Discrete Math

11 – Proofs: Basics

Outline

- Terminology
- Proof Methods
- Incorrect Proofs

Terminology

- A *theorem* is a proposition that can be shown to be true.
- A *lemma* is a preliminary proposition useful for proving later propositions.
- A *corollary* is a proposition that can be established directly from a theorem.
- A *conjecture* is a proposition that is being proposed to be a true statement.

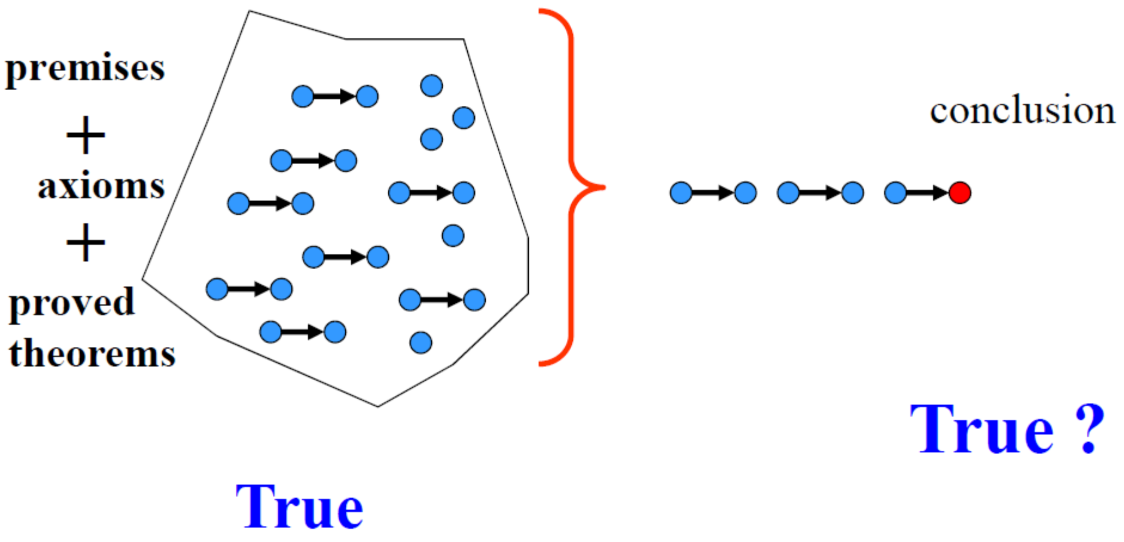
Terminology

Propositions that are simply accepted as true are called *axioms*.

- Examples:
  - For all real numbers x and y, $x + y = y + x$.
  - There is a straight-line segment between every pair of points.

Terminology

- A *proof* is a valid argument that establishes the truth of a statement.
  - can use axioms, premises (if any), and previously proved theorems.



Common Forms of Theorems

- T
- $\exists x\, T(x)$
- $\forall x\, T(x)$
  - Special cases of $T(x)$: $P(x) \longrightarrow Q(x)$ and $P \leftrightarrow Q$

Common Forms of Theorems

- T
  - Example: "$\sqrt{2}$ is not a rational number."
- $\exists x\ T(x)$
  - Example: "There exists one integer *n* such that $n^2 + n + 41$ is composite."
  - Find an element *a* in the domain such that T(*a*) is true and then apply Existential Generalization.
  - To disprove, prove that T(x) is false for all elements in the domain.

Common Forms of Theorems

- $\forall x\,(P(x) \longrightarrow Q(x))$
  - Example: “For any integer *n*, if 3*n*+2 is odd, then *n* is odd.”
  - Show that $P(c) \longrightarrow Q(c)$, where c is an arbitrary element of the domain, and then apply Universal Generalization.
  - Show that Q is true if P is true.
  - To disprove, find an element *e* such that P(*e*) is true, but Q(*e*) is false.
- $\forall x\,(P \leftrightarrow Q)$
  - Proving $P \leftrightarrow Q$ is equivalent to proving $(P \longrightarrow Q) \land (Q \longrightarrow P)$.

Proof Methods

- Direct Proof
- Proof by Contraposition
- Proof by Contradiction
- Proof by Cases

Direct Proof

- Directly show that if P is true, then Q must be true, using axioms, definitions, and previously proven theorems, together with inference rules.

Examples

1. Prove that "If n is odd, then n^2 is odd."

Proof:

- Assume that n is odd, then $n = 2k + 1$, where k is some integer.
- We have $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.
- Therefore, n^2 is an odd integer.

Examples

2. Prove that “If m and n are both perfect squares, then mn is also a perfect square.”

Proof:

- Assume that m and n are both perfect squares, then $m = s^2$ and $n = t^2$, where s and t are some integers.
- We have $mn = s^2t^2 = (st)^2$.
- Therefore, mn is a perfect square.

Exercise

Prove that “Every odd integer is the difference of two squares.”

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Proof:

- Assume that n is odd, then $n = 2k + 1$, where k is some integer.
- $n = 2k + 1 = k^2 + 2k + 1 – k^2 = (k+1)^2 - k^2$.

Proof by Contraposition

Instead of proving $P \longrightarrow Q$, prove $\neg Q \longrightarrow \neg P$

Why?

Implication Law: $P \longrightarrow Q \equiv \neg Q \longrightarrow \neg P$

Examples

1. Prove that “For any integer n, if n^2 is even, then n is even.”

Proof:

- Contraposition: “If n is odd, then n^2 is odd.”
- Example 1 of Direct Proof.
- We have proved the theorem “For any integer n, if n^2 is even, then n is even.”

Examples

2. Prove that “If $3n + 2$ is odd for an integer n, then n is odd.”

Proof:

- Contraposition: “If n is even, then $3n + 2$ is even.”
- Assume that n is even, then $n = 2k$, where k is some integer.
- We have $3n + 2 = 6k + 2 = 2(3k + 1)$.
- Therefore $3n + 2$ is even.
- We have proved the theorem “If $3n + 2$ is odd, then n is odd.”

Examples

3. Prove that “If r is irrational, then $\sqrt{r}$ is also irrational.”

Proof:

- Contraposition: “if $\sqrt{r}$ is rational, then r is rational.”
- Assume that $\sqrt{r}$ is rational.
- There exist integers p and q (no common factors), such that $\sqrt{r} = \frac{p}{q}$.
- Squaring both sides gives: $r = \frac{p^2}{q^2}$.
- Since p^2 and q^2 are integers, r is also rational.
- This completes the proof.

Exercise

Prove that "If $x + y \geq 2$, where x and y are real numbers, then $x \geq 1$ or $y \geq 1$."

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Prove that "If $x + y \geq 2$, where x and y are real numbers, then $x \geq 1$ or $y \geq 1$."

Proof:

- Contraposition: "If $x < 1$ and $y < 1$, then $x + y < 2$."
- Since $x < 1$ and $y < 1$, we have $x + y < 2$.
- Therefore we have proved the theorem.

Proof by Contradiction

- Assume we want to prove S is true.
- Suppose we can find a contradiction C such that $\neg S \longrightarrow C$ is true.
- Since C is false, but $\neg S \longrightarrow C$ is true, then S must be true.

Example

Prove that "$\sqrt{2}$ is not a rational number."

Proof:

- Assume that $\sqrt{2}$ is a rational number.
- Then $\sqrt{2} = \frac{p}{q}$, where p and q have no common factors, and $2 = \frac{p^2}{q^2}$, or $2q^2 = p^2$.
- Since p^2 is even, p is even. (Example 1 of Proof by Contraposition)
- This means that 2 is a factor of p; hence 4 is a factor of p^2, and the equation $2q^2 = p^2$ can be written as $2q^2 = 4x$ or $q^2 = 2x$ for some integer *x*.
- We have q^2 is even and thus q is even. (Example 1 of Proof by Contraposition)
- Now 2 is a factor of *q* and a factor of *p*, which contradicts the statement that *p* and *q* have no common factors.
- Hence $\sqrt{2}$ is not rational.

Proof by Contradiction

Propositions of Implication Forms: $P \longrightarrow Q$

Instead of $P \longrightarrow Q$, prove $\neg(P \longrightarrow Q) \longrightarrow \mathbf{F}$ or $(P \land \neg Q) \longrightarrow \mathbf{F}$

How to find a contradiction?

- Imply Q. Then assert $Q \land \neg Q$ as a contradiction.
- Imply $\neg P$. Then assert $P \land \neg P$ as a contradiction.
- Imply $R \land \neg R$ during the proof for some proposition R.

Examples

1. Prove that “If $3n + 2$ is odd for an integer n, then n is odd.”

Proof:

- Assume to the contrary that $3n+2$ is odd, and n is even.
- Since n is even, $n = 2k$, where k is some integer.
- We have $3n + 2 = 6k + 2 = 2(3k + 1)$.
- Thus $3n + 2$ is even, which contradicts the assumption $3n+2$ is odd.
- Therefore, we have proved the theorem “If $3n + 2$ is odd, then n is odd.”

Examples

2. Prove that “If a number added to itself gives itself, then the number is 0.”

Proof:

- Assume to the contrary that $x + x = x$ and $x \neq 0$.
- Then $2x = x$ and $x \neq 0$.
- Because $x \neq 0$, we can divide both sides of the equation by x and arrive at $2 = 1$, which is a contradiction.
- Hence $x + x = x \longrightarrow x = 0$

Exercise

Show that at least three of any 25 days chosen must fall in the same month of the year.

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Show that at least three of any 25 days chosen must fall in the same month of the year.

Proof:

- Assume to the contrary that any 25 days are chosen and at most 2 days are in the same month of the year.
- Let n_i denote the number of days in the i-th month.
- We have $n_i \leq 2$.
- The total number of chosen days is $n_1 + n_2 + \cdots + n_{12} \leq 24$.
- This contradicts the assumption that 25 days are chosen.
- This completes the proof.

Proof by Cases

- Assume that $P \equiv P_1 \vee P_2 \vee \ldots \vee P_n$.
- Instead of proving $P \longrightarrow Q$, prove

$$(P_1 \longrightarrow Q) \wedge (P_2 \longrightarrow Q) \wedge \ldots \wedge (P_n \longrightarrow Q).$$

- Why?
  - $$\begin{aligned} P_1 \vee P_2 \vee \ldots \vee P_n \longrightarrow Q &\equiv \neg(P_1 \vee P_2 \vee \ldots \vee P_n) \vee Q \\ &\equiv (\neg P_1 \wedge \neg P_2 \wedge \ldots \wedge \neg P_n) \vee Q \\ &\equiv (\neg P_1 \vee Q) \wedge (\neg P_2 \vee Q) \wedge \ldots \wedge (\neg P_n \vee Q) \\ &\equiv (P_1 \longrightarrow Q) \wedge (P_2 \longrightarrow Q) \wedge \ldots \wedge (P_n \longrightarrow Q) \end{aligned}$$

Examples

1. Prove that “If n is an even integer, $4 \le n \le 12$, then n is the sum of two prime numbers.”

Proof:

- We prove this for each value of n:
  - $n = 4 = 2 + 2$
  - $n = 6 = 3 + 3$
  - $n = 8 = 3 + 5$
  - $n = 10 = 5 + 5$
  - $n = 12 = 5 + 7$
- Therefore, this completes the proof.

Credit: Milos Hauskrecht, University of Pittsburgh

Examples

2. Prove that "For any two numbers x and y, $|x||y| = |xy|$."

Proof:

- We prove this by cases. There are 4 cases:
  - Case 1: $x \geq 0, y \geq 0$
    - $xy \geq 0$ and $|xy| = xy = |x||y|$
  - Case 2: $x \geq 0, y < 0$
    - $xy \leq 0$ and $|xy| = -xy = x(-y) = |x||y|$
  - Case 3: $x < 0, y \geq 0$
    - $xy \leq 0$ and $|xy| = -xy = (-x)y = |x||y|$
  - Case 4: $x < 0, y < 0$
    - $xy > 0$ and $|xy| = (-x)(-y) = |x||y|$
- Therefore, $|x||y| = |xy|$.

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Exercise

Prove that “If n is an integer, then $n^2 \geq n$.”

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Prove that “If n is an integer, then $n^2 \geq n$.”

Proof:

- We consider three cases: $n = 0$, $n \geq 1$, and $n \leq -1$.
- Case 1: $n = 0$
  - $n^2 = 0 = n$.
- Case 2: $n \geq 1$
  - We multiply both sizes of the inequality by n, we obtain $n^2 \geq n$.
- Case 3: $n \leq -1$
  - We have $n^2 \geq 0 \geq n$.
- Because it holds for all cases, we proved the statement.

Disproving a Statement

- Find a counterexample of the statement.
- Example: “Every positive integer is the sum of the squares of two positive integers.”

Proof:

- 3 cannot be written as the sum of the squares of two integers.
- To show this, note that the only possible integers are 0 and 1.
- However, there is no way to write 3 as the sum of two terms each of which is either 0 or 1.

What Makes a Good Proof ?

- State your game plan.
- Keep a linear flow.*
- A proof is an essay, not a calculation.
- Avoid excessive symbolism.
- Revise and simplify.
- Introduce notation thoughtfully.
- Structure long proofs.
- Be wary of the “obvious.”
- Finish.

Proof Strategies

- Understand the definitions.
- Analyze the meaning of the hypothesis and conclusion.
- Prove the statement using one of the proof methods.
- Use forward and backward reasoning.

What is Wrong with the Proof?

1. "The sum of two even numbers is a multiple of 4."

- Proof:
  - Let x and y be even numbers.
  - Then x = 2k and y = 2k, where k is an integer.
  - So x + y = 2k + 2k = 4k, which is a multiple of 4.

x and y may not be equal.

What is Wrong with the Proof?

2. “1/8 > 1/4.”

- Proof:
  - $3 > 2$
  - $\boxed{3 \log_{10}(1/2) > 2 \log_{10}(1/2)}$
  - $\log_{10}(1/2)^3 > \log_{10}(1/2)^2$
  - $(1/2)^3 > (1/2)^2$

The inequality symbol should be reversed when multiplying by a negative number.

What is Wrong with the Proof?

3. “1¢ = $1.”

- Proof:
  - 1¢ = $\boxed{\$0.01 = (\$0.1)^2}$ = (10¢)^2 = $1

$\$ \neq \$^2$