



Lecture 0C

Math Foundations Team

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Section Overview



Continuity

Derivatives

Maxima and Minima

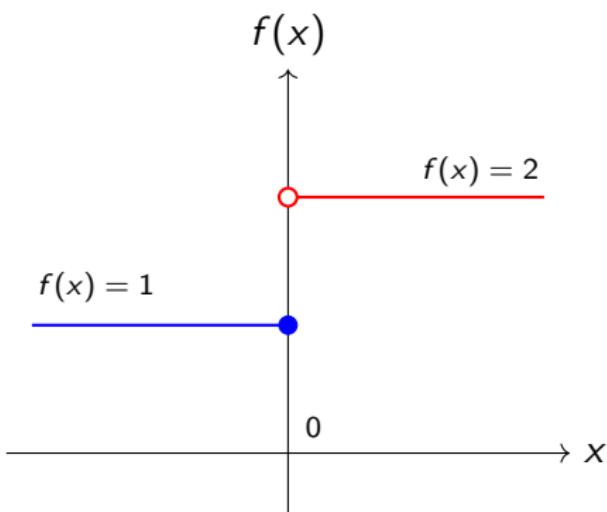
Linear Programming

We explore continuity using two informal examples.

Example 1: Piecewise Function

$$f(x) = \begin{cases} 1, & x \leq 0 \\ 2, & x > 0 \end{cases}$$

- ▶ The function is defined for all real numbers.
- ▶ Left-hand limit at $x = 0$ is 1.
- ▶ Right-hand limit at $x = 0$ is 2.
- ▶ Hence, f is **discontinuous** at $x = 0$.

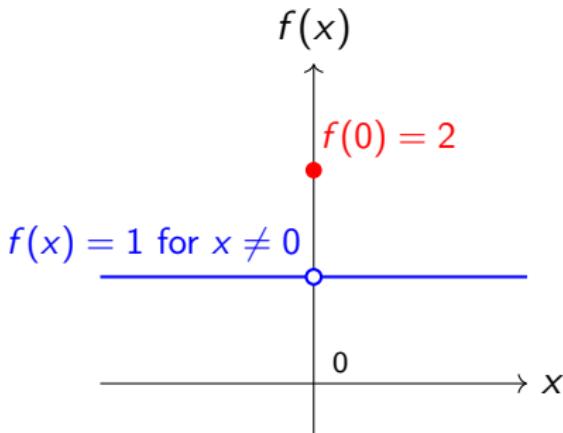


Example 2: Another Function



$$f(x) = \begin{cases} 1, & x \neq 0 \\ 2, & x = 0 \end{cases}$$

- ▶ LHL = RHL = 1
- ▶ $f(0) = 2 \Rightarrow$
Discontinuous at $x = 0$



Note: Left and right limits at $x = 0$ exist and are equal to 1, but $f(0) = 2$, so the function is **not continuous** at $x = 0$.

Naive Idea of Continuity



- ▶ A function is continuous at a point if its graph can be drawn without lifting the pen.
- ▶ Discontinuity means a break or jump in the graph.

Definition 1: Continuity at a Point



A function f is **continuous at $x = c$** if:

$$\lim_{x \rightarrow c} f(x) = f(c)$$

That is:

- ▶ Left-hand limit exists
- ▶ Right-hand limit exists
- ▶ Both limits are equal to $f(c)$

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Importance of Derivatives in Real-Life Scenarios



In many real-world situations, it is crucial to understand how one quantity changes with respect to another. Some examples include:

- ▶ **Reservoir Management:** To predict when a reservoir will overflow based on water depth measured at different times.
- ▶ **Rocket Science:** To compute the precise velocity for launching a satellite, knowing the rocket's height at various times.
- ▶ **Financial Analysis:** To forecast changes in stock value by analyzing its current and past values.

Conclusion: These scenarios require understanding how a parameter changes with respect to another.

Definition 1: Suppose f is a real-valued function and a is a point in its domain of definition. The derivative of f at a is defined by:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

provided this limit exists.

Notation: The derivative of $f(x)$ at a is denoted by $f'(a)$.

Observation: $f'(a)$ quantifies the rate of change of $f(x)$ at a with respect to x .

Geometric Interpretation of Derivative



Let $y = f(x)$ be a function and let $P = (a, f(a))$ and $Q = (a + h, f(a + h))$ be two points close to each other on the graph of this function shown in Fig1.

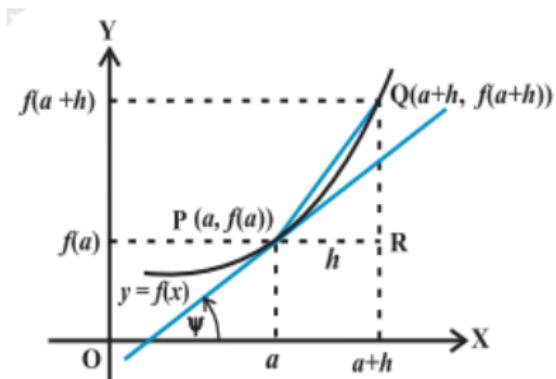


Figure 1: Geometric Interpretation

Limit Definition of Derivative



We know that the derivative of a function f at a point a is defined as:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

From triangle $\triangle PQR$, it is clear that the ratio whose limit we are taking is precisely equal to:

$$\tan(\angle QPR)$$

which represents the slope of the chord PQ .

From Chord to Tangent



As $h \rightarrow 0$, the point Q moves closer to P , and:

$$\lim_{h \rightarrow 0} \frac{QR}{PR} = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

Thus, the chord PQ tends to the tangent at point P on the curve $y = f(x)$.

Therefore, the limit gives us the slope of the tangent line:

$$f'(a) = \tan \psi$$

where ψ is the angle the tangent makes with the x -axis.

Generalization:

For a given function f , if the derivative exists at every point in its domain, it defines a new function:

$$f'(x)$$

This function describes how $f(x)$ changes with respect to x at every point, and is known as the *derivative function*.

Definition 2: First Principle of Derivative



Suppose f is a real-valued function. The function defined by:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

wherever the limit exists, is called the **derivative of f at x** .

This definition is also referred to as the **first principle of derivative**.

Note: The domain of $f'(x)$ is the set of all x for which the above limit exists.

Example



Problem: Find the derivative at $x = 2$ of the function $f(x) = 3x$.

Solution:

$$f'(2) = \lim_{h \rightarrow 0} \frac{f(2 + h) - f(2)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{3(2 + h) - 3(2)}{h} = \lim_{h \rightarrow 0} \frac{6 + 3h - 6}{h} = \lim_{h \rightarrow 0} \frac{3h}{h} = \lim_{h \rightarrow 0} 3 = 3$$

Conclusion: The derivative of the function $f(x) = 3x$ at $x = 2$ is 3.

► **General notation:**

$$f'(x), \quad \frac{d}{dx} f(x), \quad D(f(x))$$

► **If $y = f(x)$, then:**

$$\frac{dy}{dx}$$

► **At a specific point $x = a$:**

$$f'(a), \quad \left. \frac{df}{dx} \right|_{x=a}, \quad \left. \frac{d}{dx} f(x) \right|_{x=a}$$

These notations all represent the **derivative of a function with respect to x** .

Let $f(x)$ and $g(x)$ be differentiable functions, and c be a constant.
Then:

► **Sum Rule:**

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

► **Difference Rule:**

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

► **Constant Multiple Rule:**

$$\frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)$$

► **Product Rule:**

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x)g'(x) + f'(x)g(x)$$

► **Quotient Rule:**

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

These rules help us differentiate combinations of functions efficiently.

Linearity Rules with Examples



Let $f(x)$ and $g(x)$ be differentiable functions.

- ▶ **Sum Rule:** $\frac{d}{dx}(x^2 + \sin x) = 2x + \cos x$
- ▶ **Difference Rule:** $\frac{d}{dx}(e^x - \ln x) = e^x - \frac{1}{x}$
- ▶ **Constant Multiple Rule:** $\frac{d}{dx}(5x^3) = 5 \cdot 3x^2 = 15x^2$

Product and Quotient Rules with Examples



- ▶ **Product Rule:** *Example:*

$$\frac{d}{dx}(x^2 \cdot \ln x) = x^2 \cdot \frac{1}{x} + 2x \cdot \ln x = x + 2x \ln x$$

- ▶ **Quotient Rule:**

$$\frac{d}{dx} \left(\frac{x^2}{\sin x} \right) = \frac{2x \cdot \sin x - x^2 \cdot \cos x}{\sin^2 x}$$

Composite Function Derivative: Illustrative Example



Let $f(x) = (2x + 1)^3$

Method 1: Expand and Differentiate

$$f(x) = 8x^3 + 12x^2 + 6x + 1$$

$$\frac{df}{dx} = 24x^2 + 24x + 6 = 6(2x + 1)^2$$

Method 2: Use Function Composition

$$g(x) = 2x + 1, \quad h(t) = t^3, \quad f(x) = h(g(x))$$

$$\frac{df}{dx} = h'(g(x)) \cdot g'(x) = 3(2x + 1)^2 \cdot 2 = 6(2x + 1)^2$$

Observation: Using composition is more efficient, especially for $(2x + 1)^{100}$

Chain Rule (General Form): If $f(x) = h(g(x))$, then

$$\frac{df}{dx} = h'(g(x)) \cdot g'(x)$$

Why is it useful?

- ▶ Avoids expanding high-degree expressions
- ▶ Essential when working with nested functions like $\sin(x^2)$, $\ln(3x + 1)$, etc.

Example: For $f(x) = (5x - 4)^{10}$, using the Chain Rule:

$$f'(x) = 10(5x - 4)^9 \cdot 5 = 50(5x - 4)^9$$

Chain Rule for Composite Functions (Two Functions)



Let $f(x)$ be a real-valued function which is a composite of two functions u and v , i.e., $f = v \circ u$.

Suppose $t = u(x)$, and both $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist. Then, the Chain Rule states:

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$$

Example: For $f(x) = \sin(2x + 1)$, let:

$$u(x) = 2x + 1, \quad v(t) = \sin(t)$$

Thus,

$$\frac{df}{dx} = \cos(2x + 1) \cdot 2$$

Chain Rule for Composite Functions (Three Functions)



The Chain Rule can be extended to composite functions of three functions.

Suppose $f(x)$ is a composite of three functions u , v , and w , i.e., $f = (w \circ u) \circ v$.

Let $t = v(x)$ and $s = u(t)$, then:

$$\frac{df}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

Example: For $f(x) = \sin(3x + 2)^2$, we have:

$$t = 3x + 2, \quad s = t^2, \quad w(s) = \sin(s)$$

$$Thus, \frac{df}{dx} = 2 \cdot (3x + 2) \cdot \cos((3x + 2)^2) \cdot 3$$

The Chain Rule can be generalized for composite functions of more than three functions.

If $f(x) = w(u(v(x)))$, and you continue composing more functions, the derivative is:

$$\frac{df}{dx} = \frac{dw}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

Example Extension: For a composite of four functions:

$$f(x) = w(u(v(t(x))))$$

The derivative is computed by applying the Chain Rule iteratively for each composition.

Practice: Try to compute the derivative for functions such as $\cos(2x^3 + 3x)$ or $\ln(5x^2 + 2x)$ using the Chain Rule.

Useful properties of logarithms:

- ▶ **Product Rule:** $\log(ab) = \log a + \log b$
- ▶ **Quotient Rule:** $\log\left(\frac{a}{b}\right) = \log a - \log b$
- ▶ **Power Rule:** $\log(a^n) = n \log a$
- ▶ **Root Rule:** $\log\left(\sqrt[n]{a}\right) = \frac{1}{n} \log a$

Note: These are valid only when the arguments are positive real numbers.

General Formula from Logarithmic Differentiation



Given:

$$y = [u(x)]^{v(x)}$$

Taking logs:

$$\log y = v(x) \cdot \log[u(x)]$$

Differentiating:

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{v(x)u'(x)}{u(x)} + v'(x) \cdot \log[u(x)]$$

Multiplying both sides by y :

$$\frac{dy}{dx} = [u(x)]^{v(x)} \left(\frac{v(x)u'(x)}{u(x)} + v'(x) \cdot \log[u(x)] \right)$$

Illustrative Example: Setup



Differentiate:

$$y = \sqrt{\frac{(x-3)^2(x^2+4)^2}{3x^2+4x+5}}$$

Take logarithm on both sides:

$$\log y = \frac{1}{2} \cdot \log \left(\frac{(x-3)^2(x^2+4)^2}{3x^2+4x+5} \right)$$

Using log properties:

$$\log y = \frac{1}{2} [\log(x-3)^2 + \log(x^2+4)^2 - \log(3x^2+4x+5)]$$

Continue Example: Simplify Log Expression



Simplify the expression:

$$\log y = \log(x - 3) + \log(x^2 + 4) - \frac{1}{2} \log(3x^2 + 4x + 5)$$

Differentiate both sides:

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x - 3} + \frac{2x}{x^2 + 4} - \frac{6x + 4}{2(3x^2 + 4x + 5)}$$

Now multiply by y :

$$\frac{dy}{dx} = y \cdot \left[\frac{1}{x - 3} + \frac{2x}{x^2 + 4} - \frac{6x + 4}{2(3x^2 + 4x + 5)} \right]$$

Note: This method only applies where $u(x) > 0$.

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Linear Programming

We use derivatives to determine **maximum** or **minimum** values of functions by identifying the *turning points* of their graphs. These points are crucial in understanding the function's behavior and solving real-life problems.

Applications:

- ▶ The profit from a grove is given by $P(x) = ax + bx^2$. How many trees per acre maximize the profit?
- ▶ A ball is thrown from a 60 m high building. Its path is $h(x) = -x^2 + 60x + 60$. What is the **maximum height** reached?

- ▶ A helicopter follows the path $f(x) = x^2 + 7$. A soldier at (1, 2) wants to know the **nearest distance** to the helicopter.

To solve such problems, we first define:

- ▶ Local and absolute maxima/minima
- ▶ Turning points using derivatives
- ▶ Tests to identify such points

Let f be a function defined on an interval I . Then:

- (a) **Maximum Value:** f has a *maximum value* in I if there exists a point $c \in I$ such that

$$f(c) > f(x), \quad \text{for all } x \in I.$$

The number $f(c)$ is called the maximum value of f in I and c is called a point of maximum value.

- (b) **Minimum Value:** f has a *minimum value* in I if there exists a point $c \in I$ such that

$$f(c) < f(x), \quad \text{for all } x \in I.$$

The number $f(c)$ is called the minimum value of f in I and c is called a point of minimum value.

- (c) **Extreme Value:** f has an *extreme value* in I if $f(c)$ is either a maximum or a minimum value. The number $f(c)$ is then called an extreme value, and c is called an extreme point.

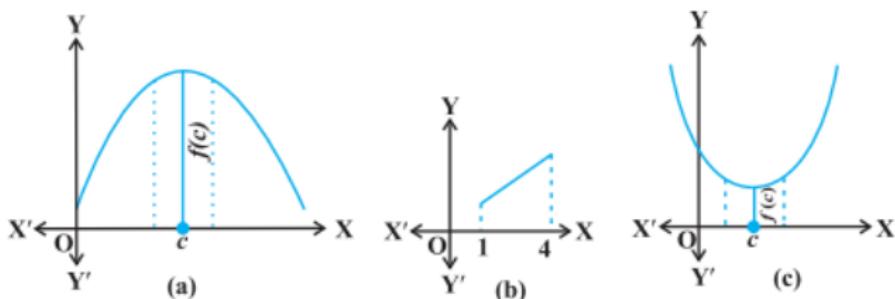


Figure 2: Graphs of Functions

- ▶ In Fig 2 (a), (b), and (c), the graphs of certain functions illustrate how maximum and minimum values can be identified visually.
- ▶ Graphs offer an intuitive way to understand the behavior of functions near a given point.
- ▶ They help us locate:
 - ▶ Local maxima
 - ▶ Local minima
 - ▶ Points of extreme values
- ▶ **Important:** Maxima and minima can sometimes be identified graphically even at points where the function is *not differentiable* (see Figure 3).

Non-Differentiable function

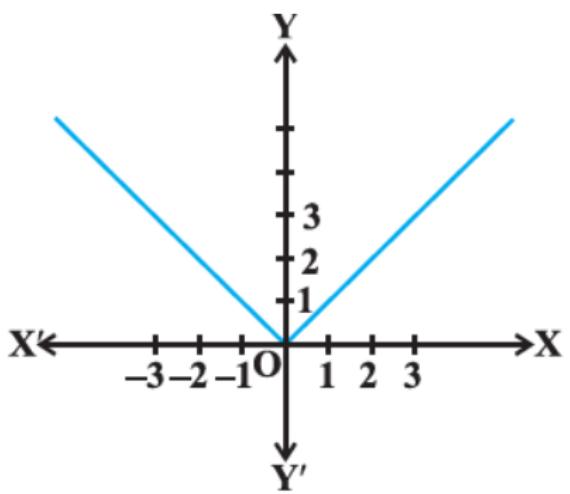


Figure 3: $f(x) = |x|$

Turning Points of a Function



- ▶ Let us examine the graph of a function as shown in Fig 4.
- ▶ Observe that at points A, B, C, and D, the function changes its nature from:
 - ▶ Decreasing to increasing or,
 - ▶ Increasing to decreasing.
- ▶ These points are called **turning points** of the function.

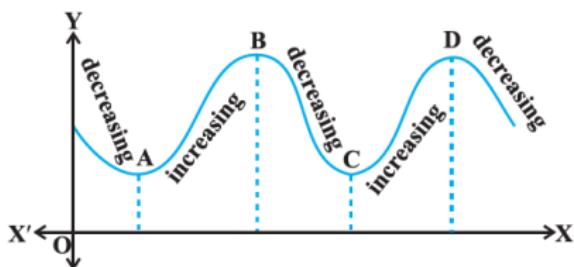


Figure 4: Turning Points of a Function

Nature of Turning Points



- ▶ At turning points, the graph has either:
 - ▶ A little hill, or
 - ▶ A little valley.
- ▶ **Points A and C** are at the bottom of valleys:
 - ▶ Function attains minimum value in their neighborhood.
- ▶ **Points B and D** are at the top of hills:
 - ▶ Function attains maximum value in their neighborhood.

- ▶ Points A and C: **Local Minima** (or Relative Minima)
- ▶ Points B and D: **Local Maxima** (or Relative Maxima)
- ▶ These are the values the function attains in a small interval around each point.
- ▶ Such local behavior helps us understand how the function behaves in parts of its domain.

Definition Let f be a real-valued function and let c be an interior point in the domain of f . Then:

- (a) c is called a **point of local maxima** if there exists $h > 0$ such that

$$f(c) \geq f(x), \quad \text{for all } x \in (c - h, c + h), \quad x \neq c$$

The value $f(c)$ is called the **local maximum value** of f .

- (b) c is called a **point of local minima** if there exists $h > 0$ such that

$$f(c) \leq f(x), \quad \text{for all } x \in (c - h, c + h)$$

The value $f(c)$ is called the **local minimum value** of f .

Geometrical Meaning of Local Extrema



- ▶ Geometrically, if $x = c$ is a point of **local maxima**, then:
 - ▶ $f'(x) > 0$ in $(c - h, c)$ (function is increasing)
 - ▶ $f'(x) < 0$ in $(c, c + h)$ (function is decreasing)
- ▶ This behavior suggests that the slope changes sign at $x = c$.

$$\Rightarrow f'(c) = 0$$

(Refer to Fig 5 for visual representation)

Geometrical Meaning of Local Extrema

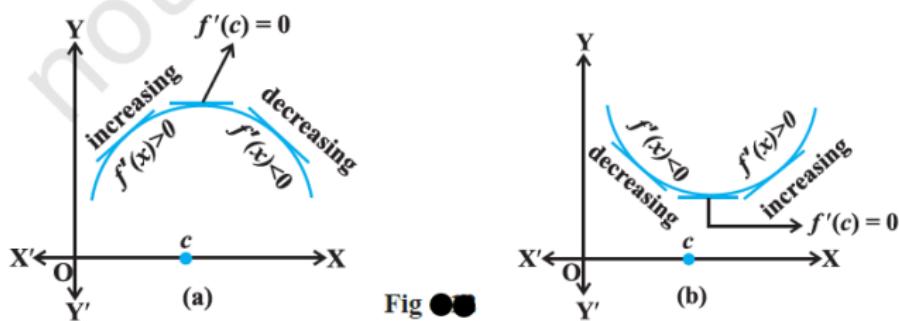


Fig 5

Figure 5: Geometrical Interpretation

First Derivative Test (Part 1)



Let f be a function defined on an open interval I , and let f be continuous at a critical point $c \in I$. Then:

- (i) If $f'(x)$ changes sign from **positive to negative** as x increases through c , i.e.,

$$f'(x) > 0 \text{ for } x < c, \quad f'(x) < 0 \text{ for } x > c,$$

then c is a point of **local maxima**.

- (ii) If $f'(x)$ changes sign from **negative to positive** as x increases through c , i.e.,

$$f'(x) < 0 \text{ for } x < c, \quad f'(x) > 0 \text{ for } x > c,$$

then c is a point of **local minima**.

First Derivative Test (Part 2)



- (iii) If $f'(x)$ does **not change sign** as x increases through c , then
- ▶ c is neither a point of local maxima nor local minima.
 - ▶ Such a point is called a **point of inflection**.

(Refer to Fig 6 for graphical illustration)

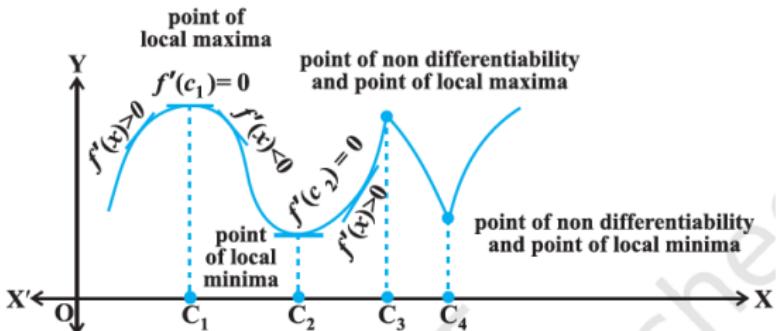


Figure 6: Derivative Test

Second Derivative Test



Let f be a function defined on an interval I and $c \in I$. Suppose f is twice differentiable at c . Then:

- (i) $x = c$ is a point of **local maxima** if

$$f'(c) = 0 \quad \text{and} \quad f''(c) < 0$$

Then $f(c)$ is the **local maximum value** of f .

- (ii) $x = c$ is a point of **local minima** if

$$f'(c) = 0 \quad \text{and} \quad f''(c) > 0$$

Then $f(c)$ is the **local minimum value** of f .

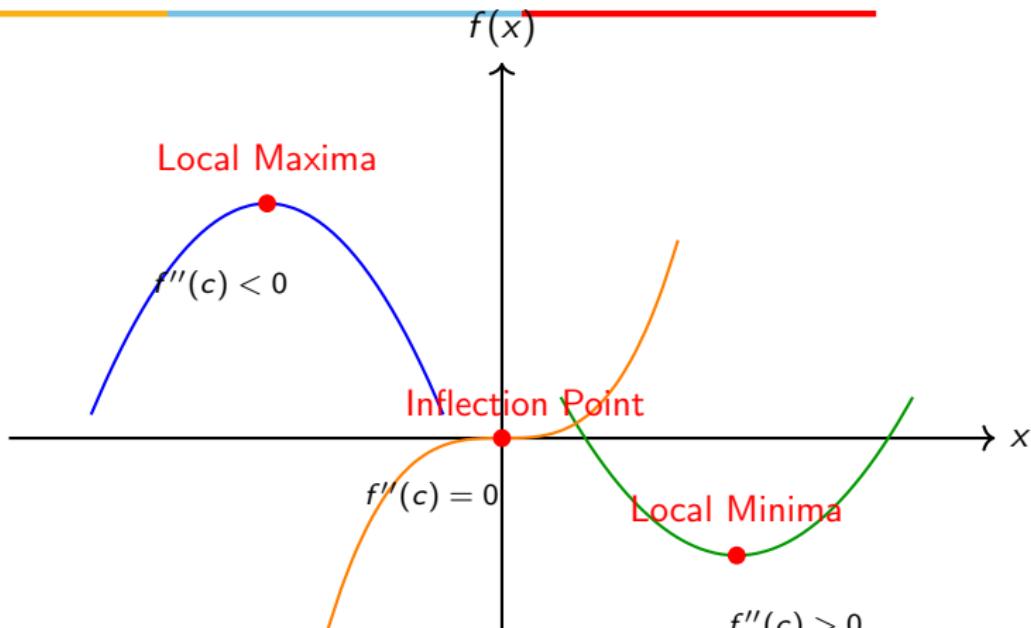
Second Derivative Test



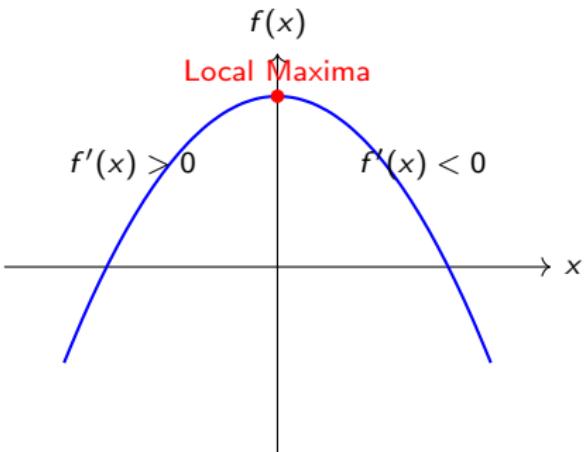
- (iii) If $f'(c) = 0$ and $f''(c) = 0$, the test **fails**.

In this case, apply the **First Derivative Test** to determine the nature of c .

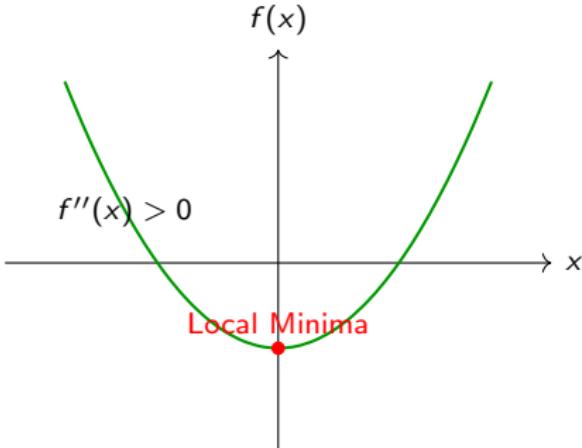
Graphical Illustration: Second Derivative Test



First Derivative Test



Second Derivative Test



First Derivative Test checks sign change in $f'(x)$; Second Derivative Test checks curvature via $f''(x)$.

Find the local maxima and minima of the function:

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 12$$

Step 1: First Derivative

$$f'(x) = \frac{d}{dx}(3x^4 + 4x^3 - 12x^2 + 12) = 12x^3 + 12x^2 - 24x$$

$$f'(x) = 12x(x - 1)(x + 2)$$

Critical Points: $x = 0, 1, -2$

Step 2: Second Derivative



Differentiate again:

$$f''(x) = \frac{d}{dx}(12x^3 + 12x^2 - 24x) = 36x^2 + 24x - 24$$

$$f''(x) = 12(3x^2 + 2x - 2)$$

Evaluate at critical points:

$$f''(0) = 12(0 - 2) = -24 \Rightarrow \text{Local Maxima}$$

$$f''(1) = 12(3 + 2 - 2) = 36 \Rightarrow \text{Local Minima}$$

$$f''(-2) = 12(12 - 4 - 2) = 72 \Rightarrow \text{Local Minima}$$

Step 3: Evaluate Values at Critical Points



$$f(0) = 3(0)^4 + 4(0)^3 - 12(0)^2 + 12 = \boxed{12}$$

$$f(1) = 3(1)^4 + 4(1)^3 - 12(1)^2 + 12 = 3 + 4 - 12 + 12 = \boxed{7}$$

$$f(-2) = 3(-2)^4 + 4(-2)^3 - 12(-2)^2 + 12 = 48 - 32 - 48 + 12 = \boxed{-20}$$

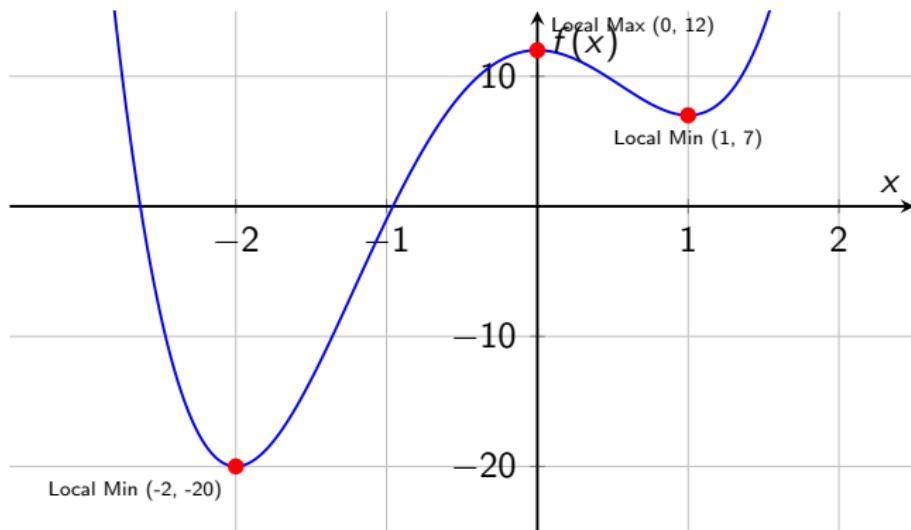
► **Local Maximum:**

- At $x = 0$, value $f(0) = 12$

► **Local Minima:**

- At $x = 1$, value $f(1) = 7$
- At $x = -2$, value $f(-2) = -20$

Graph of the Function & its Turning Points

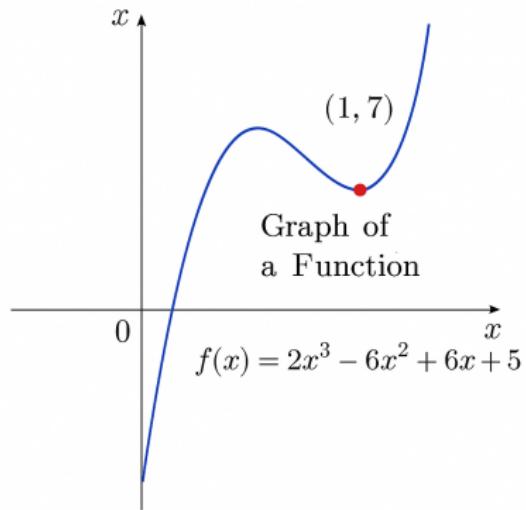


$$f(x) = 3x^4 + 4x^3 - 12x^2 + 12$$

Graph of $f(x) = 2x^3 - 6x^2 + 6x + 5$.
Find Local Maxima or Minima (if any)



Graph of a Function



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A furniture dealer deals in:

- ▶ **Tables** (Cost: Rs 2500, Profit: Rs 250)
- ▶ **Chairs** (Cost: Rs 500, Profit: Rs 75)

Resources:

- ▶ Investment: Rs 50000
- ▶ Storage space: 60 items

Let:

- ▶ x : Number of tables
- ▶ y : Number of chairs

Objective: Maximise $Z = 250x + 75y$

Let x be the number of tables and y the number of chairs the dealer buys.

Constraints:

- ▶ $x \geq 0, y \geq 0$ (Non-negativity)
- ▶ $5x + y \leq 100$ (Investment constraint)
- ▶ $x + y \leq 60$ (Storage constraint)

Optimization Problem:

Maximise $Z = 250x + 75y$ subject to:

$$5x + y \leq 100$$

$$x + y \leq 60$$

$$x, y \geq 0$$

Constraints:

$$2500x + 500y \leq 50000 \quad (\text{Investment})$$

$$x + y \leq 60 \quad (\text{Storage})$$

$$x \geq 0, y \geq 0 \quad (\text{Non-negativity})$$

Objective: Maximise Profit

$$Z = 250x + 75y$$

This is a classic **Linear Programming Problem (LPP)** used for optimisation.

The Concept: For every primal LPP, there exists a dual LPP derived systematically from it. The dual provides deep insight into the economic interpretation, bounds on the optimal value, and relationships between constraints and variables.

Primal (Standard Maximization Form):

$$\text{Maximize } Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

$$\text{subject to } a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m$$

$$x_j \geq 0, \quad j = 1, 2, \dots, n$$

Associated Dual (Minimization Form):

Minimize $W = b_1y_1 + b_2y_2 + \cdots + b_my_m$

subject to $a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \geq c_1$

$a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \geq c_2$

⋮

$a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \geq c_n$

$y_i \geq 0, i = 1, 2, \dots, m$

Key Relationships Between Primal and Dual Problems:

- ▶ The solution (objective value) of the **dual** is always **greater than or equal to** the solution of the **primal**.

- ▶ At **optimality**, the objective values of the **primal and dual** are **exactly equal**.

Note: The dual of the dual is the original primal again.

Students task: Write the dual of discussed LPP of table chair problem.

Thank You!

Questions or Discussions are Welcome.
