ECON 6351 Economic Forecasting

Lecture Nonstationary Time Series

Nonstationary Time Series

a lot of time series in economics and finance are not weakly stationary and instead

- show linear or exponential trend
- show stochastic trend - grow or fall over time or meander without a constant long-run mean
- show increasing variance over time

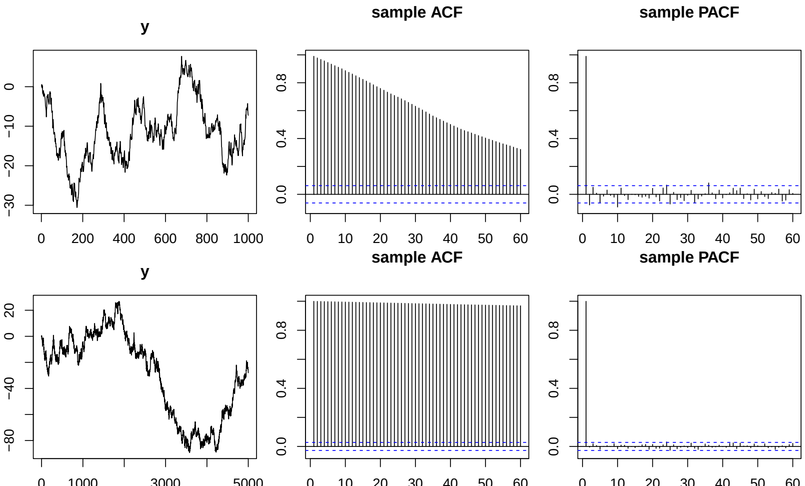
examples

- GDP, consumption, investment, exports, imports, . . .
- industrial production, retail sales, . . .
- interest rates, foreign exchange rates, stock market indices, prices of commodities,. . .
- unemployment rate, labor force participation rate, . . .
- loans, federal debt, . . .

Nonstationary Time Series

A very slowly decaying ACF suggests nonstationarity and presence of deterministic or stochastic trend in the time series, e.g. for $y_t = y_{t-1} + \varepsilon_t$

```
## Registered S3 method overwritten by 'quantmod':
##   method            from
##   as.zoo.data.frame zoo
```



Transformations

Detrending - regressing y_t on intercept and time trend - this is proper treatment if $\{y_t\}$ is trend stationary

Differencing - proper treatment if $\{y_t\}$ is difference stationary

Log transformation and differencing - proper treatment if $\{y_t\}$ grows exponentially and shows increasing variability over time

Trend-Stationary Time Series

- consider times series $\{y_t\}$ that follows
$$y_t = \alpha + \mu t + \varepsilon_t$$
where ε_t is a weakly stationary time series
- $E(y_t) = \alpha + \mu t$ and $var(y_t) = var(\varepsilon_t) = const.$
- since $E(y_t) \neq const.$ time series $\{y_t\}$ is not weakly stationary
- $\{y_t\}$ can however be made stationary by removing time trend using a regression of y_t on constant and time
- $\{y_t\}$ is **trend stationary** time series

Difference-Stationary Time Series

Random Walk

- suppose ε_t is white noise, consider a version of AR(1) model with $\phi_0 = 0$ and $\phi_1 = 1$

  $$y_t = y_{t-1} + \varepsilon_t$$

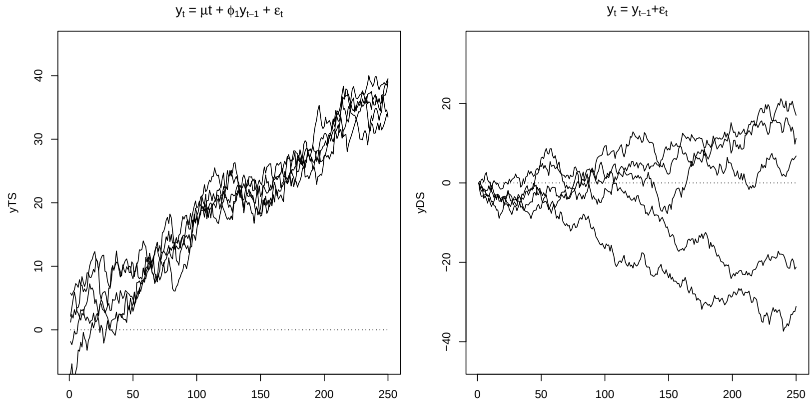
  or, by repeated substitution

  $$y_t = \alpha + \sum_{j=1}^{t} \varepsilon_j$$

  where $\alpha = y_0$
- $E(y_t) = \alpha$ and $var(y_t) = var(\sum_{j=1}^{t} \varepsilon_j) = t\sigma_\varepsilon^2$
- since $var(y_t) \neq const.$ time series $\{y_t\}$ is not weakly stationary
- $\{y_t\}$ *can not* be made difference stationary by removing time trend using a regression of y_t on constant and time
- $\{y_t\}$ can however be made stationary by differencing
- $\{y_t\}$ is **difference stationary** time series

Difference stationary series vs. Trend stationary series

five simulations of trend stationary time series vs random walk



Difference-Stationary Time Series

Random Walk with Drift

- suppose ε_t is white noise, consider a version of AR(1) model with $\phi_1 = 1$

  $$y_t = \mu + y_{t-1} + \varepsilon_t$$

  and by repeated substitution

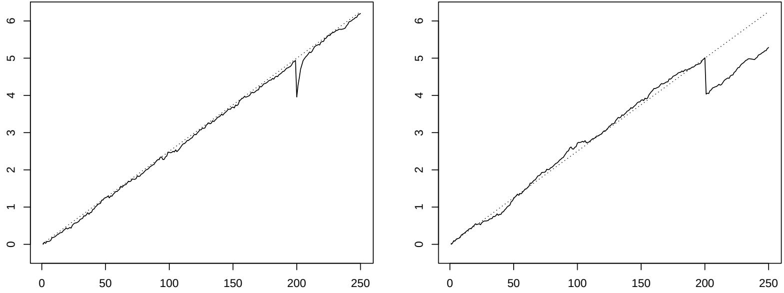
  $$y_t = \alpha + \mu t + \sum_{j=1}^{t} \varepsilon_j$$

  where $\alpha = y_0$
- $E(y_t) = \alpha + \mu t$ and $var(y_t) = var(\sum_{j=1}^{t} \varepsilon_j) = t\sigma_\varepsilon^2$
- $E(y_t) \neq const.$ and $var(y_t) \neq const.$ so $\{y_t\}$ is not weakly stationary
- $\{y_t\}$ *can not* be made difference stationary by removing time trend using a regression of y_t on constant and time
- $\{y_t\}$ can however be made stationary by differencing
- $\{y_t\}$ is **difference stationary** time series

Difference stationary series vs. Trend stationary series

It is important to be able to distinguish between the two cases:

- with trend stationary series shocks have **transitory effects**
- with difference stationary series shocks have **permanent effects**

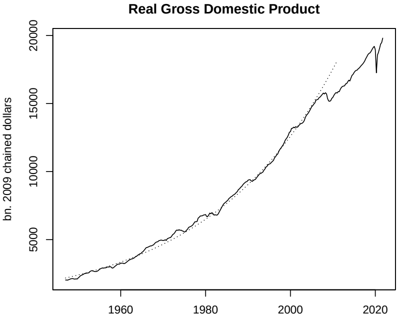


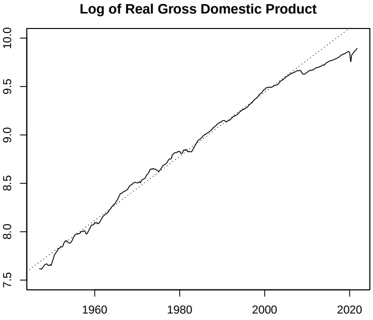
In addition, as we will see later additional issues arise with difference stationary series in the context of multivariate time series analysis

Difference stationary series vs. Trend stationary series

U.S. GDP and the effect of 2008-2009 recession

permanent effect or structural break?





Unit-root Time Series

Autoregressive Integrated Moving-Average (ARIMA) Models

- non-stationary time series is said to contain a **unit root** or to be **integrated of order one**, $I(1)$, if it can be made stationary by applying first differences
- time series $\{y_t\}$ follows an ARIMA$(p, 1, q)$ process if $\Delta y_t = (1 - L)y_t$ follows a stationary and invertible ARMA(p, q) process, so that

$$\phi(L)(1 - L)y_t = \mu + \theta(L)\varepsilon_t$$

Unit-root Time Series

Autoregressive Integrated Moving-Average (ARIMA) Models

- non-stationary time series is said to be **integrated of order** d, $I(d)$, if it can be made stationary by differencing d times
- time series $\{y_t\}$ follows an ARIMA(p, d, q) process if $\Delta^d y_t = (1 - L)^d y_t$ follows a stationary and invertible ARMA(p, q) process, thus

$$\phi(L)(1 - L)^d y_t = \mu + \theta(L)\varepsilon_t$$

- note that pure random walk and random walk with drift are special cases, an ARIMA$(0, 1, 0)$

$$(1 - L)y_t = \mu + \varepsilon_t$$

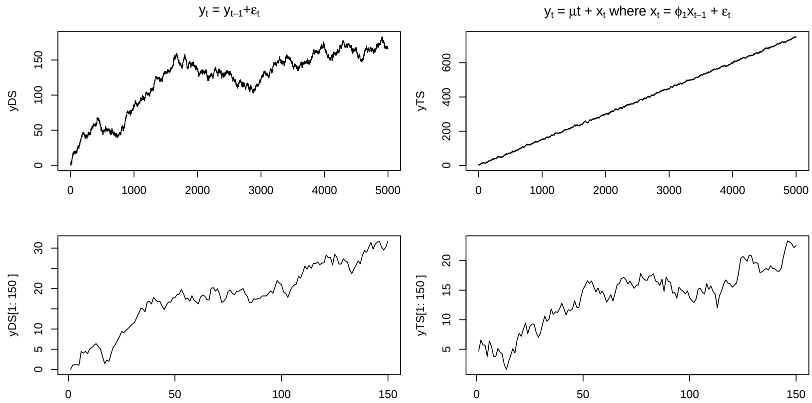
  with $\mu = 0$ in case of pure random walk and $\mu \neq 0$ in case of random walk with drift

Example 1: Difference stationary series vs. Trend stationary series

it is often very hard to distinguish random walk and trend stationary model:

150 vs 5000 observations of

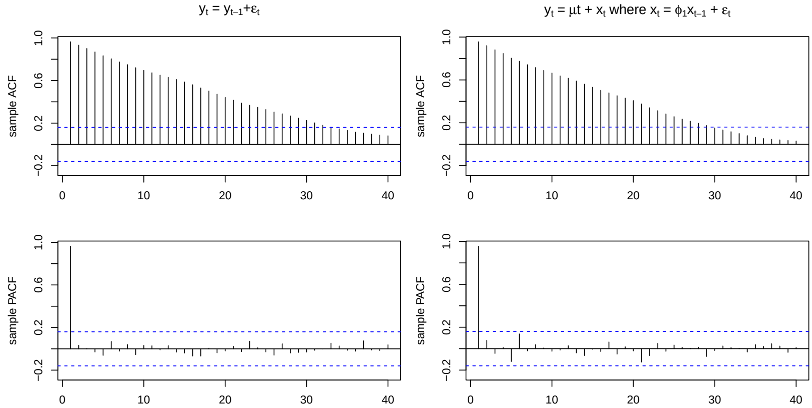
random walk vs. trend stationary AR(1) with $\mu = 0.15$, $\phi_1 = 0.95$



Example 1: Difference stationary series vs. Trend stationary series

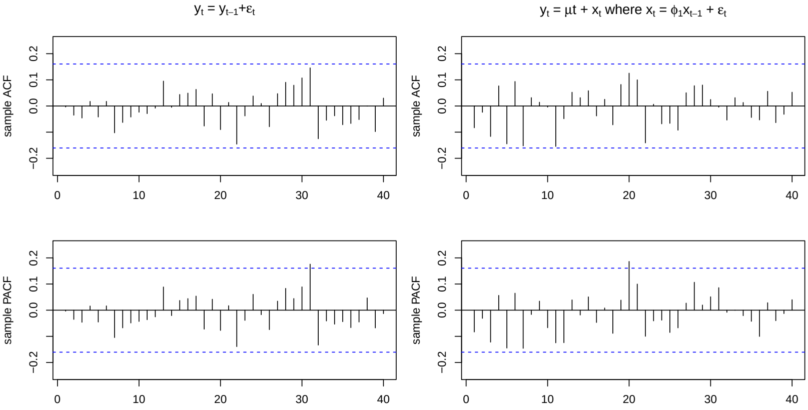
ACF and PACF for 150 observations of y_t under

random walk vs. trend stationary AR(1) with $\mu = 0.15$, $\phi_1 = 0.95$



Example 1: Difference stationary series vs. Trend stationary series

ACF and PACF for 150 observations of first difference Δy_t under random walk vs. trend stationary AR(1) with $\mu = 0.15$, $\phi_1 = 0.95$



Example 1: Difference stationary series vs. Trend stationary series

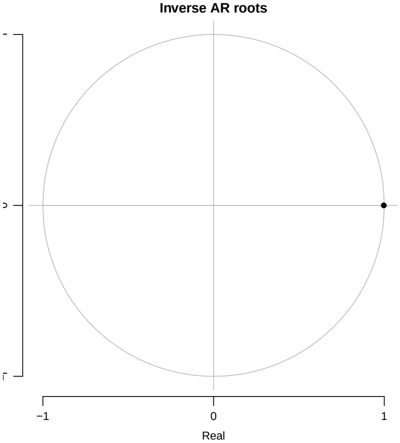
random walk vs. trend stationary AR(1) with $\mu = 0.15$, $\phi_1 = 0.95$

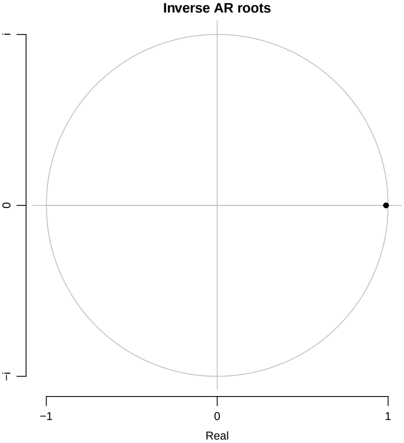
```
## Series: yDS[1:T]
## ARIMA(1,0,0) with non-zero mean
##
## Coefficients:
##          ar1    mean
##       0.9971  16.279
## s.e.  0.0038  12.711
##
## sigma^2 = 1.138:  log likelihood = -224.1
## AIC=454.19   AICc=454.36   BIC=463.22
```

```
## Series: yTS[1:T]
## ARIMA(1,0,0) with non-zero mean
##
## Coefficients:
##          ar1     mean
##       0.9878  13.7733
## s.e.  0.0123   4.7683
##
## sigma^2 = 1.065:  log likelihood = -218.44
## AIC=442.87   AICc=443.04   BIC=451.91
```

Example 1: Difference stationary series vs. Trend stationary series

random walk vs. trend stationary AR(1) with $\mu = 0.15$, $\phi_1 = 0.95$

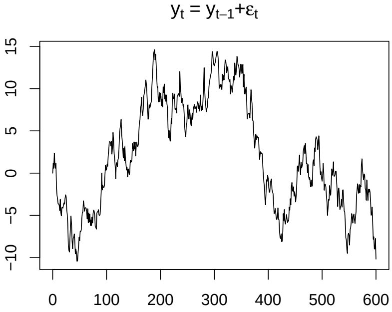


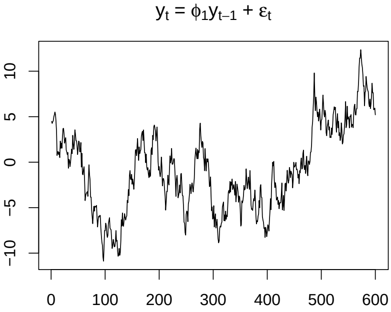


Example 2: Random Walk vs Highly Persistent AR(1)

also very hard to distinguish random walk model and highly persistent AR(1):

random walk I(1) vs. AR(1) with $\phi_1 = 0.98$

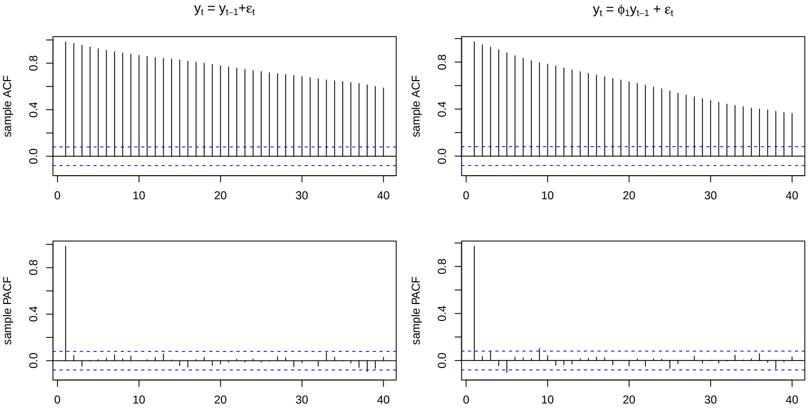




Example 2: Random Walk vs Highly Persistent AR(1)

ACF and PACF for y_t under

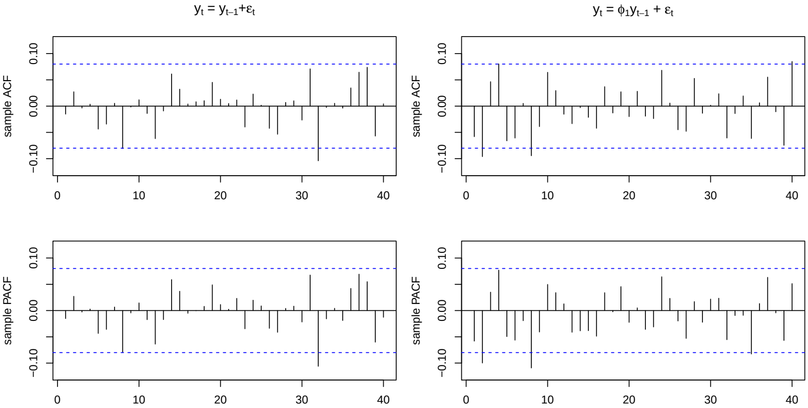
random walk vs. trend stationary AR(1) with $\phi_1 = 0.98$



Example 2: Random Walk vs Highly Persistent AR(1)

ACF and PACF for first difference Δy_t under

random walk vs. trend stationary AR(1) with $\phi_1 = 0.98$



Example 2: Random Walk vs Highly Persistent AR(1)

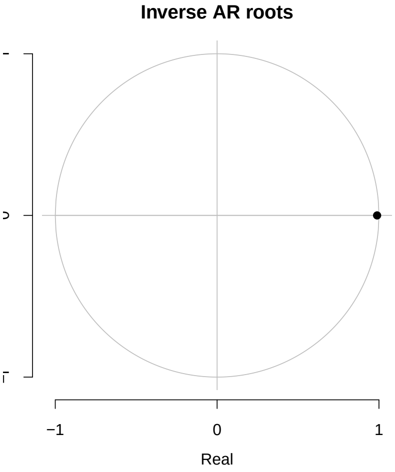
random walk vs. trend stationary AR(1) with $\phi_1 = 0.98$

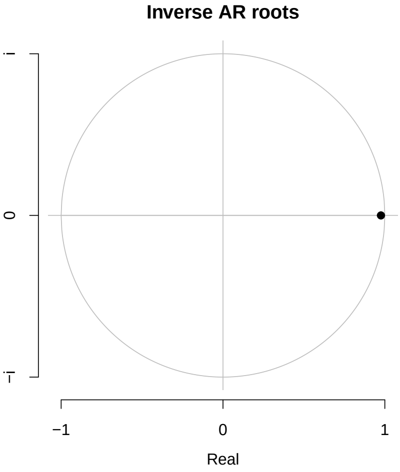
```
## Series: yI1
## ARIMA(1,0,0) with non-zero mean
##
## Coefficients:
##          ar1    mean
##       0.9885  0.4748
## s.e.  0.0060  3.2424
##
## sigma^2 = 1.034:  log likelihood = -863.67
## AIC=1733.33   AICc=1733.37   BIC=1746.53
```

```
## Series: yAR1
## ARIMA(1,0,0) with non-zero mean
##
## Coefficients:
##          ar1     mean
##       0.9760  -0.2034
## s.e.  0.0087   1.6538
##
## sigma^2 = 1.054:  log likelihood = -867.77
## AIC=1741.55   AICc=1741.59   BIC=1754.74
```

Example 1: Difference stationary series vs. Trend stationary series

random walk vs. trend stationary AR(1) with $\mu = 0.15$, $\phi_1 = 0.98$





Unit Root and Stationarity Tests

- two types of tests for nonstationarity
  - **unit root tests**: H_0 is difference stationarity, H_A is trend stationarity
  - **stationarity tests**: H_0 is trend stationary, H_A is difference stationarity
- in general, the approach of these tests is to consider $\{y_t\}$ as a sum

  $$y_t = d_t + z_t + \varepsilon_t$$

  where d_t is a deterministic component (time trend, seasonal component, etc.), z_t is a stochastic trend component and ε_t is a stationary process
- tests then investigate whether z_t is present

Augmented Dickey-Fuller (ADF) test

- main idea: suppose $\{y_t\}$ follows $AR(1)$

  $$y_t = \phi_1 y_{t-1} + \varepsilon_t$$

  then

  $$\Delta y_t = \gamma y_{t-1} + \varepsilon_t$$

  where $\gamma = \phi_1 - 1$
- if $\{y_t\}$ is $I(1)$ then $\gamma = 0$, otherwise $\gamma < 0$

Unit Root and Stationarity Tests

Augmented Dickey-Fuller (ADF) test

- unit root test H_0: time series $\{y_t\}$ has a unit root H_A: time series $\{y_t\}$ is stationary (with zero mean - model A), level stationary (with non-zero mean - model B) or trend stationary (stationary around a deterministic trend - model C)

model A (Unit root): $$\Delta y_t = \gamma y_{t-1} + \sum_{i=1}^{p-1} \rho_i \Delta y_{t-i} + e_t$$

model B (with drift): $$\Delta y_t = \gamma y_{t-1} + \mu + \sum_{i=1}^{p-1} \rho_i \Delta y_{t-i} + e_t$$

model C (with trend): $$\Delta y_t = \gamma y_{t-1} + \mu + \beta t + \sum_{i=1}^{p-1} \rho_i \Delta y_{t-i} + e_t$$

- if $\{y_t\}$ contains a unit root/is difference stationary, $\hat{\gamma}$ will be insignificant
- test $H_0 : \gamma = 0$ against $H_A : \gamma < 0$; if t-statistics for γ is lower than critical values we reject the null hypothesis of a unit root (one-sided left-tailed test)

Augmented Dickey-Fuller (ADF) test

If $\gamma < 0$ then

- under model A y_t fluctuates around zero
- under model B if $\mu \neq 0$ then y_t fluctuates around a non-zero mean
- under model C if $\mu \neq 0$, $\beta \neq 0$ then y_t fluctuates around linear deterministic trend βt

If $\gamma = 0$ then

- under model A y_t contains stochastic trend only
- under model B if $\mu \neq 0$ then y_t contains both a linear deterministic trend μt and a stochastic trend
- under model C if $\mu \neq 0$, $\beta \neq 0$ then y_t contains a quadratic deterministic trend βt^2 and a stochastic trend

Augmented Dickey-Fuller (ADF) test

- including constant and trend in the regression also weakens the test (model C is thus the weakest on, model A the strongest one)
- if possible, we want to exclude the constant and/or the trend, but if they are incorrectly excluded, the test will be biased
- in addition to providing critical values to testing whether $\gamma = 0$, Dickey and Fuller also provide critical values for the following three F tests:
    - ϕ_1 statistic for model B to test $H_0 : \gamma = \mu = 0$
    - ϕ_2 statistic for model C to test $H_0 : \gamma = \mu = \beta = 0$
    - ϕ_3 statistic for model C to test $H_0 : \gamma = \beta = 0$
- these allow us to test whether we can restrict the test

Proposed Full Procedure for ADF test

Step 1. estimate model C and use τ_3 statistic to test H_0: $\gamma = 0$
- if H_0 can not be rejected continue to Step 2
- if H_0 is rejected conclude that y_t is trend stationary

Step 2. use ϕ_3 statistic to test H_0: $\gamma = \beta = 0$
- if H_0 can not be rejected continue to step 3
- if H_0 is rejected estimate restricted model
  $\Delta y_t = \mu + \beta t + \sum_{i=1}^{p-1} \rho_i \Delta y_{t-i} + e_t$
  and use t statistic to test $H_0 : \beta = 0$
  - if H_0 can not be rejected continue to Step 3
  - if H_0 is rejected conclude that y_t is difference stationary with quadratic trend

Step 3. estimate model B and use τ_2 statistic to test H_0: $\gamma = 0$
- if H_0 can not be rejected continue to Step 4
- if H_0 is rejected conclude that y_t is trend stationary

Step 4. use ϕ_1 statistic to test H_0: $\gamma = \mu = 0$
- if H_0 can not be rejected continue to step 5
- if H_0 is rejected estimate restricted model $\Delta y_t = \mu + \sum_{i=1}^{p-1} \rho_i \Delta y_{t-i} + e_t$ and
  use standard t statistic to test $H_0 : \mu = 0$
  - if H_0 can not be rejected continue to Step 5
  - if H_0 is rejected conclude that y_t is random walk with drift

Step 5. estimate model A and use τ_1 statistic to test H_0: $\gamma = 0$

Example 1: Difference stationary series vs. Trend stationary series contd.

```
library(urca)
ur.df(yTS, type = "trend", selectlags = "AIC") %>% summary()
```

```
##
## ###############################################
## # Augmented Dickey-Fuller Test Unit Root Test #
## ###############################################
##
## Test regression trend
##
##
## Call:
## lm(formula = z.diff ~ z.lag.1 + 1 + tt + z.diff.lag)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -3.6246 -0.6734 -0.0073  0.6816  4.3585
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)  0.2156769  0.0294252   7.330 2.68e-13 ***
## z.lag.1     -0.0562692  0.0047070 -11.954  < 2e-16 ***
## tt           0.0084263  0.0007048  11.955  < 2e-16 ***
## z.diff.lag   0.0119032  0.0141433   0.842      0.4
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.018 on 4994 degrees of freedom
## Multiple R-squared:  0.02808,    Adjusted R-squared:  0.02749
## F-statistic: 48.09 on 3 and 4994 DF,  p-value: < 2.2e-16
##
##
## Value of test-statistic is: -11.9543 83.6306 71.4597
##
## Critical values for test statistics:
##       1pct  5pct 10pct
## tau3 -3.96 -3.41 -3.12
## phi2  6.09  4.68  4.03
## phi3  8.27  6.25  5.34
```

Example 1: Difference stationary series vs. Trend stationary series contd.

```
ur.df(yTS[1:150], type = "trend", selectlags = "AIC") %>% summary()
```

```
##
## ###############################################
## # Augmented Dickey-Fuller Test Unit Root Test #
## ###############################################
##
## Test regression trend
##
##
## Call:
## lm(formula = z.diff ~ z.lag.1 + 1 + tt + z.diff.lag)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -2.70057 -0.67726 -0.06942  0.71670  2.36169
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  0.657770   0.284392   2.313   0.0221 *
## z.lag.1     -0.088331   0.035947  -2.457   0.0152 *
## tt           0.009033   0.004035   2.239   0.0267 *
## z.diff.lag  -0.039590   0.082503  -0.480   0.6320
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.003 on 144 degrees of freedom
## Multiple R-squared:  0.04721,    Adjusted R-squared:  0.02736
## F-statistic: 2.378 on 3 and 144 DF,  p-value: 0.0723
##
##
## Value of test-statistic is: -2.4573 2.6964 3.0334
##
## Critical values for test statistics:
##       1pct  5pct 10pct
## tau3 -3.99 -3.43 -3.13
## phi2  6.22  4.75  4.07
## phi3  8.43  6.49  5.47
```

Example 1: Difference stationary series vs. Trend stationary series contd.

In the first set of result the absolute value of test statistics is not less than critical values ($11.9543 > 3.96$) for tau3. So we reject the null and conclude that the y_t is trend stationary.

For the second set of results, since test statistics is not less than the critical value ($2.43 < 3.43$) of tau3 we are unable to reject the null of a unit root. So, we look for Phi3, since test statistics is less than critical value ($3.03<6.49$), we reject the null for unit root and conclude that the time series is difference stationary with quadratic trend.

Unit Root and Stationarity Tests

Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test

- stationarity test H_0: $\{y_t\}$ is stationary (either mean stationary or trend stationary) H_A: $\{y_t\}$ is difference stationary (has a unit root)
- main idea: decompose time series $\{y_t\}$ as

$$y_t = d_t + z_t + \varepsilon_t$$

  where d_t is the deterministic trend, z_t is random walk $z_t = z_{t-1} + \nu_t$, ν_t is white noise (iid $E(\nu_t) = 0$, $var(\nu_t) = \sigma_\nu^2$), and ε_t stationary error (i.e. $I(0)$ but not necessarily white noise)
- stationarity of $\{y_t\}$ depends on σ_ν^2, we can run a test

$$H_0 : \sigma_\nu^2 = 0$$

  against

$$H_A : \sigma_\nu^2 > 0$$

  using Lagrange multiplier (LM) statistic

Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test

- to perform KPSS test we estimate

$$\begin{aligned} &\text{model A} \qquad y_t = \mu + e_t \\ &\text{model B} \qquad y_t = \mu + \beta t + e_t \end{aligned}$$

  model A is used if H_0 is mean stationarity, model B is used if H_0 is trend stationarity
- using residuals e_t we construct LM statistics η

$$\eta = \frac{1}{T^2}\frac{1}{s^2}\sum_{t=1}^{T} S_t^2$$

  where $S_t = \sum_{i=1}^{t} e_i$ is the partial sum process of the residuals e_t and s^2 is an estimator of the long-run variance of the residuals e_t.
- KPSS test is a one-sided right-tailed test: we reject H_0 at $\alpha\%$ level if η is greater than $100(1-\alpha)\%$ percentile from the appropriate asymptotic distribution

Example 1: Difference stationary series vs. Trend stationary series contd.

```
ur.kpss(yTS, type = "tau", lags = "long") %>% summary()
```

```
##
## #######################
## # KPSS Unit Root Test #
## #######################
##
## Test is of type: tau with 31 lags.
##
## Value of test-statistic is: 0.1483
##
## Critical value for a significance level of:
##                 10pct  5pct 2.5pct  1pct
## critical values 0.119 0.146  0.176 0.216
```

```
ur.kpss(yTS[1:150], type = "tau", lags = "long") %>% summary()
```

```
##
## #######################
## # KPSS Unit Root Test #
## #######################
##
## Test is of type: tau with 13 lags.
##
## Value of test-statistic is: 0.1809
##
## Critical value for a significance level of:
##                 10pct  5pct 2.5pct  1pct
## critical values 0.119 0.146  0.176 0.216
```

Example 1: Difference stationary series vs. Trend stationary series contd.

```r
ur.kpss(yDS, type = "tau", lags = "long") %>% summary()
```

```
##
## #######################
## # KPSS Unit Root Test #
## #######################
##
## Test is of type: tau with 31 lags.
##
## Value of test-statistic is: 1.9601
##
## Critical value for a significance level of:
##                 10pct  5pct 2.5pct  1pct
## critical values 0.119 0.146  0.176 0.216
```

```r
ur.kpss(yDS[1:150], type = "tau", lags = "long") %>% summary()
```

```
##
## #######################
## # KPSS Unit Root Test #
## #######################
##
## Test is of type: tau with 13 lags.
##
## Value of test-statistic is: 0.1412
##
## Critical value for a significance level of:
##                 10pct  5pct 2.5pct  1pct
## critical values 0.119 0.146  0.176 0.216
```

Phillips-Perron (PP) test

- an alternative to ADF test, estimates one of the models

$$\begin{array}{ll} \text{model A} & \Delta y_t = \gamma y_{t-1} + e_t \\ \text{model B} & \Delta y_t = \gamma y_{t-1} + \mu + e_t \\ \text{model C} & \Delta y_t = \gamma y_{t-1} + \mu + \beta t + e_t \end{array}$$

  and tests $H_0 : \gamma = 0$ against $H_A : \gamma < 0$
- unlike ADF uses non-parametric correction based on Newey-West heteroskedasticity and autocorrelation consistent (HAC) estimators to account for possible autocorrelation in e_t
- advantage over the ADF: PP tests are robust to general forms of heteroskedasticity and do not require to choose number of lags in the test regression
- asymptotically identical to ADF test, but likely inferior in small samples
- like ADF also not very powerful at distinguishing stationary near unit root series for unit root series