Analysis of Precoding-Based Intersession Network Coding and The Corresponding 3-Unicast Interference Alignment Scheme

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***Abstract*—Recently, a new precoding-based intersession network coding (NC) scheme has been proposed, which applies the interference alignment technique, originally devised for wireless interference channels, to the 3-unicast problem of directed acyclic networks. Motivated by the graph-theoretic characterizations of classic linear NC results, this paper investigates several key relationships between the point-to-point network channel gains and the underlying graph structure. Such relationships are critical when characterizing graph-theoretically the feasibility of precoding-based solutions. One example of the applications of our results is to answer (at least partially) the conjectures of the 3-unicast interference alignment technique and the corresponding graph-theoretic characterization conditions.**

***Index Terms*—Asymptotic interference alignment, interference channels, intersession network coding, 3-unicast networks.**

I. Introduction

Characterizing the capacity or the feasibility of satisfying the network traffic demands of multiple coexisting source-destination pairs (sessions) has been a long-standing challenge. Recently, by allowing the *coding operation* to be performed at the intermediate network nodes, a new concept of *network coding* has emerged, which is able to achieve the information-theoretic capacity for the single multicast [2] even when considering only linear network codes [3]. Several papers have since studied the network code construction problem for the above single multicast setting [4]–[7].

On the other hand, when there are multiple coexisting sessions in the network, the corresponding network code design/analysis problem, also known as the *intersession network coding* (INC) problem, becomes notoriously challenging due to the potential interference within the network. For example, *linear network coding* no longer achieves the capacity [10]. Deciding the existence of a (linear) network code that satisfies general traffic demands becomes an NP-hard problem [5], [9]. Thus, recent INC studies have focused on the optimal characterizations over some restrictive networks or limited rate constraints, including the capacity regions for *directed cycles* [14], degree-2 three-layer *directed acyclic networks* (DAG) [15], and for networks with integer link capacity and two coexisting rate-1 multicast sessions [8].

Recently, the authors in [12], [13] applied the interference alignment (IA) technique, originally developed for wireless interference channels [11], to the scenario of 3 coexisting unicast sessions in the name of 3-unicast Asymptotic Network Alignment (ANA). Their application of the interference-aligning idea leads to a new perspective to the INC problems, which enables us to focus on designing the *precoding* and *decoding mappings* at the sources and destinations while allowing randomly generated *local encoding kernels* [7] within the network. Compared to the classic algebraic framework that fully controls the encoder, decoder, and local encoding kernels [5], this precoding-based framework tradeoffs the ultimate achievable throughput with a distributed, implementation-friendly structure that allows pure random linear NC in the interior of the network. Their initial study on 3-unicast networks shows that, by performing precoding across multiple time slots and applying the IA technique, the precoding-based NC can perform strictly better than the pure routing solution in some networks, and even better than some widely-used linear NC solutions in some networks. Such results thus demonstrate a new balance between practicality and throughput enhancement. Further development of the precoding-based framework could thus have significant impact on practical network code design.

In this work, we first study several basic properties of the precoding-based framework, and then apply our results to the 3-unicast ANA scheme proposed in [12], [13], which applies the interference alignment (IA) technique to the 3-unicast network. The existing results [12], [13] show that when the network code satisfies certain algebraic conditions, the 3-unicast ANA scheme achieves asymptotically half of the interference-free throughput for each transmission pair. Note that for the wireless interference channels where the IA technique was originally developed, those algebraic feasibility conditions can be satisfied with close-to-one probability provided the channel gains are independent and continuously distributed random variables [11]. For comparison, the "network channel gains" are generally correlated and the correlation depends heavily on the underlying network topology [12], [13]. As a result, we need new efficient ways to decide whether the network of interest admits a 3-unicast ANA scheme that achieves half of the interference-free throughput. The results in this work answer this question by developing new graph-theoretic conditions which are equivalent (at least partially) to the feasibility of the 3-unicast ANA scheme. The proposed graph-theoretic conditions can be easily computed and checked within polynomial time.

II. Precoding-Based Intersession NC

A. System Model and Some Graph-Theoretic Definitions

Consider a DAG $G=(V,E)$ where V is the set of nodes and E is the set of directed edges. Each edge $e\in E$ is represented by $e=uv$, where $u=\mathsf{tail}(e)$ and $v=\mathsf{head}(e)$ are the tail and head of e, respectively. For any node $v \in V$, use $\mathsf{In}(v) \subseteq E$ to denote the collection of incoming edges $uv\in E$. Similarly, $\mathsf{Out}(v)\subseteq E$ contains all $vw\in E$.

A path P is a series of adjacent edges $e_1 e_2 \cdots e_k$ where $\mathsf{head}(e_i)=\mathsf{tail}(e_{i+1})\,\forall\, i\in\{1,\cdots,k-1\}$. We say that e_1 and e_k are the starting and ending edges of P, respectively. For any path P, we use $e\in P$ to indicate that an edge e is used by P. For a given path P, xPy denotes the path segment of P from node x to node y. A path starting from node x and ending at node y is denoted by P_{xy}. To slightly abuse the notation, we sometimes substitute the nodes x and y by the edges e_1 and e_2 and use $e_1 P e_2$ to denote the path segment from e_1 to e_2 along P. Similarly, $P_{e_1 e_2}$ denotes a path from e_1 to e_2. We say a node u is an *upstream* node of a node v (or v is a *downstream* node of u) if there exists a path P_{uv} and denote it as $u \prec v$. If neither $u \prec v$ nor $u \succ v$, then we say that u and v are *not reachable* from each other. Similarly, e_1 is an upstream edge of e_2 if $\mathsf{head}(e_1)\preceq\mathsf{tail}(e_2)$, and we denote it by $e_1 \prec e_2$. Two distinct edges e_1 and e_2 are not reachable from each other, if neither $e_1 \prec e_2$ nor $e_1 \succ e_2$. A *k-edge cut* (sometimes just the "edge cut") separating node sets $U\subseteq V$ and $W\subseteq V$ is a collection of k edges such that any path from $u\in U$ to $w\in W$ must use at least one of those k edges. The value of an edge cut is the number of edges in the cut. (A k-edge cut has value k.) We denote the minimum value among all the edge cuts separating U and W as $\mathsf{EC}(U;W)$. Then, $\mathsf{EC}(U;W)=0$ when U and W are already disconnected. By convention, if $U\cap W\neq\emptyset$, we define $\mathsf{EC}(U;W)=\infty$. We also denote the collection of all distinct 1-edge cuts separating U and W as $\mathsf{1cut}(U;W)$. Please refer to [17] for more detailed graph-theoretic definitions.

B. The Corresponding Algebraic Framework

Given a network $G=(V,E)$, we consider the multiple-unicast problem in which there are K coexisting source-destination pairs (s_k,d_k), $k=1,\cdots,K$. Let l_k denote the number of information symbols that s_k wants to transmit to d_k. Each information symbol is chosen from a finite field $\mathbb{F}_q$ with some sufficiently large q. We use $\mathbb{F}$ as shorthand of $\mathbb{F}_q$.

Following the widely-used instant-transmission model on a DAG [5], we assume that each edge is capable of transmitting one symbol in $\mathbb{F}$ in one time slot without delay. We consider *linear network coding* over the entire network, i.e., a symbol on an edge $e \in E$ is a linear combination of the symbols on its adjacent incoming edges of $\mathsf{In}(\mathsf{tail}(e))$. The collection of coefficients (i.e. network variables) used for such linear combinations are termed the local encoding kernels and can be represented by $\underline{\mathbf{x}}$, the collection of all network variables $x_{e'e''}\in\mathbb{F}$ for all adjacent edge pairs (e',e''), i.e., $\underline{\mathbf{x}}=\{x_{e'e''}:(e',e'')\in E^2 \text{ where } \mathsf{head}(e')=\mathsf{tail}(e'')\}$. See [5] for detailed discussion. Following this notation, the channel gain $m_{e_1;e_2}(\underline{\mathbf{x}})$ from an edge e_1 to an edge e_2 can be written as a polynomial with respect to $\underline{\mathbf{x}}$. More rigorously, $m_{e_1;e_2}(\underline{\mathbf{x}})$ can be rewritten as

$$m_{e_1;e_2}(\underline{\mathbf{x}}) = \sum_{\forall P_{e_1e_2}\in\mathbf{P}_{e_1e_2}} \left(\prod_{\forall\, e',\, e''\in P_{e_1e_2} \text{ where } \mathsf{head}(e')=\mathsf{tail}(e'')} x_{e'e''} \right)$$

where $\mathbf{P}_{e_1e_2}$ denotes the collection of all distinct paths from e_1 to e_2.

By convention [5], we set $m_{e_1;e_2}(\underline{\mathbf{x}})=1$ when $e_1=e_2$ and set $m_{e_1;e_2}(\underline{\mathbf{x}})=0$ when $e_1\neq e_2$ and e_2 is not downstream edge of e_1. A channel gain from a node u to a node v is defined by an $|\mathsf{In}(v)|\times|\mathsf{Out}(u)|$ polynomial matrix $\mathbf{M}_{u;v}(\underline{\mathbf{x}})$, where its (i,j)-th entry is the (edge-to-edge) channel gain from the j-th outgoing edge of u to the i-th incoming edge of v. When considering source s_i and destination d_j, we use $\mathbf{M}_{i;j}(\underline{\mathbf{x}})$ as shorthand for $\mathbf{M}_{s_i;d_j}(\underline{\mathbf{x}})$.

We allow the precoding-based NC to code across τ number of time slots, which are termed the precoding frame and τ is the frame size. The network variables corresponding to each time slot t is denoted as $\underline{\mathbf{x}}^{(t)}$, and the corresponding channel gain from s_i to d_j becomes $\mathbf{M}_{i;j}(\underline{\mathbf{x}}^{(t)})$ for all $t=1,\cdots,\tau$.

With these settings, recall that each s_k would like to send l_k symbols to d_k over a frame of τ time slots. Let $\mathbf{z}_i\in\mathbb{F}^{\,l_i\times 1}$ be the set of to-be-sent information symbols from s_i. Then, for every time slot $t=1,\cdots,\tau$, we can define the precoding matrix $\mathbf{V}_i^{(t)} \in \mathbb{F}^{|\mathsf{Out}(s_i)|\times l_i}$ for each source s_i. Given the precoding matrices, each d_j receives an $|\mathsf{In}(d_j)|$-dimensional column vector $\mathbf{y}_j^{(t)}(\underline{\mathbf{x}}^{(t)})$ as

$$\mathbf{y}_j^{(t)}(\underline{\mathbf{x}}^{(t)}) = \mathbf{M}_{j;j}(\underline{\mathbf{x}}^{(t)})\mathbf{V}_j^{(t)}\mathbf{z}_j + \sum_{\substack{i=1\\ i\neq j}}^{K}\mathbf{M}_{i;j}(\underline{\mathbf{x}}^{(t)})\mathbf{V}_i^{(t)}\mathbf{z}_i.$$

This system model can be equivalently expressed as

$$\overline{\mathbf{y}}_j = \overline{\mathbf{M}}_{j;j}\overline{\mathbf{V}}_j\,\mathbf{z}_j + \sum_{\substack{i=1\\ i\neq j}}^{K}\overline{\mathbf{M}}_{i;j}\overline{\mathbf{V}}_i\,\mathbf{z}_i \tag{1}$$

where $\overline{\mathbf{V}}_i$ is the overall precoding matrix for each source s_i by vertically concatenating $\{\mathbf{V}_i^{(t)}\}_{t=1}^{\tau}$, and $\overline{\mathbf{y}}_j$ is the vertical concatenation of $\{\mathbf{y}_j^{(t)}(\underline{\mathbf{x}}^{(t)})\}_{t=1}^{\tau}$. The overall channel matrix $\overline{\mathbf{M}}_{i;j}$ is a block-diagonal polynomial matrix with $\{\mathbf{M}_{i;j}(\underline{\mathbf{x}}^{(t)})\}_{t=1}^{\tau}$ as its diagonal blocks, thus dependent on the collection of network variables $\{\underline{\mathbf{x}}^{(t)}\}_{t=1}^{\tau}$ in the precoding frame.

After receiving packets for τ time slots, each destination d_j applies the overall decoding matrix $\overline{\mathbf{U}}_j\in\mathbb{F}^{\,l_j\times(\tau\cdot|\mathsf{In}(d_j)|)}$. Then, the decoded message vector $\hat{\mathbf{z}}_j$ can be expressed as

$$\hat{\mathbf{z}}_j = \overline{\mathbf{U}}_j\overline{\mathbf{y}}_j = \overline{\mathbf{U}}_j\overline{\mathbf{M}}_{j;j}\overline{\mathbf{V}}_j\,\mathbf{z}_j + \sum_{\substack{i=1\\ i\neq j}}^{K}\overline{\mathbf{U}}_j\overline{\mathbf{M}}_{i;j}\overline{\mathbf{V}}_i\,\mathbf{z}_i. \tag{2}$$

The combined effects of precoding, channel, and decoding from s_i to d_j is $\overline{\mathbf{U}}_j\overline{\mathbf{M}}_{i;j}\overline{\mathbf{V}}_i$, termed the *network transfer matrix* from s_i to d_j. Then, we say the precoding-based NC problem is feasible if there exists a pair of encoding and decoding matrices $\{\overline{\mathbf{V}}_i, \forall\, i\}$ and $\{\overline{\mathbf{U}}_j, \forall\, j\}$ (which may be a function of $\{\underline{\mathbf{x}}^{(t)}\}_{t=1}^{\tau}$) such that when choosing each element of the collection of network variables $\{\underline{\mathbf{x}}^{(t)}\}_{t=1}^{\tau}$ independently and uniformly randomly from $\mathbb{F}$, with high probability

$$\begin{aligned} \overline{\mathbf{U}}_j \overline{\mathbf{M}}_{i;j} \overline{\mathbf{V}}_i &= \mathbf{I}\,(\text{identity}) \quad \forall\, i = j, \\ \overline{\mathbf{U}}_j \overline{\mathbf{M}}_{i;j} \overline{\mathbf{V}}_i &= \mathbf{0} \quad \forall\, i \neq j. \end{aligned} \tag{3}$$

Remark 1: One can easily check by the cut-set bound that a necessary condition for the feasibility of a precoding-based NC problem is for the frame size $\tau \geq \max_k \{l_k / \mathsf{EC}(s_k; d_k)\}$.

Remark 2: Depending on the time relationship of $\overline{\mathbf{V}}_i$ and $\overline{\mathbf{U}}_j$ with respect to the network variables $\{\underline{\mathbf{x}}^{(t)}\}_{t=1}^{\tau}$, a precoding-based NC solution can be classified as causal vs. non-causal and time-varying vs. time-invariant schemes.

C. Comparison to the Classic NC Framework

The authors in [5] established the algebraic framework for the linear network coding problem, which has similar encoding and decoding equations as in (1) and (2) and the same algebraic feasibility conditions as in (3).[1] The main difference between the precoding-based framework and the classic framework is that the latter allows the NC designer to control the network variables $\underline{\mathbf{x}}$ while the former assumes the entries of $\underline{\mathbf{x}}$ are chosen independently and uniformly. One can thus view the precoding-based NC as a distributed version of classic NC that tradeoffs the ultimate achievable performance for more practical distributed implementation (not controlling the behavior in the interior of the network).

One challenge when using the algebraic feasibility conditions (3) is that given a network code, it is easy to verify whether (3) is satisfied or not, but it is notoriously hard to decide the existence of a NC solution satisfying (3), see [5], [9]. Only in some special scenarios, we can convert those algebraic conditions into some graph-theoretic conditions for which one can decide the existence of a feasible network code in polynomial time. For example, if there exists only a single session (s_1, d_1) in the network, then the existence of a NC solution satisfying (3) is equivalent to the time-averaged rate l_1/τ being no larger than $\mathsf{EC}(s_1; d_1)$. Moreover, if $(l_1/\tau) \leq \mathsf{EC}(s_1; d_1)$, then we can use random linear network coding [7] to construct the optimal network code. Another example is when there are only two sessions (s_1, d_1) and (s_2, d_2) with $l_1 = l_2 = \tau = 1$. Then, the existence of a network code satisfying (3) is equivalent to the conditions that the 1-edge cuts in the network who perform interference-cancelling are properly placed in certain ways [8]. Motivated by the above observation, the main focus of this work is to develop new graph-theoretic conditions for a special scenario of the precoding-based NC, the 3-unicast Asymptotic Network Alignment (ANA) scheme.

[1]The original work [5] focuses on a single time slot $\tau = 1$, although the results can be easily generalized for $\tau > 1$ as well. Note that $\tau > 1$ provides a greater degree of freedom when designing the coding matrices $\{\overline{\mathbf{V}}_i, \forall\, i\}$ and $\{\overline{\mathbf{U}}_j, \forall\, j\}$. Such *time extension* turns out to be especially critical in a precoding-based NC design as it is generally much harder to design $\{\overline{\mathbf{V}}_i, \forall\, i\}$ and $\{\overline{\mathbf{U}}_j, \forall\, j\}$ for randomly chosen $\underline{\mathbf{x}}$ when $\tau = 1$. An example of this time extension will be discussed in Section II-D.

D. A Special Example of The Precoding-Based Framework: The 3-unicast ANA Scheme

Before proceeding, we introduce some algebraic definitions. We say that a set of polynomials $\mathbf{h}(\underline{\mathbf{x}}) = \{h_1(\underline{\mathbf{x}}), ..., h_N(\underline{\mathbf{x}})\}$ is linearly dependent if and only if $\sum_{k=1}^{N} \alpha_k h_k(\underline{\mathbf{x}}) = 0$ for some coefficients $\{\alpha_k\}_{k=1}^{N}$ that are not all zeros. By treating $\mathbf{h}(\underline{\mathbf{x}}^{(k)})$ as a polynomial row vector and vertically concatenating them together, we have a $M \times N$ polynomial matrix $[\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^{M}$. We call this polynomial matrix a *row-invariant* matrix since each row is based on the same set of polynomials $\mathbf{h}(\underline{\mathbf{x}})$ but with different variables $\underline{\mathbf{x}}^{(k)}$ for each row k, respectively. We say that the row-invariant polynomial matrix $[\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^{M}$ is generated from $\mathbf{h}(\underline{\mathbf{x}})$. For two polynomials $g(\underline{\mathbf{x}})$ and $h(\underline{\mathbf{x}})$, we say $g(\underline{\mathbf{x}})$ and $h(\underline{\mathbf{x}})$ are *equivalent*, denoted by $g(\underline{\mathbf{x}}) \equiv h(\underline{\mathbf{x}})$, if $g(\underline{\mathbf{x}}) = c \cdot h(\underline{\mathbf{x}})$ for some non-zero $c \in \mathbb{F}$. If not, we say $g(\underline{\mathbf{x}})$ and $h(\underline{\mathbf{x}})$ are *not equivalent*, denoted by $g(\underline{\mathbf{x}}) \not\equiv h(\underline{\mathbf{x}})$. We use $\mathsf{GCD}(\,g(\underline{\mathbf{x}}), h(\underline{\mathbf{x}}))$ to denote the greatest common factor of the two polynomials.

We now consider a special class of networks, called the 3-unicast ANA network: A network G is a 3-unicast ANA network if (i) there are 3 source-destination pairs, $(s_i, d_i), i = 1, 2, 3$, where all source/destination nodes are distinct; (ii) the topology of G is stable over a precoding frame; (iii) $|\mathsf{In}(s_i)| = 0$ and $|\mathsf{Out}(s_i)| = 1\ \forall\, i$ (we denote the only outgoing edge of s_i as e_{s_i}.); (iv) $|\mathsf{In}(d_j)| = 1$ and $|\mathsf{Out}(d_j)| = 0\ \forall\, j$ (we denote the only incoming edge of d_j as e_{d_j}.); and (v) d_j can be reached from s_i for all (i, j) pairs (including those with $i = j$).[2] We use G_{3ANA} to emphasize that we are focusing on this 3-unicast ANA network. Note that by (iii) and (iv) the matrix $\mathbf{M}_{i;j}(\underline{\mathbf{x}})$ becomes a scalar, which we denote by $m_{ij}(\underline{\mathbf{x}})$ instead.

The authors in [12], [13] applied the interference alignment technique to construct the precoding matrices $\{\overline{\mathbf{V}}_i, \forall\, i\}$ for the above 3-unicast ANA network. Namely, consider the following parameter values: $\tau = 2n+1$, $l_1 = n + 1$, $l_2 = n$, and $l_3 = n$ for some positive integer n. The goal is thus to achieve the rate tuple $(\frac{n+1}{2n+1}, \frac{n}{2n+1}, \frac{n}{2n+1})$ in a 3-unicast ANA network by applying the following $\{\overline{\mathbf{V}}_i, \forall\, i\}$ construction method: Define $L(\underline{\mathbf{x}}) = m_{13}(\underline{\mathbf{x}}) m_{32}(\underline{\mathbf{x}}) m_{21}(\underline{\mathbf{x}})$ and $R(\underline{\mathbf{x}}) = m_{12}(\underline{\mathbf{x}}) m_{23}(\underline{\mathbf{x}}) m_{31}(\underline{\mathbf{x}})$, and consider the following 3 row vectors of dimensions $n+1$, n, and n, respectively (Each entry of these row vectors is a polynomial with respect to $\underline{\mathbf{x}}$ but we drop the input argument $\underline{\mathbf{x}}$ for simplicity.):

$$\begin{aligned} \mathbf{v}_1^{(n)}(\underline{\mathbf{x}}) &= m_{23} m_{32} \left[R^n, \quad R^{n-1}L, \quad ..., \quad RL^{n-1}, \quad L^n \right], \\ \mathbf{v}_2^{(n)}(\underline{\mathbf{x}}) &= m_{13} m_{32} \left[R^n, \quad R^{n-1}L, \quad ..., \quad RL^{n-1} \right], \text{ and} \\ \mathbf{v}_3^{(n)}(\underline{\mathbf{x}}) &= m_{12} m_{23} \left[R^{n-1}L, \quad ..., \quad RL^{n-1}, \quad L^n \right], \end{aligned}$$

where the superscript (n) is to emphasize the current n value used in the construction. The precoding matrix for each

[2]The case when $m_{ij} = 0$ for some $i \neq j$, i.e., there is some d_j who is not reachable from some s_i of $i \neq j$ (not fully-interfered), it can be easily modified to be fully-interfered artifically [12].

time slot t is thus constructed as $\mathbf{V}_i^{(t)} = \mathbf{v}_i^{(n)}(\underline{\mathbf{x}}^{(t)})$, so that their vertical concatenation, i.e., the overall precoding matrix becomes (recall $\tau = 2n+1$) $\overline{\mathbf{V}}_i = [\mathbf{v}_i^{(n)}(\underline{\mathbf{x}}^{(t)})]_{t=1}^{2n+1}$. The authors in [12], [13] further prove that the above construction achieves the desired rates $(\frac{n+1}{2n+1}, \frac{n}{2n+1}, \frac{n}{2n+1})$ if the overall precoding matrices $\{\overline{\mathbf{V}}_i, \forall\, i\}$ satisfy the following six constraints:

$$d_1 : \langle\, \overline{\mathbf{M}}_{3;1}\overline{\mathbf{V}}_3 \,\rangle = \langle\, \overline{\mathbf{M}}_{2;1}\overline{\mathbf{V}}_2 \,\rangle \tag{4}$$

$$\mathbf{S}_1^{(n)} \triangleq \big[\, \overline{\mathbf{M}}_{1;1}\overline{\mathbf{V}}_1 \;\; \overline{\mathbf{M}}_{2;1}\overline{\mathbf{V}}_2 \,\big], \text{ and } \mathsf{rank}(\mathbf{S}_1^{(n)}) = 2n+1 \tag{5}$$

$$d_2 : \langle\, \overline{\mathbf{M}}_{3;2}\overline{\mathbf{V}}_3 \,\rangle \subseteq \langle\, \overline{\mathbf{M}}_{1;2}\overline{\mathbf{V}}_1 \,\rangle \tag{6}$$

$$\mathbf{S}_2^{(n)} \triangleq \big[\, \overline{\mathbf{M}}_{2;2}\overline{\mathbf{V}}_2 \;\; \overline{\mathbf{M}}_{1;2}\overline{\mathbf{V}}_1 \,\big], \text{ and } \mathsf{rank}(\mathbf{S}_2^{(n)}) = 2n+1 \tag{7}$$

$$d_3 : \langle\, \overline{\mathbf{M}}_{2;3}\overline{\mathbf{V}}_2 \,\rangle \subseteq \langle\, \overline{\mathbf{M}}_{1;3}\overline{\mathbf{V}}_1 \,\rangle \tag{8}$$

$$\mathbf{S}_3^{(n)} \triangleq \big[\, \overline{\mathbf{M}}_{3;3}\overline{\mathbf{V}}_3 \;\; \overline{\mathbf{M}}_{1;3}\overline{\mathbf{V}}_1 \,\big], \text{ and } \mathsf{rank}(\mathbf{S}_3^{(n)}) = 2n+1 \tag{9}$$

where $\langle\, \mathbf{A} \,\rangle$ and $\mathsf{rank}(\mathbf{A})$ denote the column vector space and the rank, respectively, of a given matrix $\mathbf{A}$. Recalling the definition of $\overline{\mathbf{M}}_{i;j}$ in Section II-B, the choice of $\tau = 2n+1$ and the assumption of $|\mathsf{Out}(s_i)| = |\mathsf{In}(d_j)| = 1$ result in $\overline{\mathbf{M}}_{i;j}$ being a $(2n+1) \times (2n+1)$ diagonal matrix with the t-th diagonal element $m_{ij}(\underline{\mathbf{x}}^{(t)})$. Thus from the stable network assumption of (ii) and the constructed $\{\overline{\mathbf{V}}_i, \forall\, i\}$, the square matrices $\{\mathbf{S}_i^{(n)}, \forall\, i\}$ are all row-invariant.

The interpretation of the above constraints is straightforward. In order for the interference at d_1 to be aligned, the overall precoding matrices $\{\overline{\mathbf{V}}_i, \forall\, i\}$ must be designed such that (4) can be satisfied. Note that by simple linear algebra, $\mathsf{rank}(\overline{\mathbf{M}}_{2;1}\overline{\mathbf{V}}_2) \leq n$ and $\mathsf{rank}(\overline{\mathbf{M}}_{1;1}\overline{\mathbf{V}}_1) \leq n+1$. (5) thus guarantees that (i) the rank of $\big[\, \overline{\mathbf{M}}_{1;1}\overline{\mathbf{V}}_1 \;\; \overline{\mathbf{M}}_{2;1}\overline{\mathbf{V}}_2 \,\big]$ equals to $\mathsf{rank}(\overline{\mathbf{M}}_{1;1}\overline{\mathbf{V}}_1) + \mathsf{rank}(\overline{\mathbf{M}}_{2;1}\overline{\mathbf{V}}_2)$ and (ii) $\mathsf{rank}(\overline{\mathbf{M}}_{1;1}\overline{\mathbf{V}}_1) = n+1$. Jointly (i) and (ii) imply that d_1 can successfully remove the aligned interference while recovering all $l_1 = n+1$ information symbols intended for d_1. Similar arguments can be used to justify (6) to (9) from the perspectives of d_2 and d_3.

By noticing the special Vandermonde form when constructing $\overline{\mathbf{V}}_i$, it is shown in [12], [13] that if

$$L(\underline{\mathbf{x}}) \not\equiv R(\underline{\mathbf{x}}) \tag{10}$$

then (4), (6), and (8) hold simultaneously with high probability. The authors in [13] further prove that, assuming $L(\underline{\mathbf{x}}) \not\equiv R(\underline{\mathbf{x}})$, the constraints (5), (7), and (9) hold with high probability if the following algebraic conditions are satisfied:

$$m_{11}m_{23} \sum_{i=0}^{n} \alpha_i \big(L/R\big)^i \neq m_{21}m_{13} \sum_{j=0}^{n-1} \beta_j \big(L/R\big)^j \tag{11}$$

$$m_{22}m_{13} \sum_{i=0}^{n} \alpha_i \big(L/R\big)^i \neq m_{12}m_{23} \sum_{j=0}^{n-1} \beta_j \big(L/R\big)^j \tag{12}$$

$$m_{33}m_{12} \sum_{i=0}^{n} \alpha_i \big(L/R\big)^i \neq m_{13}m_{32} \sum_{j=0}^{n-1} \beta_j \big(L/R\big)^j \tag{13}$$

for all $\alpha_i, \beta_j \in \mathbb{F}$ such that at least one of α_i and one of β_j are not zero.

In summary, [12], [13] proves the following result.

Proposition (page 3, [13]): The aforementioned 3-unicast ANA scheme achieves the rate tuple $(\frac{n+1}{2n+1}, \frac{n}{2n+1}, \frac{n}{2n+1})$ on the $G_{3\text{ANA}}$ of interest if (10), (11), (12), and (13) hold simultaneously.

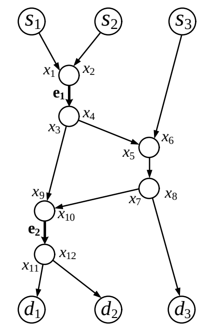

Fig. 1. Example $G_{3\text{ANA}}$ structure satisfying $L(\underline{\mathbf{x}}) \equiv R(\underline{\mathbf{x}})$ with $\underline{\mathbf{x}} = \{x_1, x_2, ..., x_{12}\}$.

It can be easily shown that directly verifying the above sufficient conditions is computationally intractable. The following conjecture is thus proposed in [13] to reduce their computational complexity when using the above proposition.

Conjecture (Page 3, [13]): For any n value used in the 3-unicast ANA scheme construction, if (10) and the following three conditions are satisfied simultaneously, then (11) to (13) must hold.

$$m_{11}m_{23} \not\equiv m_{21}m_{13} \;\text{ and }\; m_{11}m_{32} \not\equiv m_{31}m_{12}, \tag{14}$$

$$m_{22}m_{13} \not\equiv m_{12}m_{23} \;\text{ and }\; m_{22}m_{31} \not\equiv m_{32}m_{21}, \tag{15}$$

$$m_{33}m_{12} \not\equiv m_{13}m_{32} \;\text{ and }\; m_{33}m_{21} \not\equiv m_{23}m_{31}. \tag{16}$$

Whether the above conjecture is indeed true or not remains an open problem. (Currently, all numerical experiments support this conjecture [13].) Note that even if the conjecture is true, we still need to check (10), which is highly non-trivial for large networks. One main result of this work (Propositions 1 and 2) is to identify some graph-theoretic condition that is equivalent to the algebraic condition (10) and can be easily verified in polynomial time. The second main result (Proposition 3) is to prove the conjecture positively for the simplest case of $n = 1$.

Remark: In the setting of wireless interference channels, the individual channel gains are independently and continuously distributed, for which one can easily prove that $L(\underline{\mathbf{x}}) \not\equiv R(\underline{\mathbf{x}})$ with close-to-one probability. For a network setting, the channel gains $\mathbf{M}_{i;j}(\underline{\mathbf{x}})$ are no longer independent for different (i, j) pairs and the correlation depends on the underlying network topology. For example, one can easily verify that the 3-unicast ANA network described in Fig. 1 has $L(\underline{\mathbf{x}}) \equiv R(\underline{\mathbf{x}})$.

III. Properties of The Precoding-Based Framework

In this section, we provide a few fundamental relationships between the channel gains and underlying DAG G in the precoding-based framework. These newly discovered results will later be used to prove the graph-theoretic conditions of the 3-unicast ANA scheme. For the ease of exposition, we begin by simplifying the feasibility conditions of the 3-unicast ANA scheme about the row-invariant square matrices $\{\mathbf{S}_i^{(n)}\, \forall\, i\}$ in (5), (7), and (9).

A. From Non-Zero Determinant to Linear Independence

Theorem 1: Fix an arbitrary value of N. Consider any set of N polynomials $\mathbf{h}(\underline{\mathbf{x}}) = \{h_1(\underline{\mathbf{x}}), ..., h_N(\underline{\mathbf{x}})\}$ and the polynomial matrix $[\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^N$ generated from $\mathbf{h}(\underline{\mathbf{x}})$. Then, assuming sufficiently large finite field size q, $\mathbf{h}(\underline{\mathbf{x}})$ is linearly dependent *if and only if* $\mathsf{det}([\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^N) = 0$.

Proof of $\Rightarrow$: Suppose that $\mathbf{h}(\underline{\mathbf{x}})$ is linearly dependent. Then, there exists a set of coefficients $\{\alpha_k\}_{k=1}^N$ such that $\sum_{k=1}^N \alpha_k h_k(\underline{\mathbf{x}}) = 0$ and at least one of them is non-zero. Since $[\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^N$ is row-invariant, we can perform elementary column operations on $[\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^N$ using $\{\alpha_k\}_{k=1}^N$ to create an all-zero column. Thus, $\mathsf{det}([\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^N) = 0$.

Proof of $\Leftarrow$: We prove this direction by induction on the value of N. For $N = 1$, $\mathsf{det}([\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^N) = 0$ implies that $h_1(\underline{\mathbf{x}})$ is a zero polynomial, which by definition is linearly dependent.

Suppose that this holds for any $N < n_0$. When $N = n_0$, consider the (1,1)-th cofactor of $[\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^N$, which is the determinant of the submatrix of the intersection of the 2nd to n_0-th rows and the 2nd to n_0-th columns. Consider the following two cases. Case 1: the $(1,1)$-th cofactor is a zero-polynomial. Then by the induction assumption $\{h_2(\underline{\mathbf{x}}), ..., h_N(\underline{\mathbf{x}})\}$ is linearly dependent. By definition, so is $\mathbf{h}(\underline{\mathbf{x}})$. Case 2: the $(1,1)$-th cofactor is not a zero-polynomial. Since we assume a sufficiently large q, there exists an assignment $\hat{\mathbf{x}}_2 \in \mathbb{F}^{|\underline{\mathbf{x}}^{(2)}|}$ to $\hat{\mathbf{x}}_{n_0} \in \mathbb{F}^{|\underline{\mathbf{x}}^{(n_0)}|}$ such that the value of the (1,1)-th cofactor is not zero when evaluated by $\hat{\mathbf{x}}_2$ to $\hat{\mathbf{x}}_{n_0}$. But note that by the Laplace expansion, we also have $\sum_{k=1}^{n_0} h_k(\underline{\mathbf{x}}^{(1)}) C_{1k} = 0$ where C_{1k} is the $(1,k)$-th cofactor. By evaluating C_{1k} with $\{\hat{\mathbf{x}}_i\}_{i=2}^{n_0}$, we can conclude that $\mathbf{h}(\underline{\mathbf{x}})$ is linearly dependent since at least one of C_{1k} (in particular C_{11}) is not zero. ∎

Remark: Theorem 1 can be rewritten such that when $N > 1$, a set of N non-zero polynomials $\mathbf{h}(\underline{\mathbf{x}})$ is linearly independent *if and only if* there exists some assignment $\{\hat{\mathbf{x}}_k\}_{k=1}^N$ resulting $\mathsf{det}([\mathbf{h}(\hat{\mathbf{x}}_k)]_{k=1}^N) \neq 0$. By the assumption of sufficiently large field size q and by Schwartz-Zippel lemma, it is also equivalent to $\mathsf{det}([\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^N) \neq 0$ with high probability.

Theorem 1 is important in a sense that this enables us to simplify the feasibility characterization of the 3-unicast ANA scheme. From the construction in Section II-D, the square matrix $\mathbf{S}_i^{(n)}$ is in the form of $\mathbf{S}_i^{(n)} = [\mathbf{h}_i^{(n)}(\underline{\mathbf{x}}^{(t)})]_{t=1}^{(2n+1)}$. For example, for $i = 1$, $\mathbf{S}_1^{(n)} = [\mathbf{h}_1^{(n)}(\underline{\mathbf{x}}^{(t)})]_{t=1}^{(2n+1)}$ where

$$\begin{aligned}\mathbf{h}_1^{(n)}(\underline{\mathbf{x}}) &= \{\, m_{11}(\underline{\mathbf{x}})\mathbf{v}_1^{(n)}(\underline{\mathbf{x}}),\ m_{21}(\underline{\mathbf{x}})\mathbf{v}_2^{(n)}(\underline{\mathbf{x}}) \,\} \\ &= \{\, m_{11}m_{23}m_{32}R^n,\ m_{11}m_{23}m_{32}R^{n-1}L,\ \cdots, \\ &\qquad m_{11}m_{23}m_{32}L^n,\ m_{21}m_{13}m_{32}R^n, \\ &\qquad m_{21}m_{13}m_{32}R^{n-1}L,\ \cdots,\ m_{21}m_{13}m_{32}RL^{n-1} \,\}.\end{aligned}$$

Then, given the G_{3ANA} of interest satisfying (10), the linear independence of $\mathbf{h}_1^{(n)}(\underline{\mathbf{x}})$ can be used to prove that (11) is not only sufficient but also necessary for (5) with high probability. By similar arguments on $\mathbf{h}_2^{(n)}(\underline{\mathbf{x}})$ and $\mathbf{h}_3^{(n)}(\underline{\mathbf{x}})$, (12) and (13) are both necessary and sufficient for (7) and (9), respectively, with high probability.

B. Subgraph Property of the Precoding-Based Framework

Given a DAG G, recall the definition of channel gain $m_{e_1;e_2}(\underline{\mathbf{x}})$ from e_1 to e_2 in Section II-B. For a subgraph $G' \subseteq G$ containing e_1 and e_2, let $m_{e_1;e_2}(\underline{\mathbf{x}}')$ denote the channel gain from e_1 to e_2 in G'.

Theorem 2 (Subgraph Property): Given a DAG G, consider an arbitrary, but fixed, finite collection of edge pairs, $\{(e_i, e'_i) \in E^2 : i \in I\}$ where I is a finite index set, and two polynomial functions $f : \mathbb{F}^{|I|} \mapsto \mathbb{F}$ and $g : \mathbb{F}^{|I|} \mapsto \mathbb{F}$. Then, $f(\{m_{e_i;e'_i}(\underline{\mathbf{x}})\, \forall\, i \in I\}) \equiv g(\{m_{e_i;e'_i}(\underline{\mathbf{x}})\, \forall\, i \in I\})$ *if and only if* for all subgraphs $G' \subseteq G$ containing all edges in $\{e_i, e'_i\, \forall\, i \in I\}$, $f(\{m_{e_i;e'_i}(\underline{\mathbf{x}}')\, \forall\, i \in I\}) \equiv g(\{m_{e_i;e'_i}(\underline{\mathbf{x}}')\, \forall\, i \in I\})$.

Proof of $\Leftarrow$: This can be proved by simply choosing $G' = G$.

Proof of $\Rightarrow$: Since $f(\{m_{e_i;e'_i}(\underline{\mathbf{x}})\, \forall\, i \in I\}) \equiv g(\{m_{e_i;e'_i}(\underline{\mathbf{x}})\, \forall\, i \in I\})$, we can assume $f(\{m_{e_i;e'_i}(\underline{\mathbf{x}})\, \forall\, i \in I\}) = \alpha g(\{m_{e_i;e'_i}(\underline{\mathbf{x}})\, \forall\, i \in I\})$ for some $\alpha \in \mathbb{F}\backslash\{0\}$. Consider a subgraph G' containing all edges in $\{e_i, e'_i\, \forall\, i \in I\}$ and the channel gain $m_{e_i;e'_i}(\underline{\mathbf{x}}')$ on G'. Then, $m_{e_i;e'_i}(\underline{\mathbf{x}}')$ can be derived from $m_{e_i;e'_i}(\underline{\mathbf{x}})$ by substituting those $\underline{\mathbf{x}}$ variables that are not in G' by zero. As a result, we immediately have $f(\{m_{e_i;e'_i}(\underline{\mathbf{x}}')\, \forall\, i \in I\}) = \alpha g(\{m_{e_i;e'_i}(\underline{\mathbf{x}}')\, \forall\, i \in I\})$ for the same α. The proof of this direction is thus complete. ∎

Remark: Theorem 2 has a similar flavor to the result of the classic framework [5], [7], since for the single multicast from a single source s to the set of destinations $\{d_j\}$, the existence of subgraph induced by the set of edge-disjoint paths from s to d_j is enough to say that the network transfer matrix from s to d_j is full-rank with close-to-one probability.

The following corollary of Theorem 2 shows that (14), (15), and (16) can be satisfied by checking much simpler graph-theoretic conditions.

Corollary 1 (First proved in [13]): Given a G_{3ANA}, consider the corresponding channel gains $m_{ij}(\underline{\mathbf{x}})$ as defined in Section II-D. Then, $\mathsf{EC}(\{s_{i_1}, s_{i_2}\}; \{d_{j_1}, d_{j_2}\}) = 1$ *if and only if* $m_{i_1j_1}(\underline{\mathbf{x}})m_{i_2j_2}(\underline{\mathbf{x}}) \equiv m_{i_2j_1}(\underline{\mathbf{x}})m_{i_1j_2}(\underline{\mathbf{x}})$.

Proof of $\Rightarrow$: Without loss of generality, $(i_1, i_2) = (1, 2)$ and $(j_1, j_2) = (1, 3)$. Since $\mathsf{EC}(\{s_1, s_2\}; \{d_1, d_3\}) = 1$, there exists an edge e^* that separates $\{d_1, d_3\}$ from $\{s_1, s_2\}$. Therefore, we must have

$$\begin{aligned} m_{11}(\underline{\mathbf{x}}) &= m_{e_{s_1};e^*}(\underline{\mathbf{x}})\, m_{e^*;e_{d_1}}(\underline{\mathbf{x}}), \\ m_{13}(\underline{\mathbf{x}}) &= m_{e_{s_1};e^*}(\underline{\mathbf{x}})\, m_{e^*;e_{d_3}}(\underline{\mathbf{x}}), \\ m_{21}(\underline{\mathbf{x}}) &= m_{e_{s_1};e^*}(\underline{\mathbf{x}})\, m_{e^*;e_{d_1}}(\underline{\mathbf{x}}), \\ \text{and}\quad m_{23}(\underline{\mathbf{x}}) &= m_{e_{s_2};e^*}(\underline{\mathbf{x}})\, m_{e^*;e_{d_3}}(\underline{\mathbf{x}}). \end{aligned}$$

As a result, $m_{11}(\underline{\mathbf{x}})m_{23}(\underline{\mathbf{x}}) \equiv m_{21}(\underline{\mathbf{x}})m_{13}(\underline{\mathbf{x}})$.

Proof of $\Leftarrow$: We prove this direction by contradiction. Suppose $\mathsf{EC}(\{s_{i_1}, s_{i_2}\}; \{d_{j_1}, d_{j_2}\}) \geq 2$. In a 3-unicast ANA network with the network variables $\underline{\mathbf{x}}$, each source (resp. destination) has only one outgoing (resp. incoming) edge. Therefore, $\mathsf{EC}(\{s_{i_1}, s_{i_2}\}; \{d_{j_1}, d_{j_2}\}) \geq 2$ implies that at least one of the two cases must be true: Case 1: There exists a pair of edge-disjoint paths $P_{s_{i_1}d_{j_1}}$ and $P_{s_{i_2}d_{j_2}}$; Case 2: There exists a pair of edge-disjoint paths $P_{s_{i_1}d_{j_2}}$ and $P_{s_{i_2}d_{j_1}}$. For Case 1, we consider the network variables that are along the two edge-disjoint paths, i.e., consider the collection $\underline{\mathbf{x}}'$ of network variables $x_{ee'} \in \underline{\mathbf{x}}$ such that either both e and e' are used by $P_{s_{i_1}d_{j_1}}$ or both e and e' are used by $P_{s_{i_2}d_{j_2}}$. We keep those variables in $\underline{\mathbf{x}}'$ intact and set the other network variables to be zero. After hardwiring the other variables, we will have $m_{i_1j_1}(\underline{\mathbf{x}}')m_{i_2j_2}(\underline{\mathbf{x}}') = \prod_{\forall x_{ee'} \in \underline{\mathbf{x}}'} x_{ee'}$, which is not equivalent to $m_{i_2j_1}(\underline{\mathbf{x}}')m_{i_1j_2}(\underline{\mathbf{x}}') = 0$. This implies that before hardwiring some of the variables, $m_{i_1j_1}(\underline{\mathbf{x}})m_{i_2j_2}(\underline{\mathbf{x}}) \not\equiv m_{i_2j_1}(\underline{\mathbf{x}})m_{i_1j_2}(\underline{\mathbf{x}})$ as well. Case 2 can be proven by swapping the labels of j_1 and j_2. The proof is thus complete. ∎

C. New Channel Gain Property

With Theorems 1 and 2, checking whether $\mathbf{S}_i^{(n)}$ being full-rank with high probability on the $G_{3\text{ANA}}$ of interest or not can be reduced to finding one subgraph where the resulting $\mathbf{S}_i^{(n)}$ is full-rank with high probability. However, the guidance on how to search for such a subgraph is still missing. To proceed, we need deeper understanding about the channel gain property in the precoding-based framework.

Theorem 3 (Channel Gain Property): Given a DAG G and two distinct edges e_s and e_d where $s = \mathsf{head}(e_s)$ and $d = \mathsf{tail}(e_d)$, the following is true (we drop the variables $\underline{\mathbf{x}}$ for shorthand):

- If $\mathsf{EC}(s;d) = 0$, then $m_{e_s;e_d} = 0$
- If $\mathsf{EC}(s;d) = 1$, then $m_{e_s;e_d}$ is reducible and can be expressed as $m_{e_s;e_d} = m_{e_s;e_1}\left(\prod_{i=1}^{N-1} m_{e_i;e_{i+1}}\right)m_{e_N;e_d}$ where $\{e_i\}_{i=1}^N$ are all the distinct 1-edge cuts between s and d in the topological order (from the most upstream to the most downstream). Moreover, the polynomial factors $m_{e_s;e_1}$, $\{m_{e_i;e_{i+1}}\}_{i=1}^{N-1}$, and $m_{e_N;e_d}$ are all irreducible, and no two of them are equivalent.
- If $\mathsf{EC}(s;d) \geq 2$ (including ∞), then $m_{e_s;e_d}$ is irreducible.

Proof: We use the induction on the number of edges $|E|$ of $G = (V, E)$. When $|E| = 0$, then $\mathsf{EC}(s;d) = 0$ since there are no edges in G. Thus $m_{e_s;e_d} = 0$ naturally.

· Suppose that the above three claims are true for $|E| = k-1$.

· Consider $G = (V, E)$ where $|E| = k$.

Case 1 when $\mathsf{EC}(s;d) = 0$ **on** G**:** In this case, s and d are already disconnected, so are e_s and e_d since $s = \mathsf{head}(e_s)$ and $d = \mathsf{tail}(e_d)$. Thus, $m_{e_s;e_d} = 0$.

Case 2 when $\mathsf{EC}(s;d) = 1$ **on** G**:** Consider all distinct 1-edge cuts $e_1, \cdots, e_N$ between s and d from the most upstream to the most downstream. If we denote $e_0 = e_s$ and $e_{N+1} = e_d$, then we can express $m_{e_s;e_d}$ as $m_{e_s;e_d} = \prod_{i=0}^{N} m_{e_i;e_{i+1}}$. Since we considered all distinct 1-edge cuts between s and d, it must be $\mathsf{EC}(\mathsf{head}(e_i);\mathsf{tail}(e_{i+1})) \geq 2$ for $i = 0, \cdots, N$. By induction, $\{m_{e_i;e_{i+1}}\}_{i=0}^{N}$ are all irreducible and no two of them are equivalent since they cover the disjoint portions of G.

Case 3 when $\mathsf{EC}(s;d) \geq 2$ **on** G**:** If $\mathsf{EC}(s;d) = \infty$ (i.e., $s = d$), then $m_{e_s;e_d}$ is a polynomial with respect to only one variable $x_{e_se_d} \in \underline{\mathbf{x}}$ corresponding to the edge pair (e_s, e_d), thus irreducible.

We now consider the case when $\mathsf{EC}(s;d)$ has a finite value larger than or equal to two. In this case, $s \prec d$ and we can always choose an edge $e_* = uv \in E$ such that $s \preceq u \prec v \preceq d$. Consider the subgraph G' by removing e_* from G. If we temporarily define the channel gain from an edge e_1 to an edge e_2 on this subgraph $G' = G\backslash\{e_*\}$ to be $m'_{e_1;e_2}$, then the channel gain $m_{e_s;e_d}$ on G can be broken into two parts as

$$m_{e_s;e_d} = m_{e_s;e_*}m_{e_*;e_d} + m'_{e_s;e_d},$$

where both $m_{e_s;e_*}$ and $m_{e_*;e_d}$ are non-zero polynomials by induction since we chose u and v such that $u \succeq s$ and $v \preceq d$, respectively. In addition, since $G' = G\backslash\{e_*\}$, it must be $\mathsf{EC}(s;d) \geq 1$ on G'. Thus, $m'_{e_s;e_d}$ is also a non-zero polynomial by induction.

In this setting, we define the following collections of network variables related to $e_* = uv$:

$$\underline{\mathbf{x}}_u = \{\, x_{e'e_*} \in \underline{\mathbf{x}} \,:\, e' \in \mathsf{In}(u) \text{ where } e' \succeq e_s\},$$
$$\underline{\mathbf{x}}_v = \{\, x_{e_*e''} \in \underline{\mathbf{x}} \,:\, e'' \in \mathsf{Out}(v) \text{ where } e'' \preceq e_d\}.$$

From $\underline{\mathbf{x}}_u$ and $\underline{\mathbf{x}}_v$, we can define their cartesian product as $\underline{\mathbf{x}}_u \otimes \underline{\mathbf{x}}_v = \{\, x_{e'e_*}x_{e_*e''} \,|\, x_{e'e_*} \in \underline{\mathbf{x}}_u,\, x_{e_*e''} \in \underline{\mathbf{x}}_v \,\}$. Since $\underline{\mathbf{x}}_u \cap \underline{\mathbf{x}}_v = \emptyset$, we can think $\underline{\mathbf{x}}_u \otimes \underline{\mathbf{x}}_v$ as the collection of distinct monomials with respect to $\underline{\mathbf{x}}_u \cup \underline{\mathbf{x}}_v$.

Then, it is easy to see that a non-zero polynomial $m'_{e_s;e_d}$ is with respect to $\underline{\mathbf{x}}\backslash\{\underline{\mathbf{x}}_u \cup \underline{\mathbf{x}}_v\}$ since it is defined on $G' = G\backslash\{e_*\}$. Furthermore, for every monomial of $m_{e_s;e_*}$ (resp. $m_{e_*;e_d}$), exactly one of its variables must be one element of $\underline{\mathbf{x}}_u$ (resp. $\underline{\mathbf{x}}_v$) with its exponent strictly being one. Thus, every monomial of $m_{e_s;e_*}m_{e_*;e_d}$ must satisfy the following property: it contains exactly one element (not more than one) of $\underline{\mathbf{x}}_u \otimes \underline{\mathbf{x}}_v$ as a factor, with the exponent of each variable in such element being strictly one.

Since now we have $\mathsf{EC}(s;d) \geq 1$ on G', for the following, we will prove that $m_{e_s;e_d}$ is irreducible case by case depending on the property of $m'_{e_s;e_d}$.

Case 3.1 when $\mathsf{EC}(s;d) \geq 2$ **on** G'**:** In this case, the induction implies that $m'_{e_s;e_d}$ is irreducible. We now argue that $m_{e_s;e_d}$ is irreducible by contradiction. Suppose that $m_{e_s;e_d}$ is reducible. Consider first the case when $\mathsf{GCD}(\,m_{e_s;e_*}m_{e_*;e_d},\, m'_{e_s;e_d}) \neq m'_{e_s;e_d}$. Then, since $m'_{e_s;e_d}$ is irreducible, $m_{e_s;e_d}$ must have the following form to be reducible:

$$m_{e_s;e_d} = (m'_{e_s;e_d} + A)(1+B) = m'_{e_s;e_d} + m'_{e_s;e_d}B + A + AB,$$

where A and B are two polynomials such that $m_{e_s;e_*}m_{e_*;e_d} = m'_{e_s;e_d}B + A + AB$. More specifically, A must be non-zero otherwise $\mathsf{GCD}(\,m_{e_s;e_*}m_{e_*;e_d},\, m'_{e_s;e_d}) = m'_{e_s;e_d}$, and B must be non-zero for $m_{e_s;e_d}$ to be reducible.

Then, we use the monomial property of $m_{e_s;e_*}m_{e_*;e_d}$ explained above to claim that when $m_{e_s;e_d}$ is reducible and $m'_{e_s;e_d}$ is irreducible, we must have $\mathsf{GCD}(\,m_{e_s;e_*}m_{e_*;e_d},\, m'_{e_s;e_d}) = m'_{e_s;e_d}$. From the equation of $m_{e_s;e_*}m_{e_*;e_d} = m'_{e_s;e_d}B + A + AB$ and the fact that $m'_{e_s;e_d}$ is a non-zero polynomial with respect to $\underline{\mathbf{x}}\backslash\{\underline{\mathbf{x}}_u \cup \underline{\mathbf{x}}_v\}$, any monomial of A (resp. B) must satisfy such monomial property. However, they cannot hold simultaneously because the same monomial property also holds on every monomial of AB. In more details, if every monomial of A (resp. B)

satisfies such monomial property, then there must exist a monomial of AB who either contains more than one element of $\underline{\mathbf{x}}_u \otimes \underline{\mathbf{x}}_v$ as its factor, or contains exactly one element of $\underline{\mathbf{x}}_u \otimes \underline{\mathbf{x}}_v$ as its factor but with the exponent of each variable in such element being strictly larger than one. Therefore, if $m_{e_s;e_d}$ is reducible and $m'_{e_s;e_d}$ is irreducible, then we must have $\mathsf{GCD}(\,m_{e_s;e_*}m_{e_*;e_d},\, m'_{e_s;e_d}) = m'_{e_s;e_d}$.

Again by the irreducibility of $m'_{e_s;e_d}$, it must be $\mathsf{GCD}(\,m_{e_s;e_*}, m'_{e_s;e_d}) = m'_{e_s;e_d}$ or $\mathsf{GCD}(\,m_{e_*;e_d}, m'_{e_s;e_d}) = m'_{e_s;e_d}$. Then, since $m_{e_s;e_*} \neq m'_{e_s;e_d}$, $\mathsf{GCD}(\,m_{e_s;e_*}, m'_{e_s;e_d}) = m'_{e_s;e_d}$ implies that $m_{e_s;e_*}$ can be factorized and $m'_{e_s;e_d}$ is one of the irreducible factors of $m_{e_s;e_*}$. By induction, $\mathsf{EC}(s;u)=1$ and $e_d \in \mathsf{1cut}(s;u)$, which contradicts the assumption that $u \prec d$. Similar argument shows that $\mathsf{GCD}(\,m_{e_*;e_d}, m'_{e_s;e_d}) = m'_{e_s;e_d}$ contradicts $s \prec v$. Thus, the proof of this case is complete.

Case 3.2 when $\mathsf{EC}(s;d)=1$ on G': By induction, we can express that $m'_{e_s;e_d} = m'_{e_s;e_1}\left(\prod_{i=1}^{N-1} m'_{e_i;e_{i+1}}\right) m'_{e_N;e_d}$ where $\{e_i\}_{i=1}^{N}$ are the collection of all distinct 1-edge cuts on G' (in the topological order) separating s and d. Again, let $e_s = e_0$ and $e_d = e_{N+1}$, and rewrite it as $m'_{e_s;e_d} = \prod_{i=0}^{N} m'_{e_i;e_{i+1}}$. Then, $\{m'_{e_i;e_{i+1}}\}_{i=0}^{N}$ are all irreducible and no two of them are equivalent.

We now argue that $m_{e_s;e_d}$ is irreducible by contradiction. Suppose that $m_{e_s;e_d}$ is reducible. Consider first the case when $\mathsf{GCD}(\,m_{e_s;e_*}m_{e_*;e_d},\, m'_{e_i;e_{i+1}}) \neq m'_{e_i;e_{i+1}}$ for all $i = 0, \cdots, N$. Then, the similar arguments as in *Case 3.1* can be used to show that this case cannot hold when $m_{e_s;e_d}$ is reducible.

Thus, when $m_{e_s;e_d}$ is reducible, we must have $\mathsf{GCD}(\,m_{e_s;e_*}m_{e_*;e_d}, m'_{e_i;e_{i+1}}) = m'_{e_i;e_{i+1}}$ for some i. By the irreducibility of $m'_{e_i;e_{i+1}}$ from the above construction, it must be $\mathsf{GCD}(\,m_{e_s;e_*}, m'_{e_i;e_{i+1}}) = m'_{e_i;e_{i+1}}$ or $\mathsf{GCD}(\,m_{e_*;e_d}, m'_{e_i;e_{i+1}}) = m'_{e_i;e_{i+1}}$. However, we will prove by contradiction that neither $m_{e_s;e_*}$ nor $m_{e_*;e_d}$ have $m'_{e_i;e_{i+1}}$ as its factor for any $i = 0, \cdots, N$.

Suppose that $\mathsf{GCD}(\,m_{e_s;e_*}, m'_{e_i;e_{i+1}}) = m'_{e_i;e_{i+1}}$ for some i. Note that it is impossible to have $m_{e_s;e_*} = m'_{e_i;e_{i+1}}$ for any value of i, because $m_{e_s;e_*} = m'_{e_i;e_{i+1}}$ implies that $i=0$ and $e_* = e_1$ but it must be $e_* \neq e_1$ from the assumption that $G' = G\backslash\{e_*\}$ and $e_1 \in \mathsf{1cut}(s;d)$ on G'. Therefore, $\mathsf{GCD}(\,m_{e_s;e_*}, m'_{e_i;e_{i+1}}) = m'_{e_i;e_{i+1}}$ for some i implies that $m_{e_s;e_*}$ can be factorized and $m'_{e_i;e_{i+1}}$ is one of the irreducible factors of $m_{e_s;e_*}$. If $i = 0$, then $e_1 \in \mathsf{1cut}(s;u)$ on G by induction. But this, together with the assumption that $e_1 \in \mathsf{1cut}(s;d)$ on G', tells us that e_1 is also 1-edge cut separating s from d on G, which contradicts the assumption of $\mathsf{EC}(s;d) \geq 2$ on G. Similarly if $i > 0$, then $e_i \in \mathsf{1cut}(s;u)$ on G by induction but it originally belonged to $\mathsf{1cut}(s;d)$ on G'. Thus, e_i becomes also 1-edge cut separating s from d on G, contradicting $\mathsf{EC}(s;d) \geq 2$ on G. Using the similar arguments, we can also show that $m_{e_*;e_d}$ cannot contain $m'_{e_i;e_{i+1}}$ as its factor for any $i = 0, \cdots, N$. Thus, the proof of this case is complete. ∎

Remark: Theorem 3 only considers a channel gain between two distinct edges. If $e_s = e_d$, then $m_{e_s;e_d} = 1$ from the convention [5].

Theorem 3 can be used to derive the following corollary.

Corollary 2: Given a G_{3ANA}, consider the corresponding channel gains $m_{ij}(\underline{\mathbf{x}})$ as defined in Section II-D. Then, $\mathsf{GCD}(\,m_{i_1j_1}, m_{i_2j_2}) \neq m_{i_2j_2}$ unless $(i_1, j_1) = (i_2, j_2)$. Intuitively, any channel gain $m_{i_1j_1}$ cannot contain the other $m_{i_2j_2}$.

Proof: We prove this corollary by contradiction. Suppose $\mathsf{GCD}(\,m_{i_1j_1}, m_{i_2j_2}) = m_{i_2j_2}$ for some $(i_1, j_1) \neq (i_2, j_2)$. Without loss of generality, we assume that $i_1 \neq i_2$. $\mathsf{GCD}(\,m_{i_1j_1}, m_{i_2j_2}) = m_{i_2j_2}$ implies that $m_{i_1j_1}$ must be reducible. By Theorem 3, we know that $m_{i_1j_1}$ must be expressed as $m_{i_1j_1} = m_{e_{s_{i_1}};e_1}\left(\prod_{i=1}^{N-1} m_{e_i;e_{i+1}}\right) m_{e_N;e_{d_{j_1}}}$ where each term corresponds to a pair of closest 1-edge cuts separating s_{i_1} and d_{j_1}. Suppose $m_{i_2j_2}$ can also be written as $m_{i_2j_2} = m_{e_{s_{i_2}};\tilde{e}_1}\left(\prod_{i=1}^{\tilde{N}-1} m_{\tilde{e}_i;\tilde{e}_{i+1}}\right) m_{\tilde{e}_{\tilde{N}};e_{d_{j_2}}}$. Then $\mathsf{GCD}(\,m_{i_1j_1}, m_{i_2j_2}) = m_{i_2j_2}$ implies that the first term $m_{e_{s_{i_2}};\tilde{e}_1}$ must be one of the irreducible factors in $m_{i_1j_1}$. This, together with the assumption $s_{i_1} \neq s_{i_2}$, implies that $e_{s_{i_2}}$ is an edge cut separating s_{i_1} and d_{j_1}, which contradicts the assumption that in a 3-unicast ANA network $|\mathsf{In}(s_i)| = 0$ for all i. For the case that $m_{i_2j_2}$ is irreducible, we can use similar arguments to show that $e_{s_{i_2}}$ is an edge cut separating s_{i_1} and d_{j_1}, which contradicts the assumption $|\mathsf{In}(s_i)| = 0$ for all i. The proof is thus complete. ∎

IV. Detailed Studies of The 3-unicast ANA Scheme

In Section III, we investigated the key relationships between the channel gain and the underlying DAG G. We characterize graph-theoretically the feasibility of the 3-unicast ANA scheme.

A. New Graph-Theoretic Notations and The Corresponding Properties

We begin by defining new notations to be useful. Consider three indices i, j, and k taking values in $\{1, 2, 3\}$, for which the values of j and k must be distinct. Given a G_{3ANA}, let us define:

$$\overline{S}_{i;j\cap k} \triangleq \{\ e \in E\backslash\{e_{s_i}\}\ :\ e \in \mathsf{1cut}(s_i; d_j) \cap \mathsf{1cut}(s_i; d_k)\ \},$$
$$\overline{D}_{i;j\cap k} \triangleq \{\ e \in E\backslash\{e_{d_j}\}\ :\ e \in \mathsf{1cut}(s_j; d_i) \cap \mathsf{1cut}(s_k; d_i)\ \},$$

as the collection of edges except e_{s_i} (resp. e_{d_i}) which are 1-edge cuts separating s_i from $\{d_j, d_k\}$ (resp. separating $\{s_j, s_k\}$ from d_i). When the values of indices i, j, and k are all distinct, we use $\overline{S}_i$ (resp. $\overline{D}_i$) as shorthand for $\overline{S}_{i;j\cap k}$ (resp. $\overline{D}_{i;j\cap k}$). The following lemmas prove some topological relationships among $\overline{S}_i$ and $\overline{D}_j$.

Lemma 1: For all $i \neq j$, $e' \in \overline{S}_i$, and $e'' \in \overline{D}_j$, one of the following statements is true: $e' \prec e''$, $e' \succ e''$, or $e' = e''$.

Proof: All paths from s_i to d_j where $i \neq j$ must pass through all edges in $\overline{S}_i$ and all edges in $\overline{D}_j$ by definition. Thus, the relationship between any $e' \in \overline{S}_i$ and any $e'' \in \overline{D}_j$ must be either $e' \prec e''$, $e' \succ e''$, or $e' = e''$. ∎

Lemma 2: For any distinct i, j, and k in $\{1, 2, 3\}$, we have $(\overline{S}_i \cap \overline{S}_j) \subseteq \overline{D}_k$.

Proof: Consider three indices i, j, and k taking distinct values in $\{1, 2, 3\}$. Assume an edge $e \in \overline{S}_i \cap \overline{S}_j$. By definition, all paths from s_i to d_k, and all paths from s_j to d_k must pass through e, resulting $e \in \overline{D}_k$. ∎

Lemma 3: For all $i \neq j$, $e' \in \overline{S}_i \backslash \overline{D}_j$, and $e'' \in \overline{D}_j$, we have $e' \preceq e''$.

Proof: Without loss of generality, let $i = 1$ and $j = 2$. Take the most downstream edge in $\overline{S}_1 \backslash \overline{D}_2$ and denote it as e'_*. Since e'_* belongs to $\mathsf{1cut}(s_1; d_2) \cap \mathsf{1cut}(s_1; d_3)$ but not to $\mathsf{1cut}(s_3; d_2)$, there must exist a path $P_{s_3 d_2}$ from s_3 to d_2 not passing through e'_*. In addition, for any $e'' \in \overline{D}_2$, we have either $e'' \prec e'_*$, $e'' \succ e'_*$, or $e'' = e'_*$ by Lemma 1. In this setting, we prove this lemma by contradiction. Suppose there exists an edge $e'' \in \overline{D}_2$ such that $e'' \prec e'_*$. Then by definition, any path from s_1 (resp. s_3) to d_2 must pass through e''. Thus, the constructed path $P_{s_3 d_2}$ (who does not pass through e'_*) must pass through e'' beforehand. If we take an arbitrary path $P_{s_1 d_2}$ from s_1 to d_2 (who passes through e''), then $s_1 P_{s_1 d_2} e'' P_{s_3 d_2} d_2$ will be a path from s_1 to d_2 not passing through e'_*, which violates the assumption that $e'_* \in \mathsf{1cut}(s_1; d_2)$. Thus, the proof is complete. ∎

Lemma 4: For any distinct i, j, and k in $\{1, 2, 3\}$, $\overline{D}_j \cap \overline{D}_k \neq \emptyset$ *if and only if* both $\overline{S}_i \cap \overline{D}_j \neq \emptyset$ and $\overline{S}_i \cap \overline{D}_k \neq \emptyset$.

Proof of $\Rightarrow$: It trivially satisfies by applying Lemma 2.

Proof of $\Leftarrow$: Consider three indices i, j, and k taking distinct values in $\{1, 2, 3\}$. Suppose $\overline{S}_i \cap \overline{D}_j \neq \emptyset$ and $\overline{S}_i \cap \overline{D}_k \neq \emptyset$. Then, for any $e' \in \overline{S}_i \cap \overline{D}_j$ and any $e'' \in \overline{S}_i \cap \overline{D}_k$, we must have either $e' \prec e''$, $e' \succ e''$, or $e' = e''$ by Lemma 1. In this setting, we prove this lemma by contradiction. Suppose also that $\overline{D}_j \cap \overline{D}_k = \emptyset$. Then, for any $e' \in \overline{S}_i \cap \overline{D}_j$ and any $e'' \in \overline{S}_i \cap \overline{D}_k$, we must have either $e' \prec e''$ or $e' \succ e''$. However, $e' \prec e''$ results in the violation of Lemma 3 because $e' \in (\overline{S}_i \cap \overline{D}_j) \subseteq \overline{D}_j$ and $e'' \in (\overline{S}_i \cap \overline{D}_k) \subseteq (\overline{S}_i \backslash \overline{D}_j)$ by the assumption of $\overline{D}_j \cap \overline{D}_k = \emptyset$. Similarly, $e' \succ e''$ violates Lemma 3. Thus, the proof of this direction is complete. ∎

Lemma 5: For all $i \neq j$, and $e'' \in \overline{D}_i \cap \overline{D}_j$, if $\overline{S}_i \cap \overline{S}_j \neq \emptyset$, then there exists $e'_* \in \overline{S}_i \cap \overline{S}_j$ such that $e'_* \preceq e''$.

Proof: Without loss of generality, let $i = 1$ and $j = 2$. Note that any $e' \in \overline{S}_1 \cap \overline{S}_2$ and any $e'' \in \overline{D}_1 \cap \overline{D}_2$ must be either $e' \prec e''$, $e' \succ e''$, or $e' = e''$ by Lemma 1. In this setting, we prove this lemma by contradiction.

Suppose that there exists $e''_* \in \overline{D}_1 \cap \overline{D}_2$ for all $e' \in \overline{S}_1 \cap \overline{S}_2$ such that $e''_* \prec e'$. Then, we now claim that any path from $s_{i'}$ to $d_{j'}$ where $i' \neq j'$ must pass through e''_*. First notice that any path from $\{s_2, s_3\}$ to d_1, and any path from $\{s_1, s_3\}$ to d_2 must pass through $e''_* \in \overline{D}_1 \cap \overline{D}_2$ by definition. Thus, we only need to consider the path from $\{s_1, s_2\}$ to d_3.

Consider an arbitrary path $P_{s_1 d_3}$ from s_1 to d_3. Suppose by contradiction that this path $P_{s_1 d_3}$ does not pass through e''_*. But $P_{s_1 d_3}$ must pass through every edges in $\overline{S}_1 \cap \overline{S}_2$ by definition, which are all in the downstream of e''_*. Since d_2 is reachable from any $e' \in \overline{S}_1 \cap \overline{S}_2$, we fix an edge e'_* in $\overline{S}_1 \cap \overline{S}_2$ and take an arbitrary path $P_{e'_* d_2}$ from e'_* to d_2. Then, we can create a path $s_1 P_{s_1 d_3} e'_* P_{e'_* d_2} d_2$ from s_1 to d_2 not passing through e''_*, which violates the assumption of $e''_* \in (\overline{D}_1 \cap \overline{D}_2) \subseteq \overline{D}_2$. Similarly, it can be shown that any path from s_2 to d_3 must pass through e''_*.

Then, what we have shown is that e'_* belongs to $\cap_{i=1}^{3} \left(\overline{S}_i \cap \overline{D}_i\right)$. But this contradicts the assumption of the existence of e''_* such that $e''_* \prec e'$ for all $e' \in \overline{S}_1 \cap \overline{S}_2$, because e''_* cannot be in the upstream of $e''_* \in \cap_{i=1}^{3} \left(\overline{S}_i \cap \overline{D}_i\right) \subseteq \overline{S}_1 \cap \overline{S}_2$, i.e., itself. Thus, the proof of this lemma is complete. ∎

Lemma 6: Consider four indices i, j_1, j_2, and j_3 taking values in $\{1, 2, 3\}$ for which the values of j_1, j_2 and j_3 must be distinct but i may be equal to one of j_1, j_2 and j_3. If $\overline{S}_{i;j_1 \cap j_2} \neq \emptyset$ and $\overline{S}_{i;j_1 \cap j_3} \neq \emptyset$, then we have $\overline{S}_{i;j_1 \cap j_2} \cap \overline{S}_{i;j_1 \cap j_3} \neq \emptyset$, $\overline{S}_{i;j_2 \cap j_3} \neq \emptyset$, and $\overline{S}_i \neq \emptyset$.

Proof: Without loss of generality, let $i = 1$, $j_1 = 1$, $j_2 = 2$, and $j_3 = 3$. Suppose that $\overline{S}_{1;1\cap 2} \neq \emptyset$ and $\overline{S}_{1;1\cap 3} \neq \emptyset$. First notice that all the edges in $\overline{S}_{1;1\cap 2}$ (resp. $\overline{S}_{1;1\cap 3}$) belong to $\mathsf{1cut}(s_1; d_1)$ by definition. Thus if we take an arbitrary path $P_{s_1 d_1}$ from s_1 to d_1, then any $e' \in \overline{S}_{1;1\cap 2}$ and any $e'' \in \overline{S}_{1;1\cap 3}$ are used by $P_{s_1 d_1}$. In this setting, we prove this lemma by contradiction.

Suppose that $\overline{S}_{1;1\cap 2} \cap \overline{S}_{1;1\cap 3} = \emptyset$. Then, for any $e' \in \overline{S}_{1;1\cap 2}$ and any $e'' \in \overline{S}_{1;1\cap 3}$, it must be either $e' \prec e''$ or $e' \succ e''$. Take the most downstream edges $e'_* \in \overline{S}_{1;1\cap 2}$ and $e''_* \in \overline{S}_{1;1\cap 3}$, respectively, and assume without loss of generality that $e'_* \succ e''_*$. If all paths from s_1 to d_2 (who pass through e'_* by definition) pass through e''_*, then e''_* will belong to $\mathsf{1cut}(s_1; d_2)$, which violates the assumption that $\overline{S}_{1;1\cap 2} \cap \overline{S}_{1;1\cap 3} = \emptyset$. Thus, there exists a path $P_{s_1 d_2}$ from s_1 to d_2 passing through e'_* but not through e''_*. Then, we can create a path $s_1 P_{s_1 d_2} e'_* P_{s_1 d_1} d_1$ from s_1 to d_1 not passing though e''_*, which violates $e''_* \in \overline{S}_{1;1\cap 3} \subseteq \mathsf{1cut}(s_1; d_1)$. The case of $e'_* \prec e''_*$ can be proven similarly. Therefore, we have $\overline{S}_{1;1\cap 2} \cap \overline{S}_{1;1\cap 3} \neq \emptyset$, which implies that $\overline{S}_{1;2\cap 3} \neq \emptyset$ and $\overline{S}_1 \neq \emptyset$. ∎

Remark: If we swap the roles of $\overline{S}$ and $\overline{D}$, and change the upstream/downstream sign, then the corresponding statements from Lemma 1 to Lemma 6 still hold. For example, Lemma 2 also implies $(\overline{D}_i \cap \overline{D}_j) \subseteq \overline{S}_k$.

We also prove some relationship between the channel gains on $G_{3\text{ANA}}$ and the 1-edge cuts.

Lemma 7: Given a $G_{3\text{ANA}}$, consider the corresponding channel gains as defined in Section II-D. Consider three indices i, j_1, and j_2 taking values in $\{1, 2, 3\}$ for which the values of j_1 and j_2 must be distinct but i may or may not be equal to one of j_1 and j_2. If $\mathsf{GCD}(m_{ij_1}, m_{ij_2}) \not\equiv 1$, then $\overline{S}_{i;j_1 \cap j_2} \neq \emptyset$. (Similarly, if $\mathsf{GCD}(m_{j_1 i}, m_{j_2 i}) \not\equiv 1$, then $\overline{D}_{i;j_1 \cap j_2} \neq \emptyset$.)

Proof: Consider three indices i, j_1, and j_2 taking values in $\{1, 2, 3\}$ for which the values of j_1 and j_2 must be distinct but i may or may not be equal to one of j_1 and j_2. By Corollary 2, we know $m_{ij_1} \not\equiv m_{ij_2}$. Thus, $\mathsf{GCD}(m_{ij_1}, m_{ij_2}) \not\equiv 1$ implies that they are reducible and can be expressed as the product of irreducibles each corresponding to the closest 1-edge cuts in $\mathsf{1cut}(s_1; d_1)$ and $\mathsf{1cut}(s_1; d_2)$, respectively, by Theorem 3. Since they share their irreducible(s) of their own, there exists an edge $e \in \overline{S}_{i;j_1 \cap j_2}$. The case for $\mathsf{GCD}(m_{j_1 i}, m_{j_2 i}) \not\equiv 1$ can be proven similarly to imply that $\overline{D}_{i;j_1 \cap j_2} \neq \emptyset$. ∎

B. Characterizing the GTC of $L(\underline{\mathbf{x}})\not\equiv R(\underline{\mathbf{x}})$

We first prove the following graph-theoretic condition which implies $L(\underline{\mathbf{x}})\not\equiv R(\underline{\mathbf{x}})$.

Proposition 1: If there exists a distinct pair $i,j\in\{1,2,3\}$ and $i\neq j$ satisfying both $\overline{S}_i\cap\overline{S}_j\neq\emptyset$ and $\overline{D}_i\cap\overline{D}_j\neq\emptyset$ on a given G_{3ANA}, then we have $L(\underline{\mathbf{x}})\equiv R(\underline{\mathbf{x}})$ on G_{3ANA}.

Proof: Without loss of generality, suppose $\overline{S}_1\cap\overline{S}_2\neq\emptyset$ and $\overline{D}_1\cap\overline{D}_2\neq\emptyset$ (i.e., $i=1$ and $j=2$). By Lemma 5, we can find two edges $e_1\in\overline{S}_1\cap\overline{S}_2$ and $e_2\in\overline{D}_1\cap\overline{D}_2$ such that $e_1\preceq e_2$. Also note that $e_1\in\overline{D}_3$ and $e_2\in\overline{S}_3$ by Lemma 2. Then by Theorem 3, the channel gains $m_{ij}(\underline{\mathbf{x}}), i\neq j$ can be expressed by (we drop the input argument $\underline{\mathbf{x}}$ for simplicity):

$$\begin{aligned}
&m_{13}=m_{e_{s_1};e_1}m_{e_1;e_{d_3}} && m_{12}=m_{e_{s_1};e_1}m_{e_1;e_2}m_{e_2;e_{d_2}}\\
&m_{32}=m_{e_{s_3};e_2}m_{e_2;e_{d_2}} && m_{23}=m_{e_{s_2};e_1}m_{e_1;e_{d_3}}\\
&m_{21}=m_{e_{s_2};e_1}m_{e_1;e_2}m_{e_2;e_{d_1}} && m_{31}=m_{e_{s_3};e_2}m_{e_2;e_{d_1}}
\end{aligned}$$

where the expressions of m_{12} and m_{21} are derived based on Theorem 3, and the facts that $e_1\preceq e_2$ and both e_1 and e_2 belong to $\mathsf{1cut}(s_1;d_2)\cap\mathsf{1cut}(s_2;d_1)$. By the definition of $L=m_{13}m_{32}m_{21}$ and $R=m_{12}m_{23}m_{31}$, we can easily verify that the G_{3ANA} of interest satisfies $L\equiv R$. ■

Remark: In the example of Fig. 1, one can easily see that $e_1\in\overline{S}_1\cap\overline{S}_2$ and $e_2\in\overline{D}_1\cap\overline{D}_2$. Hence, Proposition 1 proves that the example network of Fig. 1 satisfies $L(\underline{\mathbf{x}})\equiv R(\underline{\mathbf{x}})$ without actually computing $L(\underline{\mathbf{x}})$ and $R(\underline{\mathbf{x}})$.

For the following, we will show that the graph-theoretic condition identified in Proposition 1 is also necessary for $L(\underline{\mathbf{x}})\equiv R(\underline{\mathbf{x}})$. Before proceeding, we prove the following graph-theoretic properties about the channel gains conditioning on $L(\underline{\mathbf{x}})\equiv R(\underline{\mathbf{x}})$. We again drop the input argument $\underline{\mathbf{x}}$ for simplicity.

Lemma 8: If the G_{3ANA} of interest satisfies $L\equiv R$, then $\overline{S}_i\neq\emptyset$ and $\overline{D}_j\neq\emptyset$ for all i and j, respectively.

Proof: Suppose the G_{3ANA} of interest satisfies $L\equiv R$. We will first prove an intermediate result: $\mathsf{EC}(\mathsf{head}(e_{s_i});\mathsf{tail}(e_{d_j}))=1$ for all $i\neq j$.

We prove the intermediate result by contradiction. Suppose there exists a pair $i\neq j$ such that $\mathsf{EC}(\mathsf{head}(e_{s_i});\mathsf{tail}(e_{d_j}))\neq1$. Without loss of generality, assume $\mathsf{EC}(\mathsf{head}(e_{s_1});\mathsf{tail}(e_{d_2}))\neq1$ (i.e., $i=1$ and $j=2$). Since we are focusing on a 3-unicast ANA network, we must have $\mathsf{EC}(\mathsf{head}(e_{s_1});\mathsf{tail}(e_{d_2}))\geq2$. By Theorem 3, m_{12} is irreducible. However, since $L\equiv R$, m_{12} must be a factor of (at least) one of the three polynomials m_{13}, m_{32}, or m_{21}. This, however, contradicts Corollary 2. As a result, $\mathsf{EC}(\mathsf{head}(e_{s_i});\mathsf{tail}(e_{d_j}))=1$ for all $i\neq j$. This shows that for all $i\neq j$, we can decompose the channel gain m_{ij} by

$$m_{ij}=m_{e_{s_i};e_1^{ij}}\left(\prod_{k=1}^{N_{ij}-1}m_{e_k^{ij};e_{k+1}^{ij}}\right)m_{e_{N_{ij}}^{ij};e_{d_j}}\tag{17}$$

where N_{ij} is the number of 1-edge cuts separating s_i and d_j; $\{e_k^{ij}\}_{k=1}^{N_{ij}}$ list all those 1-edge cuts from the most upstream to the most downstream one but not counting e_{s_i} and e_{d_j}; and $m_{e_{s_i};e_1^{ij}}$, $\{m_{e_k^{ij};e_{k+1}^{ij}}\}_{k=1}^{N_{ij}-1}$, and $m_{e_{N_{ij}}^{ij};e_{d_j}}$ are all irreducible polynomials and no two of them are equivalent to each other.

For the following, we show that for any three distinct index values i_1, i_2, and j, we must have $\mathsf{GCD}(m_{i_1j},m_{i_2j})\not\equiv1$. We prove this statement by contradiction. Suppose, say $\mathsf{GCD}(m_{21},m_{31})\equiv1$. Then by the assumption that $L\equiv R$, we must have $\mathsf{GCD}(m_{21},m_{12}m_{23})=m_{21}$. As a result, all the irreducible factors of m_{21}, see (17), must also be factors of $m_{12}m_{23}$. For example, the irreducible factor $m_{e_{N_{21}}^{21};e_{d_1}}$ of m_{21} must be a factor of $m_{12}m_{23}$. By Theorem 3, this is possible only when e_{d_1} is a 1-edge cut separating either $\{s_1\}$ and $\{d_2\}$ or $\{s_2\}$ and $\{d_3\}$. However, this contradicts the 3-unicast ANA network assumption that $|\mathsf{Out}(d_1)|=0$.

By Lemma 7 and the above discussion, we must have $\overline{D}_j\neq\emptyset$ for all j. By repeating the symmetric arguments for $\overline{S}_i$, the proof is complete. ■

We are now ready to prove the necessity counterpart of Proposition 1.

Proposition 2: If the G_{3ANA} of interest satisfies $L(\underline{\mathbf{x}})\equiv R(\underline{\mathbf{x}})$, then there exists a distinct pair $i,j\in\{1,2,3\}$ and $i\neq j$ satisfying both $\overline{S}_i\cap\overline{S}_j\neq\emptyset$ and $\overline{D}_i\cap\overline{D}_j\neq\emptyset$.

Proof: Suppose the G_{3ANA} of interest satisfies $L\equiv R$. By Lemma 8, we knew that each channel gain m_{ij} ($i\neq j$) in the expression of $L\equiv R$ can be expressed as in (17). In addition, $\overline{S}_i\neq\emptyset$ and $\overline{D}_j\neq\emptyset$ for all i and j. For the following, we consider two cases.

Case 1 when $\overline{S}_i\cap\overline{D}_j=\emptyset$ for some $i\neq j$:

Without loss of generality, assume $\overline{S}_2\cap\overline{D}_1=\emptyset$ (i.e., $i=2$ and $j=1$). Let e_2^* denote the most downstream edge in $\overline{S}_2$ and let e_1^* denote the most upstream edge in $\overline{D}_1$. Since $\overline{S}_2\cap\overline{D}_1=\emptyset$, edge e_2^* must not be in $\overline{D}_1$. By Lemma 3, we must have $e_2^*\prec e_1^*$.

For the following, we will prove that $e_2^*\in\mathsf{1cut}(s_1;d_2)$. We first notice that by definition, $e_2^*\in\overline{S}_2\subseteq\mathsf{1cut}(s_2;d_1)$ and $e_1^*\in\overline{D}_1\subseteq\mathsf{1cut}(s_2;d_1)$. Therefore, when rewriting m_{21} by (17), both e_2^* and e_1^* must participate in the form of $e_2^*=e_{\hat{N}_{21}}^{21}$ and $e_1^*=e_{\tilde{N}_{21}}^{21}$ for two integers $\hat{N}_{21}$ and $\tilde{N}_{21}$ satisfying $1\leq\hat{N}_{21}<\tilde{N}_{21}\leq N_{21}$. Define temporarily:

$$m_{e_2^*;e_1^*}=\prod_{k=\hat{N}_{21}}^{\tilde{N}_{21}-1}m_{e_k^{21};e_{k+1}^{21}}\tag{18}$$

Note that by our construction $e_2^*\prec e_1^*$ and thus $m_{e_2^*;e_1^*}\not\equiv1$.

We now claim that $\mathsf{GCD}(m_{e_2^*;e_1^*},m_{23}m_{31})\equiv1$, i.e., $m_{23}m_{31}$ cannot contain any irreducible factor of $m_{e_2^*;e_1^*}$. We will prove this claim by contradiction. Suppose m_{23} contains any irreducible factor of $m_{e_2^*;e_1^*}$, say $m_{e_{\tilde{k}}^{21};e_{\tilde{k}+1}^{21}}$ where $\hat{N}_{21}\leq\tilde{k}\leq\tilde{N}_{21}-1$. Then, by Theorem 3, $e_{\tilde{k}+1}^{21}$ must belong to $\mathsf{1cut}(s_2;d_3)$. Note that by our construction of (18), we must have $e_{\tilde{k}+1}^{21}\in\mathsf{1cut}(s_2;d_1)$ as well. Jointly it implies that $e_{\tilde{k}+1}^{21}\in\overline{S}_2$. This, however, contradicts the assumption that $e_2^*=e_{\hat{N}_{21}}^{21}\prec e_{\tilde{k}+1}^{21}$ is the most downstream edge in $\overline{S}_2$. As a result, m_{23} must not contain any irreducible factor of $m_{e_2^*;e_1^*}$. By a symmetric argument, we can also show that m_{31} must not contain any irreducible factor of $m_{e_2^*;e_1^*}$. The proof of the claim $\mathsf{GCD}(m_{e_2^*;e_1^*},m_{23}m_{31})\equiv1$ is complete. Since the assumption of $L\equiv R$ implies that $\mathsf{GCD}(m_{e_2^*;e_1^*},R)=m_{e_2^*;e_1^*}$, we must have $\mathsf{GCD}(m_{e_2^*;e_1^*}, m_{12}) = m_{e_2^*;e_1^*}$. Also note that $e_{s_1} \neq e_2^*$ otherwise s_1 is a downstream node of s_2, which contradicts the assumption $|\mathsf{In}(s_1)|=0$. Similarly, we have $e_{d_2} \neq e_1^*$. By Theorem 3, both e_1^* and e_2^* must belong to $\mathsf{1cut}(s_1;d_2)$.

For the following, we will prove that $e_2^* \in \mathsf{1cut}(s_1;d_3)$. To this end, we construct the following paths to show further arguments: A pair of edge-disjoint paths $P_{s_2\mathsf{tail}(e_1^*)}$ and $P_{s_3\mathsf{tail}(e_1^*)}$, and two other paths $P_{e_1^*d_1}$ and $P_{e_1^*d_2}$. The pair of edge-disjoint paths $P_{s_2\mathsf{tail}(e_1^*)}$ and $P_{s_3\mathsf{tail}(e_1^*)}$ always exist since e_1^* is the most upstream edge in $\overline{D}_1$. Without loss of generality,[3] we also assume that these two paths are not only edge-disjoint but also interior-vertex-disjoint. The third path $P_{e_1^*d_1}$ exists since d_1 can be reached by s_2 and $e_1^* \in \mathsf{1cut}(s_2;d_1)$. The fourth path $P_{e_1^*d_2}$ exists since d_2 can be reached by s_1 and we have just proven $e_1^* \in \mathsf{1cut}(s_1;d_2)$. Note that any $P_{s_2\mathsf{tail}(e_1^*)}$ path must use $e_2^* \in \overline{S}_2$. We construct the above paths in an arbitrary way. We will fix these four paths throughout the following discussion.

Consider any path $P_{s_1d_3}$ from s_1 to d_3. Suppose $P_{s_1d_3}$ does not use e_2^*. Consider the subgraph G' induced by $P_{s_1d_3}$ and the previously constructed four paths $P_{s_2\mathsf{tail}(e_1^*)}$, $P_{s_3\mathsf{tail}(e_1^*)}$, $P_{e_1^*d_1}$, and $P_{e_1^*d_2}$. We will argue that in the subgraph G', d_2 cannot be reached from s_1. The reason is as follows. Let $s_2P_{s_2\mathsf{tail}(e_1^*)}\mathsf{tail}(e_2^*)$ denote the path segment of $P_{s_2\mathsf{tail}(e_1^*)}$ from s_2 to $\mathsf{tail}(e_2^*)$. The $P_{s_1d_3}$ path must be vertex-disjoint from the path segment $s_2P_{s_2\mathsf{tail}(e_1^*)}\mathsf{tail}(e_2^*)$. Otherwise (if they share a vertex v), there exists a path $s_2P_{s_2\mathsf{tail}(e_1^*)}vP_{s_1d_3}d_3$ from s_2 to d_3 without using e_2^*. This contradicts the assumption that $e_2^* \in \overline{S}_2$.

Consider all the nodes u in the union of $\mathsf{head}(e_2^*)P_{s_2\mathsf{tail}(e_1^*)}$, $P_{s_3\mathsf{tail}(e_1^*)}$, $P_{e_1^*d_1}$, and $P_{e_1^*d_2}$ that can reach d_2. We argue that none of these u nodes participates in $P_{s_1d_3}$. Suppose not, then there exists a path from s_1 to u along $P_{s_1d_3}$ and then from u to d_2 along the union of $\mathsf{head}(e_2^*)P_{s_2\mathsf{tail}(e_1^*)}$, $P_{s_3\mathsf{tail}(e_1^*)}$, $P_{e_1^*d_1}$, and $P_{e_1^*d_2}$. Note that such path connects s_1 and d_2 without using e_2^*, which contradicts the fact that $e_2^* \in \mathsf{1cut}(s_1;d_2)$ we have just established. The above discussion thus proves if we walk from s_1 along path $P_{s_1d_3}$, then we will never reach d_2 even if we are allowed to switch to the other 4 existing paths at any time. Therefore, in the subgraph G' induced by the 5 paths discussed herein, d_2 cannot be reached from s_1.

However, the above result implies that in the subgraph G', $m_{12}=0$ and thus $R=0$. By noticing that in the subgraph G', d_1 is reachable from s_2, d_2 is reachable from s_3, and d_3 is reachable from s_1, we also have $L \neq 0$. This implies that $L \not\equiv R$ in G', which contradicts Theorem 2. The above argument thus shows that any path $P_{s_1d_3}$ from s_1 to d_3 must use e_2^*. Therefore, e_2^* is a 1-edge cut separating $\{s_1\}$ and $\{d_3\}$.

Thus far, we have proven $e_2^* \in \mathsf{1cut}(s_1;d_2)$ and $e_2^* \in \mathsf{1cut}(s_1;d_3)$. Jointly, we have $e_2^* \in \overline{S}_1$. Recall that e_2^* was chosen as one edge in $\overline{S}_2$. Therefore, $\overline{S}_1 \cap \overline{S}_2 \neq \emptyset$. By symmetry, we can also prove that $e_1^* \in \overline{D}_1 \cap \overline{D}_2$ and thus $\overline{D}_1 \cap \overline{D}_2 \neq \emptyset$. The proof of Case 1 is complete.

Case 2 when $\overline{S}_i \cap \overline{D}_j \neq \emptyset$ for all $i \neq j$:

By Lemma 4, we must have $\overline{S}_i \cap \overline{S}_j \neq \emptyset$ and $\overline{D}_i \cap \overline{D}_j \neq \emptyset$ $\forall\, i \neq j$. The proof of *Case 2* is complete. ∎

C. The Graph-Theoretic Condition of the Feasibility of the 3-unicast ANA Scheme with $n=1$

Propositions 1 and 2 provide the graph-theoretic condition that characterizes whether the $G_{3\text{ANA}}$ of interest satisfies (10) or not. However, to ensure the feasibility of the 3-unicast ANA scheme, $\mathbf{h}_i^{(n)}(\underline{\mathbf{x}})$ must be linearly independent on $G_{3\text{ANA}}$ for all i. In this subsection, we prove a graph-theoretic condition characterizing the linear independence of $\mathbf{h}_i^{(n)}(\underline{\mathbf{x}})$ for the simplest case of $n=1$.

Consider the following graph-theoretic conditions:

$$\overline{S}_i \cap \overline{S}_j = \emptyset \text{ or } \overline{D}_i \cap \overline{D}_j = \emptyset \quad \forall \{i,j\} \subset \{1,2,3\} \tag{19}$$

$$\mathsf{EC}(\{s_1,s_2\};\{d_1,d_3\}) \geq 2,\ \mathsf{EC}(\{s_1,s_3\};\{d_1,d_2\}) \geq 2 \tag{20}$$

$$\mathsf{EC}(\{s_1,s_2\};\{d_2,d_3\}) \geq 2,\ \mathsf{EC}(\{s_2,s_3\};\{d_1,d_2\}) \geq 2 \tag{21}$$

$$\mathsf{EC}(\{s_1,s_3\};\{d_2,d_3\}) \geq 2,\ \mathsf{EC}(\{s_2,s_3\};\{d_1,d_3\}) \geq 2 \tag{22}$$

Proposition 3: The 3-unicast ANA with $n=1$ is feasible on the $G_{3\text{ANA}}$ of interest *if and only if* $G_{3\text{ANA}}$ satisfies (19-22).

Proof: Given a 3-unicast ANA network, the first feasibility condition of the 3-unicast ANA scheme, (10), is equivalent to (19) by Proposition 1 and 2. By Theorem 1, $\mathsf{det}(\mathbf{S}_i^{(n)}) \neq 0$ with high probability whenever the set of polynomials $\mathbf{h}_i^{(n)}(\underline{\mathbf{x}})$ is linearly independent for the given $i \in \{1,2,3\}$. Furthermore, by Corollary 1, $G_{3\text{ANA}}$ satisfies both $m_{11}m_{23} \not\equiv m_{21}m_{13}$ and $m_{11}m_{32} \not\equiv m_{31}m_{12}$ *if and only if* (20) holds on $G_{3\text{ANA}}$. Similarly, jointly the two conditions $m_{22}m_{13} \not\equiv m_{12}m_{23}$ and $m_{22}m_{31} \not\equiv m_{32}m_{21}$ are equivalent to (21), and jointly the two conditions $m_{33}m_{12} \not\equiv m_{13}m_{32}$ and $m_{33}m_{21} \not\equiv m_{23}m_{31}$ are equivalent to (22).

Thus for a $G_{3\text{ANA}}$ satisfying $L \not\equiv R$, we need to show that,

(a) $\mathbf{h}_1^{(n)}(\underline{\mathbf{x}})$ is linearly independent *if and only if* $m_{11}m_{23} \not\equiv m_{21}m_{13}$ and $m_{11}m_{32} \not\equiv m_{31}m_{12}$.
(b) $\mathbf{h}_2^{(n)}(\underline{\mathbf{x}})$ is linearly independent *if and only if* $m_{22}m_{13} \not\equiv m_{12}m_{23}$ and $m_{22}m_{31} \not\equiv m_{32}m_{21}$.
(c) $\mathbf{h}_3^{(n)}(\underline{\mathbf{x}})$ is linearly independent *if and only if* $m_{33}m_{12} \not\equiv m_{13}m_{32}$ and $m_{33}m_{21} \not\equiv m_{23}m_{31}$.

For the simplest scenario of $n=1$, we now prove (a). The proof for (b) and (c) will be followed similarly. When $n=1$, we have

$$\begin{aligned}\mathbf{h}_1^{(1)}(\underline{\mathbf{x}}) &= \{\, m_{11}\mathbf{v}_1^{(1)},\ m_{21}\mathbf{v}_2^{(1)} \,\} \\ &= \{\, m_{11}m_{23}m_{32}R,\ m_{11}m_{23}m_{32}L,\ m_{21}m_{13}m_{32}R \,\}\end{aligned}$$

Proof of (a), $\Rightarrow$: We prove this direction by contradiction. Suppose $G_{3\text{ANA}}$ satisfies either $m_{11}m_{23} \equiv m_{21}m_{13}$ or $m_{11}m_{32} \equiv m_{31}m_{12}$ or both. If $m_{11}m_{23} \equiv m_{21}m_{13}$, then we notice that $m_{11}m_{23}m_{32}R \equiv m_{21}m_{13}m_{32}R$ and $\mathbf{h}_1^{(1)}(\underline{\mathbf{x}})$ is thus linearly dependent. If $m_{11}m_{32} \equiv m_{31}m_{12}$, then we notice that $m_{11}m_{23}m_{32}L \equiv m_{21}m_{13}m_{32}R$ and $\mathbf{h}_1^{(1)}(\underline{\mathbf{x}})$ is thus linearly dependent.

Proof of (a), $\Leftarrow$: Suppose that $L \not\equiv R$, $m_{11}m_{23} \not\equiv m_{21}m_{13}$, and $m_{11}m_{32} \not\equiv m_{31}m_{12}$ hold on $G_{3\text{ANA}}$. We prove this direction by contradiction. Suppose also that $\mathbf{h}_1^{(1)}(\underline{\mathbf{x}})$ is linearly

[3]This technical subtlety can be resolved by considering the *line graph* presentation of the network instead.

dependent on $G_{3\text{ANA}}$. Then by definition, there exists some coefficients $\{\alpha_0, \alpha_1, \beta_0\}$ in $\mathbb{F}$ except all zeros such that

$$\begin{aligned}\alpha_0 m_{11}m_{12}m_{23}m_{31} + \alpha_1 m_{11}m_{13}m_{32}m_{21} \\ = c_{-1}\beta_0 m_{12}m_{13}m_{21}m_{31}\end{aligned} \tag{23}$$

where c_{-1} is the additive inverse of 1 in $\mathbb{F}$. If at least one of those coefficients is zero, then one can easily show that (23) violates the assumption that $L \not\equiv R$, $m_{11}m_{23} \not\equiv m_{21}m_{13}$, and $m_{11}m_{32} \not\equiv m_{31}m_{12}$ hold on $G_{3\text{ANA}}$. Thus, we only need to consider the case when those coefficients are all non-zero. Since $m_{ij} \not\equiv 1\ \forall i, j$ on the 3-unicast ANA network $G_{3\text{ANA}}$, (23) further gives

$$\mathsf{GCD}(\, m_{11}m_{12}m_{23}m_{31},\ m_{13}m_{21}) = m_{13}m_{21} \tag{24}$$

$$\mathsf{GCD}(\, m_{11}m_{13}m_{32}m_{21},\ m_{12}m_{31}) = m_{12}m_{31} \tag{25}$$

$$\mathsf{GCD}(\, m_{12}m_{13}m_{21}m_{31},\ m_{11}) = m_{11} \tag{26}$$

By Corollary 2, we know that any channel gain cannot contain the other. Thus, the divisors in the above equations, i.e., m_{13}, m_{21}, m_{12}, m_{31}, and m_{11}, must be all reducible. By Theorem 3, they can be expressed by the product of irreducibles as in (17), respectively. Then we now show that $\mathsf{GCD}(\, m_{23}, m_{21}) \not\equiv 1$ by contradiction.

Suppose that $\mathsf{GCD}(\, m_{23}, m_{21}) \equiv 1$. Then from (24), it must be $\mathsf{GCD}(\, m_{11}m_{12}m_{31}, m_{21}) = m_{21}$. But the fact that the irreducible factor $m_{e_{s_2};e_1^{21}}$ of m_{21} is contained by $m_{11}m_{12}m_{31}$ as a factor implies that $m_{e_{s_2};e_1^{21}}$ is one of the irreducible factors of at least one channel gain among $\{m_{11},\, m_{12},\, m_{31}\}$. This further implies that s_2 must be in the downstream of s_1 or s_3, which violates the 3-unciast ANA network $G_{3\text{ANA}}$ such that $|\mathsf{In}(s_2)| = 0$. Thus, we have $\mathsf{GCD}(\, m_{23}, m_{21}) \not\equiv 1$.

Similarly, $\mathsf{GCD}(\, m_{23}, m_{13}) \not\equiv 1$ can be derived from (24). Applying Lemma 7, we have $\overline{S}_{2;1\cap 3} \neq \emptyset$ and $\overline{D}_{3;1\cap 2} \neq \emptyset$, i.e., $\overline{S}_2 \neq \emptyset$ and $\overline{D}_3 \neq \emptyset$. Following similarly from (25) using $\{m_{32},\, m_{12},\, m_{31}\}$ and applying Lemma 7, both $\overline{S}_3 \neq \emptyset$ and $\overline{D}_2 \neq \emptyset$ are also derived.

Moreover, using similar arguments, we can deduce from the above three equations such that

$$\begin{aligned} &\mathsf{GCD}(\, m_{11}m_{13},\ m_{12}) \not\equiv 1 \qquad &&\mathsf{GCD}(\, m_{11}m_{21},\ m_{31}) \not\equiv 1 \\ &\mathsf{GCD}(\, m_{11}m_{12},\ m_{13}) \not\equiv 1 \qquad &&\mathsf{GCD}(\, m_{11}m_{31},\ m_{21}) \not\equiv 1 \\ &\mathsf{GCD}(\, m_{12}m_{13},\ m_{11}) \not\equiv 1 \qquad &&\mathsf{GCD}(\, m_{21}m_{31},\ m_{11}) \not\equiv 1 \end{aligned}$$

otherwise s_1 must be in the downstream of s_2 or s_3, or d_1 must be in the upstream of d_2 or d_3, which violates the definition of the 3-unicast ANA network. Then, the left three equations imply that at least two pairs out of $\{m_{11}, m_{12}, m_{13}\}$ must satisfy $\mathsf{GCD}(\,\cdot\,,\cdot\,) \not\equiv 1$, and applying Lemmas 7 and 6 further result in $\overline{S}_1 \neq \emptyset$. Similarly, $\overline{D}_1 \neq \emptyset$ is derived from the right three equations.

Therefore, what we have extracted from (24), (25), and (26) is that $\overline{S}_i \neq \emptyset$ and $\overline{D}_i \neq \emptyset$ for all $i = 1, 2, 3$. For the following, we will further deduce that $\overline{S}_2 \cap \overline{S}_3 \neq \emptyset$ and $\overline{D}_2 \cap \overline{D}_3 \neq \emptyset$, which contradicts the assumption of $L \not\equiv R$ by Propositions 1 and 2.

The proof for $\overline{S}_2 \cap \overline{S}_3 \neq \emptyset$ from (24), (25), and (26):

We prove this claim by contradiction. Suppose $\overline{S}_2 \cap \overline{S}_3 = \emptyset$. By Lemma 4, this is equivalent to $\overline{D}_1 \cap \overline{S}_2 = \emptyset$ or $\overline{D}_1 \cap \overline{S}_3 = \emptyset$.

Case 1.1 when $\overline{D}_1 \cap \overline{S}_2 = \emptyset$ and $\overline{D}_1 \cap \overline{S}_3 = \emptyset$: Take the most downstream edge among $\overline{S}_2$ (resp. $\overline{S}_3$) and denote it as e_2^* (resp. e_3^*). By $\overline{S}_2 \cap \overline{S}_3 = \emptyset$, definitely $e_2^* \neq e_3^*$. Take also the most upstream edge among $\overline{D}_1$ and denote it as e_1^*. By Lemma 3, it is for sure that $e_1^* \succ e_2^*$ and $e_1^* \succ e_3^*$. But both e_2^* and e_3^* might be either $e_2^* \prec e_3^*$, $e_2^* \succ e_3^*$, or not reachable from each other. By definition, any path from s_2 to d_3 (resp. d_1) must pass through e_2^* (resp. both e_2^* and e_1^*), and we can express m_{23} and m_{21} as $m_{23} = m_{e_{s_2};e_2^*} m_{e_2^*;e_{d_3}}$ and $m_{21} = m_{e_{s_2};e_2^*} m_{e_2^*;e_1^*} m_{e_1^*;e_{d_1}}$, respectively. Then, it must be $\mathsf{GCD}(\, m_{e_2^*;e_{d_3}},\, m_{e_2^*;e_1^*} m_{e_1^*;e_{d_1}}) \equiv 1$ otherwise there must exist an edge $e \in \overline{S}_2$ such that $e \succ e_2^*$ by Lemma 7. This e violates that e_2^* is the most downstream edge among $\overline{S}_2$. Thus, we can further deduce that $\mathsf{GCD}(\, m_{23},\, m_{e_2^*;e_1^*}) \equiv 1$.

Similarly, any path from s_3 to d_2 (resp. d_1) must pass through e_3^* (resp. both e_3^* and e_1^*) by definition. Thus we can express m_{32} and m_{31} as $m_{32} = m_{e_{s_3};e_3^*} m_{e_3^*;e_{d_2}}$ and $m_{31} = m_{e_{s_3};e_3^*} m_{e_3^*;e_1^*} m_{e_1^*;e_{d_1}}$, respectively. Then, it must be $\mathsf{GCD}(\, m_{e_3^*;e_{d_2}},\, m_{e_3^*;e_1^*} m_{e_1^*;e_{d_1}}) \equiv 1$ otherwise there must exist an edge $e \in \overline{S}_3$ such that $e \succ e_3^*$ by Lemma 7 and this e violates that e_3^* is the most downstream edge among $\overline{S}_3$. Thus, we can further deduce that $\mathsf{GCD}(\, m_{32},\, m_{e_3^*;e_1^*}) \equiv 1$.

Moreover, notice that both m_{21} and m_{31} share the same factor $m_{e_1^*;e_{d_1}}$ from e_1^* to d_1. Then, we must have $\mathsf{GCD}(\, m_{e_{s_2};e_2^*} m_{e_2^*;e_1^*},\, m_{e_{s_3};e_3^*} m_{e_3^*;e_1^*}) \equiv 1$, otherwise there exists an edge $e \in \overline{D}_1$ such that $e \prec e_1^*$ by Lemma 7 and this e violates that e_1^* is the most upstream edge among $\overline{D}_1$. Then, this tells us that we can always find a pair of edge-disjoint paths $P_{s_2 \mathsf{tail}(e_1^*)}$ (through e_2^*) and $P_{s_3 \mathsf{tail}(e_1^*)}$ (through e_3^*). In addition, we have $\mathsf{GCD}(\, m_{e_2^*;e_1^*},\, m_{e_3^*;e_1^*}) \equiv 1$, $\mathsf{GCD}(\, m_{31},\, m_{e_2^*;e_1^*}) \equiv 1$, and $\mathsf{GCD}(\, m_{21},\, m_{e_3^*;e_1^*}) \equiv 1$.

Since we know that $\mathsf{GCD}(\, m_{11}m_{12}m_{23}m_{31},\, m_{21}) = m_{21}$ from (24) and $\mathsf{GCD}(\, m_{11}m_{13}m_{32}m_{21},\, m_{31}) = m_{31}$ from (25), if we apply the above results, then we have

$$\mathsf{GCD}(\, m_{11}m_{12},\ m_{e_2^*;e_1^*}) = m_{e_2^*;e_1^*} \tag{27}$$

$$\mathsf{GCD}(\, m_{11}m_{13},\ m_{e_3^*;e_1^*}) = m_{e_3^*;e_1^*} \tag{28}$$

For the following, we will first prove that both $\mathsf{GCD}(\, m_{11},\, m_{e_2^*;e_1^*}) \not\equiv 1$ and $\mathsf{GCD}(\, m_{11},\, m_{e_3^*;e_1^*}) \not\equiv 1$ cannot hold simultaneously. Since $m_{e_2^*;e_1^*}$ is the factor of m_{21}, Lemma 7 tells us that the former implies we have a collection of edges in $\overline{D}_{1;1\cap 2}$, which are located in-between e_2^* and e_1^*. Similarly, since $m_{e_3^*;e_1^*}$ is the factor of m_{31}, Lemma 7 gives us that the latter implies we have a collection of edges in $\overline{D}_{1;1\cap 3}$, which are located in-between e_3^* and e_1^*. But if we apply Lemma 6, then there must exist an edge $e' \in \overline{D}_{1;1\cap 2}$ where $e_2^* \preceq e' \prec e_1^*$ and an edge $e'' \in \overline{D}_{1;1\cap 3}$ where $e_3^* \preceq e'' \prec e_1^*$ such that $e' = e''$. By noticing that $m_{e_2^*;e_1^*}$ and $m_{e_3^*;e_1^*}$ can be rewritten as $m_{e_2^*;e_1^*} = m_{e_2^*;e'} m_{e';e_1^*}$ and $m_{e_3^*;e_1^*} = m_{e_3^*;e''} m_{e'';e_1^*}$, respectively, we can further deduce that $\mathsf{GCD}(\, m_{e_2^*;e_1^*},\, m_{e_3^*;e_1^*}) \not\equiv 1$. However, this contradicts the fact that we originally have $\mathsf{GCD}(\, m_{e_2^*;e_1^*},\, m_{e_3^*;e_1^*}) \equiv 1$.

Thus from (27) and (28), we must have $\mathsf{GCD}(\,m_{11},\,m_{e_2^*;e_1^*}) \equiv 1$ or $\mathsf{GCD}(\,m_{11},\,m_{e_3^*;e_1^*}) \equiv 1$. We further trim down the proof by showing that both $\mathsf{GCD}(\,m_{11},\,m_{e_2^*;e_1^*}) \equiv 1$ and $\mathsf{GCD}(\,m_{11},\,m_{e_3^*;e_1^*}) \equiv 1$ cannot hold simultaneously. Note that the former implies from (27) that $\mathsf{GCD}(\,m_{12},\,m_{e_2^*;e_1^*}) = m_{e_2^*;e_1^*}$, thereby $\{e_2^*, e_1^*\} \subset \mathsf{1cut}(s_1;d_2)$. Similarly, the latter implies from (28) that $\mathsf{GCD}(\,m_{13},\,m_{e_3^*;e_1^*}) = m_{e_3^*;e_1^*}$, thereby $\{e_3^*, e_1^*\} \subset \mathsf{1cut}(s_1;d_3)$. Since $e_3^* \in \mathsf{1cut}(s_1;d_3)$, we can take an arbitrary path $P_{s_1d_3}$ from s_1 to d_3 passing through e_3^*. Recall that we already showed $\mathsf{GCD}(\,m_{e_3^*;e_{d_2}},\,m_{e_3^*;e_1^*}) \equiv 1$, which implies that there exists a path $P_{e_3^*d_2}$ not passing through e_1^*. Then, we can create a path $s_1P_{s_1d_3}e_3^*P_{e_3^*d_2}d_2$ from s_1 to d_2 not using e_1^*, which violates $e_1^* \in \mathsf{1cut}(s_1;d_2)$.

In summary, from (27) and (28), we can deduce that either $\mathsf{GCD}(\,m_{11},\,m_{e_2^*;e_1^*}) \equiv 1$ or $\mathsf{GCD}(\,m_{11},\,m_{e_3^*;e_1^*}) \equiv 1$ is only possible. In other words, we have

(i) $\mathsf{GCD}(\,m_{11},\,m_{e_2^*;e_1^*}) \equiv 1$ and $\mathsf{GCD}(\,m_{11},\,m_{e_3^*;e_1^*}) \not\equiv 1$.
(ii) $\mathsf{GCD}(\,m_{11},\,m_{e_2^*;e_1^*}) \not\equiv 1$ and $\mathsf{GCD}(\,m_{11},\,m_{e_3^*;e_1^*}) \equiv 1$.

For the following, we first prove that the case (i) cannot hold. By $\mathsf{GCD}(\,m_{11},\,m_{e_2^*;e_1^*}) \equiv 1$, we must have $\mathsf{GCD}(\,m_{12},\,m_{e_2^*;e_1^*}) = m_{e_2^*;e_1^*}$ from (27). Thus, $\{e_2^*, e_1^*\} \subset \mathsf{1cut}(s_1;d_2)$. If $e_2^* = e_{s_1}$, i.e., $\mathsf{tail}(e_2^*) = s_1$, then the fact that $e_2^* \in \overline{S}_2$ implies that $\mathsf{In}(s_1) > 0$, which violates the definition of the 3-unicast ANA network. Hence, $e_2^* \in \mathsf{1cut}(s_1;d_2)$ implies $e_{s_1} \prec e_2^*$. Let the non-zero channel gain polynomial from e_{s_1} to e_2^* to be $m_{e_{s_1};e_2^*}$. Then, we first claim that $\mathsf{GCD}(\,m_{e_{s_1};e_2^*},\,m_{e_3^*;e_1^*}) \equiv 1$ by contradiction.

Suppose that $\mathsf{GCD}(\,m_{e_{s_1};e_2^*},\,m_{e_3^*;e_1^*}) \not\equiv 1$. This implies that both are reducible and by Theorem 3 they share their irreducible factor(s). If $e_2^* \in \mathsf{1cut}(\mathsf{head}(e_3^*);\mathsf{tail}(e_1^*))$, then $m_{e_3^*;e_1^*}$ can be broken into $m_{e_3^*;e_2^*}m_{e_2^*;e_1^*}$ but this violates the assumption that $\mathsf{GCD}(\,m_{e_2^*;e_1^*},\,m_{e_3^*;e_1^*}) \equiv 1$. Thus, there must exist an edge $e' \in \mathsf{1cut}(\mathsf{head}(e_3^*);\mathsf{tail}(e_1^*))$ where $e_{s_1} \prec e' \prec e_2^*$ such that $m_{e_3^*;e_1^*}$ can be broken into $m_{e_3^*;e'}m_{e';e_1^*}$. Then, we can draw a path from s_1 to d_2 passing through both e' and e_1^* but not through e_2^*, which violates that $e_2^* \in \mathsf{1cut}(s_1;d_2)$.

Therefore, it must be $\mathsf{GCD}(\,m_{e_{s_1};e_2^*},\,m_{e_3^*;e_1^*}) \equiv 1$. If we recall the constructed edge-disjoint paths $P_{s_2\mathsf{tail}(e_1^*)}$ (through e_2^*) and $P_{s_3\mathsf{tail}(e_1^*)}$ (through e_3^*), then $\mathsf{GCD}(\,m_{e_{s_1};e_2^*},\,m_{e_3^*;e_1^*}) \equiv 1$ further gives us that we can always find a path $Q_{s_1\mathsf{tail}(e_2^*)}$ which is edge-disjoint with the path segment $e_3^*P_{s_3\mathsf{tail}(e_1^*)}\mathsf{tail}(e_1^*)$ of $P_{s_3\mathsf{tail}(e_1^*)}$. This implies that we can always find a pair of edge-disjoint paths $P^{(1)} = s_1Q_{s_1\mathsf{tail}(e_2^*)}e_2^*P_{s_2\mathsf{tail}(e_1^*)}\mathsf{tail}(e_1^*)$ and $P^{(2)} = e_3^*P_{s_3\mathsf{tail}(e_1^*)}\mathsf{tail}(e_1^*)$. In this setting, we now prove that $\mathsf{GCD}(\,m_{11},\,m_{e_3^*;e_1^*}) \not\equiv 1$ is not possible by contradiction.

Suppose that $\mathsf{GCD}(\,m_{11},\,m_{e_3^*;e_1^*}) \not\equiv 1$. This implies that there must exist an edge e'' where $e_3^* \preceq e'' \prec e_1^*$ such that without this edge e'', s_1 and d_1 are disconnected ($e'' \in \mathsf{1cut}(s_1;d_1)$). But from the above construction, for any path $P^{(2)}$ from e_3^* to $\mathsf{tail}(e_1^*)$ using e'', we can always find a edge-disjoint path $P^{(1)}$ from s_1 to $\mathsf{tail}(e_1^*)$ through e_2^*. Considering that $e_1^* \in \overline{D}_1$, we can easily find a path from s_1 to d_1 not using e'', which violates $e'' \in \mathsf{1cut}(s_1;d_1)$. Thus, the proof of the case (i) is complete.

Using similar arguments, we can easily show that the case (ii) cannot hold, neither. Thus in summary, we have considered all the possible cases related to m_{11} from (27) and (28), and have shown that any single case cannot be hold. Therefore, what we have shown so far is that if $\mathbf{h}_1^{(1)}(\underline{\mathbf{x}})$ is linearly dependent on $G_{3\mathrm{ANA}}$ who satisfies $L \not\equiv R$, $m_{11}m_{23} \not\equiv m_{21}m_{13}$, and $m_{11}m_{32} \not\equiv m_{31}m_{12}$, then $\overline{D}_1 \cap \overline{S}_2 = \emptyset$ and $\overline{D}_1 \cap \overline{S}_3 = \emptyset$ cannot hold simultaneously. Thus, the proof of *Case 1.1* is complete.

Case 1.2 when $\overline{D}_1 \cap \overline{S}_2 \neq \emptyset$ and $\overline{D}_1 \cap \overline{S}_3 = \emptyset$: Take the most downstream edge among $\overline{S}_3$ and denote it as e_3^*. In addition, take the most upstream edge among $\overline{D}_1 \cap \overline{S}_2$ and denote it as e_{12}^*. By Lemma 3, $e_{12}^* \succ e_3^*$. By definition, any path from s_2 to d_3 (resp. d_1) must pass through e_{12}^*, and we can express m_{23} and m_{21} as $m_{23} = m_{e_{s_2};e_{12}^*}m_{e_{12}^*;e_{d_3}}$ and $m_{21} = m_{e_{s_2};e_{12}^*}m_{e_{12}^*;e_{d_1}}$, respectively.

Similarly, any path from s_3 to d_2 (resp. d_1) must pass through e_3^* (resp. both e_3^* and e_{12}^*) by definition, we can express m_{32} and m_{31} as $m_{32} = m_{e_{s_3};e_3^*}m_{e_3^*;e_{d_2}}$ and $m_{31} = m_{e_{s_3};e_3^*}m_{e_3^*;e_{12}^*}m_{e_{12}^*;e_{d_1}}$, respectively. Then, it must be $\mathsf{GCD}(\,m_{e_3^*;e_{d_2}},\,m_{e_3^*;e_{12}^*}m_{e_{12}^*;e_{d_1}}) \equiv 1$ otherwise there must exist an edge $e \in \overline{S}_3$ such that $e \succ e_3^*$ by Lemma 7 and this violates that e_3^* is the most downstream edge among $\overline{S}_3$. Thus, we can further deduce that $\mathsf{GCD}(\,m_{32},\,m_{e_3^*;e_{12}^*}) \equiv 1$.

Moreover, notice that both m_{21} and m_{31} share the same factor $m_{e_{12}^*;e_{d_1}}$ from e_{12}^* to d_1. Then, we further show that $\mathsf{GCD}(\,m_{e_{s_2};e_{12}^*},\,m_{e_{s_3};e_3^*}m_{e_3^*;e_{12}^*}) \equiv 1$ by contradiction. Suppose $\mathsf{GCD}(\,m_{e_{s_2};e_{12}^*},\,m_{e_{s_3};e_3^*}m_{e_3^*;e_{12}^*}) \not\equiv 1$. Then, there must exist an edge $e \in \overline{D}_1$ such that $e \prec e_{12}^*$ by Lemma 7. If this e does not belongs to $\overline{S}_2$, then the fact that $e \in \overline{D}_1 \backslash \overline{S}_2$ and $e_{12}^* \in (\overline{D}_1 \cap \overline{S}_2) \subseteq \overline{S}_2$ where $e \prec e_2^*$ violates Lemma 3. On the other hand, if this $e \in \overline{D}_1$ also belongs to $\overline{S}_2$, then it violates that e_{12}^* is the most upstream edge among $\overline{D}_1 \cap \overline{S}_2$. Therefore, $\mathsf{GCD}(\,m_{e_{s_2};e_{12}^*},\,m_{e_{s_3};e_3^*}m_{e_3^*;e_{12}^*}) \equiv 1$. Then, this tells us that we can always find a pair of edge-disjoint paths $P_{s_2\mathsf{tail}(e_{12}^*)}$ (not through e_3^*) and $P_{s_3\mathsf{tail}(e_{12}^*)}$ (through e_3^*). In addition, we can further deduce that $\mathsf{GCD}(\,m_{21},\,m_{e_3^*;e_{12}^*}) \equiv 1$.

Since we know that $\mathsf{GCD}(\,m_{11}m_{13}m_{32}m_{21},\,m_{31}) = m_{31}$ from (25), if we apply the above results, then we have

$$\mathsf{GCD}(\,m_{11}m_{13},\ m_{e_3^*;e_{12}^*}) = m_{e_3^*;e_{12}^*}.$$

This tells us that e_3^* belongs to either $\mathsf{1cut}(s_1;d_1)$, $\mathsf{1cut}(s_1;d_3)$, or both. Similarly, e_{12}^* can belong to either $\mathsf{1cut}(s_1;d_1)$, $\mathsf{1cut}(s_1;d_3)$, or both. However, if $e_{12}^* \in \mathsf{1cut}(s_1;d_1) \cap \mathsf{1cut}(s_1;d_3)$, then the assumption that $e_{12}^* \in (\overline{D}_1 \cap \overline{S}_2) \subseteq \overline{S}_2$ further results in violating the assumption that $m_{11}m_{23} \not\equiv m_{21}m_{13}$ by Corollary 1. Thus, e_{12}^* can belong to either $\mathsf{1cut}(s_1;d_1)$ or $\mathsf{1cut}(s_1;d_3)$.

Then, we consider the following cases:

(i) e_3^* belongs to $\mathsf{1cut}(s_1;d_1)$ (may or may not belong to $\mathsf{1cut}(s_1;d_3)$) and e_{12}^* only belongs to $\mathsf{1cut}(s_1;d_1)$
(ii) e_3^* belongs to $\mathsf{1cut}(s_1;d_3)$ (may or may not belong to $\mathsf{1cut}(s_1;d_1)$) and e_{12}^* only belongs to $\mathsf{1cut}(s_1;d_3)$
(iii) e_3^* only belongs to $\mathsf{1cut}(s_1;d_3)$ and e_{12}^* only belongs to $\mathsf{1cut}(s_1;d_1)$

(iv) e_3^* only belongs to $\mathsf{1cut}(s_1;d_1)$ and e_{12}^* only belongs to $\mathsf{1cut}(s_1;d_3)$

Case 1.2.1 when (i): In this case, we have $\{e_3^*, e_{12}^*\} \subset \mathsf{1cut}(s_1;d_1)$ such that we can express m_{11} as $m_{11} = m_{e_{s_1};e_3^*} m_{e_3^*;e_{12}^*} m_{e_{12}^*;e_{d_1}}$. Then, we first claim that $m_{e_{s_1};e_3^*}$ and $m_{e_{s_2};e_{12}^*}$ completely cover the disjoint portions of $G_{3\text{ANA}}$ by contradiction. Suppose that there exists a path from s_1 to $\mathsf{tail}(e_3^*)$ over $m_{e_{s_1};e_3^*}$ and a path from s_2 to $\mathsf{tail}(e_{12}^*)$ over $m_{e_{s_2};e_{12}^*}$, which shares a vertex u. Then, we can draw a path from s_1 to d_1 passing through both u and $e_1^* \in \mathsf{1cut}(s_1;d_1)$ but not through e_3^*, which violates $e_3^* \in \mathsf{1cut}(s_1;d_1)$. Therefore, they completely cover the disjoint portions of $G_{3\text{ANA}}$, and thus $\mathsf{GCD}(\, m_{e_{s_1};e_3^*},\, m_{e_{s_2};e_{12}^*}) \equiv 1$. Then, since $e_3^* \prec e_{12}^*$, we further have $\mathsf{GCD}(\, m_{21},\, m_{e_{s_1};e_3^*}) \equiv 1$.

We construct the following collection of paths to proceed further. Recall that $e_3^* \in \overline{S}_3 \cap \mathsf{1cut}(s_1;d_1)$. By the assumption of $m_{11}m_{32} \not\equiv m_{31}m_{12}$ and Corollary 1, there must exist a path from s_1 to d_2 not through e_3^* to make $e_3^* \notin \mathsf{1cut}(s_1;d_2)$. Consider the collection $\mathbf{P}_{12}^*$ of all paths from s_1 to d_2 not through e_3^*. Without loss of generality, suppose that $\mathsf{1cut}(s_1;\mathsf{tail}(e_3^*)) \cap \mathsf{1cut}(s_1;d_2) \neq \emptyset$ and denote its most downstream edge as e' so that $m_{e_{s_1};e_3^*}$ can be broken into $m_{e_{s_1};e_3^*} = m_{e_{s_1};e'} m_{e';e_3^*}$. Similarly, suppose without loss of generality that $\mathsf{1cut}(\mathsf{head}(e_3^*);d_2) \cap \mathsf{1cut}(s_1;d_2) \neq \emptyset$ and denote its most upstream edge as e'' so that $m_{e_3^*;e_{d_2}}$ can be broken into $m_{e_3^*;e_{d_2}} = m_{e_3^*;e''} m_{e'';e_{d_2}}$. Definitely, all paths from s_1 to d_2 (including the paths in $\mathbf{P}_{12}^*$) are passing through both e' and e''.

Then, we can deduce that $\mathsf{GCD}(\, m_{21},\, m_{e';e_3^*}) \equiv 1$ if we recall $\mathsf{GCD}(\, m_{21},\, m_{e_{s_1};e_3^*}) \equiv 1$. Moreover, any path from s_3 to $\mathsf{tail}(e_3^*)$ should not touch the vertex $\mathsf{tail}(e')$ otherwise we can always construct a path from s_3 to d_2 (through e') not passing through e_3^*, which violates $e_3^* \in \overline{S}_3$. Without loss of generality, we can assume that $\mathsf{GCD}(\, m_{e_{s_3};e_3^*},\, m_{e';e_3^*}) \equiv 1$. Then, we further have $\mathsf{GCD}(\, m_{31},\, m_{e';e_3^*}) \equiv 1$. Therefore from $\mathsf{GCD}(\, m_{12}m_{13}m_{21}m_{31},\, m_{11}) = m_{11}$ by (26), if we apply the above results, then we have

$$\mathsf{GCD}(\, m_{12}m_{13},\, m_{e';e_3^*}) = m_{e';e_3^*}. \tag{29}$$

For the following, we argue that (29) cannot hold simultaneously with $\mathsf{GCD}(\, m_{11}m_{13}m_{32}m_{21},\, m_{12}) = m_{12}$ from (25). First, let's look at the channel gain m_{12} more precisely. If we let the non-zero channel polynomial from e' to e'' over $\mathbf{P}_{12}^*$ as $m(e'\mathbf{P}_{12}^*e'')$, then we can express m_{12} as $m_{12} = m_{e_{s_1};e'}(m(e'\mathbf{P}_{12}^*e'') + m_{e';e_3^*} m_{e_3^*;e''}) m_{e'';e_{d_2}}$. If we denote the middle part $m(e'\mathbf{P}_{12}^*e'') + m_{e';e_3^*} m_{e_3^*;e''}$ of m_{12} as $m_{e';e''}$, then $m_{e';e''}$ is irreducible by Theorem 3 since $\mathsf{EC}(\mathsf{head}(e');\mathsf{tail}(e'')) \geq 2$. Moreover, $\mathsf{GCD}(\, m_{e';e_3^*} m_{e_3^*;e''},\, m(e'\mathbf{P}_{12}^*e'')) \equiv 1$ otherwise it violates the criticalness of e' or e''. Then, we have

$$\mathsf{GCD}(\, m_{e';e_3^*} m_{e_3^*;e''},\, m_{e';e''}) \equiv 1. \tag{30}$$

Any path in $\mathbf{P}_{12}^*$ should be vertex-disjoint with the paths over $m_{e_{s_2};e_{12}^*}$, $m_{e_3^*;e_{12}^*}$, and $m_{e_{12}^*;e_{d_1}}$ otherwise we can create a path from s_1 to d_1 not through $e_3^* \in \mathsf{1cut}(s_1;d_1)$. Thus, they cover the disjoint portion to that of $m(e'\mathbf{P}_{12}^*e'')$, which implies that $\mathsf{GCD}(\, m_{e_{s_2};e_{12}^*},\, m_{e';e''}) \equiv 1$, $\mathsf{GCD}(\, m_{e_3^*;e_{12}^*},\, m_{e';e''}) \equiv 1$, and $\mathsf{GCD}(\, m_{e_{12}^*;e_{d_1}},\, m_{e';e''}) \equiv 1$. Since $m_{11} = m_{e_{s_1};e'} m_{e';e_3^*} m_{e_3^*;e_{12}^*} m_{e_{12}^*;e_{d_1}}$ and from (30), we can know that $\mathsf{GCD}(\, m_{11},\, m_{e';e''}) \equiv 1$. In addition, $m_{21} = m_{e_{s_2};e_{12}^*} m_{e_{12}^*;e_{d_1}}$ tells us that $\mathsf{GCD}(\, m_{21},\, m_{e';e''}) \equiv 1$. Furthermore, it must be $\mathsf{GCD}(\, m_{e_{s_3};e_3^*},\, m_{e';e''}) \equiv 1$ because $m_{e';e''}$ is irreducible and $e' \prec e_3^* \prec e''$. Since $m_{32} = m_{e_{s_3};e_3^*} m_{e_3^*;e''} m_{e'';e_{d_2}}$ and from (30), we can deduce that $\mathsf{GCD}(\, m_{32},\, m_{e';e''}) \equiv 1$.

Applying the above reasonings to (25), we have

$$\mathsf{GCD}(\, m_{13},\, m_{e';e''}) = m_{e';e''}. \tag{31}$$

Since $\mathsf{GCD}(\, m_{12},\, m_{e';e_3^*}) \equiv 1$ by (30), (29) reduces to $\mathsf{GCD}(\, m_{13},\, m_{e';e_3^*}) = m_{e';e_3^*}$. However, it cannot hold simultaneously with (31) whether e_3^* also belongs to $\mathsf{1cut}(s_1;d_3)$ or not. More specifically, if $e_3^* \in \mathsf{1cut}(s_1;d_3)$, then it violates (31), and if $e_3^* \notin \mathsf{1cut}(s_1;d_3)$, it violates $\mathsf{GCD}(\, m_{13},\, m_{e';e_3^*}) = m_{e';e_3^*}$. Thus, the proof of (i) is complete.

Case 1.2.2 when (ii): In this case, we have $\{e_3^*, e_{12}^*\} \subset \mathsf{1cut}(s_1;d_3)$ such that we can express m_{13} as $m_{13} = m_{e_{s_1};e_3^*} m_{e_3^*;e_{12}^*} m_{e_{12}^*;e_{d_3}}$. Then, as shown in *Case 1.2.1*, we have $\mathsf{GCD}(\, m_{e_{s_1};e_3^*},\, m_{e_{s_2};e_{12}^*}) \equiv 1$. Then, since $e_3^* \prec e_{12}^*$, we further have $\mathsf{GCD}(\, m_{23},\, m_{e_{s_1};e_3^*}) \equiv 1$.

We construct the following collection of paths to proceed further. Recall that $e_3^* \in \overline{S}_3 \cap \mathsf{1cut}(s_1;d_3)$. By the assumption of $\overline{S}_2 \cap \overline{S}_3 \neq \emptyset$, there must exist a path from s_1 to d_2 not through e_3^* to make $e_3^* \notin \mathsf{1cut}(s_1;d_2)$. Consider the collection $\mathbf{P}_{12}^*$ of all paths from s_1 to d_2 not through e_3^*. Without loss of generality, suppose that $\mathsf{1cut}(s_1;\mathsf{tail}(e_3^*)) \cap \mathsf{1cut}(s_1;d_2) \neq \emptyset$ and denote its most downstream edge as e' so that $m_{e_{s_1};e_3^*}$ can be broken into $m_{e_{s_1};e_3^*} = m_{e_{s_1};e'} m_{e';e_3^*}$. Similarly, suppose without loss of generality that $\mathsf{1cut}(\mathsf{head}(e_3^*);d_2) \cap \mathsf{1cut}(s_1;d_2) \neq \emptyset$ and denote its most upstream edge as e'' so that $m_{e_3^*;e_{d_2}}$ can be broken into $m_{e_3^*;e_{d_2}} = m_{e_3^*;e''} m_{e'';e_{d_2}}$. Definitely, all paths from s_1 to d_2 (including the paths in $\mathbf{P}_{12}^*$) are passing through both e' and e''.

Then, we can deduce that $\mathsf{GCD}(\, m_{23},\, m_{e';e_3^*}) \equiv 1$ if we recall $\mathsf{GCD}(\, m_{23},\, m_{e_{s_1};e_3^*}) \equiv 1$. Moreover, any path from s_3 to $\mathsf{tail}(e_3^*)$ should not touch the vertex $\mathsf{tail}(e')$ otherwise we can always construct a path from s_3 to d_2 (through e') not passing through e_3^*, which violates $e_3^* \in \overline{S}_3$. Without loss of generality, we can assume that $\mathsf{GCD}(\, m_{e_{s_3};e_3^*},\, m_{e';e_3^*}) \equiv 1$. Then, we further have $\mathsf{GCD}(\, m_{31},\, m_{e';e_3^*}) \equiv 1$. Therefore from $\mathsf{GCD}(\, m_{11}m_{12}m_{23}m_{31},\, m_{13}) = m_{13}$ by (24), if we apply the above results, then we have

$$\mathsf{GCD}(\, m_{11}m_{12},\, m_{e';e_3^*}) = m_{e';e_3^*}. \tag{32}$$

For the following, we argue that (32) cannot hold simultaneously with $\mathsf{GCD}(\, m_{11}m_{13}m_{32}m_{21},\, m_{12}) = m_{12}$ from (25). First, let's look at the channel gain m_{12} more precisely. If we let the non-zero channel polynomial from e' to e'' over $\mathbf{P}_{12}^*$ as $m(e'\mathbf{P}_{12}^*e'')$, then we can express m_{12} as $m_{12} = m_{e_{s_1};e'}(m(e'\mathbf{P}_{12}^*e'') + m_{e';e_3^*} m_{e_3^*;e''}) m_{e'';e_{d_2}}$. If we denote the middle part $m(e'\mathbf{P}_{12}^*e'') + m_{e';e_3^*} m_{e_3^*;e''}$

of m_{12} as $m_{e';e''}$, then $m_{e';e''}$ is irreducible by Theorem 3 since $\mathsf{EC}(\mathsf{head}(e');\mathsf{tail}(e'')) \geq 2$. Moreover, $\mathsf{GCD}(m_{e';e_3^*}m_{e_3^*;e''},\, m(e'\mathbf{P}_{12}^*e''))\equiv 1$ otherwise it violates the criticalness of e' or e''. Then, we have (30).

Any path in $\mathbf{P}_{12}^*$ should be vertex-disjoint with the paths over $m_{e_{s_2};e_{12}^*}$, $m_{e_3^*;e_{12}^*}$, and $m_{e_{12}^*;e_{d_3}}$ otherwise we can create a path from s_1 to d_3 not through $e_3^* \in \mathsf{1cut}(s_1;d_3)$. Thus, they cover the disjoint portion to that of $m(e'\mathbf{P}_{12}^*e'')$, which implies that $\mathsf{GCD}(m_{e_{s_2};e_{12}^*},\, m_{e';e''})\equiv 1$, $\mathsf{GCD}(m_{e_3^*;e_{12}^*},\, m_{e';e''})\equiv 1$, and $\mathsf{GCD}(m_{e_{12}^*;e_{d_3}},\, m_{e';e''})\equiv 1$. Since $m_{13} = m_{e_{s_1};e'}m_{e';e_3^*}m_{e_3^*;e_{12}^*}m_{e_{12}^*;e_{d_3}}$ and from (30), we can know that $\mathsf{GCD}(m_{13},\, m_{e';e''}) \equiv 1$. In addition, $\mathsf{GCD}(m_{e_{12}^*;e_{d_1}},\, m_{e';e''}) \equiv 1$ due to the irreducibility of $m_{e';e''}$ and $e' \prec e_{12}^*$. Thus, $m_{21} = m_{e_{s_2};e_{12}^*}m_{e_{12}^*;e_{d_1}}$ tells us that $\mathsf{GCD}(m_{21},\, m_{e';e''}) \equiv 1$. Furthermore, it must be $\mathsf{GCD}(m_{e_{s_3};e_3^*},\, m_{e';e''})\equiv 1$ because $m_{e';e''}$ is irreducible and $e' \prec e_3^* \prec e''$. Since $m_{32} = m_{e_{s_3};e_3^*}m_{e_3^*;e''}m_{e'';e_{d_2}}$ and from (30), we can deduce that $\mathsf{GCD}(m_{32},\, m_{e';e''})\equiv 1$.

Applying the above reasonings to (25), we have

$$\mathsf{GCD}(m_{11},\, m_{e';e''}) = m_{e';e''}. \tag{33}$$

Since $\mathsf{GCD}(m_{12},\, m_{e';e_3^*}) \equiv 1$ by (30), (32) reduces to $\mathsf{GCD}(m_{11},\, m_{e';e_3^*}) = m_{e';e_3^*}$. However, it cannot hold simultaneously with (33) whether e_3^* also belongs to $\mathsf{1cut}(s_1;d_1)$ or not. More specifically, if $e_3^* \in \mathsf{1cut}(s_1;d_1)$, then it violates (33), and if $e_3^* \notin \mathsf{1cut}(s_1;d_1)$, it violates $\mathsf{GCD}(m_{11},\, m_{e';e_3^*}) = m_{e';e_3^*}$. Thus, the proof of (ii) is complete.

Case 1.2.3 when (iii): Suppose that $e_3^* \in \mathsf{1cut}(s_1;d_3)\backslash\mathsf{1cut}(s_1;d_1)$ and $e_{12}^* \in \mathsf{1cut}(s_1;d_1)\backslash\mathsf{1cut}(s_1;d_3)$. Because $e_{12}^* \in (\overline{D}_1 \cap \overline{S}_2) \subseteq \overline{S}_2$, take an arbitrary path $P_{s_2d_3}$ from s_2 to d_3 passing through e_{12}^*. Since $e_3^* \notin \mathsf{1cut}(s_1;d_1)$ and $e_{12}^* \in \mathsf{1cut}(s_1;d_1)$, there exists a path $P_{s_1d_1}$ from s_1 to d_1 not passing through e_3^* but through e_{12}^*. Then, we can create a path $s_1P_{s_1d_1}e_{12}^*P_{s_2d_3}d_3$ from s_1 to d_3 not passing through e_3^*, which violates $e_3^* \in \mathsf{1cut}(s_1;d_3)$. Thus, the proof of (iii) is complete.

Case 1.2.4 when (iv): Suppose that $e_3^* \in \mathsf{1cut}(s_1;d_1)\backslash\mathsf{1cut}(s_1;d_3)$ and $e_{12}^* \in \mathsf{1cut}(s_1;d_3)\backslash\mathsf{1cut}(s_1;d_1)$. Because $e_{12}^* \in (\overline{D}_1 \cap \overline{S}_2) \subseteq \overline{S}_2$, take an arbitrary path $P_{s_2d_1}$ from s_2 to d_1 passing through e_{12}^*. Since $e_3^* \notin \mathsf{1cut}(s_1;d_3)$ and $e_{12}^* \in \mathsf{1cut}(s_1;d_3)$, there exists a path $P_{s_1d_3}$ from s_1 to d_3 not passing through e_3^* but through e_{12}^*. Then, we can create a path $s_1P_{s_1d_3}e_{12}^*P_{s_2d_1}d_1$ from s_1 to d_1 not passing through e_3^*, which violates $e_3^* \in \mathsf{1cut}(s_1;d_1)$. Thus, the proof of (iv) is complete.

Case 1.3 when $\overline{D}_1 \cap \overline{S}_2 = \emptyset$ and $\overline{D}_1 \cap \overline{S}_3 \neq \emptyset$: If we swap the sources s_2 and s_3, and swap the destinations d_2 and d_3, then *Case 1.2* can be directly applied to say that $\overline{D}_1 \cap \overline{S}_2 = \emptyset$ and $\overline{D}_1 \cap \overline{S}_3 \neq \emptyset$ cannot hold, because the form of three equations (24), (25), and (26) do not change by such swap operations.

Therefore, this, together with *Case 1.1* and *Case 1.2*, proves that if $\mathbf{h}_1^{(1)}(\underline{\mathbf{x}})$ are linearly dependent on G_{3ANA} who satisfies $L \not\equiv R$, $m_{11}m_{23} \not\equiv m_{21}m_{13}$, and $m_{11}m_{32} \not\equiv m_{31}m_{12}$, then $\overline{S}_2 \cap \overline{S}_3 \neq \emptyset$.

The proof for $\overline{D}_2 \cap \overline{D}_3 \neq \emptyset$ from (24), (25), and (26):

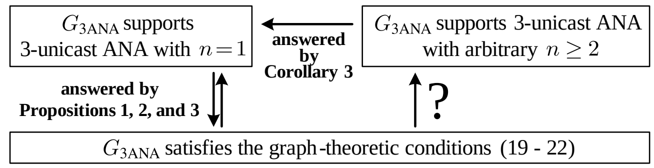

Fig. 2. Current understanding of 3-unicast ANA graph-theoretic characterization.

If we swap the roles of the source $\{s_i\}$ and the destination $\{d_i\}$ and use the proof for $\overline{S}_2 \cap \overline{S}_3$, then it is trivially proved since the assumptions, i.e., linearly dependent $\mathbf{h}_1^{(1)}(\underline{\mathbf{x}})$, $L \not\equiv R$, $m_{11}m_{23} \not\equiv m_{21}m_{13}$, and $m_{11}m_{32} \not\equiv m_{31}m_{12}$, do not change by such source-destination swap.

Therefore, what we proved so far is that $\overline{S}_2 \cap \overline{S}_3 \neq \emptyset$ and $\overline{D}_2 \cap \overline{D}_3 \neq \emptyset$, resulting in $L \equiv R$ by Propositions 1 and 2. This contradicts the assumption of G_{3ANA} satisfying $L \not\equiv R$. The proofs for (b) and (c) are followed similarly. Thus, the proof to Proposition 3 is complete. ∎

The problem of finding the graph-theoretic condition for general $n \geq 2$ remains an open problem. On the other hand, we prove the following corollary which shows that the feasibility conditions for $n = 1$ case turns out to be necessary for the cases of general n values.

Corollary 3: For a given 3-unicast ANA network, if the 3-unicast ANA with $n \geq 2$ is feasible, then the 3-unicast ANA with $n=1$ is also feasible.

The above is a corollary of Proposition 3 and Theorem 3.

Proof: If the 3-unicast ANA with $n \geq 2$ is feasible on a given G_{3ANA}, then $\mathbf{h}_i^{(n)}(\underline{\mathbf{x}})$ is linearly independent on G_{3ANA} for $i \in \{1,2,3\}$ by Theorem 1. Moreover, feasibility for general n value also means that G_{3ANA} satisfies $L(\underline{\mathbf{x}}) \not\equiv R(\underline{\mathbf{x}})$. Also note that if $\mathbf{h}_i^{(n)}(\underline{\mathbf{x}})$ is linearly independent, then $\mathbf{h}_i^{(1)}(\underline{\mathbf{x}})$, a subset of the original polynomials, is linearly independent as well. By Theorem 1, the 3-unicast ANA with $n=1$ is feasible on G_{3ANA}. ∎

Fig. 2 summarizes our knowledge about the graph-theoretic characterization of the 3-unicast ANA scheme, in which three arrows have been established, except the right one. That is, we have developed a necessary and sufficient graph-theoretic condition for the 3-unicast ANA scheme with $n = 1$. Corollary 3 shows that the graph-theoretic condition for $n = 1$ can also be viewed as a necessary condition for general $n \geq 2$. In our future work, we will investigate graph-theoretic characterization problem for general $n \geq 2$.

V. Conclusion and Future Works

The main subject of this work is the general class of precoding-based NC schemes, which focus on designing the precoding and decoding mappings at the sources and destinations while using randomly generated local encoding kernels within the network. Such a precoding-based structure includes the 3-unicast ANA scheme, originally proposed in [12], [13], as a special case. In this work, we have identified new graph-theoretic relationship for the precoding-based NC solutions. Based on the findings on the general precoding-based NC, we have further characterized the graph-theoretic feasibility conditions of the 3-unicast ANA scheme for the simplest case of $n=1$, which includes proving the open conjecture of the existing results for $n=1$. We believe that the analysis in this work will serve as a precursor to fully understand the notoriously challenging multiple-unicast NC problem and design practical, distributed NC solutions based on the precoding-based framework.

REFERENCES

[1] P. Elias, A. Feinstein, and C. E. Shannon, "A note on the maximum flow through a network," *IEEE Trans. Inform. Theory*, vol. IT-2, pp. 117-119, December 1956.

[2] R. Ahlswede, N. Cai, S.-Y. Li, and R. Yeung, "Network information flow," *IEEE Trans. Inform. Theory*, vol. 46, no. 4, pp. 1204-1216, July 2000.

[3] S.-Y. Li, R. Yeung, and N. Cai, "Linear network coding," *IEEE Trans. Inform. Theory*, vol. 49, no. 2, pp. 371-381, February 2003.

[4] P. Sanders, S. Egner, and L. Tolhuizen, "Polynomial time algorithms for network information flow," in *Proc. 15th ACM Symp. Parallel Algorithms and Architectures*, San Diego, USA, June 2003.

[5] R. Koetter and M. Medard, "An algebraic approach to network coding," *IEEE/ACM Trans. Networking*, vol. 11, no. 5, pp. 782-795, October 2003.

[6] P. Chou, Y. Wu, and K. Jain, "Practical network coding," in *Proc. 41st Annual Allerton Conf. on Comm., Contr., and Computing*, Monticello, USA, October 2003.

[7] T. Ho, M. Medard, R. Koetter, D. R. Karger, M. Effros, J. Shi, and B. Leong, "A Random Linear Network Coding Approach to Multicast", *IEEE Trans. Inform. Theory*, vol. 52, no. 10, pp. 4413-4430, October 2006.

[8] C.-C. Wang and Ness B. Shroff, "Pairwise Intersession Network Coding On Directed Networks," *IEEE Trans. Inform. Theory*, vol. 56, no. 8, pp. 3879-3900, August 2010.

[9] A. Lehman, Network coding, Ph.D. dissertation, MIT, 2005.

[10] R. Dougherty, C. Freiling, and K. Zeger, "Insufficiency of linear coding in network information flow," *IEEE Trans. Inform. Theory*, vol. 51, no. 8, pp. 2745-2759, August 2005.

[11] V. R. Cadambe and S. A. Jafar, "Interference alignment and the degrees of freedom for the k-user interference channel," *IEEE Trans. Inform. Theory*, vol. 54, no. 8, pp. 3425-3441, August 2008.

[12] A. Das, S. Vishwanath, S. Jafar and A. Markopoulou, "Network Coding for Multiple Unicasts: An Interference Alignement Approach," in *Proc. IEEE Int'l Symp. Inform. Theory*, Texas, USA, June 2010.

[13] A. Ramakrishnany, A. Dasz, H. Malekiy, A. Markopoulouy, S. Jafar and S. Vishwanath, "Network Coding for Three Unicast Sessions: Interference Alignment Approaches," in Proceedings of *48th Annual Allerton Conf. on Comm., Contr., and Computing*, Monticello, USA, September 2010.

[14] N. Harvey, R. Kleinberg, and A. Lehman, "On the capacity of information network," *IEEE Trans. Inform. Theory*, vol. 52, no. 6, pp. 2345-2364, June 2006.

[15] X. Yan, J. Yang, and Z. Zhang, "An outer bound for multisource multisink network coding with minimum cost consideration," *IEEE Trans. Inform. Theory*, vol. 52, no. 6, pp. 2373-2385, June 2006.

[16] J. Han, C.-C. Wang, and N. B. Shroff, "Analysis of Precoding-based Intersession Network Coding and The Corresponding 3-unicast Interference Alignment Scheme," *Technical Report*, *Purdue University*, *http://web.ics.purdue.edu/∼han83/*, 2011.

[17] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Springer, 2008.