Optimal Linear Network Coding When 3 Nodes Communicate Over Broadcast Erasure Channels with ACK

Jaemin Han and Chih-Chun Wang
School of Electrical and Computer Engineering, Purdue University, USA
{han83,chihw}@purdue.edu

***Abstract*—This work considers the following scenario: Three nodes $\{1, 2, 3\}$ would like to communicate with each other by sending packets through unreliable wireless medium. We consider the most general unicast traffic demands. Namely, there are six co-existing unicast flows with rates $(R_{1\to2}, R_{1\to3}, R_{2\to1}, R_{2\to3}, R_{3\to1}, R_{3\to2})$. When a node broadcasts a packet, a random subset of the other two nodes will receive the packet. After each transmission, causal ACKnowledgment is sent so that all nodes know whether the other nodes have received the packet or not.**

Such a setting has many unique features. For example, each node, say node 1, can assume many different roles: Being the transmitter of the information $R_{1\to2}$ and $R_{1\to3}$; being the receiver of the information $R_{2\to1}$ and $R_{3\to1}$; and being the relay for the information $R_{2\to3}$ and $R_{3\to2}$. This fully captures the fundamental behaviors of 3-node network communications. Allowing *network coding* (NC) to capitalize the diversity gain (i.e., overhearing packets transmitted by other nodes), this work characterizes the 6-dimensional *linear network coding* (LNC) capacity of the above erasure network. The results show that for any channel parameters, the LNC capacity can be achieved by a simple strategy that involves only a few LNC choices.

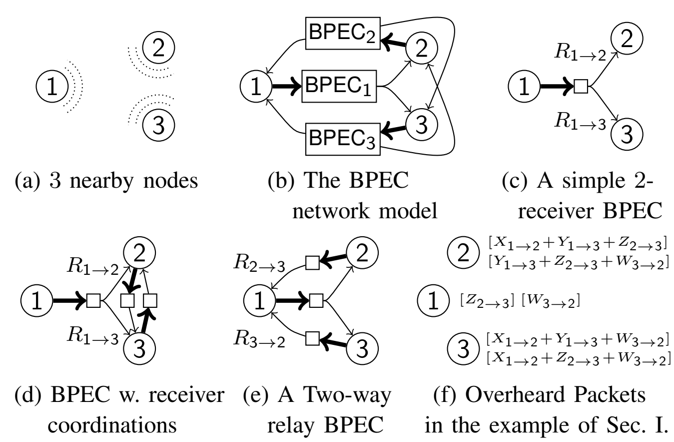

Fig. 1. Various illustrations of broadcast packet erasure channel (BPEC) networks: (a) Six $(R_{1\to2}, R_{1\to3}, R_{2\to1}, R_{2\to3}, R_{3\to1}, R_{3\to2})$ co-existing unicast flows; (b) The corresponding BPEC network model; (c) A 2-receiver BPEC scenario; (d) A 2-receiver BPEC with receiver coordinations scenario; (e) A 2-flow relaying (butterfly-style) BPEC scenario; (f) A packets-overheard scenario that node 1 can benefit 4 co-existing flows simultaneously by a single transmission of the packet $[X_{1\to2} + Y_{1\to3} + Z_{2\to3} + W_{3\to2}]$.

I. Introduction

Recently, linear network coding (LNC) has emerged as a promising technique in modern communication networks. For the single-multicast traffic over an *error-free network*, LNC strictly outperforms non-coding solutions and can achieve the multicast capacity. Even when considering *random erasure networks*, [2] characterizes the single-multicast capacity and shows that LNC is again capacity-achieving, regardless of whether we allow for causal channel state information (CSI) feedback or not.

Despite the above promising results, our understanding is still nascent when there are multiple co-existing unicast flows in the network. When there are only 2 nodes in the network, Shannon [8] characterized the capacity of two-way communication when each node serves simultaneously as a source and as a destination. Nonetheless, little is known when there are no less than three nodes [1], [3]. Moreover, if there are multiple co-existing flows in the network that go in different directions, then each node sometimes has to assume different roles (say, being a sender and/or being a relay) simultaneously, which further complicates the analysis.

In this work, we study the 3-node network, Fig. 1(a), with the most general traffic requirements. Namely, there are six co-existing unicast flows with rates $(R_{1\to2}, R_{1\to3}, R_{2\to1}, R_{2\to3}, R_{3\to1}, R_{3\to2})$ in all possible directions. To simplify the analysis, we consider the simplest non-trivial noisy channel model, the random broadcast packet erasure channel (BPEC). That is, each node is associated with a BPEC. When a node broadcasts a packet, a random subset of the other nodes will receive the packet, see Fig. 1(b). We further assume time-sharing among all three nodes so that interference is fully avoided and thus we can concentrate on the topological effects and the diversity gain of BPECs. Also, time-sharing closely matches the Wi-Fi protocols in practice [6]. Thus the theoretic understanding in this work will also benefit development of practical protocols. Motivated by the throughput benefit of CSI feedback for erasure networks [4], [5], [7], [9]–[12], this work allows for ACKnowledgment after each transmission so that all network nodes know whether the other nodes have received a certain packet or not. Using the above 3-node erasure network setting, this work characterizes the 6-dimensional LNC capacity region and finds the optimal LNC strategy.

This 3-node network contains many important practical and theoretically interesting scenarios as sub-cases. For example, if we project the 6-dimensional LNC capacity region along the 2-dimensional marginal rates $(R_{1\to2}, R_{1\to3})$ and assume that the BPECs of nodes 2 and 3 are always erasure (i.e., both nodes cannot transmit anything), then Fig. 1(b) collapses to Fig. 1(c), the 2-receiver BPEC scenario, which was studied in [4], [12] and later received many attentions (on its variants) in [4], [10]. If we further allow nodes 2 and 3 to transmit (i.e., their BPECs are not always erasure), then Fig. 1(c) evolves to the BPEC with receiver coordinations as in Fig. 1(d), for which the LNC capacity was characterized in [11]. One can easily see that Fig. 1(b) also contains Fig. 1(e) as a special example in which node 1 is a two-way relay for flows $2\rightarrow 3$ and $3\rightarrow 2$. On top of this 2-way relaying example, the 3-node 6-flow setting even contains the scenario when we allow nodes 2 and 3 to communicate directly with each other, which was extensively studied in [7]. By studying the most general 6-dimensional LNC capacity, this work explores the most fundamental behaviors of 3-node communications.

The landscape of the 3-node 6-flow problem is quite different than the existing works that involve mostly 2 co-existing flows. For example, it is known that we may sometimes benefit two destinations (two co-existing flows) simultaneously by transmitting one coded packet, see [4]. On the other hand, *a single transmission may benefit 4 co-existing flows simultaneously for the 3-node 6-flow setting.* For example, consider four information packets $X_{1\rightarrow 2}$, $Y_{1\rightarrow 3}$, $Z_{2\rightarrow 3}$, and $W_{3\rightarrow 2}$. Namely, $X_{1\rightarrow 2}$ is a packet for the flow $1\rightarrow 2$ (i.e., the packet is available at node 1 and destined for node 2) and so on so forth. Suppose node 1 has overheard $Z_{2\rightarrow 3}$ and $W_{3\rightarrow 2}$ from the past transmissions; node 2 has overheard two linear combinations $[X_{1\rightarrow 2}+Y_{1\rightarrow 3}+Z_{2\rightarrow 3}]$ and $[Y_{1\rightarrow 3}+Z_{2\rightarrow 3}+W_{3\rightarrow 2}]$; and node 3 has overheard $[X_{1\rightarrow 2}+Y_{1\rightarrow 3}+W_{3\rightarrow 2}]$ and $[X_{1\rightarrow 2}+Z_{2\rightarrow 3}+W_{3\rightarrow 2}]$. See Fig. 1(f) for illustration.

One can easily check that node 2 cannot decode any of its desired packets $W_{3\rightarrow 2}$ and $X_{1\rightarrow 2}$; and node 3 cannot decode any of its desired packets $Z_{2\rightarrow 3}$ and $Y_{1\rightarrow 3}$. Suppose node 1 now sends a linear combination $[X_{1\rightarrow 2}+Y_{1\rightarrow 3}+Z_{2\rightarrow 3}+W_{3\rightarrow 2}]$ and it is received by both nodes 2 and 3. Node 2 can now decode *both* its desired packets $W_{3\rightarrow 2}$ and $X_{1\rightarrow 2}$ by subtracting known packets $[X_{1\rightarrow 2}+Y_{1\rightarrow 3}+Z_{2\rightarrow 3}]$ and $[Y_{1\rightarrow 3}+Z_{2\rightarrow 3}+W_{3\rightarrow 2}]$ from $[X_{1\rightarrow 2}+Y_{1\rightarrow 3}+Z_{2\rightarrow 3}+W_{3\rightarrow 2}]$, respectively. Similarly, node 3 can decode *both* $Z_{2\rightarrow 3}$ and $Y_{1\rightarrow 3}$ from receiving a single packet $[X_{1\rightarrow 2}+Y_{1\rightarrow 3}+Z_{2\rightarrow 3}+W_{3\rightarrow 2}]$. A single transmission now benefits 4 co-existing flows!

The above example shows that there are many new coding choices that need to be considered for this 3-node 6-flow setting. The main contribution of this work is to first derive a 6-dimensional LNC capacity outer bound by exhaustively enumerating all possible LNC choices with the help of a linear-programming (LP) solver. We then derive an inner bound by a simple strategy that involves only 4 coding choices. By proving that the inner and outer bounds match, we have fully characterized the 6-dimensional LNC capacity and proved that the LNC capacity can be achieved by a surprisingly simple LNC solution.

II. Problem Formulation

We use node indices (i,j,k) to represent one of three cyclically shifted tuples $\{(1,2,3),(2,3,1),(3,1,2)\}$. We consider 6-dimensional traffic rates $\vec{R}\triangleq(R_{1\rightarrow 2},R_{1\rightarrow 3},R_{2\rightarrow 1},R_{2\rightarrow 3},R_{3\rightarrow 1},R_{3\rightarrow 2})$, where their total sum is denoted by R_{Σ}. We assume slotted transmissions, and within a total budget of n time slots, node i would like to send $nR_{i\rightarrow h}$ packets, denoted by a row vector $\mathbf{W}_{i\rightarrow h}$, to node $h\neq i$ (one of the other two nodes). Each uncoded packet is chosen independently and uniformly randomly from a finite field $\mathbb{F}_q$ with size $q>0$.

For any time slot $t\in\{1,\cdots,n\}$, define the *channel output vector* $\mathbf{Z}(t)=(Z_{1\rightarrow 2}(t),Z_{1\rightarrow 3}(t),Z_{2\rightarrow 1}(t),Z_{2\rightarrow 3}(t),Z_{3\rightarrow 1}(t),Z_{3\rightarrow 2}(t))\in\{1,*\}^6$, where $Z_{i\rightarrow h}(t)=1$ and $*$ represents whether node h can receive the transmission from node i or not. We assume that only one node can transmit at each time slot, and express the *scheduling decision* by $\sigma(t)\in\{1,2,3\}$. If $\sigma(t)=i$, then node i transmits a packet $X_i(t)\in\mathbb{F}_q$; and only when $Z_{i\rightarrow h}(t)=1$, node h will receive $Y_{i\rightarrow h}(t)=X_i(t)$. In all other cases, node h receives an erasure $Y_{i\rightarrow h}(t)=*$.

We further assume that $\mathbf{Z}(t)$ is memoryless and stationary, i.e., $\mathbf{Z}(t)$ is independently and identically distributed over the time axis t. We use $p_{i\rightarrow jk}\triangleq\mathsf{Prob}(Z_{i\rightarrow j}(t)=1,Z_{i\rightarrow k}(t)=1)$ to denote the probability that $X_i(t)$ is successfully received by both nodes j and k; and use $p_{i\rightarrow j\overline{k}}$ to denote the probability $\mathsf{Prob}(Z_{i\rightarrow j}(t)=1,Z_{i\rightarrow k}(t)=*)$ that $X_i(t)$ is successfully received by node j but not by node k. Probability $p_{i\rightarrow\overline{j}k}$ is defined symmetrically. We also use $p_{i\rightarrow j\vee k}=p_{i\rightarrow\overline{j}k}+p_{i\rightarrow jk}+p_{i\rightarrow j\overline{k}}$ to denote the probability that at least one of nodes j and k receives it, and use $p_{i\rightarrow j}$ (resp. $p_{i\rightarrow k}$) for the marginal reception probability from node i to node j (resp. node k).

Assuming that the 6-bit $\mathbf{Z}(t)$ vector is broadcast to all nodes after each packet transmission through a separate control channel, a *network code* is described by n scheduling functions

$$\forall\, t\in\{1,\cdots,n\},\ \ \sigma(t)=f_{\sigma,t}([\mathbf{Z}]_1^{t-1}),\tag{1}$$

plus $3n$ encoding functions: $\forall\, t\in\{1,\cdots,n\}$ and $\forall\, i\in\{1,2,3\}$,

$$X_i(t)=f_{i,t}(\mathbf{W}_{i\rightarrow\{j,k\}},[Y_{j\rightarrow i},Y_{k\rightarrow i},\mathbf{Z}]_1^{t-1})\circ 1_{\{\sigma(t)=i\}},\tag{2}$$

plus 3 decoding functions: $\forall\, i\in\{1,2,3\}$,

$$(\hat{\mathbf{W}}_{j\rightarrow i},\hat{\mathbf{W}}_{k\rightarrow i})=g_i([\sigma,Y_{j\rightarrow i},Y_{k\rightarrow i},\mathbf{Z}]_1^n),\tag{3}$$

where $\mathbf{W}_{i\rightarrow\{j,k\}}\triangleq\mathbf{W}_{i\rightarrow j}\cup\mathbf{W}_{i\rightarrow k}$ and we use brackets $[\cdot]_1^\tau$ to denote the collection from time 1 up to time τ. For example, $[\sigma,Y_{j\rightarrow i},Y_{k\rightarrow i},\mathbf{Z}]_1^n$ in (3) is a shorthand for the collection $\{\sigma(t),Y_{j\rightarrow i}(t),Y_{k\rightarrow i}(t),\mathbf{Z}(t):\forall\, t\in\{1,\cdots,n\}\}$.

Namely, at every time t, scheduling is decided based on the network-wide channel state information (CSI) up to $(t-1)$. Each node encodes based on the current scheduling decision, the information messages, what it has overheard from other nodes in the past, and the past CSI. In the end of time n, each node decodes its desired packets based on the past scheduling decisions, what it has received, and the past network-wide CSI.

III. The Space-based Formulation of Linear NC

Let $\mathbf{W}$ be an nR_{Σ}-dimensional row vector defined by

$$\mathbf{W}\triangleq(\mathbf{W}_{1\rightarrow 2},\mathbf{W}_{1\rightarrow 3},\mathbf{W}_{2\rightarrow 1},\mathbf{W}_{2\rightarrow 3},\mathbf{W}_{3\rightarrow 1},\mathbf{W}_{3\rightarrow 2}).\tag{4}$$

That is, $\mathbf{W}$ is the collection of all the information packets for the 6-dimensional traffic $\vec{R}$. Define $\Omega\triangleq(\mathbb{F}_q)^{nR_{\Sigma}}$ as the *overall message/coding space*. Then, a network code is called *linear* if (2) can be rewritten as

$$\text{If } \sigma(t)=i, \text{ then } X_i(t)=\mathbf{c}_t\mathbf{W}^\top \text{ for some } \mathbf{c}_t\in\Omega, \quad (5)$$

where $\mathbf{c}_t$ is a row coding vector in Ω. We assume that $\mathbf{c}_t$ is known causally to the entire network.[1]

We now define two important concepts: The *individual message subspace* and the *reception subspace.* To that end, we first define $\mathbf{e}_l$ as an nR_Σ-dimensional elementary row vector with its l-th coordinate being one and all the other coordinates being zero. Recall that the nR_Σ coordinates of a vector in Ω can be divided into 6 consecutive "intervals", each of them corresponds to the information packets $\mathbf{W}_{i\to h}$ for the unicast flow from node i to node $h\neq i$. For example, from (4), the third interval corresponds to the packets $\mathbf{W}_{2\to 1}$. We then define the *individual message subspace* $\Omega_{i\to j}$:

$$\Omega_{i\to j} \triangleq \mathsf{span}\{\mathbf{e}_l : l\in \text{“interval” associated to } \mathbf{W}_{i\to j}\}, \quad (6)$$

That is, $\Omega_{i\to j}$ is a linear subspace corresponding to any linear combination of $\mathbf{W}_{i\to j}$ packets. By (6), each $\Omega_{i\to j}$ is a linear subspace of Ω and $\mathsf{rank}(\Omega_{i\to j})=nR_{i\to j}$.

For each node $i\in\{1,2,3\}$, the *reception subspace* in the end of time t is defined by

$$\begin{aligned} RS_i(t) \triangleq \mathsf{span}\{\mathbf{c}_\tau : \forall \tau\leq t \text{ s.t. } \sigma(\tau)\neq i,\ Z_{\sigma(\tau)\to i}(\tau)=1,\\ \text{and } Y_{\sigma(\tau)\to i}(\tau)=X_{\sigma(\tau)}(\tau)=\mathbf{c}_\tau\mathbf{W}^\top\}. \end{aligned} \quad (7)$$

That is, $RS_i(t)$ is the linear subspace spanned by the coding vectors $\mathbf{c}_\tau$ corresponding to the packets that are sent by node $\sigma(\tau)\neq i$ and have successfully arrived at node i by the end of time t. We now define the *knowledge space* $S_i(t)$ by

$$S_i(t)\triangleq \Omega_{i\to j}\oplus\Omega_{i\to k}\oplus RS_i(t), \quad (8)$$

where $A\oplus B\triangleq \mathsf{span}\{\mathbf{v}:\mathbf{v}\in A\cup B\}$ is the *sum space* of any $A,B\subseteq\Omega$. Basically, $S_i(t)$ represents the "overall knowledge" available at node i, which contains those that are originated from node i, i.e., $\Omega_{i\to j}\oplus\Omega_{i\to k}$, and those overheard by node i until time t, i.e., $RS_i(t)$. By the above definitions, we quickly have that node i can decode the desired packets $\hat{\mathbf{W}}_{h\to i}$, $h\neq i$, as long as $S_i(n)\supseteq\Omega_{h\to i}$. That is, when the knowledge space in the end of time n contains the desired message space.

Note that each node can only send a linear mixture of the packets that it currently "knows." Therefore, we can further strengthen the encoding part (5) by the following statement:

$$\text{If } \sigma(t)=i, \text{ then } X_i(t)=\mathbf{c}_t\mathbf{W}^\top \text{ for some } \mathbf{c}_t\in S_i(t-1). \quad (9)$$

We can now define the LNC capacity region.

Definition 1: Fix the distribution of $\mathbf{Z}(t)$ and finite field $\mathbb{F}_q$. A 6-dimensional rate vector $\vec{R}$ is achievable by LNC if for any $\epsilon>0$ there exists a joint scheduling and LNC scheme with sufficiently large n such that $\mathsf{Prob}(\hat{\mathbf{W}}_{i\to h}\neq\mathbf{W}_{i\to h})<\epsilon$ for all $i\in\{1,2,3\}$ and $h\neq i$. The LNC capacity region is the closure of all LNC-achievable $\vec{R}$.

[1]Coding vector $\mathbf{c}_t$ can either be appended in the header or be computed by the network-wide causal CSI feedback $\mathbf{Z}(t)$.

IV. Main Results

Since the coding vector $\mathbf{c}_t$ has nR_Σ number of coordinates, there are exponentially many ways of jointly designing the scheduling $\sigma(t)$ and the coding vector choices $\mathbf{c}_t$ over time when sufficiently large n and $\mathbb{F}_q$ are used. We will first simplify the aforementioned design choices by comparing $\mathbf{c}_t$ to the knowledge spaces $S_i(t-1)$ described previously. Such a simplification allows us to derive Proposition 1, which uses a linear programming (LP) solver to exhaustively search over the entire coding and scheduling choices and thus computes an LNC capacity outer bound. An LNC capacity inner bound will later be derived by proposing a simple LNC solution and analyze its performance. Finally, we prove that the inner and outer bounds match.

A. The LNC Capacity outer bound

Recall that $(i,j,k)\in\{(1,2,3),(2,3,1),(3,1,2)\}$, the cyclically shifted node indices. For example, if $i=2$, then $j=3$ and $k=1$. We also use S_i as shorthand for $S_i(t-1)$, the node-i knowledge space in the end of time $t-1$. For all $i\in\{1,2,3\}$, define the following seven linear subspaces of Ω:

$$A_1^{(i)}(t)\triangleq S_i, \qquad A_2^{(i)}(t)\triangleq S_i\oplus\Omega_{j\to i}, \quad (10)$$

$$A_3^{(i)}(t)\triangleq S_i\oplus\Omega_{k\to i}, \quad A_4^{(i)}(t)\triangleq S_i\oplus\Omega_{j\to i}\oplus\Omega_{k\to i}, \quad (11)$$

$$A_1^{(i,j)}(t)\triangleq S_i\oplus S_j, \qquad A_2^{(i,j)}(t)\triangleq S_i\oplus S_j\oplus\Omega_{k\to i}, \quad (12)$$

$$A_3^{(i,j)}(t)\triangleq S_i\oplus S_j\oplus\Omega_{k\to j}. \quad (13)$$

Since the knowledge spaces S_i evolves over time, see (8), the above "A-subspaces" also evolves over time.

There are in total $7\times 3=21$ linear subspaces of Ω. We often drop the input argument "(t)" when the time instant of interest is clear in the context. We then partition the overall message space Ω into 2^{21} disjoint subsets by the *Venn diagram* generated by these 21 subspaces. That is, for any given coding vector $\mathbf{c}_t$, we can place it in exactly one of the 2^{21} disjoint subsets by testing whether it belongs to which A-subspaces.

We can further reduce the possible placement of $\mathbf{c}_t$ in the following way. By (9), we know that when $\sigma(t)=i$, node i selects $\mathbf{c}_t$ from its knowledge space $S_i(t-1)$. Hence, such $\mathbf{c}_t$ must always lie in any A-subspace that S_i appears in the definition. There are 10 such A-subspaces: $A_1^{(i)}$ to $A_4^{(i)}$; $A_1^{(i,j)}$ to $A_3^{(i,j)}$; and $A_1^{(k,i)}$ to $A_3^{(k,i)}$. As a result, for any coding vector $\mathbf{c}_t$ sent by node i, we only needs to check whether $\mathbf{c}_t$ belongs to which of the following 11 remaining A-subspaces:

$$\begin{aligned} &\ddot{\mathsf{A}}_1^{(i)}\triangleq A_1^{(j)},\ \ \ddot{\mathsf{A}}_2^{(i)}\triangleq A_2^{(j)},\ \ \ddot{\mathsf{A}}_3^{(i)}\triangleq A_3^{(j)},\ \ \ddot{\mathsf{A}}_4^{(i)}\triangleq A_4^{(j)},\\ &\ddot{\mathsf{A}}_5^{(i)}\triangleq A_1^{(k)},\ \ \ddot{\mathsf{A}}_6^{(i)}\triangleq A_2^{(k)},\ \ \ddot{\mathsf{A}}_7^{(i)}\triangleq A_3^{(k)},\ \ \ddot{\mathsf{A}}_8^{(i)}\triangleq A_4^{(k)},\\ &\ddot{\mathsf{A}}_9^{(i)}\triangleq A_1^{(j,k)},\ \ \ddot{\mathsf{A}}_{10}^{(i)}\triangleq A_2^{(j,k)},\ \ \ddot{\mathsf{A}}_{11}^{(i)}\triangleq A_3^{(j,k)}. \end{aligned} \quad (14)$$

In (14), we rename those 11 remaining A-subspace by $\ddot{\mathsf{A}}_1^{(i)}$ to $\ddot{\mathsf{A}}_{11}^{(i)}$ for easier future reference. For example when $i=3$, such 11 subspaces $\ddot{\mathsf{A}}_1^{(3)}$ to $\ddot{\mathsf{A}}_{11}^{(3)}$ are $A_1^{(1)}$ to $A_4^{(1)}$; $A_1^{(2)}$ to $A_4^{(2)}$; and $A_1^{(1,2)}$ to $A_3^{(1,2)}$, respectively. For any 11-bitstring $\mathbf{b}=b_1b_2\cdots b_{11}$, we define "the coding type-$\mathbf{b}$ of node i" by

$$\mathsf{TYPE}_{\mathbf{b}}^{(i)}\triangleq S_i\cap\left(\bigcap_{l:b_l=1}\ddot{\mathsf{A}}_l^{(i)}\right)\Big\backslash\left(\bigcup_{l:b_l=0}\ddot{\mathsf{A}}_l^{(i)}\right). \quad (15)$$

Namely, the $S_i(t-1)$ that node i can choose $\mathbf{c}_t$ from at time t is now further divided into $2^{11}=2048$ disjoint subsets, depending on whether $\mathbf{c}_t$ belongs to $\ddot{\mathsf{A}}_l^{(i)}$ or not for $l=1$ to 11. For example, $\mathsf{TYPE}_{169}^{(1)}$ (i.e., type-00010101001 of node 1) contains the $\mathbf{c}_t$ in S_1 that is in the intersection of $\{\ddot{\mathsf{A}}_4^{(1)}, \ddot{\mathsf{A}}_6^{(1)}, \ddot{\mathsf{A}}_8^{(1)}, \ddot{\mathsf{A}}_{11}^{(1)}\}$ but not in the union of $\{\ddot{\mathsf{A}}_1^{(1)}, \ddot{\mathsf{A}}_2^{(1)}, \ddot{\mathsf{A}}_3^{(1)}, \ddot{\mathsf{A}}_5^{(1)}, \ddot{\mathsf{A}}_7^{(1)}, \ddot{\mathsf{A}}_9^{(1)}, \ddot{\mathsf{A}}_{10}^{(1)}\}$. By (14) and (15), we can write

$$\begin{aligned}\mathsf{TYPE}_{169}^{(1)} \triangleq\, & S_1 \cap \left(A_4^{(2)} \cap A_2^{(3)} \cap A_4^{(3)} \cap A_3^{(2,3)}\right) \\ & \setminus \left(A_1^{(2)} \cup A_2^{(2)} \cup A_3^{(2)} \cup A_1^{(3)} \cup A_3^{(3)} \cup A_1^{(2,3)} \cup A_2^{(2,3)}\right).\end{aligned}$$

In sum, any $\mathbf{c}_t$ chosen by node i must fall into one of the $2^{11}=2048$ subsets $\mathsf{TYPE}_{\mathbf{b}}^{(i)}$ defined by (14) and (15).

We can further strengthen the above observation by proving that 1996 (out of 2048) subsets are empty. For example, $\mathsf{TYPE}_{1024}^{(i)}$ (i.e., type-10000000000) is always empty since there is no such vector that can be inside $\ddot{\mathsf{A}}_1^{(i)} \triangleq A_1^{(j)}$ but not in $\ddot{\mathsf{A}}_2^{(i)} \triangleq A_2^{(j)}$ because we clearly have $A_2^{(j)} \supset A_1^{(j)}$ by definition (10). By eliminating all the empty subsets, $\mathbf{c}_t$ chosen by node i can only be in one of 52 (out of 2048) subsets. We call those 52 subsets the *Feasible Coding Types* (FTs) and they are enumerated as follows.

$$\begin{aligned}\mathsf{FTs} \triangleq \{&0,1,2,3,7,9,11,15,31,41,43,47,63,127,130,\\ &131,135,139,143,159,171,175,191,255,386,\\ &387,391,395,399,415,427,431,447,511,647,\\ &655,671,687,703,767,903,911,927,943,959,\\ &1023,1927,1935,1951,1967,1983,2047\}. \end{aligned} \quad (16)$$

Since the coding choices are finite (52 per node and totally 3 nodes), we can derive the following upper bound using those $52\times 3=156$ feasible types that fully cover Ω at any time t.

Proposition 1: A 6-dimensional rate vector $\vec{R}$ is in the LNC capacity region only if there exists 52×3 non-negative variables $x_{\mathbf{b}}^{(i)}$ for all $\mathbf{b}\in\mathsf{FTs}$ and $i\in\{1,2,3\}$ and 7×3 non-negative y-variables, $y_1^{(i)}$ to $y_4^{(i)}$, $y_1^{(i,j)}$ to $y_3^{(i,j)}$ for all $i\in\{1,2,3\}$, such that jointly they satisfy the following three groups of linear conditions:

• Group 1, termed the *time-sharing condition*, has 1 inequality:

$$\left(\sum_{\forall \mathbf{b}\in\mathsf{FTs}} x_{\mathbf{b}}^{(1)}\right) + \left(\sum_{\forall \mathbf{b}\in\mathsf{FTs}} x_{\mathbf{b}}^{(2)}\right) + \left(\sum_{\forall \mathbf{b}\in\mathsf{FTs}} x_{\mathbf{b}}^{(3)}\right) \leq 1. \quad (17)$$

• Group 2, termed the *rank-conversion conditions*, has 21 equalities: For all $i\in\{1,2,3\}$,

$$\begin{aligned} y_1^{(i)} = &\left(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_5=0} x_{\mathbf{b}}^{(j)}\right)\cdot p_{j\to i} + \left(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_1=0} x_{\mathbf{b}}^{(k)}\right)\cdot p_{k\to i} \\ &+ R_{i\to j} + R_{i\to k}, \end{aligned} \quad (18)$$

$$\begin{aligned} y_2^{(i)} = &\left(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_6=0} x_{\mathbf{b}}^{(j)}\right)\cdot p_{j\to i} + \left(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_2=0} x_{\mathbf{b}}^{(k)}\right)\cdot p_{k\to i} \\ &+ R_{i\to j} + R_{i\to k} + R_{j\to i}, \end{aligned} \quad (19)$$

$$\begin{aligned} y_3^{(i)} = &\left(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_7=0} x_{\mathbf{b}}^{(j)}\right)\cdot p_{j\to i} + \left(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_3=0} x_{\mathbf{b}}^{(k)}\right)\cdot p_{k\to i} \\ &+ R_{i\to j} + R_{i\to k} + R_{k\to i}, \end{aligned} \quad (20)$$

$$\begin{aligned} y_4^{(i)} = &\left(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_8=0} x_{\mathbf{b}}^{(j)}\right)\cdot p_{j\to i} + \left(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_4=0} x_{\mathbf{b}}^{(k)}\right)\cdot p_{k\to i} \\ &+ R_{i\to j} + R_{i\to k} + R_{j\to i} + R_{k\to i}, \end{aligned} \quad (21)$$

$$\begin{aligned} y_1^{(i,j)} = &\left(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_9=0} x_{\mathbf{b}}^{(k)}\right)\cdot p_{k\to i\vee j} \\ &+ R_{i\to j} + R_{i\to k} + R_{j\to i} + R_{j\to k}, \end{aligned} \quad (22)$$

$$\begin{aligned} y_2^{(i,j)} = &\left(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_{10}=0} x_{\mathbf{b}}^{(k)}\right)\cdot p_{k\to i\vee j} \\ &+ R_{i\to j} + R_{i\to k} + R_{j\to i} + R_{j\to k} + R_{k\to i}, \end{aligned} \quad (23)$$

$$\begin{aligned} y_3^{(i,j)} = &\left(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_{11}=0} x_{\mathbf{b}}^{(k)}\right)\cdot p_{k\to i\vee j} \\ &+ R_{i\to j} + R_{i\to k} + R_{j\to i} + R_{j\to k} + R_{k\to j}. \end{aligned} \quad (24)$$

• Group 3, termed the *decodability conditions*, has 6 equalities:

$$\forall\, i\in\{1,2,3\}, \quad y_1^{(i)} = y_2^{(i)} = y_3^{(i)} = y_4^{(i)}, \quad (25)$$

$$\forall\, i\in\{1,2,3\}, \quad y_1^{(i,j)} = y_2^{(i,j)} = y_3^{(i,j)} = R_\Sigma. \quad (26)$$

The intuition is as follows. Consider any achievable $\vec{R}$ and the associated LNC scheme. For any time t, suppose the given scheme chooses node i to transmit a coding vector $\mathbf{c}_t$. By the previous discussions, we can examine this $\mathbf{c}_t$ to see which $\mathsf{TYPE}_{\mathbf{b}}^{(i)}$ it belongs to by looking at the corresponding A-subspaces in the end of $t-1$. Then after running the given scheme from time 1 to n, we can compute the variable $x_{\mathbf{b}}^{(i)} \triangleq \frac{1}{n}\mathbb{E}\left[\sum_{t=1}^{n} 1_{\{\mathbf{c}_t\in\mathsf{TYPE}_{\mathbf{b}}^{(i)}\}}\right]$ for each $\mathsf{TYPE}_{\mathbf{b}}^{(i)}$ as the *frequency* of scheduling node i with the chosen $\mathbf{c}_t$ happening to be in $\mathsf{TYPE}_{\mathbf{b}}^{(i)}$. Since each $\mathbf{c}_t$ belongs to exactly one of the $52\times 3=156$ feasible coding types, the time-sharing condition (17) holds naturally. We then compute the y-variables by

$$\begin{aligned} y_l^{(i)} &\triangleq \frac{1}{n}\mathbb{E}\left[\mathsf{rank}\big(A_l^{(i)}(n)\big)\right],\ \forall l\in\{1,2,3,4\}, \\ y_l^{(i,j)} &\triangleq \frac{1}{n}\mathbb{E}\left[\mathsf{rank}\big(A_l^{(i,j)}(n)\big)\right],\ \forall l\in\{1,2,3\}, \end{aligned} \quad (27)$$

as normalized expected ranks of A-subspaces in the end of time n. We now claim that these variables satisfy (18) to (26). This claim implies that for any LNC-achievable $\vec{R}$, there exists $x_{\mathbf{b}}^{(i)}$ and y-variables satisfying Proposition 1, which means that Proposition 1 constitutes an outer bound on the LNC capacity.

To prove that (18)–(24) are true,[2] consider an A-subspace, say $A_3^{(1)}(t) = S_1(t-1)\oplus\Omega_{3\to 1} = RS_1(t-1)\oplus\Omega_{1\to 2}\oplus\Omega_{1\to 3}\oplus\Omega_{3\to 1}$ as defined in (11) and (8) when $(i,j,k)=(1,2,3)$. In the beginning of time 1, node 1 has not received any packet yet, i.e., $RS_1(0)=\{\mathbf{0}\}$. Thus the rank of $A_3^{(1)}(1)$ is $\mathsf{rank}(\Omega_{1\to 2}\oplus\Omega_{1\to 3}\oplus\Omega_{3\to 1}) = nR_{1\to 2}+nR_{1\to 3}+nR_{3\to 1}$.

The fact that $S_1(t-1)$ contributes to $A_3^{(1)}(t)$ implies that $\mathsf{rank}(A_3^{(1)}(t))$ will increase by one whenever node 1 receives a packet $\mathbf{c}_t\mathbf{W}^\top$ satisfying $\mathbf{c}_t\notin A_3^{(1)}(t)$. Since $A_3^{(1)}(t)$ is labeled as $\ddot{\mathsf{A}}_7^{(2)}$, see (14) with $(i,j,k)=(2,3,1)$, whenever node 2 sends a $\mathbf{c}_t$ in $\mathsf{TYPE}_{\mathbf{b}}^{(2)}$ with $b_7=0$, such $\mathbf{c}_t$ is not in $A_3^{(1)}(t)$.

[2]For rigorous proofs, we need to invoke the law of large numbers and take care of the ϵ-error probability. For ease of discussion, the corresponding technical details are omitted when discussing the intuition of Proposition 1.

Whenever node 1 receives it, $\mathsf{rank}(A_3^{(1)}(t))$ increases by 1. On the other hand, $A_3^{(1)}(t)$ is also labeled as $\ddot{\mathsf{A}}_3^{(3)}$, see (14) with $(i,j,k)=(3,1,2)$. Hence, whenever node 3 sends a $\mathbf{c}_t$ in $\mathsf{TYPE}_{\mathbf{b}}^{(3)}$ with $b_3=0$ and node 1 receives it, $\mathsf{rank}(A_3^{(1)}(t))$ also increases by 1. Therefore, in the end of time n, we have

$$\begin{aligned}\mathsf{rank}(A_3^{(1)}(n)) =& \sum_{t=1}^{n} 1_{\left\{\substack{\text{node 2 sends } \mathbf{c}_t \in \mathsf{TYPE}_{\mathbf{b}}^{(2)} \text{ with } b_7=0,\\ \text{and node 1 receives it}}\right\}} \\ &+ \sum_{t=1}^{n} 1_{\left\{\substack{\text{node 3 sends } \mathbf{c}_t \in \mathsf{TYPE}_{\mathbf{b}}^{(3)} \text{ with } b_3=0,\\ \text{and node 1 receives it}}\right\}} \\ &+ \mathsf{rank}(A_3^{(1)}(0)).\end{aligned} \tag{28}$$

Taking the normalized expectation of (28), we have proven (20) for $i=1$. By similar *rank-conversion* arguments, (18)–(24) are true for all $i \in \{1,2,3\}$.

In the end of time n, since every node $i \in \{1,2,3\}$ can decode the desired packets $\mathbf{W}_{j\to i}$ and $\mathbf{W}_{k\to i}$, we thus have $S_i(n) \supseteq \Omega_{j\to i}$ and $S_i(n) \supseteq \Omega_{k\to i}$, or equivalently $S_i(n) = S_i(n) \oplus \Omega_{j\to i} \oplus \Omega_{k\to i}$. This implies that the ranks of $A_1^{(i)}(n)$ to $A_4^{(i)}(n)$ in (10) and (11) are all equal. Together with (27), we thus have (25). Similarly, one can prove that (26) is satisfied as well. The claim is thus proven.

B. A LNC Capacity Achieving Scheme

Proposition 2: A 6-dimensional $\vec{R}$ is LNC-achievable if there exist 15 non-negative variables $t_i^{[\mathrm{u}]}$ and $\{t_i^{[\mathrm{c},l]}\}_{l=1}^4$ for all $i \in \{1,2,3\}$ such that jointly they satisfy the following three groups of linear conditions:

- Group 1, termed the *time-sharing condition*, has 1 inequality:

$$\sum_{\forall i \in \{1,2,3\}} t_i^{[\mathrm{u}]} + t_i^{[\mathrm{c},1]} + t_i^{[\mathrm{c},2]} + t_i^{[\mathrm{c},3]} + t_i^{[\mathrm{c},4]} \leq 1. \tag{29}$$

- Group 2 has 3 inequalities: For all $i \in \{1,2,3\}$,

$$R_{i\to j} + R_{i\to k} < t_i^{[\mathrm{u}]} p_{i\to j\vee k}. \tag{30}$$

- Group 3 has 6 inequalities: For all $i \in \{1,2,3\}$,

$$R_{i\to j}\frac{p_{i\to \overline{j}k}}{p_{i\to j\vee k}} < \left(t_i^{[\mathrm{c},1]} + t_i^{[\mathrm{c},3]}\right)\cdot p_{i\to j} + \left(t_k^{[\mathrm{c},2]} + t_k^{[\mathrm{c},3]}\right)\cdot p_{k\to j}, \tag{31}$$

$$R_{i\to k}\frac{p_{i\to j\overline{k}}}{p_{i\to j\vee k}} < \left(t_i^{[\mathrm{c},1]} + t_i^{[\mathrm{c},4]}\right)\cdot p_{i\to k} + \left(t_j^{[\mathrm{c},2]} + t_j^{[\mathrm{c},4]}\right)\cdot p_{j\to k}. \tag{32}$$

Sketch of the proof: Any $\vec{R}$-satisfying Proposition 2 can be achieved by the following 2-phased scheme. Phase 1 : In the beginning of time 1, node 1 has $nR_{1\to 2} + nR_{1\to 3}$ packets (i.e., $\mathbf{W}_{1\to 2}$ and $\mathbf{W}_{1\to 3}$) that need to be sent to nodes 2 and 3, respectively. In this phase, node 1 picks one of these packets and repeatedly sends it uncodedly until at least one of nodes 2 and 3 receives it. Then node 1 picks the next packet and repeat the same process until each of these $nR_{1\to 2} + nR_{1\to 3}$ packets is heard by at least one of nodes 2 and 3. By simple analysis, see [4], node 1 can finish the transmission in $n\cdot t_i^{[\mathrm{u}]}$ slots since (30).[3] We repeat this process for nodes 2 and 3, respectively. Phase 1 can be finished in $n(\sum_i t_i^{[\mathrm{u}]})$ slots.

[3] By the law of large numbers, we can ignore the randomness of the events and treat them as deterministic when n is sufficiently large.

After Phase 1, the status of all packets is summarized as follows. Each of $\mathbf{W}_{i\to j}$ packets is heard by at least one of nodes j and k. Those that have already been heard by node j, the intended destination, is delivered successfully and thus will not be considered for future operations (Phase 2). We denote those $\mathbf{W}_{i\to j}$ packets that are overheard by node k only (not by node j) as $\mathbf{W}_{i\to j}^{(k)}$. In average, there are $nR_{i\to j}\frac{p_{i\to \overline{j}k}}{p_{i\to j\vee k}}$ number of $\mathbf{W}_{i\to j}^{(k)}$ packets. Symmetrically, we also have $nR_{i\to k}\frac{p_{i\to j\overline{k}}}{p_{i\to j\vee k}}$ number of $\mathbf{W}_{i\to k}^{(j)}$ packets that was intended for node k but was overheard only by node j in Phase 1.

Phase 2 is the LNC phase, in which each node i will send a linear combination of packets. We claim that there are (at least) 4 possible ways of sending LNC packets. That is, for each time t, node i send $X_i(t) = [W_j + W_k]$ with one of 4 possibilities of choosing W_j and W_k: (i) $W_j \in \mathbf{W}_{i\to j}^{(k)}$ and $W_k \in \mathbf{W}_{i\to k}^{(j)}$; (ii) $W_j \in \mathbf{W}_{k\to j}^{(i)}$ and $W_k \in \mathbf{W}_{j\to k}^{(i)}$; (iii) $W_j \in \mathbf{W}_{i\to j}^{(k)}$ and $W_k \in \mathbf{W}_{j\to k}^{(i)}$; and (iv) $W_j \in \mathbf{W}_{k\to j}^{(i)}$ and $W_k \in \mathbf{W}_{i\to k}^{(j)}$. Note that choice (i) is the standard LNC operation for the 2-receiver broadcast channels [4] since node i sends a linear sum that benefits both nodes j and k simultaneously. Choice (ii) is the standard LNC operation for the 2-way relay channels, since node i, as a relay for the 2-way traffic between nodes j and k, mixes the packets from two opposite directions and sends their linear sum. Choices (iii) and (iv) are the new "hybrid" cases identified in this work, for which we can mix part of the broadcast traffic and part of the 2-way traffic. One can easily prove that transmitting such a linear mixture again benefits both nodes simultaneously.

Since each node i has 4 possible coding choices, we perform coding choice l for exactly $n\cdot t_i^{[\mathrm{c},l]}$ times for $l=1$ to 4. Since $\mathbf{W}_{i\to j}^{(k)}$ participates in coding choices (i) and (iii) of node i and coding choices (ii) and (iii) of node k, (31) guarantees that we can finish sending all $\mathbf{W}_{i\to j}^{(k)}$ packets and they will all successfully arrive at node j. Symmetrically, (32) guarantees the delivery of all $\mathbf{W}_{i\to k}^{(j)}$ packets in the end of Phase 2. Finally, (29) guarantees that we can finish Phases 1 and 2 in the allotted n time slots.

Proposition 3: The outer and inner bounds in Propositions 1 and 2 match for all channel parameters and they thus describe the 6-dimensional LNC capacity region.

The proof is relegated to Appendix A. One important implication is that for the 3-node 6-flow setting, we do not need to resort to any "exotic" LNC operation as described in Section I. Instead, 4 simple coding choices (i)–(iv) are sufficient to achieve the optimal LNC capacity under *any channel parameters*.

V. Conclusion

This work characterizes the 6-dimensional LNC capacity and the optimal strategy when 3 nodes talk through erasure networks with the channel state feedback.

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APPENDIX A
PROOF OF PROPOSITION 3

For the readability, we rewrite the original 52 *Feasible Types* (FTs) defined in (16) that each node $i \in \{1,2,3\}$ can transmit:

$$\begin{aligned}\mathsf{FTs} \triangleq \{&\mathtt{000}, \mathtt{001}, \mathtt{002}, \mathtt{003}, \mathtt{007}, \mathtt{011}, \mathtt{013}, \mathtt{017}, \mathtt{037}, \mathtt{051},\\ &\mathtt{053}, \mathtt{057}, \mathtt{077}, \mathtt{0F7}, \mathtt{102}, \mathtt{103}, \mathtt{107}, \mathtt{113}, \mathtt{117}, \mathtt{137},\\ &\mathtt{153}, \mathtt{157}, \mathtt{177}, \mathtt{1F7}, \mathtt{302}, \mathtt{303}, \mathtt{307}, \mathtt{313}, \mathtt{317}, \mathtt{337},\\ &\mathtt{353}, \mathtt{357}, \mathtt{377}, \mathtt{3F7}, \mathtt{507}, \mathtt{517}, \mathtt{537}, \mathtt{557}, \mathtt{577}, \mathtt{5F7},\\ &\mathtt{707}, \mathtt{717}, \mathtt{737}, \mathtt{757}, \mathtt{777}, \mathtt{7F7}, \mathtt{F07}, \mathtt{F17}, \mathtt{F37}, \mathtt{F57},\\ &\mathtt{F77}, \mathtt{FF7}\},\end{aligned} \tag{33}$$

where each 3-digit index $\overline{\mathbf{b}}_1\overline{\mathbf{b}}_2\overline{\mathbf{b}}_3$ represent a 11-bitstring $\mathbf{b}$ of which $\overline{\mathbf{b}}_1$ is a hexadecimal of first four bits, $\overline{\mathbf{b}}_2$ is a hexadecimal of the next four bits, and $\overline{\mathbf{b}}_3$ is octal of the last three bits. It should be clear from the context whether we are representing $\mathbf{b}$ as a decimal index, e.g., $\mathsf{TYPE}^{(1)}_{169}$, or as a 3-digit index based on hexadecimal/octal, e.g., $\mathsf{TYPE}^{(1)}_{\mathtt{FF7}}$.

For the notational convenience, we often use $\mathsf{FTs}(\cdot,\cdot,\cdot)$ to denote some collection of coding types in FTs. For example, $\mathsf{FTs}(\mathtt{F},\cdot,\cdot) \triangleq \{\, \mathbf{b} \in \mathsf{FTs} \text{ with } \overline{\mathbf{b}}_1 = \mathtt{F}\,\}$, corresponding to the collection of coding types in FTs with $b_1 = b_2 = b_3 = b_4 = 1$.

Without loss of generality, we also assume that $p_{i\to j} > 0$ and $p_{i\to k} > 0$ for all $(i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\}$ since the case that any one of them is zero can be viewed as a limiting scenario and the polytope of the LP problem in Proposition 1 is continuous with respect to the channel success probability parameters.

We now introduce the following three lemmas.

Lemma 1: Given any rate vector $\vec{R}$ and the associated $\{x^{(i)}_{\mathbf{b}}\}$-variables satisfying Proposition 1, the following equalities, (E1) to (E10), always hold for all $(i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\}$.

$$R_{k\to i} + R_{k\to j} = \Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_9=0} x^{(k)}_{\mathbf{b}}\Bigg)\cdot p_{k\to i\vee j}, \tag{E1}$$

$$R_{k\to j} = \Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_{10}=0} x^{(k)}_{\mathbf{b}}\Bigg)\cdot p_{k\to i\vee j}, \tag{E2}$$

$$R_{k\to i} = \Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_{11}=0} x^{(k)}_{\mathbf{b}}\Bigg)\cdot p_{k\to i\vee j}, \tag{E3}$$

$$\Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}(\cdot,\cdot,\mathtt{0})} x^{(k)}_{\mathbf{b}}\Bigg) = \Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}(\cdot,\cdot,\mathtt{3})} x^{(k)}_{\mathbf{b}}\Bigg). \tag{E4}$$

$$\begin{aligned} R_{j\to i} + R_{k\to i} = &\Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_5=0, b_8=1} x^{(j)}_{\mathbf{b}}\Bigg)\cdot p_{j\to i}\\ &+\Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_1=0, b_4=1} x^{(k)}_{\mathbf{b}}\Bigg)\cdot p_{k\to i},\end{aligned} \tag{E5}$$

$$\begin{aligned} R_{j\to i} = &\Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_5=0, b_6=1} x^{(j)}_{\mathbf{b}}\Bigg)\cdot p_{j\to i}\\ &+\Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_1=0, b_2=1} x^{(k)}_{\mathbf{b}}\Bigg)\cdot p_{k\to i},\end{aligned} \tag{E6}$$

$$\begin{aligned} R_{k\to i} = &\Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_5=0, b_7=1} x^{(j)}_{\mathbf{b}}\Bigg)\cdot p_{j\to i}\\ &+\Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_1=0, b_3=1} x^{(k)}_{\mathbf{b}}\Bigg)\cdot p_{k\to i},\end{aligned} \tag{E7}$$

$$\begin{aligned} &\Bigg(\sum_{\mathbf{b}\in\mathsf{FTs}(\cdot,\mathtt{7},\cdot)} x^{(j)}_{\mathbf{b}}\Bigg)\cdot p_{j\to i} + \Bigg(\sum_{\mathbf{b}\in\mathsf{FTs}(\mathtt{7},\cdot,\cdot)} x^{(k)}_{\mathbf{b}}\Bigg)\cdot p_{k\to i}\\ &= \Bigg(\sum_{\mathbf{b}\in\mathsf{FTs}(\cdot,\mathtt{1},\cdot)} x^{(j)}_{\mathbf{b}}\Bigg)\cdot p_{j\to i} + \Bigg(\sum_{\mathbf{b}\in\mathsf{FTs}(\mathtt{1},\cdot,\cdot)} x^{(k)}_{\mathbf{b}}\Bigg)\cdot p_{k\to i}.\end{aligned} \tag{E8}$$

$$\begin{aligned} \Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_{10}=0} x^{(k)}_{\mathbf{b}}\Bigg)\cdot p_{k\to i\vee j} = &\Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_5=0, b_6=1} x^{(k)}_{\mathbf{b}}\Bigg)\cdot p_{k\to j}\\ &+\Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_1=0, b_2=1} x^{(i)}_{\mathbf{b}}\Bigg)\cdot p_{i\to j},\end{aligned} \tag{E9}$$

$$\begin{aligned} \Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_{11}=0} x^{(k)}_{\mathbf{b}}\Bigg)\cdot p_{k\to i\vee j} = &\Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_5=0, b_7=1} x^{(j)}_{\mathbf{b}}\Bigg)\cdot p_{j\to i}\\ &+\Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_1=0, b_3=1} x^{(k)}_{\mathbf{b}}\Bigg)\cdot p_{k\to i}.\end{aligned} \tag{E10}$$

The proof is relegated to Appendix B.

The following Lemma 2 implies that we can impose special structure on the $\{x_{\mathbf{b}}^{(i)}\}$-variables satisfying Proposition 1. For that, let us denote

$$\overline{\mathsf{FTs}} \triangleq \{\mathtt{051}, \mathtt{302}, \mathtt{337}, \mathtt{357}, \mathtt{3F7}, \mathtt{537}, \mathtt{557}, \mathtt{5F7}, \mathtt{F37}, \mathtt{F57}\}, \tag{34}$$

of which contains only 10 types out of 52 feasible coding types of the original FTs.

Lemma 2: Given any $\vec{R}$ and the associated 156 non-negative values $\{x_{\mathbf{b}}^{(i)}\}$ satisfying Proposition 1, we can always find another set of 156 non-negative values $\{\ddot{x}_{\mathbf{b}}^{(i)}\}$ such that $\vec{R}$ and $\{\ddot{x}_{\mathbf{b}}^{(i)}\}$ jointly also satisfy Proposition 1 and

$$\ddot{x}_{\mathbf{b}}^{(i)} = 0 \text{ for all } \mathbf{b} \in \mathsf{FTs}\backslash\overline{\mathsf{FTs}}. \tag{35}$$

That is, without loss of generality, we can assume only those $\{x_{\mathbf{b}}^{(i)}\}$ with $\mathbf{b} \in \overline{\mathsf{FTs}}$ may have non-zero values. The proof of this lemma is relegated to Appendix C.

Lemma 3: Given any $\vec{R}$ and the associated 156 non-negative values $\{\ddot{x}_{\mathbf{b}}^{(i)}\}$ that satisfy Proposition 1 and (35), we can always find 15 non-negative values $t_i^{[\mathrm{u}]}$ and $\{t_i^{[\mathrm{c},l]}\}_{l=1}^{4}$ for all $i \in \{1,2,3\}$ such that jointly satisfy three groups of linear conditions in Proposition 2 (when replacing all strict inequality $<$ by $\leq$).

The proof of this lemma is relegated to Appendix D.

One can clearly see that Lemmas 2 and 3 jointly imply that the outer bound in Proposition 1 matches the closure of the inner bound in Proposition 2. The proof of Proposition 3 is thus complete.

Appendix B
Proof of Lemma 1

We prove the equalities (E1) to (E4) as follows.

Proof. These equalities can be derived by using (22)–(24) and (26) in Proposition 1. Since $y_1^{(i,j)} = y_2^{(i,j)} = y_3^{(i,j)} = R_\Sigma$ by (26), substituting R_Σ to the left-hand side of (22)–(24), respectively, we have

$$R_{k\to i} + R_{k\to j} = \Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_9=0} x_{\mathbf{b}}^{(k)}\Bigg)\cdot p_{k\to i\vee j},$$

$$R_{k\to j} = \Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_{10}=0} x_{\mathbf{b}}^{(k)}\Bigg)\cdot p_{k\to i\vee j},$$

$$R_{k\to i} = \Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_{11}=0} x_{\mathbf{b}}^{(k)}\Bigg)\cdot p_{k\to i\vee j},$$

which are equivalent to (E1), (E2), and (E3), respectively.

We now prove the relationship (E4). Substituting (E2) and (E3) to the left-hand side of (E1), we then have

$$\begin{aligned}\Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_{10}=0} x_{\mathbf{b}}^{(k)} + \sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_{11}=0} x_{\mathbf{b}}^{(k)}\Bigg)\cdot p_{k\to i\vee j} \\ = \Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_9=0} x_{\mathbf{b}}^{(k)}\Bigg)\cdot p_{k\to i\vee j}.\end{aligned} \tag{36}$$

Note that for any type-$\mathbf{b}$, whenever $b_{10}=0$ (resp. $b_{11}=0$), b_9 is also zero. This is because $\ddot{\mathsf{A}}_9^{(i)} \subset \ddot{\mathsf{A}}_{10}^{(i)}$ (resp. $\ddot{\mathsf{A}}_9^{(i)} \subset \ddot{\mathsf{A}}_{11}^{(i)}$) regardless of node index i, see (14). Therefore, (36) can be further reduced to

$$\begin{aligned}\Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_9=0, b_{10}=0, b_{11}=0} x_{\mathbf{b}}^{(k)}\Bigg)\cdot p_{k\to i\vee j} \\ = \Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_9=0, b_{10}=1, b_{11}=1} x_{\mathbf{b}}^{(k)}\Bigg)\cdot p_{k\to i\vee j}.\end{aligned} \tag{37}$$

Dividing $p_{k\to i\vee j}$ on both sides of (37), we finally have (E4). The proof is thus complete. $\square$

We prove the equalities (E5) to (E8) as follows.

Proof. These equalities can be derived by using the decodability equality (25) in Proposition 1, i.e., $y_1^{(i)} = y_2^{(i)} = y_3^{(i)} = y_4^{(i)}$. First from $y_1^{(i)} = y_4^{(i)}$ and by (18) and (21), one can easily see that we have

$$\begin{aligned}R_{j\to i} + R_{k\to i} = \Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_5=0, b_8=1} x_{\mathbf{b}}^{(j)}\Bigg)\cdot p_{j\to i} \\ + \Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_1=0, b_4=1} x_{\mathbf{b}}^{(k)}\Bigg)\cdot p_{k\to i},\end{aligned}$$

which is equivalent to (E5). This is because for any type-$\mathbf{b}$, if $b_8=0$ (resp. $b_4=0$), then b_5 (resp. b_1) must be zero as well due to the fact that $\ddot{\mathsf{A}}_5^{(i)} \subset \ddot{\mathsf{A}}_8^{(i)}$ (resp. $\ddot{\mathsf{A}}_1^{(i)} \subset \ddot{\mathsf{A}}_4^{(i)}$) regardless of node index, see (14). Similarly from the facts that $\ddot{\mathsf{A}}_5^{(i)} \subset \ddot{\mathsf{A}}_6^{(i)}$, $\ddot{\mathsf{A}}_1^{(i)} \subset \ddot{\mathsf{A}}_2^{(i)}$, and by (18) and (19), $y_1^{(i)} = y_2^{(i)}$ implies

$$\begin{aligned}R_{j\to i} = \Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_5=0, b_6=1} x_{\mathbf{b}}^{(j)}\Bigg)\cdot p_{j\to i} \\ + \Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_1=0, b_2=1} x_{\mathbf{b}}^{(k)}\Bigg)\cdot p_{k\to i},\end{aligned}$$

which is equivalent to (E6).

Moreover, from the facts that $\ddot{\mathsf{A}}_5^{(i)} \subset \ddot{\mathsf{A}}_7^{(i)}$, $\ddot{\mathsf{A}}_1^{(i)} \subset \ddot{\mathsf{A}}_3^{(i)}$), and by (18) and (20), $y_1^{(i)} = y_3^{(i)}$ implies

$$\begin{aligned}R_{k\to i} = \Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_5=0, b_7=1} x_{\mathbf{b}}^{(j)}\Bigg)\cdot p_{j\to i} \\ + \Bigg(\sum_{\forall \mathbf{b}\in\mathsf{FTs}\text{ w. } b_1=0, b_3=1} x_{\mathbf{b}}^{(k)}\Bigg)\cdot p_{k\to i},\end{aligned} \tag{38}$$

which is equivalent to (E7).

We now prove the relationship (E8). Substituting (E6) and

(E7) to the left-hand side of (E5), we thus have

$$\left(\sum_{\forall \mathbf{b}\in \mathsf{FTs} \text{ w. } b_5=0,b_6=1} x_{\mathbf{b}}^{(j)} + \sum_{\forall \mathbf{b}\in \mathsf{FTs} \text{ w. } b_5=0,b_7=1} x_{\mathbf{b}}^{(j)}\right)\cdot p_{j\to i}$$
$$+\left(\sum_{\forall \mathbf{b}\in \mathsf{FTs} \text{ w. } b_1=0,b_2=1} x_{\mathbf{b}}^{(k)} + \sum_{\forall \mathbf{b}\in \mathsf{FTs} \text{ w. } b_1=0,b_3=1} x_{\mathbf{b}}^{(k)}\right)\cdot p_{k\to i}$$
$$=\left(\sum_{\forall \mathbf{b}\in \mathsf{FTs} \text{ w. } b_5=0,b_8=1} x_{\mathbf{b}}^{(j)}\right)\cdot p_{j\to i} + \left(\sum_{\forall \mathbf{b}\in \mathsf{FTs} \text{ w. } b_1=0,b_4=1} x_{\mathbf{b}}^{(k)}\right)\cdot p_{k\to i}.$$

Note that for any type-$\mathbf{b}$, whenever $b_6 = 1$ (resp. $b_7 = 1$), b_8 must be one due to the fact that $\ddot{\mathsf{A}}_6^{(i)} \subset \ddot{\mathsf{A}}_8^{(i)}$ (resp. $\ddot{\mathsf{A}}_7^{(i)} \subset \ddot{\mathsf{A}}_8^{(i)}$). The same argument holds such that for any type-$\mathbf{b}$, whenever $b_2 = 1$ (resp. $b_3 = 1$), we have $b_4 = 1$. Then the above equality further reduces to

$$\left(\sum_{\forall \mathbf{b}\in \mathsf{FTs} \text{ w. } b_5=0,b_6=1,b_7=1,b_8=1} x_{\mathbf{b}}^{(j)}\right)\cdot p_{j\to i} + \left(\sum_{\forall \mathbf{b}\in \mathsf{FTs} \text{ w. } b_1=0,b_2=1,b_3=1,b_4=1} x_{\mathbf{b}}^{(k)}\right)\cdot p_{k\to i}$$
$$=\left(\sum_{\forall \mathbf{b}\in \mathsf{FTs} \text{ w. } b_5=0,b_6=0,b_7=0,b_8=1} x_{\mathbf{b}}^{(j)}\right)\cdot p_{j\to i} + \left(\sum_{\forall \mathbf{b}\in \mathsf{FTs} \text{ w. } b_1=0,b_2=0,b_3=0,b_4=1} x_{\mathbf{b}}^{(k)}\right)\cdot p_{k\to i},$$

which is equivalent to (E8). The proof is thus compelte. □

We prove the equalities (E9) and (E10) as follows.

Proof. By cyclic symmetry, we can rewrite (E6) as follows.

$$R_{k\to j} = \left(\sum_{\forall \mathbf{b}\in \mathsf{FTs} \text{ w. } b_5=0,b_6=1} x_{\mathbf{b}}^{(k)}\right)\cdot p_{k\to j} + \left(\sum_{\forall \mathbf{b}\in \mathsf{FTs} \text{ w. } b_1=0,b_2=1} x_{\mathbf{b}}^{(i)}\right)\cdot p_{i\to j}. \tag{39}$$

Then, (E9) is a direct result of (E2) and (39). Similarly, (E10) is a direct result of (E3) and (E7). The proof is thus complete. □

Appendix C
Proof of Lemma 2

Before proving this lemma, we introduce the following "weight-movement" operator.

1) For any 2 non-negative values a and b, the operator $a \rightsquigarrow b$ implies that we keep decreasing a and increasing b by the same amount until $a = 0$. Namely, after the operator, the new a and b values are
$$a_{\text{new}} = 0, \qquad b_{\text{new}} = b + a.$$
2) For any 3 non-negative values a, b, and c, the operator $\{a, b\} \rightsquigarrow c$ implies that we keep decreasing a and b simultaneously and keep increasing c by the same amount until at least one of a and b being 0. Namely, after the operator, the new a, b, and c values are
$$a_{\text{new}} = a - \min\{a, b\}, \quad b_{\text{new}} = b - \min\{a, b\},$$
$$c_{\text{new}} = c + \min\{a, b\}.$$
3) For any 4 non-negative values a, b, c, and d, the operator $\{a, b\} \rightsquigarrow \{c, d\}$ implies that we keep decreasing a and b simultaneously and keep increasing c and d simultaneously by the same amount until at least one of a and b being 0. Namely, after the operator, we have
$$a_{\text{new}} = a - \min\{a, b\}, \quad b_{\text{new}} = b - \min\{a, b\},$$
$$c_{\text{new}} = c + \min\{a, b\}, \quad d_{\text{new}} = d + \min\{a, b\}.$$
4) We can also concatenate the operators. For example, for any three non-negative values a, b, and c, the operator $a \rightsquigarrow b \rightsquigarrow c$ implies that
$$a_{\text{new}} = 0, \quad b_{\text{new}} = 0, \quad c_{\text{new}} = c + (a + b).$$
5) Sometimes, we do not want to "move the weight to the largest possible degree" as was defined previously. To that end, we define the operator $a \overset{\Delta}{\rightsquigarrow} b$:
$$a_{\text{new}} = a - \Delta, \qquad b_{\text{new}} = b + \Delta.$$
where Δ ($\leq a$) is the amount of weight being moved from a to b.
6) Finally, $a \rightsquigarrow \emptyset$ means $a_{\text{new}} = 0$ and $a \overset{\Delta}{\rightsquigarrow} \emptyset$ means $a_{\text{new}} = a - \Delta$.

We now prove Lemma 2. Given $\vec{R}$ and $\{x_{\mathbf{b}}^{(i)}\}$-values satisfying Proposition 1, let us denote the corresponding values of y-variables in the rank-conversion conditions (18)–(24) as $\{y\}$.

Recall that each coding type $\mathsf{TYPE}_{\mathbf{b}}^{(i)}$ of node i corresponds to a specific subset of its knowledge space S_i, governed by 11 A-subspaces $\ddot{\mathsf{A}}_1^{(i)}$ to $\ddot{\mathsf{A}}_{11}^{(i)}$, see (14). As a result, by the rank conversion equalities (18)–(24), the bitstring $\mathbf{b}$ of each $\mathsf{TYPE}_{\mathbf{b}}^{(i)}$ will determine the contribution from the value $x_{\mathbf{b}}^{(i)}$ to the associated 11 y-values: $y_1^{(j)}$ to $y_4^{(j)}$; $y_1^{(k)}$ to $y_4^{(k)}$; and $y_1^{(j,k)}$ to $y_3^{(j,k)}$. For example, any vector $\mathbf{c}_t$ of $\mathsf{TYPE}_{\mathsf{7F7}}^{(i)}$ (i.e., type-01111111111 of node i), does not belong to $\ddot{\mathsf{A}}_1^{(i)}$. By (14) and (10)–(13), we know that $\ddot{\mathsf{A}}_1^{(i)} = A_1^{(j)}(t) = S_j(t-1)$. As a result, whenever a $\mathsf{TYPE}_{\mathsf{7F7}}^{(i)}$ coding vector, sent by node i at time t, is succesfully received by node j, the rank of $S_j(t-1)$ will increase by 1. Therefore, the value $x_{\mathsf{7F7}}^{(i)}$ (the frequency of using type-7F7 of node i) contributes to $y_1^{(j)}$ (the normalized expected rank of $A_1^{(j)}(n)$ in the end of time n) by $x_{\mathsf{7F7}}^{(i)} \cdot p_{i\to j}$. Any change of the value $x_{\mathsf{7F7}}^{(i)}$ will thus change the corresponding value $y_1^{(j)}$ accordingly as described in the rank conversion equalities (18)–(24) in Proposition 1.

The above intuition/explanation turns out to be very helpful when discussing the LP problem. Also, since all $\{y\}$-values can always be calculated from the given $\{x_{\mathbf{b}}^{(i)}\}$-values by (18)–(24), all our discussion can be focused on the given $\{x_{\mathbf{b}}^{(i)}\}$-values, and all $\{y\}$-values can be automatically computed.

The proof of Lemma 2 is done by proving the following intermediate claims.

Intermediate Claim 1*:* For any $\vec{R}$ and the corresponding 156 non-negative values $\{x_{\mathbf{b}}^{(i)}\}$ satisfying Proposition 1, we can always find another set of 156 non-negative values $\{\ddot{x}_{\mathbf{b}}^{(i)}\}$ such that $\vec{R}$ and $\{\ddot{x}_{\mathbf{b}}^{(i)}\}$ jointly satisfy Proposition 1 and

$$\begin{aligned} \ddot{x}_{\mathbf{b}}^{(i)} = 0, \quad & \forall\, i \in \{1,2,3\} \text{ and} \\ & \forall\, \mathbf{b} \in \{\mathtt{FF7}, \mathtt{F07}, \mathtt{0F7}, \mathtt{007}\}\,. \end{aligned} \tag{C1}$$

Proof of Intermediate Claim 1: The proof is done by explicit construction. We sequentially perform the following weight movement operations for all $i \in \{1,2,3\}$: $x_{\mathtt{FF7}}^{(i)} \rightsquigarrow \emptyset$; $x_{\mathtt{F07}}^{(i)} \rightsquigarrow \emptyset$; $x_{\mathtt{0F7}}^{(i)} \rightsquigarrow \emptyset$; and $x_{\mathtt{007}}^{(i)} \rightsquigarrow \emptyset$. After the weight movement, (C1) is obviously true for the new values of $\{x_{\mathbf{b}}^{(i)}\}$. What remains to prove that the time-sharing condition (17) and the decodability conditions (25)–(26) still hold (when computing the new $\{y\}$-values using the new $\{x_{\mathbf{b}}^{(i)}\}$-values) after the weight movement.

To that end, we prove that (17), (25), and (26) hold after each of the weight movement operations. We first observe that $x_{\mathtt{FF7}}^{(i)} \rightsquigarrow \emptyset$ does not change any y-value because the coding type-11111111111 does not participate in the rank conversion process. As a result, after $x_{\mathtt{FF7}}^{(i)} \rightsquigarrow \emptyset$, the decodability conditions (25)–(26) still hold. Since $x_{\mathtt{FF7}}^{(i)} \rightsquigarrow \emptyset$ reduces the value of $x_{\mathtt{FF7}}^{(i)}$, the time sharing condition (17) still holds.

We now consider $x_{\mathtt{F07}}^{(i)} \rightsquigarrow \emptyset$. Since $\mathtt{F07} = 11110000111$ in 11-bitstring, it means that $x_{\mathtt{F07}}^{(i)}$ contributes to the ranks of $\ddot{\mathsf{A}}_5^{(i)}$ to $\ddot{\mathsf{A}}_8^{(i)}$. By (14), $x_{\mathtt{F07}}^{(i)}$ contributes[4] to the values of $y_1^{(k)}$ to $y_4^{(k)}$, the ranks of $A_1^{(k)}$ to $A_4^{(k)}$ in the end of time n, respectively. By (18)–(21), the operation $x_{\mathtt{F07}}^{(i)} \rightsquigarrow \emptyset$ will decrease each of $y_1^{(k)}$ to $y_4^{(k)}$ by the same amount $(x_{\mathtt{F07}}^{(i)} \cdot p_{i\to k})$. Therefore, after $x_{\mathtt{F07}}^{(i)} \rightsquigarrow \emptyset$, the new values of $y_1^{(k)}$ to $y_4^{(k)}$ still satisfy the decodability equality (25). Note that $x_{\mathtt{F07}}^{(i)}$ does not contribute to any of $y_1^{(j,k)}$ to $y_3^{(j,k)}$ and therefore (26) still holds after $x_{\mathtt{F07}}^{(i)} \rightsquigarrow \emptyset$.

By similar arguments, the operation $x_{\mathtt{0F7}}^{(i)} \rightsquigarrow \emptyset$ will decrease $y_1^{(j)}$ to $y_4^{(j)}$ by the same amount $(x_{\mathtt{0F7}}^{(i)} \cdot p_{i\to j})$ while keeping all $y_1^{(k)}$ to $y_4^{(k)}$ and $y_1^{(j,k)}$ to $y_3^{(j,k)}$ unchanged. Therefore the decodability condition (25) still holds. By similar arguments, the operation $x_{\mathtt{007}}^{(i)} \rightsquigarrow \emptyset$ will decrease $y_1^{(j)}$ to $y_4^{(j)}$ by the same amount of $(x_{\mathtt{007}}^{(i)} \cdot p_{i\to j})$ and decrease $y_1^{(k)}$ to $y_4^{(k)}$ by the same amount $(x_{\mathtt{007}}^{(i)} \cdot p_{i\to k})$ while keeping all $y_1^{(j,k)}$ to $y_3^{(j,k)}$ unchanged. Therefore the decodability conditions (25) and (26) still hold. Intermediate Claim 1 is thus proven. □

Intermediate Claim 2*:* For any $\vec{R}$ vector and the 156 corresponding non-negative $\{x_{\mathbf{b}}^{(i)}\}$-values satisfying Proposition 1 and (C1), we can always find another set of 156 non-negative values $\{\ddot{x}_{\mathbf{b}}^{(i)}\}$ such that $\vec{R}$ and $\{\ddot{x}_{\mathbf{b}}^{(i)}\}$ jointly satisfy Proposition 1 and (C1), plus

$$\begin{aligned} \ddot{x}_{\mathbf{b}}^{(i)} = 0, \quad & \forall\, i \in \{1,2,3\} \text{ and} \\ & \forall\, \mathbf{b} \in \left\{ \begin{matrix} \mathtt{000}, \mathtt{003}, \mathtt{013}, \mathtt{053}, \mathtt{103}, \\ \mathtt{113}, \mathtt{153}, \mathtt{303}, \mathtt{313}, \mathtt{353} \end{matrix} \right\}. \end{aligned} \tag{C2}$$

Proof of Intermediate Claim 2*:* Consider any $\{x_{\mathbf{b}}^{(i)}\}$-values satisfying Proposition 1 and (C1). Since Proposition 1 holds, Lemma 1 implies that (E4) holds as well. When we count the non-zero coding types in (E4) (those not in (C1)), we immediately have

$$\begin{aligned} x_{\mathtt{000}}^{(i)} = {} & x_{\mathtt{003}}^{(i)} + x_{\mathtt{013}}^{(i)} + x_{\mathtt{053}}^{(i)} + x_{\mathtt{103}}^{(i)} \\ & + x_{\mathtt{113}}^{(i)} + x_{\mathtt{153}}^{(i)} + x_{\mathtt{303}}^{(i)} + x_{\mathtt{313}}^{(i)} + x_{\mathtt{353}}^{(i)}. \end{aligned} \tag{40}$$

Then, we sequentially perform the following operations:

$$\begin{aligned} &\{x_{\mathtt{003}}^{(i)}, x_{\mathtt{000}}^{(i)}\} \rightsquigarrow \{x_{\mathtt{001}}^{(i)}, x_{\mathtt{002}}^{(i)}\}, \\ &\{x_{\mathtt{013}}^{(i)}, x_{\mathtt{000}}^{(i)}\} \rightsquigarrow \{x_{\mathtt{002}}^{(i)}, x_{\mathtt{011}}^{(i)}\}, \\ &\{x_{\mathtt{053}}^{(i)}, x_{\mathtt{000}}^{(i)}\} \rightsquigarrow \{x_{\mathtt{002}}^{(i)}, x_{\mathtt{051}}^{(i)}\}, \\ &\{x_{\mathtt{103}}^{(i)}, x_{\mathtt{000}}^{(i)}\} \rightsquigarrow \{x_{\mathtt{001}}^{(i)}, x_{\mathtt{102}}^{(i)}\}, \\ &\{x_{\mathtt{113}}^{(i)}, x_{\mathtt{000}}^{(i)}\} \rightsquigarrow \{x_{\mathtt{011}}^{(i)}, x_{\mathtt{102}}^{(i)}\}, \\ &\{x_{\mathtt{153}}^{(i)}, x_{\mathtt{000}}^{(i)}\} \rightsquigarrow \{x_{\mathtt{051}}^{(i)}, x_{\mathtt{102}}^{(i)}\}, \\ &\{x_{\mathtt{303}}^{(i)}, x_{\mathtt{000}}^{(i)}\} \rightsquigarrow \{x_{\mathtt{001}}^{(i)}, x_{\mathtt{302}}^{(i)}\}, \\ &\{x_{\mathtt{313}}^{(i)}, x_{\mathtt{000}}^{(i)}\} \rightsquigarrow \{x_{\mathtt{011}}^{(i)}, x_{\mathtt{302}}^{(i)}\}, \\ &\{x_{\mathtt{353}}^{(i)}, x_{\mathtt{000}}^{(i)}\} \rightsquigarrow \{x_{\mathtt{051}}^{(i)}, x_{\mathtt{302}}^{(i)}\}. \end{aligned}$$

By (40), one can easily verify that after the above operations, we have (C2). Thus it is left to show that after these operations the linear conditions of Proposition 1 are still satisfied.

First notice that the time-sharing condition (17) is still satisfied since weight-moving operation decreases weights of two entries and increases the weights of another two entries by the same amount. We now argue that after each of the totally 9 weight-moving operations, the associated y-values remain unchanged. Take the last weight-moving operation $\{x_{\mathtt{353}}^{(i)}, x_{\mathtt{000}}^{(i)}\} \rightsquigarrow \{x_{\mathtt{051}}^{(i)}, x_{\mathtt{302}}^{(i)}\}$ for example. The corresponding coding types are

$$\begin{aligned} \mathsf{TYPE}_{\mathtt{353}}^{(i)} &\text{ in 11-bitstring} = 0011\,0101\,011, \\ \mathsf{TYPE}_{\mathtt{000}}^{(i)} &\text{ in 11-bitstring} = 0000\,0000\,000, \\ \mathsf{TYPE}_{\mathtt{051}}^{(i)} &\text{ in 11-bitstring} = 0000\,0101\,001, \\ \mathsf{TYPE}_{\mathtt{302}}^{(i)} &\text{ in 11-bitstring} = 0011\,0000\,010. \end{aligned}$$

Let $b_l(\mathtt{353})$ denote the l-th bit of the 11-bitstring $\mathtt{353} = \mathtt{00110101011}$, and similarly $b_l(\mathtt{000})$, $b_l(\mathtt{051})$, and $b_l(\mathtt{302})$ denote the l-th bit of 11-bitstrings $\mathtt{000}$, $\mathtt{051}$, and $\mathtt{302}$, respectively. One can see that for any l, the set $\{b_l(\mathtt{353}), b_l(\mathtt{000})\}$ is identical, as a set, to the set $\{b_l(\mathtt{051}), b_l(\mathtt{302})\}$ for all $l = 1$ to 11. Namely, when performing $\{x_{\mathtt{353}}^{(i)}, x_{\mathtt{000}}^{(i)}\} \rightsquigarrow \{x_{\mathtt{051}}^{(i)}, x_{\mathtt{302}}^{(i)}\}$, for all $l = 1$ to 11, the impact on the rank of $\ddot{\mathsf{A}}_l^{(i)}$ by decreasing simultaneously the two entries $\{x_{\mathtt{353}}^{(i)}, x_{\mathtt{000}}^{(i)}\}$ is offset completely by increasing simultaneously the two entries $\{x_{\mathtt{051}}^{(i)}, x_{\mathtt{302}}^{(i)}\}$. For example, bit b_1 (when $l = 1$) corresponds to $\ddot{\mathsf{A}}_1^{(i)} = A_1^{(j)}$

[4]This argument can also be made by directly examining equalities (18)–(24). In (18)–(24), we can see that only in (18)–(21) we use the b_5 to b_8 values to determine the contribution of $\{x_{\mathbf{b}}^{(i)}, x_{\mathbf{b}}^{(j)}, x_{\mathbf{b}}^{(k)}\}$. Since $y_1^{(i)}$ to $y_4^{(i)}$ are contributed by $x_{\mathtt{F07}}^{(j)}$, we thus know that only $y_1^{(k)}$ to $y_4^{(k)}$ are contributed by $x_{\mathtt{F07}}^{(i)}$.

and we have $b_1(\mathsf{353}) = 0$ and $b_1(\mathsf{000}) = 0$. Therefore, if we separate the weight-moving operation $\{x^{(i)}_{\mathsf{353}}, x^{(i)}_{\mathsf{000}}\} \rightsquigarrow \{x^{(i)}_{\mathsf{051}}, x^{(i)}_{\mathsf{302}}\}$ into the decreasing half and the increasing half, then during the decreasing half, the $y^{(j)}_1$-value will decrease by $\min\{x^{(i)}_{\mathsf{353}}, x^{(i)}_{\mathsf{000}}\} \cdot p_{i\to j}$ due to the decrease of $x^{(i)}_{\mathsf{353}}$ and then decrease by another $\min\{x^{(i)}_{\mathsf{353}}, x^{(i)}_{\mathsf{000}}\} \cdot p_{i\to j}$ due to the decrease of $x^{(i)}_{\mathsf{000}}$. On the other hand, during the increasing half, the $y^{(j)}_1$ value will increase by $\min\{x^{(i)}_{\mathsf{353}}, x^{(i)}_{\mathsf{000}}\} \cdot p_{i\to j}$ due to the increase of $x^{(i)}_{\mathsf{051}}$ and then increase by another $\min\{x^{(i)}_{\mathsf{353}}, x^{(i)}_{\mathsf{000}}\} \cdot p_{i\to j}$ due to the increase of $x^{(i)}_{\mathsf{302}}$. The amounts of increase and decrease perfectly offset each other since $\{b_1(\mathsf{353}), b_1(\mathsf{000})\} = \{0, 0\} = \{b_1(\mathsf{051}), b_1(\mathsf{302})\}$.

In sum, by similar reasoning, all the y-values will remain the same after each of the above 9 weight-moving operations. The proof is thus complete. $\square$

Intermediate Claim 3: For any $\vec{R}$ vector and the 156 corresponding non-negative $\{x^{(i)}_{\mathbf{b}}\}$-values satisfying Proposition 1 and (C1) to (C2), we can always find another set of 156 non-negative values $\{\ddot{x}^{(i)}_{\mathbf{b}}\}$ such that $\vec{R}$ and $\{\ddot{x}^{(i)}_{\mathbf{b}}\}$ jointly satisfy Proposition 1 and (C1) to (C2), plus for all $i \in \{1, 2, 3\}$,

$$\left(\sum_{\mathbf{b}\in\mathsf{FTs}(\cdot,7,\cdot)} x^{(i)}_{\mathbf{b}}\right) = \left(\sum_{\mathbf{b}\in\mathsf{FTs}(\cdot,1,\cdot)} x^{(i)}_{\mathbf{b}}\right), \quad \left(\sum_{\mathbf{b}\in\mathsf{FTs}(7,\cdot,\cdot)} x^{(i)}_{\mathbf{b}}\right) = \left(\sum_{\mathbf{b}\in\mathsf{FTs}(1,\cdot,\cdot)} x^{(i)}_{\mathbf{b}}\right). \tag{C3}$$

Proof of Intermediate Claim 3: Since the node indices are cyclically decided, we will prove the following equivalent forms:

$$\left(\sum_{\mathbf{b}\in\mathsf{FTs}(\cdot,7,\cdot)} x^{(j)}_{\mathbf{b}}\right) = \left(\sum_{\mathbf{b}\in\mathsf{FTs}(\cdot,1,\cdot)} x^{(j)}_{\mathbf{b}}\right), \tag{41}$$

$$\left(\sum_{\mathbf{b}\in\mathsf{FTs}(7,\cdot,\cdot)} x^{(k)}_{\mathbf{b}}\right) = \left(\sum_{\mathbf{b}\in\mathsf{FTs}(1,\cdot,\cdot)} x^{(k)}_{\mathbf{b}}\right), \tag{42}$$

based on the equality (E8) of Lemma 1. For shorthand, define the following 4 non-negative terms of (E8) as follows:

$$\mathsf{term}_1 \triangleq \left(\sum_{\mathbf{b}\in\mathsf{FTs}(\cdot,7,\cdot)} x^{(j)}_{\mathbf{b}}\right) \cdot p_{j\to i},$$

$$\mathsf{term}_2 \triangleq \left(\sum_{\mathbf{b}\in\mathsf{FTs}(7,\cdot,\cdot)} x^{(k)}_{\mathbf{b}}\right) \cdot p_{k\to i},$$

$$\mathsf{term}_3 \triangleq \left(\sum_{\mathbf{b}\in\mathsf{FTs}(\cdot,1,\cdot)} x^{(j)}_{\mathbf{b}}\right) \cdot p_{j\to i},$$

$$\mathsf{term}_4 \triangleq \left(\sum_{\mathbf{b}\in\mathsf{FTs}(1,\cdot,\cdot)} x^{(k)}_{\mathbf{b}}\right) \cdot p_{k\to i}.$$

Using the above 4 terms, (E8) can be rewritten by

$$\mathsf{term}_1 + \mathsf{term}_2 = \mathsf{term}_3 + \mathsf{term}_4. \tag{43}$$

Recall that we assume both $p_{j\to i} > 0$ and $p_{k\to i} > 0$. Consider the following three cases depending on the values of term_1 and term_3.

Case 1: $\mathsf{term}_1 = \mathsf{term}_3$. By (43), we also have $\mathsf{term}_2 = \mathsf{term}_4$. By the definitions of term_1 to term_4, both (41) and (42) hold automatically.

Case 2: $\mathsf{term}_1 < \mathsf{term}_3$. Since each term is strictly non-negative, we thus have $\mathsf{term}_3 > 0$. Also by (43), we must also have $\mathsf{term}_2 > \mathsf{term}_4$ and thus $\mathsf{term}_2 > 0$. In the following, we will describe a set of weight-moving operations such that after moving the weights among $\{x^{(j)}_{\mathbf{b}}, x^{(k)}_{\mathbf{b}}\}$, the new $\{x^{(j)}_{\mathbf{b}}, x^{(k)}_{\mathbf{b}}\}$ satisfy Proposition 1, (C1), and (C2); and the gap $\mathsf{term}_3 - \mathsf{term}_1$ computed using the new $\{x^{(j)}_{\mathbf{b}}\}$ is strictly smaller than the gap computed by the old $\{x^{(j)}_{\mathbf{b}}\}$ while $\mathsf{term}_3 \geq \mathsf{term}_1$. We can thus iteratively perform the weight movements until $\mathsf{term}_1 = \mathsf{term}_3$. The final $\{x^{(j)}_{\mathbf{b}}, x^{(k)}_{\mathbf{b}}\}$ then satisfy (C3) now.

The desired weight-moving operations are described as follows. Since $\mathsf{term}_3 > 0$, we can find an 11-bitstring $\mathbf{b}^{\mathsf{term}_3} \in \mathsf{FTs}(\cdot, 1, \cdot)$ such that $x^{(j)}_{\mathbf{b}^{\mathsf{term}_3}} > 0$. Similarly, since $\mathsf{term}_2 > 0$, we can find a $\mathbf{b}^{\mathsf{term}_2} \in \mathsf{FTs}(7, \cdot, \cdot)$ such that $x^{(k)}_{\mathbf{b}^{\mathsf{term}_2}} > 0$. We then define

$$\Delta = \min\left\{ x^{(j)}_{\mathbf{b}^{\mathsf{term}_3}} \cdot p_{j\to i},\ x^{(k)}_{\mathbf{b}^{\mathsf{term}_2}} \cdot p_{k\to i},\ \mathsf{term}_3 - \mathsf{term}_1 \right\}.$$

Obviously, we have $\Delta > 0$ since we assume $p_{j\to i} > 0$ and $p_{k\to i} > 0$ for all (i, j, k). We then compute $\Delta^{\mathsf{term}_3} = \Delta / p_{j\to i}$ and $\Delta^{\mathsf{term}_2} = \Delta / p_{k\to i}$. By the definition of Δ, we have $0 < \Delta^{\mathsf{term}_3} \leq x^{(j)}_{\mathbf{b}^{\mathsf{term}_3}}$ and $0 < \Delta^{\mathsf{term}_2} \leq x^{(k)}_{\mathbf{b}^{\mathsf{term}_2}}$.

Then, we perform the following weight-moving operations:

$$x^{(j)}_{\mathbf{b}^{\mathsf{term}_3}} \overset{\Delta^{\mathsf{term}_3}}{\rightsquigarrow} x^{(j)}_{\mathbf{b}^{\mathsf{term}_3} \oplus \mathsf{040}}, \tag{OP1}$$

$$x^{(k)}_{\mathbf{b}^{\mathsf{term}_2}} \overset{\Delta^{\mathsf{term}_2}}{\rightsquigarrow} x^{(k)}_{\mathbf{b}^{\mathsf{term}_2} \oplus \mathsf{400}}, \tag{OP2}$$

where $\oplus$ is bit-wise exclusive or. For example, if $\mathbf{b}^{\mathsf{term}_3} = \mathsf{117}$ which belongs to $\mathsf{FTs}(\cdot, 1, \cdot)$, then $\mathbf{b}^{\mathsf{term}_3} \oplus \mathsf{040} = \mathsf{157}$ which now belongs to $\mathsf{FTs}(\cdot, 5, \cdot)$ instead. Similarly, if $\mathbf{b}^{\mathsf{term}_2} = \mathsf{737}$, then $\mathbf{b}^{\mathsf{term}_2} \oplus \mathsf{400} = \mathsf{337}$, which now belongs to $\mathsf{FTs}(3, \cdot, \cdot)$.

We now argue that after moving the weights among $\{x^{(j)}_{\mathbf{b}}, x^{(k)}_{\mathbf{b}}\}$, the new $\{x^{(j)}_{\mathbf{b}}, x^{(k)}_{\mathbf{b}}\}$ satisfy Proposition 1, (C1), and (C2); and the gap $\mathsf{term}_3 - \mathsf{term}_1$ computed using the new $\{x^{(j)}_{\mathbf{b}}\}$ is strictly smaller than the gap computed by the old $\{x^{(j)}_{\mathbf{b}}\}$ while $\mathsf{term}_3 \geq \mathsf{term}_1$. To that end, we first argue that after the above weight movements, both (C1) and (C2) still hold. The reason is that since $\mathbf{b}^{\mathsf{term}_2} \oplus \mathsf{400} \in \mathsf{FTs}(3, \cdot, \cdot)$ and $\mathbf{b}^{\mathsf{term}_3} \oplus \mathsf{040} \in \mathsf{FTs}(\cdot, 5, \cdot)$, we never move any weight to the frequencies $\{x^{(j)}_{\mathbf{b}}, x^{(k)}_{\mathbf{b}}\}$ satisfying (C1). As a result, (C1) still holds after the above weight movements. Since $\mathbf{b}^{\mathsf{term}_2} \oplus \mathsf{400} \in \mathsf{FTs}(3, \cdot, \cdot)$, it may look possible that we can increase the weight of $x^{(k)}_{\mathsf{303}}$, $x^{(k)}_{\mathsf{313}}$, and $x^{(k)}_{\mathsf{353}}$ in (C2) by the weight-moving operation (OP2). However, to increase the weight of $x^{(k)}_{\mathsf{303}}$, $x^{(k)}_{\mathsf{313}}$, and $x^{(k)}_{\mathsf{353}}$, it means that we must have $\mathbf{b}^{\mathsf{term}_2} \in \{\mathsf{703}, \mathsf{713}, \mathsf{753}\}$ to begin with. However, they are not in the feasible coding types FTs, see (33). As a result, after (OP2) movement, (C2) still holds. Since $x^{(j)}_{\mathbf{b}^{\mathsf{term}_3}} \oplus \mathsf{040} \in \mathsf{FTs}(\cdot, 5, \cdot)$, it may look possible that we can increase the weight of $x^{(j)}_{\mathsf{053}}$, $x^{(j)}_{\mathsf{153}}$, and $x^{(j)}_{\mathsf{353}}$ in (C2) by the weight-moving operation (OP1). However, to increase the weight of $x^{(j)}_{\mathsf{053}}$, $x^{(j)}_{\mathsf{153}}$, and $x^{(j)}_{\mathsf{353}}$, it means that we must have $\mathbf{b}^{\mathsf{term}_3} \in \{\mathtt{013}, \mathtt{113}, \mathtt{313}\}$ to begin with. However, since we choose $\mathbf{b}^{\mathsf{term}_3}$ such that $x^{(j)}_{\mathbf{b}^{\mathsf{term}_3}} > 0$, and the original $\{x^{(j)}_{\mathbf{b}}\}$-values satisfy (C2), it is impossible to have $\mathbf{b}^{\mathsf{term}_3} \in \{\mathtt{013}, \mathtt{113}, \mathtt{313}\}$. As a result, after (OP1) movement, (C2) still holds.

We now consider the conditions in Proposition 1. We first notice that it is clear that after moving the weights, the time-sharing condition of Proposition 1 still holds because at every iteration we only "move" the weights on the frequencies $\{x^{(j)}_{\mathbf{b}}, x^{(k)}_{\mathbf{b}}\}$ without changing the overall sum. We now examine whether other conditions of Proposition 1 are still satisfied after the above modification process. For that, we argue that the above process keeps all the y-values unchanged. To see that, suppose $(i, j, k) = (1, 2, 3)$ without loss of generality. Since the 11-bitstring $\mathtt{040}$ has only 6-th bit being 1 and all the other bits being 0, the (OP1) operation will change only the rank of $\ddot{\mathsf{A}}^{(j)}_6$, i.e., $\ddot{\mathsf{A}}^{(2)}_6$ when $(i, j, k) = (1, 2, 3)$. By (14), $\ddot{\mathsf{A}}^{(2)}_6 = A^{(1)}_2$ and thus only $y^{(1)}_2$ will be affected by this operation. Since we are moving the weight of Δ^{term_3} from $x^{(2)}_{\mathbf{b}^{\mathsf{term}_3}}$ (the 6-th bit of $\mathbf{b}^{\mathsf{term}_3}$ is 0 since $\mathbf{b}^{\mathsf{term}_3} \in \mathsf{FTs}(\cdot, \mathtt{1}, \cdot)$) to $x^{(2)}_{\mathbf{b}^{\mathsf{term}_3} \oplus \mathtt{040}}$ (the 6-th bit of $\mathbf{b}^{\mathsf{term}_3} \oplus \mathtt{040}$ is 1), $y^{(1)}_2$ will be decreased by $(\Delta^{\mathsf{term}_3} \cdot p_{2\to 1})$, which is equal to Δ. On the other hand since the 11-bitstring $\mathtt{400}$ has only the 2nd bit being 1 and all the other bits being 0, the (OP2) operation will change only the rank of $\ddot{\mathsf{A}}^{(k)}_2$, i.e., $\ddot{\mathsf{A}}^{(3)}_2$ when $(i, j, k) = (1, 2, 3)$. By (14), $\ddot{\mathsf{A}}^{(3)}_2 = A^{(1)}_2$ and thus again only $y^{(1)}_2$ will be affected by this operation. Since we are moving the weight of Δ^{term_2} from $x^{(3)}_{\mathbf{b}^{\mathsf{term}_2}}$ (the 2nd bit of $\mathbf{b}^{\mathsf{term}_2}$ is 1 since $\mathbf{b}^{\mathsf{term}_2} \in \mathsf{FTs}(\mathtt{7}, \cdot, \cdot)$) to $x^{(3)}_{\mathbf{b}^{\mathsf{term}_2} \oplus \mathtt{400}}$ (the 2nd bit of $\mathbf{b}^{\mathsf{term}_2} \oplus \mathtt{400}$ is 0), $y^{(1)}_2$ will be increased by $(\Delta^{\mathsf{term}_2} \cdot p_{3\to 1})$, which is equal to Δ. The impacts of the two weight-moving operations (OP1) and (OP2) on $y^{(1)}_2$ perfectly offset each other. As a result, any of y-values are unchanged.

In the following, we will prove that (OP1) will decrease the value of term_3 by Δ while keeping the values of term_1, term_2, and term_4 unchanged; and (OP2) will decrease the value of term_2 by Δ while keeping the values of term_1, term_3, and term_4 unchanged. Thus after performing (OP1) and (OP2), the gap $\mathsf{term}_3 - \mathsf{term}_1$ computed by the new $\{x^{(j)}_{\mathbf{b}}\}$-values decreases by Δ and we still have $\mathsf{term}_3 \geq \mathsf{term}_1$ by the definition of Δ while satisfying (43). We first observe that (OP1) manipulates only $\{x^{(j)}_{\mathbf{b}}\}$, thus term_2 and term_4 will not be affected since both are based on $\{x^{(k)}_{\mathbf{b}}\}$ of another node index. Also notice that $\mathbf{b}^{\mathsf{term}_3} \in \mathsf{FTs}(\cdot, \mathtt{1}, \cdot)$ if and only if $\mathbf{b}^{\mathsf{term}_3} \oplus \mathtt{040} \in \mathsf{FTs}(\cdot, \mathtt{5}, \cdot)$. Therefore, the weight movement (OP1) does not change the value of term_1 since term_1 involves only those frequencies with $\mathbf{b} \in \mathsf{FTs}(\cdot, \mathtt{7}, \cdot)$. Finally, since $\mathbf{b}^{\mathsf{term}_3} \in \mathsf{FTs}(\cdot, \mathtt{1}, \cdot)$ and $\mathbf{b}^{\mathsf{term}_3} \oplus \mathtt{040} \in \mathsf{FTs}(\cdot, \mathtt{5}, \cdot)$, the (OP1) movement will decrease the value of term_3 and the decrease amount will be $\Delta^{\mathsf{term}_3} \cdot p_{j\to i} = \Delta$. The statement that (OP2) decreases the value of term_2 by Δ while keeping the values of term_1, term_3, and term_4 unchanged can be proved similarly. The proof of Case 2 is thus complete.

Case 3: $\mathsf{term}_1 > \mathsf{term}_3$. Since each term is strictly non-negative, we thus have $\mathsf{term}_1 > 0$ and by (43), we must also have $\mathsf{term}_4 > 0$. Again, we will describe a set of weight-moving operations such that after moving the weights among $\{x^{(j)}_{\mathbf{b}}, x^{(k)}_{\mathbf{b}}\}$, the new $\{x^{(j)}_{\mathbf{b}}, x^{(k)}_{\mathbf{b}}\}$ satisfy Proposition 1, (C1), and (C2); and the gap $\mathsf{term}_1 - \mathsf{term}_3$ computed using the new $\{x^{(j)}_{\mathbf{b}}\}$ is strictly smaller than the gap computed by the old $\{x^{(j)}_{\mathbf{b}}\}$ while satisfying (43) and $\mathsf{term}_1 \geq \mathsf{term}_3$. We can thus iteratively perform the weight movements until $\mathsf{term}_1 = \mathsf{term}_3$. The final $\{x^{(j)}_{\mathbf{b}}, x^{(k)}_{\mathbf{b}}\}$ thus satisfy (C3).

The desired weight-moving operations are described as follows. Since $\mathsf{term}_1 > 0$, we can find an 11-bitstring $\mathbf{b}^{\mathsf{term}_1} \in \mathsf{FTs}(\cdot, \mathtt{7}, \cdot)$ such that $x^{(j)}_{\mathbf{b}^{\mathsf{term}_1}} > 0$. Similarly, since $\mathsf{term}_4 > 0$, we can find a $\mathbf{b}^{\mathsf{term}_4} \in \mathsf{FTs}(\mathtt{1}, \cdot, \cdot)$ such that $x^{(k)}_{\mathbf{b}^{\mathsf{term}_4}} > 0$. We then define

$$\Delta = \min\left\{ x^{(j)}_{\mathbf{b}^{\mathsf{term}_1}} \cdot p_{j\to i},\ \ x^{(k)}_{\mathbf{b}^{\mathsf{term}_4}} \cdot p_{k\to i},\ \ \mathsf{term}_1 - \mathsf{term}_3 \right\}.$$

We then compute $\Delta^{\mathsf{term}_1} = \Delta / p_{j\to i}$ and $\Delta^{\mathsf{term}_4} = \Delta / p_{k\to i}$. Then, we perform the following weight-moving operations:

$$x^{(j)}_{\mathbf{b}^{\mathsf{term}_1}} \xrightarrow{\Delta^{\mathsf{term}_1}} x^{(j)}_{\mathbf{b}^{\mathsf{term}_1} \oplus \mathtt{040}}, \qquad x^{(k)}_{\mathbf{b}^{\mathsf{term}_4}} \xrightarrow{\Delta^{\mathsf{term}_4}} x^{(k)}_{\mathbf{b}^{\mathsf{term}_4} \oplus \mathtt{400}}.$$

By almost identical reasonings as in the discussion of Case 2, we can prove that after the above modification process, we have that the new $\{x^{(j)}_{\mathbf{b}}, x^{(k)}_{\mathbf{b}}\}$ satisfy Proposition 1, (C1), and (C2); and the gap $\mathsf{term}_1 - \mathsf{term}_3$ computed using the new $\{x^{(j)}_{\mathbf{b}}\}$ is strictly smaller than the gap computed by the old $\{x^{(j)}_{\mathbf{b}}\}$ while satisfying (43) and $\mathsf{term}_1 \geq \mathsf{term}_3$. The proof of Case 3 is thus complete. □

Intermediate Claim 4*:* For any $\vec{R}$ vector and the 156 corresponding non-negative $\{x^{(i)}_{\mathbf{b}}\}$-values satisfying Proposition 1 and (C1) to (C3), we can always find another set of 156 non-negative values $\{\ddot{x}^{(i)}_{\mathbf{b}}\}$ such that $\vec{R}$ and $\{\ddot{x}^{(i)}_{\mathbf{b}}\}$ jointly satisfy Proposition 1 and (C1) to (C3), plus

$$\ddot{x}^{(i)}_{\mathbf{b}} = 0, \forall\, i \in \{1,2,3\} \text{ and}$$
$$\forall\, \mathbf{b} \in \left\{ \begin{matrix} \mathtt{011}, \mathtt{017}, \mathtt{037}, \mathtt{057}, \mathtt{077}, \mathtt{102}, \mathtt{107}, \\ \mathtt{117}, \mathtt{137}, \mathtt{157}, \mathtt{177}, \mathtt{1F7}, \mathtt{307}, \mathtt{317}, \\ \mathtt{377}, \mathtt{507}, \mathtt{517}, \mathtt{577}, \mathtt{707}, \mathtt{717}, \mathtt{737}, \\ \mathtt{757}, \mathtt{777}, \mathtt{7F7}, \mathtt{F17}, \mathtt{F77} \end{matrix} \right\}. \quad \text{(C4)}$$

Proof of Intermediate Claim 4*:* We simultaneously perform the weight-moving operations in the first column of Table I for all nodes $i \in \{1, 2, 3\}$. For each operation, we also present how the associated y-values are affected after each operation. As described in the proof of *Intermediate Claim* 1, one can verify the variations of y-values by each operation in Table I. For example, the first operation $x^{(i)}_{\mathtt{011}} \rightsquigarrow x^{(i)}_{\mathtt{051}}$ moves all the weight from $x^{(i)}_{\mathtt{011}}$ to $x^{(i)}_{\mathtt{051}}$. Since

$$\mathsf{TYPE}^{(i)}_{\mathtt{011}} \text{ in 11-bitstring} = 0000\,0001\,001,$$
$$\mathsf{TYPE}^{(i)}_{\mathtt{051}} \text{ in 11-bitstring} = 0000\,0101\,001,$$

one can easily see that only the rank of $\ddot{\mathsf{A}}^{(i)}_6$ will be affected since the only different bit between $\mathtt{011}$ and $\mathtt{051}$ is the 6-th bit. By (14), $\ddot{\mathsf{A}}^{(i)}_6 = A^{(k)}_2$ and thus only $y^{(k)}_2$ will be affected by $x^{(i)}_{\mathtt{011}} \rightsquigarrow x^{(i)}_{\mathtt{051}}$ operation. We observe that $\mathsf{TYPE}^{(i)}_{\mathtt{011}}$ participates in the increase of $y^{(k)}_2$ (the 6-th bit of $\mathtt{011}$ being 0) but $\mathsf{TYPE}^{(i)}_{\mathtt{051}}$ (the 6-th bit of $\mathtt{051}$ being 1) does not. Thus after

The underlying y-values are associated to 11-bitstring of node i's coding type-$\mathbf{b} \in \mathsf{FTs}$. See (14) for conversion.
For shorthand, we define $p \triangleq p_{i\to j}$ and $q \triangleq p_{i\to k}$.

	$y_1^{(j)}$	$y_2^{(j)}$	$y_3^{(j)}$	$y_4^{(j)}$	$y_1^{(k)}$	$y_2^{(k)}$	$y_3^{(k)}$	$y_4^{(k)}$	$y_1^{(j,k)}$	$y_2^{(j,k)}$	$y_3^{(j,k)}$
$x_{\mathtt{011}}^{(i)} \rightsquigarrow x_{\mathtt{051}}^{(i)}$						$-x_{\mathtt{011}}^{(i)} \cdot q$					
$x_{\mathtt{102}}^{(i)} \rightsquigarrow x_{\mathtt{302}}^{(i)}$			$-x_{\mathtt{102}}^{(i)} \cdot p$								
$x_{\mathtt{137}}^{(i)} \rightsquigarrow x_{\mathtt{177}}^{(i)} \rightsquigarrow x_{\mathtt{377}}^{(i)} \rightsquigarrow x_{\mathtt{337}}^{(i)}$			$-x_{\mathtt{137}}^{(i)} \cdot p$ $-x_{\mathtt{177}}^{(i)} \cdot p$			$+x_{\mathtt{177}}^{(i)} \cdot q$ $+x_{\mathtt{377}}^{(i)} \cdot q$					
$x_{\mathtt{117}}^{(i)} \rightsquigarrow x_{\mathtt{157}}^{(i)} \rightsquigarrow x_{\mathtt{317}}^{(i)} \rightsquigarrow x_{\mathtt{357}}^{(i)}$			$-x_{\mathtt{117}}^{(i)} \cdot p$ $-x_{\mathtt{157}}^{(i)} \cdot p$			$-x_{\mathtt{117}}^{(i)} \cdot q$ $-x_{\mathtt{317}}^{(i)} \cdot q$					
$x_{\mathtt{107}}^{(i)} \rightsquigarrow x_{\mathtt{307}}^{(i)} \rightsquigarrow x_{\mathtt{1F7}}^{(i)} \rightsquigarrow x_{\mathtt{3F7}}^{(i)}$			$-x_{\mathtt{107}}^{(i)} \cdot p$ $-x_{\mathtt{1F7}}^{(i)} \cdot p$		$-x_{\mathtt{107}}^{(i)} \cdot q$ $-x_{\mathtt{307}}^{(i)} \cdot q$	$-x_{\mathtt{107}}^{(i)} \cdot q$ $-x_{\mathtt{307}}^{(i)} \cdot q$	$-x_{\mathtt{107}}^{(i)} \cdot q$ $-x_{\mathtt{307}}^{(i)} \cdot q$	$-x_{\mathtt{107}}^{(i)} \cdot q$ $-x_{\mathtt{307}}^{(i)} \cdot q$			
$x_{\mathtt{577}}^{(i)} \rightsquigarrow x_{\mathtt{737}}^{(i)} \rightsquigarrow x_{\mathtt{777}}^{(i)} \rightsquigarrow x_{\mathtt{537}}^{(i)}$			$+x_{\mathtt{737}}^{(i)} \cdot p$ $+x_{\mathtt{777}}^{(i)} \cdot p$			$+x_{\mathtt{577}}^{(i)} \cdot q$ $+x_{\mathtt{777}}^{(i)} \cdot q$					
$x_{\mathtt{517}}^{(i)} \rightsquigarrow x_{\mathtt{717}}^{(i)} \rightsquigarrow x_{\mathtt{757}}^{(i)} \rightsquigarrow x_{\mathtt{557}}^{(i)}$			$+x_{\mathtt{717}}^{(i)} \cdot p$ $+x_{\mathtt{757}}^{(i)} \cdot p$			$-x_{\mathtt{517}}^{(i)} \cdot q$ $-x_{\mathtt{717}}^{(i)} \cdot q$					
$x_{\mathtt{507}}^{(i)} \rightsquigarrow x_{\mathtt{707}}^{(i)} \rightsquigarrow x_{\mathtt{7F7}}^{(i)} \rightsquigarrow x_{\mathtt{5F7}}^{(i)}$			$+x_{\mathtt{707}}^{(i)} \cdot p$ $+x_{\mathtt{7F7}}^{(i)} \cdot p$		$-x_{\mathtt{507}}^{(i)} \cdot q$ $-x_{\mathtt{707}}^{(i)} \cdot q$	$-x_{\mathtt{507}}^{(i)} \cdot q$ $-x_{\mathtt{707}}^{(i)} \cdot q$	$-x_{\mathtt{507}}^{(i)} \cdot q$ $-x_{\mathtt{707}}^{(i)} \cdot q$	$-x_{\mathtt{507}}^{(i)} \cdot q$ $-x_{\mathtt{707}}^{(i)} \cdot q$			
$x_{\mathtt{037}}^{(i)} \rightsquigarrow x_{\mathtt{077}}^{(i)} \rightsquigarrow x_{\mathtt{F77}}^{(i)} \rightsquigarrow x_{\mathtt{F37}}^{(i)}$	$-x_{\mathtt{037}}^{(i)} \cdot p$ $-x_{\mathtt{077}}^{(i)} \cdot p$	$-x_{\mathtt{037}}^{(i)} \cdot p$ $-x_{\mathtt{077}}^{(i)} \cdot p$	$-x_{\mathtt{037}}^{(i)} \cdot p$ $-x_{\mathtt{077}}^{(i)} \cdot p$	$-x_{\mathtt{037}}^{(i)} \cdot p$ $-x_{\mathtt{077}}^{(i)} \cdot p$		$+x_{\mathtt{077}}^{(i)} \cdot q$ $+x_{\mathtt{F77}}^{(i)} \cdot q$					
$x_{\mathtt{017}}^{(i)} \rightsquigarrow x_{\mathtt{057}}^{(i)} \rightsquigarrow x_{\mathtt{F17}}^{(i)} \rightsquigarrow x_{\mathtt{F57}}^{(i)}$	$-x_{\mathtt{017}}^{(i)} \cdot p$ $-x_{\mathtt{057}}^{(i)} \cdot p$	$-x_{\mathtt{017}}^{(i)} \cdot p$ $-x_{\mathtt{057}}^{(i)} \cdot p$	$-x_{\mathtt{017}}^{(i)} \cdot p$ $-x_{\mathtt{057}}^{(i)} \cdot p$	$-x_{\mathtt{017}}^{(i)} \cdot p$ $-x_{\mathtt{057}}^{(i)} \cdot p$		$-x_{\mathtt{017}}^{(i)} \cdot q$ $-x_{\mathtt{F17}}^{(i)} \cdot q$					

TABLE I
THE WEIGHT-MOVING OPERATIONS AND THE CORRESPONDING VARIATIONS OF THE ASSOCIATED y-VALUES FOR *Intermediate Claim* 4.

the weight movement, $y_2^{(k)}$ will be decreased by the amount of $(x_{\mathtt{011}}^{(i)} \cdot p_{i\to k})$ as indicated in Table I. The rest of Table I is populated by examining all 10 weight-moving operations (the 10 rows) and their corresponding impact on the y-values.

One can easily see from Table I that after completing all 10 weight-moving operations, for each node i, 26 coding types (enumerated in (C4)) of the new values $\{x_{\mathbf{b}}^{(i)}\}$ will be set to zeros.

We now argue that after completing all 10 operations, the linear conditions of Proposition 1 plus (C1) to (C3) are still satisfied. To that end, we first notice that only those $\{x_{\mathbf{b}}^{(i)}\}$ with $\mathbf{b} \in \{\mathtt{051}, \mathtt{302}, \mathtt{337}, \mathtt{357}, \mathtt{3F7}, \mathtt{537}, \mathtt{557}, \mathtt{5F7}, \mathtt{F37}, \mathtt{F57}\}$ will increase after the weight movements. Since those coding types do not participate in any of the terms in (C1) to (C3), the conditions (C1) to (C3) still hold after the weight movements.

We now observe that the time-sharing conditions (17) are still satisfied since we only "move" the weights. We now argue that after completing all 10 operations, all $y_1^{(j)}$ to $y_4^{(j)}$ will decrease by the same amount $(x_{\mathtt{037}}^{(i)}+x_{\mathtt{077}}^{(i)}+x_{\mathtt{017}}^{(i)}+x_{\mathtt{057}}^{(i)})\cdot p_{i\to j}$. The fact that $y_1^{(j)}$, $y_2^{(j)}$ and $y_4^{(j)}$ all decrease by the same amount $(x_{\mathtt{037}}^{(i)}+x_{\mathtt{077}}^{(i)}+x_{\mathtt{017}}^{(i)}+x_{\mathtt{057}}^{(i)})\cdot p_{i\to j}$ can be easily verified by summing up the "impact" of the 10 weight movement operations over each column of Table I, for columns 1, 2, and 4, respectively. To prove that $y_3^{(j)}$ also decreases by the same amount, we need to prove that

$$\begin{aligned}&\left(x_{\mathtt{737}}^{(i)} + x_{\mathtt{777}}^{(i)} + x_{\mathtt{717}}^{(i)} + x_{\mathtt{757}}^{(i)} + x_{\mathtt{707}}^{(i)} + x_{\mathtt{7F7}}^{(i)}\right) \cdot p_{i\to j} \\ &= \left(x_{\mathtt{102}}^{(i)} + x_{\mathtt{137}}^{(i)} + x_{\mathtt{177}}^{(i)} + x_{\mathtt{117}}^{(i)} + x_{\mathtt{157}}^{(i)} + x_{\mathtt{107}}^{(i)} + x_{\mathtt{1F7}}^{(i)}\right) \cdot p_{i\to j}.\end{aligned} \tag{44}$$

We can prove that (44) holds by noticing that (44) is equivalent to the second equality in (C3) when removing the zero terms specified in (C1) and (C2).

We now argue that after completing all 10 operations, all $y_1^{(k)}$ to $y_4^{(k)}$ will decrease by the same amount $(x_{\mathtt{107}}^{(i)} + x_{\mathtt{307}}^{(i)} + x_{\mathtt{507}}^{(i)}+x_{\mathtt{707}}^{(i)})\cdot p_{i\to k}$. The fact that $y_1^{(k)}$, $y_3^{(k)}$ and $y_4^{(k)}$ all increase by the same amount $(x_{\mathtt{107}}^{(i)}+x_{\mathtt{307}}^{(i)}+x_{\mathtt{507}}^{(i)}+x_{\mathtt{707}}^{(i)})\cdot p_{i\to k}$ can be easily verified by summing up the "impact" of the 10 weight movement operations over each column, for columns 5, 7, and 8, respectively. To prove that $y_2^{(k)}$ also increases by the same amount, we need to prove that

$$\begin{aligned}&\left(x_{\mathtt{177}}^{(i)} + x_{\mathtt{377}}^{(i)} + x_{\mathtt{577}}^{(i)} + x_{\mathtt{777}}^{(i)} + x_{\mathtt{077}}^{(i)} + x_{\mathtt{F77}}^{(i)}\right) \cdot p_{i\to k} \\ &= \left(x_{\mathtt{011}}^{(i)} + x_{\mathtt{117}}^{(i)} + x_{\mathtt{317}}^{(i)} + x_{\mathtt{517}}^{(i)} + x_{\mathtt{717}}^{(i)} + x_{\mathtt{017}}^{(i)} + x_{\mathtt{F17}}^{(i)}\right) \cdot p_{i\to k}.\end{aligned} \tag{45}$$

We can prove that (45) holds by noticing that (45) is equivalent to the first equality in (C3) when removing the zero terms specified in (C1) and (C2).

From Table I, one can also prove that $y_1^{(j,k)}$ to $y_3^{(j,k)}$ remain unchanged since the 10 weight movement operations have no impact on these three y-values. Since $y_1^{(j)}$ to $y_4^{(j)}$ all decrease by the same amount; $y_1^{(k)}$ to $y_4^{(k)}$ all decrease by the same amount; and $y_1^{(j,k)}$ to $y_3^{(j,k)}$ all remain the same, then the decodability conditions (25) and (26) must hold after the 10 weight movement operations. The proof of Intermediate Case 4 is thus complete. □

Intermediate Claim 5*:* For any $\vec{R}$ vector and the 156 corresponding non-negative $\{x_{\mathbf{b}}^{(i)}\}$-values satisfying Proposition 1 and (C1) to (C4), we can always find another set of 156 non-negative values $\{\ddot{x}_{\mathbf{b}}^{(i)}\}$ such that $\vec{R}$ and $\{\ddot{x}_{\mathbf{b}}^{(i)}\}$ jointly satisfy Proposition 1 and (C1) to (C4), plus for all $i \in \{1,2,3\}$,

$$\ddot{x}_{\mathbf{b}}^{(i)} = 0, \ \forall \mathbf{b} \in \{\mathsf{001}, \mathsf{002}\}. \tag{C5}$$

Proof of Intermediate Claim 5*:* We now provide an explicit weight movement such that after the weight-moving process, Proposition 1 and (C1) to (C4) hold, and additionally (C5) holds for the case when $i=1$, i.e., $(i,j,k)=(1,2,3)$. Then by applying the cyclically symmetric weight-moving process to the cases of $(i,j,k)=(2,3,1)$ and $(i,j,k)=(3,1,2)$, we can construct the new values $\{\ddot{x}_{\mathbf{b}}^{(i)}\}$ that satisfy Proposition 1, (C1) to (C4), and (C5) for all i.

The weight movements for the case of $(i,j,k)=(1,2,3)$ consist of two steps: Firstly, we make $x_{\mathsf{001}}^{(1)}=0$, and then secondly, we make $x_{\mathsf{002}}^{(1)}=0$. For the first step, we assume $x_{\mathsf{001}}^{(1)}>0$. Otherwise, we can skip to the second step directly. We now perform the following six operations:

$$\{x_{\mathsf{001}}^{(1)}, x_{\mathsf{357}}^{(1)}\} \rightsquigarrow \{x_{\mathsf{051}}^{(1)}, x_{\mathsf{3F7}}^{(1)}\}, \tag{OP3}$$

$$\{x_{\mathsf{001}}^{(1)}, x_{\mathsf{557}}^{(1)}\} \rightsquigarrow \{x_{\mathsf{051}}^{(1)}, x_{\mathsf{5F7}}^{(1)}\}, \tag{OP4}$$

$$\{x_{\mathsf{001}}^{(1)}, x_{\mathsf{F57}}^{(1)}\} \rightsquigarrow x_{\mathsf{051}}^{(1)}, \tag{OP5}$$

$$x_{\mathsf{001}}^{(1)} \overset{\Delta/p_{1\to 3}}{\rightsquigarrow} x_{\mathsf{051}}^{(1)} \quad \text{and} \quad x_{\mathsf{537}}^{(2)} \overset{\Delta/p_{2\to 3}}{\rightsquigarrow} x_{\mathsf{F37}}^{(2)} \quad \text{where } \Delta = \mathsf{min}\{x_{\mathsf{001}}^{(1)} \cdot p_{1\to 3},\ x_{\mathsf{537}}^{(2)} \cdot p_{2\to 3}\}, \tag{OP6}$$

$$x_{\mathsf{001}}^{(1)} \overset{\Delta/p_{1\to 3}}{\rightsquigarrow} x_{\mathsf{051}}^{(1)} \quad \text{and} \quad x_{\mathsf{557}}^{(2)} \overset{\Delta/p_{2\to 3}}{\rightsquigarrow} x_{\mathsf{F57}}^{(2)} \quad \text{where } \Delta = \mathsf{min}\{x_{\mathsf{001}}^{(1)} \cdot p_{1\to 3},\ x_{\mathsf{557}}^{(2)} \cdot p_{2\to 3}\}, \tag{OP7}$$

$$x_{\mathsf{001}}^{(1)} \overset{\Delta/p_{1\to 3}}{\rightsquigarrow} x_{\mathsf{051}}^{(1)} \quad \text{and} \quad x_{\mathsf{5F7}}^{(2)} \overset{\Delta/p_{2\to 3}}{\rightsquigarrow} \emptyset \quad \text{where } \Delta = \mathsf{min}\{x_{\mathsf{001}}^{(1)} \cdot p_{1\to 3},\ x_{\mathsf{5F7}}^{(2)} \cdot p_{2\to 3}\}. \tag{OP8}$$

We now argue that after these operations, (i) Proposition 1 and (C1) to (C4) still hold; and (ii) the new value of $x_{\mathsf{001}}^{(1)}$ is zero. To prove (i), we note that after the above weight movements, the time-sharing condition (17) of Proposition 1 still holds because except for the operations (OP5) and (OP8), we only "move" the weight between different frequencies while keeping the overall sum. And both (OP5) and (OP8) decrease the total sum. As a result, the time-sharing condition still holds. Moreover, since none of the coding types involved in (OP3) to (OP8) participate in any of the terms in (C1) to (C4), the conditions (C1) to (C4) still hold after these operations.

In the following, we prove that the decodability conditions (25) and (26) of Proposition 1 still hold after performing any one of the above 6 weight-moving operations. For example, we claim that the decodability conditions still hold after (OP3). For that, we first notice that

$$\begin{aligned}
\mathsf{TYPE}_{\mathsf{001}}^{(1)} \text{ in 11-bitstring} &= 0000\,0000\,001,\\
\mathsf{TYPE}_{\mathsf{357}}^{(1)} \text{ in 11-bitstring} &= 0011\,0101\,111,\\
\mathsf{TYPE}_{\mathsf{051}}^{(1)} \text{ in 11-bitstring} &= 0000\,0101\,001,\\
\mathsf{TYPE}_{\mathsf{3F7}}^{(1)} \text{ in 11-bitstring} &= 0011\,1111\,111,
\end{aligned}$$

where each bit is associated to one y-value and the associated 11 y-values are $y_1^{(2)}$ to $y_4^{(2)}$, $y_1^{(3)}$ to $y_4^{(3)}$, and $y_1^{(2,3)}$ to $y_3^{(2,3)}$ in the order of 11-bitstring, see (14). For shorthand, we denote the collection of these y-values corresponding to the first four bits, the second four bits, and the last three bits as $\vec{y}_{1-4}^{(2)}$, $\vec{y}_{1-4}^{(3)}$, and $\vec{y}_{1-3}^{(2,3)}$, respectively. Then by the same arguments as used in the proof of *Intermediate Claim* 2, one can easily prove that the 7 different y-values: $\vec{y}_{1-4}^{(2)}$ and $\vec{y}_{1-3}^{(2,3)}$, remain unchanged after (OP3). If we apply the same arguments as used in the proof of *Intermediate Claim* 2, we can also prove that all y-values in the collection $\vec{y}_{1-4}^{(3)}$ (the second four) decrease by the same amount of $\left(\mathsf{min}\{x_{\mathsf{001}}^{(1)}, x_{\mathsf{357}}^{(1)}\} \cdot p_{1\to 3}\right)$. Since other y-values were intact, the decodability equalities (25) and (26) are still satisfied after (OP3).

For the weight movement (OP4), we can prove by similar arguments that after (OP4), all $\vec{y}_{1-4}^{(2)}$ and $\vec{y}_{1-3}^{(2,3)}$ remain the same and all $\vec{y}_{1-4}^{(3)}$ decrease by the same amount of $\left(\mathsf{min}\{x_{\mathsf{001}}^{(1)}, x_{\mathsf{557}}^{(1)}\} \cdot p_{1\to 3}\right)$. Similarly, after the weight movement (OP5), all $\vec{y}_{1-4}^{(2)}$ and $\vec{y}_{1-3}^{(2,3)}$ remain the same and all $\vec{y}_{1-4}^{(3)}$ decrease by the same amount of $\left(\mathsf{min}\{x_{\mathsf{001}}^{(1)}, x_{\mathsf{F57}}^{(1)}\} \cdot p_{1\to 3}\right)$. Since other y-values were intact, the decodability equalities (25) and (26) still hold after these operations.

We now prove that after (OP6), the decodability conditions in Proposition 1 still hold. Since (OP6) involves the frequencies of different node indices $\{x_{\mathsf{001}}^{(1)}, x_{\mathsf{051}}^{(1)}, x_{\mathsf{537}}^{(2)}, x_{\mathsf{F37}}^{(2)}\}$, we first provide the following table that summarizes the contributions of these frequencies to the y-values:

	$\vec{y}_{1-4}^{(1)}$	$\vec{y}_{1-4}^{(2)}$	$\vec{y}_{1-4}^{(3)}$	$\vec{y}_{1-3}^{(1,2)}$	$\vec{y}_{1-3}^{(2,3)}$	$\vec{y}_{1-3}^{(3,1)}$
$x_{\mathsf{001}}^{(1)}$		0000	0000		001	
$x_{\mathsf{537}}^{(2)}$	0011		0101			111
$x_{\mathsf{051}}^{(1)}$		0000	0101		001	
$x_{\mathsf{F37}}^{(2)}$	0011		1111			111

TABLE II
THE CONTRIBUTIONS OF $x_{\mathsf{001}}^{(1)}$, $x_{\mathsf{537}}^{(2)}$, $x_{\mathsf{051}}^{(1)}$, AND $x_{\mathsf{F37}}^{(2)}$ TO THE y-VALUES.

For example, since $\mathsf{537} = 01010011111$ in 11-bitstring and $x_{\mathsf{537}}^{(2)}$ contributes to $\{\vec{y}_{1-4}^{(3)}, \vec{y}_{1-4}^{(1)}, \vec{y}_{1-3}^{(3,1)}\}$, we can thus list the contribution of $x_{\mathsf{537}}^{(2)}$ to all the y-values as in the second row of Table II. The first, third, and fourth rows of Table II can be populated similarly. If we compare the first and the third rows of Table II, we can see that the operation of $x_{\mathsf{001}}^{(1)} \overset{\Delta/p_{1\to 3}}{\rightsquigarrow} x_{\mathsf{051}}^{(1)}$ in (OP6) will decrease both $y_2^{(3)}$ and $y_4^{(3)}$ by the same amount Δ

while all the other 19 y-values remain the same. If we compare the second and the fourth rows of Table II, we can see that the operation of $x^{(2)}_{537} \overset{\Delta/p_{2\to 3}}{\rightsquigarrow} x^{(2)}_{\mathsf{F37}}$ will decrease both $y^{(3)}_1$ and $y^{(3)}_3$ by the same amount Δ while all the other 19 y-values remain the same. Since (OP6) performs both $x^{(1)}_{001} \overset{\Delta/p_{1\to 3}}{\rightsquigarrow} x^{(1)}_{051}$ and $x^{(2)}_{537} \overset{\Delta/p_{2\to 3}}{\rightsquigarrow} x^{(2)}_{\mathsf{F37}}$ simultaneously, in the end we will have all four values of $\vec{y}^{(3)}_{1-4}$ decrease by the same amount of Δ while the rest 17 y-values remain the same. As a result, the decodability equalities (25) and (26) of Proposition 1 are still satisfied after (OP6). Similar arguments can be used to prove that after (OP7) and (OP8), the decodability equalities of Proposition 1 still hold.

To prove (ii), we notice that after the above 6 weight movements (OP3) to (OP8), the final $\{x^{(i)}_{\mathbf{b}}\}$-values satisfy Proposition 1. Then Lemma 1 implies that (E1) to (E10) must hold. Since (C1), (C2), and (C4) are true, if we only count the coding types that may have non-zero value, then (E9) can be written as follows.

$$\left(x^{(i)}_{001} + x^{(i)}_{051}\right) \cdot p_{i\to j\vee k} = \left(x^{(i)}_{051} + x^{(i)}_{357} + x^{(i)}_{557} + x^{(i)}_{\mathsf{F57}}\right) \cdot p_{i\to k} + \left(x^{(j)}_{537} + x^{(j)}_{557} + x^{(j)}_{\mathsf{5F7}}\right) \cdot p_{j\to k}, \tag{46}$$

Eq. (46) further implies the following inequality:

$$\begin{aligned} x^{(i)}_{001} \cdot p_{i\to j\vee k} \leq & \left(x^{(i)}_{357} + x^{(i)}_{557} + x^{(i)}_{\mathsf{F57}}\right) \cdot p_{i\to k} \\ & + \left(x^{(j)}_{537} + x^{(j)}_{557} + x^{(j)}_{\mathsf{5F7}}\right) \cdot p_{j\to k}, \end{aligned} \tag{47}$$

because we always have $x^{(i)}_{051} \cdot p_{i\to j\vee k} \geq x^{(i)}_{051} \cdot p_{i\to k}$.

Then notice that after performing (OP3) to (OP8), we will have either $x^{(1)}_{001} = 0$ or the total sum $x^{(1)}_{357} + x^{(1)}_{557} + x^{(1)}_{\mathsf{F57}} + x^{(2)}_{537} + x^{(2)}_{557} + x^{(2)}_{\mathsf{5F7}} = 0$. Note that whenever the latter sum is zero, by (47) when $(i, j, k) = (1, 2, 3)$, we also have $x^{(1)}_{001} = 0$. As a result, we must have $x^{(1)}_{001} = 0$ after the above 6 weight movements.

We now present the second step, which makes $x^{(1)}_{002} = 0$. To that end, we perform the following six operations:

$$\{x^{(1)}_{002}, x^{(1)}_{337}\} \rightsquigarrow \{x^{(1)}_{302}, x^{(1)}_{\mathsf{F37}}\}, \tag{OP9}$$

$$\{x^{(1)}_{002}, x^{(1)}_{357}\} \rightsquigarrow \{x^{(1)}_{302}, x^{(1)}_{\mathsf{F57}}\}, \tag{OP10}$$

$$\{x^{(1)}_{002}, x^{(1)}_{\mathsf{3F7}}\} \rightsquigarrow x^{(1)}_{302}, \tag{OP11}$$

$$\begin{aligned} & x^{(1)}_{002} \overset{\Delta/p_{1\to 2}}{\rightsquigarrow} x^{(1)}_{302} \quad \text{and} \quad x^{(3)}_{337} \overset{\Delta/p_{3\to 2}}{\rightsquigarrow} x^{(3)}_{\mathsf{3F7}} \\ & \quad \text{where } \Delta = \mathsf{min}\{x^{(1)}_{002} \cdot p_{1\to 2},\ x^{(3)}_{337} \cdot p_{3\to 2}\}, \end{aligned} \tag{OP12}$$

$$\begin{aligned} & x^{(1)}_{002} \overset{\Delta/p_{1\to 2}}{\rightsquigarrow} x^{(1)}_{302} \quad \text{and} \quad x^{(3)}_{537} \overset{\Delta/p_{3\to 2}}{\rightsquigarrow} x^{(3)}_{\mathsf{5F7}} \\ & \quad \text{where } \Delta = \mathsf{min}\{x^{(1)}_{002} \cdot p_{1\to 2},\ x^{(3)}_{537} \cdot p_{3\to 2}\}, \end{aligned} \tag{OP13}$$

$$\begin{aligned} & x^{(1)}_{002} \overset{\Delta/p_{1\to 2}}{\rightsquigarrow} x^{(1)}_{302} \quad \text{and} \quad x^{(3)}_{\mathsf{F37}} \overset{\Delta/p_{3\to 2}}{\rightsquigarrow} \emptyset \\ & \quad \text{where } \Delta = \mathsf{min}\{x^{(1)}_{002} \cdot p_{1\to 2},\ x^{(3)}_{\mathsf{F37}} \cdot p_{3\to 2}\}. \end{aligned} \tag{OP14}$$

Again, we will prove that after these 6 weight movements, (i) Proposition 1 and (C1) to (C4) hold; and (ii) the new value of $x^{(1)}_{002}$ is zero. The proof of (i) is almost identical to that of the first step and we thus omit the detailed derivations. To prove (ii), we notice that after these weight-moving operations, the final $\{x^{(i)}_{\mathbf{b}}\}$-values still satisfy Proposition 1. Then Lemma 1 implies that (E1) to (E10) must hold. Since (C1), (C2), and (C4) are true, if we only count the coding types that may have non-zero value, then (E10) can be written as follows.

$$\left(x^{(i)}_{002} + x^{(i)}_{302}\right) \cdot p_{i\to j\vee k} = \left(x^{(i)}_{302} + x^{(i)}_{337} + x^{(i)}_{357} + x^{(i)}_{\mathsf{3F7}}\right) \cdot p_{i\to j} + \left(x^{(k)}_{337} + x^{(k)}_{537} + x^{(k)}_{\mathsf{F37}}\right) \cdot p_{k\to j},$$

which in turn implies when $(i, j, k) = (1, 2, 3)$,

$$\begin{aligned} x^{(1)}_{002} \cdot p_{1\to 2\vee 3} \leq & \left(x^{(1)}_{337} + x^{(1)}_{357} + x^{(1)}_{\mathsf{3F7}}\right) \cdot p_{1\to 2} \\ & + \left(x^{(3)}_{337} + x^{(3)}_{537} + x^{(3)}_{\mathsf{F37}}\right) \cdot p_{3\to 2}. \end{aligned} \tag{48}$$

We then observe that after the above 6 operations (OP9) to (OP14), we will have either $x^{(1)}_{002} = 0$ or $x^{(1)}_{337} + x^{(1)}_{357} + x^{(1)}_{\mathsf{3F7}} + x^{(3)}_{337} + x^{(3)}_{537} + x^{(3)}_{\mathsf{F37}} = 0$. The by (48), we must have $x^{(1)}_{002} = 0$ after the above 6 weight-moving process.

Thus far, we have proven (C5) for the case of $i = 1$ while satisfying the linear conditions of Proposition 1 and (C1) to (C4). Note that in our weight movements (OP3)–(OP8) and (OP9)–(OP14), we never increase $x^{(2)}_{001}$, $x^{(2)}_{002}$, $x^{(3)}_{001}$, and $x^{(3)}_{002}$. Therefore, we can simply apply the above 2-step procedure to the cases of $(i, j, k) = (2, 3, 1)$ and $(i, j, k) = (3, 1, 2)$, sequentially. In the end, the final $\{x^{(i)}_{\mathbf{b}}\}$-values satisfy Proposition 1 and the conditions (C1) to (C5). The proof is thus complete. $\square$

APPENDIX D
PROOF OF LEMMA 3

Given $\vec{R}$ and the reception probabilities, consider 156 non-negative values $\{\ddot{x}^{(i)}_{\mathbf{b}}\}$ such that jointly they satisfy Proposition 1 and (35). Since by (35) all the $\{\ddot{x}^{(i)}_{\mathbf{b}}\}$-values with $\mathbf{b} \in \mathsf{FTs}\backslash\overline{\mathsf{FTs}}$ are zeros, we only consider the 30 non-negative values $\{\ddot{x}^{(i)}_{\mathbf{b}}\}$ with $\mathbf{b} \in \overline{\mathsf{FTs}}$ for the ongoing discussions.

For the proof of Lemma 3, we first prove the following claim.

Claim: The above 30 non-negative values $\{\ddot{x}^{(i)}_{\mathbf{b}}\}$ for all $\mathbf{b} \in \overline{\mathsf{FTs}}$ jointly satisfy the following equalities: for all $(i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$,

$$R_{i\to j} + R_{i\to k} = \left(\ddot{x}^{(i)}_{051} + \ddot{x}^{(i)}_{302}\right) p_{i\to j\vee k}, \tag{49}$$

$$\begin{aligned} R_{i\to j} \frac{p_{i\to \overline{j}k}}{p_{i\to j\vee k}} = & \left(\ddot{x}^{(i)}_{337} + \ddot{x}^{(i)}_{357} + \ddot{x}^{(i)}_{\mathsf{3F7}}\right) \cdot p_{i\to j} \\ & + \left(\ddot{x}^{(k)}_{337} + \ddot{x}^{(k)}_{537} + \ddot{x}^{(k)}_{\mathsf{F37}}\right) \cdot p_{k\to j}, \end{aligned} \tag{50}$$

$$\begin{aligned} R_{i\to k} \frac{p_{i\to j\overline{k}}}{p_{i\to j\vee k}} = & \left(\ddot{x}^{(i)}_{357} + \ddot{x}^{(i)}_{557} + \ddot{x}^{(i)}_{\mathsf{F57}}\right) \cdot p_{i\to k} \\ & + \left(\ddot{x}^{(j)}_{537} + \ddot{x}^{(j)}_{557} + \ddot{x}^{(j)}_{\mathsf{5F7}}\right) \cdot p_{j\to k}. \end{aligned} \tag{51}$$

Proof of Claim. Since node indices are cyclically decided, we prove (49)–(51) only for the case when $(i, j, k) = (1, 2, 3)$.

That is,

$$R_{1\to 2}+R_{1\to 3}=\left(\ddot{x}^{(1)}_{\mathsf{051}}+\ddot{x}^{(1)}_{\mathsf{302}}\right)\cdot p_{1\to 2\vee 3}, \tag{52}$$

$$\begin{aligned}R_{1\to 2}\frac{p_{1\to 2\overline{3}}}{p_{1\to 2\vee 3}}&=\left(\ddot{x}^{(1)}_{\mathsf{337}}+\ddot{x}^{(1)}_{\mathsf{357}}+\ddot{x}^{(1)}_{\mathsf{3F7}}\right)\cdot p_{1\to 2}\\&\quad+\left(\ddot{x}^{(3)}_{\mathsf{337}}+\ddot{x}^{(3)}_{\mathsf{537}}+\ddot{x}^{(3)}_{\mathsf{F37}}\right)\cdot p_{3\to 2},\end{aligned} \tag{53}$$

$$\begin{aligned}R_{1\to 3}\frac{p_{1\to \overline{2}3}}{p_{1\to 2\vee 3}}&=\left(\ddot{x}^{(1)}_{\mathsf{357}}+\ddot{x}^{(1)}_{\mathsf{557}}+\ddot{x}^{(1)}_{\mathsf{F57}}\right)\cdot p_{1\to 3}\\&\quad+\left(\ddot{x}^{(2)}_{\mathsf{537}}+\ddot{x}^{(2)}_{\mathsf{557}}+\ddot{x}^{(2)}_{\mathsf{5F7}}\right)\cdot p_{2\to 3}.\end{aligned} \tag{54}$$

We now make the following observations. Since the above $\{\ddot{x}^{(i)}_{\mathbf{b}}:\forall\, i\in\{1,2,3\}$ and $\mathbf{b}\in\overline{\mathsf{FTs}}\}$ satisfy Proposition 1, Lemma 1 implies that they satisfies (E1) as well. We then note that (52) is a direct result of the equality (E1) of Lemma 1 when $(i,j,k)=(2,3,1)$.

We now use the equalities (E2) and (E3) when $(i,j,k)=(2,3,1)$. Since type-$\mathsf{051}$ (resp. type-$\mathsf{302}$) is the only coding type in $\overline{\mathsf{FTs}}$ with $b_{10}=0$ (resp. $b_{11}=0$), we thus have, respectively,

$$R_{1\to 3}=\ddot{x}^{(1)}_{\mathsf{051}}\cdot p_{1\to 2\vee 3}, \tag{55}$$

$$R_{1\to 2}=\ddot{x}^{(1)}_{\mathsf{302}}\cdot p_{1\to 2\vee 3}. \tag{56}$$

Then, (53) can be derived as follows. From the equality (E9) when $(i,j,k)=(2,3,1)$, we have

$$\begin{aligned}\ddot{x}^{(1)}_{\mathsf{302}}\cdot p_{1\to 2\vee 3}&=\left(\ddot{x}^{(3)}_{\mathsf{337}}+\ddot{x}^{(3)}_{\mathsf{537}}+\ddot{x}^{(3)}_{\mathsf{F37}}\right)\cdot p_{3\to 2}\\&\quad+\left(\ddot{x}^{(1)}_{\mathsf{302}}+\ddot{x}^{(1)}_{\mathsf{337}}+\ddot{x}^{(1)}_{\mathsf{357}}+\ddot{x}^{(1)}_{\mathsf{3F7}}\right)\cdot p_{1\to 2}.\end{aligned}$$

By simple probability manipulation, the above equality is equivalent to

$$\begin{aligned}\ddot{x}^{(1)}_{\mathsf{302}}\cdot p_{1\to\overline{2}3}&=\left(\ddot{x}^{(1)}_{\mathsf{337}}+\ddot{x}^{(1)}_{\mathsf{357}}+\ddot{x}^{(1)}_{\mathsf{3F7}}\right)\cdot p_{1\to 2}\\&\quad+\left(\ddot{x}^{(3)}_{\mathsf{337}}+\ddot{x}^{(3)}_{\mathsf{537}}+\ddot{x}^{(3)}_{\mathsf{F37}}\right)\cdot p_{3\to 2}.\end{aligned} \tag{57}$$

Then (53) is derived by substituting $\ddot{x}^{(1)}_{\mathsf{302}}=R_{1\to 2}/p_{1\to 2\vee 3}$ (see (56) again) on the LHS of (57).

Similarly, one can derive (54) by using (55) and the equality (E10) when $(i,j,k)=(2,3,1)$. The claim is thus proven.. □

Using the above claim, we will prove Lemma 3 by explicitly constructing $t_i^{[\mathrm{u}]}$ and $t_i^{[\mathrm{c},1]}$ to $t_i^{[\mathrm{c},4]}$ values as follows.

$$t_i^{[\mathrm{u}]}=\ddot{x}^{(i)}_{\mathsf{051}}+\ddot{x}^{(i)}_{\mathsf{302}}, \tag{58}$$

$$t_i^{[\mathrm{c},1]}=\ddot{x}^{(i)}_{\mathsf{357}}+\ddot{x}^{(i)}_{\mathsf{3F7}}, \tag{59}$$

$$t_i^{[\mathrm{c},2]}=\ddot{x}^{(i)}_{\mathsf{537}}+\ddot{x}^{(i)}_{\mathsf{5F7}}, \tag{60}$$

$$t_i^{[\mathrm{c},3]}=\ddot{x}^{(i)}_{\mathsf{337}}+\ddot{x}^{(i)}_{\mathsf{F37}}, \tag{61}$$

$$t_i^{[\mathrm{c},4]}=\ddot{x}^{(i)}_{\mathsf{557}}+\ddot{x}^{(i)}_{\mathsf{F57}}. \tag{62}$$

In the following, we prove that the above $\{t_i\}$-values satisfy the linear conditions of Proposition 2 (when $<$ being replaced by $\leq$).

Since the $\{\ddot{x}^{(i)}_{\mathbf{b}}\}$-values satisfy the time-sharing condition (17) of Proposition 1, the $\{t_i\}$-values in the above construction also satisfy the time-sharing condition (29).

By (49) and (58), we have

$$R_{i\to j}+R_{i\to k}=t_i^{[\mathrm{u}]}\cdot p_{i\to j\vee k},$$

which implies (30).

We now show that our construction also satisfies (31) and (32). By our construction (59)–(62), the followings are always true: for all $i\in\{1,2,3\}$,

$$\begin{aligned}\left(\ddot{x}^{(i)}_{\mathsf{337}}+\ddot{x}^{(i)}_{\mathsf{357}}+\ddot{x}^{(i)}_{\mathsf{3F7}}\right)&\leq\left(t_i^{[\mathrm{c},1]}+t_i^{[\mathrm{c},3]}\right),\\\left(\ddot{x}^{(i)}_{\mathsf{337}}+\ddot{x}^{(i)}_{\mathsf{537}}+\ddot{x}^{(i)}_{\mathsf{F37}}\right)&\leq\left(t_i^{[\mathrm{c},2]}+t_i^{[\mathrm{c},3]}\right),\\\left(\ddot{x}^{(i)}_{\mathsf{357}}+\ddot{x}^{(i)}_{\mathsf{557}}+\ddot{x}^{(i)}_{\mathsf{F57}}\right)&\leq\left(t_i^{[\mathrm{c},1]}+t_i^{[\mathrm{c},4]}\right),\\\left(\ddot{x}^{(j)}_{\mathsf{537}}+\ddot{x}^{(j)}_{\mathsf{557}}+\ddot{x}^{(j)}_{\mathsf{5F7}}\right)&\leq\left(t_i^{[\mathrm{c},2]}+t_i^{[\mathrm{c},4]}\right).\end{aligned}$$

Since we have already shown that (50) and (51) are true, one can easily verify by direct substitutions that (31) and (32) are satisfied as well. The proof of Lemma 3 is thus complete.