Graph-Theoretic Characterization of The Feasibility of The Precoding-Based 3-Unicast Interference Alignment Scheme

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***Abstract*—A new precoding-based intersession network coding (NC) scheme has recently been proposed, which applies the interference alignment technique, originally devised for wireless interference channels, to the 3-unicast problem of directed acyclic networks. The main result of this work is a graph-theoretic characterization of the feasibility of the 3-unicast interference alignment scheme. To that end, we first investigate several key relationships between the point-to-point network channel gains and the underlying graph structure. Such relationships turn out to be critical when characterizing graph-theoretically the feasibility of precoding-based solutions.**

***Index Terms*—Asymptotic interference alignment, interference channels, intersession network coding, 3-unicast networks.**

I. Introduction

Deciding whether there exists a *network code* [1] that satisfies the network traffic demands has been a long-standing open challenge when there are multiple coexisting source-destination pairs (sessions) in the network. For the degenerate case in which there is only one multicast session in the network, also termed the *single-multicast* setting, it is known that linear network coding [10] is capable of achieving the information-theoretic capacity. Several papers have since studied the network code construction problem for the single multicast setting [3], [7], [8], [15].

On the other hand, when there are multiple coexisting sessions in the network, the corresponding network code design/analysis problem, also known as the *intersession network coding* (INC) problem, becomes highly challenging due to the potential interference within the network. For example, *linear network coding* no longer achieves the capacity [5]. Deciding the existence of a (linear) network code that satisfies general traffic demands becomes an NP-hard problem [8], [9]. Thus, recent INC studies have focused on the optimal characterizations over some special networks or under some restrictive rate constraints. The results along this direction include the *index coding* problem [14], finding the capacity regions for *directed cycles* [6], degree-2 three-layer *directed acyclic networks* (DAG) [18], *node-constrained* line and star networks [19], and the 1-hop broadcast *packet erasure channel* with feedback [16], and for networks with integer link capacity and two coexisting rate-1 multicast sessions [17].

Recently, the authors in [4], [13] have applied interference alignment (IA), originally developed for wireless interference channels [2], to the scenario of 3 coexisting unicast sessions called 3-unicast Asymptotic Network Alignment (ANA). The concept of interference alignment leads to a new perspective on INC problems. Namely, the network designer focuses on designing the *precoding* and *decoding mappings* at the source and destination node while allowing randomly generated *local encoding kernels* [7] within the network. Compared to the classic algebraic framework that fully controls the encoder, decoder, and local encoding kernels [8], this precoding-based framework trades off the ultimate achievable throughput with a distributed, implementation-friendly structure that exploits pure random linear NC in the interior of the network. Their initial study on 3-unicast networks shows that, under certain network topology and traffic demand, the precoding-based NC can perform better than the pure routing (non-coding) solution and a few widely-used simple linear NC solutions. Such results strike a new balance between practicality and throughput enhancement.

This work, motivated by its practical advantages over the classic network coding framework, focuses exclusively on the precoding-based framework and characterizes its corresponding properties. We then use the newly developed results to analyze the 3-unicast ANA scheme proposed in [4], [13]. Specifically, the existing results [4], [13] show that the 3-unicast ANA scheme achieves asymptotically half of the interference-free throughput for each transmission pair when a set of algebraic conditions on the *channel gains* of the networks are satisfied. Note that for the case of *wireless interference channels*, these algebraic feasibility conditions can be satisfied with close-to-one probability provided the channel gains are independently and continuously distributed random variables [2]. For comparison, the "network channel gains" are usually highly correlated[1] discrete random variables and thus the algebraic channel gain conditions do not always hold with close-to-one probability. Moreover, except for some very simple networks, checking whether the algebraic channel gain conditions hold turns out to be computationally prohibitive. As a result, we need new and efficient ways to decide whether

[1]The correlation depends heavily on the underlying network topology.

the network of interest admits a 3-unicast ANA scheme that achieves half of the interference-free throughput. The results in this work answer this question by developing new graph-theoretic conditions that characterize the feasibility of the 3-unicast ANA scheme. The proposed graph-theoretic conditions can be easily computed and checked within polynomial time.

The key contributions of this work are:

- We formulate the precoding-based framework and identify several fundamental properties (Propositions 1 to 3), which allow us to bridge the gap between the algebraic feasibility of the precoding-based NC problem and the underlying network topology.
- Using these relationships, we characterize the graph-theoretic conditions for the feasibility of the 3-unicast ANA scheme.

The rest of the paper is organized as follows: Section II introduces some useful graph-theoretic definitions, and compares the algebraic formulation of the proposed precoding-based framework to that of the classic NC framework [8]. In addition, the 3-unicast ANA scheme proposed by [4], [13] is introduced in the context of the precoding-based framework. Section II also discusses its algebraic feasibility conditions and the graph-theoretic conjectures proposed in [13]. Section III identifies several key properties of the precoding-based framework and provides the corresponding proofs. Based on the new fundamental properties, in Section IV we develop the graph-theoretic necessary and sufficient conditions for the feasibility of the 3-unicast ANA scheme. Section V concludes this work.

II. Precoding-Based Intersession NC

A. System Model and Some Graph-Theoretic Definitions

Consider a DAG $G=(V,E)$ where V is the set of nodes and E is the set of directed edges. Each edge $e\in E$ is represented by $e=uv$, where $u=\mathsf{tail}(e)$ and $v=\mathsf{head}(e)$ are the tail and head of e, respectively. For any node $v\in V$, we use $\mathsf{In}(v)\subset E$ to denote the collection of its incoming edges $uv\in E$. Similarly, $\mathsf{Out}(v)\subset E$ contains all the outgoing edges $vw\in E$.

A path P is a series of adjacent edges $e_1e_2\cdots e_k$ where $\mathsf{head}(e_i)=\mathsf{tail}(e_{i+1})\,\forall\, i\in\{1,\cdots,k-1\}$. We say that e_1 and e_k are the starting and ending edges of P, respectively. For any path P, we use $e\in P$ to indicate that an edge e is used by P. For a given path P, xPy denotes the path segment of P from node x to node y. A path starting from node x and ending at node y is sometimes denoted by P_{xy}. By slightly abusing the notation, we sometimes substitute the nodes x and y by the edges e_1 and e_2 and use e_1Pe_2 to denote the path segment from $\mathsf{tail}(e_1)$ to $\mathsf{head}(e_2)$ along P. Similarly, $P_{e_1e_2}$ denotes a path from $\mathsf{tail}(e_1)$ to $\mathsf{head}(e_2)$. We say a node u is an *upstream* node of a node v (or v is a *downstream* node of u) if $u\neq v$ and there exists a path P_{uv}, and we denote it as $u\prec v$. If neither $u\prec v$ nor $u\succ v$, then we say that u and v are *not reachable* from each other. Similarly, e_1 is an upstream edge of e_2 if $\mathsf{head}(e_1)\preceq\mathsf{tail}(e_2)$ (where $\preceq$ means either $\mathsf{head}(e_1)\prec\mathsf{tail}(e_2)$ or $\mathsf{head}(e_1)=\mathsf{tail}(e_2)$), and we denote it by $e_1\prec e_2$. Two distinct edges e_1 and e_2 are not reachable from each other, if neither $e_1\prec e_2$ nor $e_1\succ e_2$. Given any edge set E_1, we say an edge e is one of the most upstream edges in E_1 if (i) $e\in E_1$; and (ii) e is not reachable from any other edge $e'\in E_1\backslash e$. One can easily see that the most upstream edge may not be unique. The collection of the most upstream edges of E_1 is denoted by $\mathsf{upstr}(E_1)$. A *k-edge cut* (sometimes just the "edge cut") separating node sets $U\subset V$ and $W\subset V$ is a collection of k edges such that any path from any $u\in U$ to any $w\in W$ must use at least one of those k edges. The value of an edge cut is the number of edges in the cut. (A k-edge cut has value k.) We denote the minimum value among all the edge cuts separating U and W as $\mathsf{EC}(U;W)$. By definition, we have $\mathsf{EC}(U;W)=0$ when U and W are already disconnected. By convention, if $U\cap W\neq\emptyset$, we define $\mathsf{EC}(U;W)=\infty$. We also denote the collection of all distinct 1-edge cuts separating U and W as $\mathsf{1cut}(U;W)$.

B. The Algebraic Framework of Linear Network Coding

Given a network $G=(V,E)$, we consider the multiple-unicast problem in which there are K coexisting source-destination pairs (s_k,d_k), $k=1,\cdots,K$. Let l_k denote the number of information symbols that s_k wants to transmit to d_k. Each information symbol is chosen independently and uniformly from a finite field $\mathbb{F}_q$ with some sufficiently large q.

Following the widely-used instantaneous transmission model for DAGs [8], we assume that each edge is capable of transmitting one symbol in $\mathbb{F}_q$ in one time slot without delay. We consider *linear network coding* over the entire network, i.e., a symbol on an edge $e\in E$ is a linear combination of the symbols on its adjacent incoming edges $\mathsf{In}(\mathsf{tail}(e))$. The coefficients (also known as the network variables) used for such linear combinations are termed local encoding kernels. The collection of all local kernels $x_{e'e''}\in\mathbb{F}_q$ for all adjacent edge pairs (e',e'') is denoted by $\underline{\mathbf{x}}=\{x_{e'e''}:(e',e'')\in E^2 \text{ where } \mathsf{head}(e')=\mathsf{tail}(e'')\}$. See [8] for detailed discussion. Following this notation, the channel gain $m_{e_1;e_2}(\underline{\mathbf{x}})$ from an edge e_1 to an edge e_2 can be written as a polynomial with respect to $\underline{\mathbf{x}}$. More rigorously, $m_{e_1;e_2}(\underline{\mathbf{x}})$ can be rewritten as

$$m_{e_1;e_2}(\underline{\mathbf{x}})=\sum_{\forall P_{e_1e_2}\in\mathbf{P}_{e_1e_2}}\left(\prod_{\substack{\forall\, e',\,e''\in P_{e_1e_2}\\ \text{where } \mathsf{head}(e')=\mathsf{tail}(e'')}} x_{e'e''}\right)$$

where $\mathbf{P}_{e_1e_2}$ denotes the collection of all distinct paths from e_1 to e_2.

By convention [8], we set $m_{e_1;e_2}(\underline{\mathbf{x}})=1$ when $e_1=e_2$ and set $m_{e_1;e_2}(\underline{\mathbf{x}})=0$ when $e_1\neq e_2$ and e_2 is not a downstream edge of e_1. The channel gain from a node u to a node v is defined by an $|\mathsf{In}(v)|\times|\mathsf{Out}(u)|$ polynomial matrix $\mathbf{M}_{u;v}(\underline{\mathbf{x}})$, where its (i,j)-th entry is the (edge-to-edge) channel gain from the j-th outgoing edge of u to the i-th incoming edge of v. When considering source s_i and destination d_j, we use $\mathbf{M}_{i;j}(\underline{\mathbf{x}})$ as shorthand for $\mathbf{M}_{s_i;d_j}(\underline{\mathbf{x}})$.

We allow the precoding-based NC to code across τ number of time slots, which are termed the precoding frame and τ is the frame size. The network variables used in time slot t is denoted as $\underline{\mathbf{x}}^{(t)}$, and the corresponding channel gain from s_i to d_j becomes $\mathbf{M}_{i;j}(\underline{\mathbf{x}}^{(t)})$ for all $t=1,\cdots,\tau$.

With these settings, let $\mathbf{z}_i\in\mathbb{F}_q^{l_i\times 1}$ be the set of to-be-sent information symbols from s_i. Then, for every time slot

$t = 1, \cdots, \tau$, we can define the precoding matrix $\mathbf{V}_i^{(t)} \in \mathbb{F}_q^{|\mathsf{Out}(s_i)| \times l_i}$ for each source s_i. Given the precoding matrices, each d_j receives an $|\mathsf{In}(d_j)|$-dimensional column vector $\mathbf{y}_j^{(t)}$ at time t:

$$\mathbf{y}_j^{(t)}(\underline{\mathbf{x}}^{(t)}) = \mathbf{M}_{j;j}(\underline{\mathbf{x}}^{(t)})\mathbf{V}_j^{(t)}\mathbf{z}_j + \sum_{\substack{i=1 \\ i \neq j}}^{K} \mathbf{M}_{i;j}(\underline{\mathbf{x}}^{(t)})\mathbf{V}_i^{(t)}\mathbf{z}_i.$$

where we use the input argument "$(\underline{\mathbf{x}}^{(t)})$" to emphasize that $\mathbf{M}_{j;j}$ and $\mathbf{y}_j^{(t)}$ are functions of the network variables $\underline{\mathbf{x}}^{(t)}$.

This system model can be equivalently expressed as

$$\overline{\mathbf{y}}_j = \overline{\mathbf{M}}_{j;j}\overline{\mathbf{V}}_j\,\mathbf{z}_j + \sum_{\substack{i=1 \\ i \neq j}}^{K} \overline{\mathbf{M}}_{i;j}\overline{\mathbf{V}}_i\,\mathbf{z}_i, \tag{1}$$

where $\overline{\mathbf{V}}_i$ is the overall precoding matrix for each source s_i by vertically concatenating $\{\mathbf{V}_i^{(t)}\}_{t=1}^{\tau}$, and $\overline{\mathbf{y}}_j$ is the vertical concatenation of $\{\mathbf{y}_j^{(t)}(\underline{\mathbf{x}}^{(t)})\}_{t=1}^{\tau}$. The overall channel matrix $\overline{\mathbf{M}}_{i;j}$ is a block-diagonal polynomial matrix with $\{\mathbf{M}_{i;j}(\underline{\mathbf{x}}^{(t)})\}_{t=1}^{\tau}$ as its diagonal blocks. Note that $\overline{\mathbf{M}}_{i;j}$ is a polynomial matrix with respect to the network variables $\{\underline{\mathbf{x}}^{(t)}\}_{t=1}^{\tau}$.

After receiving packets for τ time slots, each destination d_j applies the overall decoding matrix $\overline{\mathbf{U}}_j \in \mathbb{F}_q^{l_j \times (\tau \cdot |\mathsf{In}(d_j)|)}$. Then, the decoded message vector $\hat{\mathbf{z}}_j$ can be expressed as

$$\hat{\mathbf{z}}_j = \overline{\mathbf{U}}_j\overline{\mathbf{y}}_j = \overline{\mathbf{U}}_j\overline{\mathbf{M}}_{j;j}\overline{\mathbf{V}}_j\,\mathbf{z}_j + \sum_{\substack{i=1 \\ i \neq j}}^{K} \overline{\mathbf{U}}_j\overline{\mathbf{M}}_{i;j}\overline{\mathbf{V}}_i\,\mathbf{z}_i. \tag{2}$$

The combined effects of precoding, channel, and decoding from s_i to d_j is $\overline{\mathbf{U}}_j\overline{\mathbf{M}}_{i;j}\overline{\mathbf{V}}_i$, which is termed the *network transfer matrix* from s_i to d_j. We say that the precoding-based NC problem is feasible if there exists a pair of encoding and decoding matrices $\{\overline{\mathbf{V}}_i, \forall\, i\}$ and $\{\overline{\mathbf{U}}_j, \forall\, j\}$ (which may be a function of $\{\underline{\mathbf{x}}^{(t)}\}_{t=1}^{\tau}$) such that when choosing each element of the collection of network variables $\{\underline{\mathbf{x}}^{(t)}\}_{t=1}^{\tau}$ independently and uniformly randomly from $\mathbb{F}_q$, with high probability,

$$\begin{aligned} &\overline{\mathbf{U}}_j\overline{\mathbf{M}}_{i;j}\overline{\mathbf{V}}_i = \mathbf{I} \quad \text{(the identity matrix)} \quad \forall\, i = j, \\ &\overline{\mathbf{U}}_j\overline{\mathbf{M}}_{i;j}\overline{\mathbf{V}}_i = \mathbf{0} \quad \forall\, i \neq j. \end{aligned} \tag{3}$$

Remark 1: One can easily check by the cut-set bound that a necessary condition for the feasibility of a precoding-based NC problem is for the frame size $\tau \geq \max_k\{l_k/\mathsf{EC}(s_k; d_k)\}$.

Remark 2: Depending on the time relationship of $\overline{\mathbf{V}}_i$ and $\overline{\mathbf{U}}_j$ with respect to the network variables $\{\underline{\mathbf{x}}^{(t)}\}_{t=1}^{\tau}$, a precoding-based NC solution can be classified as causal vs. non-causal and time-varying vs. time-invariant schemes.

For convenience to the reader, we have summarized in Table I several key definitions used in the precoding-based framework.

Notations for the precoding-based framework

K	The number of coexisting unicast sessions
l_i	The number of information symbols sent from s_i to d_i
$\underline{\mathbf{x}}$	The network variables / local encoding kernels
$m_{e_1;e_2}(\underline{\mathbf{x}})$	The channel gain from an edge e_1 to an edge e_2, which is a polynomial with respect to $\underline{\mathbf{x}}$
$\mathbf{M}_{u;v}(\underline{\mathbf{x}})$	The channel gain matrix from a node u to a node v where its (i, j)-th entry is the channel gain from j-th outgoing edge of u to i-th incoming edge of v
τ	The precoding frame size (number of time slot)
$\underline{\mathbf{x}}^{(t)}$	The network variables corresponding to time slot t
$\mathbf{V}_i^{(t)}$	The precoding matrix for s_i at time slot t
$\mathbf{M}_{i;j}(\underline{\mathbf{x}}^{(t)})$	The channel gain matrix from s_i to d_j at time slot t, shorthand for $\mathbf{M}_{s_i;d_j}(\underline{\mathbf{x}}^{(t)})$
$\mathbf{U}_j^{(t)}$	The decoding matrix for d_j at time slot t
$\overline{\mathbf{V}}_i$	The overall precoding matrix for s_i for the entire precoding frame $t = 1, \cdots, \tau$.
$\overline{\mathbf{M}}_{i;j}$	The overall channel gain matrix from s_i to d_j for the entire precoding frame $t = 1, \cdots, \tau$.
$\overline{\mathbf{U}}_j$	The overall decoding matrix for d_j for the entire precoding frame $t = 1, \cdots, \tau$.

Notations for the 3-unicast ANA network

$m_{ij}(\underline{\mathbf{x}})$	The channel gain from s_i to d_j
$L(\underline{\mathbf{x}})$	The product of three channel gains: $m_{13}(\underline{\mathbf{x}})m_{32}(\underline{\mathbf{x}})m_{21}(\underline{\mathbf{x}})$
$R(\underline{\mathbf{x}})$	The product of three channel gains: $m_{12}(\underline{\mathbf{x}})m_{23}(\underline{\mathbf{x}})m_{31}(\underline{\mathbf{x}})$

TABLE I
KEY DEFINITIONS OF THE PRECODING-BASED FRAMEWORK

C. Comparison to the Existing Linear NC Framework

The authors in [8] established the algebraic framework for linear network coding, which admits similar encoding and decoding equations as in (1) and (2) and the same algebraic feasibility conditions as in (3). This original work focuses on a single time slot $\tau = 1$ while the corresponding results can be easily generalized for $\tau > 1$ as well. Note that $\tau > 1$ provides a greater degree of freedom when designing the coding matrices $\{\overline{\mathbf{V}}_i, \forall\, i\}$ and $\{\overline{\mathbf{U}}_j, \forall\, j\}$. Such *time extension* turns out to be especially critical in a precoding-based NC design as it is generally much harder (sometimes impossible) to design $\{\overline{\mathbf{V}}_i, \forall\, i\}$ and $\{\overline{\mathbf{U}}_j, \forall\, j\}$ when $\tau = 1$. An example of this time extension will be discussed in Section II-D.

The main difference between the precoding-based framework and the classic framework is that the latter allows the NC designer to control the network variables $\underline{\mathbf{x}}$ while the former assumes that the entries of $\underline{\mathbf{x}}$ are chosen independently and uniformly randomly. One can thus view the precoding-based NC as a distributed version of classic NC schemes that trades off the ultimate achievable performance for more practical distributed implementation (not controlling the behavior in the interior of the network).

One challenge when using algebraic feasibility conditions (3) is that given a network code, it is easy to verify whether or not (3) is satisfied, but it is difficult to decide whether there exists a NC solution satisfying (3), see [8], [9]. Only in some special scenarios can we convert those algebraic conditions into some graph-theoretic conditions for which one can decide the existence of a feasible network code in polynomial time. For example, if there exists only a single session (s_1, d_1) in the network, then the existence of a NC solution satisfying (3) is equivalent to the time-averaged rate l_1/τ being no larger than $\mathsf{EC}(s_1; d_1)$. Moreover, if $(l_1/\tau) \leq \mathsf{EC}(s_1; d_1)$, then we can use random linear network coding [7] to construct the optimal network code. Another example is when there are only two sessions (s_1, d_1) and (s_2, d_2) with $l_1 = l_2 = \tau = 1$. Then, the existence of a network code satisfying (3) is equivalent to the conditions that the 1-edge cuts in the network are properly placed in certain ways [17]. Motivated by the above observation, the main focus of this work is to develop new graph-theoretic conditions for a special scenario of the precoding-based NC, the 3-unicast Asymptotic Network Alignment (ANA) scheme, which will be introduced in the next subsection.

D. The 3-Unicast Asymptotic Network Alignment (ANA) Scheme

Before proceeding, we introduce some algebraic definitions. We say that a set of polynomials $\mathbf{h}(\underline{\mathbf{x}})=\{h_1(\underline{\mathbf{x}}), ..., h_N(\underline{\mathbf{x}})\}$ is linearly dependent if and only if $\sum_{k=1}^{N} \alpha_k h_k(\underline{\mathbf{x}})=0$ for some coefficients $\{\alpha_k\}_{k=1}^{N}$ that are not all zeros. By treating $\mathbf{h}(\underline{\mathbf{x}}^{(k)})$ as a polynomial row vector and vertically concatenating them together, we have an $M\times N$ polynomial matrix $[\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^{M}$. We call this polynomial matrix a *row-invariant* matrix since each row is based on the same set of polynomials $\mathbf{h}(\underline{\mathbf{x}})$ but with different variables $\underline{\mathbf{x}}^{(k)}$ for each row k, respectively. We say that the row-invariant polynomial matrix $[\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^{M}$ is generated from $\mathbf{h}(\underline{\mathbf{x}})$. For two polynomials $g(\underline{\mathbf{x}})$ and $h(\underline{\mathbf{x}})$, we say $g(\underline{\mathbf{x}})$ and $h(\underline{\mathbf{x}})$ are *equivalent*, denoted by $g(\underline{\mathbf{x}})\equiv h(\underline{\mathbf{x}})$, if $g(\underline{\mathbf{x}})= c\cdot h(\underline{\mathbf{x}})$ for some non-zero $c\in\mathbb{F}_q$. If not, we say that $g(\underline{\mathbf{x}})$ and $h(\underline{\mathbf{x}})$ are *not equivalent*, denoted by $g(\underline{\mathbf{x}})\not\equiv h(\underline{\mathbf{x}})$. We use $\mathsf{GCD}(\,g(\underline{\mathbf{x}}), h(\underline{\mathbf{x}}))$ to denote the greatest common factor of the two polynomials.

We now consider a special class of networks, called the 3-unicast ANA network: A network G is a 3-unicast ANA network if (i) there are 3 source-destination pairs, $(s_i, d_i), i= 1, 2, 3$, where all source/destination nodes are distinct; (ii) $|\mathsf{In}(s_i)|=0$ and $|\mathsf{Out}(s_i)|=1\ \forall\, i$ (We denote the only outgoing edge of s_i as e_{s_i}, termed the s_i-source edge.); (iii) $|\mathsf{In}(d_j)|=1$ and $|\mathsf{Out}(d_j)|=0\ \forall\, j$ (We denote the only incoming edge of d_j as e_{d_j}, termed the d_j-destination edge.); and (iv) d_j can be reached from s_i for all (i, j) pairs (including those with $i=j$).[2] We use the notation G_{3ANA} to emphasize that we are focusing on this 3-unicast ANA network. Note that by (ii) and (iii) the matrix $\mathbf{M}_{i;j}(\underline{\mathbf{x}})$ becomes a scalar, which we denote by $m_{ij}(\underline{\mathbf{x}})$ instead.

The authors in [4], [13] applied interference alignment to construct the precoding matrices $\{\overline{\mathbf{V}}_i, \forall\, i\}$ for the above 3-unicast ANA network. Namely, consider the following parameter values: $\tau = 2n+1$, $l_1 = n + 1$, $l_2 = n$, and $l_3 = n$ for some positive integer n termed symbol extension parameter, and assume that all the network variables $\underline{\mathbf{x}}^{(1)}$ to $\underline{\mathbf{x}}^{(\tau)}$ are chosen independently and uniformly randomly from $\mathbb{F}_q$. The goal is to achieve the rate tuple $(\frac{n+1}{2n+1}, \frac{n}{2n+1}, \frac{n}{2n+1})$ in a 3-unicast ANA network by applying the following $\{\overline{\mathbf{V}}_i, \forall\, i\}$ construction method: Define $L(\underline{\mathbf{x}}) = m_{13}(\underline{\mathbf{x}})m_{32}(\underline{\mathbf{x}})m_{21}(\underline{\mathbf{x}})$ and $R(\underline{\mathbf{x}}) = m_{12}(\underline{\mathbf{x}})m_{23}(\underline{\mathbf{x}})m_{31}(\underline{\mathbf{x}})$, and consider the following 3 row vectors of dimensions $n+1$, n, and n, respectively. (Each entry of these row vectors is a polynomial with respect to $\underline{\mathbf{x}}$ but we drop the input argument $\underline{\mathbf{x}}$ for simplicity.)

$$\mathbf{v}_1^{(n)}(\underline{\mathbf{x}}) = m_{23}m_{32}\left[R^n,\ R^{n-1}L,\ \cdots,\ RL^{n-1},\ L^n\right], \quad (4)$$
$$\mathbf{v}_2^{(n)}(\underline{\mathbf{x}}) = m_{13}m_{32}\left[R^n,\ R^{n-1}L,\ \cdots,\ RL^{n-1}\right], \quad (5)$$
$$\mathbf{v}_3^{(n)}(\underline{\mathbf{x}}) = m_{12}m_{23}\left[R^{n-1}L,\ \cdots,\ RL^{n-1},\ L^n\right], \quad (6)$$

where the superscript "(n)" is to emphasize the value of the symbol extension parameter n used in the construction. The precoding matrix for each time slot t is designed to be $\mathbf{V}_i^{(t)} = \mathbf{v}_i^{(n)}(\underline{\mathbf{x}}^{(t)})$. The overall precoding matrix (the vertical concatenation of $\mathbf{V}_i^{(1)}$ to $\mathbf{V}_i^{(\tau)}$) is thus $\overline{\mathbf{V}}_i = [\mathbf{v}_i^{(n)}(\underline{\mathbf{x}}^{(t)})]_{t=1}^{2n+1}$.

The authors in [4], [13] prove that the above construction achieves the desired rates $(\frac{n+1}{2n+1}, \frac{n}{2n+1}, \frac{n}{2n+1})$ if the overall precoding matrices $\{\overline{\mathbf{V}}_i, \forall\, i\}$ satisfy the following six constraints:[3]

$$d_1 : \langle\, \overline{\mathbf{M}}_{3;1}\overline{\mathbf{V}}_3\,\rangle = \langle\, \overline{\mathbf{M}}_{2;1}\overline{\mathbf{V}}_2\,\rangle \quad (7)$$
$$\mathbf{S}_1^{(n)} \triangleq \left[\,\overline{\mathbf{M}}_{1;1}\overline{\mathbf{V}}_1\ \ \overline{\mathbf{M}}_{2;1}\overline{\mathbf{V}}_2\,\right], \text{ and } \mathsf{rank}(\mathbf{S}_1^{(n)})=2n+1 \quad (8)$$
$$d_2 : \langle\, \overline{\mathbf{M}}_{3;2}\overline{\mathbf{V}}_3\,\rangle \subseteq \langle\, \overline{\mathbf{M}}_{1;2}\overline{\mathbf{V}}_1\,\rangle \quad (9)$$
$$\mathbf{S}_2^{(n)} \triangleq \left[\,\overline{\mathbf{M}}_{2;2}\overline{\mathbf{V}}_2\ \ \overline{\mathbf{M}}_{1;2}\overline{\mathbf{V}}_1\,\right], \text{ and } \mathsf{rank}(\mathbf{S}_2^{(n)})=2n+1 \quad (10)$$
$$d_3 : \langle\, \overline{\mathbf{M}}_{2;3}\overline{\mathbf{V}}_2\,\rangle \subseteq \langle\, \overline{\mathbf{M}}_{1;3}\overline{\mathbf{V}}_1\,\rangle \quad (11)$$
$$\mathbf{S}_3^{(n)} \triangleq \left[\,\overline{\mathbf{M}}_{3;3}\overline{\mathbf{V}}_3\ \ \overline{\mathbf{M}}_{1;3}\overline{\mathbf{V}}_1\,\right], \text{ and } \mathsf{rank}(\mathbf{S}_3^{(n)})=2n+1 \quad (12)$$

with close-to-one probability, where $\langle\, \mathbf{A}\,\rangle$ and $\mathsf{rank}(\mathbf{A})$ denote the column vector space and the rank, respectively, of a given matrix $\mathbf{A}$. The overall channel matrix $\overline{\mathbf{M}}_{i;j}$ is a $(2n+1)\times(2n+1)$ diagonal matrix with the t-th diagonal element $m_{ij}(\underline{\mathbf{x}}^{(t)})$ due to the assumption of $|\mathsf{Out}(s_i)| = |\mathsf{In}(d_j)| = 1$. We also note that the construction in (8), (10), and (12) ensures that the square matrices $\{\mathbf{S}_i^{(n)}, \forall\, i\}$ are row-invariant.

The intuition behind (7) to (12) is straightforward. Whenever (7) is satisfied, the interference from s_2 and from s_3 are aligned from the perspective of d_1. Further, by simple linear algebra we must have $\mathsf{rank}(\overline{\mathbf{M}}_{2;1}\overline{\mathbf{V}}_2) \leq n$ and $\mathsf{rank}(\overline{\mathbf{M}}_{1;1}\overline{\mathbf{V}}_1) \leq n + 1$. (8) thus guarantees that (i) the rank of $\left[\,\overline{\mathbf{M}}_{1;1}\overline{\mathbf{V}}_1\ \ \overline{\mathbf{M}}_{2;1}\overline{\mathbf{V}}_2\,\right]$ equals to $\mathsf{rank}(\overline{\mathbf{M}}_{1;1}\overline{\mathbf{V}}_1) + \mathsf{rank}(\overline{\mathbf{M}}_{2;1}\overline{\mathbf{V}}_2)$ and (ii) $\mathsf{rank}(\overline{\mathbf{M}}_{1;1}\overline{\mathbf{V}}_1) = n+1$. Jointly (i) and (ii) imply that d_1 can successfully remove the aligned interference while recovering all $l_1=n+1$ information symbols intended for d_1. Similar arguments can be used to justify (9) to (12) from the perspectives of d_2 and d_3, respectively.

By noticing the special Vandermonde form of $\overline{\mathbf{V}}_i$, it is shown in [4], [13] that (7), (9), and (11) always hold. The authors in [13] further prove that if

$$L(\underline{\mathbf{x}}) \not\equiv R(\underline{\mathbf{x}}) \quad (13)$$

and the following algebraic conditions are satisfied:

$$m_{11}m_{23}\sum_{i=0}^{n}\alpha_i\big(L/R\big)^i \neq m_{21}m_{13}\sum_{j=0}^{n-1}\beta_j\big(L/R\big)^j \quad (14)$$

[2]The above *fully interfered* setting is the worst case scenario. For the scenario in which there is some d_j who is not reachable from some s_i, one can devise an achievable solution by modifying the solution for the worst-case fully interfered 3-ANA networks [4].

[3]Here the interference alignment is performed based on (s_1, d_1)-pair who achieves larger rate than others. Basically, any transmission pair can be chosen as an alignment-basis achieving $\frac{n+1}{2n+1}$, and the corresponding precoding matrices and six constraints can be constructed accordingly.

$$m_{22}m_{13}\sum_{i=0}^{n-1}\alpha_i\big(L/R\big)^i \neq m_{12}m_{23}\sum_{j=0}^{n}\beta_j\big(L/R\big)^j \qquad (15)$$

$$m_{33}m_{12}\sum_{i=1}^{n}\alpha_i\big(L/R\big)^i \neq m_{13}m_{32}\sum_{j=0}^{n}\beta_j\big(L/R\big)^j \qquad (16)$$

for all $\alpha_i, \beta_j \in \mathbb{F}_q$ with at least one of α_i and at least one of β_j being non-zero, then the constraints (8), (10), and (12) hold with close-to-one probability (recalling that the network variables $\underline{\mathbf{x}}^{(1)}$ to $\underline{\mathbf{x}}^{(\tau)}$ are chosen independently and uniformly randomly).

In summary, [4], [13] proves the following result.

Proposition (page 3, [13]): For a sufficiently large finite field $\mathbb{F}_q$, the 3-unicast ANA scheme described in (4) to (6) achieves the rate tuple $(\frac{n+1}{2n+1}, \frac{n}{2n+1}, \frac{n}{2n+1})$ with close-to-one probability if (13), (14), (15), and (16) hold simultaneously.

It can be easily seen that directly verifying the above sufficient conditions is computationally intractable. The following conjecture is thus proposed in [13] to reduce the computational complexity when using the above proposition.

Conjecture (Page 3, [13]): For any n value used in the 3-unicast ANA scheme construction, if (13) and the following three conditions are satisfied simultaneously, then (14) to (16) must hold.

$$m_{11}m_{23} \not\equiv m_{21}m_{13} \quad \text{and} \quad m_{11}m_{32} \not\equiv m_{31}m_{12}, \qquad (17)$$

$$m_{22}m_{13} \not\equiv m_{12}m_{23} \quad \text{and} \quad m_{22}m_{31} \not\equiv m_{32}m_{21}, \qquad (18)$$

$$m_{33}m_{12} \not\equiv m_{13}m_{32} \quad \text{and} \quad m_{33}m_{21} \not\equiv m_{23}m_{31}. \qquad (19)$$

Note that even if the conjecture is true, checking whether (13), (17)–(19) are satisfied is still highly non-trivial for large networks. Moreover, recent results in [11] showed that the above conjecture is false. The main contribution of this work is to work on the original conditions (13)–(16) directly and provide an easily verifiable graph-theoretic characterization that supersedes their original algebraic forms.

Remark: In the setting of wireless interference channels, the individual channel gains are independently and continuously distributed, for which one can prove that the feasibility conditions (13), (8), (10), and (12) hold with probability one [2]. For a network setting, the channel gains $m_{i;j}(\underline{\mathbf{x}})$ are no longer independently distributed for different (i, j) pairs and the correlation depends on the underlying network topology. For example, one can verify that the 3-unicast ANA network described in Fig. 1 always leads to $L(\underline{\mathbf{x}}) \equiv R(\underline{\mathbf{x}})$ even when all network variables $\underline{\mathbf{x}}$ are chosen uniformly randomly from an arbitrarily large finite field $\mathbb{F}_q$.

III. Properties of The Precoding-Based Framework

In this section, we characterize a few fundamental relationships between the channel gains and the underlying DAG G, which bridge the gap between the algebraic feasibility of the precoding-based NC problem and the underlying network structure. These properties hold for any precoding-based schemes and can be of benefit to future development of any precoding-based solution. These newly discovered results will later be used to prove the graph-theoretic characterizations of the 3-unicast ANA scheme. In Sections III-A to III-C we state

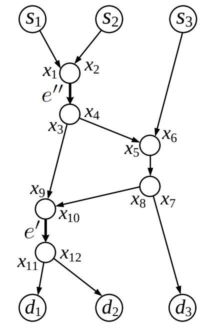

Fig. 1. Example $G_{3\text{ANA}}$ structure satisfying $L(\underline{\mathbf{x}}) \equiv R(\underline{\mathbf{x}})$ with $\underline{\mathbf{x}} = \{x_1, x_2, ..., x_{12}\}$.

Propositions 1 to 3, respectively. In Section III-D, we discuss how these results can be applied to the existing 3-unicast ANA scheme.

A. From Non-Zero Determinant to Linear Independence

Proposition 1: Fix an arbitrary value of N. Consider any set of N polynomials $\mathbf{h}(\underline{\mathbf{x}}) = \{h_1(\underline{\mathbf{x}}), ..., h_N(\underline{\mathbf{x}})\}$ and the polynomial matrix $[\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^N$ generated from $\mathbf{h}(\underline{\mathbf{x}})$. Then, assuming sufficiently large finite field size q, $\det([\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^N)$ is non-zero polynomial if and only if $\mathbf{h}(\underline{\mathbf{x}})$ is linearly independent.

The proof of Proposition 1 is relegated to Appendix A.

Remark: Suppose a sufficiently large finite field $\mathbb{F}_q$ is used. If we choose the variables $\underline{\mathbf{x}}^{(1)}$ to $\underline{\mathbf{x}}^{(N)}$ independently and uniformly randomly from $\mathbb{F}_q$, by Schwartz-Zippel lemma, we have $\det([\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^N) \neq 0$ with close-to-one probability if and only if $\mathbf{h}(\underline{\mathbf{x}})$ is linearly independent.

The implication of Proposition 1 is as follows. Similar to the seminal work [8], most algebraic characterization of the precoding-based framework involves checking whether or not a determinant is non-zero. For example, the first feasibility condition of (3) is equivalent to checking whether or not the determinant of the network transfer matrix is non-zero. Also, (8), (10), and (12) are equivalent to checking whether or not the determinant of the row-invariant matrix $\mathbf{S}_i^{(n)}$ is non-zero. Proposition 1 says that as long as we can formulate the corresponding matrix in a row-invariant form, then checking whether the determinant is non-zero is equivalent to checking whether the corresponding set of polynomials is linearly independent. As will be shown shortly after, the latter task admits more tractable analysis.

B. The Subgraph Property of the Precoding-Based Framework

Consider a DAG G and recall the definition of the channel gain $m_{e_1;e_2}(\underline{\mathbf{x}})$ from e_1 to e_2 in Section II-B. For a subgraph $G' \subseteq G$ containing e_1 and e_2, let $m_{e_1;e_2}(\underline{\mathbf{x}}')$ denote the channel gain from e_1 to e_2 in G'.

Proposition 2 (Subgraph Property): Given a DAG G, consider an arbitrary, but fixed, finite collection of edge pairs, $\{(e_i, e_i') \in E^2 : i \in I\}$ where I is a finite index set, and consider two arbitrary polynomial functions $f : \mathbb{F}_q^{|I|} \mapsto \mathbb{F}_q$ and $g : \mathbb{F}_q^{|I|} \mapsto \mathbb{F}_q$. Then, $f(\{m_{e_i;e_i'}(\underline{\mathbf{x}}) : \forall\, i \in I\}) \equiv g(\{m_{e_i;e_i'}(\underline{\mathbf{x}}) : \forall\, i \in I\})$ if and only if for all subgraphs $G' \subseteq G$ containing all edges in $\{e_i, e'_i : \forall i \in I\}$, $f(\{m_{e_i;e'_i}(\underline{\mathbf{x}}') : \forall i \in I\}) \equiv g(\{m_{e_i;e'_i}(\underline{\mathbf{x}}') : \forall i \in I\})$.

The proof of Proposition 2 is relegated to Appendix A.

Remark: Proposition 2 has a similar flavor to the classic results [8] and [7]. More specifically, for the single multicast setting from a source s to the destinations $\{d_j\}$, the transfer matrix $\mathbf{U}_{d_j}\mathbf{M}_{d_j;s}(\underline{\mathbf{x}})\mathbf{V}_s$ from s to d_j is of full rank (i.e., the polynomial $\det(\mathbf{U}_{d_j}\mathbf{M}_{d_j;s}(\underline{\mathbf{x}})\mathbf{V}_s)$ is non-zero in the original graph G) is equivalent to the existence of a subgraph G' (usually being chosen as the subgraph induced by a set of edge-disjoint paths from s to d_j) satisfying the polynomial $\det(\mathbf{U}_{d_j}\mathbf{M}_{d_j;s}(\underline{\mathbf{x}}')\mathbf{V}_s)$ being non-zero.

Compared to Proposition 1, Proposition 2 further connects the linear dependence of the polynomials to the subgraph properties of the underlying network. For example, to prove that a set of polynomials over a given arbitrary network is linearly independent, we only need to construct a (much smaller) subgraph and prove that the corresponding set of polynomials is linearly independent.

C. The Channel Gain Property

Both Propositions 1 and 2 have a similar flavor to the classic results of the LNC framework [8]. The following channel gain property, on the other hand, is unique to the precoding-based framework.

Proposition 3 (The Channel Gain Property): Consider a DAG G and two distinct edges e_s and e_d. For notational simplicity, we denote $\mathsf{head}(e_s)$ by s and denote $\mathsf{tail}(e_d)$ by d. Then, the following statements must hold (we drop the variables $\underline{\mathbf{x}}$ for shorthand):

- If $\mathsf{EC}(s;d) = 0$, then $m_{e_s;e_d} = 0$
- If $\mathsf{EC}(s;d) = 1$, then $m_{e_s;e_d}$ is reducible. Moreover, let $N \triangleq |\mathsf{1cut}(s;d)|$ denote the number of 1-edge cuts separating s and d, and we sort the 1-edge cuts by their topological order with e_1 being the most upstream and e_N being the most downstream. The channel gain $m_{e_s;e_d}$ can now be expressed as $m_{e_s;e_d} = m_{e_s;e_1}\left(\prod_{i=1}^{N-1} m_{e_i;e_{i+1}}\right) m_{e_N;e_d}$ and all the polynomial factors $m_{e_s;e_1}$, $\{m_{e_i;e_{i+1}}\}_{i=1}^{N-1}$, and $m_{e_N;e_d}$ are irreducible, and no two of them are equivalent.
- If $\mathsf{EC}(s;d) \geq 2$ (including ∞), then $m_{e_s;e_d}$ is irreducible.

The proof of Proposition 3 is relegated to Appendix C.

Remark: Proposition 3 only considers a channel gain between two distinct edges. If $e_s = e_d$, then by convention [8], we have $m_{e_s;e_d} = 1$.

Proposition 3 relates the factoring problem of the channel gain polynomial to the graph-theoretic edge cut property. As will be shown afterwards, this observation enables us to tightly connect the algebraic and graph-theoretic conditions for the precoding-based solutions.

D. Application of The Properties of The Precoding-based Framework to The 3-unicast ANA Schene

In this subsection, we discuss how the properties of the precoding-based framework, Propositions 1 to 3, can benefit our understanding of the 3-unicast ANA scheme.

Proposition 1 enables us to simplify the feasibility characterization of the 3-unicast ANA scheme in the following way. From the construction in Section II-D, the square matrix $\mathbf{S}_i^{(n)}$ can be written as a row-invariant matrix $\mathbf{S}_i^{(n)} = [\mathbf{h}_i^{(n)}(\underline{\mathbf{x}}^{(t)})]_{t=1}^{(2n+1)}$ for some set of polynomials $\mathbf{h}_i(\underline{\mathbf{x}})$. For example, by (4), (5), and (8) we have $\mathbf{S}_1^{(n)} = [\mathbf{h}_1^{(n)}(\underline{\mathbf{x}}^{(t)})]_{t=1}^{(2n+1)}$ where

$$\begin{aligned}\mathbf{h}_1^{(n)}(\underline{\mathbf{x}}) = \{\, & m_{11}m_{23}m_{32}R^n,\ m_{11}m_{23}m_{32}R^{n-1}L, \\ & \cdots,\ m_{11}m_{23}m_{32}L^n,\ m_{21}m_{13}m_{32}R^n, \\ & m_{21}m_{13}m_{32}R^{n-1}L,\ \cdots,\ m_{21}m_{13}m_{32}RL^{n-1}\,\}.\end{aligned} \tag{20}$$

Proposition 1 implies that (8) being true is equivalent to the set of polynomials $\mathbf{h}_1^{(n)}(\underline{\mathbf{x}})$ is linearly independent. Assuming the $G_{3\text{ANA}}$ of interest satisfies (13), $\mathbf{h}_1^{(n)}(\underline{\mathbf{x}})$ being linearly independent is equivalent to (14) being true. As a result, (14) is not only sufficient but also necessary for (8) to hold with close-to-one probability. By similar arguments (15) (resp. (16)) is both necessary and sufficient for (10) (resp. (12)) to hold with high probability.

Proposition 2 enables us to find the graph-theoretic equivalent counterparts of (14)–(16) of the *Conjecture* (p. 3, [13]).

Corollary 1 (First stated in [13]): Consider a $G_{3\text{ANA}}$ and four indices i_1, i_2, j_1, and j_2 satisfying $i_1 \neq i_2$ and $j_1 \neq j_2$. We have $\mathsf{EC}(\{s_{i_1}, s_{i_2}\}; \{d_{j_1}, d_{j_2}\}) = 1$ if and only if $m_{i_1j_1}m_{i_2j_2} \equiv m_{i_2j_1}m_{i_1j_2}$.

The main intuition behind Corollary 1 is as follows. When $\mathsf{EC}(\{s_{i_1}, s_{i_2}\}; \{d_{j_1}, d_{j_2}\}) = 1$, one can show that we must have $m_{i_1j_1}(\underline{\mathbf{x}})m_{i_2j_2}(\underline{\mathbf{x}}) = m_{i_2j_1}(\underline{\mathbf{x}})m_{i_1j_2}(\underline{\mathbf{x}})$ by analyzing the underlying graph structure. When $\mathsf{EC}(\{s_{i_1}, s_{i_2}\}; \{d_{j_1}, d_{j_2}\}) \neq 1$, we can construct a subgraph G' satisfying $m_{i_1j_1}(\underline{\mathbf{x}}')m_{i_2j_2}(\underline{\mathbf{x}}') \not\equiv m_{i_2j_1}(\underline{\mathbf{x}}')m_{i_1j_2}(\underline{\mathbf{x}}')$. Proposition 2 thus implies $m_{i_1j_1}(\underline{\mathbf{x}})m_{i_2j_2}(\underline{\mathbf{x}}) \not\equiv m_{i_2j_1}(\underline{\mathbf{x}})m_{i_1j_2}(\underline{\mathbf{x}})$. A detailed proof of Corollary 1 is relegated to Appendix B.

Proposition 3 can be used to derive the following corollary, which studies the relationship of the channel polynomials m_{ij}.

Corollary 2: Given a $G_{3\text{ANA}}$, consider a source s_i to destination d_j channel gain m_{ij}. Then, $\mathsf{GCD}(m_{i_1j_1}, m_{i_2j_2}) \equiv m_{i_2j_2}$ if and only if $(i_1, j_1) = (i_2, j_2)$. Intuitively, any channel gain $m_{i_1j_1}$ from source s_{i_1} to destination d_{j_1} cannot contain another source-destination channel gain $m_{i_2j_2}$ as its factor.

The intuition behind Corollary 2 is as follows. For example, suppose we actually have $\mathsf{GCD}(m_{11}, m_{12}) \equiv m_{12}$ and assume that $\mathsf{EC}(\mathsf{head}(e_{s_1}); \mathsf{tail}(e_{d_2})) \geq 2$. Then we must have the d_2-destination edge e_{d_2} being an edge cut separating s_1 and d_1. The reason is that (i) Proposition 3 implies that any irreducible factor of the channel gain m_{11} corresponds to the channel gain between two consecutive 1-edge cuts separating s_1 and d_1; and (ii) The assumption $\mathsf{EC}(\mathsf{head}(e_{s_1}); \mathsf{tail}(e_{d_2})) \geq 2$ implies that m_{12} is irreducible. Thus (i), (ii), and $\mathsf{GCD}(m_{11}, m_{12}) \equiv m_{12}$ together imply that $e_{d_2} \in \mathsf{1cut}(s_1; d_1)$. This, however, contradicts the assumption of $|\mathsf{Out}(d_2)| = 0$ for any 3-unicast ANA network $G_{3\text{ANA}}$. The detailed proof of Corollary 2, which studies more general case in which $\mathsf{EC}(\mathsf{head}(e_{s_1}); \mathsf{tail}(e_{d_2})) = 1$, is relegated to Appendix B.

IV. Detailed Studies of The 3-unicast ANA Scheme

In Section III, we investigated the basic relationships between the channel gain polynomials and the underlying DAG

G for arbitrary precoding-based solutions. In this section, we turn our attention to a specific precoding-based solution, the 3-unicast ANA scheme, and characterize graph-theoretically its feasibility conditions.

A. New Graph-Theoretic Notations and The Corresponding Properties

We begin by defining some new notations. Consider three indices i, j, and k in $\{1,2,3\}$ satisfying $j \neq k$ but i may or may not be equal to j (resp. k). Given a G_{3ANA}, define:

$$\overline{S}_{i;\{j,k\}} \triangleq \mathsf{1cut}(s_i;d_j) \cap \mathsf{1cut}(s_i;d_k) \backslash \{e_{s_i}\}$$
$$\overline{D}_{i;\{j,k\}} \triangleq \mathsf{1cut}(s_j;d_i) \cap \mathsf{1cut}(s_k;d_i) \backslash \{e_{d_i}\}$$

as the 1-edge cuts separating s_i and $\{d_j, d_k\}$ minus the s_i-source edge e_{s_i} and the 1-edge cuts separating $\{s_j, s_k\}$ and d_i minus the d_i-destination edge e_{d_i}. When the values of indices are all distinct, we use $\overline{S}_i$ (resp. $\overline{D}_i$) as shorthand for $\overline{S}_{i;\{j,k\}}$ (resp. $\overline{D}_{i;\{j,k\}}$). The following lemmas prove some topological relationships between the edge sets $\overline{S}_i$ and $\overline{D}_j$ and the corresponding proofs are relegated to Appendix D.

Lemma 1: For all $i \neq j$, $e' \in \overline{S}_i$, and $e'' \in \overline{D}_j$, one of the following three statements is true: $e' \prec e''$, $e' \succ e''$, or $e' = e''$.

Lemma 2: For any distinct i, j, and k in $\{1,2,3\}$, we have $(\overline{D}_i \cap \overline{D}_j) \subset \overline{S}_k$.

Lemma 3: For all $i \neq j$, $e' \in \overline{S}_i \backslash \overline{D}_j$, and $e'' \in \overline{D}_j$, we have $e' \prec e''$.

Lemma 4: For any distinct i, j, and k in $\{1,2,3\}$, $\overline{D}_j \cap \overline{D}_k \neq \emptyset$ if and only if both $\overline{S}_i \cap \overline{D}_j \neq \emptyset$ and $\overline{S}_i \cap \overline{D}_k \neq \emptyset$.

Lemma 5: For all $i \neq j$ and $e'' \in \overline{D}_i \cap \overline{D}_j$, if $\overline{S}_i \cap \overline{S}_j \neq \emptyset$, then there exists $e' \in \overline{S}_i \cap \overline{S}_j$ such that $e' \preceq e''$.

Lemma 6: Consider four indices i, j_1, j_2, and j_3 taking values in $\{1,2,3\}$ for which the values of j_1, j_2 and j_3 must be distinct and i is equal to one of j_1, j_2 and j_3. If $\overline{S}_{i;\{j_1,j_2\}} \neq \emptyset$ and $\overline{S}_{i;\{j_1,j_3\}} \neq \emptyset$, then the following three statements are true: (i) $\overline{S}_{i;\{j_1,j_2\}} \cap \overline{S}_{i;\{j_1,j_3\}} \neq \emptyset$; (ii) $\overline{S}_{i;\{j_2,j_3\}} \neq \emptyset$; and (iii) $\overline{S}_i \neq \emptyset$.

Remark: All the above lemmas are purely graph-theoretic. If we swap the roles of sources and destinations, then we can also derive the (s,d)-*symmetric version* of these lemmas. For example, the (s,d)-symmetric version of Lemma 2 becomes $(\overline{S}_i \cap \overline{S}_j) \subseteq \overline{D}_k$. The (s,d)-symmetric version of Lemma 5 is: For all $i \neq j$ and $e'' \in \overline{S}_i \cap \overline{S}_j$, if $\overline{D}_i \cap \overline{D}_j \neq \emptyset$, then there exists $e' \in \overline{D}_i \cap \overline{D}_j$ such that $e' \succeq e''$.

Lemmas 1 to 6 discuss the topological relationship between the edge sets $\overline{S}_i$ and $\overline{D}_j$. The following lemma establishes the relationship between $\overline{S}_i$ (resp. $\overline{D}_j$) and the channel gains.

Lemma 7: Given a G_{3ANA}, consider the corresponding channel gains as defined in Section II-D. Consider three indices i, j_1, and j_2 taking values in $\{1,2,3\}$ for which the values of j_1 and j_2 must be distinct. Then, $\mathsf{GCD}(m_{ij_1}, m_{ij_2}) \equiv 1$ if and only if $\overline{S}_{i;\{j_1,j_2\}} = \emptyset$. Symmetrically, $\mathsf{GCD}(m_{j_1 i}, m_{j_2 i}) \equiv 1$ if and only if $\overline{D}_{i;\{j_1,j_2\}} = \emptyset$.

The proof of Lemma 7 is relegated to Appendix D.

B. The Graph-Theoretic Characterization of $L(\underline{\mathbf{x}}) \not\equiv R(\underline{\mathbf{x}})$

A critical condition of the 3-unicast ANA scheme [4], [13] is the assumption that $L(\underline{\mathbf{x}}) \not\equiv R(\underline{\mathbf{x}})$, which is the fundamental reason why the Vandermonde precoding matrix $\overline{\mathbf{V}}_i$ is of full (column) rank. However, for some networks we may have $L(\underline{\mathbf{x}}) \equiv R(\underline{\mathbf{x}})$, for which the 3-unicast ANA scheme does not work (see Fig. 1). Next, we prove the following graph-theoretic condition that fully characterizes whether $L(\underline{\mathbf{x}}) \equiv R(\underline{\mathbf{x}})$.

Proposition 4: For a given G_{3ANA}, we have $L(\underline{\mathbf{x}}) \equiv R(\underline{\mathbf{x}})$ if and only if there exists a pair of distinct indices $i, j \in \{1,2,3\}$ satisfying both $\overline{S}_i \cap \overline{S}_j \neq \emptyset$ and $\overline{D}_i \cap \overline{D}_j \neq \emptyset$.

Proof of the "$\Leftarrow$" direction. Without loss of generality, suppose $\overline{S}_1 \cap \overline{S}_2 \neq \emptyset$ and $\overline{D}_1 \cap \overline{D}_2 \neq \emptyset$ (i.e., $i=1$ and $j=2$). By Lemma 5, we can find two edges $e' \in \overline{S}_1 \cap \overline{S}_2$ and $e'' \in \overline{D}_1 \cap \overline{D}_2$ such that $e' \preceq e''$. Also note that Lemma 2 and its (s,d)-symmetric version imply that $e' \in \overline{D}_3$ and $e'' \in \overline{S}_3$. Then by Proposition 3, the channel gains $m_{ij}(\underline{\mathbf{x}})$ for all $i \neq j$ can be expressed by (we omit the variables $\underline{\mathbf{x}}$ for simplicity):

$$m_{13} = m_{e_{s_1};e'}\, m_{e';e_{d_3}} \qquad m_{12} = m_{e_{s_1};e'}\, m_{e';e''}\, m_{e'';e_{d_2}}$$
$$m_{32} = m_{e_{s_3};e''}\, m_{e'';e_{d_2}} \qquad m_{23} = m_{e_{s_2};e'}\, m_{e';e_{d_3}}$$
$$m_{21} = m_{e_{s_2};e'}\, m_{e';e''}\, m_{e'';e_{d_1}} \qquad m_{31} = m_{e_{s_3};e''}\, m_{e'';e_{d_1}}$$

where the expressions of m_{12} and m_{21} are derived based on the facts that $e' \preceq e''$ and $\{e', e''\} \subset \mathsf{1cut}(s_1;d_2) \cap \mathsf{1cut}(s_2;d_1)$. By plugging in the above 6 equalities to the definitions of $L = m_{13}m_{32}m_{21}$ and $R = m_{12}m_{23}m_{31}$, we can easily verify that $L \equiv R$. The proof of this direction is complete. $\square$

Remark: In the example of Fig. 1, one can easily see that $e' \in \overline{S}_1 \cap \overline{S}_2$ and $e'' \in \overline{D}_1 \cap \overline{D}_2$. Hence, the above proof shows that the example network in Fig. 1 satisfies $L(\underline{\mathbf{x}}) \equiv R(\underline{\mathbf{x}})$ without actually computing the polynomials $L(\underline{\mathbf{x}})$ and $R(\underline{\mathbf{x}})$.

We will now focus on proving the necessity. Before proceeding, we state and prove the following lemma.

Lemma 8: If the G_{3ANA} of interest satisfies $L(\underline{\mathbf{x}}) \equiv R(\underline{\mathbf{x}})$, then $\overline{S}_i \neq \emptyset$ and $\overline{D}_j \neq \emptyset$ for all i and j, respectively.

Proof. We prove this by contradiction. Suppose $\overline{S}_1 = \emptyset$. Denote the most upstream 1-edge cut separating $\mathsf{head}(e_{s_1})$ and d_2 by e_{12} (we have at least the d_2-destination edge e_{d_2}). Also denote the most upstream 1-edge cut separating $\mathsf{head}(e_{s_1})$ and d_3 by e_{13} (we have at least the d_3-destination edge e_{d_3}). Since $\overline{S}_1 = \emptyset$ and by the definition of the 3-unicast ANA network, it is obvious that $e_{12} \neq e_{13}$. Moreover, both of the two polynomials $m_{e_{s_1};e_{12}}$ (a factor of m_{12}) and $m_{e_{s_1};e_{13}}$ (a factor of m_{13}) are irreducible and non-equivalent to each other. Therefore, these two polynomials are coprime. If we plug in the two polynomials into $L(\underline{\mathbf{x}}) \equiv R(\underline{\mathbf{x}})$, then it means that one of the following three cases must be true: (i) $m_{e_{13};e_{d_3}}$ contains $m_{e_{s_1};e_{12}}$ as a factor; (ii) m_{32} contains $m_{e_{s_1};e_{12}}$ as a factor; or (iii) m_{21} contains $m_{e_{s_1};e_{12}}$ as a factor. However, (i), (ii), and (iii) cannot be true as $|\mathsf{In}(s_1)| = 0$ and by Proposition 3. The proof is thus complete by applying symmetry. $\square$

Proof of the "$\Rightarrow$" direction of Proposition 4. Suppose the G_{3ANA} of interest satisfies $L(\underline{\mathbf{x}}) \equiv R(\underline{\mathbf{x}})$. By Lemma 8, we know that $\overline{S}_i \neq \emptyset$ and $\overline{D}_j \neq \emptyset$ for all i and j. Then it is obvious that $\mathsf{EC}(\mathsf{head}(e_{s_i}); \mathsf{tail}(e_{d_j})) = 1$ for all $i \neq j$ because if (for example) $\mathsf{EC}(\mathsf{head}(e_{s_1}); \mathsf{tail}(e_{d_2})) \geq 2$ then both $\overline{S}_1$ and $\overline{D}_2$ will be empty by definition. Thus by Proposition 3, we can express each channel gain m_{ij} $(i \neq j)$ as a product of irreducibles, each corresponding to the channel gain between two consecutive 1-edge cuts (including e_{s_i} and e_{d_j}) separating s_i and d_j. We now consider two cases.

Case 1: $\overline{S}_i \cap \overline{D}_j = \emptyset$ for some $i \neq j$. Assume without loss of generality that $\overline{S}_2 \cap \overline{D}_1 = \emptyset$ (i.e., $i=2$ and $j=1$). Let e_2^* denote the most downstream edge in $\overline{S}_2$ and let e_1^* denote the most upstream edge in $\overline{D}_1$. Since $\overline{S}_2 \cap \overline{D}_1 = \emptyset$, the edge e_2^* must not be in $\overline{D}_1$. By Lemma 3, we have $e_2^* \prec e_1^*$.

For the following, we will prove $\{e_2^*, e_1^*\} \subset 1\mathsf{cut}(s_1; d_2)$. We first notice that by definition, $e_2^* \in \overline{S}_2 \subset 1\mathsf{cut}(s_2; d_1)$ and $e_1^* \in \overline{D}_1 \subset 1\mathsf{cut}(s_2; d_1)$. Hence by Proposition 3, we can express m_{21} as $m_{21} = m_{e_{s_2}; e_2^*} m_{e_2^*; e_1^*} m_{e_1^*; e_{d_1}}$. Note that by our construction $e_2^* \prec e_1^*$ we have $m_{e_2^*; e_1^*} \not\equiv 1$.

We now claim $\mathsf{GCD}(\, m_{e_2^*; e_1^*},\, m_{23} m_{31}) \equiv 1$, i.e., $m_{23} m_{31}$ cannot contain any factor of $m_{e_2^*; e_1^*}$. We will prove this claim by contradiction. Suppose $\mathsf{GCD}(\, m_{e_2^*; e_1^*},\, m_{23}) \not\equiv 1$, i.e., m_{23} contains an irreducible factor of $m_{e_2^*; e_1^*}$. Since that factor is also a factor of m_{21}, by Proposition 3, there must exist at least one edge e satisfying (i) $e_2^* \prec e \preceq e_1^*$; and (ii) $e \in 1\mathsf{cut}(s_2; d_1) \cap 1\mathsf{cut}(s_2; d_3)$. These jointly implies that we have an $\overline{S}_2$ edge in the downstream of e_2^*. This, however, contradicts the assumption that e_2^* is the most downstream edge of $\overline{S}_2$. By a symmetric argument, we can also show that m_{31} must not contain any irreducible factor of $m_{e_2^*; e_1^*}$. The proof of the claim $\mathsf{GCD}(\, m_{e_2^*; e_1^*},\, m_{23} m_{31}) \equiv 1$ is complete. Since the assumption $L(\underline{\mathbf{x}}) \equiv R(\underline{\mathbf{x}})$ implies that $\mathsf{GCD}(\, m_{e_2^*; e_1^*},\, R) = m_{e_2^*; e_1^*}$, we must have $\mathsf{GCD}(\, m_{e_2^*; e_1^*},\, m_{12}) = m_{e_2^*; e_1^*}$. This implies by Proposition 3 that $\{e_2^*, e_1^*\} \subset 1\mathsf{cut}(s_1; d_2)$.

For the following, we will prove that $e_2^* \in 1\mathsf{cut}(s_1; d_3)$. To that end, we consider the factor $m_{e_2^*; e_{d_3}}$ of the channel gain m_{23}. This is possible by Proposition 3 because $e_2^* \in \overline{S}_2 \subset 1\mathsf{cut}(s_2; d_3)$. Then similarly following the above discussion, we must have $\mathsf{GCD}(\, m_{21},\, m_{e_2^*; e_{d_3}}) \equiv 1$ otherwise there will be an $\overline{S}_2$ edge in the downstream of e_2^*. Since the assumption $L(\underline{\mathbf{x}}) \equiv R(\underline{\mathbf{x}})$ means that $\mathsf{GCD}(\, L,\, m_{e_2^*; e_{d_3}}) = m_{e_2^*; e_{d_3}}$, this further implies that $\mathsf{GCD}(\, m_{13} m_{32},\, m_{e_2^*; e_{d_3}}) = m_{e_2^*; e_{d_3}}$.

Now consider the most upstream $1\mathsf{cut}(s_2; d_3)$ edge that is in the downstream of e_2^*, and denote it as e_u (we have at least the d_3-destination edge e_{d_3}). Obviously, $e_2^* \prec e_u \preceq e_{d_3}$ and $m_{e_2^*; e_u}$ is an irreducible factor of $m_{e_2^*; e_{d_3}}$. Then we must have $\mathsf{GCD}(\, m_{32},\, m_{e_2^*; e_u}) \equiv 1$ and the reason is as follows. If not, then by $m_{e_2^*; e_u}$ being irreducible we have $e_2^* \in 1\mathsf{cut}(s_3; d_2)$. Then every path from s_3 to $\mathsf{tail}(e_1^*)$ must use e_2^*, otherwise s_3 can reach e_1^* without using e_2^* and finally arrive at d_2 since e_1^* can reach d_2 (we showed in the above discussion that $e_1^* \in 1\mathsf{cut}(s_1; d_2)$). This contradicts the previously constructed $e_2^* \in 1\mathsf{cut}(s_3; d_2)$. Therefore, we must have $e_2^* \in 1\mathsf{cut}(s_3; \mathsf{tail}(e_1^*))$. Since $e_1^* \in \overline{D}_1 \subset 1\mathsf{cut}(s_3; d_1)$, this in turn implies that e_2^* is also an 1-edge cut separating s_3 and d_1. However, note by the assumption that $e_2^* \in \overline{S}_2 \subset 1\mathsf{cut}(s_2; d_1)$. Thus, e_2^* will belong to $\overline{D}_1$, which contradicts the assumption that e_1^* is the most upstream $\overline{D}_1$ edge. We thus have proven $\mathsf{GCD}(\, m_{32},\, m_{e_2^*; e_u}) \equiv 1$. Since we showed that $\mathsf{GCD}(\, m_{13} m_{32},\, m_{e_2^*; e_{d_3}}) = m_{e_2^*; e_{d_3}}$, this further implies that the irreducible factor $m_{e_2^*; e_u}$ of $m_{e_2^*; e_{d_3}}$ must be contained by m_{13} as a factor. Therefore, we have proven that $e_2^* \in 1\mathsf{cut}(s_1; d_3)$. Symmetrically applying the above argument using the factor $m_{e_{s_3}; e_1^*}$ of the channel gain m_{31}, we can also prove that $e_1^* \in 1\mathsf{cut}(s_3; d_2)$.

Thus far, we have proven that $e_2^* \in 1\mathsf{cut}(s_1; d_2)$ and $e_2^* \in 1\mathsf{cut}(s_1; d_3)$. However, $e_2^* = e_{s_1}$ is not possible since e_2^*, by our construction, is a downstream edge of e_{s_2} but e_{s_1} is not (since $|\mathsf{In}(s_1)| = 0$). As a result, we have proven $e_2^* \in \overline{S}_1$. Recall that e_2^* was chosen as one edge in $\overline{S}_2$. Therefore, $\overline{S}_1 \cap \overline{S}_2 \neq \emptyset$. Similarly, we can also prove that $e_1^* \in \overline{D}_1 \cap \overline{D}_2$ and thus $\overline{D}_1 \cap \overline{D}_2 \neq \emptyset$. The proof of **Case 1** is complete.

Case 2: $\overline{S}_i \cap \overline{D}_j \neq \emptyset$ for all $i \neq j$. By Lemma 4 and its (s,d)-symmetric version, we must have $\overline{S}_i \cap \overline{S}_j \neq \emptyset$ and $\overline{D}_i \cap \overline{D}_j \neq \emptyset$ $\forall\, i \neq j$. The proof of **Case 2** is complete. $\square$

C. The Graph-Theoretic Conditions of the Feasibility of the 3-unicast ANA Scheme

Proposition 4 provides the graph-theoretic condition that characterizes whether or not the $G_{3\text{ANA}}$ of interest satisfies the algebraic condition of (13), which implies that (7), (9), and (11) hold simultaneously with close-to-one probability. However, to further ensure the feasibility of the 3-unicast ANA scheme, $\det(\mathbf{S}_i^{(n)})$ must be non-zero polynomial (see (8), (10), and (12)) for all $i \in \{1,2,3\}$. As a result, we need to prove the graph-theoretic characterization for the inequalities $\det(\mathbf{S}_i^{(n)}) \neq 0$. Note by Proposition 1 that the condition $\det(\mathbf{S}_i^{(n)}) \neq 0$ is equivalent to for all $i \in \{1,2,3\}$ the set of polynomials $\mathbf{h}_i^{(n)}(\underline{\mathbf{x}})$ is linearly independent, where $\mathbf{h}_1^{(n)}(\underline{\mathbf{x}})$ is defined in (20) and $\mathbf{h}_2^{(n)}(\underline{\mathbf{x}})$ and $\mathbf{h}_3^{(n)}(\underline{\mathbf{x}})$ are defined as follows:

$$\begin{aligned}\mathbf{h}_2^{(n)}(\underline{\mathbf{x}}) = \{\, & m_{22} m_{13} m_{32} R^n,\ m_{22} m_{13} m_{32} R^{n-1} L, \\ & \cdots, m_{22} m_{13} m_{32} R L^{n-1},\ m_{12} m_{23} m_{32} R^n, \\ & m_{12} m_{23} m_{32} R^{n-1} L,\ \cdots,\ m_{12} m_{23} m_{32} L^n \,\},\end{aligned} \tag{21}$$

$$\begin{aligned}\mathbf{h}_3^{(n)}(\underline{\mathbf{x}}) = \{\, & m_{33} m_{12} m_{23} R^{n-1} L,\ \cdots, \\ & m_{33} m_{12} m_{23} R L^{n-1},\ m_{33} m_{12} m_{23} L^n, \\ & m_{13} m_{23} m_{32} R^n,\ m_{13} m_{23} m_{32} R^{n-1} L, \\ & \cdots,\ m_{13} m_{23} m_{32} L^n \,\}.\end{aligned} \tag{22}$$

Thus in this subsection, we prove a graph-theoretic condition that characterizes the linear independence of $\mathbf{h}_i^{(n)}(\underline{\mathbf{x}})$ for all $i \in \{1,2,3\}$ when $n=1$ and $n \geq 2$, respectively. Consider the following graph-theoretic conditions:

$$\overline{S}_i \cap \overline{S}_j = \emptyset \text{ or } \overline{D}_i \cap \overline{D}_j = \emptyset \ \ \forall\, i,j \in \{1,2,3\},\ i \neq j, \tag{23}$$

$$\mathsf{EC}(\{s_1,s_2\};\{d_1,d_3\}) \geq 2,\ \mathsf{EC}(\{s_1,s_3\};\{d_1,d_2\}) \geq 2, \tag{24}$$

$$\mathsf{EC}(s_1;d_1) \geq 1 \text{ on } G_{3\text{ANA}} \backslash \big\{\mathsf{upstr}\big((\overline{S}_2 \cap \overline{D}_3) \cup (\overline{S}_3 \cap \overline{D}_2)\big)\big\}, \tag{25}$$

$$\mathsf{EC}(\{s_1,s_2\};\{d_2,d_3\}) \geq 2,\ \mathsf{EC}(\{s_2,s_3\};\{d_1,d_2\}) \geq 2, \tag{26}$$

$$\mathsf{EC}(s_2;d_2) \geq 1 \text{ on } G_{3\text{ANA}} \backslash \big\{\mathsf{upstr}\big((\overline{S}_1 \cap \overline{D}_3) \cup (\overline{S}_3 \cap \overline{D}_1)\big)\big\}, \tag{27}$$

$$\mathsf{EC}(\{s_1,s_3\};\{d_2,d_3\}) \geq 2,\ \mathsf{EC}(\{s_2,s_3\};\{d_1,d_3\}) \geq 2, \tag{28}$$

$$\mathsf{EC}(s_3;d_3) \geq 1 \text{ on } G_{3\text{ANA}} \backslash \big\{\mathsf{upstr}\big((\overline{S}_1 \cap \overline{D}_2) \cup (\overline{S}_2 \cap \overline{D}_1)\big)\big\}. \tag{29}$$

Note that (i) (23) is equivalent to $L(\underline{\mathbf{x}}) \not\equiv R(\underline{\mathbf{x}})$ by Proposition 4; (ii) (24), (26), and (28) are equivalent to (17) to (19) by Corollary 1; and (iii) (25), (27), and (29) are the new conditions that help characterize (14) to (16).

To further simplify the analysis, we consider the following set of polynomials:

$$\mathbf{k}_1^{(n)}(\underline{\mathbf{x}}) = \{\, m_{11}m_{23}m_{31}L^n,\ m_{11}m_{23}m_{31}L^{n-1}R,\\ \cdots, m_{11}m_{23}m_{31}LR^{n-1},\ m_{21}m_{13}m_{31}L^n,\\ m_{21}m_{13}m_{31}L^{n-1}R,\ \cdots,\ m_{21}m_{13}m_{31}R^n \,\}, \tag{30}$$

where $\mathbf{k}_1^{(n)}(\underline{\mathbf{x}})$ is obtained by swapping the roles of s_1 and s_2 (resp. s_3), and the roles of d_1 and d_2 (resp. d_3) to the expression of $\mathbf{h}_2^{(n)}(\underline{\mathbf{x}})$ in (21) (resp. $\mathbf{h}_3^{(n)}(\underline{\mathbf{x}})$ in (22)). Note that $R = m_{12}m_{23}m_{31}$ becomes $L = m_{13}m_{32}m_{21}$ and vice versa by such swap operation. Once we characterize the graph-theoretic conditions for the linear independence of $\mathbf{k}_1^{(n)}(\underline{\mathbf{x}})$, then the characterization for $\mathbf{h}_2^{(n)}(\underline{\mathbf{x}})$ and $\mathbf{h}_3^{(n)}(\underline{\mathbf{x}})$ being linearly independent will be followed symmetrically.[4]

Proposition 5: For a given $G_{3\text{ANA}}$, when $n=1$, we have

(H1) $\mathbf{h}_1^{(1)}(\underline{\mathbf{x}})$ is linearly independent if and only if $G_{3\text{ANA}}$ satisfies (23) and (24).

(K1) $\mathbf{k}_1^{(1)}(\underline{\mathbf{x}})$ is linearly independent if and only if $G_{3\text{ANA}}$ satisfies (23), (24), and (25).

Moreover when $n \geq 2$, we have

(H2) $\mathbf{h}_1^{(n)}(\underline{\mathbf{x}})$ is linearly independent if and only if $G_{3\text{ANA}}$ satisfies (23), (24), and (25).

(K2) $\mathbf{k}_1^{(n)}(\underline{\mathbf{x}})$ is linearly independent if and only if $G_{3\text{ANA}}$ satisfies (23), (24), and (25).

Remark: Proposition 5 proves that the conjecture in [13] holds only for the linearly independent $\mathbf{h}_1^{(1)}(\underline{\mathbf{x}})$. In general, it is no longer true for the case of $n \geq 2$ and even for $n = 1$. This coincides with the recent results [12], which show that for the case of $n \geq 2$, the conjecture in [13] no longer holds.

Proof. Similar to most graph-theoretic proofs, the proofs of (H1), (K1), (H2), and (K2) involve detailed discussion of several subcases. To structure our proof, we first define the following logic statements. Each statement could be true or false. We will later use these statements to complete the proof.

- **H1:** $\mathbf{h}_1^{(n)}(\underline{\mathbf{x}})$ is linearly independent for $n=1$.
- **K1:** $\mathbf{k}_1^{(n)}(\underline{\mathbf{x}})$ is linearly independent for $n=1$.
- **H2:** $\mathbf{h}_1^{(n)}(\underline{\mathbf{x}})$ is linearly independent for some $n \geq 2$.
- **K2:** $\mathbf{k}_1^{(n)}(\underline{\mathbf{x}})$ is linearly independent for some $n \geq 2$.
- **LNR:** $L(\underline{\mathbf{x}}) \not\equiv R(\underline{\mathbf{x}})$.
- **G1:** $m_{11}m_{23} \not\equiv m_{21}m_{13}$ and $m_{11}m_{32} \not\equiv m_{31}m_{12}$.
- **G2:** $\mathsf{EC}(s_1; d_1) \geq 1$ on $G_{3\text{ANA}} \backslash \{\mathsf{upstr}((\overline{S}_2 \cap \overline{D}_3) \cup (\overline{S}_3 \cap \overline{D}_2))\}$.

One can clearly see that proving Statement (H1) is equivalent to proving "**LNR** $\wedge$ **G1** $\Leftrightarrow$ **H1**" where "$\wedge$" is the AND operator. Similarly, proving Statements (K1), (H2), and (K2) is equivalent to proving "**LNR** $\wedge$ **G1** $\wedge$ **G2** $\Leftrightarrow$ **K1**", "**LNR** $\wedge$ **G1** $\wedge$ **G2** $\Leftrightarrow$ **H2**", and "**LNR** $\wedge$ **G1** $\wedge$ **G2** $\Leftrightarrow$ **K2**", respectively.

The reason why we use the notation of "logic statements" (e.g., **H1**, **LNR**, etc.) is that it enables us to break down the overall proof into proving several smaller "logic relationships" (e.g, "**LNR** $\wedge$ **G1** $\Leftrightarrow$ **H1**", etc.) and later assemble all the logic relationships to derive the final results. The interested readers can thus separate the verification of the proof of each individual logic relationship from the examination of the overall structure of the proof of the main results. The proof of each logic relationship is kept no longer than one page and is independent from the proof of any other logic relationship. This allows the readers to set their own pace when going through the proofs.

To give an insight how the proof works, here we provide the proof of "**LNR** $\wedge$ **G1** $\Leftarrow$ **H1**" at the bottom. All the other proofs are relegated to the appendices. Specifically, we provide the general structured proofs for the necessity direction "$\Leftarrow$" in Appendix E. Applying this result, the proofs of "**LNR** $\wedge$ **G1** $\wedge$ **G2** $\Leftarrow$ **H2**, **K1**, **K2**" are provided in Appendix G. Similarly, the general structured proofs for the sufficiency direction "$\Rightarrow$" is provided in Appendix H. The proofs of "**LNR** $\wedge$ **G1** $\Rightarrow$ **H1**" and "**LNR** $\wedge$ **G1** $\wedge$ **G2** $\Rightarrow$ **K1**, **H2**, **K2**" are provided in Appendix I.

*The proof of "***LNR** $\wedge$ **G1** $\Leftarrow$ **H1***":* We prove the following statement instead: $(\neg$ **LNR**$) \vee (\neg$ **G1**$) \Rightarrow (\neg$ **H1**$)$ where $\neg$ is the NOT logic operator and "$\vee$" is the OR operator. From the expression of $\mathbf{h}_1^{(n)}(\underline{\mathbf{x}})$ in (20), consider $\mathbf{h}_1^{(1)}(\underline{\mathbf{x}})$ which contains 3 polynomials:

$$\mathbf{h}_1^{(1)}(\underline{\mathbf{x}}) = \{\, m_{11}m_{23}m_{32}R,\ m_{11}m_{23}m_{32}L,\ m_{21}m_{13}m_{32}R \,\}. \tag{31}$$

Suppose $G_{3\text{ANA}}$ satisfies $(\neg$ **LNR**$) \vee (\neg$ **G1**$)$, which means $G_{3\text{ANA}}$ satisfies either $L(\underline{\mathbf{x}}) \equiv R(\underline{\mathbf{x}})$ or $m_{11}m_{23} \equiv m_{21}m_{13}$ or $m_{11}m_{32} \equiv m_{31}m_{12}$. If $L(\underline{\mathbf{x}}) \equiv R(\underline{\mathbf{x}})$, then we notice that $m_{11}m_{23}m_{32}R \equiv m_{11}m_{23}m_{32}L$ and $\mathbf{h}_1^{(1)}(\underline{\mathbf{x}})$, defined in (31), is thus linearly dependent. If $m_{11}m_{23} \equiv m_{21}m_{13}$, then we notice that $m_{11}m_{23}m_{32}R \equiv m_{21}m_{13}m_{32}R$. Similarly if $m_{11}m_{32} \equiv m_{31}m_{12}$, then we have $m_{11}m_{23}m_{32}L \equiv m_{21}m_{13}m_{32}R$. The proof is thus complete. $\square$

V. Conclusion and Future Works

The main subject of this work is the general class of precoding-based NC schemes, which focus on designing the precoding and decoding mappings at the sources and destinations while using randomly generated local encoding kernels within the network. One example of the precoding-based structure is the 3-unicast ANA scheme, originally proposed in [4], [13]. In this work, we have identified new graph-theoretic relationships for the precoding-based NC solutions. Based on the findings on the general precoding-based NC, we have further characterized the graph-theoretic feasibility conditions of the 3-unicast ANA scheme. We believe that the analysis in this work will serve as a precursor to fully understand the notoriously challenging multiple-unicast NC problem and design practical, distributed NC solutions based on the precoding-based framework.

Appendix A
Proofs of Propositions 1 and 2

We prove Proposition 1 as follows.

Proof of $\Rightarrow$. We prove this direction by contradiction. Suppose that $\mathbf{h}(\underline{\mathbf{x}})$ is linearly dependent. Then, there exists a set of coefficients $\{\alpha_k\}_{k=1}^N$ such that $\sum_{k=1}^N \alpha_k h_k(\underline{\mathbf{x}}) = 0$ and at least one of them is non-zero. Since $[\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^N$ is row-invariant, we can perform elementary column operations

[4] In Section II-D, (s_1, d_1)-pair was chosen to achieve larger rate than other pairs when aligning the interference. Thus the feasibility characterization for the other transmission pairs, (s_2, d_2) and (s_3, d_3) who achieve the same rate, becomes symmetric.

on $[\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^{N}$ using $\{\alpha_k\}_{k=1}^{N}$ to create an all-zero column. Thus, $\mathsf{det}([\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^{N})$ is a zero polynomial. □

Proof of $\Leftarrow$. This direction is also proven by contradiction. Suppose that $\mathsf{det}([\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^{N})$ is a zero polynomial. We will prove that $\mathbf{h}(\underline{\mathbf{x}})$ is linearly dependent by induction on the value of N. For $N=1$, $\mathsf{det}([\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^{N})=0$ implies that $h_1(\underline{\mathbf{x}})$ is a zero polynomial, which by definition is linearly dependent.

Suppose that the statement holds for any $N<n_0$. When $N=n_0$, consider the (1,1)-th cofactor of $[\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^{N}$, which is the determinant of the submatrix of the intersection of the 2nd to N-th rows and the 2nd to N-th columns. Consider the following two cases. Case 1: the $(1,1)$-th cofactor is a zero polynomial. Then by the induction assumption $\{h_2(\underline{\mathbf{x}}), ..., h_N(\underline{\mathbf{x}})\}$ is linearly dependent. By definition, so is $\mathbf{h}(\underline{\mathbf{x}})$. Case 2: the $(1,1)$-th cofactor is a non-zero polynomial. Since we assume a sufficiently large q, there exists an assignment $\hat{\mathbf{x}}_2\in\mathbb{F}_q^{|\underline{\mathbf{x}}|}$ to $\hat{\mathbf{x}}_N\in\mathbb{F}_q^{|\underline{\mathbf{x}}|}$ such that the value of the (1,1)-th cofactor is non-zero when evaluated by $\hat{\mathbf{x}}_2$ to $\hat{\mathbf{x}}_N$. But note that by the Laplace expansion, we also have $\sum_{k=1}^{N} h_k(\underline{\mathbf{x}}^{(1)})\, C_{1k} = 0$ where C_{1k} is the $(1,k)$-th cofactor. By evaluating C_{1k} with $\{\hat{\mathbf{x}}_i\}_{i=2}^{N}$, we can conclude that $\mathbf{h}(\underline{\mathbf{x}})$ is linearly dependent since at least one of C_{1k} (specifically C_{11}) is non-zero. □

We prove Proposition 2 as follows.

Proof of $\Leftarrow$. This can be proved by simply choosing $G'=G$. □

Proof of $\Rightarrow$. Since $f(\{m_{e_i;e'_i}(\underline{\mathbf{x}}) : \forall\, i\in I\}) \equiv g(\{m_{e_i;e'_i}(\underline{\mathbf{x}}) : \forall\, i\in I\})$, we can assume $f(\{m_{e_i;e'_i}(\underline{\mathbf{x}}) : \forall\, i\in I\}) = \alpha g(\{m_{e_i;e'_i}(\underline{\mathbf{x}}) : \forall\, i\in I\})$ for some non-zero $\alpha\in\mathbb{F}_q$. Consider any subgraph G' containing all edges in $\{e_i, e'_i : \forall\, i\in I\}$ and the channel gain $m_{e_i;e'_i}(\underline{\mathbf{x}}')$ on G'. Then, $m_{e_i;e'_i}(\underline{\mathbf{x}}')$ can be derived from $m_{e_i;e'_i}(\underline{\mathbf{x}})$ by substituting those $\underline{\mathbf{x}}$ variables that are not in G' by zero. As a result, we immediately have $f(\{m_{e_i;e'_i}(\underline{\mathbf{x}}') : \forall\, i\in I\}) = \alpha g(\{m_{e_i;e'_i}(\underline{\mathbf{x}}') : \forall\, i\in I\})$ for the same α. The proof of this direction is thus complete. □

Appendix B
Proofs of Corollaries 1 and 2

We prove Corollary 1 as follows.

Proof of $\Rightarrow$. We assume $(i_1,i_2)=(1,2)$ and $(j_1,j_2)=(1,3)$ without loss of generality. Since $\mathsf{EC}(\{s_1,s_2\};\{d_1,d_3\}) = 1$, there exists an edge e^* that separates $\{d_1,d_3\}$ from $\{s_1,s_2\}$. Therefore, we must have $m_{11}=m_{e_{s_1};e^*}m_{e^*;e_{d_1}}$, $m_{13}=m_{e_{s_1};e^*}m_{e^*;e_{d_3}}$, $m_{21}=m_{e_{s_2};e^*}\ m_{e^*;e_{d_1}}$, and $m_{23}=m_{e_{s_2};e^*}m_{e^*;e_{d_3}}$. As a result, $m_{11}m_{23}\equiv m_{21}m_{13}$. □

Proof of $\Leftarrow$. We prove this direction by contradiction. Suppose $\mathsf{EC}(\{s_{i_1},s_{i_2}\};\{d_{j_1},d_{j_2}\})\geq 2$. In a $G_{3\text{ANA}}$ network, each source (resp. destination) has only one outgoing (resp. incoming) edge. Therefore, $\mathsf{EC}(\{s_{i_1},s_{i_2}\};\{d_{j_1},d_{j_2}\})\geq 2$ implies that at least one of the following two cases must be true: Case 1: There exists a pair of edge-disjoint paths $P_{s_{i_1}d_{j_1}}$ and $P_{s_{i_2}d_{j_2}}$; Case 2: There exists a pair of edge-disjoint paths $P_{s_{i_1}d_{j_2}}$ and $P_{s_{i_2}d_{j_1}}$. For Case 1, we consider the network variables that are along the two edge-disjoint paths, i.e., consider the collection $\underline{\mathbf{x}}'$ of network variables $x_{ee'}\in\underline{\mathbf{x}}$ such that either both e and e' are used by $P_{s_{i_1}d_{j_1}}$ or both e and e' are used by $P_{s_{i_2}d_{j_2}}$. We keep those variables in $\underline{\mathbf{x}}'$ intact and set the other network variables to be zero. As a result, we will have $m_{i_1j_1}(\underline{\mathbf{x}}')m_{i_2j_2}(\underline{\mathbf{x}}') = \prod_{\forall x_{ee'}\in\underline{\mathbf{x}}'} x_{ee'}$ and $m_{i_2j_1}(\underline{\mathbf{x}}')m_{i_1j_2}(\underline{\mathbf{x}}')=0$ where the latter is due the edge-disjointness between two paths $P_{s_{i_1}d_{j_1}}$ and $P_{s_{i_2}d_{j_2}}$. This implies that before hardwiring the variables outside $\underline{\mathbf{x}}'$, we must have $m_{i_1j_1}(\underline{\mathbf{x}})m_{i_2j_2}(\underline{\mathbf{x}})\not\equiv m_{i_2j_1}(\underline{\mathbf{x}})m_{i_1j_2}(\underline{\mathbf{x}})$. The proof of Case 1 is complete. Case 2 can be proven by swapping the labels of j_1 and j_2. □

We prove Corollary 2 as follows.

Proof. When $(i_1,j_1)=(i_2,j_2)$, obviously $m_{i_1j_1}=m_{i_2j_2}$ and $\mathsf{GCD}(m_{i_1j_1}, m_{i_2j_2})\equiv m_{i_2j_2}$. Suppose that for some $(i_1,j_1)\neq(i_2,j_2)$, $\mathsf{GCD}(m_{i_1j_1}, m_{i_2j_2})\equiv m_{i_2j_2}$. Without loss of generality, we assume $i_1\neq i_2$. Since the channel gains are defined for two distinct sources, we must have $m_{i_1j_1}\not\equiv m_{i_2j_2}$. As a result, $\mathsf{GCD}(m_{i_1j_1}, m_{i_2j_2})\equiv m_{i_2j_2}$ implies that $m_{i_1j_1}$ must be reducible. By Proposition 3, $m_{i_1j_1}$ must be expressed as $m_{i_1j_1}=m_{e_{s_{i_1}};e_1}\Big(\prod_{i=1}^{N-1}m_{e_i;e_{i+1}}\Big)m_{e_N;e_{d_{j_1}}}$ where each term corresponds to a pair of consecutive 1-edge cuts separating s_{i_1} and d_{j_1}. For $m_{i_1j_1}$ to contain $m_{i_2j_2}$ as a factor, the source edge $e_{s_{i_2}}$ must be one of the 1-edge cuts separating s_{i_1} and d_{j_1}. This contradicts the assumption that in a 3-unicast ANA network $|\mathsf{In}(s_i)|=0$ for all i. The proof is thus complete. □

Appendix C
Proof of Proposition 3

Proposition 3 will be proven through the concept of the line graph, which is defined as follows: The line graph of a DAG $G=(V,E)$ is represented as $\mathcal{G}=(\mathcal{V},\mathcal{E})$, with the vertex set $\mathcal{V}=E$ and edge set $\mathcal{E}=\{(e',e'')\in E^2 : \mathsf{head}(e')=\mathsf{tail}(e'')\}$ (the set representing the adjacency relationships between the edges of E). Provided that G is directed acyclic, its line graph $\mathcal{G}$ is also directed acyclic. The graph-theoretic notations for G defined in Section II-A are applied in the same way as in $\mathcal{G}$.

Note that the line graph translates the edges into vertices. Thus, a *vertex cut* in the line graph is the counterpart of the edge cut in a normal graph. Specifically, a *k-vertex cut* separating vertex sets U and W is a collection of k vertices other than the vertices in U and W such that any path from any $u\in U$ to any $w\in W$ must use at least one of those k vertices. Moreover, the minimum value (number of vertices) of all the possible vertex cuts between vertex sets U and W is termed $\mathsf{VC}(U;W)$. For any nodes u and v in V, one can easily see that $\mathsf{EC}(u;v)$ in G is equal to $\mathsf{VC}(\tilde{u};\tilde{v})$ in $\mathcal{G}$ where $\tilde{u}$ and $\tilde{v}$ are the vertices in $\mathcal{G}$ corresponding to any incoming edge of u and any outgoing edge of v, respectively.

Once we focus on the line graph $\mathcal{G}$, the network variables $\underline{\mathbf{x}}$, originally defined over the (e',e'') pairs of the normal graph, are now defined on the edges of the line graph. We can thus define the channel gain from a vertex u to a vertex v on $\mathcal{G}$ as

$$\mathring{m}_{u;v} = \sum_{\forall\, P_{uv}\in\mathbf{P}_{uv}}\ \prod_{\forall\, e\in P_{uv}} x_e, \tag{32}$$

where $\mathbf{P}_{uv}$ denotes the collection of all distinct paths from u to v. For notational simplicity, we sometimes simply use "an edge e" to refer to the corresponding network variable x_e. Each x_e (or e) thus takes values in $\mathbb{F}_q$. When $u=v$, simply set $\mathring{m}_{u;v}=1$.

The line-graph-based version of Proposition 3 is described as follows:

Corollary 3: Given the line graph $\mathcal{G}$ of a DAG G, $\mathring{m}$ defined above, and two distinct vertices s and d, the following is true:

- If $\mathsf{VC}(s;d)=0$, then $\mathring{m}_{s;d}=0$
- If $\mathsf{VC}(s;d)=1$, then $\mathring{m}_{s;d}$ is reducible and can be expressed as $\mathring{m}_{s;d}=\mathring{m}_{s;u_1}\left(\prod_{i=1}^{N-1}\mathring{m}_{u_i;u_{i+1}}\right)\mathring{m}_{u_N;d}$ where $\{u_i\}_{i=1}^{N}$ are all the distinct 1-vertex cuts between s and d in the topological order (from the most upstream to the most downstream). Moreover, the polynomial factors $\mathring{m}_{s;u_1}$, $\{\mathring{m}_{u_i;u_{i+1}}\}_{i=1}^{N-1}$, and $\mathring{m}_{u_N;d}$ are all irreducible, and no two of them are equivalent.
- If $\mathsf{VC}(s;d)\geq 2$ (including ∞), then $\mathring{m}_{s;d}$ is irreducible.

Proof. We use the induction on the number of edges $|\mathcal{E}|$ of $\mathcal{G}=(\mathcal{V},\mathcal{E})$. When $|\mathcal{E}|=0$, then $\mathsf{VC}(s;d)=0$ since there are no edges in $\mathcal{G}$. Thus $\mathring{m}_{s;d}=0$ naturally.

Suppose that the above three claims are true for $|\mathcal{E}|=k-1$. We would like to prove that those claims also hold for the line graph $\mathcal{G}$ with $|\mathcal{E}|=k$.

Case 1: $\mathsf{VC}(s;d)=0$ on $\mathcal{G}$. In this case, s and d are already disconnected. Therefore, $\mathring{m}_{s;d}=0$.

Case 2: $\mathsf{VC}(s;d)=1$ on $\mathcal{G}$. Consider all distinct 1-vertex cuts $u_1,\cdots,u_N$ between s and d in the topological order. If we define $u_0\triangleq s$ and $u_{N+1}\triangleq d$, then we can express $\mathring{m}_{s;d}$ as $\mathring{m}_{s;d}=\prod_{i=0}^{N}\mathring{m}_{u_i;u_{i+1}}$. Since we considered all distinct 1-vertex cuts between s and d, we must have $\mathsf{VC}(u_i;u_{i+1})\geq 2$ for $i=0,\cdots,N$. By induction, $\{\mathring{m}_{u_i;u_{i+1}}\}_{i=0}^{N}$ are all irreducible. Also, since each sub-channel gain $\mathring{m}_{u_i;u_{i+1}}$ covers a disjoint portion of $\mathcal{G}$, no two of them are equivalent.

Case 3: $\mathsf{VC}(s;d)\geq 2$ on $\mathcal{G}$. Without loss of generality, we can also assume that s can reach any vertex $u\in\mathcal{V}$ and d can be reached from any vertex $u\in\mathcal{V}$. Consider two subcases: Case 3.1: all edges in $\mathcal{E}$ have their tails being s and their heads being d. In this case, $\mathring{m}_{s;d}=\sum_{e\in\mathcal{E}}x_e$. Obviously $\mathring{m}_{s;d}$ is irreducible. Case 3.2: at least one edge in $\mathcal{E}$ is not directly connecting s and d. In this case, there must exist an edge e' such that $s\prec\mathsf{tail}(e')$ and $\mathsf{head}(e')=d$. Arbitrarily pick one such edge e' and fix it. We denote the tail vertex of the chosen e' by w. By the definition of (32), we have

$$\mathring{m}_{s;d}=\mathring{m}_{s;w}x_{e'}+\mathring{m}'_{s;d}, \tag{33}$$

where $\mathring{m}_{s;w}$ is the channel gain from s to w, and $\mathring{m}'_{s;d}$ is the channel gain from s to d on the subgraph $\mathcal{G}'=\mathcal{G}\backslash\{e'\}$ that removes e' from $\mathcal{G}$. Note that there always exists a path from s to d not using w on $\mathcal{G}'$ otherwise w will be a cut separating s and d on $\mathcal{G}$, contradicting the assumption that $\mathsf{VC}(s;d)\geq 2$.

We now argue by contradiction that $\mathring{m}_{s;d}$ must be irreducible. Suppose not, then $\mathring{m}_{s;d}$ can be written as a product of two polynomials A and B with the degrees of A and B being larger than or equal to 1. We can always write $A=x_{e'}A_1+A_2$ by singling out the portion of A that has $x_{e'}$ as a factor. Similarly we can write $B=x_{e'}B_1+B_2$. We then have

$$\mathring{m}_{s;d}=(x'_eA_1+A_2)(x'_eB_1+B_2). \tag{34}$$

We first notice that by (33) there is no quadratic term of $x_{e'}$ in $\mathring{m}_{s;d}$. Therefore, one of A_1 and B_1 must be a zero polynomial. Assume $B_1=0$. Comparing (33) and (34) shows that $\mathring{m}_{s;w}=A_1B_2$ and $\mathring{m}'_{s;d}=A_2B_2$. Since the degree of B is larger than or equal to 1 and $B_1=0$, the degree of B_2 must be larger than equal to 1. As a result, we have $\mathsf{GCD}(\mathring{m}_{s;w},\mathring{m}'_{s;d})\not\equiv 1$ (having at least a non-zero polynomial B_2 as its common factor).

The facts that $\mathsf{GCD}(\mathring{m}_{s;w},\mathring{m}'_{s;d})\not\equiv 1$ and $w\prec d$ imply that one of the following three cases must be true: (i) Both $\mathring{m}_{s;w}$ and $\mathring{m}'_{s;d}$ are reducible; (ii) $\mathring{m}_{s;w}$ is reducible but $\mathring{m}'_{s;d}$ is not; and (iii) $\mathring{m}'_{s;d}$ is reducible but $\mathring{m}_{s;w}$ is not. For Case (i), by applying Proposition 3 to the subgraph $\mathcal{G}'=\mathcal{G}\backslash\{e'\}$, we know that $\mathsf{VC}(s;w)=\mathsf{VC}(s;d)=1$ and both polynomials $\mathring{m}_{s;w}$ and $\mathring{m}'_{s;d}$ can be factorized according to their 1-vertex cuts, respectively. Since $\mathring{m}_{s;w}$ and $\mathring{m}'_{s;d}$ have a common factor, there exists a vertex u that is both a 1-vertex cut separating s and w and a 1-vertex cut separating s and d when focusing on $\mathcal{G}'$. As a result, such u is a 1-vertex cut separating s and d in the original graph $\mathcal{G}$. This contradicts the assumption $\mathsf{VC}(s;d)\geq 2$ in $\mathcal{G}$. For Case (ii), by applying Proposition 3 to $\mathcal{G}'$, we know that $\mathsf{VC}(s;w)=1$ and $\mathring{m}_{s;w}$ can be factorized according to their 1-vertex cuts. Since $\mathring{m}_{s;w}$ and the irreducible $\mathring{m}'_{s;d}$ have a common factor, $\mathring{m}_{s;w}$ must contain $\mathring{m}'_{s;d}$ as a factor, which implies that d is a 1-vertex cut separating s and w in $\mathcal{G}'$. This contradicts the construction of $\mathcal{G}'$ where $w\prec d$. For Case (iii), by applying Proposition 3 to $\mathcal{G}'$, we know that $\mathsf{VC}(s;d)=1$ and $\mathring{m}'_{s;d}$ can be factorized according to their 1-vertex cuts. Since $\mathring{m}'_{s;d}$ and the irreducible $\mathring{m}_{s;w}$ have a common factor, $\mathring{m}'_{s;d}$ must contain $\mathring{m}_{s;w}$ as a factor, which implies that w is a 1-vertex cut separating s and d in $\mathcal{G}'$. As a result, w is a 1-vertex cut separating s and d in the original graph $\mathcal{G}$. This contradicts the assumption $\mathsf{VC}(s;d)\geq 2$ in $\mathcal{G}$. ☐

Appendix D
Proofs of Lemmas 1 to 7

We prove Lemma 1 as follows.

Proof. Consider indices $i\neq j$. By the definition, all paths from s_i to d_j must use all edges in $\overline{S}_i$ and all edges in $\overline{D}_j$. Thus, for any $e'\in\overline{S}_i$ and any $e''\in\overline{D}_j$, one of the following statements must be true: $e'\prec e''$, $e'\succ e''$, or $e'=e''$. ☐

We prove Lemma 2 as follows.

Proof. Consider three indices i, j, and k taking distinct values in $\{1,2,3\}$. Consider an arbitrary edge $e\in\overline{D}_i\cap\overline{D}_j$. By definition, all paths from s_k to d_i, and all paths from s_k to d_j must use e. Therefore, $e\in\overline{S}_k$. ☐

We prove Lemma 3 as follows.

Proof. Without loss of generality, let $i=1$ and $j=2$. Choose the most downstream edge in $\overline{S}_1\backslash\overline{D}_2$ and denote it as e'_*. Since e'_* belongs to $\mathsf{1cut}(s_1;d_2)\cap\mathsf{1cut}(s_1;d_3)$ but not to $\mathsf{1cut}(s_3;d_2)$, there must exist a s_3-to-d_2 path P_{32} not using e'_*. In addition, for any $e''\in\overline{D}_2$, we have either $e''\prec e'_*$, $e''\succ e'_*$, or $e''=e'_*$ by Lemma 1. Suppose there exists an edge $e''\in\overline{D}_2$ such that $e''\prec e'_*$. Then by definition, any s_3-to-d_2 path must use e''. Also note that since $e''\in\overline{D}_2$, there exists a path $P_{s_1\mathsf{tail}(e'')}$

from s_1 to $\mathsf{tail}(e'')$. Consider the concatenated s_1-to-d_2 path $P_{s_1\mathsf{tail}(e'')}e''P_{32}$. We first note that since $e'' \prec e'_*$, the path segment $P_{s_1\mathsf{tail}(e'')}e''$ does not use e'_*. By our construction, P_{32} also does not use e'_*. Jointly, the above observations contradict the fact that $e'_* \in \overline{S}_1$ is a 1-edge cut separating s_1 and d_2. By contradiction, we must have $e'_* \preceq e''$. Note that since by our construction e'_* must not be in $\overline{D}_2$ while e'' is in $\overline{D}_2$, we must have $e'_* \neq e''$ and thus $e'_* \prec e''$. Since e'_* was chosen as the most downstream edge of $\overline{S}_1 \backslash \overline{D}_2$, we have $e' \prec e''$ for all $e' \in \overline{S}_1 \backslash \overline{D}_2$ and $e'' \in \overline{D}_2$. The proof is thus complete. □

We prove Lemma 4 as follows.

Proof of $\Rightarrow$. We note that $(\overline{S}_i \cap \overline{D}_j) \supset (\overline{S}_i \cap \overline{D}_j \cap \overline{D}_k) = (\overline{D}_j \cap \overline{D}_k)$ where the equality follows from Lemma 2. As a result, when $\overline{D}_j \cap \overline{D}_k \neq \emptyset$, we also have $\overline{S}_i \cap \overline{D}_j \neq \emptyset$. □

Proof of $\Leftarrow$. Consider three indices i, j, and k taking distinct values in $\{1,2,3\}$. Suppose that $\overline{S}_i \cap \overline{D}_j \neq \emptyset$ and $\overline{S}_i \cap \overline{D}_k \neq \emptyset$. Then, for any $e' \in \overline{S}_i \cap \overline{D}_j$ and any $e'' \in \overline{S}_i \cap \overline{D}_k$, we must have either $e' \prec e''$, $e' \succ e''$, or $e' = e''$ by Lemma 1. Suppose that $\overline{D}_j \cap \overline{D}_k = \emptyset$. Then we must have $e' \neq e''$, which leaves only two possibilities: either $e' \prec e''$ or $e' \succ e''$. However, $e' \prec e''$ contradicts Lemma 3 because $e' \in (\overline{S}_i \cap \overline{D}_j) \subset \overline{D}_j$ and $e'' \in (\overline{S}_i \cap \overline{D}_k) \subset (\overline{S}_i \backslash \overline{D}_j)$, the latter of which is due to the assumption of $\overline{D}_j \cap \overline{D}_k = \emptyset$. By swapping the roles of j and k, one can also show that it is impossible to have $e' \succ e''$. By contradiction, we must have $\overline{D}_j \cap \overline{D}_k \neq \emptyset$. The proof is thus complete. □

We prove Lemma 5 as follows.

Proof. Without loss of generality, consider $i = 1$ and $j = 2$. Note that by Lemma 1 any $e' \in \overline{S}_1 \cap \overline{S}_2$ and any $e'' \in \overline{D}_1 \cap \overline{D}_2$ must satisfy either $e' \prec e''$, $e' \succ e''$, or $e' = e''$. For the following, we prove this lemma by contradiction.

Suppose that there exists an edge $e''_* \in \overline{D}_1 \cap \overline{D}_2$ such that for all $e' \in \overline{S}_1 \cap \overline{S}_2$ we have $e''_* \prec e'$. For the following, we first prove that any path from s_i to d_j where $i, j \in \{1,2,3\}$ and $i \neq j$ must pass through e''_*. To that end, we first notice that by the definition of $\overline{D}_1$ and $\overline{D}_2$ and by the assumption $e''_* \in \overline{D}_1 \cap \overline{D}_2$, any path from $\{s_2, s_3\}$ to d_1, and any path from $\{s_1, s_3\}$ to d_2 must use e''_*. Thus, we only need to prove that any path from $\{s_1, s_2\}$ to d_3 must use e''_* as well.

Suppose there exists a s_1-to-d_3 path P_{13} that does not use e''_*. By the definition of $\overline{S}_1$, P_{13} must use all edges of $\overline{S}_1 \cap \overline{S}_2$, all of which are in the downstream of e''_* by the assumption. Also d_2 is reachable from any $e' \in \overline{S}_1 \cap \overline{S}_2$. Choose arbitrarily one edge $e'_* \in \overline{S}_1 \cap \overline{S}_2$ and a path $P_{\mathsf{head}(e'_*)d_2}$ from $\mathsf{head}(e'_*)$ to d_2. Then, we can create an path $P_{13}\, e'_*\, P_{\mathsf{head}(e'_*)d_2}$ from s_1 to d_2 without using e''_*. The reason is that P_{13} does not use e''_* by our construction and $e'_* P_{\mathsf{head}(e'_*)d_2}$ does not use e''_* since $e''_* \prec e'_*$. Such an s_1-to-d_2 path not using e''_* thus contradicts the assumption of $e''_* \in (\overline{D}_1 \cap \overline{D}_2) \subset \mathsf{1cut}(s_1; d_2)$. Symmetrically, any s_2-to-d_3 path must use e''_*.

In summary, we have shown that $e''_* \in \cap_{i=1}^{3} \left(\overline{S}_i \cap \overline{D}_i\right)$. However, this contradicts the assumption that e''_* is in the upstream of all $e' \in \overline{S}_1 \cap \overline{S}_2$, because we can simply choose $e' = e''_* \in \cap_{i=1}^{3} \left(\overline{S}_i \cap \overline{D}_i\right) \subset (\overline{S}_1 \cap \overline{S}_2)$ and e''_* cannot be an upstream edge of itself $e' = e''_*$. The proof is thus complete. □

We prove Lemma 6 as follows.

Proof. Without loss of generality, let $i = 1$, $j_1 = 1$, $j_2 = 2$, and $j_3 = 3$. Suppose that $\overline{S}_{1;\{1,2\}} \neq \emptyset$ and $\overline{S}_{1;\{1,3\}} \neq \emptyset$. For the following, we prove this lemma by contradiction.

Suppose that $\overline{S}_{1;\{1,2\}} \cap \overline{S}_{1;\{1,3\}} = \emptyset$. For any $e' \in \overline{S}_{1;\{1,2\}}$ and any $e'' \in \overline{S}_{1;\{1,3\}}$, since both e' and e'' are 1-edge cuts separating s_1 and d_1, it must be either $e' \prec e''$ or $e' \succ e''$, or $e' = e''$. The last case is not possible since we assume $\overline{S}_{1;\{1,2\}} \cap \overline{S}_{1;\{1,3\}} = \emptyset$. Consider the most downstream edges $e'_* \in \overline{S}_{1;\{1,2\}}$ and $e''_* \in \overline{S}_{1;\{1,3\}}$, respectively. We first consider the case $e'_* \prec e''_*$. If all paths from s_1 to d_3 use e'_*, which, by definition, use e''_*, then e'_* will belong to $\mathsf{1cut}(s_1; d_3)$, which contradicts the assumption that $\overline{S}_{1;\{1,2\}} \cap \overline{S}_{1;\{1,3\}} = \emptyset$. Thus, there exists a s_1-to-d_3 path P_{13} using e''_* but not e'_*. Then, s_1 can follow P_{13} and reach d_1 via e''_* without using e'_*. Such a s_1-to-d_1 path contradicts the definition $e'_* \in \overline{S}_{1;\{1,2\}} \subset \mathsf{1cut}(s_1; d_1)$. Therefore, it is impossible to have $e'_* \prec e''_*$. By symmetric arguments, it is also impossible to have $e'_* \succ e''_*$. By definition, any edge in $\overline{S}_{1;\{1,2\}} \cap \overline{S}_{1;\{1,3\}}$ is a 1-edge cut separating s_1 and $\{d_2, d_3\}$, which implies that $\overline{S}_{1;\{2,3\}} \neq \emptyset$ and $\overline{S}_1 \neq \emptyset$. □

We prove Lemma 7 as follows.

Proof of $\Rightarrow$. Suppose $\overline{S}_{i;\{j_1,j_2\}} \neq \emptyset$. By definition, there exists an edge $e \in \mathsf{1cut}(s_i; d_{j_1}) \cap \mathsf{1cut}(s_i; d_{j_2})$ in the downstream of the s_i-source edge e_{s_i}. Then, the channel gains m_{ij_1} and m_{ij_2} have a common factor $m_{e_{s_i};e}$ and we thus have $\mathsf{GCD}(\, m_{ij_1},\, m_{ij_2}) \not\equiv 1$. □

Proof of $\Leftarrow$. We prove this direction by contradiction. Suppose $\mathsf{GCD}(\, m_{ij_1}, m_{ij_2}) \not\equiv 1$. By Corollary 2, we know that $\mathsf{GCD}(\, m_{ij_1},\, m_{ij_2})$ must not be m_{ij_1} nor m_{ij_2}. Thus, both must be reducible and by Proposition 3 can be expressed as the product of irreducibles, for which each factor corresponds to the consecutive 1-edge cuts in $\mathsf{1cut}(s_i; d_{j_1})$ and $\mathsf{1cut}(s_i; d_{j_2})$, respectively. Since they have at least one common irreducible factor, there exists an edge $e \in \mathsf{1cut}(s_i; d_{j_1}) \cap \mathsf{1cut}(s_i; d_{j_2})$ in the downstream of the s_i-source edge e_{s_i}. Thus, $e \in \overline{S}_{i;\{j_1,j_2\}}$. The case for $\mathsf{GCD}(\, m_{j_1 i},\, m_{j_2 i}) \equiv 1$ can be proven symmetrically. The proof is thus complete. □

Appendix E
General Structured Proof for the Necessity

In this appendix, we provide Corollary 4, which will be used to prove the graph-theoretic necessary direction of 3-unicast ANA network for arbitrary n values. Since we already provided the proof for "**LNR** $\wedge$ **G1** $\Leftarrow$ **H1**" in Proposition 5, here we focus on proving "**LNR** $\wedge$ **G1** $\wedge$ **G2** $\Leftarrow$ **H2**, **K1**, **K2**". After introducing Corollary 4, the main proof of "**LNR** $\wedge$ **G1** $\wedge$ **G2** $\Leftarrow$ **H2**, **K1**, **K2**" will be provided in Appendix G.

Before proceeding, we need the following additional logic statements to describe the general proof structure.

E-1. The first set of logic statements

Consider the following logic statements.

- **G0:** $m_{11}m_{23}m_{32} = R + L$.
- **G3:** $\overline{S}_2 \cap \overline{D}_3 = \emptyset$.

• **G4:** $\overline{S}_3 \cap \overline{D}_2 = \emptyset$.

Several implications can be made when **G3** is true. We term those implications *the properties of* **G3**. Several properties of **G3** are listed as follows, for which their proofs are provided in Appendix J.

Consider the case in which **G3** *is true.* Use e_2^* to denote the most downstream edge in $\mathsf{1cut}(s_2; d_1) \cap \mathsf{1cut}(s_2; d_3)$. Since the source edge e_{s_2} belongs to both $\mathsf{1cut}(s_2; d_1)$ and $\mathsf{1cut}(s_2; d_3)$, such e_2^* always exists. Similarly, use e_3^* to denote the most upstream edge in $\mathsf{1cut}(s_1; d_3) \cap \mathsf{1cut}(s_2; d_3)$. The properties of **G3** can now be described as follows.

⋄ **Property 1 of G3:** $e_2^* \prec e_3^*$ and the channel gains m_{13}, m_{21}, and m_{23} can be expressed as $m_{13} = m_{e_{s_1}; e_3^*} m_{e_3^*; e_{d_3}}$, $m_{21} = m_{e_{s_2}; e_2^*} m_{e_2^*; e_{d_1}}$, and $m_{23} = m_{e_{s_2}; e_2^*} m_{e_2^*; e_3^*} m_{e_3^*; e_{d_3}}$.

⋄ **Property 2 of G3:** $\mathsf{GCD}(m_{e_{s_1}; e_3^*},\ m_{e_{s_2}; e_2^*} m_{e_2^*; e_3^*}) \equiv 1$, $\mathsf{GCD}(m_{e_2^*; e_3^*} m_{e_3^*; e_{d_3}},\ m_{e_2^*; e_{d_1}}) \equiv 1$, $\mathsf{GCD}(m_{13},\ m_{e_2^*; e_3^*}) \equiv 1$, and $\mathsf{GCD}(m_{21},\ m_{e_2^*; e_3^*}) \equiv 1$.

On the other hand, when **G3** is false, or equivalently when $\neg$ **G3** is true where "$\neg$" is the NOT operator, we can also derive several implications, termed *the properties of* $\neg$ **G3**.

Consider the case in which **G3** *is false*. Use e_u^{23} (resp. e_v^{23}) to denote the most upstream (resp. the most downstream) edge in $\overline{S}_2 \cap \overline{D}_3$. By definition, it must be $e_u^{23} \preceq e_v^{23}$. We now describe the following properties of $\neg$ **G3**.

⋄ **Property 1 of** $\neg$ **G3:** The channel gains m_{13}, m_{21}, and m_{23} can be expressed as $m_{13} = m_{e_{s_1}; e_u^{23}} m_{e_u^{23}; e_v^{23}} m_{e_v^{23}; e_{d_3}}$, $m_{21} = m_{e_{s_2}; e_u^{23}} m_{e_u^{23}; e_v^{23}} m_{e_v^{23}; e_{d_1}}$, and $m_{23} = m_{e_{s_2}; e_u^{23}} m_{e_u^{23}; e_v^{23}} m_{e_v^{23}; e_{d_3}}$.

⋄ **Property 2 of** $\neg$ **G3:** $\mathsf{GCD}(m_{e_{s_1}; e_u^{23}},\ m_{e_{s_2}; e_u^{23}}) \equiv 1$ and $\mathsf{GCD}(m_{e_v^{23}; e_{d_1}},\ m_{e_v^{23}; e_{d_3}}) \equiv 1$.

⋄ **Property 3 of** $\neg$ **G3:** $\{e_u^{23}, e_v^{23}\} \subset \mathsf{1cut}(s_1; \mathsf{head}(e_v^{23}))$ and $\{e_u^{23}, e_v^{23}\} \subset \mathsf{1cut}(\mathsf{tail}(e_u^{23}); d_1)$. This further implies that for any s_1-to-d_1 path P, if there exists a vertex $w \in P$ satisfying $\mathsf{tail}(e_u^{23}) \preceq w \preceq \mathsf{head}(e_v^{23})$, then we must have $\{e_u^{23}, e_v^{23}\} \subset P$.

Symmetrically, we define the following properties of **G4** and $\neg$ **G4**.

Consider the case in which **G4** *is true.* Use e_3^* to denote the most downstream edge in $\mathsf{1cut}(s_3; d_1) \cap \mathsf{1cut}(s_3; d_2)$, and use e_2^* to denote the most upstream edge in $\mathsf{1cut}(s_1; d_2) \cap \mathsf{1cut}(s_3; d_2)$. We now describe the following properties of **G4**.

⋄ **Property 1 of G4:** $e_3^* \prec e_2^*$ and the channel gains m_{12}, m_{31}, and m_{32} can be expressed as $m_{12} = m_{e_{s_1}; e_2^*} m_{e_2^*; e_{d_2}}$, $m_{31} = m_{e_{s_3}; e_3^*} m_{e_3^*; e_{d_1}}$, and $m_{32} = m_{e_{s_3}; e_3^*} m_{e_3^*; e_2^*} m_{e_2^*; e_{d_2}}$.

⋄ **Property 2 of G4:** $\mathsf{GCD}(m_{e_{s_1}; e_2^*},\ m_{e_{s_3}; e_3^*} m_{e_3^*; e_2^*}) \equiv 1$, $\mathsf{GCD}(m_{e_3^*; e_2^*} m_{e_2^*; e_{d_2}},\ m_{e_3^*; e_{d_1}}) \equiv 1$, $\mathsf{GCD}(m_{12},\ m_{e_3^*; e_2^*}) \equiv 1$, and $\mathsf{GCD}(m_{31},\ m_{e_3^*; e_2^*}) \equiv 1$.

Consider the case in which **G4** *is false.* Use e_u^{32} (resp. e_v^{32}) to denote the most upstream (resp. the most downstream) edge in $\overline{S}_3 \cap \overline{D}_2$. By definition, it must be $e_u^{32} \preceq e_v^{32}$. We now describe the following properties of $\neg$ **G4**.

⋄ **Property 1 of** $\neg$ **G4:** The channel gains m_{12}, m_{31}, and m_{32} can be expressed as $m_{12} = m_{e_{s_1}; e_u^{32}} m_{e_u^{32}; e_v^{32}} m_{e_v^{32}; e_{d_2}}$, $m_{31} = m_{e_{s_3}; e_u^{32}} m_{e_u^{32}; e_v^{32}} m_{e_v^{32}; e_{d_1}}$, and $m_{32} = m_{e_{s_3}; e_u^{32}} m_{e_u^{32}; e_v^{32}} m_{e_v^{32}; e_{d_2}}$.

⋄ **Property 2 of** $\neg$ **G4:** $\mathsf{GCD}(m_{e_{s_1}; e_u^{32}},\ m_{e_{s_3}; e_u^{32}}) \equiv 1$ and $\mathsf{GCD}(m_{e_v^{32}; e_{d_1}},\ m_{e_v^{32}; e_{d_2}}) \equiv 1$

⋄ **Property 3 of** $\neg$ **G4:** $\{e_u^{32}, e_v^{32}\} \subset \mathsf{1cut}(s_1; \mathsf{head}(e_v^{32}))$ and $\{e_u^{32}, e_v^{32}\} \subset \mathsf{1cut}(\mathsf{tail}(e_u^{32}); d_1)$. This further implies that for any s_1-to-d_1 path P, if there exists a vertex $w \in P$ satisfying $\mathsf{tail}(e_u^{32}) \preceq w \preceq \mathsf{head}(e_v^{32})$, then we must have $\{e_u^{32}, e_v^{32}\} \subset P$.

The following logic statements are well-defined if and only if $(\neg$ **G3**$) \wedge (\neg$ **G4**$)$ is true. Recall the definition of e_u^{23}, e_v^{23}, e_u^{32}, and e_v^{32} when $(\neg$ **G3**$) \wedge (\neg$ **G4**$)$ is true.

• **G5:** Either $e_u^{23} \prec e_u^{32}$ or $e_u^{23} \succ e_u^{32}$.

• **G6:** Any vertex w' where $\mathsf{tail}(e_u^{23}) \preceq w' \preceq \mathsf{head}(e_v^{23})$ and any vertex w'' where $\mathsf{tail}(e_u^{32}) \preceq w'' \preceq \mathsf{head}(e_v^{32})$ are not reachable from each other. (That is, neither $w' \preceq w''$ nor $w'' \preceq w'$.)

It is worth noting that a *statement* being well-defined does not mean that it is true. Any well-defined logic statement can be either true or false. For comparison, a *property* of **G3** is both well-defined and true whenever **G3** is true.

E-2. General Necessity Proof Structure

The following "logic relationships" are proved in Appendix K, which will be useful for the proof of the following Corollary 4.

• **N1: H2** $\Rightarrow$ **LNR** $\wedge$ **G1**.

• **N2: K1** $\Rightarrow$ **LNR** $\wedge$ **G1**.

• **N3: K2** $\Rightarrow$ **LNR** $\wedge$ **G1**.

• **N4:** $(\neg$ **G2**$) \wedge$ **G3** $\wedge$ **G4** $\Rightarrow$ false.

• **N5: G1** $\wedge (\neg$ **G2**$) \wedge (\neg$ **G3**$) \wedge$ **G4** $\Rightarrow$ false.

• **N6: G1** $\wedge (\neg$ **G2**$) \wedge$ **G3** $\wedge (\neg$ **G4**$) \Rightarrow$ false.

• **N7: LNR** $\wedge (\neg$ **G3**$) \wedge (\neg$ **G4**$) \wedge (\neg$ **G5**$) \Rightarrow$ **G6**.

• **N8: G1** $\wedge (\neg$ **G2**$) \wedge (\neg$ **G3**$) \wedge (\neg$ **G4**$) \wedge$ **G5** $\Rightarrow$ false.

• **N9:** $(\neg$ **G2**$) \wedge (\neg$ **G3**$) \wedge (\neg$ **G4**$) \wedge (\neg$ **G5**$) \wedge$ **G6** $\Rightarrow$ **G0**.

Corollary 4: Let $\mathbf{h}(\underline{\mathbf{x}})$ be a set of (arbitrarily chosen) polynomials based on the 9 channel gains m_{ij} of the 3-unicast ANA network, and define **X** to be the logic statement that $\mathbf{h}(\underline{\mathbf{x}})$ is linearly independent. If we can prove that **X** $\Rightarrow$ **LNR** $\wedge$ **G1** and **X** $\wedge$ **G0** $\Rightarrow$ false, then the logic relationship **X** $\Rightarrow$ **LNR** $\wedge$ **G1** $\wedge$ **G2** must hold.

Proof. Suppose **X** $\Rightarrow$ **LNR** $\wedge$ **G1** and **X** $\wedge$ **G0** $\Rightarrow$ false. We first see that **N7** and **N9** jointly imply

$$\mathbf{LNR} \wedge (\neg\, \mathbf{G2}) \wedge (\neg\, \mathbf{G3}) \wedge (\neg\, \mathbf{G4}) \wedge (\neg\, \mathbf{G5}) \Rightarrow \mathbf{G0}.$$

Combined with **N8**, we thus have

$$\mathbf{LNR} \wedge \mathbf{G1} \wedge (\neg\, \mathbf{G2}) \wedge (\neg\, \mathbf{G3}) \wedge (\neg\, \mathbf{G4}) \Rightarrow \mathbf{G0}.$$

This, jointly with **N4**, **N5**, and **N6**, further imply

$$\mathbf{LNR} \wedge \mathbf{G1} \wedge (\neg\, \mathbf{G2}) \Rightarrow \mathbf{G0}.$$

Together with the assumption that **X** $\wedge$ **G0** $\Rightarrow$ false, we have **X** $\wedge$ **LNR** $\wedge$ **G1** $\wedge (\neg$ **G2**$) \Rightarrow$ false. Combining with the assumption that **X** $\Rightarrow$ **LNR** $\wedge$ **G1** then yields

$$\mathbf{X} \wedge (\neg\, \mathbf{G2}) \Rightarrow \text{false},$$

which equivalently implies that **X** $\Rightarrow$ **G2**. The proof is thus complete. □

Appendix F
The Reference Table

For the ease of exposition, we provide the Table II, the reference table. The reference table helps finding where to look for the individual logic statements and relationships for the entire proof of Proposition 5.

The Logic Statements for the Proof of Proposition 5

C0 to **C6**	defined in p. 15.	**G7** to **G15**	defined in p. 19.
D1 to **D6**	defined in p. 15.	**G16** to **G26**	defined in p. 23.
E0 to **E2**	defined in p. 14.	**G27** to **G31**	defined in p. 29.
G0	defined in p. 12.	**G32** to **G36**	defined in p. 30.
G1, **G2**	defined in p. 9.	**G37** to **G43**	defined in p. 32.
G3, **G4**	defined in p. 12.	**H1**, **H2**, **K1**, **K2**	defined in p. 9.
G5, **G6**	defined in p. 13.	**LNR**	defined in p. 9.

The Logic Relationships for the Proof of Proposition 5

N1 to **N9**	defined in p. 13, to help proving Corollary 4, the general structured proof for the necessity of Proposition 5.
R1 to **R10**	defined in p. 19, to help proving **S11**.
R11 to **R25**	defined in p. 23, to help proving **S13**.
R26 to **R33**	defined in p. 29, to help proving **S14**.
R34 to **R40**	defined in p. 31, to help proving **R28**.
R41 to **R47**	defined in p. 32, to help proving **R29**.
S1 to **S14**	defined in p. 15, to help proving Corollary 5, the general structured proof for the sufficiency of Proposition 5.

TABLE II

THE REFERENCE TABLE FOR THE PROOF OF PROPOSITION 5.

APPENDIX G
PROOF OF "LNR ∧ G1 ∧ G2 ⇐ K1 ∨ H2 ∨ K2"

Thanks to Corollary 4 and the logic relationships **N1**, **N2**, and **N3** in Appendix E, we only need to show that (i) **K1** $\wedge$ **G0** $\Rightarrow$ false; (ii) **H2** $\wedge$ **G0** $\Rightarrow$ false; and (iii) **K2** $\wedge$ **G0** $\Rightarrow$ false.

We prove "**K1** $\wedge$ **G0** $\Rightarrow$ false" as follows.

Proof. We prove an equivalent form: **G0** $\Rightarrow$ ($\neg$ **K1**). Suppose **G0** is true. Consider $\mathbf{k}_1^{(1)}(\underline{\mathbf{x}})$ which contains 3 polynomials (see (30) when $n=1$):

$$\mathbf{k}_1^{(1)}(\underline{\mathbf{x}}) = \{\, m_{11}m_{23}m_{31}L,\ m_{21}m_{13}m_{31}L,\ m_{21}m_{13}m_{31}R\,\}. \tag{35}$$

Since $L = m_{13}m_{32}m_{21}$, the first polynomial in $\mathbf{k}_1^{(1)}(\underline{\mathbf{x}})$ is equivalent to $m_{11}m_{23}m_{32}m_{21}m_{13}m_{31}$. Then $\mathbf{k}_1^{(1)}(\underline{\mathbf{x}})$ becomes linearly dependent by substituting $R+L$ for $m_{11}m_{23}m_{32}$ (from **G0** being true). The proof is thus complete. □

We prove "**H2** $\wedge$ **G0** $\Rightarrow$ false" as follow.

Proof. We prove an equivalent form: **G0** $\Rightarrow$ ($\neg$ **H2**). Suppose **G0** is true. Consider $\mathbf{h}_1^{(n)}(\underline{\mathbf{x}})$ in (20). Substituting $R+L$ for $m_{11}m_{23}m_{32}$ (from **G0** being true) and $L = m_{21}m_{13}m_{32}$ to the expression of $\mathbf{h}_1^{(n)}(\underline{\mathbf{x}})$, then we have

$$\begin{aligned}\mathbf{h}_1^{(n)}(\underline{\mathbf{x}}) = \{\, &(R+L)R^n,\ (R+L)R^{n-1}L,\ \cdots,\ (R+L)L^n,\\ &,\ R^nL,\ R^{n-1}L^2,\ \cdots,\ RL^n\,\}.\end{aligned}$$

One can see that $\mathbf{h}_1^{(n)}(\underline{\mathbf{x}})$ becomes linearly dependent when $n\geq 2$. The proof is thus complete. □

We prove "**K2** $\wedge$ **G0** $\Rightarrow$ false" as follow.

Proof. Similarly following the proof of "**K1** $\wedge$ **G0** $\Rightarrow$ false", we further have

$$\begin{aligned}\mathbf{k}_1^{(n)}(\underline{\mathbf{x}}) = m_{21}m_{13}m_{31}\,\{\, &(R+L)L^{n-1},\ (R+L)L^{n-2}R,\\ &\cdots,(R+L)R^{n-1},\ L^n,\ L^{n-1}R,\ \cdots,LR^{n-1},\ R^n\,\},\end{aligned}$$

which becomes linearly dependent when $n\geq 2$. The proof is thus complete. □

APPENDIX H
GENERAL STRUCTURED PROOF FOR THE SUFFICIENCY

In this appendix, we provide Corollary 5, which will be used to prove the graph-theoretic sufficient direction of 3-unicast ANA network for arbitrary $n>0$ values. We need the following additional logic statements to describe the general proof structure.

H-1. The second set of logic statements

Given a 3-unicast ANA network G_{3ANA}, recall the definitions $L = m_{13}m_{32}m_{21}$ and $R = m_{12}m_{23}m_{31}$ (we drop the input argument $\underline{\mathbf{x}}$ for simplicity). By the definition of G_{3ANA}, any channel gains are non-trivial, and thus R and L are non-zero polynomials. Let $\psi_\alpha^{(n)}(R,L)$ and $\psi_\beta^{(n)}(R,L)$ to be some polynomials with respect to $\underline{\mathbf{x}}$, represented by

$$\psi_\alpha^{(n)}(R,L)=\sum_{i=0}^{n}\alpha_i R^{n-i}L^i,\quad \psi_\beta^{(n)}(R,L)=\sum_{j=0}^{n}\beta_j R^{n-j}L^j,$$

with some set of coefficients $\{\alpha_i\}_{i=0}^n$ and $\{\beta_j\}_{j=0}^n$, respectively. Basically, given a value of n and the values of $\{\alpha_i\}_{i=0}^n$ and $\{\beta_j\}_{j=0}^n$, $\psi_\alpha^{(n)}(R,L)$ (resp. $\psi_\beta^{(n)}(R,L)$) represents a linear combination of $\{R^n,\ R^{n-1}L,\ \cdots,\ RL^{n-1},\ L^n\}$, the set of Vandermonde polynomials

We need the following additional logic statements.

- **E0:** Let I_{3ANA} be a finite index set defined by $I_{\text{3ANA}} = \{(i,j):\ \ i,j\in\{1,2,3\} \text{ and } i\neq j\}$. Consider two non-zero polynomial functions $f:\mathbb{F}_q^{|I_{\text{3ANA}}|}\mapsto\mathbb{F}_q$ and $g:\mathbb{F}_q^{|I_{\text{3ANA}}|}\mapsto\mathbb{F}_q$. Then given a G_{3ANA} of interest, there exists some coefficient values $\{\alpha_i\}_{i=0}^n$ and $\{\beta_j\}_{j=0}^n$ such that

$$\begin{aligned}&m_{11}\, f(\{m_{ij}:\forall\,(i,j)\in I_{\text{3ANA}}\})\,\psi_\alpha^{(n)}(R,L)\\ &\qquad = g(\{m_{ij}:\forall\,(i,j)\in I_{\text{3ANA}}\})\,\psi_\beta^{(n)}(R,L),\end{aligned}$$

with (i) At least one of coefficients $\{\alpha_i\}_{i=0}^n$ is non-zero; and (ii) At least one of coefficients $\{\beta_j\}_{j=0}^n$ is non-zero.

Among $\{\alpha_i\}_{i=0}^n$ and $\{\beta_j\}_{j=0}^n$, define i_{st} (resp. j_{st}) as the smallest i (resp. j) such that $\alpha_i\neq 0$ (resp. $\beta_j\neq 0$). Similarly, define i_{end} (resp. j_{end}) as the largest i (resp. j) such that $\alpha_i\neq 0$ (resp. $\beta_j\neq 0$).[5] Then, we can rewrite the above equation as follows:

$$\begin{aligned}&\sum_{i=i_{\text{st}}}^{i_{\text{end}}}\alpha_i\, m_{11}\, f(\{m_{ij}:\forall\,(i,j)\in I_{\text{3ANA}}\})\,R^{n-i}L^i\\ &\quad = \sum_{j=j_{\text{st}}}^{j_{\text{end}}}\beta_j\, g(\{m_{ij}:\forall\,(i,j)\in I_{\text{3ANA}}\})\,R^{n-j}L^j.\end{aligned} \tag{36}$$

- **E1:** Continue from the definition of **E0**. The considered G_{3ANA} satisfies (36) with (i) $f(\{m_{ij}:\forall\,(i,j)\in I_{\text{3ANA}}\}) = m_{23}$; and (ii) $g(\{m_{ij}:\forall\,(i,j)\in I_{\text{3ANA}}\}) = m_{13}m_{21}$. Then, (36) reduces to

$$\sum_{i=i_{\text{st}}}^{i_{\text{end}}}\alpha_i m_{11}m_{23}R^{n-i}L^i = \sum_{j=j_{\text{st}}}^{j_{\text{end}}}\beta_j m_{13}m_{21}R^{n-j}L^j. \tag{37}$$

- **E2:** Continue from the definition of **E0**. The chosen coefficients $\{\alpha_i\}_{i=0}^n$ and $\{\beta_j\}_{j=0}^n$ which satisfy (36) in the given

[5]From definition, $0\leq i_{\text{st}}\leq i_{\text{end}}\leq n$ and $0\leq j_{\text{st}}\leq j_{\text{end}}\leq n$.

$G_{3\text{ANA}}$ also satisfy (i) $\alpha_k \neq \beta_k$ for some $k \in \{0,...,n\}$; and (ii) either $\alpha_0 \neq 0$ or $\beta_n \neq 0$ or $\alpha_k \neq \beta_{k-1}$ for some $k \in \{1,...,n\}$.

One can see that whether the above logic statements are true or false depends on the polynomials m_{ij} and on the $\{\alpha_i\}_{i=0}^n$ and $\{\beta_j\}_{j=0}^n$ values being considered.

The following logic statements are well-defined if and only if **E0** is true. Whether the following logic statements are true depends on the values of i_{st}, i_{end}, j_{st}, and j_{end}.

- **C0:** $i_{\text{st}} > j_{\text{st}}$ and $i_{\text{end}} = j_{\text{end}}$.
- **C1:** $i_{\text{st}} < j_{\text{st}}$.
- **C2:** $i_{\text{st}} > j_{\text{st}}$.
- **C3:** $i_{\text{st}} = j_{\text{st}}$.
- **C4:** $i_{\text{end}} < j_{\text{end}}$.
- **C5:** $i_{\text{end}} > j_{\text{end}}$.
- **C6:** $i_{\text{end}} = j_{\text{end}}$.

We also define the following statements for the further organization.

- **D1:** $\mathsf{GCD}(m_{12}^{l_1} m_{23}^{l_1} m_{31}^{l_1},\, m_{32}) = m_{32}$ for some integer $l_1 > 0$.
- **D2:** $\mathsf{GCD}(m_{13}^{l_2} m_{32}^{l_2} m_{21}^{l_2},\, m_{23}) = m_{23}$ for some integer $l_2 > 0$.
- **D3:** $\mathsf{GCD}(\, m_{11} m_{13}^{l_3} m_{32}^{l_3} m_{21}^{l_3},\, m_{12} m_{31}) = m_{12} m_{31}$ for some integer $l_3 > 0$.
- **D4:** $\mathsf{GCD}(\, m_{11} m_{12}^{l_4} m_{23}^{l_4} m_{31}^{l_4},\, m_{13} m_{21}) = m_{13} m_{21}$ for some integer $l_4 > 0$.
- **D5:** $\mathsf{GCD}(\, m_{11} m_{12}^{l_5} m_{23}^{l_5} m_{31}^{l_5},\, m_{32}) = m_{32}$ for some integer $l_5 > 0$.
- **D6:** $\mathsf{GCD}(\, m_{11} m_{13}^{l_6} m_{32}^{l_6} m_{21}^{l_6},\, m_{23}) = m_{23}$ for some integer $l_6 > 0$.

H-2. General Sufficiency Proof Structure

We prove the following "logic relationships," which will be used for the proof of Corollary 5.

- **S1**: **D1** $\Rightarrow$ **D5**.
- **S2**: **D2** $\Rightarrow$ **D6**.
- **S3**: **E0** $\wedge$ **E1** $\wedge$ **C1** $\Rightarrow$ **D4** $\wedge$ **D5**.
- **S4**: **E0** $\wedge$ **E1** $\wedge$ **C2** $\Rightarrow$ **D1**.
- **S5**: **G1** $\wedge$ **E0** $\wedge$ **E1** $\wedge$ **C3** $\Rightarrow$ **D4**.
- **S6**: **E0** $\wedge$ **E1** $\wedge$ **C4** $\Rightarrow$ **D2** $\wedge$ **D3**.
- **S7**: **E0** $\wedge$ **E1** $\wedge$ **C5** $\Rightarrow$ **D3**.
- **S8**: **G1** $\wedge$ **E0** $\wedge$ **E1** $\wedge$ **C6** $\Rightarrow$ **D2**.
- **S9**: **E0** $\wedge$ **E1** $\wedge$ **C0** $\Rightarrow$ **E2**.
- **S10**: **G1** $\wedge$ **E0** $\wedge$ **E1** $\wedge$ ($\neg$ **C0**) $\Rightarrow$ (**D1** $\wedge$ **D3**) $\vee$ (**D2** $\wedge$ **D4**) $\vee$ (**D3** $\wedge$ **D4**).
- **S11**: **LNR** $\wedge$ **G1** $\wedge$ **E0** $\wedge$ **D1** $\wedge$ **D3** $\Rightarrow$ false.
- **S12**: **LNR** $\wedge$ **G1** $\wedge$ **E0** $\wedge$ **D2** $\wedge$ **D4** $\Rightarrow$ false.
- **S13**: **LNR** $\wedge$ **G1** $\wedge$ **G2** $\wedge$ **E0** $\wedge$ **E1** $\wedge$ **E2** $\wedge$ **D1** $\wedge$ **D2** $\Rightarrow$ false.
- **S14**: **LNR** $\wedge$ **G1** $\wedge$ **E0** $\wedge$ **D3** $\wedge$ **D4** $\Rightarrow$ false.

The proofs of **S1** to **S10** are relegated to Appendix L. The proofs of **S11** to **S14** are relegated to Appendices M, O, P, and R, respectively. Note that the above **S1** to **S14** relationships greatly simplify the analysis of finding the graph-theoretic conditions for the feasibility of the 3-unicast ANA network. This observation is summarized in Corollary 5.

Corollary 5: Let $\mathbf{h}(\underline{\mathbf{x}})$ be a set of (arbitrarily chosen) polynomials based on the 9 channel gains m_{ij} of the 3-unicast ANA network, and define **X** to be the logic statement that $\mathbf{h}(\underline{\mathbf{x}})$ is linearly independent. Let **G** to be an arbitrary logic statement in the 3-unicast ANA network. If we can prove that

(A) **G** $\wedge$ ($\neg$ **X**) $\Rightarrow$ **E0** $\wedge$ **E1** $\wedge$ ($\neg$ **C0**),

then the logic relationship **LNR** $\wedge$ **G1** $\wedge$ **G** $\wedge$ ($\neg$ **X**) $\Rightarrow$ false must also hold.

Also, if we can prove that

(B) **G** $\wedge$ ($\neg$ **X**) $\Rightarrow$ **E0** $\wedge$ **E1** $\wedge$ **C0**,

then the logic relationship **LNR** $\wedge$ **G1** $\wedge$ **G2** $\wedge$ **G** $\wedge$ ($\neg$ **X**) $\Rightarrow$ false must also hold.

Proof. First, notice that **S11**, **S12**, and **S14** jointly imply

$$\mathbf{LNR} \wedge \mathbf{G1} \wedge \mathbf{E0} \wedge \{(\mathbf{D1} \wedge \mathbf{D3}) \vee (\mathbf{D2} \wedge \mathbf{D4}) \vee (\mathbf{D3} \wedge \mathbf{D4})\} \Rightarrow \text{ false.} \tag{38}$$

Then, (38), jointly with **S10** further imply

$$\mathbf{LNR} \wedge \mathbf{G1} \wedge \mathbf{E0} \wedge \mathbf{E1} \wedge (\neg\, \mathbf{C0}) \Rightarrow \text{ false.} \tag{39}$$

Note that by definition **C0** is equivalent to **C2** $\wedge$ **C6**. Then **S4** and **S8** jointly imply

$$\mathbf{G1} \wedge \mathbf{E0} \wedge \mathbf{E1} \wedge \mathbf{C0} \Rightarrow \mathbf{D1} \wedge \mathbf{D2}. \tag{40}$$

Then, (40), **S9**, and **S13** jointly imply

$$\mathbf{LNR} \wedge \mathbf{G1} \wedge \mathbf{G2} \wedge \mathbf{E0} \wedge \mathbf{E1} \wedge \mathbf{C0} \Rightarrow \text{ false.} \tag{41}$$

Now we prove the result using (39) and (41). Suppose we can also prove (A) **G** $\wedge$ ($\neg$ **X**) $\Rightarrow$ **E0** $\wedge$ **E1** $\wedge$ ($\neg$ **C0**). Then, one can see that this, jointly with (39), implies **LNR** $\wedge$ **G1** $\wedge$ **G** $\wedge$ ($\neg$ **X**) $\Rightarrow$ false. Similarly, (B) **G** $\wedge$ ($\neg$ **X**) $\Rightarrow$ **E0** $\wedge$ **E1** $\wedge$ **C0** and (41) jointly imply **LNR** $\wedge$ **G1** $\wedge$ **G2** $\wedge$ **G** $\wedge$ ($\neg$ **X**) $\Rightarrow$ false. The proof is thus complete. ☐

H-3. The insight on proving the sufficiency

To prove the sufficiency directions, we need to show that a set of polynomials is linearly independent given any 3-unicast ANA network, for example, "**LNR** $\wedge$ **G** $\Rightarrow$ **X**". To that end, we prove the equivalent relationship "**LNR** $\wedge$ **G** $\wedge$ ($\neg$ **X**) $\Rightarrow$ false." Focusing on the linear dependence condition $\neg$ **X**, although there are many possible cases, allows us to use the subgraph property (Proposition 2) to simplify the proof. Further, we use the logic statements **S3** to **S10** to convert all the cases of the linear dependence condition into the greatest common divisor statements **D1** to **D6**, for which the channel gain property (Proposition 3) further helps us to find the corresponding graph-theoretic implication.

APPENDIX I
PROOFS OF "LNR $\wedge$ G1 $\Rightarrow$ H1" AND "LNR $\wedge$ G1 $\wedge$ G2 $\Rightarrow$ K1 $\vee$ H2 $\vee$ K2"

As discussed in Appendix H, we use Corollary 5 to prove the sufficiency directions. We first show that (i) **LNR** $\wedge$ **G1** $\wedge$ **G2** $\Rightarrow$ **H2**; and (ii) **LNR** $\wedge$ **G1** $\wedge$ **G2** $\Rightarrow$ **K2**. Then the remaining sufficiency directions "**LNR** $\wedge$ **G1** $\Rightarrow$ **H1**" and "**LNR** $\wedge$ **G1** $\wedge$ **G2** $\Rightarrow$ **K1**" are derived using simple facts of "**H2** $\Rightarrow$ **H1**" and "**K2** $\Rightarrow$ **K1**", respectively. Note that **H2** $\Rightarrow$ **H1** is straightforward since $\mathbf{h}_1^{(1)}(\underline{\mathbf{x}})$ is a subset of the polynomials $\mathbf{h}_1^{(n)}(\underline{\mathbf{x}})$ (multiplied by a common factor) and whenever $\mathbf{h}_1^{(n)}(\underline{\mathbf{x}})$ is linearly independent, so is $\mathbf{h}_1^{(1)}(\underline{\mathbf{x}})$. Similarly, we have **K2** $\Rightarrow$ **K1**.

We prove "**LNR** $\wedge$ **G1** $\wedge$ **G2** $\Rightarrow$ **H2**" as follows.

Proof. By the definition of linear dependence, $\neg$ **H2** implies that there exist two sets of coefficients $\{\alpha_i\}_{i=0}^n$ and $\{\beta_j\}_{j=0}^{n-1}$ such that

$$\sum_{i=0}^{n} \alpha_i m_{11} m_{23} R^{n-i} L^i = \sum_{j=0}^{n-1} \beta_j m_{13} m_{21} R^{n-j} L^j. \quad (42)$$

We will now argue that at least one of $\{\alpha_i\}_{i=0}^n$ and at least one of $\{\beta_j\}_{j=0}^{n-1}$ are non-zero if $L \not\equiv R$. The reason is as follows. For example, suppose that all $\{\beta_j\}_{j=0}^{n-1}$ are zero. By definition (iv) of the 3-unicast ANA network, any channel gain is non-trivial. Thus $m_{11}m_{23}$ is a non-trivial polynomial. Then, (42) becomes $\sum_{i=0}^{n} \alpha_i R^{n-i} L^i = 0$, which implies that the set of $(n+1)$ polynomials, $\tilde{\mathbf{h}}(\underline{\mathbf{x}}) = \{R^n, R^{n-1}L, ..., RL^{n-1}, L^n\}$, is linearly dependent. By Proposition 1, the determinant of the Vandermonde matrix $[\tilde{\mathbf{h}}(\underline{\mathbf{x}}^{(k)})]_{k=1}^{n+1}$ is thus zero, which implies $L(\underline{\mathbf{x}}) \equiv R(\underline{\mathbf{x}})$. This contradicts the assumption **LNR**. The fact that not all $\{\alpha_i\}_{i=0}^n$ are zero can be proven similarly.

As a result, there exist two sets of coefficients $\{\alpha_i\}_{i=0}^n$ and $\{\beta_j\}_{j=0}^{n-1}$ with at least one of each group being non-zero such that the following logic relationship holds:

$$\mathbf{LNR} \wedge (\neg\, \mathbf{H2}) \Rightarrow \mathbf{E0} \wedge \mathbf{E1}. \quad (43)$$

Then, note that (43) implies

$$\mathbf{LNR} \wedge (\neg\, \mathbf{C0}) \wedge (\neg\, \mathbf{H2}) \Rightarrow \mathbf{E0} \wedge \mathbf{E1} \wedge (\neg\, \mathbf{C0}),$$
$$\text{and} \quad \mathbf{LNR} \wedge \mathbf{C0} \wedge (\neg\, \mathbf{H2}) \Rightarrow \mathbf{E0} \wedge \mathbf{E1} \wedge \mathbf{C0}.$$

Applying Corollary 5(A) (substituting **G** by **LNR** $\wedge$ ($\neg$ **C0**) and **X** by **H2**, respectively), the former implies **LNR** $\wedge$ **G1** $\wedge$ ($\neg$ **C0**) $\wedge$ ($\neg$ **H2**) $\Rightarrow$ false. By Corollary 5(B), the latter implies **LNR** $\wedge$ **G1** $\wedge$ **G2** $\wedge$ **C0** $\wedge$ ($\neg$ **H2**) $\Rightarrow$ false. These jointly imply

$$\mathbf{LNR} \wedge \mathbf{G1} \wedge \mathbf{G2} \wedge (\neg\, \mathbf{H2}) \Rightarrow \text{false},$$

which is equivalent to **LNR** $\wedge$ **G1** $\wedge$ **G2** $\Rightarrow$ **H2**. The proof is thus complete. □

We prove "**LNR** $\wedge$ **G1** $\wedge$ **G2** $\Rightarrow$ **K2**" as follows.

Proof. We will only show the logic relationship "**LNR** $\wedge$ ($\neg$ **K2**) $\Rightarrow$ **E0** $\wedge$ **E1**" so that the rest can be proved by Corollary 5 as in the proof of "**LNR** $\wedge$ **G1** $\wedge$ **G2** $\Rightarrow$ **H2**". Suppose $\neg$ **K2** is true. Then, there exists two sets of coefficients $\{\alpha_i\}_{i=1}^n$ and $\{\beta_j\}_{j=0}^n$ such that

$$\sum_{i=1}^{n} \alpha_i m_{11} m_{23} R^{n-i} L^i = \sum_{j=0}^{n} \beta_j m_{13} m_{21} R^{n-j} L^j. \quad (44)$$

One can easily see that, similarly to the above proof, the assumption **LNR** results in the not-being-all-zero condition on both $\{\alpha_i\}_{i=1}^n$ and $\{\beta_j\}_{j=0}^n$, which in turn implies that "**LNR** $\wedge$ ($\neg$ **K2**) $\Rightarrow$ **E0** $\wedge$ **E1**". The proof is thus complete. □

Appendix J
Proofs of the properties of G3, G4, ¬G3, and ¬G4

We prove Properties 1 and 2 of **G3** as follows.

Proof. Suppose **G3** is true, that is, $\overline{S}_2 \cap \overline{D}_3 = \emptyset$. Consider e_2^*, the most downstream edge of $\mathsf{1cut}(s_2; d_1) \cap \mathsf{1cut}(s_2; d_3)$ and e_3^*, the most upstream edge of $\mathsf{1cut}(s_1; d_3) \cap \mathsf{1cut}(s_2; d_3)$. If either $e_2^* = e_{s_2}$ or $e_3^* = e_{d_3}$ (or both), we must have $e_2^* \prec e_3^*$ otherwise it contradicts definitions (ii) and (iii) of the 3-unicast ANA network. Consider the case in which both $e_2^* \neq e_{s_2}$ and $e_3^* \neq e_{d_3}$. Recall the definitions of $\overline{S}_2 \triangleq \mathsf{1cut}(s_2; d_1) \cap \mathsf{1cut}(s_2; d_3) \backslash \{e_{s_2}\}$ and $\overline{D}_3 \triangleq \mathsf{1cut}(s_1; d_3) \cap \mathsf{1cut}(s_2; d_3) \backslash \{e_{d_3}\}$. We thus have $e_2^* \in \overline{S}_2$ and $e_3^* \in \overline{D}_3$. By the assumption $\overline{S}_2 \cap \overline{D}_3 = \emptyset$ and Lemma 3, we must have $e_2^* \prec e_3^*$ as well.

From the construction of e_2^* and e_3^*, the channel gains m_{13}, m_{21}, and m_{23} can be expressed as $m_{13} = m_{e_{s_1}; e_3^*} m_{e_3^*; e_{d_3}}$, $m_{21} = m_{e_{s_2}; e_2^*} m_{e_2^*; e_{d_1}}$, and $m_{23} = m_{e_{s_2}; e_2^*} m_{e_2^*; e_3^*} m_{e_3^*; e_{d_3}}$. Moreover, we have both $\mathsf{GCD}(\, m_{e_{s_1}; e_3^*},\ m_{e_{s_2}; e_2^*} m_{e_2^*; e_3^*}) \equiv 1$ and $\mathsf{GCD}(\, m_{e_2^*; e_3^*} m_{e_3^*; e_{d_3}},\ m_{e_2^*; e_{d_1}}) \equiv 1$ otherwise it violates that e_2^* (resp. e_3^*) is the most downstream (resp. upstream) edge of $\overline{S}_2$ (resp. $\overline{D}_3$). The same argument also leads to $\mathsf{GCD}(\, m_{13},\ m_{e_2^*; e_3^*}) \equiv 1$ and $\mathsf{GCD}(\, m_{21},\ m_{e_2^*; e_3^*}) \equiv 1$. □

We prove Properties 1, 2, and 3 of $\neg$ **G3** as follows.

Proof. Suppose $\neg$ **G3** is true, i.e., $\overline{S}_2 \cap \overline{D}_3 \neq \emptyset$. Choose the most upstream e_u^{23} and most downstream e_v^{23} edges in $\overline{S}_2 \cap \overline{D}_3$. Then, the channel gains m_{13}, m_{21}, and m_{23} can be expressed as $m_{13} = m_{e_{s_1}; e_u^{23}} m_{e_u^{23}; e_v^{23}} m_{e_v^{23}; e_{d_3}}$, $m_{21} = m_{e_{s_2}; e_u^{23}} m_{e_u^{23}; e_v^{23}} m_{e_v^{23}; e_{d_1}}$, and $m_{23} = m_{e_{s_2}; e_u^{23}} m_{e_u^{23}; e_v^{23}} m_{e_v^{23}; e_{d_3}}$. Moreover, we must have both $\mathsf{GCD}(m_{e_{s_1}; e_u^{23}}, m_{e_{s_2}; e_u^{23}}) \equiv 1$ and $\mathsf{GCD}(m_{e_v^{23}; e_{d_3}}, m_{e_v^{23}; e_{d_1}}) \equiv 1$ otherwise it violates Lemma 3 and/or e_u^{23} (resp. e_v^{23}) being the most upstream (resp. downstream) edge among $\overline{S}_2 \cap \overline{D}_3$. For example, if $\mathsf{GCD}(\, m_{e_{s_1}; e_u^{23}},\ m_{e_{s_2}; e_u^{23}}) \not\equiv 1$, then by Lemma 7 and the assumption $e_u^{23} \in \overline{S}_2 \cap \overline{D}_3 \subset \overline{D}_3$, there must exist an edge $e \in \overline{D}_3$ such that $e \prec e_u^{23}$. If such edge e is also in $\overline{S}_2$, then this e violates the construction that e_u^{23} is the most upstream edge of $\overline{S}_2 \cap \overline{D}_3$. If such edge e is not in $\overline{S}_2$, then it contradicts the conclusion in Lemma 3.

We now prove Property 3 of $\neg$ **G3**. Suppose that at least one of $\{e_u^{23}, e_v^{23}\}$ is not an 1-edge cut separating s_1 and $\mathsf{head}(e_v^{23})$. Say $e_u^{23} \notin \mathsf{1cut}(s_1; \mathsf{head}(e_v^{23}))$, then s_1 can reach $\mathsf{head}(e_v^{23})$ without using e_u^{23}. Since $\mathsf{head}(e_v^{23})$ reaches d_3, we can create an s_1-to-d_3 path not using e_u^{23}. This contradicts the construction that $e_u^{23} \in \overline{S}_2 \cap \overline{D}_3 \subset \overline{D}_3$. Similarly, we can also prove that $e_v^{23} \notin \mathsf{1cut}(s_1; \mathsf{head}(e_v^{23}))$ leads to a contradiction. Therefore, we have proven $\{e_u^{23}, e_v^{23}\} \subset \mathsf{1cut}(s_1; \mathsf{head}(e_v^{23}))$. Symmetrically applying the above arguments, we can also prove that $\{e_u^{23}, e_v^{23}\} \subset \mathsf{1cut}(\mathsf{tail}(e_u^{23}); d_1)$.

Now consider an s_1-to-d_1 path P such that there exists one vertex $w \in P$ satisfying $\mathsf{tail}(e_u^{23}) \preceq w \preceq \mathsf{head}(e_v^{23})$. If the path of interest P does not use e_u^{23} and $w = \mathsf{tail}(e_u^{23})$, then $\mathsf{tail}(e_u^{23})$ can follow P to d_1 without using e_u^{23}, which contradicts $e_u^{23} \in \mathsf{1cut}(\mathsf{tail}(e_u^{23}); d_1)$. If P does not use e_u^{23} and $\mathsf{tail}(e_u^{23}) \prec w \preceq \mathsf{head}(e_v^{23})$, then s_1 can follow P to w and reach $\mathsf{head}(e_v^{23})$ without using e_u^{23}, which contradicts $e_u^{23} \in \mathsf{1cut}(s_1; \mathsf{head}(e_v^{23}))$. By the similar arguments, we can also prove the case when P does not use e_v^{23} leads to a contradiction. Therefore, we must have $\{e_u^{23}, e_v^{23}\} \subset P$. The proof is complete. □

By swapping the roles of s_2 and s_3, and the roles of d_2 and d_3, the above proofs can also be used to prove Properties 1 and 2 of **G4** and Properties 1, 2, and 3, of $\neg$ **G4**.

Appendix K
Proofs of N1 to N9

We prove **N1** as follows.

Proof. Instead of proving directly, we prove **H2** $\Rightarrow$ **H1** and use the existing result of "**LNR** $\wedge$ **G1** $\Leftarrow$ **H1**" established in the proof of Proposition 5. **H2** $\Rightarrow$ **H1** is straightforward since $\mathbf{h}_1^{(1)}(\underline{\mathbf{x}})$ is a subset of the polynomials $\mathbf{h}_1^{(n)}(\underline{\mathbf{x}})$ (multiplied by a common factor) and whenever $\mathbf{h}_1^{(n)}(\underline{\mathbf{x}})$ is linearly independent, so is $\mathbf{h}_1^{(1)}(\underline{\mathbf{x}})$. The proof is thus complete. ☐

We prove **N2** as follows.

Proof. We prove an equivalent relationship: $(\neg$ **LNR**$)\vee(\neg$ **G1**$)\Rightarrow(\neg$ **K1**$)$. Consider $\mathbf{k}_1^{(1)}(\underline{\mathbf{x}})$ as in (35). Suppose $G_{3\text{ANA}}$ satisfies $(\neg$ **LNR**$)\vee(\neg$ **G1**$)$, which means $G_{3\text{ANA}}$ satisfies either $L(\underline{\mathbf{x}})\equiv R(\underline{\mathbf{x}})$ or $m_{11}m_{23}\equiv m_{21}m_{13}$ or $m_{11}m_{32}\equiv m_{31}m_{12}$. If $L(\underline{\mathbf{x}})\equiv R(\underline{\mathbf{x}})$, then we notice that $m_{21}m_{13}m_{31}L\equiv m_{21}m_{13}m_{31}R$ and $\mathbf{k}_1^{(1)}(\underline{\mathbf{x}})$ is thus linearly dependent. If $m_{11}m_{23}\equiv m_{21}m_{13}$, then we notice $m_{11}m_{23}m_{31}L\equiv m_{21}m_{13}m_{31}L$. Similarly if $m_{11}m_{32}\equiv m_{31}m_{12}$, then we have $m_{11}m_{23}m_{31}L\equiv m_{21}m_{13}m_{31}R$. The proof is thus complete. ☐

Following similar arguments used in proving **N2**, i.e., **K2** $\Rightarrow$ **K1**, one can easily prove **N3**.

We prove **N4** as follows.

Proof. $(\neg$ **G2**$)\wedge$ **G3** $\wedge$ **G4** implies that s_1 cannot reach d_1 on $G_{3\text{ANA}}$. This violates the definition (iv) of the 3-unicast ANA network. ☐

We prove **N5** as follow.

Proof. We prove an equivalent relationship: $(\neg$ **G2**$)\wedge(\neg$ **G3**$)\wedge$ **G4** $\Rightarrow(\neg$ **G1**$)$. Suppose $(\neg$ **G2**$)\wedge(\neg$ **G3**$)\wedge$ **G4** is true. Then the most upstream edge of $\overline{S}_2\cap\overline{D}_3$ is an 1-edge cut separating s_1 and d_1. Therefore we have $\mathsf{EC}(\{s_1,s_2\};\{d_1,d_3\})=1$ and thus by Corollary 1, $m_{11}m_{23}\equiv m_{21}m_{13}$. This further implies that **G1** is false. ☐

By swapping the roles of s_2 and s_3, and the roles of d_2 and d_3, the above **N5** proof can also be used to prove **N6**.

We prove **N7** as follows.

Proof. Suppose **LNR** $\wedge(\neg$ **G3**$)\wedge(\neg$ **G4**$)\wedge(\neg$ **G5**$)$ is true. From **LNR** being true, any $\overline{S}_2\cap\overline{D}_3$ edge and any $\overline{S}_3\cap\overline{D}_2$ edge must be distinct, otherwise (if there exists an edge $e\in\overline{S}_2\cap\overline{S}_3\cap\overline{D}_2\cap\overline{D}_3$) it contradicts the assumption **LNR** by Proposition 4. From **G5** being false, we have either $e_u^{23}=e_u^{32}$ or both e_u^{23} and e_u^{32} are not reachable from each other. But $e_u^{23}=e_u^{32}$ cannot be true by the assumption **LNR**.

Now we prove **G6**, i.e., any vertex w' where $\mathsf{tail}(e_u^{23})\preceq w'\preceq\mathsf{head}(e_v^{23})$ and any vertex w'' where $\mathsf{tail}(e_u^{32})\preceq w''\preceq\mathsf{head}(e_v^{32})$ are not reachable from each other. Suppose not and assume that some vertex w' satisfying $\mathsf{tail}(e_u^{23})\preceq w'\preceq\mathsf{head}(e_v^{23})$ and some vertex w'' satisfying $\mathsf{tail}(e_u^{32})\preceq w''\preceq\mathsf{head}(e_v^{32})$ are reachable from each other. Since s_1 can reach $\mathsf{tail}(e_u^{23})$ or $\mathsf{tail}(e_u^{32})$ and d_1 can be reached from $\mathsf{head}(e_v^{23})$ or $\mathsf{head}(e_v^{32})$ by Property 1 of $\neg$ **G3** and $\neg$ **G4**, we definitely have an s_1-to-d_1 path P who uses both w' and w''. The reason is that if $w'\preceq w''$, then s_1 can first reach $\mathsf{tail}(e_u^{23})$, visit w', w'', and $\mathsf{head}(e_v^{32})$, and finally arrive at d_1. The case when $w''\preceq w'$ can be proven by symmetry. By Property 3 of $\neg$ **G3**, such path must use $\{e_u^{23},e_v^{23}\}$. Similarly by Property 3 of $\neg$ **G4**, such path must also use $\{e_u^{32},e_v^{32}\}$. Together with the above discussion that any $\overline{S}_2\cap\overline{D}_3$ edge and any $\overline{S}_3\cap\overline{D}_2$ edge are distinct, this implies that all four edges $\{e_u^{23},e_v^{23},e_u^{32},e_v^{32}\}$ are not only distinct but also used by a single path P. However, this contradicts the assumption **LNR** $\wedge(\neg$ **G5**$)$ that e_u^{23} and e_u^{32} are not reachable from each other. ☐

We prove **N8** as follows.

Proof. Suppose **G1** $\wedge(\neg$ **G2**$)\wedge(\neg$ **G3**$)\wedge(\neg$ **G4**$)\wedge$ **G5** is true. Consider e_u^{23} and e_u^{32}, the most upstream edges of $\overline{S}_2\cap\overline{D}_3$ and $\overline{S}_3\cap\overline{D}_2$, respectively. Say we have $e_u^{23}\prec e_u^{32}$. Then $\neg$ **G2** implies that removing e_u^{23} will disconnect s_1 and d_1. Therefore, $e_u^{23}\in\overline{S}_2\cap\overline{D}_3$ also belongs to $\mathsf{1cut}(s_1;d_1)$. This further implies that we have $\mathsf{EC}(\{s_1,s_2\};\{d_1,d_3\})=1$ and thus $G_{3\text{ANA}}$ satisfies $m_{11}m_{23}\equiv m_{13}m_{21}$. However, this contradicts the assumption that **G1** is true. Similar arguments can be applied to show that the case when $e_u^{32}\prec e_u^{23}$ also contradicts **G1**. The proof of **N8** is thus complete. ☐

We prove **N9** as follows.

Proof. Suppose that $(\neg$ **G2**$)\wedge(\neg$ **G3**$)\wedge(\neg$ **G4**$)\wedge(\neg$ **G5**$)\wedge$ **G6** is true. Consider e_u^{23} and e_u^{32}, the most upstream edges of $\overline{S}_2\cap\overline{D}_3$ and $\overline{S}_3\cap\overline{D}_2$, respectively. From $(\neg$ **G5**$)\wedge$ **G6** being true, one can see that e_u^{23} and e_u^{32} are not only distinct but also not reachable from each other. Thus by $\neg$ **G2** being true, $\{e_u^{23},e_u^{32}\}$ constitutes an edge cut separating s_1 and d_1. Note from Property 1 of $\neg$ **G3** and $\neg$ **G4** that s_1 can reach d_1 through either e_u^{23} or e_u^{32}. Since e_u^{23} and e_u^{32} are not reachable from each other, both have to be removed to disconnect s_1 and d_1 (removing only one of them is not enough).

From **G6** being true, any vertex w' where $\mathsf{tail}(e_u^{23})\preceq w'\preceq\mathsf{head}(e_v^{23})$ and any vertex w'' where $\mathsf{tail}(e_u^{32})\preceq w''\preceq\mathsf{head}(e_v^{32})$ are not reachable from each other. Thus e_u^{23} (resp. e_u^{32}) cannot reach e_v^{32} (resp. e_v^{23}). Moreover, e_v^{23} and e_v^{32} are not only distinct but also not reachable from each other. This implies that e_u^{23} (resp. e_u^{32}) can only reach e_v^{23} (resp. e_v^{32}) if $e_u^{23}\neq e_v^{23}$ (resp. $e_u^{32}\neq e_v^{32}$). Then the above discussions further that imply $\{e_v^{23},e_v^{32}\}$ is also an edge cut separating s_1 and d_1.

Let $m'_{11}=m_{e_{s_1};e_u^{23}}m_{e_u^{23};e_v^{23}}m_{e_v^{23};e_{d_1}}$, which takes into account the overall path gain from s_1 to d_1 for all paths that use both e_u^{23} and e_v^{23}. Similarly denote $m''_{11}=m_{e_{s_1};e_u^{32}}m_{e_u^{32};e_v^{32}}m_{e_v^{32};e_{d_1}}$ to be the overall path gain from s_1 to d_1 for all paths that use both e_u^{32} and e_v^{32}. Then the discussions so far imply that the channel gain m_{11} consists of two polynomials: $m_{11}=m'_{11}+m''_{11}$. Then, it follows that

$$
\begin{aligned}
&m_{11}m_{23}m_{32}=(m'_{11}+m''_{11})\,m_{23}m_{32}\\
&=(m_{e_{s_1};e_u^{23}}m_{e_u^{23};e_v^{23}}m_{e_v^{23};e_{d_1}})\,m_{23}m_{32}\\
&\;+(m_{e_{s_1};e_u^{32}}m_{e_u^{32};e_v^{32}}m_{e_v^{32};e_{d_1}})\,m_{23}m_{32}\\
&=(m_{e_{s_1};e_u^{23}}m_{e_u^{23};e_v^{23}}m_{e_v^{23};e_{d_1}})(m_{e_{s_2};e_u^{23}}m_{e_u^{23};e_v^{23}}m_{e_v^{23};e_{d_3}})\,m_{32}\\
&\;+(m_{e_{s_1};e_u^{32}}m_{e_u^{32};e_v^{32}}m_{e_v^{32};e_{d_1}})\,m_{23}\,(m_{e_{s_3};e_u^{32}}m_{e_u^{32};e_v^{32}}m_{e_v^{32};e_{d_2}})\\
&=(m_{e_{s_1};e_u^{23}}m_{e_u^{23};e_v^{23}}m_{e_v^{23};e_{d_3}})\,m_{32}\,(m_{e_{s_2};e_u^{23}}m_{e_u^{23};e_v^{23}}m_{e_v^{23};e_{d_1}})\\
&\;+(m_{e_{s_1};e_u^{32}}m_{e_u^{32};e_v^{32}}m_{e_v^{32};e_{d_2}})\,m_{23}\,(m_{e_{s_3};e_u^{32}}m_{e_u^{32};e_v^{32}}m_{e_v^{32};e_{d_1}})\\
&=m_{13}m_{32}m_{21}+m_{12}m_{23}m_{31}=L+R.
\end{aligned}
$$

where the third and fourth equalities follow from the Property 1 of both $\neg$ **G3** and $\neg$ **G4**. The proof is thus complete. $\square$

Appendix L
Proofs of S1 to S10

We prove **S1** as follows.

Proof. Suppose **D1** is true, that is, $G_{3\text{ANA}}$ satisfies $\text{GCD}(m_{12}^{l_1}m_{23}^{l_1}m_{31}^{l_1},\, m_{32}) = m_{32}$ for some integer $l_1 > 0$. Then $G_{3\text{ANA}}$ also satisfies $\text{GCD}(m_{11}m_{12}^{l_1}m_{23}^{l_1}m_{31}^{l_1},\, m_{32}) = m_{32}$ obviously. Thus we have **D5**. $\square$

By swapping the roles of s_2 and s_3, and the roles of d_2 and d_3, the proof for **S1** can be applied symmetrically to the proof for **S2**.

We prove **S3** as follows.

Proof. Suppose **E0** $\wedge$ **E1** $\wedge$ **C1** is true. By **E0** $\wedge$ **E1** being true, $G_{3\text{ANA}}$ of interest satisfies (37). By the definition of **C1**, we have $i_{\text{st}} < j_{\text{st}}$.

By (37), we can divide $L^{i_{\text{st}}}$ on both sides. Then we have

$$\sum_{i=i_{\text{st}}}^{i_{\text{end}}} \alpha_i m_{11}m_{23}R^{n-i}L^{i-i_{\text{st}}} = \sum_{j=j_{\text{st}}}^{j_{\text{end}}} \beta_j m_{13}m_{21}R^{n-j}L^{j-i_{\text{st}}}.$$

Since $i_{\text{st}} < j_{\text{st}}$, each term with non-zero β_j in the right-hand side (RHS) has L as a common factor. Similarly, each term with non-zero α_i on the left-hand side (LHS) has L as a common factor except for the first term (since $\alpha_{i_{\text{st}}} \neq 0$). Therefore the first term $\alpha_{i_{\text{st}}}m_{11}m_{23}R^{n-i_{\text{st}}}$ must contain $L = m_{13}m_{32}m_{21}$ as a factor, which implies $\text{GCD}(m_{11}m_{12}^{n-i_{\text{st}}}m_{23}^{n-i_{\text{st}}+1}m_{31}^{n-i_{\text{st}}},\, m_{13}m_{32}m_{21}) = m_{13}m_{32}m_{21}$. Since $i_{\text{st}} < j_{\text{st}} \leq n$, we have $n - i_{\text{st}} \geq 1$. Hence, we have $\text{GCD}(m_{11}m_{12}^{k}m_{23}^{k+1}m_{31}^{k},\, m_{13}m_{32}m_{21}) = m_{13}m_{32}m_{21}$ for some integer $k \geq 1$. This observation implies the following two statements. Firstly, $\text{GCD}(m_{11}m_{12}^{l_4}m_{23}^{l_4}m_{31}^{l_4},\, m_{13}m_{21}) = m_{13}m_{21}$ when $l_4 = k+1 \geq 2$ and thus we have proven **D4**. Secondly, $\text{GCD}(m_{11}m_{12}^{l_5}m_{23}^{l_5}m_{31}^{l_5},\, m_{32}) = m_{32}$ when $l_5 = k+1 \geq 2$ and thus we have proven **D5**. The proof is thus complete. $\square$

We prove **S4** as follows.

Proof. Suppose **E0** $\wedge$ **E1** $\wedge$ **C2** is true. Then $G_{3\text{ANA}}$ of interest satisfies (37) and we have $i_{\text{st}} > j_{\text{st}}$.

We now divide $L^{j_{\text{st}}}$ on both sides of (37), which leads to

$$\sum_{i=i_{\text{st}}}^{i_{\text{end}}} \alpha_i m_{11}m_{23}R^{n-i}L^{i-j_{\text{st}}} = \sum_{j=j_{\text{st}}}^{j_{\text{end}}} \beta_j m_{13}m_{21}R^{n-j}L^{j-j_{\text{st}}}.$$

Each term with non-zero α_i on the LHS has L as a common factor. Similarly, each term with non-zero β_j on the RHS has L as a common factor except for the first term (since $\beta_{j_{\text{st}}} \neq 0$). As a result, the first term $\beta_{j_{\text{st}}}m_{13}m_{21}R^{n-j_{\text{st}}}$ must contain $L = m_{13}m_{32}m_{21}$ as a factor. This implies that $\text{GCD}(R^{n-j_{\text{st}}},\, m_{32}) = m_{32}$. Since $j_{\text{st}} < i_{\text{st}} \leq n$, we have $n - j_{\text{st}} \geq 1$ and thus $\text{GCD}(R^k,\, m_{32}) = m_{32}$ for some positive integer k, which is equivalent to **D1**. The proof is thus complete. $\square$

We prove **S5** as follows.

Proof. Suppose **G1** $\wedge$ **E0** $\wedge$ **E1** $\wedge$ **C3** is true. By **E0** $\wedge$ **E1** being true, $G_{3\text{ANA}}$ of interest satisfies (37). Since $i_{\text{st}} = j_{\text{st}}$, we can divide $L^{i_{\text{st}}} = L^{j_{\text{st}}}$ on both sides of (37), which leads to

$$\sum_{i=i_{\text{st}}}^{i_{\text{end}}} \alpha_i m_{11}m_{23}R^{n-i}L^{i-i_{\text{st}}} = \sum_{j=j_{\text{st}}}^{j_{\text{end}}} \beta_j m_{13}m_{21}R^{n-j}L^{j-j_{\text{st}}}.$$

Note that if $i_{\text{st}} = j_{\text{st}} = n$ meaning that $i_{\text{st}} = j_{\text{st}} = i_{\text{end}} = j_{\text{end}} = n$, then (37) reduces to $m_{11}m_{23} \equiv m_{13}m_{21}$ (since $\alpha_{i_{\text{st}}} \neq 0$ and $\beta_{j_{\text{st}}} \neq 0$). This contradicts the assumption **G1**.

Thus for the following, we only consider the case when $i_{\text{st}} = j_{\text{st}} \leq n-1$. Note that each term with non-zero β_j on the RHS has a common factor $m_{13}m_{21}$. Similarly, each term with non-zero α_i on the LHS has a common factor $L = m_{13}m_{32}m_{21}$ except for the first term ($i = i_{\text{st}}$). As a result, the first term $\alpha_{i_{\text{st}}}m_{11}m_{23}R^{n-i_{\text{st}}}$ must contain $m_{13}m_{21}$ as a factor. Since $i_{\text{st}} \leq n-1$, we have $\text{GCD}(m_{11}m_{12}^{k}m_{23}^{k+1}m_{31}^{k},\, m_{13}m_{21}) = m_{13}m_{21}$ for some integer $k \geq 1$. Therefore, we have **D4**. $\square$

We prove **S6** as follows.

Proof. Suppose **E0** $\wedge$ **E1** $\wedge$ **C4** is true. By **E0** $\wedge$ **E1** being true, $G_{3\text{ANA}}$ of interest satisfies (37). Since $i_{\text{end}} < j_{\text{end}}$, we can divide $R^{n-j_{\text{end}}}$ on both sides of (37). Then, we have

$$\sum_{i=i_{\text{st}}}^{i_{\text{end}}} \alpha_i m_{11}m_{23}R^{j_{\text{end}}-i}L^{i} = \sum_{j=j_{\text{st}}}^{j_{\text{end}}} \beta_j m_{13}m_{21}R^{j_{\text{end}}-j}L^{j}.$$

Each term with non-zero α_i on the LHS has R as a common factor. Similarly, each term with non-zero β_j on the RHS has R as a common factor except for the last term (since $\beta_{j_{\text{end}}} \neq 0$). Thus, the last term $\beta_{j_{\text{end}}}m_{13}m_{21}L^{j_{\text{end}}}$ must be divisible by $R = m_{12}m_{23}m_{31}$, which implies that $\text{GCD}(m_{13}^{k+1}m_{32}^{k}m_{21}^{k+1},\, m_{12}m_{23}m_{31}) = m_{12}m_{23}m_{31}$ for some integer $k = j_{\text{end}} \geq i_{\text{end}} + 1 \geq 1$. This observation has two implications. Firstly, $\text{GCD}(m_{11}m_{13}^{l_3}m_{32}^{l_3}m_{21}^{l_3},\, m_{12}m_{31}) = m_{12}m_{31}$ for some positive integer $l_3 = k+1$ and thus we have proven **D3**. Secondly, we also have $\text{GCD}(m_{13}^{l_2}m_{32}^{l_2}m_{21}^{l_2},\, m_{23}) = m_{23}$ for some positive integer $l_2 = k+1$ and thus we have proven **D2**. The proof is thus complete. $\square$

We prove **S7** as follows.

Proof. Suppose **E0** $\wedge$ **E1** $\wedge$ **C5** is true. By **E0** $\wedge$ **E1** being true, $G_{3\text{ANA}}$ of interest satisfies (37). Since $i_{\text{end}} > j_{\text{end}}$, we can divide $R^{n-i_{\text{end}}}$ on both sides of (37). Then we have

$$\sum_{i=i_{\text{st}}}^{i_{\text{end}}} \alpha_i m_{11}m_{23}R^{i_{\text{end}}-i}L^{i} = \sum_{j=j_{\text{st}}}^{j_{\text{end}}} \beta_j m_{13}m_{21}R^{i_{\text{end}}-j}L^{j}.$$

Each term on the RHS has R as a common factor. Similarly, each term on the LHS has R as a common factor except for the last term (since $\alpha_{i_{\text{end}}} \neq 0$). Thus, the last term $\alpha_{i_{\text{end}}}m_{11}m_{23}L^{i_{\text{end}}}$ must be divisible by $R = m_{12}m_{23}m_{31}$, which implies that $\text{GCD}(m_{11}L^{k},\, m_{12}m_{31}) = m_{12}m_{31}$ for some integer $k = i_{\text{end}} \geq j_{\text{end}} + 1 \geq 1$. This further implies **D3**. $\square$

We prove **S8** as follows.

Proof. Suppose **G1** $\wedge$ **E0** $\wedge$ **E1** $\wedge$ **C6** is true. By **E0** $\wedge$ **E1** being true, $G_{3\text{ANA}}$ of interest satisfies (37). Since **G5**, $i_{\text{end}} = j_{\text{end}}$, is true, define $t=i_{\text{end}}=j_{\text{end}}$ and $m=\min\{i_{\text{st}},j_{\text{st}}\}$. Then by dividing R^{n-t} and L^m from both sides of (37), we have

$$\sum_{i=i_{\text{st}}}^{t}\alpha_i m_{11}m_{23}R^{t-i}L^{i-m}=\sum_{j=j_{\text{st}}}^{t}\beta_j m_{13}m_{21}R^{t-j}L^{j-m}. \tag{45}$$

Each term with non-zero α_i on the LHS has a common factor m_{23}. We first consider the case of $m<t$. Then each term with non-zero β_j on the RHS has a common factor $R=m_{12}m_{23}m_{31}$ except the last term $\beta_t m_{13}m_{21}L^{t-m}$. As a result, $\beta_t m_{13}m_{21}L^{t-m}$ must be divisible by m_{23}, which implies that $\mathsf{GCD}(\,m_{13}^{k+1}m_{32}^{k}m_{21}^{k+1},\,m_{23})=m_{23}$ for some $k=t-m\geq 1$. This implies **D2**.

On the other hand, we argue that we cannot have $m=t$. If so, then $i_{\text{st}}=j_{\text{st}}=i_{\text{end}}=j_{\text{end}}$ and (37) reduces to $m_{11}m_{23}\equiv m_{13}m_{21}$. However, this contradicts the assumption **G1**. The proof is thus complete. $\square$

We prove **S9** as follows.

Proof. Suppose **E0** $\wedge$ **E1** $\wedge$ **C0** is true. By **E0** $\wedge$ **E1** being true, $G_{3\text{ANA}}$ of interest satisfies (37) with not-being-all-zero coefficients $\{\alpha_i\}_{i=0}^n$ and $\{\beta_j\}_{j=0}^n$. Our goal is to prove that, when $i_{\text{st}}>j_{\text{st}}$ and $i_{\text{end}}=j_{\text{end}}$, we have **E2**: (i) $\alpha_k\neq\beta_k$ for some $k\in\{0,...,n\}$; and (ii) either $\alpha_0\neq 0$ or $\beta_n\neq 0$ or $\alpha_k\neq\beta_{k-1}$ for some $k\in\{1,...,n\}$.

Note that (i) is obvious since $i_{\text{st}}>j_{\text{st}}$. Note by definition that i_{st} (resp. j_{st}) is the smallest i (resp. j) among $\alpha_i\neq 0$ (resp. $\beta_j\neq 0$). Then, $i_{\text{st}}>j_{\text{st}}$ implies that $\alpha_{j_{\text{st}}}=0$ while $\beta_{j_{\text{st}}}\neq 0$. Thus simply choosing $k=j_{\text{st}}$ proves (i).

We now prove (ii). Suppose (ii) is false such that $\alpha_0=0$; $\beta_n=0$; and $\alpha_k=\beta_{k-1}$ for all $k\in\{1,...,n\}$. Since $\beta_n=0$, by definition, j_{end} must be less than or equal to $n-1$. Since we assumed $i_{\text{end}}=j_{\text{end}}$, this in turn implies that $\alpha_n=0$. Then β_{n-1} must be zero because $\beta_{n-1}=\alpha_n$. Again this implies $j_{\text{end}}\leq n-2$. Applying iteratively, we have all zero coefficients $\{\alpha_i\}_{i=0}^n$ and $\{\beta_j\}_{j=0}^n$. However, this contradicts the assumption **E0** since we assumed that at least one of each coefficient group is non-zero. The proof of **S9** is thus complete. $\square$

We prove **S10** as follows.

Proof. Suppose **G1** $\wedge$ **E0** $\wedge$ **E1** $\wedge$ ($\neg$ **C0**) is true. By **E0** $\wedge$ **E1** being true, $G_{3\text{ANA}}$ of interest satisfies (37) with some values of i_{st}, j_{st}, i_{end}, and j_{end}. Investigating their relationships, there are total 9 possible cases that $G_{3\text{ANA}}$ can satisfy (37): (i) $i_{\text{st}}<j_{\text{st}}$ and $i_{\text{end}}<j_{\text{end}}$; (ii) $i_{\text{st}}<j_{\text{st}}$ and $i_{\text{end}}>j_{\text{end}}$; (iii) $i_{\text{st}}<j_{\text{st}}$ and $i_{\text{end}}=j_{\text{end}}$; (iv) $i_{\text{st}}>j_{\text{st}}$ and $i_{\text{end}}<j_{\text{end}}$; (v) $i_{\text{st}}>j_{\text{st}}$ and $i_{\text{end}}>j_{\text{end}}$; (vi) $i_{\text{st}}>j_{\text{st}}$ and $i_{\text{end}}=j_{\text{end}}$; (vii) $i_{\text{st}}=j_{\text{st}}$ and $i_{\text{end}}<j_{\text{end}}$; (viii) $i_{\text{st}}=j_{\text{st}}$ and $i_{\text{end}}>j_{\text{end}}$; and (ix) $i_{\text{st}}=j_{\text{st}}$ and $i_{\text{end}}=j_{\text{end}}$.

Note that **C0** is equivalent to (vi). Since we assumed that **C0** is false, $G_{3\text{ANA}}$ can satisfy (37) with all the possible cases except (vi). We also note that (i) is equivalent to **C1** $\wedge$ **C4**, (ii) is equivalent to **C1** $\wedge$ **C5**, etc. By applying **S3** and **S6**, we have

- **E0** $\wedge$ **E1** $\wedge$ (i) $\Rightarrow$ (**D4** $\wedge$ **D5**) $\wedge$ (**D2** $\wedge$ **D3**).

By similarly applying **S3** to **S8**, we have the following relationships:

- **E0** $\wedge$ **E1** $\wedge$ (ii) $\Rightarrow$ (**D4** $\wedge$ **D5**) $\wedge$ **D3**.
- **G1** $\wedge$ **E0** $\wedge$ **E1** $\wedge$ (iii) $\Rightarrow$ (**D4** $\wedge$ **D5**) $\wedge$ **D2**.
- **E0** $\wedge$ **E1** $\wedge$ (iv) $\Rightarrow$ **D1** $\wedge$ (**D2** $\wedge$ **D3**).
- **E0** $\wedge$ **E1** $\wedge$ (v) $\Rightarrow$ **D1** $\wedge$ **D3**.
- **G1** $\wedge$ **E0** $\wedge$ **E1** $\wedge$ (vii) $\Rightarrow$ **D4** $\wedge$ (**D2** $\wedge$ **D3**).
- **G1** $\wedge$ **E0** $\wedge$ **E1** $\wedge$ (viii) $\Rightarrow$ **D4** $\wedge$ **D3**.
- **G1** $\wedge$ **E0** $\wedge$ **E1** $\wedge$ (ix) $\Rightarrow$ **D4** $\wedge$ **D2**.

Then, the above relationships jointly imply **G1** $\wedge$ **E0** $\wedge$ **E1** $\wedge$ ($\neg$ **C0**) $\Rightarrow$ (**D1** $\wedge$ **D3**) $\vee$ (**D2** $\wedge$ **D4**) $\vee$ (**D3** $\wedge$ **D4**). The proof of **S10** is thus complete. $\square$

Appendix M
Proof of S11

M-1. The third set of logic statements

To prove **S11**, we need the third set of logic statements.

- **G7:** There exists an edge $\tilde{e}$ such that both the following conditions are satisfied: (i) $\tilde{e}$ can reach d_1 but cannot reach any of d_2 and d_3; and (ii) $\tilde{e}$ can be reached from s_1 but not from any of s_2 nor s_3.
- **G8:** $\overline{S}_3\neq\emptyset$ and $\overline{D}_2\neq\emptyset$.

The following logic statements are well-defined if and only if **G4** $\wedge$ **G8** is true. Recall the definition of e_3^* and e_2^* when **G4** is true.

- **G9:** $\{e_3^*,e_2^*\}\subset\mathsf{1cut}(s_2;d_3)$.
- **G10:** $e_3^*\in\mathsf{1cut}(s_2;d_1)$.
- **G11:** $e_3^*\in\mathsf{1cut}(s_1;d_1)$.
- **G12:** $e_2^*\in\mathsf{1cut}(s_1;d_3)$.
- **G13:** $e_2^*\in\mathsf{1cut}(s_1;d_1)$.

The following logic statements are well-defined if and only if $\neg$ **G4** is true. Recall the definition of e_u^{32} and e_v^{32} when $\neg$ **G4** is true.

- **G14:** $e_u^{32}\notin\mathsf{1cut}(s_1;d_1)$.
- **G15:** Let $\tilde{e}_u$ denote the most downstream edge among $\mathsf{1cut}(s_1;d_1)\cap\mathsf{1cut}(s_1;\mathsf{tail}(e_u^{32}))$. Also let $\tilde{e}_v$ denote the most upstream edge among $\mathsf{1cut}(s_1;d_1)\cap\mathsf{1cut}(\mathsf{head}(e_v^{32});d_1)$. Then we have (a) $\mathsf{head}(\tilde{e}_u)\prec\mathsf{tail}(e_u^{32})$ and $\mathsf{head}(e_v^{32})\prec\mathsf{tail}(\tilde{e}_v)$; there exists a s_1-to-d_1 path P_{11}^* through $\tilde{e}_u$ and $\tilde{e}_v$ satisfying the following two conditions: (b) P_{11}^* is vertex-disjoint from any s_3-to-d_2 path; and (c) there exists an edge $\tilde{e}\in P_{11}^*$ where $\tilde{e}_u\prec\tilde{e}\prec\tilde{e}_v$ that is not reachable from any of $\{e_u^{32},e_v^{32}\}$.

M-2. The skeleton of proving **S11**

We prove the following relationships, which jointly prove **S11**. The proofs for the following statements are relegated to Appendix N.

- **R1: D1** $\Rightarrow$ **G8**.
- **R2: G4** $\wedge$ **G8** $\wedge$ **D1** $\Rightarrow$ **G9**.
- **R3: G4** $\wedge$ **G8** $\wedge$ **G9** $\wedge$ **D3** $\Rightarrow$ (**G10** $\vee$ **G11**) $\wedge$ (**G12** $\vee$ **G13**).
- **R4: G4** $\wedge$ **G8** $\wedge$ **G9** $\wedge$ ($\neg$ **G10**) $\wedge$ **G11** $\wedge$ **E0** $\Rightarrow$ false.
- **R5: G4** $\wedge$ **G8** $\wedge$ **G9** $\wedge$ ($\neg$ **G12**) $\wedge$ **G13** $\wedge$ **E0** $\Rightarrow$ false.
- **R6: G4** $\wedge$ **G8** $\wedge$ **G9** $\wedge$ **G10** $\wedge$ **G12** $\Rightarrow$ ($\neg$ **LNR**).
- **R7: G1** $\wedge$ ($\neg$ **G4**) $\Rightarrow$ **G14**.
- **R8:** ($\neg$ **G4**) $\wedge$ **G14** $\Rightarrow$ **G15**.
- **R9:** ($\neg$ **G4**) $\wedge$ **G14** $\wedge$ **D3** $\Rightarrow$ **G7**.
- **R10: G7** $\wedge$ **E0** $\Rightarrow$ false.

One can easily verify that jointly **R4** to **R6** imply

$$\begin{aligned}&\mathbf{LNR}\wedge\mathbf{G4}\wedge\mathbf{G8}\wedge\mathbf{G9}\wedge\mathbf{E0}\wedge(\mathbf{G10}\vee\mathbf{G11})\wedge\\&\qquad(\mathbf{G12}\vee\mathbf{G13})\Rightarrow\ \text{false}.\end{aligned}\tag{46}$$

Together with **R3**, (46) reduces to

$$\mathbf{LNR}\wedge\mathbf{G4}\wedge\mathbf{G8}\wedge\mathbf{G9}\wedge\mathbf{E0}\wedge\mathbf{D3}\Rightarrow \text{ false.} \tag{47}$$

Jointly with **R1** and **R2**, (47) further reduces to

$$\mathbf{LNR}\wedge\mathbf{G4}\wedge\mathbf{E0}\wedge\mathbf{D1}\wedge\mathbf{D3}\Rightarrow \text{ false.} \tag{48}$$

In addition, **R7**, **R9**, and **R10** jointly imply

$$\mathbf{G1}\wedge(\neg\,\mathbf{G4})\wedge\mathbf{E0}\wedge\mathbf{D3}\Rightarrow \text{ false.} \tag{49}$$

One can easily verify that jointly (48) and (49) imply **S11**. The skeleton of the proof of **S11** is complete.

Appendix N
Proofs of R1 to R10

We prove **R1** as follows.

Proof. Suppose **D1** is true. By Corollary 2, any channel gain cannot have the other channel gain as a factor. Therefore, m_{32} must be reducible. Furthermore we must have $\mathsf{GCD}(m_{12},\, m_{32})\not\equiv 1$ since m_{12} is the only channel gain in the LHS of **D1** that reaches d_2. (See the proof of Lemma 8 for detailed discussion). Similarly, we must have $\mathsf{GCD}(m_{31},\, m_{32})\not\equiv 1$. Lemma 7 then implies $\overline{S}_3\neq\emptyset$ and $\overline{D}_2\neq\emptyset$. ☐

We prove **R2** as follows.

Proof. Suppose $\mathbf{G4}\wedge\mathbf{G8}\wedge\mathbf{D1}$ is true. From $\mathbf{G4}\wedge\mathbf{G8}$ being true, by definition, e_3^* (resp. e_2^*) is the most downstream (resp. upstream) edge of $\overline{S}_3$ (resp. $\overline{D}_2$) and $e_3^*\prec e_2^*$. For the following, we will prove that $\{e_3^*, e_2^*\}\subset \mathsf{1cut}(s_2;d_3)$.

We now consider $m_{e_3^*;e_2^*}$, a part of m_{32}. From **D1** and Property 2 of **G4**, we have

$$\mathsf{GCD}(\,m_{23}^{l_1},\; m_{e_3^*;e_2^*}) = m_{e_3^*;e_2^*}, \tag{50}$$

for some positive integer l_1. This implies that $m_{e_3^*;e_2^*}$ is a factor of m_{23}. By Proposition 3, we have $\{e_3^*, e_2^*\}\subset \mathsf{1cut}(s_2;d_3)$. The proof is thus complete. ☐

We prove **R3** as follows.

Proof. Suppose $\mathbf{G4}\wedge\mathbf{G8}\wedge\mathbf{G9}\wedge\mathbf{D3}$ is true. Therefore, the e_3^* (resp. e_2^*) defined in the properties of **G4** must also be the most downstream (resp. upstream) edge of $\overline{S}_3$ (resp. $\overline{D}_2$). Moreover, since $\{e_3^*, e_2^*\}\subset \mathsf{1cut}(s_2;d_3)$, we can express m_{23} as $m_{23} = m_{e_{s_2};e_3^*}m_{e_3^*;e_2^*}m_{e_2^*;e_{d_3}}$. For the following, we will prove that $e_3^*\in\mathsf{1cut}(s_1;d_1)\cup\mathsf{1cut}(s_2;d_1)$.

We use the following observation: For any edge $e'\in\mathsf{1cut}(s_3;d_2)$ that is in the upstream of e_2^*, there must exist a path from s_1 to $\mathsf{tail}(e_2^*)$ that does not use such e'. Otherwise, $e'\in\mathsf{1cut}(s_3;d_2)$ is also a 1-edge cut separating s_1 and d_2, which contradicts that e_2^* is the most upstream edge of $\overline{D}_2$.

We now consider $m_{e_3^*;e_{d_1}}$, a factor of m_{31}. From **D3** and Property 2 of **G4**, we have $\mathsf{GCD}(\,m_{11}m_{13}^{l_3}m_{21}^{l_3},\, m_{e_3^*;e_{d_1}}) = m_{e_3^*;e_{d_1}}$. By Proposition 3, we must have $e_3^*\in\mathsf{1cut}(s_1;d_1)\cup\mathsf{1cut}(s_1;d_3)\cup\mathsf{1cut}(s_2;d_1)$. We also note that by the observation in the beginning of this proof, there exists a path from s_1 to $\mathsf{tail}(e_2^*)$ not using e_3^*. Furthermore, $e_2^*\in\mathsf{1cut}(s_2;d_3)$ implies that e_2^* can reach d_3. These jointly shows that there exists a path from s_1 through e_2^* to d_3 without using e_3^*,

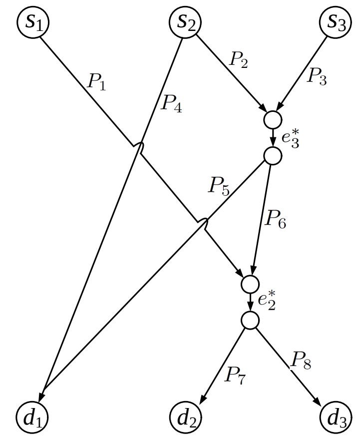

Fig. 2. The subgraph G' of the 3-unicast ANA network $G_{3\text{ANA}}$ induced by the union of the 8 paths plus two edges e_3^* and e_2^* in the proof of **R4**.

which means $e_3^*\notin\mathsf{1cut}(s_1;d_3)$. Therefore, e_3^* belongs to $\mathsf{1cut}(s_1;d_1)\cup\mathsf{1cut}(s_2;d_1)$. The proof of $e_2^*\in\mathsf{1cut}(s_1;d_1)\cup\mathsf{1cut}(s_1;d_3)$ can be derived similarly. The proof **R3** is thus complete. ☐

We prove **R4** as follows.

Proof. Assume $\mathbf{G4}\wedge\mathbf{G8}\wedge\mathbf{G9}\wedge(\neg\,\mathbf{G10})\wedge\mathbf{G11}\wedge\mathbf{E0}$ is true. Recall that e_3^* is the most downstream edge in $\overline{S}_3$ and e_2^* is the most upstream edge in $\overline{D}_2$. For the following we construct 8 path segments that interconnects s_1 to s_3, d_1 to d_3, and two edges e_3^* and e_2^*.

- P_1: a path from s_1 to $\mathsf{tail}(e_2^*)$ without using e_3^*. This is always possible due to Properties 1 and 2 of **G4**.
- P_2: a path from s_2 to $\mathsf{tail}(e_3^*)$. This is always possible due to **G8** and **G9** being true.
- P_3: a path from s_3 to $\mathsf{tail}(e_3^*)$. This is always possible due to **G4** and **G8** being true.
- P_4: a path from s_2 to d_1 without using e_3^*. This is always possible due to **G10** being false.
- P_5: a path from $\mathsf{head}(e_3^*)$ to d_1 without using e_2^*. This is always possible due to Properties 1 and 2 of **G4**.
- P_6: a path from $\mathsf{head}(e_3^*)$ to $\mathsf{tail}(e_2^*)$. This is always possible due to Property 1 of **G4**.
- P_7: a path from $\mathsf{head}(e_2^*)$ to d_2. This is always possible due to **G4** and **G8** being true.
- P_8: a path from $\mathsf{head}(e_2^*)$ to d_3. This is always possible due to **G8** and **G9** being true.

Fig. 2 illustrates the relative topology of these 8 paths. We now consider the subgraph G' induced by the 8 paths and two edges e_3^* and e_2^*. One can easily check that s_i can reach d_j for any $i\neq j$. In particular, s_1 can reach d_2 through $P_1e_2^*P_7$; s_1 can reach d_3 through $P_1e_2^*P_8$; s_2 can reach d_1 through either P_4 or $P_2e_3^*P_5$; s_2 can reach d_3 through $P_2e_3^*P_6e_2^*P_8$; s_3 can reach d_1 through $P_3e_3^*P_5$; and s_3 can reach d_2 through $P_3e_3^*P_6e_2^*P_7$.

We first show the following topological relationships: P_1 is vertex-disjoint with P_2, P_3, and P_4, respectively, in the induced subgraph G'. From **G9**, $\{P_1, P_2\}$ must be vertex-disjoint paths otherwise s_2 can reach d_3 without using $e_3^*\in\mathsf{1cut}(s_2;d_3)$. Similarly from the fact that $e_3^*\in\overline{S}_3$, $\{P_1, P_3\}$ must be vertex-disjoint paths. Also notice that by **G11**, e_3^*

is a 1-edge cut separating s_1 and d_1 in the original graph. Therefore any s_1-to-d_1 path in the subgraph must use e_3^* as well. But by definition, both P_1 and P_4 do not use e_3^* and s_1 can reach d_1 if they share a vertex. This thus implies that $\{P_1, P_4\}$ are vertex-disjoint paths.

The above topological relationships further imply that s_1 cannot reach d_1 in the induced subgraph G'. The reason is as follows. We first note that P_1 is the only path segment that s_1 can use to reach other destinations, and any s_1-to-d_1 path, if exists, must use path segment P_1 in the very beginning. Since P_1 ends at $\mathsf{tail}(e_2^*)$, using path segment P_1 alone is not possible to reach d_1. Therefore, if a s_1-to-d_1 path exists, then at some point, it must use one of the other 7 path segments P_2 to P_8. On the other hand, we also note that $e_3^* \in \mathsf{1cut}(s_1; d_1)$ and the path segments P_5 to P_8 are in the downstream of e_3^*. Therefore, for any s_1-to-d_1 path, if it uses any of the vertices of P_5 to P_8, it must first go through $\mathsf{tail}(e_3^*)$, the end point of path segments P_2 and P_3. As a result, we only need to consider the scenario in which one of $\{P_2, P_3, P_4\}$ is used by the s_1-to-d_1 path when this path switches from P_1 to a new path segment. But we have already showed that P_1 and $\{P_2, P_3, P_4\}$ are vertex-disjoint with each other. As a result, no s_1-to-d_1 path can exist. Thus s_1 cannot reach d_1 on the induced graph G'.

By **E0** being true and Proposition 2, any subgraph who contains the source and destination edges (hence G') must satisfy **E0**. Note that we already showed there is no s_1-to-d_1 path on G'. Recalling (36), its LHS becomes zero. Thus, we have $g(\{m_{ij} : \forall\,(i,j) \in I_{3\text{ANA}}\})\,\psi_\beta^{(n)}(R, L) = 0$ with at least one non-zero coefficient β_j. But note also that any channel gain m_{ij} where $i \neq j$ is non-trivial on G'. Thus R, L, and $g(\{m_{ij} : \forall\,(i,j) \in I_{3\text{ANA}}\})$ are all non-zero polynomials. Therefore, G' must satisfy $\psi_\beta^{(n)}(R, L) = 0$ with at least one non-zero coefficient β_j and this further implies that the set of polynomials $\{R^n, R^{n-1}L, \cdots, RL^{n-1}, L^n\}$ is linearly dependent on G'. Since this is the Vandermonde form, it is equivalent to that $L \equiv R$ holds on G'.

For the following, we further show that in the induced graph G', the following three statements are true: (a) $\overline{S}_2 \cap \overline{S}_3 = \emptyset$; (b) $\overline{S}_1 \cap \overline{S}_2 = \emptyset$; and (c) $\overline{S}_1 \cap \overline{S}_3 = \emptyset$, which implies by Proposition 4 that G' must have $L \not\equiv R$. We thus have a contradiction.

(a) $\overline{S}_2 \cap \overline{S}_3 = \emptyset$ on G': Suppose there is an edge $e \in \overline{S}_2 \cap \overline{S}_3$ on G'. Since $e \in \overline{S}_2$, such e must belong to P_4 and any s_2-to-d_3 path. Since both $e \in P_4$ and $e_3^* \notin P_4$ belong to $\mathsf{1cut}(s_2; d_3)$, we have either $e \prec e_3^*$ or $e \succ e_3^*$. We first note that e must not be in the downstream of e_3^*. Otherwise, s_2 can use P_4 to reach e without using e_3^* and finally to d_3 (since $e \in \overline{S}_2$), which contradicts the assumption of **G9** that $e_3^* \in \mathsf{1cut}(s_2; d_3)$. As a result, $e \prec e_3^*$ and any path from s_2 to $\mathsf{tail}(e_3^*)$ must use e. This in turn implies that P_2 uses e. We now argue that P_3 must also use e. The reason is that the s_3-to-d_1 path $P_3 e_3^* P_5$ must use e since $e \in \overline{S}_3$ and $e \prec e_3^*$. Then these jointly contradict that $e_3^* \in \overline{S}_3$ since s_3 can follow P_3, switch to P_4 through e, and reach d_1 without using e_3^*.

(b) $\overline{S}_1 \cap \overline{S}_2 = \emptyset$ on G': Suppose there is an edge $e \in \overline{S}_1 \cap \overline{S}_2$. Since $e \in \overline{S}_2$, by the same arguments as used in proving (a), we know that $e \prec e_3^*$ and e must be used by both P_2 and P_4. We then note that e must also be used by the s_1-to-d_3 path $P_1 e_2^* P_8$ since $e \in \overline{S}_1$. This in turn implies that P_1 must use e since $e \prec e_3^* \prec e_2^*$. However, these jointly contradict the fact P_1 and $\{P_2, P_3, P_4\}$ being vertex-disjoint, which were proved previously. The proof of (b) is complete.

(c) $\overline{S}_1 \cap \overline{S}_3 = \emptyset$ on G': Suppose there is an edge $e \in \overline{S}_1 \cap \overline{S}_3$. We then note that e must be used by the s_1-to-d_3 path $P_1 e_2^* P_8$ since $e \in \overline{S}_1$. Then e must be either e_3^* or used by P_3 since e_3^* is the most downstream edge of $\overline{S}_3$. Therefore, P_1 must use e (since $e_3^* \prec e_2^*$). In addition, since by our construction P_1 does not use e_3^*, it is P_3 who uses e. However, P_1 and P_3 are vertex-disjoint with each other, which contradicts what we just derived $e \in P_1 \cap P_3$. The proof of (c) is complete. ☐

We prove **R5** as follows.

Proof. We notice that **R5** is a symmetric version of **R4** by simultaneously reversing the roles of sources and destinations and relabeling flow 2 by flow 3, i.e., we swap the roles of the following three pairs: (s_1, d_1), (s_2, d_3), and (s_3, d_2). We can then reuse the proof of **R4**. ☐

We prove **R6** as follows.

Proof. Assume **G4** $\wedge$ **G8** is true and recall that e_3^* is the most downstream edge in $\overline{S}_3$ and e_2^* is the most upstream edge in $\overline{D}_2$. From **G9** $\wedge$ **G10** $\wedge$ **G12** being true, we further have $e_3^* \in \mathsf{1cut}(s_2; d_1) \cap \mathsf{1cut}(s_2; d_3)$ and $e_2^* \in \mathsf{1cut}(s_1; d_3) \cap \mathsf{1cut}(s_2; d_3)$. This implies that e_3^* (resp. e_2^*) belongs to $\overline{S}_2 \cap \overline{S}_3$ (resp. $\overline{D}_2 \cap \overline{D}_3$). We thus have $\neg$ **LNR** by Proposition 4. ☐

We prove **R7** as follows.

Proof. We prove an equivalent relationship: $(\neg$ **G4**$)$ $\wedge$ $(\neg$ **G14**$)$ $\Rightarrow$ $(\neg$ **G1**$)$. From **G4** being false, we have $e_u^{32} \in \overline{S}_3 \cap \overline{D}_2 \subset \mathsf{1cut}(s_3; d_2) \cap \mathsf{1cut}(s_1; d_2) \cap \mathsf{1cut}(s_3; d_1)$. From **G14** being false, we have $e_u^{32} \in \mathsf{1cut}(s_1; d_1)$. As a result, e_u^{32} is a 1-edge cut separating $\{s_1, s_3\}$ and $\{d_1, d_2\}$. This implies $m_{11} m_{32} \equiv m_{12} m_{31}$ and thus $\neg$ **G1**. The proof of **R7** is thus complete. ☐

We prove **R8** as follows.

Proof. Suppose that $(\neg$ **G4**$)$ $\wedge$ **G14** is true. From Property 3 of $\neg$ **G4**, any s_1-to-d_1 path who uses a vertex w where $\mathsf{tail}(e_u^{32}) \preceq w \preceq \mathsf{head}(e_v^{32})$ must use both e_u^{32} and e_v^{32}. Since we have $e_u^{32} \notin \mathsf{1cut}(s_1; d_1)$ from **G14**, there must exist a s_1-to-d_1 path not using e_u^{32}. Then, these jointly imply that there exists a s_1-to-d_1 path which does not use any vertex in-between $\mathsf{tail}(e_u^{32})$ and $\mathsf{head}(e_v^{32})$. Fix arbitrarily one such path as P_{11}^*.

If the chosen P_{11}^* shares a vertex with any path segment from s_3 to $\mathsf{tail}(e_u^{32})$, then s_3 can reach d_1 without using e_u^{32}, contradicting $e_u^{32} \in \overline{S}_3 \cap \overline{D}_2 \subset \mathsf{1cut}(s_3; d_1)$. By the similar argument, P_{11}^* should not share a vertex with any path segment from $\mathsf{head}(e_v^{32})$ to d_2. Then jointly with the above discussion, we can conclude that P_{11}^* is vertex-disjoint with any s_3-to-d_2 path. We thus have proven (b) of **G15**.

Now consider $\tilde{e}_u$ (we have at least the s_1-source edge e_{s_1}) and $\tilde{e}_v$ (we have at least the d_1-destination edge e_{d_1}) defined in **G15**. By definition, $\tilde{e}_u \prec e_u^{32}$ and $e_v^{32} \prec \tilde{e}_v$, and the chosen P_{11}^* must use both $\tilde{e}_u$ and $\tilde{e}_v$. Thus if $\mathsf{head}(\tilde{e}_u) = \mathsf{tail}(e_u^{32})$, then this contradicts the above discussion since $\mathsf{tail}(e_u^{32}) \in P_{11}^*$.

Therefore, it must be $\mathsf{head}(\tilde{e}_u) \prec \mathsf{tail}(e_u^{32})$. Similarly, it must also be $\mathsf{head}(e_v^{32}) \prec \mathsf{tail}(\tilde{e}_v)$. Thus we have proven (a) of **G15**.

We now prove (c) of **G15**. We prove this by contradiction. Fix arbitrarily one edge $e \in P_{11}^*$ where $\tilde{e}_u \prec e \prec \tilde{e}_v$ and assume that this edge e is reachable from either e_u^{32} or e_v^{32} or both. We first prove that whenever e_u^{32} reaches e, then e must be in the downstream of e_v^{32}. The reason is as follows. If e_u^{32} reaches e, then $e \in P_{11}^*$ should not reach e_v^{32} because it will be located in-between e_u^{32} and e_v^{32}, and this contradicts the above discussion. The case when e are e_v^{32} are not reachable from each other is also not possible because s_1 can first reach e through e_u^{32} and follow P_{11}^* to d_1 without using e_v^{32}, which contradicts the Property 3 of $\neg$**G4**. Thus, if $e_u^{32} \prec e$, then it must be $e_v^{32} \prec e$. By the similar argument, we can show that if $e \prec e_v^{32}$, it must be $e \prec e_u^{32}$. Therefore, only two cases are possible when e is reachable from either e_u^{32} or e_v^{32} or both: either $e \prec e_u^{32}$ or $e_v^{32} \prec e$. Extending this result to every edges of P_{11}^* from $\tilde{e}_u$ to $\tilde{e}_v$, we can group them into two: edges in the upstream of e_u^{32}; and edges in the downstream of e_v^{32}. Since $\tilde{e}_u \prec e_u^{32} \prec e_v^{32} \prec \tilde{e}_v$, this further implies that the chosen P_{11}^* must be disconnected. This, however, contradicts the construction P_{11}^*. Therefore, there must exist an edge $\tilde{e} \in P_{11}^*$ where $\tilde{e}_u \prec e \prec \tilde{e}_v$ that is not reachable from any of $\{e_u^{32}, e_v^{32}\}$. We thus have proven (c) of **G15**. The proof of **R8** is complete. ☐

We prove **R9** as follows.

Proof. Suppose $(\neg$**G4**$) \wedge$ **G14** $\wedge$ **D3** is true. From **R8**, **G15** must also be true, and we will use the s_1-to-d_1 path P_{11}^*, the two edges $\tilde{e}_u$ and $\tilde{e}_v$, and the edge $\tilde{e} \in P_{11}^*$ defined in **G15**. For the following, we will prove that the specified $\tilde{e}$ satisfies **G7**. Since $\tilde{e} \in P_{11}^*$, we only need to prove that $\tilde{e}$ cannot be reached by any of $\{s_2, s_3\}$ and cannot reach any of $\{d_2, d_3\}$.

We first claim that $\tilde{e}$ cannot be reached from s_3. Suppose not. Then we can consider a new s_3-to-d_1 path: s_3 can reach $\tilde{e}$ and follow P_{11}^* to d_1. Since $\tilde{e}$ is not reachable from any of $\{e_u^{32}, e_v^{32}\}$ by (c) of **G15**, this new s_3-to-d_1 path must not use any of $\{e_u^{32}, e_v^{32}\}$. However, this contradicts the construction $\{e_u^{32}, e_v^{32}\} \subset \overline{S}_3 \cap \overline{D}_2 \subset \mathsf{1cut}(s_3; d_1)$. We thus have proven the first claim that $\tilde{e}$ cannot be reached from s_3. Symmetrically, we can also prove that $\tilde{e}$ cannot reach d_2.

What remains to be proven is that $\tilde{e}$ cannot be reached from s_2 and cannot reach d_3. Since **D3** is true, there exists a positive integer l_3 satisfying $\mathsf{GCD}(m_{11}m_{13}^{l_3}m_{32}^{l_3}m_{21}^{l_3},\ m_{12}m_{31}) = m_{12}m_{31}$. Consider $m_{e_{s_1}; e_u^{32}}$, a part of m_{12}, and $m_{e_v^{32}; e_{d_1}}$, a part of m_{31}. By Property 1 of $\neg$**G4**, we have

$$\mathsf{GCD}(m_{11}m_{13}^{l_3}m_{21}^{l_3},\ m_{e_{s_1}; e_u^{32}} m_{e_v^{32}; d_{d_1}}) = m_{e_{s_1}; e_u^{32}} m_{e_v^{32}; d_{d_1}}.$$

Recall the definition of $\tilde{e}_u$ (resp. $\tilde{e}_v$) being the most downstream (resp. upstream) edge among $\mathsf{1cut}(s_1; \mathsf{tail}(e_v^{32})) \cap \mathsf{1cut}(s_1; d_1)$ (resp. $\mathsf{1cut}(\mathsf{head}(e_v^{32}); d_1) \cap \mathsf{1cut}(s_1; d_1)$). Then we can further factorize $m_{e_{s_1}; e_u^{32}} = m_{e_{s_1}; \tilde{e}_u} m_{\tilde{e}_u; e_u^{32}}$ and $m_{e_v^{32}; e_{d_1}} = m_{e_v^{32}; \tilde{e}_v} m_{\tilde{e}_v; e_{d_1}}$, respectively. Since both $\tilde{e}_u$ and $\tilde{e}_v$ separate s_1 and d_1, we can express m_{11} as $m_{11} = m_{e_{s_1}; \tilde{e}_u} m_{\tilde{e}_u; \tilde{e}_v} m_{\tilde{e}_v; e_{d_1}}$. Then one can see that the middle part of m_{11}, i.e., $m_{\tilde{e}_u; \tilde{e}_v}$, must be co-prime to both $m_{\tilde{e}_u; e_u^{32}}$ and $m_{e_v^{32}; \tilde{e}_v}$, otherwise it violates the construction of $\tilde{e}_u$ (resp. $\tilde{e}_v$) being the most downstream (resp. upstream) edge among $\mathsf{1cut}(s_1; \mathsf{tail}(e_v^{32})) \cap \mathsf{1cut}(s_1; d_1)$ (resp. $\mathsf{1cut}(\mathsf{head}(e_v^{32}); d_1) \cap \mathsf{1cut}(s_1; d_1)$). The above equation thus reduces to

$$\mathsf{GCD}(m_{13}^{l_3}m_{21}^{l_3},\ m_{\tilde{e}_u; e_u^{32}} m_{e_v^{32}; \tilde{e}_v}) = m_{\tilde{e}_u; e_u^{32}} m_{e_v^{32}; \tilde{e}_v}. \quad (51)$$

Using (51) and the previous constructions, we first prove that $\tilde{e}$ cannot reach d_3. Since $\mathsf{head}(\tilde{e}_u) \prec \mathsf{tail}(e_u^{32})$ by (a) of **G15**, we must have $0 < \mathsf{EC}(\mathsf{head}(\tilde{e}_u); \mathsf{tail}(e_u^{32})) < \infty$. By Proposition 3, $m_{\tilde{e}_u; e_u^{32}}$ is either irreducible or the product of irreducibles corresponding to the consecutive edges among $\tilde{e}_u$, $\mathsf{1cut}(\mathsf{head}(\tilde{e}_u); \mathsf{tail}(e_u^{32}))$, and e_u^{32}. Consider the following edge set $E_u = \{\tilde{e}_u\} \cup \mathsf{1cut}(\mathsf{head}(\tilde{e}_u); \mathsf{tail}(e_u^{32})) \cup \{e_u^{32}\}$, the collection of $\mathsf{1cut}(\mathsf{head}(\tilde{e}_u); \mathsf{tail}(e_u^{32}))$ and two edges $\tilde{e}_u$ and e_u^{32}. Note that in the proof of **R8**, P_{11}^* was chosen arbitrarily such that $\tilde{e}_u \in P_{11}^*$ and $e_u^{32} \notin P_{11}^*$ but there was no consideration for the 1-edge cuts from $\mathsf{head}(\tilde{e}_u)$ to $\mathsf{tail}(e_u^{32})$ if non-empty. In other words, when s_1 follow the chosen P_{11}^*, it is obvious that it first meets $\tilde{e}_u$ but it is not sure when it starts to deviate not to use e_u^{32} if we have non-empty $\mathsf{1cut}(\mathsf{head}(\tilde{e}_u); \mathsf{tail}(e_u^{32}))$. Let e_1^u denote the most downstream edge of $E_u \cap P_{11}^*$ (we have at least $\tilde{e}_u$) and let e_2^u denote the most upstream edge of $E_u \backslash P_{11}^*$ (we have at least e_u^{32}). From the constructions of P_{11}^* and E_u, the defined edges $e_1^u \in P_{11}^*$ and $e_2^u \notin P_{11}^*$ are edges of E_u such that $\tilde{e}_u \preceq e_1^u \prec e_2^u \preceq e_u^{32}$; $m_{e_1^u; e_2^u}$ is irreducible; and $m_{\tilde{e}_u; e_u^{32}}$ contain $m_{e_1^u; e_2^u}$ as a factor. By doing this way, we can clearly specify the location (in-between $e_1^u \in P_{11}^*$ and $e_2^u \notin P_{11}^*$) when P_{11}^* starts to deviate not to use e_u^{32}.

For the following, we first argue that $\mathsf{GCD}(m_{13},\ m_{e_1^u; e_2^u}) \not\equiv 1$. Suppose not then we have $\mathsf{GCD}(m_{21},\ m_{e_1^u; e_2^u}) = m_{e_1^u; e_2^u}$ from (51). By Proposition 3, we have $\{e_1^u, e_2^u\} \subset \mathsf{1cut}(s_2; d_1)$. However from the above construction, $e_1^u \in \mathsf{1cut}(s_2; d_1)$ implies that s_2 can first reach $e_1^u \in P_{11}^*$ and then follow P_{11}^* to d_1 without using e_2^u since $e_1^u \prec e_2^u$ and $e_2^u \notin P_{11}^*$. This contradicts $e_2^u \in \mathsf{1cut}(s_2; d_1)$ that we just established. This thus proves that $\mathsf{GCD}(m_{13},\ m_{e_1^u; e_2^u}) \not\equiv 1$. Since $m_{e_1^u; e_2^u}$ is irreducible, again by Proposition 3, we further have $\{e_1^u, e_2^u\} \subset \mathsf{1cut}(s_1; d_3)$.

We now argue that $\tilde{e}$ cannot reach d_3. Suppose not and assume that there exists a path segment Q from $\tilde{e}$ to d_3. Since $\tilde{e} \in P_{11}^*$ is not reachable from any of $\{e_u^{32}, e_v^{32}\}$ by (c) of **G15**, it is obvious that $\tilde{e}$ must be in the downstream of $e_1^u \in P_{11}^*$ since $e_1^u \prec e_u^{32}$ from the above construction. Then when s_1 follow P_{11}^* to $\tilde{e}$ (through e_1^u) and switch to Q to reach d_3, it will not use e_2^u unless $\tilde{e} \prec e_2^u$ and $e_2^u \in Q$, but $\tilde{e}$ cannot be in the upstream of e_2^u since $e_2^u \preceq e_u^{32}$ from the above construction. Therefore, this s_1-to-d_3 path $P_{11}^* \tilde{e} Q$ will not use e_2^u and thus contradicts $e_2^u \in \mathsf{1cut}(s_1; d_3)$ that we just established. As a result, $\tilde{e}$ cannot reach d_3.

The proof that $\tilde{e}$ cannot be reached from s_2 can be derived symmetrically. In particular, we can apply the above proof arguments ($\tilde{e}$ cannot reach d_3) by symmetrically using the following: the edge set $E_v = \{e_v^{32}\} \cup \mathsf{1cut}(\mathsf{head}(e_v^{32}); \mathsf{tail}(\tilde{e}_v)) \cup \{\tilde{e}_v\}$ and denote e_1^v (resp. e_2^v) be the most downstream (resp. upstream) edge of $E_v \backslash P_{11}^*$ (resp. $E_v \cap P_{11}^*$) such that $\{e_1^v, e_2^v\} \subset \mathsf{1cut}(s_2; d_1)$ from (51).

Therefore we have proven that $\tilde{e}$ cannot be reached from s_2 and cannot reach d_3. The proof of **R9** is thus complete. ☐

We prove **R10** as follows.

Proof. We prove an equivalent relationship: **G7** $\Rightarrow$ ($\neg$ **E0**). Suppose **G7** is true and consider the edge $\tilde{e}$ defined in **G7**. Consider an s_1-to-d_1 path P_{11} that uses $\tilde{e}$ and an edge $e \in P_{11}$ that is immediate downstream of $\tilde{e}$ along this path, i.e., $\mathsf{head}(\tilde{e}) = \mathsf{tail}(e)$. Such edge e always exists since $\tilde{e}$ cannot be the d_1-destination edge e_{d_1}. (Recall that $\tilde{e}$ cannot be reached by s_2.) We now observe that since **G7** is true, such e cannot reach any of $\{d_2, d_3\}$ (otherwise $\tilde{e}$ can reach one of $\{d_2, d_3\}$). Now consider a local kernel $x_{\tilde{e}e}$ from $\tilde{e}$ to e. Then, one can see that by the facts that $\tilde{e}$ cannot be reached by any of $\{s_2, s_3\}$ and e cannot reach any of $\{d_2, d_3\}$, any channel gain m_{ij} where $i \neq j$ cannot depend on $x_{\tilde{e}e}$. On the other hand, the channel gain polynomial m_{11} has degree 1 in $x_{\tilde{e}e}$ since both $\tilde{e}$ and e are used by a path P_{11}.

Since any channel gain m_{ij} where $i \neq j$ is non-trivial on a given $G_{3\text{ANA}}$, the above discussion implies that $f(\{m_{ij} : \forall (i,j) \in I_{3\text{ANA}}\})$, $g(\{m_{ij} : \forall (i,j) \in I_{3\text{ANA}}\})$, R, and L become all non-zero polynomials, any of which does not depend on $x_{\tilde{e}e}$. Thus recalling (36), its RHS does not depend on $x_{\tilde{e}e}$. However, the LHS of (36) has a common factor m_{11} and thus has degree 1 in $x_{\tilde{e}e}$. This implies that $G_{3\text{ANA}}$ does not satisfy (36) if we have at least one non-zero coefficient α_i and β_j, respectively. This thus implies $\neg$ **E0**. □

APPENDIX O
PROOF OF **S12**

If we swap the roles of sources and destinations, then the proof of **S11** in Appendix M can be directly applied to show **S12**. More specifically, note that **D1** (resp. **D3**) are converted back and forth from **D2** (resp. **D4**) by such (s, d)-swapping. Also, one can easily verify that **LNR**, **G1**, and **E0** remain the same after the index swapping. Thus we can see that **S11** becomes **S12** after reverting flow indices. The proofs of **S11** in Appendix M can thus be used to prove **S12**.

APPENDIX P
PROOF OF **S13**

P-1. The fourth set of logic statements

To prove **S13**, we need the fourth set of logic statements.

- **G16:** There exists a subgraph $G' \subset G_{3\text{ANA}}$ such that in G' both the following conditions are true: (i) s_i can reach d_j for all $i \neq j$; and (ii) s_1 can reach d_1.
- **G17:** Continue from the definition of **G16**. The considered subgraph G' also contains an edge $\tilde{e}$ such that both the following conditions are satisfied: (i) $\tilde{e}$ can reach d_1 but cannot reach any of $\{d_2, d_3\}$; (ii) $\tilde{e}$ can be reached from s_1 but not from any of $\{s_2, s_3\}$.
- **G18:** Continue from the definition of **G16**. There exists a subgraph $G'' \subset G'$ such that (i) s_i can reach d_j for all $i \neq j$; and (ii) s_1 can reach d_1. Moreover, the considered subgraph G'' also satisfies (iii) $m_{11}m_{23} = m_{13}m_{21}$; and (iv) $L \not\equiv R$.
- **G19:** Continue from the definition of **G16**. There exists a subgraph $G'' \subset G'$ such that (i) s_i can reach d_j for all $i \neq j$; and (ii) s_1 can reach d_1. Moreover, the considered subgraph G'' also satisfies (iii) $m_{11}m_{32} = m_{12}m_{31}$; and (iv) $L \not\equiv R$.
- **G20:** $\overline{S}_2 \neq \emptyset$ and $\overline{D}_3 \neq \emptyset$.

The following logic statements are well-defined if and only if **G3** $\wedge$ **G20** is true. Recall the definition of e_2^* and e_3^* when **G3** is true.

- **G21:** $\{e_2^*, e_3^*\} \subset \mathsf{1cut}(s_3; d_2)$.

The following logic statements are well-defined if and only if ($\neg$ **G3**) $\wedge$ ($\neg$ **G4**) is true. Recall the definition of e_u^{23}, e_v^{23}, e_u^{32}, and e_v^{32} when ($\neg$ **G3**) $\wedge$ ($\neg$ **G4**) is true.

- **G22:** There exists a path P_{11}^* from s_1 to d_1 who does not use any vertex in-between $\mathsf{tail}(e_u^{23})$ and $\mathsf{head}(e_v^{23})$, and any vertex in-between $\mathsf{tail}(e_u^{32})$ and $\mathsf{head}(e_v^{32})$.
- **G23:** $e_u^{23} \prec e_u^{32}$.
- **G24:** $e_u^{32} \prec e_u^{23}$.
- **G25:** $e_u^{32} \prec e_v^{23}$.
- **G26:** $e_u^{23} \prec e_v^{32}$.

P-2. The skeleton of proving **S13**

We prove the following relationships, which jointly prove **S13**. The proofs for the following statements are relegated to Appendix Q.

- **R11: D1** $\Rightarrow$ **G8** (identical to **R1**).
- **R12: G4** $\wedge$ **G8** $\wedge$ **D1** $\Rightarrow$ **G9** (identical to **R2**).
- **R13: LNR** $\wedge$ **G4** $\wedge$ **G8** $\wedge$ **G9** $\wedge$ **D2** $\Rightarrow$ false.
- **R14: D2** $\Rightarrow$ **G20**.
- **R15: G3** $\wedge$ **G20** $\wedge$ **D2** $\Rightarrow$ **G21**.
- **R16: LNR** $\wedge$ **G3** $\wedge$ **G20** $\wedge$ **G21** $\wedge$ **D1** $\Rightarrow$ false.
- **R17: LNR** $\wedge$ **G2** $\wedge$ ($\neg$ **G3**) $\wedge$ ($\neg$ **G4**) $\wedge$ ($\neg$ **G5**) $\Rightarrow$ **G7**.
- **R18: G16** $\wedge$ **G17** $\wedge$ **E0** $\Rightarrow$ false.
- **R19: G16** $\wedge$ (**G18** $\vee$ **G19**) $\wedge$ **E0** $\wedge$ **E1** $\wedge$ **E2** $\Rightarrow$ false.
- **R20: G1** $\wedge$ ($\neg$ **G3**) $\wedge$ ($\neg$ **G4**) $\wedge$ ($\neg$ **G22**) $\wedge$ **G23** $\Rightarrow$ **G16** $\wedge$ **G18**.
- **R21: LNR** $\wedge$ ($\neg$ **G3**) $\wedge$ ($\neg$ **G4**) $\wedge$ **G22** $\wedge$ **G23** $\wedge$ **G25** $\Rightarrow$ **G16** $\wedge$ **G17**.
- **R22: LNR** $\wedge$ ($\neg$ **G3**) $\wedge$ ($\neg$ **G4**) $\wedge$ **G22** $\wedge$ **G23** $\wedge$ ($\neg$ **G25**) $\Rightarrow$ **G16** $\wedge$ (**G17** $\vee$ **G18**).
- **R23: G1** $\wedge$ ($\neg$ **G3**) $\wedge$ ($\neg$ **G4**) $\wedge$ ($\neg$ **G22**) $\wedge$ **G24** $\Rightarrow$ **G16** $\wedge$ **G19**.
- **R24: LNR** $\wedge$ ($\neg$ **G3**) $\wedge$ ($\neg$ **G4**) $\wedge$ **G22** $\wedge$ **G24** $\wedge$ **G26** $\Rightarrow$ **G16** $\wedge$ **G17**.
- **R25: LNR** $\wedge$ ($\neg$ **G3**) $\wedge$ ($\neg$ **G4**) $\wedge$ **G22** $\wedge$ **G24** $\wedge$ ($\neg$ **G26**) $\Rightarrow$ **G16** $\wedge$ (**G17** $\vee$ **G19**).

One can easily verify that jointly **R11** to **R13** imply

$$\mathbf{LNR} \wedge \mathbf{G4} \wedge \mathbf{D1} \wedge \mathbf{D2} \Rightarrow \text{ false.} \tag{52}$$

Similarly, **R14** to **R16** jointly imply

$$\mathbf{LNR} \wedge \mathbf{G3} \wedge \mathbf{D1} \wedge \mathbf{D2} \Rightarrow \text{ false.} \tag{53}$$

Thus, (52) and (53) together imply

$$\mathbf{LNR} \wedge (\mathbf{G3} \vee \mathbf{G4}) \wedge \mathbf{D1} \wedge \mathbf{D2} \Rightarrow \text{ false.} \tag{54}$$

Now recall **R10**, i.e., **G7** $\wedge$ **E0** $\Rightarrow$ false. Then, jointly **R10** and **R17** imply

$$\mathbf{LNR} \wedge \mathbf{G2} \wedge (\neg\, \mathbf{G3}) \wedge (\neg\, \mathbf{G4}) \wedge (\neg\, \mathbf{G5}) \wedge \mathbf{E0} \Rightarrow \text{false.} \tag{55}$$

One can easily verify that jointly **R18** and **R19** imply

$$\mathbf{G16} \wedge (\mathbf{G17} \vee \mathbf{G18} \vee \mathbf{G19}) \wedge \mathbf{E0} \wedge \mathbf{E1} \wedge \mathbf{E2} \Rightarrow \text{false.} \tag{56}$$

One can see that jointly (56), **R20**, **R21**, and **R22** imply

$$\begin{aligned} \mathbf{LNR} \wedge \mathbf{G1} \wedge (\neg\, \mathbf{G3}) \wedge (\neg\, \mathbf{G4}) \wedge \mathbf{G23}& \\ \wedge\, \mathbf{E0} \wedge \mathbf{E1} \wedge \mathbf{E2} \Rightarrow \text{false.}& \end{aligned} \tag{57}$$

By similar arguments as used in deriving (57), jointly (56), **R23**, **R24**, and **R25** imply

$$\begin{aligned}\mathbf{LNR}\wedge\mathbf{G1}\wedge(\neg\,\mathbf{G3})\wedge(\neg\,\mathbf{G4})\wedge\mathbf{G24}&\\ \wedge\,\mathbf{E0}\wedge\mathbf{E1}\wedge\mathbf{E2}\Rightarrow\text{false}.&\end{aligned}\tag{58}$$

Since by definition $(\neg\,\mathbf{G3})\wedge(\neg\,\mathbf{G4})\wedge\mathbf{G5}\Rightarrow(\neg\,\mathbf{G3})\wedge(\neg\,\mathbf{G4})\wedge(\mathbf{G23}\vee\mathbf{G24})$, jointly (57) and (58) imply

$$\begin{aligned}\mathbf{LNR}\wedge\mathbf{G1}\wedge(\neg\,\mathbf{G3})\wedge(\neg\,\mathbf{G4})\wedge\mathbf{G5}&\\ \wedge\,\mathbf{E0}\wedge\mathbf{E1}\wedge\mathbf{E2}\Rightarrow\text{false}.&\end{aligned}\tag{59}$$

By similar arguments as used in deriving (57), (59) and (55) further imply

$$\begin{aligned}\mathbf{LNR}\wedge\mathbf{G1}\wedge\mathbf{G2}\wedge(\neg\,\mathbf{G3})\wedge(\neg\,\mathbf{G4})&\\ \wedge\,\mathbf{E0}\wedge\mathbf{E1}\wedge\mathbf{E2}\Rightarrow\text{false}.&\end{aligned}\tag{60}$$

Finally, one can easily verify that jointly (54) and (60) imply that we have $\mathbf{LNR}\wedge\mathbf{G1}\wedge\mathbf{G2}\wedge\mathbf{E0}\wedge\mathbf{E1}\wedge\mathbf{E2}\wedge\mathbf{D1}\wedge\mathbf{D2}\Rightarrow$ false, which proves **S13**. The skeleton of the proof of **S13** is complete.

APPENDIX Q
PROOFS OF R11 TO R25

Since **R11** and **R12** is identical to **R1** and **R2**, respectively, see Appendix N for their proofs.

We prove **R13** as follows.

Proof. We prove an equivalent relationship: $\mathbf{G4}\wedge\mathbf{G8}\wedge\mathbf{G9}\wedge\mathbf{D2}\Rightarrow\neg\,\mathbf{LNR}$. Suppose $\mathbf{G4}\wedge\mathbf{G8}\wedge\mathbf{G9}$ is true. The e_3^* (resp. e_2^*) defined in the properties of **G4** must be the most downstream (resp. upstream) edge of $\overline{S}_3$ (resp. $\overline{D}_2$), both of which belongs to $\mathsf{1cut}(s_2;d_3)$.

For the following, we will prove that there exists an edge in-between $\{e_{s_2},e_{s_3}\}$ and e_3^* who belongs to $\overline{S}_2\cap\overline{S}_3$. We will also prove that there exists an edge in-between e_2^* and $\{e_{d_2},e_{d_3}\}$ who belongs to $\overline{D}_2\cap\overline{D}_3$. By Proposition 4 we thus have **LNR** being false.

Define a node $u=\mathsf{tail}(e_3^*)$. Since $e_3^*\in\mathsf{1cut}(s_2;d_3)$, u is reachable from s_2. Since $e_3^*\in\overline{S}_3$, u is also reachable form s_3. Consider the set of edges $\{\mathsf{1cut}(s_2;u)\cap\mathsf{1cut}(s_3;u)\}\cup\{e_3^*\}$ and choose e'' as the most upstream one (we have at least e_3^*). Let e' denote the most downstream edge of $\mathsf{1cut}(s_2;\mathsf{tail}(e''))$ (we have at least the s_2-source edge e_{s_2}). Since we choose e' to be the most downstream one, by Proposition 3 the channel gain $m_{e';e''}$ must be irreducible.

Moreover, since $e_3^*\in\mathsf{1cut}(s_2;d_3)$, both e' and e'' must be in $\mathsf{1cut}(s_2;d_3)$. The reason is that by $e_3^*\in\mathsf{1cut}(s_2;d_3)$ any path from s_2 to d_3 must use e_3^*, which in turn implies that any path from s_2 to d_3 must use e'' since e'' separates s_2 and $\mathsf{tail}(e_3^*)$. Therefore $e''\in\mathsf{1cut}(s_2;d_3)$. Similarly, any s_2-to-d_3 path must use e'', which means any s_2-to-d_3 path must use e' as well since $e'\in\mathsf{1cut}(s_2;\mathsf{tail}(e''))$. As a result, the channel gain m_{23} contains $m_{e';e''}$ as a factor.

Since **D2** is true, it implies that $m_{e';e''}$ must be a factor of one of the following three channel gains m_{13}, m_{32}, and m_{21}. We first argue that $m_{e';e''}$ is not a factor of m_{32}. The reason is that if $m_{e';e''}$ is a factor of m_{32}, then $e'\in\mathsf{1cut}(s_3;d_2)$, which means that $e'\in\mathsf{1cut}(s_3;\mathsf{tail}(e_3^*))$. Since e' is also in $\mathsf{1cut}(s_2;\mathsf{tail}(e_3^*))$, this contradicts the construction that e'' is the most upstream edge of $\mathsf{1cut}(s_2;\mathsf{tail}(e_3^*))\cap\mathsf{1cut}(s_3;\mathsf{tail}(e_3^*))$.

Now we argue that $\mathsf{GCD}(m_{13},m_{e';e''})\equiv 1$. Suppose not. Then since $m_{e';e''}$ is irreducible, Proposition 3 implies that $\{e',e''\}$ are 1-edge cuts separating s_1 and d_3. Also from Property 1 of **G4**, there always exists a path segment from s_1 to e_2^* without using e_3^*. Since $e_2^*\in\mathsf{1cut}(s_2;d_3)$, e_2^* can reach d_3 and we thus have a s_1-to-d_3 path without using e_3^*. However by the assumption that $e'\in\mathsf{1cut}(s_1;d_3)$, this chosen path must use e'. As a result, s_2 can first reach e' and then reach d_3 through the chosen path without using e_3^*, which contradicts the assumption **G9**, i.e., $e_3^*\in\mathsf{1cut}(s_2;d_3)$.

From the above discussion $m_{e';e''}$ must be a factor of m_{21}, which by Proposition 3 implies that $\{e',e''\}$ also belong to $\mathsf{1cut}(s_2;d_1)$. Since by our construction e'' satisfies $e''\in\overline{S}_3\cap\mathsf{1cut}(s_2;d_3)$, we have thus proved that $e''\in\overline{S}_2\cap\overline{S}_3$. The proof for the existence of an edge satisfying $\overline{D}_2\cap\overline{D}_3$ can be followed symmetrically. The proof of **R12** is thus complete. $\square$

By swapping the roles of s_2 and s_3, and the roles of d_2 and d_3, the proofs of **R11** to **R13** can also be used to prove **R14** to **R16**, respectively. More specifically, **D1** and **D2** are converted back and forth from each other when swapping the flow indices. The same thing happens between **G3** and **G4**; between **G20** and **G8**; and between **G21** and **G9**. Moreover, **LNR** remains the same after the index swapping. The above proofs can thus be used to prove **R14** to **R16**.

We prove **R17** as follows.

Proof. Suppose $\mathbf{LNR}\wedge\mathbf{G2}\wedge(\neg\,\mathbf{G3})\wedge(\neg\,\mathbf{G4})\wedge(\neg\,\mathbf{G5})$ is true. Recall the definitions of e_u^{23}, e_u^{32}, e_v^{23}, and e_v^{32} from Properties of both $\neg\,\mathbf{G3}$ and $\neg\,\mathbf{G4}$. Since $\mathbf{LNR}\wedge(\neg\,\mathbf{G3})\wedge(\neg\,\mathbf{G4})$ is true, we have **G6** if we recall **N7**. Together with $\neg\,\mathbf{G5}$, e_u^{23} and e_u^{32} are distinct and not reachable from each other. Thus from **G2** being true, there must exist a s_1-to-d_1 path who does not use any of $\{e_u^{23},e_u^{32}\}$. Combined with Property 3 of $\neg\,\mathbf{G3}$ and $\neg\,\mathbf{G4}$, this further implies that such s_1-to-d_1 path does not use any of $\{e_v^{23},e_v^{32}\}$. Fix one such s_1-to-d_1 path as P_{11}^*.

We will now show that there exists an edge in P_{11}^* satisfying **G7**. To that end, we will show that if an edge $e\in P_{11}^*$ can be reached from s_2, then it must be in the downstream of e_v^{23}. We first argue that e_v^{23} and e are reachable from each other. The reason is that we now have a s_2-to-d_1 path by first going from s_2 to $e\in P_{11}^*$ and then use P_{11}^* to d_1. Since $e_v^{23}\in\mathsf{1cut}(s_2;d_1)$ by definition, such path must use e_v^{23}. As a result, we either have $e_v^{23}\prec e$ or $e\prec e_v^{23}$. ($e=e_v^{23}$ is not possible since $e_v^{23}\notin P_{11}^*$.) We then prove that $e\prec e_v^{23}$ is not possible. The reason is that P_{11}^* does not use e_u^{23} and thus s_1 must not reach e_v^{23} through P_{11}^* due to Property 3 of $\neg\,\mathbf{G3}$. As a result, we must have $e_v^{23}\prec e$. By symmetric arguments, any $e\in P_{11}^*$ that can be reached from reach s_3 must be in the downstream of e_v^{32} and any $e\in P_{11}^*$ that can reach d_3 (resp. d_2) must be in the upstream of e_u^{23} (resp. e_u^{32}).

For the following, we prove that there exists an edge $\tilde{e}\in P_{11}^*$ that cannot reach any of $\{d_2,d_3\}$, and that cannot be reached from any of $\{s_2,s_3\}$. Since $\tilde{e}\in P_{11}^*$, this will imply **G7**. Let e' denote the most downstream edge of P_{11}^* that can reach at least one of $\{d_2,d_3\}$ (we have at least the s_1-source edge

e_{s_1}). Among all the edges in P_{11}^* that are downstream of e', let e'' denote the most upstream one that can be reached by at least one of $\{s_2, s_3\}$ (we have at least the d_1-destination edge e_{d_1}). In the next paragraph, we argue that e'' is not the immediate downstream edge of e', i.e., $\mathsf{head}(e') \prec \mathsf{tail}(e'')$. This conclusion directly implies that we have at least one edge $\tilde{e}$ that satisfies **G7** (which is in-between e' and e'').

Without loss of generality, assume that $\mathsf{head}(e') = \mathsf{tail}(e'')$ and e' can reach d_2. Then, by our previous arguments, e' is an upstream edge of e_u^{32}. Consider two cases: Case 1: Suppose e'' is reachable from s_3, then by our previous arguments, e'' is a downstream edge of e_v^{32}. However, this implies that we can go from $\mathsf{head}(e')$ through e_u^{32} to e_v^{32} and then back to $\mathsf{tail}(e'') = \mathsf{head}(e')$, which contradicts the assumption that G is acyclic. Consider the Case 2: e'' is reachable from s_2. Then by our previous arguments, e'' is a downstream edge of e_v^{23}. Then we can go from e_u^{23} to e_v^{23}, then to $\mathsf{tail}(e'') = \mathsf{head}(e')$ and then to e_u^{32}. This contradicts the assumption of $\neg$ **G5**. The proof of **R17** is thus complete. □

We prove **R18** as follows.

Proof. Suppose **G16** $\wedge$ **G17** $\wedge$ **E0** is true. From **E0** being true, $G_{3\text{ANA}}$ satisfies (36) with at least two non-zero coefficients α_i and β_j. From **G16** being true, the considered subgraph G' has the non-trivial channel gain polynomials m_{ij} for all $i \neq j$ and m_{11}. By Proposition 2, G' also satisfies (36) with the same set of non-zero coefficients α_i and β_j.

From **G17** being true, consider the defined edge $\tilde{e} \in G'$ that cannot reach any of $\{d_2, d_3\}$ (but reaches d_1) and cannot be reached by any of $\{s_2, s_3\}$ (but reached from s_1). This chosen $\tilde{e}$ must not be the s_1-source edge e_{s_1} otherwise ($\tilde{e} = e_{s_1}$) $\tilde{e}$ will reach d_2 or d_3 and thus contradict the assumption **G17**.

Choose an edge $e \in G'$ such that $e_{s_1} \preceq e$ and $\mathsf{head}(e) = \mathsf{tail}(\tilde{e})$. This is always possible because s_1 can reach $\tilde{e}$ and $e_{s_1} \prec \tilde{e}$ on G'. Then, this chosen edge e should not be reached from s_2 or s_3 otherwise s_2 or s_3 can reach $\tilde{e}$ and this contradicts the assumption **G17**. Now consider a local kernel $x_{e\tilde{e}}$ from e to $\tilde{e}$. Then, one can quickly see that the channel gains m_{21}, m_{23}, m_{31}, and m_{32} must not have $x_{e\tilde{e}}$ as a variable since e is not reachable from s_2 nor s_3. Also m_{12} and m_{13} must not have $x_{e\tilde{e}}$ as a variable since $\tilde{e}$ doe not reach any of $\{d_2, d_3\}$.

This further implies that the RHS of (36) does not depend on $x_{e\tilde{e}}$. However, the LHS of (36) has a common factor m_{11} and thus has degree 1 in $x_{e\tilde{e}}$. This contradicts the above discussion that G' also satisfies (36). □

We prove **R19** as follows.

Proof. Equivalently, we prove the following two relationships: **G16** $\wedge$ **G18** $\wedge$ **E0** $\wedge$ **E1** $\wedge$ **E2** $\Rightarrow$ false; and **G16** $\wedge$ **G19** $\wedge$ **E0** $\wedge$ **E1** $\wedge$ **E2** $\Rightarrow$ false.

We first prove the former. Suppose that **G16** $\wedge$ **G18** $\wedge$ **E0** $\wedge$ **E1** $\wedge$ **E2** is true. From **E0** $\wedge$ **E1** $\wedge$ **E2** being true, there exists some coefficient values $\{\alpha_i\}_{i=0}^n$ and $\{\beta_j\}_{j=0}^n$ such that $G_{3\text{ANA}}$ of interest satisfies

$$m_{11} m_{23}\, \psi_\alpha^{(n)}(R, L) = m_{13} m_{21}\, \psi_\beta^{(n)}(R, L), \tag{61}$$

with (i) At least one of α_i is non-zero; (ii) At least one of β_j is non-zero; (iii) $\alpha_k \neq \beta_k$ for some $k \in \{0, ..., n\}$; and (iv) either $\alpha_0 \neq 0$ or $\beta_n \neq 0$ or $\alpha_k \neq \beta_{k-1}$ for some $k \in \{1, ..., n\}$.

From the assumption that **G16** is true, consider a subgraph G' which has the non-trivial channel gain polynomials m_{ij} for all $i \neq j$ and m_{11}. Thus by Proposition 2, G' also satisfies (61) with the same coefficient values.

Now from **G18** being true, we will prove the first relationship, i.e., **G16** $\wedge$ **G18** $\wedge$ **E0** $\wedge$ **E1** $\wedge$ **E2** $\Rightarrow$ false. Since **G18** is true, there exists a subgraph $G'' \subset G'$ which also has the non-trivial channel gains m_{ij} for all $i \neq j$ and m_{11}. Thus again by Proposition 2, G'' satisfies (61) with the same coefficients. Since G'' also satisfies $m_{11} m_{23} = m_{13} m_{21}$, by (61), we know that G'' satisfies

$$\psi_\alpha^{(n)}(R, L) = \psi_\beta^{(n)}(R, L). \tag{62}$$

Note that by (iii), the coefficient values were chosen such that $\alpha_k \neq \beta_k$ for some $k \in \{0, ..., n\}$. Then (62) further implies that G'' satisfies $\sum_{k=0}^n \gamma_k R^{n-k} L^k = 0$ with at least one non-zero γ_k. Equivalently, this means that the set of polynomials $\{R^n, R^{n-1}L, \cdots, RL^{n-1}, L^n\}$ is linearly dependent. Since this is the Vandermonde form, it is equivalent to that $L \equiv R$ holds on G''. However, this contradicts the assumption **G18** that G'' satisfies $L \not\equiv R$.

To prove the second relationship, i.e., **G16** $\wedge$ **G19** $\wedge$ **E0** $\wedge$ **E1** $\wedge$ **E2** $\Rightarrow$ false, we assume **G19** is true. Since **G19** is true, there exists a subgraph $G'' \subset G'$ which also has the non-trivial channel gains m_{ij} for all $i \neq j$ and m_{11}. Thus again by Proposition 2, G'' satisfies (61) with the same coefficients. Moreover, G'' satisfies $m_{11} m_{32} = m_{12} m_{31}$, which together with (61) imply that G'' also satisfies

$$R\, \psi_\alpha^{(n)}(R, L) = L\, \psi_\beta^{(n)}(R, L), \tag{63}$$

where we first multiply m_{32} on both sides of (61).

Expanding (63), we have

$$\begin{aligned} & R\, \psi_\alpha^{(n)}(R, L) - L\, \psi_\beta^{(n)}(R, L) \\ & = \alpha_0 R^{n+1} + \sum_{k=1}^{n} (\alpha_k - \beta_{k-1}) R^{n+1-k} L^k + \beta_n L^{n+1} \\ & = \sum_{k=0}^{n+1} \gamma_k R^{n+1-k} L^k = 0 \end{aligned} \tag{64}$$

By (iv), the coefficient values were chosen such that either $\alpha_0 \neq 0$ or $\beta_n \neq 0$ or $\alpha_k \neq \beta_{k-1}$ for some $k \in \{1, ..., n\}$. Then (64) further implies that G'' satisfies $\sum_{k=0}^{n+1} \gamma_k R^{n+1-k} L^k = 0$ with some non-zero γ_k. Equivalently, this means that the set of polynomials $\{R^{n+1}, R^n L, \cdots, RL^n, L^{n+1}\}$ is linearly dependent, and thus G'' satisfies $L \equiv R$. This contradicts the assumption **G19** that $L \not\equiv R$ holds on G''. The proof of **R19** is thus complete. □

We prove **R20** as follows.

Proof. Suppose **G1** $\wedge$ ($\neg$ **G3**) $\wedge$ ($\neg$ **G4**) $\wedge$ ($\neg$ **G22**) $\wedge$ **G23** is true. Recall the definitions of e_u^{23}, e_u^{32}, e_v^{23}, and e_v^{32} when ($\neg$ **G3**) $\wedge$ ($\neg$ **G4**) is true. From Property 1 of both $\neg$ **G3** and $\neg$ **G4**, we know that s_1 can reach e_u^{23} (resp. e_u^{32}) and then use e_v^{23} (resp. e_v^{32}) to arrive at d_1. Note that $\neg$ **G22** being

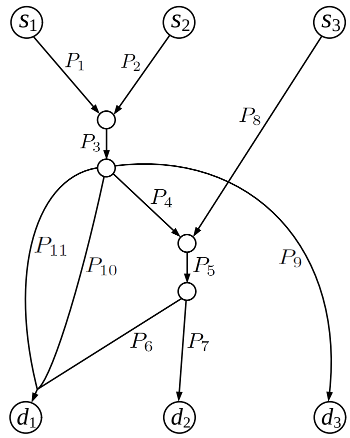

Fig. 3. The subgraph G' of the 3-unicast ANA network $G_{3\text{ANA}}$ induced by the union of the 11 paths in the proof of **R20**.

true implies that every s_1-to-d_1 path must use a vertex w in-between $\mathsf{tail}(e_u^{23})$ and $\mathsf{head}(e_v^{23})$ or in-between $\mathsf{tail}(e_u^{32})$ and $\mathsf{head}(e_v^{32})$ or both. Combined with Property 3 of both $\neg$ **G3** and $\neg$ **G4**, this further implies that every s_1-to-d_1 path must use $\{e_u^{23}, e_v^{23}\}$ or $\{e_u^{32}, e_v^{32}\}$ or both.

From **G23** being true, we have $e_u^{23} \prec e_u^{32}$. For the following we prove that (i) $\mathsf{head}(e_v^{23}) \prec \mathsf{tail}(e_u^{32})$; and (ii) there exists a path segment from $\mathsf{head}(e_v^{23})$ to d_1 which is vertex-disjoint with any vertex in-between $\mathsf{tail}(e_u^{32})$ and $\mathsf{head}(e_v^{32})$. First we note that e_u^{23} is not an 1-edge cut separating s_1 and $\mathsf{tail}(e_u^{32})$. The reason is that if $e_u^{23} \in \mathsf{1cut}(s_1; \mathsf{tail}(e_u^{32}))$, then e_u^{23} must be an 1-edge cut separating s_1 and d_1 since any s_1-to-d_1 path must use $\{e_u^{23}, e_v^{23}\}$ or $\{e_u^{32}, e_v^{32}\}$ or both. However, since $e_u^{23} \in \overline{S}_2 \cap \overline{D}_3$, this implies $e_u^{23} \in \mathsf{1cut}(\{s_1, s_2\}; \{d_1, d_3\})$. This contradicts the assumption **G1**. We now consider all the possible cases: either $e_v^{23} \prec e_u^{32}$ or $e_u^{32} \preceq e_v^{23}$ or not reachable from each other. We first show that the last case is not possible. The reason is that suppose e_v^{23} and e_u^{32} are not reachable from each other, then s_1 can first reach e_u^{23}, then reach e_u^{32} to d_1 without using e_v^{23}. This contradicts Property 3 of $\neg$ **G3**. Similarly, the second case is not possible because when $e_u^{32} \preceq e_v^{23}$, we can find a path from s_1 to e_u^{32} to e_v^{23} to d_1 not using e_u^{23} since $e_u^{23} \notin \mathsf{1cut}(s_1; \mathsf{tail}(e_u^{32}))$. This also contradicts Property 3 of $\neg$ **G3**. We thus have shown $e_v^{23} \prec e_u^{32}$. Now we still need to show that e_v^{23} and e_u^{32} are not immediate neighbors: $\mathsf{head}(e_v^{23}) \prec \mathsf{tail}(e_u^{32})$. Suppose not, i.e., $\mathsf{head}(e_v^{23}) = \mathsf{tail}(e_u^{32})$. Then by Property 3 of $\neg$ **G3**, we know that any path from $\mathsf{head}(e_v^{23}) = \mathsf{tail}(e_u^{32})$ to d_1 must use both e_u^{32} and e_v^{32}. By the conclusion in the first paragraph of this proof, we know that this implies $\{e_u^{32}, e_v^{32}\} \subset \mathsf{1cut}(s_1; d_1)$. However, this further implies that $\{e_u^{32}, e_v^{32}\} \subset \mathsf{1cut}(\{s_1, s_3\}; \{d_1, d_2\})$, which contradicts **G1**. The proof of (i) is complete.

We now prove (ii). Suppose that every path from $\mathsf{head}(e_v^{23})$ to d_1 has at least one vertex w that satisfies $\mathsf{tail}(e_u^{32}) \preceq w \preceq \mathsf{head}(e_v^{32})$. Then by Property 3 of $\neg$ **G3**, every s_1-to-d_1 path that uses e_v^{23} must use both e_u^{32} and e_v^{32}. By the findings in the first paragraph of this proof, this also implies that any s_1-to-d_1 path must use both e_u^{32} and e_v^{32}. However, this further implies that $\{e_u^{32}, e_v^{32}\} \subset \mathsf{1cut}(\{s_1, s_3\}; \{d_1, d_2\})$. This contradicts **G1**. We have thus proven (ii).

Using the assumptions and the above discussions, we construct the following 11 path segments.

- P_1: a path from s_1 to $\mathsf{tail}(e_u^{23})$. This is always possible due to **G3** being false.
- P_2: a path from s_2 to $\mathsf{tail}(e_u^{23})$, which is edge-disjoint with P_1. This is always possible due to **G3** being false.
- P_3: a path starting from e_u^{23} and ending at e_v^{23}. This is always possible due to **G3** being false.
- P_4: a path from $\mathsf{head}(e_v^{23})$ to $\mathsf{tail}(e_u^{23})$. This is always possible since we showed (i) in the above discussion.
- P_5: a path starting from e_u^{32} and ending at e_v^{32}. This is always possible due to **G4** being false.
- P_6: a path from $\mathsf{head}(e_v^{32})$ to d_1. This is always possible due to **G4** being false.
- P_7: a path from $\mathsf{head}(e_v^{32})$ to d_2, which is edge-disjoint with P_6. This is always possible due to **G4** being false and Property 2 of $\neg$ **G4**.
- P_8: a path from s_3 to $\mathsf{tail}(e_u^{32})$. This is always possible due to **G4** being false.
- P_9: a path from $\mathsf{head}(e_v^{23})$ to d_3. This is always possible due to **G3** being false.
- P_{10}: a path from $\mathsf{head}(e_v^{23})$ to d_1, which is vertex-disjoint with P_5. This is always possible since we showed (ii) in the above discussion.
- P_{11}: a path from $\mathsf{head}(e_v^{23})$ to d_1, which is edge-disjoint with P_9. This is always possible due to **G3** being false.

Fig. 3 illustrates the relative topology of these 11 paths. We now consider the subgraph G' induced by the above 11 path segments. First, one can see that s_i can reach d_j for all $i \neq j$. In particular, s_1 can reach d_2 through $P_1P_3P_4P_5P_7$; s_1 can reach d_3 through $P_1P_3P_9$; s_2 can reach d_1 through either $P_2P_3P_4P_5P_6$ or $P_2P_3P_{10}$ or $P_2P_3P_{11}$; s_2 can reach d_3 through $P_2P_3P_9$; s_3 can reach d_1 through $P_8P_5P_6$; and s_3 can reach d_2 through $P_8P_5P_7$. Moreover, s_1 can reach d_1 through either $P_1P_3P_4P_5P_6$ or $P_1P_3P_{10}$ or $P_1P_3P_{11}$. Thus we showed **G16**.

For the following, we will prove that $m_{11}m_{23} = m_{13}m_{21}$ and $L \not\equiv R$ hold in the above G'. Note that $\{P_1, P_2, P_3, P_{10}\}$ must be vertex-disjoint with P_8, otherwise s_3 can reach d_1 without using P_5 and this contradicts $\{e_u^{32}, e_v^{32}\} \subset \overline{S}_3 \cap \overline{D}_2 \subset \mathsf{1cut}(s_3; d_1)$. Since P_8 is vertex-disjoint from $\{P_1, P_2\}$, one can easily see that removing P_3 separates $\{s_1, s_2\}$ and $\{d_1, d_3\}$. Thus G' satisfies $m_{11}m_{23} = m_{13}m_{21}$.

To show that $L \not\equiv R$ holds on G', we make the following arguments. First, we show that G' satisfies $\overline{S}_2 \cap \overline{S}_3 = \emptyset$. Note that any $\overline{S}_2$ edge can exist only as one of four cases: (i) P_2; (ii) P_3; (iii) an edge that P_4, P_9, P_{10}, and P_{11} share; and (iv) an edge that P_6, P_9, P_{10}, and P_{11} share. Note also that any $\overline{S}_3$ edge can exist only as one of three cases: (i) P_8; (ii) P_5; and (iii) an edge that P_6 and P_7 shares. But since P_6 and P_7 were chosen to be edge-disjoint with each other from the above construction, any $\overline{S}_3$ edge can exist on either P_8 or P_5. However, P_5 was chosen to be vertex-disjoint with P_{10} from the above construction and we also showed that P_8 is vertex-disjoint with $\{P_2, P_3, P_{10}\}$. Thus, $\overline{S}_2 \cap \overline{S}_3 = \emptyset$ on G'.

Second, we show that G' satisfies $\overline{D}_1 \cap \overline{D}_2 = \emptyset$. Note that any $\overline{D}_1$ edge can exist on an edge that all P_6, P_{10}, and P_{11} share since P_6 cannot share an edge with any of its upstream paths (in particular P_2, P_3, P_4, and P_5); P_5 cannot share an edge with P_{10} due to vertex-disjointness; and P_8 cannot share edge with $\{P_2, P_3, P_{10}\}$ otherwise there will be an s_3-to-d_1 path not using P_5. Note also that any $\overline{D}_2$ edge can exist on (i) an edge that both P_4 and P_8 share; (ii) P_5; and (iii) P_7. However, P_7 was chosen to be edge-disjoint with P_6, and P_5 was chosen to be vertex-disjoint with P_{10}. Moreover, we already showed that P_8 is vertex-disjoint with P_{10}. Thus, $\overline{D}_1 \cap \overline{D}_2{=}\emptyset$ on G'.

Third, we show that G' satisfies $\overline{D}_1 \cap \overline{D}_3{=}\emptyset$. Note that any $\overline{D}_1$ edge can exist on an edge that P_6, P_{10} and P_{11} share. Note also that any $\overline{D}_3$ edge can exist on (i) P_3; and (ii) P_9. However, all P_6, P_{10} and P_{11} are the downstream paths of P_3. Moreover, P_9 was chosen to be edge-disjoint with P_{11} by our construction. Thus, $\overline{D}_1 \cap \overline{D}_3{=}\emptyset$ on G'.

Hence, the above discussions, together with Proposition 4, implies that the considered G' satisfies $L \not\equiv R$. Thus we have proven **G18** being true. The proof is thus complete. □

We prove **R21** as follows.

Proof. Suppose **LNR** $\wedge (\neg$ **G3**$) \wedge (\neg$ **G4**$) \wedge$ **G22** $\wedge$ **G23** $\wedge$ **G25** is true. Recall the definitions of e_u^{23}, e_u^{32}, e_v^{23}, and e_v^{32} when $(\neg$ **G3**$) \wedge (\neg$ **G4**$)$ is true. From Property 1 of both $\neg$ **G3** and $\neg$ **G4**, s_1 reaches e_u^{23} and e_u^{32}, respectively. From **G22** being true, there exists a s_1-to-d_1 path P_{11}^* who does not use any vertex in-between $\mathsf{tail}(e_u^{23})$ and $\mathsf{head}(e_v^{23})$, and any vertex in-between $\mathsf{tail}(e_u^{32})$ and $\mathsf{head}(e_v^{32})$.

Note that **G23** $\wedge$ **G25** implies $e_u^{23} \prec e_u^{32} \prec e_v^{23}$. For the following, we prove that $e_v^{32} \prec e_v^{23}$. Note that by our construction $e_u^{32} \preceq e_v^{32}$. As a result, we have $e_u^{23} \prec e_u^{32} \preceq e_v^{32} \prec e_v^{23}$. To that end, we consider all the possible cases between e_v^{32} and e_v^{23}: $e_v^{32} \prec e_v^{23}$; or $e_v^{23} \prec e_v^{32}$; or $e_v^{32} = e_v^{23}$; or they are not reachable from each other. We first show that the third case is not possible. The reason is that if $e_v^{32} = e_v^{23}$, then we have $\overline{S}_2 \cap \overline{S}_3 \cap \overline{D}_2 \cap \overline{D}_3 \neq \emptyset$, which contradicts the assumption **LNR**. The last case in which e_v^{32} and e_v^{23} are not reachable from each other is also not possible. The reason is that by our construction, there is always an s_1-to-d_1 path through e_u^{23}, e_u^{32}, and e_v^{32} without using e_v^{23}. Note that by Property 3 of $\neg$ **G3**, such s_1-to-d_1 path must use e_v^{23}, which is a contradiction. We also claim that the second case, $e_v^{23} \prec e_v^{32}$, is not possible. The reason is that if $e_v^{23} \prec e_v^{32}$, then together with the assumption **G23** $\wedge$ **G25** we have $e_u^{23} \prec e_u^{32} \prec e_v^{23} \prec e_v^{32}$. We also note that e_u^{32} must be an 1-edge cut separating s_1 and $\mathsf{tail}(e_v^{23})$, otherwise s_1 can reach $\mathsf{tail}(e_v^{23})$ without using e_u^{32} and then use e_v^{23} and e_v^{32} to arrive at d_2. This contradicts the construction $e_u^{32} \in \overline{S}_3 \cap \overline{D}_2 \subset \mathsf{1cut}(s_1; d_2)$. Since $e_v^{23} \in \overline{S}_2 \cap \overline{D}_3$ is also an 1-edge cut separating s_1 and d_3, this in turn implies that $e_u^{32} \in \mathsf{1cut}(s_1; d_3)$. Symmetrically following this argument, we can also prove that $e_v^{23} \in \mathsf{1cut}(s_3; d_1)$. Since $e_u^{32} \in \overline{S}_3 \cap \overline{D}_2$ and $e_v^{23} \in \overline{S}_2 \cap \overline{D}_3$, these further imply that $e_u^{32} \in \overline{S}_1 \cap \overline{S}_3 \cap \overline{D}_2$ and $e_v^{23} \in \overline{S}_2 \cap \overline{D}_1 \cap \overline{D}_3$, which contradicts the assumption **LNR** by Proposition 4. We have thus established $e_v^{32} \prec e_v^{23}$ and together with the assumption **G23** $\wedge$ **G25**, we have $e_u^{23} \prec e_u^{32} \preceq e_v^{32} \prec e_v^{23}$.

Using the assumptions and the above discussions, we construct the following 7 path segments.

- P_1: a path from s_1 to $\mathsf{tail}(e_u^{23})$. This is always possible due to **G3** being false.
- P_2: a path from s_2 to $\mathsf{tail}(e_u^{23})$ which is edge-disjoint with P_1. This is always possible due to **G3** being false and Property 2 of $\neg$ **G3**.
- P_3: a path starting from e_u^{23}, using e_u^{32} and e_v^{32}, and ending at e_v^{23}. This is always possible from the above discussion.
- P_4: a path from $\mathsf{head}(e_v^{23})$ to d_1. This is always possible due to **G3** being false.
- P_5: a path from $\mathsf{head}(e_v^{23})$ to d_3 which is edge-disjoint with P_4. This is always possible due to **G3** being false and Property 2 of $\neg$ **G3**.
- P_6: a path from s_3 to $\mathsf{tail}(e_u^{32})$. This is always possible due to **G4** being false.
- P_7: a path from $\mathsf{head}(e_v^{32})$ to d_2. This is always possible due to **G4** being false.

We now consider the subgraph G' induced by the above 7 path segments and P_{11}^*. First, one can easily check that s_i can reach d_j for all $i \neq j$. In particular, s_1 can reach d_2 through $P_1 P_3 e_v^{32} P_7$; s_1 can reach d_3 through $P_1 P_3 P_5$; s_2 can reach d_1 through $P_2 P_3 P_4$; s_2 can reach d_3 through $P_2 P_3 P_5$; s_3 can reach d_1 through $P_6 e_u^{32} P_3 P_4$; and s_3 can reach d_2 through $P_6 e_u^{32} P_3 e_v^{32} P_7$. Moreover, s_1 can reach d_1 through either P_{11}^* or $P_1 P_3 P_4$. As a result, **G16** must hold.

We now prove **G17**. To that end, we will show that there exists an edge $\tilde{e} \in P_{11}^*$ that cannot reach any of $\{d_2, d_3\}$, and cannot be reached from any of $\{s_2, s_3\}$. Note from **G22** being true that P_{11}^* was chosen to be vertex-disjoint with P_3. Note that P_{11}^* must also be vertex-disjoint with P_2 (resp. P_6) otherwise s_2 (resp. s_3) can reach d_1 without using P_3 (resp. $e_u^{32} P_3 e_v^{32}$). Similarly, P_{11}^* must also be vertex-disjoint with P_5 (resp. P_7) otherwise s_1 can reach d_3 (resp. d_2) without using P_3 (resp. $e_u^{32} P_3 e_v^{32}$). Hence, among 7 path segments constructed above, the only path segments that can share a vertex with P_{11}^* are P_1 and P_4. Without loss of generality, we also assume that P_1 is chosen such that it overlaps with P_{11}^* in the beginning but then "branches out". That is, let u^* denote the most downstream vertex among those who are used by both P_1 and P_{11}^* and we can then replace P_1 by $s_1 P_{11}^* u^* P_1 \mathsf{tail}(e_u^{23})$. Note that the new construction still satisfies the requirement that P_1 and P_2 are edge-disjoint since P_{11}^* is vertex-disjoint with P_2. Similarly, we also assume that P_4 is chosen such that it does not overlap with P_{11}^* in the beginning but then "merges" with P_{11}^* whenever P_4 shares a vertex v^* with P_{11}^* for the first time. The new construction of P_4, i.e., $\mathsf{head}(e_v^{23}) P_4 v^* P_{11}^* d_1$ is still edge-disjoint from P_5. Then in the considered subgraph G', in order for an edge $e \in P_{11}^*$ to reach d_2 or d_3, we must have $\mathsf{head}(e) \preceq u^*$. Similarly, in order for an edge $e \in P_{11}^*$ to be reached from s_2 or s_3, this edge e must satisfy $v^* \preceq \mathsf{tail}(e)$. If there does not exist such an edge $\tilde{e} \in P_{11}^*$ satisfying **G17**, then it means that $u^* = v^*$. This, however, contradicts the assumption that G is acyclic because now we can walk from u^* through $P_1 P_3 P_4$ back to $v^* = u^*$. Therefore, we thus have **G17**. The proof of **R21** is thus complete. □

We prove **R22** as follows.

Proof. Suppose **LNR** $\wedge (\neg$ **G3**$) \wedge (\neg$ **G4**$) \wedge$ **G22** $\wedge$ **G23** $\wedge$ $(\neg$ **G25**$)$ is true. Recall the definitions of e_u^{23}, e_u^{32}, e_v^{23}, and e_v^{32} when $(\neg$ **G3**$) \wedge (\neg$ **G4**$)$ is true. From Property 1 of both $\neg$ **G3** and $\neg$ **G4**, s_1 reaches e_u^{23} and e_u^{32}, respectively. From **G22** being true, there exists a s_1-to-d_1 path P_{11}^* who does not use any vertex in-between $\mathsf{tail}(e_u^{23})$ and $\mathsf{head}(e_v^{23})$, and any vertex in-between $\mathsf{tail}(e_u^{32})$ and $\mathsf{head}(e_v^{32})$.

Note that **G23** implies $e_u^{23} \prec e_u^{32}$. For the following, we prove that $\mathsf{head}(e_v^{23}) \prec \mathsf{tail}(e_u^{32})$. To that end, we consider all the possible cases by $\neg$ **G25** being true: either $e_v^{23} \prec e_u^{32}$ or $e_v^{23} = e_u^{32}$ or not reachable from each other. We first show that the second case is not possible. The reason is that if $e_v^{23} = e_u^{32}$, then we have $\overline{S}_2 \cap \overline{S}_3 \cap \overline{D}_2 \cap \overline{D}_3 \neq \emptyset$, which contradicts the assumption **LNR**. The third case in which e_v^{23} and e_u^{32} are not reachable from each other is also not possible. The reason is that by our construction, there is always an s_1-to-d_1 path through e_u^{23}, e_u^{32}, and e_v^{32} without using e_v^{23}. Note that by Property 3 of $\neg$ **G3**, such s_1-to-d_1 path must use e_v^{23}, which is a contradiction. We have thus established $e_v^{23} \prec e_u^{32}$. We still need to show that e_v^{23} and e_u^{32} are not immediate neighbors since we are proving $\mathsf{head}(e_v^{23}) \prec \mathsf{tail}(e_u^{32})$. We prove this by contradiction. Suppose not, i.e., $w = \mathsf{head}(e_v^{23}) = \mathsf{tail}(e_u^{32})$. Since $e_u^{32} \in \overline{S}_3 \cap \overline{D}_2 \subset \mathsf{1cut}(s_1; d_2)$, any s_1-to-d_2 path must use its tail w. By Property 3 of $\neg$ **G3** we have $e_v^{23} \in \mathsf{1cut}(s_1; w)$. This in turn implies that e_v^{23} is also an 1-edge cut separating s_1 and d_2. By symmetry, we can also prove $e_u^{32} \in \mathsf{1cut}(s_2; d_1)$. Jointly the above argument implies that $e_v^{23} \in \overline{S}_1 \cap \overline{S}_2 \cap \overline{D}_3$ and $e_u^{32} \in \overline{S}_3 \cap \overline{D}_1 \cap \overline{D}_2$, which contradicts the assumption **LNR** by Proposition 4.

Based on the above discussions, we construct the following 9 path segments.

- P_1: a path from s_1 to $\mathsf{tail}(e_u^{23})$. This is always possible due to **G3** being false.
- P_2: a path from s_2 to $\mathsf{tail}(e_u^{23})$ which is edge-disjoint with P_1. This is always possible due to **G3** being false and Property 2 of $\neg$ **G3**.
- P_3: a path starting from e_u^{23} and ending at e_v^{23}. This is always possible due to **G3** being false.
- P_4: a path from $\mathsf{head}(e_v^{23})$ to $\mathsf{tail}(e_u^{32})$. This is always possible from the above discussion.
- P_5: a path starting from e_u^{32} and ending at e_v^{32}. This is always possible due to **G4** being false.
- P_6: a path from $\mathsf{head}(e_v^{32})$ to d_1. This is always possible due to **G4** being false.
- P_7: a path from $\mathsf{head}(e_v^{32})$ to d_2 which is edge-disjoint with P_6. This is always possible due to **G4** being false and Property 2 of $\neg$ **G4**.
- P_8: a path from s_3 to $\mathsf{tail}(e_u^{32})$. This is always possible due to **G4** being false.
- P_9: a path from $\mathsf{head}(e_v^{23})$ to d_3. This is always possible due to **G3** being false.

From **G22** being true, P_{11}^* was chosen to be vertex-disjoint with $\{P_3, P_5\}$. Note that P_{11}^* must also be vertex-disjoint with P_2 (resp. P_8) otherwise s_2 (resp. s_3) can reach d_1 without using P_3 (resp. P_5). Similarly, P_{11}^* must also be vertex-disjoint with P_7 (resp. P_9) otherwise s_1 can reach d_2 (resp. d_3) without using P_5 (resp. P_3). Hence, among 9 path segments constructed above, the only path segments that can share a vertex with P_{11}^* are P_1, P_4, and P_6.

We now consider the subgraph G' induced by the above 9 path segments and P_{11}^*. First, one can easily check that s_i can reach d_j for all $i \neq j$. In particular, s_1 can reach d_2 through $P_1P_3P_4P_5P_7$; s_1 can reach d_3 through $P_1P_3P_9$; s_2 can reach d_1 through $P_2P_3P_4P_5P_6$; s_2 can reach d_3 through $P_2P_3P_9$; s_3 can reach d_1 through $P_8P_5P_6$; and s_3 can reach d_2 through $P_8P_5P_7$. Moreover, s_1 can reach d_1 through either P_{11}^* or $P_1P_3P_4P_5P_6$. Thus we showed **G16**.

Case 1: P_{11}^* is also vertex-disjoint with P_4. In this case, we will prove that **G17** is satisfied. Namely, we claim that there exists an edge $\tilde{e} \in P_{11}^*$ that cannot reach any of $\{d_2, d_3\}$, and cannot be reached from any of $\{s_2, s_3\}$. Note that only path segments that P_{11}^* can share a vertex with are P_1 and P_6. Without loss of generality, we assume that P_1 is chosen such that it overlaps with P_{11}^* in the beginning but then "branches out". That is, let u^* denote the most downstream vertex among those who are used by both P_1 and P_{11}^* and we can then replace P_1 by $s_1P_{11}^*u^*P_1\mathsf{tail}(e_u^{23})$. Note that the new construction still satisfies the requirement that P_1 and P_2 are edge-disjoint since P_{11}^* is vertex-disjoint with P_2. Similarly, we also assume that P_6 is chosen such that it does not overlap with P_{11}^* in the beginning but then "merges" with P_{11}^* whenever P_6 shares a vertex v^* with P_{11}^* for the first time. The new construction of P_6, i.e, $\mathsf{head}(e_v^{32})P_6v^*P_{11}^*d_1$, is still edge-disjoint from P_7. Then in the considered subgraph G', in order for an edge $e \in P_{11}^*$ to reach d_2 or d_3, we must have $\mathsf{head}(e) \preceq u^*$. Similarly, in order for an edge $e \in P_{11}^*$ to be reached from s_2 or s_3, this edge e must satisfy $v^* \preceq \mathsf{tail}(e)$. If there does not exist such an edge $\tilde{e} \in P_{11}^*$ satisfying **G17**, then it means that $u^* = v^*$. This, however, contradicts the assumption that G is acyclic because now we can walk from u^* through $P_1P_3P_4P_5P_6$ back to $v^* = u^*$. Therefore, we thus have **G17** for **Case 1**.

Case 2: P_{11}^* shares a vertex with P_4. In this case, we will prove that **G18** is true. Since P_{11}^* is vertex-disjoint with $\{P_3, P_5\}$, P_{11}^* must share a vertex w with P_4 where $\mathsf{head}(e_v^{23}) \prec w \prec \mathsf{tail}(e_u^{32})$. Choose the most downstream vertex among those who are used by both P_{11}^* and P_4 and denote it as w'. Then, denote the path segment $\mathsf{head}(e_v^{23})P_4w'P_{11}^*d_1$ by P_{10}. Note that we do not introduce new paths but only introduce a new notation as shorthand for a combination of some existing path segments. We observe that there may be some edge overlap between P_4 and P_9 since both starts from $\mathsf{head}(e_v^{23})$. Let $\tilde{w}$ denote the most downstream vertex that is used by both P_4 and P_9. We then replace P_9 by $\tilde{w}P_9d_3$, i.e., we truncate P_9 so that P_9 is now edge-disjoint from P_4.

Since the path segment $w'P_{10}d_1$ originally comes from P_{11}^*, $w'P_{10}d_1$ is also vertex-disjoint with $\{P_2, P_3, P_5, P_7, P_8, P_9\}$. In addition, P_8 must be vertex-disjoint with $\{P_1, P_2, P_3, P_{10}\}$, otherwise s_3 can reach d_1 without using P_5.

Now we consider the another subgraph $G'' \subset G'$ induced by the path segments P_1 to P_8, the redefined P_9, and newly constructed P_{10}, i.e., when compared to G', we replace P_{11}^* by P_{10}. One can easily verify that s_i can reach d_j for all $i \neq j$, and s_1 can reach d_1 on this new subgraph G''. Using the above topological relationships between these constructed path segments, we will further show that the induced G'' satisfies $m_{11}m_{23} = m_{13}m_{21}$ and $L \not\equiv R$.

Since P_8 is vertex-disjoint from $\{P_1, P_2\}$, one can see that removing P_3 separates $\{s_1, s_2\}$ and $\{d_1, d_3\}$. Thus, the considered G'' also satisfies $m_{11}m_{23} = m_{13}m_{21}$.

To prove $L \not\equiv R$, we first show that G'' satisfies $\overline{S}_2 \cap \overline{S}_3 = \emptyset$.

Note that any $\overline{S}_2$ edge can exist only as one of three cases: (i) P_2; (ii) P_3; (iii) an edge that P_4 and P_{10} share, whose head is in the upstream of or equal to $\tilde{w}$, i.e., $\{e \in P_4 \cap P_{10} : \mathsf{head}(e) \preceq \tilde{w}\}$ (may or may not be empty); and (iv) an edge that P_6, P_9, and P_{10} share. Note also that any $\overline{S}_3$ edge can exist only as on of three cases: (i) P_8; (ii) P_5; and (iii) an edge that P_6 and P_7 share. But since P_6 and P_7 were chosen to be edge-disjoint from the above construction, any $\overline{S}_3$ edge can exist on either P_8 or P_5. We then notice that P_8 is vertex-disjoint with $\{P_2, P_3, P_{10}\}$. Also, P_5 was chosen to be vertex-disjoint with P_{10} and both P_2 and P_3 are in the upstream of P_5. The above arguments show that no edge can be simultaneously in $\overline{S}_2$ and $\overline{S}_3$. We thus have $\overline{S}_2 \cap \overline{S}_3 = \emptyset$ on G''.

Second, we show that G'' satisfies $\overline{D}_1 \cap \overline{D}_2 = \emptyset$. Note that any $\overline{D}_1$ edge can exist only an edge that both P_6 and P_{10} share since any of $\{P_5, P_8\}$ does not share an edge with any of $\{P_2, P_3, P_{10}\}$. Note also that any $\overline{D}_2$ edge can exist only as one of three cases: (i) an edge that both P_4 and P_8 share; (ii) P_5; and (iii) P_7. However, P_7 was chosen to be edge-disjoint with P_6, and we have shown that P_5 is vertex-disjoint with P_{10}. Moreover, we already showed that P_8 is vertex-disjoint with P_{10}. Thus, $\overline{D}_1 \cap \overline{D}_2 = \emptyset$ on G''.

Third, we show that G'' satisfies $\overline{D}_1 \cap \overline{D}_3 = \emptyset$. Note that any $\overline{D}_1$ edge can exist only on an edge that both P_{10} and P_6 share. Note also that any $\overline{D}_3$ edge can exist only as one of three cases: (i) a P_3 edge; (ii) a P_4 edge whose head is in the upstream of or equal to $\tilde{w}$, i.e., $\{e \in P_4 : \mathsf{head}(e) \preceq \tilde{w}\}$ (may or may not be empty); and (iii) P_9. However, P_6 is in the downstream of P_3 and P_4. Moreover, P_9 is edge-disjoint with P_{11}^* and thus edge-disjoint with $w'P_{10}d_1$. As a result, no edge can be simultaneously in $\overline{D}_1$ and $\overline{D}_3$. Thus $\overline{D}_1 \cap \overline{D}_3 = \emptyset$ on G''.

Hence, the above discussions, together with Proposition 4, implies that the considered G'' satisfies $L \not\equiv R$. We thus have proven **G18** being true for **Case 2**. □

By swapping the roles of s_2 and s_3, and the roles of d_2 and d_3, the proofs of **R20** to **R22** can also be used to prove **R23** to **R25**, respectively. More specifically, **G3** and **G4** are converted back and forth from each other when swapping the flow indices. The same thing happens between **G23** and **G24**; between **G25** and **G26**; and between **G18** and **G19**. Moreover, **LNR**, **G1**, **G16**, **G17**, and **G22** remain the same after the index swapping. Thus the above proofs of **R20** to **R22** can thus be used to prove **R23** to **R25**.

APPENDIX R
PROOF OF S14

R-1. The fifth set of logic statements

To prove **S14**, we need the fifth set of logic statements.

- **G27:** $\overline{S}_2 \cap \overline{D}_1 = \emptyset$.
- **G28:** $\overline{S}_3 \cap \overline{D}_1 = \emptyset$.
- **G29:** $\overline{D}_2 \cap \overline{S}_1 = \emptyset$.
- **G30:** $\overline{D}_3 \cap \overline{S}_1 = \emptyset$.
- **G31:** $\overline{S}_i \neq \emptyset$ and $\overline{D}_i \neq \emptyset$ for all $i \in \{1, 2, 3\}$.

Several implications can be made when **G27** is true. We term those implications *the properties of* **G27**. Several properties of **G27** are listed as follows, for which their proofs are provided in Appendix S.

Consider the case in which **G27** *is true.* Use e_2^* to denote the most downstream edge in $\mathsf{1cut}(s_2; d_1) \cap \mathsf{1cut}(s_2; d_3)$. Since the source edge e_{s_2} belongs to both $\mathsf{1cut}(s_2; d_1)$ and $\mathsf{1cut}(s_2; d_3)$, such e_2^* always exists. Similarly, use e_1^* to denote the most upstream edge in $\mathsf{1cut}(s_2; d_1) \cap \mathsf{1cut}(s_3; d_1)$. The properties of **G27** can now be described as follows.

⋄ **Property 1 of G27:** $e_2^* \prec e_1^*$ and the channel gains m_{21}, m_{23}, and m_{31} can be expressed as $m_{21} = m_{e_{s_2}; e_2^*} m_{e_2^*; e_1^*} m_{e_1^*; e_{d_1}}$, $m_{23} = m_{e_{s_2}; e_2^*} m_{e_2^*; e_{d_3}}$, and $m_{31} = m_{e_{s_3}; e_1^*} m_{e_1^*; e_{d_1}}$.

⋄ **Property 2 of G27:** $\mathsf{GCD}(m_{e_{s_3}; e_1^*},\ m_{e_{s_2}; e_2^*} m_{e_2^*; e_1^*}) \equiv 1$, $\mathsf{GCD}(m_{e_2^*; e_1^*} m_{e_1^*; e_{d_1}},\ m_{e_2^*; e_{d_3}}) \equiv 1$, $\mathsf{GCD}(m_{31},\ m_{e_2^*; e_1^*}) \equiv 1$, and $\mathsf{GCD}(m_{23},\ m_{e_2^*; e_1^*}) \equiv 1$.

On the other hand, when **G27** is false, we can also derive several implications, which are termed *the properties of* $\neg$ **G27**.

Consider the case in which **G27** *is false*. Use e_u^{21} (resp. e_v^{21}) to denote the most upstream (resp. the most downstream) edge in $\overline{S}_2 \cap \overline{D}_1$. By definition, it must be $e_u^{21} \preceq e_v^{21}$. We now describe the following properties of $\neg$ **G27**.

⋄ **Property 1 of ¬ G27:** The channel gains m_{21}, m_{23}, and m_{31} can be expressed as $m_{21} = m_{e_{s_2}; e_u^{21}} m_{e_u^{21}; e_v^{21}} m_{e_v^{21}; e_{d_1}}$, $m_{23} = m_{e_{s_2}; e_u^{21}} m_{e_u^{21}; e_v^{21}} m_{e_v^{21}; e_{d_3}}$, and $m_{31} = m_{e_{s_3}; e_u^{21}} m_{e_u^{21}; e_v^{21}} m_{e_v^{21}; e_{d_1}}$.

⋄ **Property 2 of ¬ G27:** $\mathsf{GCD}(m_{e_{s_2}; e_u^{21}},\ m_{e_{s_3}; e_u^{21}}) \equiv 1$ and $\mathsf{GCD}(m_{e_v^{21}; e_{d_1}},\ m_{e_v^{21}; e_{d_3}}) \equiv 1$.

Symmetrically, we define the following properties of **G28** and $\neg$ **G28**.

Consider the case in which **G28** *is true.* Use e_3^* to denote the most downstream edge in $\mathsf{1cut}(s_3; d_1) \cap \mathsf{1cut}(s_3; d_2)$, and use e_1^* to denote the most upstream edge in $\mathsf{1cut}(s_2; d_1) \cap \mathsf{1cut}(s_3; d_1)$. We now describe the following properties of **G28**.

⋄ **Property 1 of G28:** $e_3^* \prec e_1^*$ and the channel gains m_{31}, m_{32}, and m_{21} can be expressed as $m_{31} = m_{e_{s_3}; e_3^*} m_{e_3^*; e_1^*} m_{e_1^*; e_{d_1}}$, $m_{32} = m_{e_{s_3}; e_3^*} m_{e_3^*; e_{d_2}}$, and $m_{21} = m_{e_{s_2}; e_1^*} m_{e_1^*; e_{d_1}}$.

⋄ **Property 2 of G28:** $\mathsf{GCD}(m_{e_{s_2}; e_1^*},\ m_{e_{s_3}; e_3^*} m_{e_3^*; e_1^*}) \equiv 1$, $\mathsf{GCD}(m_{e_3^*; e_1^*} m_{e_1^*; e_{d_1}},\ m_{e_3^*; e_{d_2}}) \equiv 1$, $\mathsf{GCD}(m_{21},\ m_{e_3^*; e_1^*}) \equiv 1$, and $\mathsf{GCD}(m_{32},\ m_{e_3^*; e_1^*}) \equiv 1$.

Consider the case in which **G28** *is false.* Use e_u^{31} (resp. e_v^{31}) to denote the most upstream (resp. the most downstream) edge in $\overline{S}_3 \cap \overline{D}_1$. By definition, it must be $e_u^{31} \preceq e_v^{31}$. We now describe the following properties of $\neg$ **G28**.

⋄ **Property 1 of ¬ G28:** The channel gains m_{31}, m_{32}, and m_{21} can be expressed as $m_{31} = m_{e_{s_3}; e_u^{31}} m_{e_u^{31}; e_v^{31}} m_{e_v^{31}; e_{d_1}}$, $m_{32} = m_{e_{s_3}; e_u^{31}} m_{e_u^{31}; e_v^{31}} m_{e_v^{23}; e_{d_2}}$, and $m_{21} = m_{e_{s_2}; e_u^{31}} m_{e_u^{31}; e_v^{31}} m_{e_v^{31}; e_{d_1}}$.

⋄ **Property 2 of ¬ G28:** $\mathsf{GCD}(m_{e_{s_2}; e_u^{31}},\ m_{e_{s_3}; e_u^{31}}) \equiv 1$ and $\mathsf{GCD}(m_{e_v^{31}; e_{d_1}},\ m_{e_v^{31}; e_{d_2}}) \equiv 1$.

R-2. The skeleton of proving **S14**

We prove the following relationships, which jointly prove **S14**.

- **R26: D3** $\wedge$ **D4** $\Rightarrow$ **G31**.
- **R27: LNR** $\wedge$ ($\neg$ **G27**) $\wedge$ ($\neg$ **G28**) $\wedge$ ($\neg$ **G29**) $\wedge$ ($\neg$ **G30**) $\Rightarrow$ false.
- **R28: D3** $\wedge$ **D4** $\wedge$ **G27** $\wedge$ **G28** $\Rightarrow$ false.

- **R29: LNR** $\wedge$ **G1** $\wedge$ **E0** $\wedge$ **D3** $\wedge$ **D4** $\wedge$ ($\neg$ **G27**) $\wedge$ **G28** $\Rightarrow$ false.
- **R30: LNR** $\wedge$ **G1** $\wedge$ **E0** $\wedge$ **D3** $\wedge$ **D4** $\wedge$ **G27** $\wedge$ ($\neg$ **G28**) $\Rightarrow$ false.
- **R31: D3** $\wedge$ **D4** $\wedge$ **G29** $\wedge$ **G30** $\Rightarrow$ false.
- **R32: LNR** $\wedge$ **G1** $\wedge$ **E0** $\wedge$ **D3** $\wedge$ **D4** $\wedge$ ($\neg$ **G29**) $\wedge$ **G30** $\Rightarrow$ false.
- **R33: LNR** $\wedge$ **G1** $\wedge$ **E0** $\wedge$ **D3** $\wedge$ **D4** $\wedge$ **G29** $\wedge$ ($\neg$ **G30**) $\Rightarrow$ false.

One can see that **R28** and **R31** imply, respectively,

$$\textbf{LNR} \wedge \textbf{G1} \wedge \textbf{E0} \wedge \textbf{D3} \wedge \textbf{D4} \wedge \textbf{G27} \wedge \textbf{G28} \Rightarrow \text{ false}, \quad (65)$$

$$\textbf{LNR} \wedge \textbf{G1} \wedge \textbf{E0} \wedge \textbf{D3} \wedge \textbf{D4} \wedge \textbf{G29} \wedge \textbf{G30} \Rightarrow \text{ false}. \quad (66)$$

Also **R27** implies

$$\begin{aligned}&\textbf{LNR} \wedge \textbf{G1} \wedge \textbf{E0} \wedge \textbf{D3} \wedge \textbf{D4} \wedge (\neg\, \textbf{G27}) \wedge \\ &\qquad (\neg\, \textbf{G28}) \wedge (\neg\, \textbf{G29}) \wedge (\neg\, \textbf{G30}) \Rightarrow \text{ false}.\end{aligned} \quad (67)$$

R29, **R30**, **R32**, **R33**, (65), (66), and (67) jointly imply

$$\textbf{LNR} \wedge \textbf{G1} \wedge \textbf{E0} \wedge \textbf{D3} \wedge \textbf{D4} \Rightarrow \text{ false},$$

which proves **S14**. The proofs of **R26** and **R27** are relegated to Appendix T. The proofs of **R28**, **R29**, and **R30** are provided in Appendices U, V, and W, respectively.

The logic relationships **R31** to **R33** are the symmetric versions of **R28** to **R30**. Specifically, if we swap the roles of sources and destinations, then the resulting graph is still a 3-unicast ANA network; **D3** is now converted to **D4**; **D4** is converted to **D3**; **G27** is converted to **G29**; and **G28** is converted to **G30**. Therefore, the proof of **R28** can serve as a proof of **R31**. Further, after swapping the roles of sources and destinations, the **LNR** condition (see (23)) remains the same; **G1** remains the same (see (24)); and **E0** remains the same. Therefore, the proof of **R29** (resp. **R30**) can serve as a proof of **R32** (resp. **R33**).

Appendix S
Proofs of the properties of G27, G28, ¬ G27, and ¬ G28

We prove Properties 1 and 2 of **G27** as follows.

Proof. By swapping the roles of s_1 and s_3, and the roles of d_1 and d_3, the proof of the properties of **G3** in Appendix J can be used to prove the properties of **G27**. ☐

We prove Properties 1 and 2 of $\neg$ **G27** as follows.

Proof. By swapping the roles of s_1 and s_3, and the roles of d_1 and d_3, the proof of Properties 1 and 2 of $\neg$ **G3** in Appendix J can be used to prove the properties of $\neg$ **G27**. ☐

By swapping the roles of s_2 and s_3, and the roles of d_2 and d_3, the above proofs can also be used to prove Properties 1 and 2 of **G28** and Properties 1 and 2 of $\neg$ **G28**.

Appendix T
Proofs of R26 and R27

We prove **R26** as follows.

Proof. Suppose **D3** $\wedge$ **D4** is true. By Corollary 2, we know that any channel gain cannot have any other channel gain as a factor. Since **D3** $\wedge$ **D4** is true, any one of the four channel gains m_{12}, m_{31}, m_{13}, and m_{21} must be reducible.

Since **D4** is true, we must also have for some positive integer l_4 such that

$$\mathsf{GCD}(\, m_{11}m_{12}^{l_4}m_{23}^{l_4}m_{31}^{l_4},\; m_{21}) = m_{21}. \quad (68)$$

We first note that m_{23} is the only channel gain starting from s_2 out of the four channel gains $\{m_{11}, m_{12}, m_{23}, m_{31}\}$. Therefore, we must have $\mathsf{GCD}(\, m_{23}, m_{21}) \not\equiv 1$ since "we need to cover the factor of m_{21} that emits from s_2." Lemma 7 then implies that $\overline{S}_2 \neq \emptyset$.

Further, **D4** implies $\mathsf{GCD}(\, m_{11}m_{12}^{l_4}m_{23}^{l_4}m_{31}^{l_4},\; m_{13}) = m_{13}$ for some positive integer l_4, which, by similar arguments, implies $\mathsf{GCD}(\, m_{23}, m_{13}) \not\equiv 1$. Lemma 7 then implies that $\overline{D}_3 \neq \emptyset$. By similar arguments but focusing on **D3** instead, we can also prove that $\overline{S}_3 \neq \emptyset$ and $\overline{D}_2 \neq \emptyset$.

We also notice that out of the four channel gains $\{m_{11}, m_{12}, m_{23}, m_{31}\}$, both m_{11} and m_{12} are the only channel gains starting from s_1. By **D4**, we thus have for some positive integer l_4 such that

$$\mathsf{GCD}(\, m_{11}m_{12}^{l_4},\; m_{13}) \not\equiv 1. \quad (69)$$

Similarly, by **D3** and **D4**, we have for some positive integers l_2 and l_4 such that

$$\mathsf{GCD}(\, m_{11}m_{31}^{l_4},\; m_{21}) \not\equiv 1, \quad (70)$$

$$\mathsf{GCD}(\, m_{11}m_{13}^{l_2},\; m_{12}) \not\equiv 1, \quad (71)$$

$$\mathsf{GCD}(\, m_{11}m_{21}^{l_2},\; m_{31}) \not\equiv 1. \quad (72)$$

For the following, we will prove $\overline{S}_1 \neq \emptyset$. Consider the following subcases: Subcase 1: If $\mathsf{GCD}(\, m_{12},\, m_{13}) \not\equiv 1$, then by Lemma 7, $\overline{S}_1 \neq \emptyset$. Subcase 2: If $\mathsf{GCD}(\, m_{12},\, m_{13}) \equiv 1$, then (69) and (71) jointly imply both $\mathsf{GCD}(\, m_{11},\, m_{13}) \not\equiv 1$ and $\mathsf{GCD}(\, m_{11},\, m_{12}) \not\equiv 1$. Then by first applying Lemma 7 and then applying Lemma 6, we have $\overline{S}_1 \neq \emptyset$. The proof of $\overline{D}_1 \neq \emptyset$ can be derived similarly by focusing on (70) and (72). The proof of **R26** is complete. ☐

We prove **R27** as follows.

Proof. We prove an equivalent relationship: ($\neg$ **G27**) $\wedge$ ($\neg$ **G28**) $\wedge$ ($\neg$ **G29**) $\wedge$ ($\neg$ **G30**) $\Rightarrow$ $\neg$ **LNR**. Suppose ($\neg$ **G27**) $\wedge$ ($\neg$ **G28**) $\wedge$ ($\neg$ **G29**) $\wedge$ ($\neg$ **G30**) is true. By Lemma 4, we know that ($\neg$ **G27**) $\wedge$ ($\neg$ **G28**) is equivalent to $\overline{S}_2 \cap \overline{S}_3 \neq \emptyset$. Similarly, ($\neg$ **G29**) $\wedge$ ($\neg$ **G30**) is equivalent to $\overline{D}_2 \cap \overline{D}_3 \neq \emptyset$. By Proposition 4, we have $L \equiv R$. The proof is thus complete. ☐

Appendix U
Proof of R28

U-1. The additional set of logic statements

To prove **R28**, we need an additional set of logic statements. The following logic statements are well-defined if and only if **G27** $\wedge$ **G28** is true. Recall the definition of e_2^*, e_3^*, and e_1^* in Appendix R when **G27** $\wedge$ **G28** is true.

- **G32:** $e_2^* \neq e_3^*$ and $\mathsf{GCD}(\, m_{e_{s_2};e_2^*}m_{e_2^*;e_1^*},\; m_{e_{s_3};e_3^*}m_{e_3^*;e_1^*}) \equiv 1$.
- **G33:** $\mathsf{GCD}(\, m_{11},\; m_{e_2^*;e_1^*}) \equiv 1$.
- **G34:** $\mathsf{GCD}(\, m_{11},\; m_{e_3^*;e_1^*}) \equiv 1$.

The following logic statements are well-defined if and only if **G27** $\wedge$ **G28** $\wedge$ **G31** is true.

- **G35:** $\{e_2^*, e_1^*\} \subset \mathsf{1cut}(s_1; d_2)$.
- **G36:** $\{e_3^*, e_1^*\} \subset \mathsf{1cut}(s_1; d_3)$.

U-2. The skeleton of proving **R28**

We prove the following logic relationships, which jointly proves **R28**.

- **R34: G27** $\wedge$ **G28** $\Rightarrow$ **G32**.
- **R35: D4** $\wedge$ **G27** $\wedge$ **G28** $\wedge$ **G31** $\wedge$ **G33** $\Rightarrow$ **G35**.
- **R36: D3** $\wedge$ **G27** $\wedge$ **G28** $\wedge$ **G31** $\wedge$ **G34** $\Rightarrow$ **G36**.
- **R37: G27** $\wedge$ **G28** $\wedge$ ($\neg$ **G33**) $\wedge$ ($\neg$ **G34**) $\Rightarrow$ false.
- **R38: G27** $\wedge$ **G28** $\wedge$ **G31** $\wedge$ ($\neg$ **G33**) $\wedge$ **G36** $\Rightarrow$ false.
- **R39: G27** $\wedge$ **G28** $\wedge$ **G31** $\wedge$ ($\neg$ **G34**) $\wedge$ **G35** $\Rightarrow$ false.
- **R40: G27** $\wedge$ **G28** $\wedge$ **G31** $\wedge$ **G35** $\wedge$ **G36** $\Rightarrow$ false.

Specifically, **R35** and **R39** jointly imply that

$$\mathbf{D3} \wedge \mathbf{D4} \wedge \mathbf{G27} \wedge \mathbf{G28} \wedge \mathbf{G31} \wedge \mathbf{G33} \wedge (\neg\, \mathbf{G34}) \Rightarrow \text{ false}.$$

Moreover, **R36** and **R38** jointly imply that

$$\mathbf{D3} \wedge \mathbf{D4} \wedge \mathbf{G27} \wedge \mathbf{G28} \wedge \mathbf{G31} \wedge (\neg\, \mathbf{G33}) \wedge \mathbf{G34} \Rightarrow \text{ false}.$$

Furthermore, **R35**, **R36**, and **R40** jointly imply that

$$\mathbf{D3} \wedge \mathbf{D4} \wedge \mathbf{G27} \wedge \mathbf{G28} \wedge \mathbf{G31} \wedge \mathbf{G33} \wedge \mathbf{G34} \Rightarrow \text{ false}.$$

Finally, **R37** implies that

$$\mathbf{D3} \wedge \mathbf{D4} \wedge \mathbf{G27} \wedge \mathbf{G28} \wedge \mathbf{G31} \wedge (\neg\, \mathbf{G33}) \wedge (\neg\, \mathbf{G34}) \Rightarrow \text{ false}.$$

The above four relationships jointly imply **D3** $\wedge$ **D4** $\wedge$ **G27** $\wedge$ **G28** $\wedge$ **G31** $\Rightarrow$ false. By **R26** in Appendix R, i.e., **D3** $\wedge$ **D4** $\Rightarrow$ **G31**, we thus have **D3** $\wedge$ **D4** $\wedge$ **G27** $\wedge$ **G28** $\Rightarrow$ false. The proof of **R28** is thus complete. The detailed proofs of **R34** to **R40** are provided in the next subsection.

U-3. The proofs of **R34** *to* **R40**

We prove **R34** as follows.

Proof. Suppose **G27** $\wedge$ **G28** is true. Since e_1^* is the most upstream 1-edge cut separating d_1 from $\{s_2, s_3\}$, there must exist two edge-disjoint paths connecting $\{s_2, s_3\}$ and $\mathsf{tail}(e_1^*)$. By Property 1 of **G27** and **G28**, one path must use e_2^* and the other must use e_3^*. Due to the edge-disjointness, $e_2^* \neq e_3^*$. Since we have two edge-disjoint paths from s_2 (resp. s_3) to $\mathsf{tail}(e_1^*)$, we also have $\mathsf{GCD}(\, m_{e_{s_2};e_2^*} m_{e_2^*;e_1^*},\, m_{e_{s_3};e_3^*} m_{e_3^*;e_1^*}) \equiv 1$. $\square$

We prove **R35** as follows.

Proof. Suppose **D4** $\wedge$ **G27** $\wedge$ **G28** $\wedge$ **G31** $\wedge$ **G33** is true. By the Properties of **G27** and **G28** and by **G31**, e_2^* (resp. e_3^*) is the most downstream edge of $\overline{S}_2$ (resp. $\overline{S}_3$). And both e_2^* and e_3^* are in the upstream of e_1^* where e_1^* is the most upstream edge of $\overline{D}_1$. Consider $m_{e_2^*;e_1^*}$, a factor of m_{21}. From Property 2 of **G27**, we have $\mathsf{GCD}(\, m_{23},\, m_{e_2^*;e_1^*}) \equiv 1$. In addition, since **G27** $\wedge$ **G28** $\Rightarrow$ **G32** as established in **R34**, we have $\mathsf{GCD}(\, m_{31},\, m_{e_2^*;e_1^*}) \equiv 1$. Together with the assumption that **D4** is true, we have for some positive integer l_4 such that

$$\mathsf{GCD}(\, m_{11} m_{12}^{l_4},\, m_{e_2^*;e_1^*}) = m_{e_2^*;e_1^*}. \tag{73}$$

Since we assume that **G33** is true, (73) further implies $\mathsf{GCD}(\, m_{12}^{l_4},\, m_{e_2^*;e_1^*}) = m_{e_2^*;e_1^*}$. By Proposition 3, we must have **G35**: $\{e_2^*, e_1^*\} \subset \mathsf{1cut}(s_1; d_2)$. The proof is thus complete. $\square$

R36 is a symmetric version of **R35** and can be proved by relabeling (s_2, d_2) as (s_3, d_3), and relabeling (s_3, d_3) as (s_2, d_2) in the proof of **R35**.

We prove **R37** as follows.

Proof. Suppose **G27** $\wedge$ **G28** $\wedge$ ($\neg$ **G33**) $\wedge$ ($\neg$ **G34**) is true. Since **G27** $\wedge$ **G28** is true, we have two edge-disjoint paths $P_{s_2\mathsf{tail}(e_1^*)}$ through e_2^* and $P_{s_3\mathsf{tail}(e_1^*)}$ through e_3^* if we recall **R34**. Consider $m_{e_2^*;e_1^*}$, a factor of m_{21}, and $m_{e_3^*;e_1^*}$, a factor of m_{31}. Since $\neg$ **G33** is true, there is an irreducible factor of $m_{e_2^*;e_1^*}$ that is also a factor of m_{11}. Since that factor is also a factor of m_{21}, by Proposition 3 and Property 1 of **G27**, there must exist at least one edge e' satisfying (i) $e_2^* \preceq e' \prec e_1^*$; (ii) $e' \in \overline{D}_{1;\{1,2\}}$; and (iii) $e' \in P_{s_2\mathsf{tail}(e_1^*)}$. Similarly, $\neg$ **G34** implies that there exists at least one edge e'' satisfying (i) $e_3^* \preceq e'' \prec e_1^*$; (ii) $e'' \in \overline{D}_{1;\{1,3\}}$; and (iii) $e'' \in P_{s_3\mathsf{tail}(e_1^*)}$. Then the above observation implies that $e' \in P_{s_2\mathsf{tail}(e_1^*)} \cap \mathsf{1cut}(s_1; d_1)$ and $e'' \in P_{s_3\mathsf{tail}(e_1^*)} \cap \mathsf{1cut}(s_1; d_1)$. Since $P_{s_2\mathsf{tail}(e_1^*)}$ and $P_{s_3\mathsf{tail}(e_1^*)}$ are edge-disjoint paths, it must be $e' \neq e''$. But both e' and e'' are 1-edge cuts separating s_1 and d_1. Thus e' and e'' must be reachable from each other: either $e' \prec e''$ or $e'' \prec e'$. However, both cases are impossible because one in the upstream can always follow the corresponding $P_{s_2\mathsf{tail}(e_1^*)}$ or $P_{s_3\mathsf{tail}(e_1^*)}$ path to e_1^* without using the one in the downstream. For example, if $e' \prec e''$, then s_1 can first reach e' and follow $P_{s_2\mathsf{tail}(e_1^*)}$ to arrive at $\mathsf{tail}(e_1^*)$ without using e''. Since $e_1^* \in \overline{D}_1$ reaches d_1, this contradicts $e'' \in \mathsf{1cut}(s_1; d_1)$. Since neither case can be true, the proof is thus complete. $\square$

We prove **R38** as follows.

Proof. Suppose **G27** $\wedge$ **G28** $\wedge$ **G31** $\wedge$ ($\neg$ **G33**) $\wedge$ **G36** is true. By the Properties of **G27** and **G28** and by **G31**, e_2^* (resp. e_3^*) is the most downstream edge of $\overline{S}_2$ (resp. $\overline{S}_3$). And both e_2^* and e_3^* are in the upstream of e_1^* where e_1^* is the most upstream edge of $\overline{D}_1$. Since e_1^* is the most upstream $\overline{D}_1$ edge, there exist three edge-disjoint paths $P_{s_2\mathsf{tail}(e_1^*)}$, $P_{s_3\mathsf{tail}(e_1^*)}$, and $P_{\mathsf{head}(e_1^*)d_1}$. Fix any arbitrary construction of these paths. Obviously, $P_{s_2\mathsf{tail}(e_1^*)}$ uses e_2^* and $P_{s_3\mathsf{tail}(e_1^*)}$ uses e_3^*.

Since $\neg$ **G33** is true, there is an irreducible factor of $m_{e_2^*;e_1^*}$ that is also a factor of m_{11}. Since that factor is also a factor of m_{21}, by Proposition 3, there must exist an edge e satisfying (i) $e_2^* \preceq e \prec e_1^*$; (ii) $e \in \mathsf{1cut}(s_1; d_1) \cap \mathsf{1cut}(s_2; d_1)$. By (i), (ii), and the construction $e_1^* \in \overline{D}_1 \subset \mathsf{1cut}(s_2; d_1)$, the pre-defined path $P_{s_2\mathsf{tail}(e_1^*)}$ must use such e.

Since **G36** is true, e_3^* is reachable from s_1 and e_1^* reaches to d_3. Choose arbitrarily one path $P_{s_1\mathsf{tail}(e_3^*)}$ from s_1 to $\mathsf{tail}(e_3^*)$ and one path $P_{\mathsf{head}(e_1^*)d_3}$ from $\mathsf{head}(e_1^*)$ to d_3. We argue that $P_{s_1\mathsf{tail}(e_3^*)}$ must be vertex-disjoint with $P_{s_2\mathsf{tail}(e_1^*)}$. Suppose not and let v denote a vertex shared by $P_{s_1\mathsf{tail}(e_3^*)}$ and $P_{s_2\mathsf{tail}(e_1^*)}$. Then there is a s_1-to-d_3 path $P_{s_1\mathsf{tail}(e_3^*)} v P_{s_2\mathsf{tail}(e_1^*)} e_1^* P_{\mathsf{head}(e_1^*)d_3}$ without using e_3^*. This contradicts the assumption **G36** since **G36** implies $e_3^* \in \mathsf{1cut}(s_1; d_3)$. However, if $P_{s_1\mathsf{tail}(e_3^*)}$ is vertex-disjoint with $P_{s_2\mathsf{tail}(e_1^*)}$, then there is an s_1-to-d_1 path $P_{s_1\mathsf{tail}(e_3^*)} e_3^* P_{s_3\mathsf{tail}(e_1^*)} e_1^* P_{\mathsf{head}(e_1^*)d_1}$ not using the edge e defined in the previous paragraph since $e \in P_{s_2\mathsf{tail}(e_1^*)}$ and $P_{s_2\mathsf{tail}(e_1^*)}$ is edge-disjoint with $P_{s_3\mathsf{tail}(e_1^*)}$. This also contradicts (ii). Since neither case can be true, the proof of **R38** is thus complete. $\square$

R39 is a symmetric version of **R38** and can be proved by swapping the roles of s_2 and s_3, and the roles of d_2 and d_3 in the proof of **R38**.

We prove **R40** as follows.

Proof. Suppose **G27** $\wedge$ **G28** $\wedge$ **G31** $\wedge$ **G35** $\wedge$ **G36** is true. By the Properties of **G27** and **G28** and by **G31**, e_2^* (resp. e_3^*) is the most downstream edge of $\overline{S}_2$ (resp. $\overline{S}_3$). Also $e_2^* \prec e_1^*$ and $e_3^* \prec e_1^*$ where e_1^* is the most upstream $\overline{D}_1$ edge.

By **G36**, there exists a path from s_1 to e_3^*. Since $e_3^* \in \overline{S}_3$, there exists a path from e_3^* to d_2 without using e_1^*. As a result, there exists a path from s_1 to d_2 through e_3^* without using e_1^*. This contradicts the assumption **G35** since **G35** implies $e_1^* \in \mathsf{1cut}(s_1; d_2)$. The proof is thus complete. $\square$

APPENDIX V
PROOF OF R29

V-1. The additional set of logic statements

To prove **R29**, we need some additional sets of logic statements. The following logic statements are well-defined if and only if **G28** is true. Recall the definition of e_3^* and e_1^* when **G28** is true.

- **G37:** $e_3^* \in \mathsf{1cut}(s_1; d_1)$.
- **G38:** $e_3^* \in \mathsf{1cut}(s_1; d_3)$.
- **G39:** $e_1^* \in \mathsf{1cut}(s_1; d_1)$.
- **G40:** $e_1^* \in \mathsf{1cut}(s_1; d_3)$.
- **G41:** $e_3^* \in \mathsf{1cut}(s_1; d_2)$.

The following logic statements are well-defined if and only if $(\neg$ **G27**$)$ $\wedge$ **G28** is true. Recall the definition of e_u^{21}, e_v^{21}, e_3^*, and e_1^* when $(\neg$ **G27**$)$ $\wedge$ **G28** is true.

- **G42:** $e_1^* = e_u^{21}$.
- **G43:** Let e' be the most downstream edge of $\mathsf{1cut}(s_1; d_2) \cap \mathsf{1cut}(s_1; \mathsf{tail}(e_3^*))$ and also let e'' be the most upstream edge of $\mathsf{1cut}(s_1; d_2) \cap \mathsf{1cut}(\mathsf{head}(e_3^*); d_2)$. Then, e' and e'' simultaneously satisfy the following two conditions: (i) both e' and e'' belong to $\mathsf{1cut}(s_1; d_3)$; and (ii) $e'' \in \mathsf{1cut}(\mathsf{head}(e_v^{21}); \mathsf{tail}(e_{d_3}))$ and $e'' \prec e_{d_2}$.

V-2. The skeleton of proving R29

We prove the following relationships, which jointly proves **R29**.

- **R41:** $(\neg$ **G27**$)$ $\wedge$ **G28** $\Rightarrow$ **G42**.
- **R42:** **D3** $\wedge$ $(\neg$ **G27**$)$ $\wedge$ **G28** $\wedge$ **G31** $\Rightarrow$ (**G37** $\vee$ **G38**) $\wedge$ (**G39** $\vee$ **G40**).
- **R43:** **G1** $\wedge$ **G28** $\wedge$ **G31** $\wedge$ **G37** $\Rightarrow$ $\neg$ **G41**.
- **R44:** **D3** $\wedge$ $(\neg$ **G27**$)$ $\wedge$ **G28** $\wedge$ **G31** $\wedge$ **G37** $\wedge$ $(\neg$ **G41**$)$ $\Rightarrow$ **G43**.
- **R45:** **G1** $\wedge$ **E0** $\wedge$ **D3** $\wedge$ $(\neg$ **G27**$)$ $\wedge$ **G28** $\wedge$ **G31** $\wedge$ **G37** $\Rightarrow$ false.
- **R46:** $(\neg$ **G27**$)$ $\wedge$ **G28** $\wedge$ **G31** $\wedge$ $(\neg$ **G37**$)$ $\wedge$ **G38** $\wedge$ **G39** $\Rightarrow$ false.
- **R47:** **LNR** $\wedge$ **D4** $\wedge$ $(\neg$ **G27**$)$ $\wedge$ **G28** $\wedge$ **G31** $\wedge$ $(\neg$ **G37**$)$ $\wedge$ **G38** $\wedge$ **G40** $\Rightarrow$ false.

One can easily verify that jointly **R46** and **R47** imply

$$\mathbf{LNR} \wedge \mathbf{D4} \wedge (\neg\,\mathbf{G27}) \wedge \mathbf{G28} \wedge \mathbf{G31} \wedge (\neg\,\mathbf{G37}) \wedge \mathbf{G38} \wedge (\mathbf{G39} \vee \mathbf{G40}) \Rightarrow \text{false}.$$

From the above logic relationship and by **R42**, we have

$$\mathbf{LNR} \wedge \mathbf{D3} \wedge \mathbf{D4} \wedge (\neg\,\mathbf{G27}) \wedge \mathbf{G28} \wedge \mathbf{G31} \wedge (\neg\,\mathbf{G37}) \wedge \mathbf{G38} \Rightarrow \text{false}.$$

From the above logic relationship and by **R45**, we have

$$\mathbf{LNR} \wedge \mathbf{G1} \wedge \mathbf{E0} \wedge \mathbf{D3} \wedge \mathbf{D4} \wedge (\neg\,\mathbf{G27}) \wedge \mathbf{G28} \wedge \mathbf{G31} \wedge (\mathbf{G37} \vee \mathbf{G38}) \Rightarrow \text{false}.$$

By applying **R42** and **R26**, we have **LNR** $\wedge$ **G1** $\wedge$ **E0** $\wedge$ **D3** $\wedge$ **D4** $\wedge$ $(\neg$ **G27**$)$ $\wedge$ **G28** $\Rightarrow$ false, which proves **R29**. The detailed proofs for **R41** to **R47** are provided in the next subsection.

V-3. The proofs of R41 *to* R47

We prove **R41** as follows.

Proof. Suppose $(\neg$ **G27**$)$ $\wedge$ **G28** is true. By $\neg$ **G27** being true and its Property 1, we have e_u^{21} (resp. e_v^{21}), the most upstream (resp. downstream) edge of $\overline{S}_2 \cap \overline{D}_1$. Since $\neg$ **G27** implies that $\overline{D}_1 \neq \emptyset$, by Property 1 of **G28**, we also have e_1^*, the most upstream $\overline{D}_1$ edge.

Since $\overline{D}_1 \cap \overline{S}_2 \neq \emptyset$, we can partition the non-empty $\overline{D}_1$ by $\overline{D}_1 \backslash \overline{S}_2$ and $\overline{D}_1 \cap \overline{S}_2$. By the (s, d)-symmetric version of Lemma 3, if $\overline{D}_1 \backslash \overline{S}_2 \neq \emptyset$, then any $\overline{D}_1 \backslash \overline{S}_2$ edge must be in the downstream of $e_v^{21} \in \overline{D}_1 \cap \overline{S}_2 \subset \overline{S}_2$. Thus, e_u^{21}, the most upstream $\overline{D}_1 \cap \overline{S}_2$ edge, must also be the most upstream edge of $\overline{D}_1$. Therefore, $e_1^* = e_u^{21}$. The proof is thus complete. $\square$

We prove **R42** as follows.

Proof. Suppose **D3** $\wedge$ $(\neg$ **G27**$)$ $\wedge$ **G28** $\wedge$ **G31** is true. Since $(\neg$ **G27**$)$ $\wedge$ **G28** $\wedge$ **G31** is true, e_3^* (resp. e_1^*) is the most downstream (resp. upstream) edge of $\overline{S}_3$ (resp. $\overline{D}_1$) and $e_3^* \prec e_1^*$. By **R41**, **G42** is also true and thus e_1^* is also the most upstream edge of $\overline{S}_2 \cap \overline{D}_1$.

Consider $m_{e_3^*;e_1^*}$, a factor of m_{31}. From Property 2 of **G28**, $\mathsf{GCD}(m_{32}, m_{e_3^*;e_1^*}) \equiv 1$. By **G42** being true and Property 2 of $\neg$ **G27**, we also have $\mathsf{GCD}(m_{e_{s_2};e_1^*}, m_{e_{s_3};e_3^*} m_{e_3^*;e_1^*}) \equiv 1$, which implies that $\mathsf{GCD}(m_{21}, m_{e_3^*;e_1^*}) \equiv 1$. Then since **D3** is true, we have for some positive integer l_2 such that

$$\mathsf{GCD}(m_{11} m_{13}^{l_2},\ m_{e_3^*;e_1^*}) = m_{e_3^*;e_1^*}.$$

Proposition 3 then implies that both e_3^* and e_1^* must be in $\mathsf{1cut}(s_1; d_1) \cup \mathsf{1cut}(s_1; d_3)$. This is equivalent to (**G37** $\vee$ **G38**) $\wedge$ (**G39** $\vee$ **G40**) being true. The proof of **R42** is complete. $\square$

We prove **R43** as follows.

Proof. We prove an equivalent form: **G28** $\wedge$ **G31** $\wedge$ **G37** $\wedge$ **G41** $\Rightarrow$ $\neg$ **G1**. Suppose **G28** $\wedge$ **G31** $\wedge$ **G37** $\wedge$ **G41** is true. Since **G28** $\wedge$ **G31** is true, we have e_3^* being the most downstream edge of $\overline{S}_3$. Therefore $e_3^* \in \mathsf{1cut}(s_3; d_1) \cap \mathsf{1cut}(s_3; d_2)$. Since **G37** $\wedge$ **G41** is also true, e_3^* belongs to $\mathsf{1cut}(s_1; d_1) \cap \mathsf{1cut}(s_1; d_2)$ as well. As a result, $\mathsf{EC}(\{s_1, s_3\}; \{d_1, d_2\}) = 1$, which, by Corollary 2 implies $\neg$ **G1**. $\square$

We prove **R44** as follows.

Proof. Suppose that **D3** $\wedge$ $(\neg$ **G27**$)$ $\wedge$ **G28** $\wedge$ **G31** $\wedge$ **G37** $\wedge$ $(\neg$ **G41**$)$ is true, which by **R41** implies that **G42** is true as well. Since **G28** $\wedge$ **G31** is true, e_3^* (resp. e_1^*) is the most downstream (resp. upstream) edge of $\overline{S}_3$ (resp. $\overline{D}_1$) and $e_3^* \prec e_1^*$. Recall the definition in **G43** that e' is the most downstream edge of $\mathsf{1cut}(s_1; d_2) \cap \mathsf{1cut}(s_1; \mathsf{tail}(e_3^*))$ and e'' is the most upstream edge of $\mathsf{1cut}(s_1; d_2) \cap \mathsf{1cut}(\mathsf{head}(e_3^*); d_2)$. By the constructions of e' and e'', we must have $e_{s_1} \preceq e' \prec e_3^* \prec e'' \preceq e_{d_2}$. Then, we claim that the above construction together with $\neg$ **G41** implies $\mathsf{EC}(\mathsf{head}(e'); \mathsf{tail}(e'')) \geq 2$. The reason is that if $\mathsf{EC}(\mathsf{head}(e');\mathsf{tail}(e''))=1$, then we can find an 1-edge cut separating $\mathsf{head}(e')$ and $\mathsf{tail}(e'')$ and by $\neg$ **G41** such edge cut must not be e_3^*. Hence, such edge cut is either an upstream or a downstream edge of e_3^*. However, either case is impossible, because the edge cut being in the upstream of e_3^* will contradict that e' is the most downstream one during its construction. Similarly, the edge cut being in downstream of e_3^* will contradict the construction of e''. The conclusion $\mathsf{EC}(\mathsf{head}(e');\mathsf{tail}(e''))\geq 2$ further implies $m_{e';e''}$ is irreducible.

Further, because e_3^* is the most downstream $\overline{S}_3$ edge and e'', by construction, satisfies $e''\in 1\mathsf{cut}(s_3;d_2)$, e'' must not belong to $1\mathsf{cut}(s_3;d_1)$, which in turn implies $e''\notin 1\mathsf{cut}(\mathsf{head}(e_3^*);d_1)$. Since **G37** is true, s_1 can reach e_3^*. Therefore, there exists an s_1-to-d_1 path using e_3^* but not using e''. As a result, $e''\notin 1\mathsf{cut}(s_1;d_1)$. Together with $m_{e';e''}$ being irreducible, we thus have $\mathsf{GCD}(\,m_{11},\, m_{e';e''})\equiv 1$ by Proposition 3.

Now we argue that $\mathsf{GCD}(\,m_{21},\, m_{e';e''})\equiv 1$. Suppose not. Since $m_{e';e''}$ is irreducible, we must have e' being an 1-edge cut separating s_2 and d_1. Since e_1^* is the most upstream $\overline{D}_1$ edge, by Property 2 of **G28**, there exists a s_2-to-d_1 path P_{21} not using e_3^*. By the construction of e', s_1 reaches e'. Choose arbitrarily a path $P_{s_1e'}$ from s_1 to e'. Then, the following s_1-to-d_1 path $P_{s_1e'}e'P_{21}$ does not use e_3^*, which contradicts **G37**. As a result, we must have $\mathsf{GCD}(\,m_{21},\, m_{e';e''})\equiv 1$.

Now we argue that $\mathsf{GCD}(\,m_{32},\, m_{e';e''})\equiv 1$. Suppose not. Since $m_{e';e''}$ is irreducible, both e' and e'' must belong to $1\mathsf{cut}(s_3;d_2)$ and there is no 1-edge cut of $1\mathsf{cut}(s_3;d_2)$ that is strictly being downstream to e' and being upstream to e''. This, however, contradicts the above construction that $e'\prec e_3^*\prec e''$ and $e_3^*\in\overline{S}_3\subset 1\mathsf{cut}(s_3;d_2)$. As a result, we must have $\mathsf{GCD}(\,m_{32},\, m_{e';e''})\equiv 1$.

Together with the assumption that **D3** is true and the fact that $m_{e';e''}$ is a factor of m_{12}, we have for some positive integer l_2 such that

$$\mathsf{GCD}(\,m_{13}^{l_2},\, m_{e';e''})=m_{e';e''}.$$

Proposition 3 then implies $\{e',e''\}\subset 1\mathsf{cut}(s_1;d_3)$, which shows the first half of **G43**.

Therefore, any s_1-to-d_3 path must use e''. Since $e_3^*\prec e''$ and s_1 can reach e_3^*, any path from $\mathsf{head}(e_3^*)$ to d_3 must use e''. Note that when we establish $\mathsf{GCD}(\,m_{11},\, m_{e';e''})\equiv 1$ in the beginning of this proof, we also proved that $e''\notin 1\mathsf{cut}(s_1;d_1)$. Thus, there exists a path from $\mathsf{head}(e_3^*)$ to d_1 not using e''. Then such path must use e_v^{21} because e_v^{21} is also an 1-edge cut separating $\mathsf{head}(e_3^*)$ and d_1 by the facts that $e_v^{21}\in\overline{S}_2\cap\overline{D}_1\subset 1\mathsf{cut}(s_3;d_1)$; $e_3^*\prec e_v^{21}$; s_3 reaches e_3^*. Moreover, since $e_v^{21}\in\overline{S}_2\cap\overline{D}_1\subset 1\mathsf{cut}(s_2;d_3)$, $\mathsf{head}(e_v^{21})$ can reach d_3. In sum, we have shown that (i) any path from $\mathsf{head}(e_3^*)$ to d_3 must use e''; (ii) there exists a path from e_3^* to e_v^{21} not using e''; (iii) $\mathsf{head}(e_v^{21})$ can reach d_3. Jointly (i) to (iii) imply that any path from $\mathsf{head}(e_v^{21})$ to d_3 must use e''. As a result, we have $e''\in 1\mathsf{cut}(\mathsf{head}(e_v^{21});d_3)$. Also e'' must not be the d_3-destination edge e_{d_3} since by construction $e''\preceq e_{d_2}$, $e_{d_2}\neq e_{d_3}$, and $|\mathsf{Out}(d_3)|=0$. This further implies that e'' must not be the d_2-destination edge e_{d_2} since $e''\prec e_{d_3}$ and $|\mathsf{Out}(d_2)|=0$. We have thus proven the second half of **G43**: $e''\in 1\mathsf{cut}(\mathsf{head}(e_v^{21});\mathsf{tail}(e_{d_3}))$ and $e''\prec e_{d_2}$. The proof of **R44** is complete. □

We prove **R45** as follows.

Proof. Suppose **G1** $\wedge$ **E0** $\wedge$ **D3** $\wedge(\neg$ **G27**$)\wedge$ **G28** $\wedge$ **G31** $\wedge$ **G37** is true. By **R41**, **R43**, and **R44**, we know that **G42**, $\neg$ **G41**, and **G43** are true as well. For the following we construct 10 path segments that interconnects s_1 to s_3, d_1 to d_3, and three edges e'', e_3^*, and e_1^*.

- P_1: a path starting from e_{s_1} and ending at e'. This is always possible due to **G43** being true.
- P_2: a path from s_2 to $\mathsf{tail}(e_1^*)$ without using e_3^*. This is always possible due to the properties of **G28**.
- P_3: a path from s_3 to $\mathsf{tail}(e_3^*)$. This is always possible due to **G28** and **G31** being true. We also impose that P_3 is edge-disjoint with P_2. Again, this is always possible due to Property 2 of **G28**.
- P_4: a path from $\mathsf{head}(e')$ to $\mathsf{tail}(e'')$. This is always possible due to **G43** being true. We also impose the condition that $e_3^*\notin P_4$. Again this is always possible since $\neg$ **G41** being true, which implies that one can always find a path from s_1 to d_2 not using e_3^* but uses both e' and e'' (due to the construction of e' and e'' of **G43**).
- P_5: a path from $\mathsf{head}(e_3^*)$ to $\mathsf{tail}(e_1^*)$. We also impose the condition that P_5 is edge-disjoint with P_2. The construction of such P_5 is always possible due to the Properties of **G28**.
- P_6: a path from $\mathsf{head}(e_1^*)$ to d_1. This is always possible due to $(\neg$ **G27**$)\wedge$ **G28** being true. We also impose the condition that $e''\notin P_6$. Again this is always possible. The reason is that e_3^* is the most downstream $\overline{S}_3$ edge and thus there are two edge-disjoint paths connecting $\mathsf{head}(e_3^*)$ and $\{d_1,d_2\}$. By our construction e'' must be in the latter path while we can choose P_6 to be part of the first path.
- P_7: a path from $\mathsf{head}(e_3^*)$ to $\mathsf{tail}(e'')$, which is edge-disjoint with $\{P_5,e_1^*,P_6\}$. This is always possible due to the property of e_3^* and the construction of **G43**.
- P_8: a path from $\mathsf{head}(e'')$ to d_2, which is edge-disjoint with $\{P_5,e_1^*,P_6\}$. This is always possible due to the property of e_3^* and the construction of **G43**.
- P_9: a path from $\mathsf{head}(e_1^*)$ to $\mathsf{tail}(e'')$. This is always possible due to **G43** being true (in particular the (ii) condition of **G43**).
- P_{10}: a path from $\mathsf{head}(e'')$ to d_3. This is always possible due to **G43** being true (in particular the (ii) condition of **G43**).

Fig. 4 illustrates the relative topology of these 10 paths. We now consider the subgraph G' induced by the 10 paths plus the three edges e'', e_3^*, and e_1^*. One can easily check that s_i can reach d_j for all $i\neq j$. In particular, s_1 can reach d_2 through $P_1P_4e''P_8$; s_1 can reach d_3 through $P_1P_4e''P_{10}$; s_2 can reach d_1 through $P_2e_1^*P_6$; s_2 can reach d_3 through $P_2e_1^*P_9e''P_{10}$; s_3 can reach d_1 through $P_3e_3^*P_5e_1^*P_6$; and s_3 can reach d_2 through either $P_3e_3^*P_5e_1^*P_9e''P_8$ or $P_3e_3^*P_7e''P_8$. Furthermore, topologically, the 6 paths P_5 to P_{10} are all in the downstream of e_3^*.

For the following we argue that s_1 cannot reach d_1 in the induced subgraph G'. To that end, we first notice that by **G37**, $e_3^*\in 1\mathsf{cut}(s_1;d_1)$ in the original graph. Therefore any s_1-to-d_1

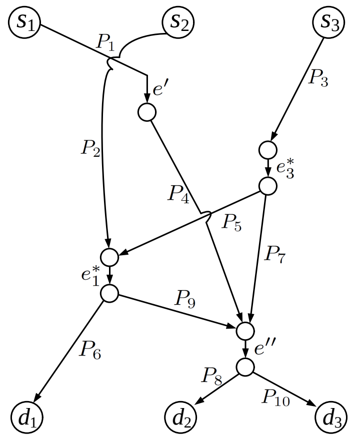

Fig. 4. The subgraph G' of the 3-unicast ANA network G_{3ANA} induced by 10 paths and three edges e'', e_3^*, and e_1^* in the proof of **R45**.

path in the subgraph must use e_3^* as well. Since P_5 to P_{10} are in the downstream of e_3^*, we only need to consider P_1 to P_4.

By definition, P_3 reaches e_3^*. We now like to show that $e_3^* \notin P_2$, and $\{P_2, P_3\}$ are vertex-disjoint paths. The first statement is done by our construction. Suppose P_2 and P_3 share a common vertex v (v can possibly be $\mathsf{tail}(e_3^*)$), then there exists a s_3-to-d_1 path $P_3 v P_2 e_1^* P_6$ not using e_3^*. This contradicts **G28** (specifically $e_3^* \in \overline{S}_3 \subset 1\mathsf{cut}(s_3; d_1)$). The above arguments show that the first time a path enters/touches part of P_3 (including $\mathsf{tail}(e_3^*)$) must be along either P_1 or P_4 (cannot be along P_2). As a result, when deciding whether there exists an s_1-to-d_1 path using e_3^*, we only need to consider whether P_1 (and/or P_4) can share a vertex with P_3. To that end, we will prove that (i) $e_3^* \notin P_1$; (ii) $\{P_1, P_3\}$ are vertex-disjoint paths; (iii) $e_3^* \notin P_4$; and (iv) $\{P_3, P_4\}$ are vertex-disjoint paths. Once (i) to (iv) are true, then there is no s_1-to-d_1 path in the subgraph G'.

We now notice that (i) is true since $e' \prec e_3^*$; (iii) is true due to our construction; (ii) is true otherwise let v denote the shared vertex and there will exist a s_3-to-d_2 path $P_3 v P_1 P_4 e'' P_8$ without using e_3^*, which contradicts **G28** ($e_3^* \in \overline{S}_3 \subset 1\mathsf{cut}(s_3; d_2)$); and by the same reason, (iv) is true otherwise let v denote the shared vertex and there will exist a s_3-to-d_2 path $P_3 v P_4 e'' P_8$ without using e_3^*. We have thus proven that there is no s_1-to-d_1 path in G'.

Since **E0** is true, G_{3ANA} must satisfy (36) with at least one non-zero coefficients α_i and β_j, respectively. Applying Proposition 2 implies that the subgraph G' must satisfy (36) with the same coefficient values. Note that there is no path from s_1 to d_1 on G' but any channel gain m_{ij} for all $i \neq j$ is non-trivial on G'. Recalling the expression of (36), its LHS becomes zero since it contains the zero polynomial m_{11} as a factor. We have $g(\{m_{ij} : \forall (i,j) \in I_{\text{3ANA}}\})\, \psi_\beta^{(n)}(R, L) = 0$ and thus $\psi_\beta^{(n)}(R, L) = 0$ with at least one non-zero coefficients β_j. This further implies that the set of polynomials $\{R^n, R^{n-1}L, \cdots, RL^{n-1}, L^n\}$ is linearly dependent on G'. Since this is the Vandermonde form, it is equivalent to that $L \equiv R$ holds on G'. However for the following, we will show that (a) $\overline{D}_1 \cap \overline{D}_2 = \emptyset$; (b) $\overline{S}_1 \cap \overline{S}_3 = \emptyset$; and (c) $\overline{S}_2 \cap \overline{S}_3 = \emptyset$ on G', which implies by Proposition 4 that G' indeed satisfies $L \not\equiv R$. This is a contradiction and thus proves **R45**.

(a) $\overline{D}_1 \cap \overline{D}_2 = \emptyset$ on G': Note that any $\overline{D}_1$ edge can exist on (i) e_1^*; and (ii) P_6. Note also that any $\overline{D}_2$ edge can exist on (i) e''; and (ii) P_8. But from the above constructions, P_6 was chosen not to use e''. In addition, P_8 was chosen to be edge-disjoint with $\{e_1^*, P_6\}$. Moreover, $e_1^* \prec e''$. Thus, we must have $\overline{D}_1 \cap \overline{D}_2 = \emptyset$ on G'.

(b) $\overline{S}_1 \cap \overline{S}_3 = \emptyset$ on G': Note that any $\overline{S}_1$ edge can exist on (i) P_1; (ii) P_4; (iii) e''; and (iv) an edge that P_8 and P_{10} shares. Note also that any $\overline{S}_3$ can exist on (i) P_3; and (ii) e_3^*. But e_3^* is in the upstream of e'', P_8, and P_{10}. Also, e_3^* is in the downstream of e', ending edge of P_1. In addition, P_4 was chosen not to use e_3^*. Moreover, we already showed that $\{P_1, P_3\}$ are vertex-disjoint paths; and $\{P_3, P_4\}$ are vertex-disjoint paths. Thus, we must have $\overline{S}_1 \cap \overline{S}_3 = \emptyset$ on G'.

(c) $\overline{S}_2 \cap \overline{S}_3 = \emptyset$ on G': Note that any $\overline{S}_2$ edge can exist on (i) P_2; (ii) e_1^*; (iii) an edge that P_6 and P_9 shares; and (iv) an edge that P_6 and P_{10} share. Note also that any $\overline{S}_3$ edge can exist on (i) P_3; and (ii) e_3^*. However, e_3^* is in the upstream of e_1^*, P_6, P_9, and P_{10}. In addition, P_2 was chosen not to use e_3^*. Moreover, we already showed that $\{P_2, P_3\}$ are vertex-disjoint paths. Thus, we must have $\overline{S}_2 \cap \overline{S}_3 = \emptyset$ on G'. ☐

We prove **R46** as follows.

Proof. Suppose that $(\neg\,$**G27**$) \wedge$ **G28** $\wedge$ **G31** $\wedge (\neg\,$**G37**$) \wedge$ **G38** $\wedge$ **G39** is true. By **R41**, **G42** is true as well. Since **G28** $\wedge$ **G31** is true, e_3^* (resp. e_1^*) is the most downstream (resp. upstream) edge of $\overline{S}_3$ (resp. $\overline{D}_1$). From $(\neg\,$**G37**$) \wedge$ **G38** $\wedge$ **G39** being true, we also have $e_3^* \in 1\mathsf{cut}(s_1; d_3) \backslash 1\mathsf{cut}(s_1; d_1)$ and $e_1^* \in 1\mathsf{cut}(s_1; d_1)$.

Since **G42** is true, we have $e_1^* = e_u^{21}$ is in $\overline{S}_2$. Any arbitrary s_2-to-d_3 path P_{23} thus must use e_1^*. Since $e_3^* \notin 1\mathsf{cut}(s_1; d_1)$ and $e_1^* \in 1\mathsf{cut}(s_1; d_1)$, there exists an s_1-to-d_1 path Q_{11} using e_1^* but not using e_3^*. Then, we can create a s_1-to-d_3 path $Q_{11} e_1^* P_{23}$ not using e_3^*, which contradicts $e_3^* \in 1\mathsf{cut}(s_1; d_3)$. The proof of **R46** is complete. ☐

We prove **R47** as follows.

Proof. Suppose that **LNR** $\wedge$ **D4** $\wedge (\neg\,$**G27**$) \wedge$ **G28** $\wedge$ **G31** $\wedge$ $(\neg\,$**G37**$) \wedge$ **G38** $\wedge$ **G40** is true. Since **G28** $\wedge$ **G31** is true, e_3^* (resp. e_1^*) is the most downstream (resp. upstream) edge of $\overline{S}_3$ (resp. $\overline{D}_1$). Since $(\neg\,$**G27**$) \wedge$ **G28** implies **G42**, e_1^* also belongs to $\overline{S}_2$, which implies that $e_1^* \in 1\mathsf{cut}(s_2; d_3)$. Since **G40** is true, we have $e_1^* \in 1\mathsf{cut}(s_1; d_3)$. Jointly the above arguments imply $e_1^* \in \overline{D}_1 \cap \overline{D}_3$. Also, **G38** being true implies $e_3^* \in \overline{S}_3 \cap 1\mathsf{cut}(s_1; d_3)$. Since **LNR** is true and $\overline{D}_1 \cap \overline{D}_3 \neq \emptyset$, by Proposition 4 we must have $\overline{S}_1 \cap \overline{S}_3 = \emptyset$, which implies that e_3^* cannot belong to $1\mathsf{cut}(s_1; d_2)$.

Let a node u be the tail of the edge e_3^*. Since $e_3^* \in 1\mathsf{cut}(s_1; d_3)$, u is reachable from s_1. Since $e_3^* \in \overline{S}_3$, u is also reachable form s_3. Consider the collection of edges, $1\mathsf{cut}(s_1; u) \cap 1\mathsf{cut}(s_3; u)$ (may be empty), all edges of which are in the upstream of e_3^* if non-empty. Note that $(1\mathsf{cut}(s_1; u) \cap 1\mathsf{cut}(s_3; u)) \cup \{e_3^*\}$ is always non-empty (since it contains at least e_3^*). Then, we use e'' to denote the most upstream edge of $(1\mathsf{cut}(s_1; u) \cap 1\mathsf{cut}(s_3; u)) \cup \{e_3^*\}$. Let e' denote the most downstream edge among all edges in $1\mathsf{cut}(s_1; \mathsf{tail}(e''))$. Such choice is always possible since

$\mathsf{1cut}(s_1; \mathsf{tail}(e''))$ contains at least one edge (the s_1-source edge e_{s_1}) and thus we have $e_{s_1} \preceq e' \prec e'' \preceq e_3^*$. Since we choose e' to be the most downstream one, by Proposition 3 the channel gain $m_{e';e''}$ must be irreducible. Moreover, since $e_3^* \in \mathsf{1cut}(s_1; d_3)$, any path from s_1 to d_3 must use e_3^*. Consequently since $e'' \in \mathsf{1cut}(s_1; u) \cup \{e_3^*\}$, any path from s_1 to d_3 must also use e''. Consequently since $e' \in \mathsf{1cut}(s_1; \mathsf{tail}(e''))$, any path from s_1 to d_3 must also use e'. As a result, $\{e', e''\} \subset \mathsf{1cut}(s_1; d_3)$. Therefore $m_{e';e''}$ is a factor of m_{13}.

Now we argue that $\mathsf{GCD}(\, m_{31},\, m_{e';e''}) \equiv 1$. Suppose not. Since $m_{e';e''}$ is irreducible, by Proposition 3 we must have $e' \in \mathsf{1cut}(s_3; d_1)$. Note that $e' = e_{s_1}$ cannot be a 1-edge cut separating s_3 and d_1 from the definitions (i) and (ii) of the 3-unicast ANA network. Thus, we only need to consider the case when $e_{s_1} \prec e'$ since $e_{s_1} \preceq e'$ from the construction of e'. Since $e_3^* \in \mathsf{1cut}(s_3; d_1)$ and $e' \prec e_3^*$ is an 1-edge cut separating s_3 and d_1, we must have $e' \in \mathsf{1cut}(s_3; u)$. Note that the most downstream $\mathsf{1cut}(s_1; \mathsf{tail}(e''))$ edge e' also belongs to $\mathsf{1cut}(s_1; u)$ from our construction. Therefore, jointly, this contradicts the construction that e'' is the most upstream edge of $(\mathsf{1cut}(s_1; u) \cap \mathsf{1cut}(s_3; u)) \cup \{e_3^*\}$ since $e' \prec e''$.

Now we argue that $\mathsf{GCD}(\, m_{23},\, m_{e';e''}) \equiv 1$. Suppose not. Since $m_{e';e''}$ is irreducible, we must have $e' \in \mathsf{1cut}(s_2; d_3)$ and thus $e_{s_1} \prec e'$. Choose arbitrarily a path from s_1 to e'. Since we have already established $e_3^* \prec e_1^*$ and e_1^* is the most upstream edge of $\overline{D}_1$, there exists a path $P_{s_2 \mathsf{tail}(e_1^*)}$ from s_2 to $\mathsf{tail}(e_1^*)$ not using e_3^*. Since e_1^* is also in $\overline{D}_3$, $\mathsf{head}(e_1^*)$ can reach d_3. Note that the chosen path $P_{s_2 \mathsf{tail}(e_1^*)}$ must use e' since $e' \in \mathsf{1cut}(s_2; d_3)$. As a result, s_1 can reach d_3 by going to e' first, and then following $P_{s_2 \mathsf{tail}(e_1^*)}$ to e_1^*, and then going to d_3, without using e_3^*. This contradicts the assumption that $e_3^* \in \mathsf{1cut}(s_1; d_3)$.

Now we argue that $\mathsf{GCD}(\, m_{12},\, m_{e';e''}) \equiv 1$. Suppose not. Since $m_{e';e''}$ is irreducible, we must have $e'' \in \mathsf{1cut}(s_1; d_2)$. Since we have established $\neg$ **G41** (i.e., $e_3^* \notin \mathsf{1cut}(s_1; d_2)$), we only need to consider the case when $e'' \prec e_3^*$. Then by construction there exists a s_1-to-d_2 path P_{12} going through e'' but not e_3^*. However, since by construction e'' is reachable from s_3, there exists a path from s_3 to e'' first and then use P_{12} to arrive at d_2. Such a s_3-to-d_2 path does not use e_3^*, which contradicts the assumption that $e_3^* \in \overline{S}_3 \subset \mathsf{1cut}(s_3; d_2)$.

Now we argue that $\mathsf{GCD}(\, m_{11},\, m_{e';e''}) \equiv 1$. Suppose not. Since $m_{e';e''}$ is irreducible, we must have $e'' \in \mathsf{1cut}(s_1; d_1)$. Since $\neg$ **G37** is true (i.e., $e_3^* \notin \mathsf{1cut}(s_1; d_1)$), we only need to consider the case when $e'' \prec e_3^*$. Then by construction there exists a s_1-to-d_1 path P_{11} going through e'' but not e_3^*. However, since by construction e'' is reachable from s_3, there exists a path from s_3 to e'' first and then use P_{11} to arrive at d_1. Such a s_3-to-d_1 path does not use e_3^*, which contradicts the assumption that $e_3^* \in \overline{S}_3 \subset \mathsf{1cut}(s_3; d_1)$.

The four statements in the previous paragraphs shows that

$$\mathsf{GCD}(\, m_{11} m_{12} m_{23} m_{31},\ m_{e';e''}) \equiv 1.$$

This, however, contradicts the assumption that **D4** is true since we have shown that $m_{e';e''}$ is a factor of m_{13}. The proof of **R47** is thus complete. □

Appendix W
Proof of R30

If we swap the roles of s_2 and s_3, and the roles of d_2 and d_3, then the proof of **R29** in Appendix V can be directly applied to show **R30**. More specifically, note that both **D3** and **D4** are converted back and forth from each other when swapping the flow indices. Similarly, the index swapping also converts **G27** to **G28** and vice versa. Since **LNR**, **G1**, and **E0** remain the same after swapping the flow indices, we can see that **R29** becomes **R30** after swapping the flow indices. The proofs of **R29** in Appendix V can thus be used to prove **R30**.

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