The Weil Representation

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This writeup addresses the construction of the Weil representation, a tool used to construct automorphic representations of $G_{\mathbf{A}} = \mathrm{SL}(2, \mathbf{A})$, where $\mathbf{A}$ is defined over some global field $\mathbf{k}$. Let G_v denote a local component of $G_{\mathbf{A}}$, and recall the following subgroups of G_v:

$$P_v = \left\{ \begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} \right\}, \quad M_v = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\}, \quad N_v = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\},$$

where $P_v = M_v N_v = N_v M_v$ is the *standard parabolic subgroup*, M_v is the *Levi component*, and N_v is the *unipotent radical*. These subgroups decompose G_v via the *Bruhat decomposition*,

$$G_v = P_v \sqcup P_v w N_v, \quad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where w is the usual *Weyl element*. We frequently denote elements of M and N by indicating the nontrivial entry as a subscript, i.e. $m_a = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ and $n_b = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$. Let $H_{\mathbf{A}}$ be the *adelic orthogonal group* (defined in section 1) associated to some quadratic Q, and let H_v denote it's local completion at v. For a quadratic extension field $\mathbf{K}$ of $\mathbf{k}$, the *Weil* representation ω is defined locally over the Schwartz space $S(\mathbf{A}_{\mathbf{K}})$:

$$\begin{aligned} \omega(h)\varphi(t) &= \varphi(ht) && h \in H_v \\ \omega(m_a)\varphi(t) &= \chi(a)|a|_{\mathbf{k}}\varphi(ta) && m_a \in M_v \\ \omega(n_b)\varphi(t) &= \psi(xN_{\mathbf{K}/\mathbf{k}}(t))\varphi(t) && n_b \in N_v \\ \omega(w)\varphi(t) &= c_w \mathcal{F}\varphi(t) && w = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \end{aligned}$$

for $\varphi \in S(\mathbf{K}_v)$. Above, ψ denotes the usual adelic additive character, χ is a quadratic character, and c_w is some eight root of unity. Technically, we should be writing ω_v, but I will tend to omit the v when context dictates we are working locally.

The above definition is considerably difficult. There is no guarantee that using the Bruhat decomposition to extend this collection of actions to the full group $G_{\mathbf{A}} \times H_{\mathbf{A}}$ yields a well-defined representation. The goal of this writeup is to show that the Weil representation is well-defined on $G_{\mathbf{A}} \times H_{\mathbf{A}}$. I closely follow [1], an unpublished writeup by Jerry Shurman.

1 Initial definitions

Let $\mathbf{k}$ be a global number field and $\mathbf{K} = \mathbf{k}(\sqrt{\alpha})$ a quadratic extension of $\mathbf{k}$. We define ψ to be the usual adelic additive character. Let σ be the non-trivial automorphism in $\mathrm{Aut}(\mathbf{K}/\mathbf{k})$. We may induce a pairing from σ:

$$\sigma : \mathbf{K} \times \mathbf{K} \to \mathbf{k}, \qquad \langle x, y \rangle = \frac{1}{2}(x^\sigma y + x y^\sigma).$$

Given $h, x, y \in \mathbf{K}$, the above pairing satisfies

$$\langle hx, y \rangle = \langle x, h^\sigma y \rangle.$$

For any $x \in \mathbf{K}$, define the quadratic form Q,

$$Q(x) = \langle x, x \rangle = x^\sigma x = N_{\mathbf{K}/\mathbf{k}}(x),$$

where $N_{\mathbf{K}/\mathbf{k}}$ is the field norm. Define the *special orthogonal group*

$$H = \{h \in \mathbf{K} : \langle hx, hy \rangle = \langle x, y \rangle, \quad \text{for all } x, y \in \mathbf{K}\} = \{h \in \mathbf{K} : N_{\mathbf{K}/\mathbf{k}}(h) = 1\}.$$

For all $h \in H$,

$$|h|_{\mathbf{K}} = |N_{\mathbf{K}/\mathbf{k}}(h)|_{\mathbf{k}} = 1.$$

To eventually adelize H, we must consider the completion of $\mathbf{K}$ at a place v of $\mathbf{k}$. To localize the orthogonal group, let

$$\mathbf{K}_v = \mathbf{K} \otimes_{\mathbf{k}} \mathbf{k}_v$$

for each place v of $\mathbf{k}$. Recalling that $\mathbf{K} = \mathbf{k}(\sqrt{\alpha})$, basic results in algebraic number theory give

$$\mathbf{K}_v = \mathbf{k}_v[X]/\langle X^2 - \alpha \rangle = \begin{cases} \mathbf{k}_v(\sqrt{\alpha}) & \text{if } \alpha \text{ is not a square in } \mathbf{k}_v \\ \mathbf{k}_v \times \mathbf{k}_v & \text{else.} \end{cases}$$

Quadratic reciprocity tells us that $\mathbf{K}_v$ is only a field half the time, but in any case it is a two-dimensional vector space over $\mathbf{k}_v$. Thus σ extends to a nontrivial automorphism of $\mathbf{K}_v$ over $\mathbf{k}_v$, and we may define the *local orthogonal group*

$$H_v = \{h \in \mathbf{K}_v : N_{\mathbf{K}_v/\mathbf{k}_v}(h) = 1\}$$

for each place v of $\mathbf{k}$. Likewise, for each place v of $\mathbf{K}$, there exists a *local quadratic character*,

$$\chi_v : \mathbf{k}_v \to \{\pm 1\}, \qquad \chi_v(x) = \begin{cases} 1 & \text{if } x \text{ is a local norm} \\ -1 & \text{else} \end{cases}.$$

The character is nontrivial if and only if $\mathbf{K}_v$ is a field. We may now define the *adelic orthogonal group*:

$$H_{\mathbf{A}} = \prod_v H_v \subset \mathbf{J}_{\mathbf{K}},$$

where $\mathbf{J}_{\mathbf{K}}$ is the idele group. The product of the local quadratic characters is defined on the idele classes,

$$\chi : \mathbf{J}_{\mathbf{k}}/\mathbf{k}^{\times} \to \{\pm 1\}.$$

Recall that we want to make the Schwartz space $S(\mathbf{A}_{\mathbf{K}})$ a representation of the group $H_{\mathbf{A}} \times G_{\mathbf{A}}$, where $G_{\mathbf{A}} = \mathrm{SL}_2(\mathbf{A}_{\mathbf{k}})$. Once again, the Weil representation is defined in fragments:

$$\begin{aligned} \omega(h)\varphi(t) &= \varphi(ht) \\ \omega(m_a)\varphi(t) &= \chi(a)|a|_{\mathbf{k}}\varphi(ta) \\ \omega(n_b)\varphi(t) &= \psi(xN_{\mathbf{K}/\mathbf{k}}(t))\varphi(t) \\ \omega(w)\varphi(t) &= c_w\mathcal{F}\varphi(t), \end{aligned}$$

where $c_w^2 = \chi(-1)$. We understand that $|a|_{\mathbf{k}}$ is shorthand for $|a|_{\mathbf{J}_{\mathbf{k}}}$, and $N_{\mathbf{K}/\mathbf{k}}$ is shorthand for $N_{\mathbf{J}_{\mathbf{K}}/\mathbf{J}_{\mathbf{k}}}$. Because we are working over a general number field, the Fourier transform is defined

$$\mathcal{F} : S(\mathbf{A}_{\mathbf{K}}) \to S(A_{\mathbf{K}}), \qquad \mathcal{F}\varphi(x) = \int_{\mathbf{A}_{\mathbf{K}}} \varphi(\xi)\psi(-\mathrm{tr}\mathbf{K}/\mathbf{k}\langle x, \xi\rangle)\ \mathrm{d}\xi,$$

where $\mathrm{tr}\mathbf{K}/\mathbf{k}$ is the field trace. For the remainder of the appendix, it will be assumed that $g \cdot \varphi(x) = \omega(g)\varphi(x)$.

2 The Weil representation is well-defined

Proving ω is a well-defined representation is non-trivial. Due to the piecewise nature of the representation, we must break the proof into several lemmas, first working with the isolated actions, and then stitching them together with the help of the Bruhat decomposition. The first step is showing that each of the *individual* actions is well-defined.

Lemma 2.1. *For any $\varphi \in S(\mathbf{A_K})$,*

$$\begin{aligned} h \cdot (h' \cdot \varphi) &= (hh') \cdot \varphi \\ m_a \cdot (m_{a'} \cdot \varphi) &= (mm_a) \cdot \varphi \\ n_b \cdot (n_{b'} \cdot \varphi) &= (n_b n_{b'}) \cdot \varphi \\ w \cdot (w \cdot \varphi) &= (ww) \cdot \varphi \end{aligned}$$

Proof. We prove the result for $n \in N$ and the Weyl element w – the other cases follow in a similar fashion. For the unipotent radical, observe that

$$n_b n_{b'} = n_{b+b'}.$$

The computation follows quickly:

$$\begin{aligned} n_b \cdot (n_{b'} \cdot \varphi)(t) &= n_b \cdot (\psi(b' N_{\mathbf{K}/\mathbf{k}}(t))\varphi(t)) \\ &= \psi(b N_{\mathbf{K}/\mathbf{k}}(t))\psi(b' N_{\mathbf{K}/\mathbf{k}}(t))\varphi(t) \\ &= \psi((b+b') N_{\mathbf{K}/\mathbf{k}})\varphi(t) \\ &= n_{b+b'} \cdot \varphi(t). \end{aligned}$$

For w, compute

$$\begin{aligned} w \cdot (w \cdot \varphi(t)) &= c_w^2 \mathcal{F}(\mathcal{F}\varphi)(t) \\ &= \chi(-1)\varphi(-t) \\ &= m_{-1} \cdot \varphi(t). \end{aligned}$$

Because $ww = m_{-1}$, the result follows. ∎

The inner product over Schwartz space is defined

$$\langle \varphi_1, \varphi_2 \rangle = \int_{\mathbf{A}} \varphi_1(\xi)\overline{\varphi}_2(\xi) \, \mathrm{d}\xi.$$

The Weil representation preserves the inner product:

$$\begin{aligned}\langle h\cdot\varphi_1, h\cdot\varphi_2\rangle &= \langle\varphi_1,\varphi_2\rangle\\ \langle m_a\cdot\varphi_1, m_{a'}\cdot\varphi_2\rangle &= \langle\varphi_1,\varphi_2\rangle\\ \langle n_b\cdot h\varphi_1, n_{b'}\cdot h\varphi_2\rangle &= \langle\varphi_1,\varphi_2\rangle\\ \langle w\cdot h\varphi_1, w\cdot h\varphi_2\rangle &= \langle\varphi_1,\varphi_2\rangle.\end{aligned}$$

Each identity is easy to verify the the exception of the last. Note that

$$\begin{aligned}\langle w\cdot h\varphi_1, w\cdot h\varphi_2\rangle &= \langle\mathcal{F}(\varphi_1)(t),\mathcal{F}(\varphi_2)(t)\rangle\\ &= \langle\varphi_1,\varphi_2\rangle\end{aligned}$$

The last equality holds because the Fourier transform is an isometry of the Schwartz space. Next, we must show that the $H_{\mathbf{A}}$-action commutes with the $G_{\mathbf{A}}$-action.

Lemma 2.2. *For any $\varphi\in S(\mathbf{A})$,*

$$\begin{aligned}h\cdot m_a\cdot\varphi &= m_a\cdot h\cdot\varphi\\ h\cdot n_b\cdot\varphi &= n_b\cdot h\cdot\varphi\\ h\cdot w\cdot\varphi &= w\cdot h\cdot\varphi\end{aligned}$$

Proof. The first two are rote computations similar to those in the previous lemma, so we only concern ourselves with the Weyl element. We begin with a preliminary computation:

$$\begin{aligned}\mathcal{F}(ht) &= \int_{A_{\mathbf{K}}}\varphi(\xi)\psi(-\mathrm{tr}\mathbf{K}/\mathbf{k}\langle ht,\xi\rangle)\,\mathrm{d}\xi\\ &= \int_{A_{\mathbf{K}}}\varphi(h\xi)\psi(-\mathrm{tr}\mathbf{K}/\mathbf{k}\langle ht,h\xi\rangle)\,\mathrm{d}h\xi\\ &= \int_{A_{\mathbf{K}}}h\cdot\varphi(\xi)\psi(-\mathrm{tr}\mathbf{K}/\mathbf{k}\langle t,\xi\rangle)\,\mathrm{d}\xi \qquad \text{because } \langle ht,h\xi\rangle = \langle t,\xi\rangle\\ &= \mathcal{F}(h\varphi)(\xi).\end{aligned}$$

It follows that

$$\begin{aligned}(h\cdot w\cdot\varphi)(t) &= (w\cdot\varphi)(ht)\\ &= c_w\mathcal{F}\varphi(ht)\\ &= c_w\mathcal{F}(h\cdot\varphi)(t)\\ &= (w\cdot h\cdot\varphi)(t).\end{aligned}$$

□

We now work locally at a finite place v of $\mathbf{k}$, but suppress v from the notation. We verify that the Weil representation respects multiplication within M and N. Note that M normalizes N,

$$m_a n_b = n_{a^2 b} m_a.$$

Compute that

$$\begin{aligned} m_a \cdot (n_b \cdot \varphi)(t) &= \chi(a)|a|_{\mathbf{k}} \psi(b N_{\mathbf{K}/\mathbf{k}}(ta))\varphi(ta) \\ &= \psi(a^2 b N_{\mathbf{K}/\mathbf{k}}(t))\chi(a)|a|_{\mathbf{k}}\varphi(ta) \qquad \text{because } N_{\mathbf{K}/\mathbf{k}}(a) = a^2 \\ &= n_{a^2 b} \cdot (m_a \cdot \varphi)(t) \end{aligned}$$

So ω indeed respects multiplication in $P = MN$. Next, observe that w normalizes M:

$$w m_a = m_{a^{-1}} w.$$

Again, compute

$$\begin{aligned} w \cdot (m_a \cdot \varphi)(t) &= c_w \chi(a)|a|_{\mathbf{k}}|a|_{\mathbf{K}}^{-1} \mathcal{F}\varphi(ta^{-1}) \\ &= \chi(a)|a|_{\mathbf{k}}^{-1} c_w \mathcal{F}\varphi(ta^{-1}) \qquad \text{because } |a|_{\mathbf{K}} = |a|_{\mathbf{k}}^2 \\ &= \chi(a^{-1})|a|_{\mathbf{k}}^{-1} c_w \mathcal{F}\varphi(ta^{-1}) \qquad \text{because } \chi \text{ is quadratic} \\ &= m_{a^{-1}} \cdot (w \cdot \varphi)(t) \end{aligned}$$

Thus ω respects multiplication in the group generated by M and w as well.

Still working over a finite place v, the final piece of the argument is showing that the Weil representation is well-defined on all of G. To this end, recall the Bruhat decomposition

$$G = P \sqcup PwN.$$

The action of elements in the big Bruhat cell is defined as

$$(pwn) \cdot \varphi = p \cdot (w \cdot (n \cdot \varphi)).$$

We need to verify that ω respects the various products of elements in the small and big Bruhat cells:

$$\begin{aligned} p_1 \cdot p_2 &= p_1 \cdot p_2 \\ p_1 \cdot p_2 w n_2 &= p_1 p_2 w n_2 \\ p_1 w n_1 \cdot p_2 &= p_1 w n_1 m_2 n_2 = p_1 w m_2 n_1' n_2 = p_1 m_2' w n_1' n_2 = p_3 w n_3 \\ p_1 w n_1 \cdot p_2 w n_2 &= p_3 \\ p_1 w n_1 \cdot p_2 w n_2 &= p_3 w n_3. \end{aligned}$$

Fortunately, our previous calculations immediately dispatch the first three cases.

Consider the fourth case, the product of two big cell elements which is itself a small cell element. We may set $p_2 = nm$. Observing that $mw = -wm^{-1}$, the relationship on line 4 becomes

$$wn_1nw = -p_1^{-1}p_3n_2^{-1}m \in P.$$

In order for $wn_1nw \in P$, we must have $n_1n = 1$. Thus the relationship becomes

$$p_1w \cdot mwn_2 = p_3.$$

We can easily check this:

$$\begin{aligned} p \cdot (w \cdot (m_a \cdot (w \cdot (n_b \cdot \varphi)))) &= p \cdot (w \cdot (w \cdot (m_{a^{-1}} \cdot (n_b \cdot \varphi)))) \\ &= p \cdot (m_{-a^{-1}} \cdot (n_b \cdot \varphi)) \\ &= p' \cdot \varphi. \end{aligned}$$

Verifying the fifth and final relationship proves that the Weil representation is well-defined. Naturally, this equivalence is quite difficult to verify, so we must address some technical issues before proceeding. Let

$$\gamma_b = \int_{\mathbf{K}} S_b(\xi)\ \mathrm{d}\xi,$$

where $S_b(\xi) = \psi(bN_{\mathbf{K}/\mathbf{k}}(\xi))$. The Fourier transform of S_b is

$$\mathcal{F}(S_b)(t) = \int_{\mathbf{K}} \psi(bN_{\mathbf{K}/\mathbf{k}}(\xi) - \mathrm{tr}\mathbf{K}/\mathbf{k}\langle t, \xi\rangle)\ \mathrm{d}\xi.$$

Note that $bN_{\mathbf{K}/\mathbf{k}}(\xi) - \mathrm{tr}\mathbf{K}/\mathbf{k}\langle t, \xi\rangle = bN_{\mathbf{K}/\mathbf{k}}(\xi - t/b) - (1/b)N_{\mathbf{K}/\mathbf{k}}(t)$ by completing the square, so

$$\begin{aligned} \mathcal{F}(S_b)(t) &= \psi(-(1/b)N_{\mathbf{K}/\mathbf{k}}(t)) \int_{\mathbf{K}} \psi(bN_{\mathbf{K}/\mathbf{k}}(\xi - t/b))\ \mathrm{d}\xi \\ &= S_{-1/b}(\xi) \int_{\mathbf{K}} \psi(bN_{\mathbf{K}/\mathbf{k}}(\xi))\ \mathrm{d}\xi \\ &= S_{-1/b}\gamma_b \end{aligned}$$

That is,

$$\mathcal{F}S_b = S_{-1/b}\gamma_b,$$

and by taking the Fourier transform of both sides,

$$\gamma_{-1/b} = \gamma_b.$$

Lemma 2.3. *For any $b \in \mathbf{k}$,*

$$\gamma_b = \chi(b)|b|_{\mathbf{k}}^{-1}\gamma_1$$

Proof. First, suppose $b = N_{\mathbf{K}/\mathbf{k}}(y)$ for some $y \in \mathbf{K}$. Compute

$$\gamma_b = \int_{\mathbf{K}} \psi(bN_{\mathbf{K}/\mathbf{k}}(\xi))\ \mathrm{d}\xi = |y|_{\mathbf{K}}^{-1} \int_{\mathbf{K}} \psi(N_{\mathbf{K}/\mathbf{k}}(y\xi))\ \mathrm{d}(y\xi) = |y|_{\mathbf{K}}^{-1}\gamma_1.$$

Recall that $|y|_{\mathbf{K}} = |yy^\sigma|_{\mathbf{k}} = |b|_{\mathbf{k}}$. Because b is a local norm, we have $\chi(b) = 1$, so we may write

$$\gamma_b = \chi(b)|b|_{\mathbf{k}}^{-1}\gamma_1.$$

Thus the relationship holds when b is a local norm.

Next, suppose b is not a local norm. Take $H = \ker(N_{\mathbf{K}/\mathbf{k}})$, so $N(hx) = 0$ for all $h \in H$. The multiplicative group $H \setminus \mathbf{K}^\times$ is isomorphic to $N(\mathbf{K}^\times)$, where

$$N(\mathbf{K}^\times) = \{u \in \mathbf{k}^\times : u = N_{\mathbf{K}/\mathbf{k}}(x) \text{ for some } x \in \mathbf{K}^\times\}.$$

A small Haar measure computation shows that for $x \in \mathbf{K}^\times$ and $u = N_{\mathbf{K}}/\mathbf{k}(x) \in \mathbf{k}^\times$,

$$\mathrm{d}u = |u|_{\mathbf{k}}\ \mathrm{d}^\times u = |xx^\sigma|_{\mathbf{k}}\ \mathrm{d}^\times(xx^\sigma) = |x|_{\mathbf{K}}\ 2\mathrm{d}^\times x = 2\ \mathrm{d}x.$$

So, letting Θ have measure 1 and halving $\mathrm{d}u$,

$$\begin{aligned}
\gamma_b &= \int_{H\setminus\mathbf{K}} \int_H \psi(bN(hx))\ \mathrm{d}h\mathrm{d}\overline{x} \\
&= \int_{H\setminus\mathbf{K}} \psi(bN(\overline{x}))\ \mathrm{d}\overline{x} \\
&= \int_{N(\mathbf{K}^\times)} \psi(tu)\ \mathrm{d}u \\
&= |t|_{\mathbf{k}}^{-1} \int_{tN(\mathbf{K}^\times)} \psi(u)\ \mathrm{d}u.
\end{aligned}$$

Specifically, for $b = 1$,

$$\gamma_1 = \int_{N(\mathbf{K}^\times)} \psi(u)\ \mathrm{d}u.$$

So we need only show that

$$\int_{\mathbf{k}} \psi(u)\ \mathrm{d}u = 0,$$

but this is immediate since ψ is a nontrivial character on $\mathbf{k}$. $\square$

Because $\gamma_{-1/b} = \gamma_b^{-1}$,

$$\gamma_1\gamma_{-1} = \gamma_1\gamma_1^{-1} = 1.$$

Setting $b = -1$ in the Lemma 2.3,

$$\begin{aligned} \gamma_{-1} = \chi(-1)\gamma_1 \;&\Leftrightarrow\; \gamma_{-1} = \chi(-1)\gamma_{-1}^{-1} \\ &\Leftrightarrow\; \gamma_{-1}^2 = \chi(-1) \end{aligned}$$

Therefore defining $c_w = \gamma_{-1}$ is indeed appropriate. We are now ready to prove the desired result.

Theorem 2.1. *The Weil representation ω is well-defined.*

Proof. Given Lemma 2.1, Lemma 2.2, and the previous discussion, the proof reduces to showing that the Weil representation respects a product of two big cell elements which is itself a big cell element,

$$p_1wn_1 \cdot p_2wn_2 = p_3wn_3.$$

Using our observations that M normalizes N and w normalizes M, the above relationship reduces to

$$wn_{-1/x}w^{-1} = n_xm_xwn_x.$$

So we need only verify that both of these elements have equivalent actions under ω.

Recall that

$$\gamma_b = \int_{\mathbf{K}} S_b(\xi)\ \mathrm{d}\xi,$$

where $S_b(\xi) = \psi(bN_{\mathbf{K}/\mathbf{k}}(\xi))$. Furthermore, recall the relationships

$$\begin{aligned} \mathcal{F}S_b &= S_{-1/b}\gamma_b \\ \gamma_{-1/b} &= \gamma_b^{-1}. \end{aligned}$$

Compute

$$\begin{aligned} (w \cdot n_{-1/b} \cdot w^{-1} \cdot \varphi)(x) &= (-w \cdot n_{-1/b} \cdot w \cdot \varphi)(x) \\ &= \chi(-1)c_w\mathcal{F}(S_{-1/b} \cdot c_w\mathcal{F}\varphi)(x) \\ &= \gamma_b^{-1}\mathcal{F}(\mathcal{F}(S_b) \cdot \mathcal{F}\varphi)(x) \\ &= \gamma_b^{-1}(S_b * \varphi)(x). \end{aligned}$$

We may simplify the convolution:

$$
\begin{aligned}
(S_b * \varphi)(x) &= \int_{\mathbf{K}} S_b(x-\xi)\varphi(\xi)\ \mathrm{d}\xi \\
&= \int_{\mathbf{K}} \psi(bN_{\mathbf{K}/\mathbf{k}}(x-\xi))\varphi(\xi)\ \mathrm{d}\xi \\
&= \psi(bN_{\mathbf{K}/\mathbf{k}}(x)) \int_{\mathbf{K}} \psi(-b\langle x,\xi\rangle)\psi(bN_{\mathbf{K}/\mathbf{k}}(\xi))\varphi(\xi)\ \mathrm{d}\xi \\
&= S_b(x) \int_{\mathbf{K}} (S_b \cdot \varphi)(\xi)\psi(-\langle bx,\xi\rangle)\ \mathrm{d}\xi \\
&= S_b(x)\mathcal{F}(S_b \cdot \varphi)(bx).
\end{aligned}
$$

Putting the pieces together,

$$(w \cdot n_{-1/b} \cdot w^{-1} \cdot \varphi)(x) = \gamma_b^{-1} S_b(x)\mathcal{F}(S_b \cdot \varphi)(bx).$$

Likewise, we find

$$(n_b \cdot m_b \cdot w \cdot n_b \cdot \varphi)(x) = \chi(b)|b|_{\mathbf{k}} c_w S_b(x)\mathcal{F}(S_b \cdot \varphi)(bx).$$

By Lemma 2.3, $\chi(b)|b|_{\mathbf{k}} c_w = \gamma_b^{-1}$, so the two displays are equal. □

I have been slightly cavalier about working locally at v, making Theorem 2.1 misleading. We really proved that ω_v is well-defined at every place v. That said, results in representation theory show that any unitary representation factors over local places. In other words, the *global* Weil representation is well-defined because it can be written as the product of *local* Weil representations,

$$\omega = \bigotimes_v \omega_v,$$

where each ω_v is well-defined.

References

[1] Jerry Shurman, "Transition to the adeles".