MATH 300 **NAME:**
WORKSHEET 6 (4.1-3)

Prove the following using either the first or second principle of mathematical induction.

1. For every natural number n, it follows that $\sum_{i=1}^{n}(8i-5) = 4n^2 - n$.

   Proof: We will prove that for all natural numbers n, $\sum_{i=1}^{n}(8i-5) = 4n^2 - n$ using the First Principle of Mathematical Induction. We will let $P(n)$ be the statement "$\sum_{i=1}^{n}(8i-5) = 4n^2 - n$".

   For the base case, we will show that $P(1)$ is true. The left side of the equation evaluates out to:

$$\sum_{i=1}^{1}(8i-5) = 8(1) - 5 = 8 - 5 = 3$$

   and the right side of the equation evaluates out to:

$$4(1)^2 - 1 = 4(1) - 1 = 4 - 1 = 3$$

   and since both are equal, $P(n)$ holds in the base case of $n = 1$.
   For the inductive step, we will show that for all natural numbers n, if $P(n)$ is true, then $P(n+1)$ must be true by implication. We assume n is a natural number, and that $P(n)$ holds: $\sum_{i=1}^{n}(8i-5) = 4n^2 - n$. To show that $P(n+1)$ is true, we must show that both sides of the equal sign with input $n+1$ are equivalent.

$$\begin{aligned}
\sum_{i=1}^{n+1}(8i-5) &= \sum_{i=1}^{n}(8i-5) + (8(n+1)-5) \\
&= (4n^2 - n) + (8(n+1)-5) && \text{(by inductive hypothesis)} \\
&= 4n^2 - n + 8n + 8 - 5 \\
&= 4n^2 + 7n + 3 && (1)
\end{aligned}$$

$$\begin{aligned}
4(n+1)^2 - (n+1) &= 4(n^2 + 2n + 1) - n - 1 \\
&= 4n^2 + 8n + 4 - n - 1 \\
&= 4n^2 + 7n + 3 && (2)
\end{aligned}$$

   Since expressions (1) and (2) are equivalent, the statement holds for $n+1$. Thus by the First Principle of Mathematical Induction, the statement $\sum_{i=1}^{n}(8i-5) = 4n^2 - n$ is true for all natural numbers n.

2. For any integer $n \geq 0$, it follows that $3|(5^{2n} - 1)$.

   Proof: We will prove that for all integers $n \geq 0$, $3|(5^{2n} - 1)$ using the First Principle of Mathematical Induction. We will let $P(n)$ be the statement "$3|(5^{2n} - 1)$".

   For the base case, we will show that $P(0)$ is true. The expression evaluates out to

$$5^{2(0)} - 1 = 5^0 - 1 = 1 - 1 = 0$$

which is divisible by 3 ($n|0; \forall n > 0 \in \mathbb{N}$). Thus $P(n)$ holds in the base case of $n = 0$.

For the inductive step, we will show that for all integers $n \geq 0$, if $P(n)$ is true, $P(n+1)$ must be true by implication. We assume that n is an integer and that $P(n)$ holds: $3|(5^{2n} - 1)$. To show that $P(n+1)$ is true, we must show that $3|(5^{2(n+1)} - 1)$. Since we are assuming $P(n)$ is true, it follows that there exists an integer m such that $5^{2n} - 1 = 3m$.

$$\begin{aligned}
5^{2(n+1)} - 1 &= 5^{2n+2} - 1 \\
&= 5^2(5^{2n}) \\
&= 25(5^{2n} \\
&= 24(5^{2n}) + 5^{2n} - 1 \\
&= 24(5^{2n}) + 3m && \text{(by inductive hypothesis)} \\
&= 3(8(5^{2n}) + m)
\end{aligned}$$

This is clearly divisible by 3, meaning $P(n+1)$ holds. Thus, by the First Principle of Mathematical Induction, the statement $3|(5^{2n} - 1)$ is true for all integers n.

3. For each natural number n,

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}.$$

Proof: We will prove using the First Principle of Mathematical Induction that for every natural number n, the above equation holds. We will let equation (1) be the proposed equivalence, for simplicity sake. We will also let $P(n)$ be the statement "$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}$."

For the base case, we will show that $P(1)$ is true. The equation simplifies to

$$\frac{1}{\sqrt{1}} = \frac{1}{1} = 1 \geq \sqrt{1} = 1$$

and since this is true, $P(n)$ holds in the base case of $n = 1$.

For the inductive step, we will show that for all natural numbers $n \geq 1$, if $P(n)$ is true, $P(n+1)$ must be true by implication. We assume that n is a natural number and that $P(n)$ holds: $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}$. To show that $P(n+1)$ is true, we must show that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} \geq \sqrt{n+1}$$

We can do this by rewriting the equation with a $\sqrt{n}$:

$$\begin{aligned}
(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}) + \frac{1}{\sqrt{n+1}} &\geq \sqrt{n} + \frac{1}{\sqrt{n+1}} && \text{by inductive hypothesis} \\
\sqrt{n} + \frac{1}{\sqrt{n+1}} &\geq \sqrt{n+1}
\end{aligned}$$

so that now we just need to prove that $\sqrt{n}+\dfrac{1}{\sqrt{n+1}} \geq \sqrt{n+1}$. We can do this by rearranging the equation:

$$\begin{aligned}
\sqrt{n}+\frac{1}{\sqrt{n+1}} &\geq \sqrt{n+1} \\
\frac{1}{\sqrt{n+1}} &\geq \sqrt{n+1}-\sqrt{n} \\
\frac{1}{\sqrt{n+1}} &\geq \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} \\
\frac{1}{\sqrt{n+1}} &\geq \frac{(k+1)+(\sqrt{k}\sqrt{k+1})-(\sqrt{k}\sqrt{k+1})-(k)}{\sqrt{n+1}+\sqrt{n}} \\
\frac{1}{\sqrt{n+1}} &\geq \frac{1}{\sqrt{n+1}+\sqrt{n}} \\
\sqrt{n+1}+\sqrt{n} &\geq \sqrt{n+1} \\
\sqrt{n} &\geq 0
\end{aligned}$$

and since this is clearly true since $n \geq 1$, $P(n+1)$ holds for all natural numbers n. Thus, by the First Principle of Mathematical Induction, the statement $\dfrac{1}{\sqrt{1}}+\dfrac{1}{\sqrt{2}}+\cdots+\dfrac{1}{\sqrt{n}} \geq \sqrt{n}$ is true for all natural numbers n.

4. Let $a_1 = 5$ and $a_2 = 13$. For each natural number n, let $a_{n+2} = 5a_{n+1} - 6a_n$, and prove that $a_n = 2^n + 3^n$.

   Proof: We will prove using the First Principle of Mathematical Induction that for all natural numbers n, if $a_1 = 5$, $a_2 = 13$, and $a_{n+2} = 5a_{n+1} - 6a_n$, then $a_n = 2^n + 3^n$. We will let $P(n)$ be the statement "$a_n = 2^n + 3^n$".

   For the base case, we will show that $P(1)$ is true. If $a_1 = 5$, $a_1 = 2^1 + 3^1 = 5$ proves the statement correct for the base case of $n = 1$.

   For the inductive step, we will show that for all natural numbers n, the proposed sequence is equivalent to $a_n = 2^n + 3^n$. We assume that n is a natural number and that $P(n)$ holds: $a_n = 2^n + 3^n$. To show that $P(n+1)$ is true, we must show that $a_{n+1} = 2^{n+1} + 3^{n+1}$.

$$\begin{aligned}
a_{n+1} &= 5a_n - 6a_{n-1} \\
&= 5(2^n+3^n) - 6(2^{n-1}+3^{n-1}) && \text{by inductive hypothesis} \\
&= 5(2^n+3^n) - 6(\frac{1}{2}*2^n + \frac{1}{3}*3^n) \\
&= 5*2^n + 5*3^n - 3*2^n - 2*3^n \\
&= 2*2^n + 3*3^n \\
&= 2^{n+1} + 3^{n+1}
\end{aligned}$$

   Since we have proven that for all natural numbers $P(n+1)$ holds, we have proven via the First Principle of Mathematical Induction that for all natural numbers n, if $a_1 = 5$, $a_2 = 13$, and $a_{n+2} = 5a_{n+1} - 6a_n$, then $a_n = 2^n + 3^n$.