MATH 300 **NAME:**
WORKSHEET 7 (Chapter 5)

1. Prove or disprove each of the following:

   (a) $\{9^n \mid n \in \mathbb{N}\} = \{3^n \mid n \in \mathbb{N}\}$

   Proof: We will disprove the proposition by counterexample. We understand that the set $\{9^n \mid n \in \mathbb{N}\}$ is the set of all of the natural powers of 9: $\{1, 9, 81, 729, 6561, \ldots\}$, and that $\{3^n \mid n \in \mathbb{N}\}$ is the set of all of the natural powers of 3: $\{1, 3, 9, 27, \ldots\}$

   9^n can be rewritten as $(3^2)^n = 3^{2n}$, forming the equivalence $\{9^n \mid n \in \mathbb{N}\} = \{3^{2n} \mid n \in \mathbb{N}\}$. This makes the proposition equivalent to $\{3^{2n} \mid n \in \mathbb{N}\} = \{3^n \mid n \in \mathbb{N}\}$. However, this obviously does not cover the same range of values, as the left hand side contains only the even natural powers of 3, while the right hand side contains all natural powers of 3. For example, 3^2 is in both sets, but 3^3 is not, as there is no natural number m such that $3^3 = 3^{2m}$. Since these two sets are not equivalent, the proposition is false.

   (b) $\{9^n \mid n \in \mathbb{Q}\} = \{3^n \mid n \in \mathbb{Q}\}$

   Proof: We will prove the proposition directly. We understand that the set $\{9^n \mid n \in \mathbb{Q}\}$ is the set of all rational powers of 9, and that $\{3^n \mid n \in \mathbb{Q}\}$ is the set of all of rational powers of 3. Again, we will rewrite $\{9^n \mid n \in \mathbb{Q}\}$ as $\{3^{2n} \mid n \in \mathbb{Q}\}$.

   In this case, we have the equivalence $\{3^{2n} \mid n \in \mathbb{Q}\} = \{3^n \mid n \in \mathbb{Q}\}$. If we let $m = 2n \in \mathbb{Q}$, then $n = \dfrac{m}{2}$ and our proposition is equivalent to $\{3^m \mid m \in \mathbb{Q}\} = \{3^n \mid n \in \mathbb{Q}\}$. Now, as long as $n \in \mathbb{Q}$, then it follows that $\dfrac{m}{2} \in \mathbb{Q}$. Thus, these two sets represent the same values, since there will always be a corresponding value for m with input n that allows the same value to be present in both sets. Thus, we have proven that for all $n \in \mathbb{Q}$, $\{9^n \mid n \in \mathbb{Q}\} = \{3^n \mid n \in \mathbb{Q}\}$.

2. Prove that $\cap_{x\in\mathbb{R}}[3 - x^2, 5 + x^2] = [3, 5]$.

   Proof: We will prove the proposition directly. To restate it: "for all real numbers x, the intersection of the range of the functions $3 - x^2$ and $5 + x^2$ is equal to $[3, 5]$.

   To prove this, we can figure out key features of the functions (notably their derivatives, maximum/minimum values, and concavity), find the maximum of the negatively concave function and the minimum of the positively concave function, then find the region that neither function covers. To find the derivatives of both functions:

$$f(x) = 3 - x^2$$
$$f'(x) = \frac{d}{dx}(3 - x^2)$$
$$f'(x) = -2x$$

$$g(x) = 5 + x^2$$
$$g'(x) = \frac{d}{dx}(5 + x^2)$$
$$g'(x) = 2x$$

   $f'(x)$ is a negative function, indicating that $f(x)$ opens downwards, which makes sense due to the x^2 being negative. On the other hand, $g(x)$ opens upwards for the opposite reason. The extrema of both functions occur at $x = 0$ ($f'(0) = g'(0) = 0$), so we can find the extreme values for both:

$$f(0) = 3 - (0)^2 = 3$$

$$g(0) = 5 + (0)^2 = 5$$

Since $f(x)$ is negatively concave, $g(x)$ is positively concave, and the minimum value of $g(x)$ is greater than the maximum of $f(x)$, there is a gap $\mathbb{D} = [3, 5]$ that neither function can reach when given a real number. Thus we have proven the proposition to be true: it is the case that $\cap_{x\in\mathbb{R}}[3 - x^2, 5 + x^2] = [3, 5]$.

3. For sets A, B, C, D, prove that $(A\times B)\cup(C\times D) \subseteq (A\cup C)\times(B\cup D)$, but $(A\times B)\cup(C\times D) \neq (A\cup C)\times(B\cup D)$.

   Proof: We start by proving the subset inclusion in cases. Let there be an element $(x, y) \in (A\times B)\cup(C\times D)$. By the definition of union, this means $(x, y) \in A \times B$ or $(x, y) \in C \times D$.

   Case 1: If $(x, y) \in A\times B$, then $x \in A$ and $y \in B$. Since $A \subseteq A\cup C$ and $B \subseteq B\cup D$, it follows that $x \in A\cup C$ and $y \in B \cup D$, so $(x, y) \in (A \cup C) \times (B \cup D)$.

   Case 2: If $(x, y) \in C \times D$, then $x \in C$ and $y \in D$, which implies $x \in A \cup C$ and $y \in B \cup D$, thus $(x, y) \in (A \cup C) \times (B \cup D)$. In either case, $(x, y) \in (A \cup C) \times (B \cup D)$.

   Therefore, we have shown that $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

   To prove that the sets are not equal, consider an example with specific sets: $A = \{a\}, B = \{b\}, C = \{c\}, D = \{d\}$. Then $A \times B = \{(a, b)\}$ and $C \times D = \{(c, d)\}$, so

   $$(A \times B) \cup (C \times D) = \{(a, b), (c, d)\}$$

   Meanwhile, $A \cup C = \{a, c\}$ and $B \cup D = \{b, d\}$ giving

   $$(A \cup C) \times (B \cup D) = \{(a, b), (a, d), (c, b), (c, d)\}$$

   Note that $(a, d) \in (A \cup C) \times (B \cup D)$ but $(a, d) \notin (A \times B) \cup (C \times D)$. Therefore, $(A \times B) \cup (C \times D) \neq (A \cup C) \times (B \cup D)$. Thus, we conclude that $(A \times B) \cup (C \times D) \subset (A \cup C) \times (B \cup D)$.