $G_{\mathbf{Q}}$ acts on $\mathcal{H}$ through a Möbius transformation:

$$\gamma \cdot z = \frac{az+b}{cz+d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathbf{Q}}.$$

We define the *standard factor of automorphy* as a function

$$j : G_{\mathbf{Q}} \times \mathcal{H} \longrightarrow \mathbf{C}, \qquad j(\gamma, z) = cz + d.$$

Note that for $\gamma, \gamma' \in G_{\mathbf{Q}}$, j satisfies the *cocycle condition*,

$$j(\gamma\gamma', z) = j(\gamma, \gamma' z) j(\gamma', z).$$

Let $f : \mathcal{H} \to \mathbf{C}$. For any $\gamma \in G_{\mathbf{Q}}$ and $\kappa \in \mathbf{Z}^+$, we define the *weight-κ operator* as

$$[\gamma]_\kappa : \{f : \mathcal{H} \to \mathbf{C}\} \longrightarrow \{f : \mathcal{H} \to \mathbf{C}\}, \quad f[\gamma]_\kappa = f(\gamma z) j(\gamma, z)^{-\kappa}.$$

We call f $[\Gamma_L]_\kappa$*-invariant* if $f[\gamma]_\kappa = f$ for $\gamma \in \Gamma_L$. This invariance is also sometimes referred to as an *automorphy condition*.

Definition 1.1 (Modular form)**.** Let $\kappa \in \mathbf{Z}$. A function $f : \mathcal{H} \to \mathbf{C}$ is a *modular form of weight* κ with respect to Γ_L if

1. f is holomorphic on $\mathcal{H}$.

2. f is $[\Gamma_L]_\kappa$-invariant.

3. $f[\alpha]_\kappa$ is holomorphic at ∞ for all $\alpha \in G_{\mathbf{Z}}$.

While our definition of modular form restricts to a principal congruence subgroup Γ_L, we may define a modular form for any congruence subgroup Γ of $G_{\mathbf{Z}}$.

Suppose $f : \mathcal{H} \longrightarrow \mathbf{C}$ is L-periodic, so $f(z+L) = f(z)$. Think of $f(z) = f(x+iy)$ as periodic in the variable x with parameter y. The set of functions

$$B = \{\psi_{\ell/L}(x) : \ell \in \mathbf{Z} \text{ and } x \in \mathbf{R}\}, \qquad \psi_{\ell/L}(x) = e^{2\pi i\ell x/L}$$

is a complete orthonormal set for $\mathbf{R}/L\mathbf{Z}$. Thus we may define the Fourier series of f,

$$f(z) = \sum_{\ell \in Z} \psi_{\ell/L}(x) c_{\ell/L}(f, y),$$

where

$$c_{\ell/L}(f, y) = L^{-1} \int_{\mathbf{R}/L\mathbf{Z}} \psi_{-\ell/L}(\xi) f(\xi + iy)\, \mathrm{d}\xi.$$

If the Fourier expansion of $f[\alpha]_\kappa$ has constant term $c_0 = 0$ for all $\alpha \in G_{\mathbf{Z}}$, we call f a *cusp form*.

1.1 The classical Eisenstein series

Before defining the Eisenstein series, introduce a normalizing constant for $L \in \mathbf{Z}^+$,

$$C_L = \begin{cases} 1/2 & \text{if } L \leq 2 \\ 1 & \text{if } L > 2, \end{cases}$$

and the congruence subgroup,

$$N_{\mathbf{Z},L} = \left\{ \gamma \in G_{\mathbf{Z}} : \gamma = \begin{pmatrix} 1 & kL \\ 0 & 1 \end{pmatrix},\ k \in \mathbf{Z} \right\}.$$

The *Eisenstein series* on $\mathcal{H}$ of weight κ and level L is defined

$$E_{\mathcal{H},\kappa}(z) = C_L \sum_{\substack{(c,d)\equiv(0,1)\ (L)\\ (c,d)\neq 0\\ \gcd(c,d)=1}} (cz+d)^{-\kappa} = C_L \sum_{\gamma\in N_{\mathbf{Z},L}\backslash\Gamma_L} (1[\gamma]_\kappa)(z), \quad z\in\mathcal{H}.$$

The series $E_{\mathcal{H}}$ converges absolutely and uniformly on compact subsets for all weights $\kappa \geq 3$, though we will not prove this. The normalizing constant C_L is a necessary inconvenience. At levels $L \leq 2$, the set of summation is closed under negation, doubling the terms for even κ. Should κ be odd, the terms cancel pairwise and the Eisenstein series is 0.

Our goal is not to dwell on this elementary definition of Eisenstein series, but one should be provided with some indication as to why people studied such things. Fixing $L = 1$ for the remainder of the section (so $\Gamma_L = \mathrm{SL}(2,\mathbf{Z})$), $E_{\mathcal{H},\kappa}$ has Fourier expansion

$$E_{\mathcal{H},\kappa}(z) = 1 + \frac{(-2\pi i)^\kappa}{\zeta(\kappa)(\kappa-1)!}\sum_{i=1}^{\infty}\sigma_{\kappa-1}(m)q^m, \quad q = e^{2\pi iz}.$$

The Fourier expansions contain the divisor sums σ_κ, so the expansions can be used to derive a variety of interesting divisor relationships. That is, the Eisenstein series immediately reveals itself to have a host of number-theoretic properties. The space of weight-κ modular forms (res. cusp forms) with respect to Γ_L forms a vector space over $\mathbf{C}$, denoted $\mathcal{M}_\kappa(\Gamma_L)$ (res. $\mathcal{S}_\kappa(\Gamma_L)$). Because $\dim(M_\kappa(\Gamma_L)) = 1$ for $\kappa = 4, 6, 8, 10, 14$, the Eisenstein series must satisfy

$$\begin{aligned} E_{\mathcal{H},4}^2 &= E_{\mathcal{H},8}\\ E_{\mathcal{H},4}E_{\mathcal{H},6} &= E_{\mathcal{H},10}\\ E_{\mathcal{H},4}E_{\mathcal{H},10} &= E_{\mathcal{H},14}. \end{aligned}$$

Define the *theta series* as

$$\vartheta(z;\tau) = \sum_{n\in\mathbf{Z}} e^{\pi i n^2\tau + 2\pi i n z}.$$

The theta series ϑ is interesting for a number of reasons, its involvement in the solution to the one-dimensional heat equation being one example. Returning to the relevant discussion, setting $z = 0$ yields a function of

τ defined on the complex upper half-plane, and this series is intimately related with the Eisenstein series. To see this, first consider the *auxiliary theta series*

$$\begin{aligned}\vartheta_{01}(z;\tau) &= \vartheta\big(z+\frac{1}{2};\tau\big)\\ \vartheta_{10}(z;\tau) &= e^{\frac{1}{4}\pi i\tau+\pi iz}\vartheta\big(z+\frac{1}{2}\tau;\tau\big)\end{aligned}$$

and let $\vartheta_{00}(z;\tau) = \vartheta(z;\tau)$ denote the original theta series. For notational convenience, let $x = \vartheta_{00}$, $y = \vartheta_{10}$, and $z = \vartheta_{01}$. We may write certain Eisenstein series as weighted combinations of theta series:

$$\begin{aligned}E_4(\tau) &= \frac{1}{2}(x^8+y^8+z^8)\\ E_6(\tau) &= \frac{1}{2}\sqrt{\frac{(x^8+y^8+z^8)^3-54(xyz)^8}{2}}.\end{aligned}$$

While perhaps not immediately clear why these relationships are interesting, we claim that they may be used to solve several problems in number theory. Indeed, the *four squares problem* asks for any integer n, how many ways may we write n as the sum of four squares? For example, if $n = 4$, there are two possibilities:

$$\begin{aligned}4 &= 1^2+1^2+1^2+1^2\\ 4 &= 2^2+0^2+0^2+0^2.\end{aligned}$$

The relationships between the theta and Eisenstein series can actually be used to derive a counting formula that solves this problem. Moreover, there exists a scaled-up automorphic version of the formulae above known as the *Siegel-Weil formula*. While space limitations restrict us from exploring this formula, the rest of this essay is devoted to developing the automorphic equivalent of the classical Eisenstein series.

2 Decompositions of $\mathrm{SL}(2,\mathbf{k})$

Before we may "adelize" classical notions from the study of modular forms, we must shift our attention from special, symmetric functions on a domain $\mathcal{H}$, to suitably invariant functions on an appropriate group G. As it turns out, the latter class of functions subsumes the former, as we will soon see. Before we can do any of this, however, we must establish two important decompositions of the special linear group. Let $G_{\mathbf{Q}} = \mathrm{SL}(2, \mathbf{Q})$, as before. Consider the subgroups

$$N_{\mathbf{Q}} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\}, \quad M_{\mathbf{Q}} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\}, \quad P_{\mathbf{Q}} = \left\{ \begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} \right\}.$$

We refer to these subgroups as the *unipotent radical*, the *Levi component*, and the *standard parabolic subgroup*, respectively. We frequently denote elements of these subgroups by indicating their relevant entries in the indices, i.e. $m_a = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$, $n_b = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$, etc. Furthermore, we define the Weyl element,

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The *Bruhat decomposition* states that

$$G_{\mathbf{Q}} = P_{\mathbf{Q}} \sqcup P_{\mathbf{Q}} w N_{\mathbf{Q}}.$$

To see this, consider $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_{\mathbf{Q}}$. If $c = 0$, then $\gamma \in P_{\mathbf{Q}}$. Suppose $c \neq 0$, and observe that for fixed $a, d, c \in \mathbf{Q}$, $ad - bc = 1$ implies $b = (ad-1)/c$. With this in mind, compute

$$\begin{pmatrix} 1/c & a \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & (ad-1)/c \\ c & d \end{pmatrix}.$$

So for $c = 0$ or $c \neq 0$, the Bruhat decomposition holds.

Another useful decomposition of $G_v = \mathrm{SL}(2, \mathbf{Q}_v)$, where the v is an archimedean or non-archimedean completion of $\mathbf{Q}$, is the *Iwasawa decomposition*,

$$G_v = N_v M_v K_v,$$

where K_v is the *maximal compact* subgroup of G_v. If $\mathbf{Q}_v = \mathbf{R}$, then $K_v = K_{\mathbf{R}} = \mathrm{SO}(2, \mathbf{R})$; for a nonarchimedean place v, $K_v = \mathrm{SL}(2, \mathcal{O}_v)$, where $\mathcal{O}_v$ denotes the local integers. For the archimedean case, the decomposition specifies to

$$G_{\mathbf{R}} = N_{\mathbf{R}} M_{\mathbf{R}} K_{\mathbf{R}}.$$

Let $z_0 = i$, and let $g \cdot z_0$ denote the usual Möbius transformation of $G_{\mathbf{R}}$ on $\mathbf{C}$. Then if $g \cdot z_0 = z = x + iy$, we also have

$$n_x m_{\sqrt{y}} \cdot z_0 = n_x \cdot iy = x + iy = z.$$

Thus $g \cdot z_0 = n_x m_y \cdot z_0$, so $(n_x m_y)^{-1} g$ fixes z_0, meaning it takes the form $rk \in \mathbf{R}^+ K_{\mathbf{R}}$, thus $g = n_x \cdot r m_y \cdot k$, proving the Iwasawa decomposition. Moreover, the derivation shows that the Iwasawa decomposition is unique.

Next, consider a nonarchimedean place v,

$$G_v = N_v M_v K_v,$$

where $K_v = \mathrm{SL}(2, \mathcal{O}_v)$. To verify this, we check that the following product is an integral matrix:

$$m_y^{-1} n_x^{-1} g = \begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} y^{-1}(a - xc) & y^{-1}(b - xd) \\ yc & yd \end{pmatrix}.$$

Both c and d cannot be zero, so at least one of cd^{-1} or dc^{-1} is defined and integral. In the former case, set $y = 1/d$ and $x = b/d$ to make the right side

$$m_{1/d}^{-1} n_{b/d}^{-1} g = \begin{pmatrix} 1 & 0 \\ c/d & 1 \end{pmatrix},$$

which is integral. In the latter case, we similarly set $y = -1/c$ and $x = a/c$ to make the right side

$$m_{-1/c}^{-1} n_{a/c} g = \begin{pmatrix} 0 & 1 \\ -1 & d/c \end{pmatrix},$$

which is also integral, proving the Iwasawa decomposition. Note that the decomposition is no longer unique. That is, if both c and d are local units, then both c/d and d/c are integral and either decomposition is valid.

3 Automorphic forms over a topological group

This section aims to scale the classical theory to a more modern environment. Although much of the content in a classical modular forms course is interesting in and of itself, the machinery of complex analysis quickly becomes unwieldy if one tries to generalize $\mathcal{H}$ in an inappropriate manner. Once we begin considering functions on certain groups G rather than the domain $\mathcal{H}$, methods from functional analysis, representation theory, and Lie theory help to clarify and generalize arcane, complex-analytic arguments. Even though we do not technically *need* this machinery for the case of $G = \mathrm{SL}(2, \mathbf{Q})$ (the group we will address), the methods discussed in this section generally apply to an arbitrary reductive or semi-simple group.

3.1 Automorphic forms over $\mathrm{SL}(2, \mathbf{R})$

Let $G_{\mathbf{R}} = \mathrm{SL}(2, \mathbf{R})$ and $z_0 = i$. One can transition a function $f : \mathcal{H} \to \mathbf{C}$ from the upper half complex plane to the group $G_{\mathbf{R}}$. Indeed, note that $K_{\mathbf{R}} = \mathrm{SO}(2, \mathbf{R}) \subset G_{\mathbf{R}}$ fixes z_0:

$$\gamma \cdot i = \frac{i \cos\theta - \sin\theta}{i \sin\theta + \cos\theta} = \frac{ie^{i\theta}}{e^{i\theta}} = i, \quad \text{for all } \gamma \in K_{\mathbf{R}}.$$

Moreover, $K_{\mathbf{R}}$ is the *maximal compact* fixing subgroup of z_0. Thus we have an isomorphism,

$$G_{\mathbf{R}}/K_{\mathbf{R}} \hookrightarrow\!\!\!\!\rightarrow \mathcal{H}, \quad gK_{\mathbf{R}} = gz_0,$$

where gz_0 is the usual Möbius transformation of z_0. In conclusion, any function $f : \mathcal{H} \to \mathbf{C}$ can be converted to a function $f : G_{\mathbf{R}}/K_{\mathbf{R}}$ in the following manner:

$$\phi : \mathcal{C}(\mathcal{H}) \longrightarrow \mathcal{C}(G_{\mathbf{R}}), \quad \phi(f_{\mathcal{H}}(gz_0)) = f_{\mathbf{R}}(g).$$

The remainder of the discussion is dedicated to transitioning the notions of weight and level to the Lie group environment.

To that end, tweak the map ϕ to incorporate weight,

$$\phi_\kappa : \mathcal{C}(\mathcal{H}) \longrightarrow \mathcal{C}(G_{\mathbf{R}}), \quad \phi_\kappa(f_{\mathcal{H}}) = f_{\mathbf{R},\kappa},$$

where

$$f_{\mathbf{R},\kappa}(g) = f_{\mathcal{H}}(gz_0) j(g, z_0)^{-\kappa}.$$

Define a function

$$\chi_\kappa : K_{\mathbf{R}} \to \mathbf{C}, \qquad \chi_\kappa(k) = j(k, z_0)^{-k}.$$

The fact that $K_{\mathbf{R}}$ fixes z_0 and the cocycle condition combine to show that χ is a well-defined multiplicative character:

$$\chi(kk') = j(kk', z_0) = j(k, k'z_0) j(k', z_0) = j(k, z_0) j(k', z_0) = \chi(k)\chi(k').$$

Additionally, the Lie group function $\phi_\kappa(f_{\mathcal{H}}) = f_{\mathbf{R},\kappa}$ satisfies right $K_{\mathbf{R}}$-equivariance with respect to χ_κ:

$$\begin{aligned} f_{\mathbf{R},\kappa}(gk) &= f_{\mathcal{H}}(gkz_0)j(gk,z_0)^{-\kappa} \\ &= f_{\mathcal{H}}(gz_0)j(g,kz_0)^{-\kappa}j(k,z_0)^{-\kappa} \\ &= f_{\mathbf{R},\kappa}(g)\chi_\kappa(k). \end{aligned}$$

In fact, ϕ_κ establishes a bijection between functions on $\mathcal{H}$ and weight-κ right $K_{\mathbf{R}}$-equivariant functions on $G_{\mathbf{R}}$. Let $\mathcal{C}_{K,\kappa}(G_{\mathbf{R}})$ denote the space of such functions.

We now consider the equivalent condition of $[\Gamma_L]_\kappa$-invariance for a function $f_{\mathbf{R}} : G_{\mathbf{R}} \longrightarrow \mathbf{C}$ over the Lie group. Indeed, for $\gamma \in G_{\mathbf{R}}$, define the composition operator

$$\circ\gamma : \mathcal{C}(G_{\mathbf{R}}) \longrightarrow \mathcal{C}(G_{\mathbf{R}}), \qquad (f \circ \gamma)(g) = f(\gamma g).$$

Let $\mathcal{C}_{[\Gamma_L]_\kappa}(\mathcal{H})$ denote the set of continuous $[\Gamma_L]_\kappa$-invariant functions from $\mathcal{H}$ to $\mathbf{C}$. Then the following diagram commutes:

$$\begin{tikzcd} \mathcal{C}_{[\Gamma_L]_\kappa}(\mathcal{H}) \arrow[r, "\phi_\kappa"] \arrow[d, "{[\gamma]_\kappa}"'] & \mathcal{C}_{K,\kappa}(G_{\mathbf{R}}) \arrow[d, "\circ\gamma"] \\ \mathcal{C}_{[\Gamma_L]_\kappa}(\mathcal{H}) \arrow[r, "\phi_\kappa"'] & \mathcal{C}_{K,\kappa}(G_{\mathbf{R}}) \end{tikzcd}$$

Because the diagram commutes, $[\Gamma_L]_\kappa$-invariant functions on $\mathcal{H}$ are sent to left Γ_L-invariant functions on $G_{\mathbf{R}}$, $f_{\mathbf{R},\kappa}(\gamma g) = f_{\mathbf{R},\kappa}(g)$. More explicitly,

$$\begin{aligned} f_{\mathbf{R},\kappa}(\gamma g) &= f_{\mathcal{H}}(\gamma g z_0)j(\gamma g,z_0)^{-\kappa} \\ &= f_{\mathcal{H}}[\gamma]_\kappa(gz_0)j(g,z_0)^{-\kappa} \\ &= f_{\mathcal{H}}(gz_0)j(g,z_0)^{-\kappa} \\ &= f_{\mathbf{R},\kappa}(g). \end{aligned}$$

In other words, for all $\gamma \in \Gamma_L$, we have a bijective map $\phi_{\kappa,L}$:

$$\phi_{\kappa,L} : \{f_{\mathcal{H}} \in \mathcal{C}_{[\Gamma_L]_\kappa}(\mathcal{H})\} \hookrightarrow\!\!\!\!\rightarrow \{f_{\mathbf{R},\kappa} \in \mathcal{C}_{K,\kappa}(G_{\mathbf{R}}) : f_{\mathbf{R},\kappa} \circ \gamma = f_{\mathbf{R},\kappa}\}$$

The right side is a significantly nicer environment, splitting up the awkward condition of $[\Gamma_L]_\kappa$-invariance into left Γ_L-invariance and right κ-equivariance under $K_{\mathbf{R}}$. Moreover, we may enlarge the Lie group environment by collecting all of the individual weights κ into a single function space.

Observe that $\phi_{\kappa,L}$ is an injective map of $\mathcal{C}_{[\Gamma_L]_\kappa}(\mathcal{H})$ into the space $\mathcal{C}(\Gamma_L\backslash G_{\mathbf{R}})$ of left Γ_L-invariant functions on $G_{\mathbf{R}}$. Because $\mathcal{C}(\Gamma_L\backslash G_{\mathbf{R}})$ is totally independent of κ, this becomes our new object of study. In sum, we have lifted $[\Gamma_L]_\kappa$ functions on $\mathcal{H}$ to functions on $\Gamma_L\backslash G_{\mathbf{R}}$:

$$\begin{tikzcd}
\Gamma_L\backslash G_{\mathbf{R}} \arrow[dr, "f_{\mathbf{R}}"] & \\
\mathcal{H} \arrow[u, "\phi_{\kappa,L}(f_{\mathcal{H}})"] \arrow[r, "f_{\mathcal{H}}"'] & \mathbf{C}
\end{tikzcd} \tag{1}$$

From now on, we denote a Lie group function $f_{\mathbf{R}} \in \mathcal{C}(\Gamma_L\backslash G_{\mathbf{R}})$, omitting the κ. This diagram constitutes a slight abuse of notation, as $\phi_{\kappa,L}$ is a mapping from a function space to a function space rather than a domain to a quotient, but this is a trivial issue.

Deriving the Lie group version of the Eisenstein reduces to quickly applying the map $\phi_{\kappa,L}$. Recall our earlier definition of the Eisenstein series on the upper half-plane,

$$E_{\mathcal{H}}(z) = C_L \sum_{\gamma\in N_{\mathbf{Z},L}\backslash\Gamma_L} (1[\gamma]_\kappa)(z),$$

where C_L is a normalizing constant:

$$C_L = \begin{cases} 1/2 & \text{if } L \le 2 \\ 1 & \text{if } L > 2. \end{cases}$$

In the case of $f = 1$, the map $\phi_{\kappa,L}$ specializes to

$$\phi_{\kappa,L}(1) = \delta,$$

where

$$\delta(g) = j(g, z_0)^{-\kappa}, \qquad \text{for all } g \in G_{\mathbf{R}}.$$

Thus we replace $1[\gamma]_\kappa$ with the equivalent δ, producing the *Lie group Eisenstein series*,

$$E_{\mathbf{R}}(g) = C_L \sum_{\gamma \in N_{\mathbf{Z},L} \backslash \Gamma_L} \delta(\gamma g).$$

Given the Iwasawa decomposition $G_{\mathbf{R}} = N_{\mathbf{R}} M_{\mathbf{R}} K_{\mathbf{R}}$, we may compute δ directly for any $g = n_x m_a k_\theta \in G_{\mathbf{R}}$:

$$\begin{aligned}
\delta(n_x m_a k_\theta) &= j(n_x m_a k_\theta, z_0)^{-\kappa} \\
&= j(n_x, m_a k_\theta \cdot z_0)^{-\kappa} j(m_a, k_\theta \cdot z_0)^{-\kappa} j(k_\theta, z_0)^{-\kappa} \\
&= a^\kappa e^{-i\kappa\theta}.
\end{aligned}$$

This formula will be helpful when constructing the adelic Eisenstein series.

3.2 Automorphic forms over the adeles

We now expand the Lie group environment to the adele group $G_{\mathbf{A}} = \mathrm{SL}(2, \mathbf{A})$. Let $G_v = \mathrm{SL}(2, \mathbf{Q}_v)$, and let $K_v = \mathrm{SL}(2, \mathbf{Z}_v)$ be a compact subgroup of G_v. We then have the global compact subgroup

$$K_{\mathbf{A}} = K_{\mathbf{f}} \times K_{\mathbf{R}} = \big(\prod_{v \neq \infty} K_v\big) \times K_{\mathbf{R}},$$

where $K_{\mathbf{R}} = \mathrm{SO}(2, \mathbf{R})$, as before. Define the L-dependent compact subgroup $K_{v,L}$ as

$$K_{v,L} = \left\{ g \in K_v : g \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{L} \right\}.$$

Note that $K_{v,L} = K_v$ if $v \nmid L$. Define the corresponding global compact subgroup attached to L,

$$K_{\mathbf{A},L} = K_{\mathbf{f},L} \times K_{\mathbf{R}} = \big(\prod_{v \neq \infty} K_{v,L}\big) \times K_{\mathbf{R}}$$

Before proceeding, recall the standard diagonal embedding of $G_{\mathbf{Q}}$ in $G_{\mathbf{A}}$,

$$G_{\mathbf{Q}} \longrightarrow G_{\mathbf{A}}, \quad g \mapsto g_{\mathbf{Q}} = (\gamma, \gamma, \dots, \gamma)$$

where each γ on the right is actually the v-adic representation of the γ on the left. This convention creates certain notational hazards, as $G_{\mathbf{Q}}$ can be interpreted as itself or its embedded counterpart. We will try to be clear about when we are working with an object or its embedding, but this can almost always be deduced from context.

Proposition 3.1. *The standard embedding of $G_{\mathbf{R}}$ into $G_{\mathbf{A}}$ induces an injection*

$$\iota_L : \Gamma_L \backslash G_{\mathbf{R}} \hookrightarrow G_{\mathbf{Q}} \backslash G_{\mathbf{A}} / K_{\mathbf{f},L}, \quad \Gamma_L g = G_{\mathbf{Q}} g K_{\mathbf{f},L}$$

Proof. First, we confirm that $\iota_L(g) = \iota_L(\gamma g)$ for all $\gamma \in \Gamma_L$ and $g \in G_{\mathbf{R}}$. Let $\gamma = \gamma_{\mathbf{f}} \gamma_{\mathbf{R}}$ be the embedded image of γ in $G_{\mathbf{A}}$. Because $\gamma^{-1} \in G_{\mathbf{Q}}$, we have

$$\begin{aligned} G_{\mathbf{Q}} \gamma_{\mathbf{R}} g K_{\mathbf{f},L} &= G_{\mathbf{Q}} \gamma^{-1} \gamma_{\mathbf{R}} g \gamma_{\mathbf{f}} K_{\mathbf{f},L} \\ &= G_{\mathbf{Q}} \gamma_f^{-1} g \gamma_{\mathbf{f}} k_{\mathbf{f},L} \\ &= G_{\mathbf{Q}} g K_{\mathbf{f},L}. \end{aligned}$$

Thus the map is well-defined. Next we confirm the map is injective. Suppose for $g, g' \in G_{\mathbf{R}}$, $\iota_L(\Gamma_L g) = \iota_L(\Gamma_L g')$. Working at a finite place v, this means that there exists $\gamma \in G_v$ and $k \in K_{v,L}$ such that

$$g = \gamma g' k.$$

Thus $\gamma = k^{-1}$, forcing $\gamma \in G_{\mathcal{O}_v}$ and $\gamma \equiv 1 \pmod{L}$, meaning $\gamma \in \Gamma_{v,L}$. At the ∞-place, we have $g' = \gamma g$, forcing $\gamma \in \Gamma_{\infty,L}$, thus $\Gamma_L g' = \Gamma_L g$. $\square$

There exists a corresponding surjection π_L:

$$\pi_L : \mathcal{C}(G_{\mathbf{Q}} \backslash G_{\mathbf{A}} / K_{\mathbf{f},L}) \twoheadrightarrow \mathcal{C}(\Gamma_L \backslash G_{\mathbf{R}}).$$

Let $f_{\mathbf{R}} : G_{\mathbf{R}} \to \mathbf{C}$ be a continuous, left Γ_L-invariant function. The inverse image of $f_{\mathbf{R}}$ is the set of all $f_{\mathbf{A}} : G_{\mathbf{A}} \to \mathbf{C}$ that agree with $f_{\mathbf{R}}$ on the embedded image of $g \in G_{\mathbf{R}}$,

$$f_{\mathbf{A}}(g_{\mathbf{R}}) = f_{\mathbf{R}}(g).$$

In general π_L is not bijective, but because we are working over $G = \mathrm{SL}(2, \cdot)$, the following theorem implies that π_L is indeed a bijection.

Theorem 3.1 (Strong approximation). *Let S be a finite set of places, and let $G_{\mathbf{Q},S}$ and $G_{\mathbf{A}}^S$ temporarily denote*

$$G_{\mathbf{Q},S} = \prod_{v \in S} G_v$$

$$G_{\mathbf{A}}^S = \left\{ (x_v) \in \prod_{v \notin S} G_v : x_v \in \mathcal{O}_v \text{ for almost all } v \right\}.$$

Suppose that $G_{\mathbf{Q},S}$ is not compact. Then $G_{\mathbf{Q},S}$ is dense in $G_{\mathbf{A}}^S$. Equivalently, given any open compact subgroup $K^S \subset G_{\mathbf{A}}^S$, we have

$$G_{\mathbf{A}} = G_{\mathbf{Q}} \cdot G_{\mathbf{Q},S} \cdot K^S.$$

Proof. See [1] for a proof. □

As a consequence of Theorem 3.1,

$$\Gamma_L \backslash G_{\mathbf{R}} \approx G_{\mathbf{Q}} \backslash G_{\mathbf{A}} / K_{\mathbf{f},L}. \tag{2}$$

To see this, first note that for $\Gamma = G_{\mathbf{Q}} \cap K^S$, where S is a set of places as described above, we have

$$G_{\mathbf{Q}} \backslash G_{\mathbf{A}} / K^S \approx \Gamma \backslash G_{\mathbf{Q},S}.$$

Setting $\Gamma = \Gamma_L = K_{\mathbf{f},L} \cap G_{\mathbf{Q}}$ and $S = \{\infty\}$ immediately gives us the isomorphism on line (2). That is, any left Γ_L-invariant function $f_{\mathbf{R}}$ on the Lie group corresponds to a left $G_{\mathbf{Q}}$-invariant, right $K_{\mathbf{f},L}$-invariant function on the adele group. Again enlarging the group as we did in the Lie group section, there exists an injection for any level L:

$$\varrho : \mathcal{C}(\Gamma_L \backslash G_{\mathbf{R}}) \hookrightarrow \mathcal{C}(G_{\mathbf{Q}} \backslash G_{\mathbf{A}}).$$

Thus we may revise Diagram 1 to include a lift to left $G_{\mathbf{Q}}$-invariant functions on $G_{\mathbf{A}}$.

$$\begin{tikzcd}
G_{\mathbf{Q}} \backslash G_{\mathbf{A}} \arrow[ddr, "f_{\mathbf{A}}"] & \\
\Gamma_L \backslash G_{\mathbf{R}} \arrow[u, "\varrho(f_{\mathbf{R}})"] \arrow[dr, "f_{\mathbf{R}}"] & \\
\mathcal{H} \arrow[u, "\phi_{\kappa,L}(f_{\mathcal{H}})"] \arrow[r, "f_{\mathcal{H}}"'] & \mathbf{C}
\end{tikzcd} \tag{3}$$

Note that our diagram would be messier were we not working over $G = \mathrm{SL}(2, \cdot)$, as ϱ depends on strong approximation. Still, for an arbitrary reductive group G, this does not impede the development of automorphic forms over $G_{\mathbf{A}}$. Indeed, the classical theory is usually defined over $\mathrm{GL}(2, \cdot)$ rather than $\mathrm{SL}(2, \cdot)$, and strong approximation does not hold for the former group. That said, $\mathrm{GL}(2, \cdot)$ accommodates generalizations of Hecke operators, which are deeply connected with the representation theory of $\mathrm{GL}(2, \mathbf{A})$.

Returning to the relevant environment $G_{\mathbf{A}} = \mathrm{SL}(2, \mathbf{A})$, we provide the modern definition of an automorphic form over the topological group $G_{\mathbf{A}}$.

Definition 3.1 (Automorphic form, version 1)**.** An *automorphic form* is a smooth function $f : G_{\mathbf{A}} \to \mathbf{C}$ that is left $G_{\mathbf{Q}}$-invariant. We call an automorphic form a *cusp form* if

$$\int_{N_{\mathbf{Q}} \backslash N_{\mathbf{A}}} f(\nu g)\, \mathrm{d}\nu \overset{\text{call}}{=} \lambda_f(g) = 0.$$

The function $\lambda_f(g)$ is referred to as the *constant term* of f. We use $\mathcal{A}(G_{\mathbf{Q}} \backslash G_{\mathbf{A}})$ to denote the space of automorphic forms over $G_{\mathbf{Q}} \backslash G_{\mathbf{A}}$.

While Definition 3.1 appears different from Definition 1.1, hopefully the previous pages have made clear why the two are closely related. There are some technicalities we have omitted from the definition, but they are not relevant to our current discussion.

3.3 The adelic Eisenstein series

Section 3.1 defined the Lie group Eisenstein series,

$$E_{\mathbf{R}}(g) = C_L \sum_{\gamma \in N_{\mathbf{Z},L} \backslash \Gamma_L} \delta(\gamma g),$$

where $\delta(g) = j(g, z_0)^{-\kappa}$. We found that for $g \in G_{\mathbf{R}}$ with Iwasawa decomposition $g = n_x m_a k_\theta$,

$$\delta(g) = a^\kappa e^{-i\kappa\theta}.$$

An Eisenstein series on $G_{\mathbf{A}}$ corresponding to $E_{\mathbf{R}}$ is a left $G_{\mathbf{Q}}$-invariant, right $K_{\mathbf{f},L}$-invariant function $E_{\mathbf{A}} : G_{\mathbf{A}} \longrightarrow \mathbf{C}$ satisfying

$$E_{\mathbf{A}}(g_{\mathbf{R}}) = E_{\mathbf{R}}(g), \quad g \in G_{\mathbf{R}}.$$

To begin, we find the natural domain of summation for $E_{\mathbf{A}}$. Because automorphic forms over $G_{\mathbf{A}}$ are left $G_{\mathbf{Q}}$-invariant, $E_{\mathbf{A}}$ will sum over some quotient $\cdot\backslash G_{\mathbf{Q}}$. The goal is to show that the set of summation $N_{\mathbf{Z},L}\backslash\Gamma_L$ for $E_{\mathbf{R}}$ can generally be viewed as a subspace of $P_{\mathbf{Q}}\backslash G_{\mathbf{Q}}$, over which we will define $E_{\mathbf{A}}$. To that end, recall the L-subgroups

$$\Gamma_L = \{\gamma \in \mathrm{SL}_2(\mathbf{Z}) : \gamma \equiv 1 \pmod{L\mathbf{Z}}\}$$
$$N_{\mathbf{Z},L} = \left\{\gamma \in \mathrm{SL}_2(Z) : \gamma = \begin{pmatrix} 1 & bL \\ 0 & 1 \end{pmatrix},\ b \in \mathbf{Z}\right\},$$

and let $P_{\mathbf{Z}}$ be the usual parabolic subgroup over the integers. Consider the map

$$\Gamma_L \longrightarrow G_{\mathbf{Z}} \longrightarrow P_Z\backslash G_{\mathbf{Z}}.$$

Note that

$$\Gamma_L \cap P_{\mathbf{Z}} = \begin{cases} \pm N_{\mathbf{Z},L} & \text{for } L \le 2 \\ N_{\mathbf{Z},L} & \text{for } L > 2, \end{cases}$$

giving us a surjection followed by a natural injection for $L \le 2$,

$$N_{\mathbf{Z},L}\backslash\Gamma_L \twoheadrightarrow \pm N_{\mathbf{Z},L}\backslash\Gamma_L \hookrightarrow P_{\mathbf{Z}}\backslash G_{\mathbf{Z}}. \tag{4}$$

For all $L > 2$, we simply have a natural injection,

$$N_{\mathbf{Z},L}\backslash\Gamma_L \hookrightarrow P_{\mathbf{Z}}\backslash G_{\mathbf{Z}}. \tag{5}$$

So for all levels, we have a well-defined map $N_{\mathbf{Z},L}\backslash\Gamma_L \to P_{\mathbf{Z}}\backslash G_{\mathbf{Z}}$. If we can show that $P_{\mathbf{Z}}\backslash G_{\mathbf{Z}} \approx P_{\mathbf{Q}}\backslash G_{\mathbf{Q}}$, we will have a well-defined domain of summation for $E_{\mathbf{A}}$.

Proposition 3.2. *There exists a bijection,*

$$P_{\mathbf{Z}}\backslash G_{\mathbf{Z}} \hookrightarrow\!\!\!\!\rightarrow P_{\mathbf{Q}}\backslash G_{\mathbf{Q}}. \tag{6}$$

Thus the map

$$N_{\mathbf{Z},L}\backslash\Gamma_L \longrightarrow P_{\mathbf{Q}}\backslash G_{\mathbf{Q}},$$

is well-defined.

Proof. The natural map $P_{\mathbf{Z}}\backslash G_{\mathbf{Z}} \to P_{\mathbf{Q}}\backslash G_{\mathbf{Q}}$ is injective. To see this, note that the composite

$$P_{\mathbf{Z}} \longrightarrow G_{\mathbf{Q}} \longrightarrow P_{\mathbf{Q}}\backslash G_{\mathbf{Q}}$$

is clear, and since $G_{\mathbf{Z}} \cap P_{\mathbf{Q}} = P_{\mathbf{Z}}$, our map is both well-defined and injective.

Next we show surjectivity. Let $g \in G_{\mathbf{Q}}$ – there exists a (unique up to sign) $s \in \mathbf{Q}^\times$ such that the product

$$\begin{pmatrix} s^{-1} & 0 \\ 0 & s \end{pmatrix} g = \begin{pmatrix} \alpha & \beta \\ c & d \end{pmatrix},$$

has an integral bottom row $c, d \in \mathbf{Z}$. Let $x = (b-\beta)/d$ for some b such that $(d-1)|bc$. Using the fact that $\alpha d - \beta c = 1$,

$$\alpha + xc = \frac{1+\beta c}{d} + \frac{bc - \beta c}{d} = \frac{1+bc}{d} \overset{\text{call}}{=} a \in \mathbf{Z}$$

and

$$\beta + xd = \beta + b - \beta = b \in \mathbf{Z}.$$

Thus there exists $p \in P_{\mathbf{Q}}$ such that

$$pg = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z}),$$

proving surjectivity. Now that we know map (6) is bijective, the final statement follows for $L \le 2$ and $L > 2$ by composing maps (4) or (5) with map (6), respectively. □

By Proposition 3.2, we may take the adelic Eisenstein series to be defined over $P_{\mathbf{Q}}\backslash G_{\mathbf{Q}}$,

$$E_\varepsilon(g) = \sum_{\gamma \in P_{\mathbf{Q}}\backslash G_{\mathbf{Q}}} \varepsilon(\gamma g), \quad g \in G_{\mathbf{A}}$$

where $\varepsilon : G_{\mathbf{A}} \to \mathbf{C}$ is some function yet to be determined. Per usual, the sum is symmeterized to be left $G_{\mathbf{Q}}$-invariant. Note the absence of C_L in the definition of E_ε. At $L \leq 2$, the $2-1$ collapsing by map (4) subsumes this constant. Obviously we have not gone through all of this work just to have a slightly nicer expression for the Eisenstein series, but the absorption of C_L into algebraic structure is indicative of how working over the adeles, at least in many cases, will simplify computations.

The last step in ensuring $E_{\mathbf{A}}$ is well-defined is producing an appropriate, left $P_{\mathbf{Q}}$-invariant ε that agrees with δ on $G_{\mathbf{R}}$. To do so, for each place v define

$$\chi_{\kappa,v} : P_v \longrightarrow \mathbf{C}, \quad \chi_{\kappa,v}\begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} = |a|_v^\kappa.$$

Likewise, consider the local functions

$$\varphi_v : K_{v,L} \to \mathbf{C},$$

where φ_v is the characteristic function of the subgroup $\Gamma_L \subset K_{v,L}$ for finite primes v, and $\varphi_\infty(k_\theta) = e^{-2\kappa\theta}$. We now define ε as

$$\varepsilon = \prod_v \varepsilon_v, \quad \text{where } \varepsilon_v(g_v) = \chi_{\kappa,v}(p_v)\varphi_v(k_v).$$

Proposition 3.3. *Let ε be as just defined. Then the adele group Eisenstein series attached to ε,*

$$E_\varepsilon(g) = \sum_{\gamma \in P_{\mathbf{Q}} \backslash G_{\mathbf{Q}}} \varepsilon(\gamma g)$$

is compatible with the Lie group Eisenstein series and left $P_{\mathbf{Q}}$-invariant.

Proof. First we show left $P_{\mathbf{Q}}$-invariance. Let $p \in P_{\mathbf{Q}}$, $\gamma \in P_{\mathbf{Q}} \backslash G_{\mathbf{Q}}$, and $g \in G_{\mathbf{A}}$. By the given definition of ε,

$$\varepsilon(p\gamma g) = \prod_v \varepsilon_v(p_v \gamma_v g_v) = \prod_v \chi_{\kappa,v}(p_v) \prod_v \varepsilon(\gamma_v g_v) = \prod_v \varepsilon(\gamma_v g_v) = \varepsilon(\gamma g),$$

where the second to last equality follows from the adele product formula, $\prod_v |a|_v = 1$. Thus the local P_v-equivariance of ε_v forces ε to be left $P_{\mathbf{Q}}$-invariant. The left $P_{\mathbf{Q}}$-invariance of E_ε immediately follows.

Next we show that E_ε agrees with its Lie group counterpart. Let $g \in G_{\mathbf{R}}$, where $g_\infty = n_x m_a k_\theta$ is the obvious local embedding. Observe that ε has been defined so that,

$$\varepsilon_\infty(g_\infty) = a^\kappa e^{-i\kappa\theta} = \delta(g).$$

Let $\gamma \in G_{\mathbf{Z}}$, and consider $\gamma_v = p_v k_v$ at a finite place v. Because γ has integral entries, p_v must be trivial, meaning $\varepsilon_v(\gamma_v) = \chi_{\kappa,v}(p_v)\varphi_v(k_v) = \varphi_v(k_v)$. In other words, $\varepsilon_v(\gamma_v)$ is the characteristic function of Γ_L for all finite places v. Therefore,

$$\varepsilon(\gamma g_{\mathbf{R}}) = \begin{cases} \delta(\gamma g) & \text{if } \gamma \in \Gamma_L \\ 0 & \text{if } \gamma \notin \Gamma_L. \end{cases}$$

Thus for $L \leq 2$,

$$E_\varepsilon(g_{\mathbf{R}}) = \sum_{\gamma \in \pm N_{\mathbf{Z},L} \backslash \Gamma_L} \delta(\gamma g) = \frac{1}{2} \sum_{\gamma \in N_{\mathbf{Z},L} \backslash \Gamma_L} \delta(\gamma g) = E_{\mathbf{R}}(g)$$

and for $L > 2$,

$$E_\varepsilon(g_{\mathbf{R}}) = \sum_{\gamma \in N_{\mathbf{Z},L} \backslash \Gamma_L} \delta(\gamma g) = E_{\mathbf{R}}(g).$$

We conclude that $E_\varepsilon(g_{\mathbf{R}}) = E_\varepsilon(g)$ for all L. $\square$

Thus far, generalizing the Lie group Eisenstein series has motivated our discussion of the adelic equivalent, but we can discuss E_ε intrinsically, without any reference to the Lie group definition. Consider a left $G_{\mathbf{Q}}$-invariant function,

$$E : G_{\mathbf{A}} \to \mathbf{C}.$$

We argue that any such function can be described as a sum. To see this, consider a left $P_{\mathbf{Q}}$-invariant function $h : G_{\mathbf{A}} \longrightarrow \mathbf{C}$ with the property

$$\sum_{\gamma \in P_{\mathbf{Q}} \backslash G_{\mathbf{Q}}} h(\gamma g) = 1 \quad \text{for any } g \in G_{\mathbf{A}}.$$

Define $\varepsilon(g) = E(g)h(g)$ for $g \in G_{\mathbf{A}}$. Then ε is left $P_{\mathbf{Q}}$-invariant and for any $g \in G_{\mathbf{A}}$,

$$E(g) = \sum_{\gamma \in P_{\mathbf{Q}} \backslash G_{\mathbf{Q}}} h(\gamma g) E(g) = \sum_{\gamma \in P_{\mathbf{Q}} \backslash G_{\mathbf{Q}}} h(\gamma g) E(\gamma g) = \sum_{\gamma \in P_{\mathbf{Q}} \backslash G_{\mathbf{Q}}} \varepsilon(\gamma g).$$

Adelic Eisenstein series are constructed as above by taking functions $\varepsilon : G_{\mathbf{A}} \to \mathbf{C}$ satisfying some form of invariance, and then further symmetrizing them. To that end, we briefly discuss *Hecke characters*.

Definition 3.2 (Hecke character)**.** A *Hecke character* χ over $\mathbf{Q}$ is a smooth character on the idele group $\mathbf{J}$ that is trivial on $\mathbf{Q}^\times$,

$$\chi : \mathbf{J} \longrightarrow \mathbf{C}^\times, \quad \chi(x) = 1 \text{ for all } x \in \mathbf{Q}^\times.$$

Many of the interesting things to be said about Hecke characters and the ideles are contingent upon us working over an extension field of $\mathbf{Q}$, which we are not. For our purposes, a few quick remarks will suffice. The nature of the idele topology forces any continuous homomorphism out of $\mathbf{J}$ to contain almost all of the local unit groups $\mathcal{O}_v^\times$. Article [4] gives a more thorough account of why this is the case. Noting that every unramified character on $\mathbf{Q}_v^\times$ takes the form $|\cdot|_v^s$, we have that every Hecke character is locally identical to

$$\chi_v(x) = |x|_v^s, \quad x \in \mathbf{J}$$

for almost all nonarchimedean v, where $s \in \mathbf{C}$.

Returning to our original discussion, suppose some function $\varepsilon : G_{\mathbf{A}} \to \mathbf{C}$ is $P_{\mathbf{A}}$-equivariant with respect to a Hecke character χ,

$$\varepsilon(pg) = \chi(p)\varepsilon(g), \quad p \in P_{\mathbf{A}}, g \in G_{\mathbf{A}}$$

Let $p \in P_{\mathbf{Q}}$. Because χ is a Hecke character, we have $\chi_v(p) = |p|_v^s = 1$ at almost all places v, making ε left $P_{\mathbf{Q}}$-invariant. Averaging over χ in a suitable manner, we may define a left $G_{\mathbf{Q}}$-invariant function.

Definition 3.3 (Eisenstein series, Version 1)**.** Letting ε be as above, the adele group Eisenstein series associated to ε is

$$E_\varepsilon : G_{\mathbf{A}} \longrightarrow \mathbf{C}, \quad E_\varepsilon(g) = \sum_{\gamma \in P_{\mathbf{Q}} \backslash G_{\mathbf{Q}}} \varepsilon(\gamma g), \quad g \in G_{\mathbf{A}}.$$

As the (Version 1) suggests, we can view the above object in a number of ways. The view we take here is the natural extension of the classical viewpoint, that the Eisenstein series is a function on $G_{\mathbf{A}}$ satisfying a number of invariance properties. But this function actually takes two parameters, $\varepsilon \in V_s$ and $g \in G_{\mathbf{A}}$, where V_s is some sort of mystery object. As we will find out, $\varepsilon_v \in I_{s,v}$ is actually a member of something known as a *principal series representation*, and for almost all places v, ε_v is of special significance within the principal series. In other words, letting $\mathcal{A}(G_{\mathbf{Q}}\backslash G_{\mathbf{A}})$ be the space of automorphic forms over G, we can view the Eisenstein series as a function on V_s:

$$E : V_s \longrightarrow \mathcal{A}(G_{\mathbf{Q}}\backslash G_{\mathbf{A}}), \quad E(\cdot, g) \longmapsto E(\varepsilon, g),$$

altering notation in the obvious way. It is this interpretation of E that has tremendous significance in modern number theory. For more information, see [1].

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