Lecture 0a

Math Foundations Team

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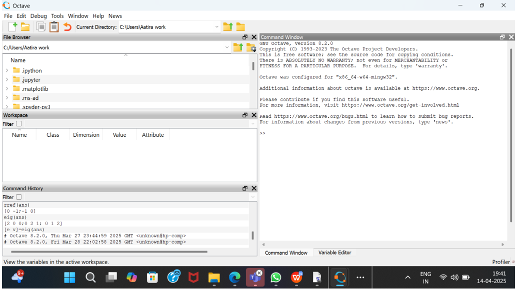
Octave

- Octave is very similar to MATLAB — often used as a free alternative.
- Ideal for linear algebra, calculus, differential equations, and data visualization.
- The syntax for octave is MATLAB compatible.
- Download from: https://www.gnu.org/software/octave/

Getting Started with Octave

To start, when we double click on the icon, what we get is



One can start typing commands in command prompt or script editor.

Example

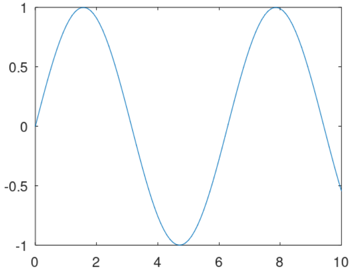
Let us type this in the command prompt.

x = 0:0.1:10;

y = sin(x);

plot(x, y);

Then, we get



Matrices

- A **matrix** is a rectangular array of numbers or functions which we will enclose in brackets. For example,

$$\begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix}, \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \qquad \textbf{(1)}$$

$$\begin{bmatrix} e^{-x} & 2x^2 \\ e^{6x} & 4x \end{bmatrix}, \quad \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ \frac{1}{2} \end{bmatrix}$$

- The numbers (or functions) are called **entries** or, less commonly, *elements* of the matrix.
- The first matrix in (1) has two **rows**, which are the horizontal lines of entries.

Matrix-Notations

- We shall denote matrices by capital boldface letters **A**, **B**, **C**, … , or by writing the general entry in brackets; thus **A** = $[a_{jk}]$, and so on.
- By an $m \times n$ **matrix** (read *m by n matrix*) we mean a matrix with m rows and n columns—rows always come first! $m \times n$ is called the **size** of the matrix. Thus an $m \times n$ matrix is of the form

$$\mathbf{A} = \left[a_{jk}\right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}. \qquad \textbf{(2)}$$

Octave: A=[1 2 3; 4 5 6]

Vector

- A **vector** is a matrix with only one row or column. Its entries are called the **components** of the vector.
- We shall denote vectors by ***lowercase*** boldface letters **a**, **b**, … or by its general component in brackets, $\mathbf{a} = [a_j]$, and so on. Our special vectors in (1) suggest that a (general) **row vector** is of the form

$$\mathbf{a} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}. \quad \text{For instance,} \quad \mathbf{a} = \begin{bmatrix} -2 & 5 & 0.8 & 0 & 1 \end{bmatrix}.$$

A **column vector**

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}. \quad \text{For instance,} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix}.$$

Equality of matrices

- Two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ are **equal**, written $\mathbf{A} = \mathbf{B}$, if and only if (1) they have the same size and (2) the corresponding entries are equal, that is, $a_{11} = b_{11}$, $a_{12} = b_{12}$, and so on.

- Matrices that are not equal are called **different**. Thus, matrices of different sizes are always different.

Algebra of Matrices

1. Addition of Matrices

- The **sum** of two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ ***of the same size*** is written $\mathbf{A} + \mathbf{B}$ and has the entries $a_{jk} + b_{jk}$ obtained by adding the corresponding entries of $\mathbf{A}$ and $\mathbf{B}$. Matrices of different sizes cannot be added.

2. Scalar Multiplication (Multiplication by a Number)

- The **product** of any $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ and any **scalar** c (number c) is written $c\mathbf{A}$ and is the $m \times n$ matrix $c\mathbf{A} = [ca_{jk}]$ obtained by multiplying each entry of $\mathbf{A}$ by c.

(a) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

(b) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ (written $\mathbf{A} + \mathbf{B} + \mathbf{C}$)

(c) $\mathbf{A} + \mathbf{0} = \mathbf{A}$

(d) $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$.

(a) $c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$

(b) $(c + k)\mathbf{A} = c\mathbf{A} + k\mathbf{A}$

(c) $c(k\mathbf{A}) = (ck)\mathbf{A}$ (written $ck\mathbf{A}$)

(d) $1\mathbf{A} = \mathbf{A}$.

- Here $\mathbf{0}$ denotes the **zero matrix** (of size $m \times n$), that is, the $m \times n$ matrix with all entries zero.

Multiplication of a Matrix by a Matrix

- The **product** $\mathbf{C} = \mathbf{AB}$ (in this order) of an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ times an $r \times p$ matrix $\mathbf{B} = [b_{jk}]$ is defined if and only if $r = n$ and is then the $m \times p$ matrix $\mathbf{C} = [c_{jk}]$ with entries

(3) $$c_{jk} = \sum_{l=1}^{n} a_{jl} b_{lk} = a_{j1}b_{1k} + a_{j2}b_{2k} + \cdots + a_{jn}b_{nk} \qquad \begin{matrix} j = 1, \cdots, m \\ k = 1, \cdots, p. \end{matrix}$$

- The condition $r = n$ means that the second factor, **B**, must have as many rows as the first factor has columns, namely n. A diagram of sizes that shows when matrix multiplication is possible is as follows:

$$\begin{matrix} \mathbf{A} & \mathbf{B} & = & \mathbf{C} \\ [m \times n] & [n \times p] & = & [m \times p]. \end{matrix}$$

Matrix Multiplication

EXAMPLE 1

$$AB = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 22 & -2 & 43 & 42 \\ 26 & -16 & \boxed{14} & 6 \\ -9 & 4 & -37 & -28 \end{bmatrix}$$

- Here $c_{11} = 3 \cdot 2 + 5 \cdot 5 + (-1) \cdot 9 = 22$, and so on. The entry in the box is $c_{23} = 4 \cdot 3 + 0 \cdot 7 + 2 \cdot 1 = 14$.

- The product **BA** is not defined.

Matrix Multiplication

Matrix Multiplication Is *Not Commutative*, AB ≠ BA in General

- This is illustrated by Example 1, where one of the two products is not even defined. But it also holds for square matrices. For instance,

$$\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{but} \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} = \begin{bmatrix} 99 & 99 \\ -99 & -99 \end{bmatrix}.$$

- It is interesting that this also shows that $\mathbf{AB} = \mathbf{0}$ does *not* necessarily imply $\mathbf{BA} = \mathbf{0}$ or $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$.

Transposition of Matrices

- The transpose of an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ is the $n \times m$ matrix $\mathbf{A}^\mathsf{T}$ (read *A transpose*) that has the first *row* of $\mathbf{A}$ as its first *column*, the second *row* of $\mathbf{A}$ as its second *column*, and so on. Thus the transpose of $\mathbf{A}$ in (2) is $\mathbf{A}^\mathsf{T} = [a_{kj}]$, written out

$$\mathbf{A}^\mathsf{T} = [a_{kj}] = \begin{bmatrix} a_{11} & \text{a21} & \cdots & a_{m1} \\ \text{a12} & a_{22} & \cdots & a_{m2} \\ \cdot & \cdot & \cdots & \cdot \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}.$$

- As a special case, transposition converts row vectors to column vectors and conversely.

Transposition of Matrices

- <u>Rules for transposition are</u>

$$
\begin{aligned}
&\text{(a)} \quad (\mathbf{A}^{\mathsf{T}})^{\mathsf{T}} = \mathbf{A}\\
&\text{(b)} \quad (\mathbf{A}+\mathbf{B})^{\mathsf{T}} = \mathbf{A}^{\mathsf{T}} + \mathbf{B}^{\mathsf{T}}\\
&\text{(c)} \quad (c\mathbf{A})^{\mathsf{T}} = c\mathbf{A}^{\mathsf{T}}\\
&\text{(d)} \quad (\mathbf{AB})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}.
\end{aligned}
\tag{5}
$$

CAUTION! Note that in (5d) the transposed matrices are ***in reversed order.***

Octave for Matrices

```
A=[1 2 3; 4 5 6];
B=[2 4 0; 6 7 8];
C=[1 2; 3 4; 4 1];
disp('The sum of 2 matrices is')
A+B
disp('4A is given by')
4*A
disp('The product AB')
A*B
disp('The product BA')
B*A
disp('The transpose of A' )
A'
```

Special Matrices

- Symmetric: $a_{ij} = a_{ji}$ Eg: $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 5 \end{bmatrix}$
- Skew Symmetric : $a_{ij} = -\, a_{ji}$ Eg: $\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$
- Triangular: Upper Triangular → $a_{ij} = 0$ for all $i > j$
  Lower Triangular → $a_{ij} = 0$ for all $i < j$

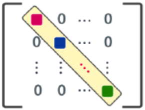
Upper triangular matrix: U

$$\begin{bmatrix} 1 & 1/2 & 3 & 0 \\ 0 & 5 & 0 & 1 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Lower triangular matrix: L

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 1 & -2 & 0 \end{bmatrix}$$

- Diagonal Matrix: $a_{ij} = 0$ for all $i \neq j$ Eg:



- Sparse Matrix: Many zeroes and few non-zero entities

$$\begin{bmatrix} 0 & 0 & 3 & 0 & 4 \\ 0 & 0 & 5 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 6 & 0 & 0 \end{bmatrix}$$

Positive Definite Matrix

- Let **A** be a real symmetric matrix. Then **A** is positive definite if for any x ≠ 0,

$$\boldsymbol{x^T A x > 0}$$

- Example:
- A = $\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}$ $x^T A x$ = $\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

  $=2x_1^2 + 12x_1x_2 + 20x_2^2 = 2(x_1 + 3x_2)^2 + 2x_2^2 > 0$

- **A** is positive semi-definite **if** $\boldsymbol{x^T A x \geq 0}$
- A = $\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}$ $\Rightarrow$ $x^T A x$ = $2(x_1 + 3x_2)^2$ = 0 when x_1=3 and $x_2 = -1$

Elementary Row Operations

Given a matrix A, the following operations are called Elementary Row Operations

- *Interchange of two rows*
- *Addition of a constant multiple of one row to another row*
- *Multiplication of a row by a **non-zero** constant c*

CAUTION! These operations are for rows, ***not for columns***!

Row Echelon Form (REF)

- Any rows of all zeros are below any other non zero rows.
- The first non zero entry in each row is called as the leading entry of that row.
- Each leading entry of a row is in a column to the right of the leading entry of the row above it.
- All entries in a column below a leading entry are zeros.

Example

$$\begin{bmatrix} 3 & 2 & 0 & 7 & 9 \\ 0 & 4 & 5 & 10 & 0 \\ 0 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Row Echelon Form (RREF)

innovate achieve lead

- We say that a matrix is in Reduced Row Echelon Form if it is in Echelon form and additionally,
  1. The leading entry in each row is 1.
  2. Each leading 1 is the only non zero entry in its column

- Example

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 9 \\ 0 & 1 & 4 & 0 & -6 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Uniqueness of RREF

- We can transform any matrix into a matrix in reduced row echelon form by using elementary row operations.
- No matter what sequence of row operations we use each matrix is row equivalent to one and only one reducedrow echelon matrix

REF

$$\begin{bmatrix} 0 & 1 & 2 & 0 & -1 \\ -2 & 2 & 0 & 3 & 2 \\ 2 & 8 & 20 & 0 & 6 \end{bmatrix} \xrightarrow{\text{Swap rows 1 and 2}} \begin{bmatrix} -2 & 2 & 0 & 3 & 2 \\ 0 & 1 & 2 & 0 & -1 \\ 2 & 8 & 20 & 0 & 6 \end{bmatrix}$$

Replace R3 by R3+1.R1

$$\begin{bmatrix} -2 & 2 & 0 & 3 & 2 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 10 & 20 & 3 & 8 \end{bmatrix}$$

Replace R3 by R3+(-10).R2

$$\begin{bmatrix} -2 & 2 & 0 & 3 & 2 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 3 & 18 \end{bmatrix}$$

Row Echelon Form

Rank of a Matrix

- The number of non zero rows, r in the reduced row or row echelon form $\boldsymbol{R}$ of matrix is called the rank of $\boldsymbol{R}$ and also the rank of $\boldsymbol{A}$.
- The rank is invariant under elelmentary row operations.
- In the last example, number of nonzero rows is 3 and hence rank is 3.

Rank

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix} \quad \text{(given)}$$

$$= \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix} \quad \begin{matrix} \text{Row 2} + 2\,\text{Row 1} \\ \text{Row 3} - 7\,\text{Row 1} \end{matrix}$$

$$= \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Row 3} + \frac{1}{2}\,\text{Row 2.}$$

- The last matrix is in row-echelon form and has two nonzero rows. Hence rank $\mathbf{A} = 2$.

Minor

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Each element in **A** has a minor.

Delete first row and column from **A**. The determinant of the remaining 2x2 submatrix is the minor of a_{11} which is $a_{22}a_{33} - a_{23}a_{32}$

$$m_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Minor

$$m_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

And the minor for a_{13} is:

$$m_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Cofactor

The cofactor C_{ij} of an element a_{ij} is defined as:

$$C_{ij} = (-1)^{i+j} m_{ij}$$

When the sum of a row number *i* and column *j* is even, $c_{ij} = m_{ij}$ and when *i+j* is odd, $c_{ij} = -m_{ij}$

$$c_{11}(i=1, j=1) = (-1)^{1+1} m_{11} = +m_{11}$$

$$c_{12}(i=1, j=2) = (-1)^{1+2} m_{12} = -m_{12}$$

$$c_{13}(i=1, j=3) = (-1)^{1+3} m_{13} = +m_{13}$$

Determinant

The determinant of an n x n matrix **A** can now be defined as

$$|A| = \det A = a_{11}c_{11} + a_{12}c_{12} + \ldots + a_{1n}c_{1n}$$

The determinant of **A** is therefore the sum of the products of the elements of the first row of **A** and their corresponding cofactors.

(It is possible to define |**A**| in terms of any other row or column but for simplicity, the first row only is used)

Determinant

Therefore the 2 x 2 matrix :

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Example 1:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

Has cofactors :

$$c_{11} = m_{11} = |a_{22}| = a_{22}$$

$$|A| = (3)(2) - (1)(1) = 5$$

And:

$$c_{12} = -m_{12} = -|a_{21}| = -a_{21}$$

And the determinant of **A** is:

$$|A| = a_{11}c_{11} + a_{12}c_{12} = a_{11}a_{22} - a_{12}a_{21}$$

Determinant

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The cofactors of the first row are:

$$c_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

$$c_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -(a_{21}a_{33} - a_{23}a_{31})$$

$$c_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{22}a_{31}$$

Determinant

$$|A| = a_{11}c_{11} + a_{12}c_{12} = a_{11}a_{22} - a_{12}a_{21}$$

Which by substituting for the cofactors in this case is:

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Example 2:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$$

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$|A| = (1)(2-0) - (0)(0+3) + (1)(0+2) = 4$$

Properties of Determinants

1. $\det(AB) = \det(A) * \det(B)$
2. $\det(A)$ nonzero implies there exists a matrix B such that $AB=BA=I$
3. Two Rows Equal $\rightarrow$ $\det = 0$(Singular)
4. R_i and R_j swapped $\rightarrow$ det gets a minus sign ($i \neq j$)
5. $\det(A) = \det(A^T)$
6. $R_i \leftarrow cR_j \rightarrow \det A \leftarrow c \det A$

Adjoint

The adjoint matrix of **A**, denoted by adj **A**, is the transpose of its cofactor matrix

$$adj A = C^T$$

It can be shown that:

$$\mathbf{A}(\text{adj } \mathbf{A}) = (\text{adj}\mathbf{A})\ \mathbf{A} = |\mathbf{A}|\ \mathbf{I}$$

Example: $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$

$$|A| = (1)(4) - (2)(-3) = 10$$

$$adj A = C^T = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}$$

Adjoint

$$A(adj\,A) = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = 10I$$

$$(adj\,A)A = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = 10I$$

Inverse

- $A^{-1} = \dfrac{adj(A)}{det(A)}$ where $det(A) \neq 0$

Reiterate $det(A) \neq 0 \rightarrow$ A is Non singular

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

Inverse

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

The determinant of **A** is

$$|\mathbf{A}| = (3)(-1-0)-(-1)(-2-0)+(1)(4-1) = -2$$

The elements of the cofactor matrix are

$$c_{11} = +(-1), \quad c_{12} = -(-2), \quad c_{13} = +(3),$$
$$c_{21} = -(-1), \quad c_{22} = +(-4), \quad c_{23} = -(7),$$
$$c_{31} = +(-1), \quad c_{32} = -(-2), \quad c_{33} = +(5),$$

Inverse

The cofactor matrix is therefore

$$C = \begin{bmatrix} -1 & 2 & 3 \\ 1 & -4 & -7 \\ -1 & 2 & 5 \end{bmatrix}$$

so

$$adj A = C^T = \begin{bmatrix} -1 & 1 & -1 \\ 2 & -4 & 2 \\ 3 & -7 & 5 \end{bmatrix}$$

and

$$A^{-1} = \frac{adj A}{|A|} = \frac{1}{-2}\begin{bmatrix} -1 & 1 & -1 \\ 2 & -4 & 2 \\ 3 & -7 & 5 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ -1.0 & 2.0 & -1.0 \\ -1.5 & 3.5 & -2.5 \end{bmatrix}$$

Inverse

The result can be checked using

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

The determinant of a matrix must not be zero for the inverse to exist as there will not be a solution

Nonsingular matrices have non-zero determinants

Singular matrices have zero determinants

Inverse

So that for a 2×2 matrix the inverse can be constructed in a simple fashion as

$$\boldsymbol{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{|\boldsymbol{A}|} & \frac{-b}{|\boldsymbol{A}|} \\ \frac{-c}{|\boldsymbol{A}|} & \frac{a}{|\boldsymbol{A}|} \end{bmatrix} = \frac{1}{|\boldsymbol{A}|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Exchange elements of diagonal.
- Change sign of non main diagonal elements.
- Divide resulting matrix by the determinant of the original matrix.

Some More Octave Command

- A = zeros(m,n) creates a zero matrix A of size m x n
- A = eye(k) creates an identity matrix A of size k
- $[r, c]$ = size(A) gives the size of A and the number of rows is stored in r and number of columns in c
- rref(A) gives the reduced row echelon form of A
- rank(A) gives the rank of A
- det(A) gives the determinant of A
- inv(A) gives the inverse of A
- issymmetric(A) returns either 0 (not symmetric) or 1 (symmetric)

Linear System

- A **linear system of** m **equations in** n **unknowns** $x_1, \dots,$ x_n is a set of equations of the form

$$
\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\
&\cdots\cdots\cdots\cdots \\
a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m.
\end{aligned}
\qquad \textbf{(1)}
$$

- The system is called *linear* because each variable x_j appears in the first power only, just as in the equation of a straight line.

Linear System

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots&\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m . \end{aligned} \qquad \textbf{(1)}$$

- $a_{11}, \ldots, a_{mn}$ are given numbers, called the **coefficients** of the system.
- $b_1, \ldots, b_m$ on the right are also given numbers.
- If all the bj are zero, then **(1)** is called a **homogeneous system**.
- If at least one bj is not zero, then **(1)** is called a **non-homogeneous system**.

Matrix Form of the Linear System (1). (continued)

- We assume that the coefficients a_{jk} are not all zero, so that **A** is not a zero matrix. Note that **x** has n components, whereas **b** has m components. The matrix

$$\tilde{\mathbf{A}} = \left[\begin{array}{ccc:c} a_{11} & \cdots & a_{1n} & b_1 \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

is called the **augmented matrix** of the system (1).

- The dashed vertical line could be omitted, as we shall do later. It is merely a reminder that the last column of $\tilde{\mathbf{A}}$ did not come from matrix **A** but came from vector **b**. Thus, we *augmented* the matrix **A**.

Example

Consider the folllowing system of linear equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 - x_2 + 2x_3 &= 2 \\ 2x_1 + 3x_3 &= 1 \end{aligned}$$

Adding the first and second equations gives $2x_1 + 3x_3 = 5$ which contradicts the third equation. Thus there is no set of values for the variables x_1, x_2, x_3 such that the equations above are simultaneously satisfied.

Modified example

Consider a slightly modified example

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 3 \\
x_1 - x_2 + 2x_3 &= 2 \\
x_2 + x_3 &= 2
\end{aligned}
$$

In this case we can see from the first and third equations that $x_1 = 1$. Substituting this value of x1 into equation (2), we get $-x_2 + 2x_3 = 1$. Adding this equation to equation (3), we get $3x_3 = 3$ which means $x_3 = 1$. Substituting $x_3 = 1$ into equation (3) shows $x_2 = 1$, so the overall solution is $x_1 = x_2 = x_3 = 1$. This is the unique solution to the problem

Infinite solutions

Now consider another modification to the original set of equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 - x_2 + 2x_3 &= 2 \\ 2x_1 + 3x_3 &= 5 \end{aligned}$$

Adding the first and second equations gives $2x_1 + 3x_3 = 5$ which is the same as the third equation. Thus the solution to the three equations is any tuple x_1, x_2, x_3 which satisfies $2x_1 + 3x_3 = 5$, and there are infinite solutions. We now express these solutions in a way whose motivation will become clear later: adding equations (1) and (2) above we get $2x_1 = 5 - 3x_3$.

Infinite solutions

- Subtracting equation (2) from (1) we get $2x_2 - x_3 = 1$, so we can write

$$\begin{aligned} x_1 &= \frac{5}{2} - \frac{3}{2}x_3 \\ x_2 &= \frac{1}{2} + \frac{x_3}{2} \end{aligned}$$

- For the previous problem we can express the set of infinite solutions in terms of the free variable x_3.
- Once x_3 is fixed, the other two variables have to take on specific values - they are known as pivot variables.

- To solve we always try to eliminate variables and then solve.
- To do that we have to add a multiple of one equation to the other etc.
- So, if we consider the matrix form of the system of transformed equations for which solutions could be obtained essentially by inspection.
- To make this work we need to preserve solutions of the original system of equations. So whatever elementary transformations we perform on coefficient matrix, same row transformation needs to be performed on the constant vector. So, we perform all elementary row transformation on the augmented matrix of the linear system.

- The system is inconsistent if the rank of the coefficient matrix is less than the rank of augmented matrix.
- The system is consistent if the rank of the coefficient matrix is same as the rank of augmented matrix.
- For the consistent system, if there are no free variables, the solution is unique.
- For the consistent system, if there are free variables, there are infinitely many solutions.