UNIT-IV
LINEAR ALGEBRA

1. Eigen values & vectors. } they are inter-related 8 marks Ques.
2. Diagonalization.
3. Symmetric & Skew-symmetric matrices
4. Orthogonal matrices
5. Similar matrices } 4 marks Ques.

Definitions :-

(1) **Characteristic Eqn.**

Let A be a square matrix
Then the equation

$$\text{determinant } [A - \lambda I] = 0$$

is called characteristic eq. of A

(2) **Eigen value or latent root**

The roots of characteristic eqn. of A are called Eigen values of A.

An $n \times n$ ordered matrix has n eigen values.

If A is 3×3 order, then A will have three Eigen values. The Eigen values of A are unique.

Eigen value is a stretching factor.
Eigen value has some resemblence to wavelength.

(3) **Eigen Vector :-**

Let λ be an eigen value of A.

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A column vector X satisfying the eqn. $AX = \lambda X$ is called eigen vector of A corresponding to eigen value λ.

Eigen vectors ~~can~~ are not unique. Corresponding to one eigen value, there can be infinitely many eigen vectors.

(4) <u>Modal Matrix</u>:- Let A be a matrix of order $n \times n$. Then A will have n eigen values $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$.

Let $X_1, X_2, \ldots, X_n$ be n linearly independent eigen vectors of A corresponding to eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively. Then the matrix $P = [X_1, X_2, \ldots, X_n]$

is called modal matrix of A.

(5) <u>Diagonalization of A</u>

$P^{-1}AP$ is a diagonal matrix whose diagonal matrix come out to be equal to eigen values of A. This process is called diagonalization of A.

(6) <u>Cayley Hamilton theorem</u>:-

Every sq. matrix satisfies its own characteristic eqn.

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Que Diagonalise $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Verify Cayley Hamilton theorem for A.

Hence find A^{-1}.

Sol

Step-I Characteristic eqn. of A is $\det[A - \lambda I] = 0$

$$\Rightarrow \left| \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 7\lambda + 10 - 4 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 6 = 0$$

This is characteristic eqn. of A

Step-II Find Eigen values of A

$\lambda^2 - 7\lambda + 6 = 0 \Rightarrow \lambda = 1$ or 6 are eigen values of A

Step III Eigen vectors

when $\lambda = 1$, Let $X_1 = \begin{bmatrix} x \\ y \end{bmatrix}$

be the eigen vector corresponding to eigen value $\lambda = 1$

$\therefore AX_1 = \lambda X_1$

$\Rightarrow [A - \lambda I] X_1 = 0$

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$\Rightarrow [A-(1)I]X_1 = 0 \Rightarrow \begin{bmatrix} 5-1 & 4 \\ 1 & 2-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow 4x + 4y = 0$
$x + y = 0 \Rightarrow x + y = 0$

Let $x = 1 \Rightarrow y = -1$

$\therefore X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

When $\lambda = 6$, Let $X_2 = \begin{bmatrix} x \\ y \end{bmatrix}$

be the eigen vector corresponding to eigen value $\lambda = 6$

$\therefore AX_2 = \lambda X_2$

$\Rightarrow [A - \lambda I]X_2 = 0$

$\Rightarrow [A - 6I]X_2 = 0 \Rightarrow \begin{bmatrix} 5-6 & 4 \\ 1 & 2-6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-x + 4y = 0 \Rightarrow x - 4y = 0$
$x - 4y = 0$

Let $y = 1 \Rightarrow x = 4$

$\therefore X_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

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Step-IV Modal Matrix of A is

$$P = [X_1 \; X_2] = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}$$

$$|P| = 1+4 = 5 \neq 0$$

Step-V Diagonalisation of A.

$$\therefore P^{-1} = \frac{1}{|P|} \text{adj} \cdot P = \frac{1}{5}\begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{5}\begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 5-4 & 4-8 \\ 5+1 & 4+2 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 1 & -4 \\ 6 & 6 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 1+4 & 4-4 \\ 6-6 & 24+6 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 5 & 0 \\ 0 & 30 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

Step-VI Verification of Cayley Hamilton Thm.

Ch. eq. of A is $\lambda^2 - 7\lambda + 6 = 0$

By Cayley Hamilton Thm,

$$A^2 - 7A + 6I = 0 \quad \text{--- (1)}$$

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L.H.S. $A^2 = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 29 & 28 \\ 7 & 8 \end{bmatrix}$

$7A = 7\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 35 & 28 \\ 7 & 14 \end{bmatrix}$

$\therefore A^2 - 7A + 6I = \begin{bmatrix} 29 & 28 \\ 7 & 8 \end{bmatrix} - \begin{bmatrix} 35 & 28 \\ 7 & 14 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$

$= \begin{bmatrix} 29-35+6 & 28-28-0 \\ 7-7+0 & 8-14+6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$=$ R.H.S.

Hence Cayley Hamilton Thm. has been verified for A.

Multiplying both sides of ① by A^{-1}

$A - 7I + 6A^{-1} = 0$

$\Rightarrow A^{-1} = \frac{1}{6}[7I - A] = \frac{1}{6}\left[\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} - \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\right]$

$= \frac{1}{6}\begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix}$

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Que Diagonalise $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

Sol Step I Characteristic eqn. of A is

$$\det[A - \lambda I] = 0$$

$$\Rightarrow \left| \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

$$3-\lambda[(2-\lambda)(5-\lambda) - 0] - 1(0) + 4(0) = 0$$

Right $(3-\lambda)(2-\lambda)(5-\lambda) = 0$

$(6 - 3\lambda - 2\lambda + \lambda^2)(5-\lambda) = 0$

$30 - 6\lambda - 15\lambda + 3\lambda^2 - 10\lambda + 2\lambda^2 + 5\lambda^2 - \lambda^3 = 0$

$-\lambda^3 + 10\lambda^2 - 31\lambda + 30 = 0$

$\lambda^3 - 10\lambda^2 + 31\lambda - 30 = 0$

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$$(3-\lambda)(2-\lambda)(5-\lambda)=0$$

Step-II Eigen values $\lambda = 2, 3, 5$

Step-III Eigen vectors when $\lambda=2$ Let $X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be

eigen vector corresponding to $\lambda=2$

$$\therefore [A-\lambda I]X_1 = 0$$

$$\Rightarrow [A-2I]X_1 = 0$$

$$\left[\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x+y+4z=0$$
$$0+0+6z=0$$
$$0+0+3z=0$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 4 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right] \quad R_3 \to R_3 - \frac{1}{2}R_2$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 4 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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$\rho(A) = 2 < $ no of unknowns $= 3$

Thus the system has infinitely many soln.

From R_2, $6z = 0 \Rightarrow z = 0$
Let $y = 1$

From R_1, $x + y + 4z = 0$
$\Rightarrow x + 1 + 0 = 0$
$\Rightarrow x = -1$

$$\therefore X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

When $\lambda = 3$, Let $X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

eigen vector corresponding to $\lambda = 2$

$\therefore [A - \lambda I] X_2 = 0$

$\Rightarrow [A - 3I] X_2 = 0$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$y + 4z = 0$
$-y + 6z = 0$
$2z = 0$

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From third eqn. , $z=0$

From 2nd , $-y+6(0)=0$
$\Rightarrow y=0$

Let x be 1 , $x=1$

$$X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

when $\lambda=5$, Let $X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

eigen vector corresponding to $\lambda=5$

$\therefore [A-\lambda I]X_3=0$

$\Rightarrow [A-5I]X_3=0$

$$\Rightarrow \left[\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}\right]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x+y+4z=0$
$0-3y+6z=0$
$0+0+0=0$

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$$\Rightarrow \left[\begin{array}{ccc:c} -2 & 1 & 4 & 0 \\ 0 & -3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rank is $2 <$ no of unknowns.

$\therefore$ The system has infinitely many solns.

Let $z = 1$

$\therefore$ From R_2, $-3y + 6z = 0$

$$-3y + 6(1) = 0$$

$$y = 2$$

From R_1, $-2x + 1y + 4z = 0$

$$-2x + 2 + 4 = 0$$

$$-2x = -6$$

$$x = 3$$

$$X_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Step-IV Modal matrix is $P = \begin{bmatrix} -1 & 1 & 3 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Step-V (B) Diagonalisation of A

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$P = IP$

$$\begin{bmatrix} -1 & 1 & 3 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} P$$

Apply $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} P$$

Apply $R_2 \to R_2 + R_1$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} P$$

Apply $R_1 \to R_1 - 2R_3$
$R_2 \to R_2 - 5R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} P$$

$$P^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} -1 & 1 & 3 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$A^2 = \begin{bmatrix} 0 & 2 & -4 \\ 3 & 3 & -15 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} -1 & 1 & 3 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Verification of Cayley thm.

Ch-eq. of A is $\lambda^3 - 10\lambda^2 + 31\lambda - 30 = 0$

By Hamilton thm. $A^3 - 10A^2 + 31A - 30 = 0$

$$\text{LHS } A^3 = \begin{bmatrix} 3 & 1 & 9 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 5 & 38 \\ 0 & 4 & 42 \\ 0 & 0 & 25 \end{bmatrix} \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 27 & 19 \end{bmatrix}$$

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Que Diagonalize $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Sol

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

Step-I Characteristic eqn. of A is given by

$\det.(A - \lambda I) = 0$

$$\left| \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$\cancel{1-\lambda(6-3\lambda-2+\lambda)+1(2-\lambda-4+2\lambda)=0}$

$\cancel{(1-\lambda)(}$ $\Rightarrow +\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\cancel{\Rightarrow \lambda^3 - 6\lambda^2 - 11\lambda + 6 = 0}$

By hit and trial $\lambda=1$ is root of characteristic eqn.

$\therefore$ $(\lambda - 1)$ is a factor

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$$\begin{array}{r|l} & \lambda^2 - 5\lambda + 6 \\ \hline \lambda - 1 & \lambda^3 - 6\lambda^2 + 11\lambda - 6 \\ & \lambda^3 - \lambda^2 \\ & - \quad + \\ \hline & -5\lambda^2 + 11\lambda \\ & -5\lambda^2 + 5\lambda \\ & + \quad - \\ \hline & 6\lambda - 6 \\ & 6\lambda - 6 \\ & + \quad - \\ \hline & 0 \end{array}$$

$\lambda^2 - 5\lambda + 6 = 0$

$\Rightarrow \lambda = 2$ or 3

$\therefore$ $\lambda = 1, 2, 3$ are the eigen values of A

Eigen vectors

When $\lambda = 3$, Let $X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigen vector corresponding to eigen value $\lambda = 3$

$\therefore$ $[A - \lambda I] X_3 = 0$

$\Rightarrow [A - 3I] X_3 = 0$

$$\Rightarrow \begin{bmatrix} 1-3 & 0 & -1 \\ 1 & 2-3 & 1 \\ 2 & 2 & 3-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$\begin{bmatrix} -2x - z \\ x - y + z \\ 2x + 2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2.x + 0.y - z = 0$$
$$x - y + z = 0$$
$$2x + 2y + 0.z = 0$$

$$\begin{array}{cccc} 2 & 3 & 1 & 2 \\ \hline 0 & -1 & -2 & 0 \\ 2 & 0 & 2 & 2 \end{array}$$

From 1st & 3rd

$$\frac{x}{0 \times -1 \atop 2 \times 0} = \frac{y}{-1 \times -2 \atop 0 \times 2} = \frac{z}{-2 \times 0 \atop 0 \times 2}$$

$$\Rightarrow \frac{x}{2} = \frac{y}{-2} = \frac{z}{-4}$$

$$\Rightarrow \frac{x}{1} = \frac{y}{-1} = \frac{z}{-2} \Rightarrow$$

$$\therefore X_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

$\Rightarrow$ when $\lambda = 2$, Let $X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigen vector

corresponding to eigen value $\lambda = 2$

$$\therefore [A - \lambda I] X_2 = 0$$

$$\Rightarrow [A - 2I] X_2 = 0$$

$$\Rightarrow \begin{bmatrix} 1-2 & 0 & -1 \\ 1 & 2-2 & 1 \\ 2 & 2 & 3-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x + 0.y - z = 0$$
$$x + 0y + z = 0$$
$$2x + 2y + z = 0$$

From (1)' & 3rd eqn.

$$\begin{array}{cccccc} 2 & & 3 & & 1 & & 2 \\ \hline 0 & & -1 & & -1 & & 0 \\ 2 & & 1 & & 2 & & 2 \end{array}$$

$$\begin{array}{cccccc} 2 & & 3 & & 1 & & 2 \\ \hline 0 & & -1 & & -1 & & 0 \\ 2 & & 1 & & 1 & & 2 \end{array}$$

$$\frac{x}{\begin{matrix} 0 & 1 \\ 2 & 1 \end{matrix}} = \frac{y}{\begin{matrix} -1 & -1 \\ 1 & 2 \end{matrix}} = \frac{z}{\begin{matrix} -1 & 0 \\ 2 & 2 \end{matrix}}$$

$$\frac{x}{2} = \frac{y}{-1} = \frac{z}{-2}$$

$$\therefore X_2 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

When $\lambda = 1$, $X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigen vector corresponding to eigen value $\lambda = 1$

as $[A - \lambda I] X_1 = 0$

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$$[A - 1I]X_1 = 0$$

$$\begin{bmatrix} 1-1 & 0 & -1 \\ 1 & 2-1 & 1 \\ 2 & 2 & 3-1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0.x + 0.y - z = 0$$
$$1.x + 1.y + 1.z = 0$$
$$2x + 2y + 2z = 0$$

from 2nd & 3rd eqn.

$$\begin{array}{cccc} 2 & 3 & 1 & 2 \\ \hline 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{array} \qquad \begin{array}{c} 2 \\ 0 \\ 2 \end{array} \quad \begin{array}{c} 3 \\ 0 \\ 2 \end{array} \qquad 1 \,|\, 2$$

$$\begin{array}{cccc} 2 & 3 & 1 & 2 \\ \hline 0 & -1 & 0 & 0 \\ 2 & 2 & 2 & 2 \end{array}$$

$$\frac{x}{\begin{matrix} 0 & -1 \\ 2 & 2 \end{matrix}} = \frac{y}{\begin{matrix} -1 & 0 \\ 2 & 2 \end{matrix}} = \frac{z}{\begin{matrix} 0 & 0 \\ 2 & 2 \end{matrix}}$$

$$\frac{x}{2} = \frac{y}{-2} = \frac{z}{0}$$

$$\frac{x}{1} = \frac{y}{-1} = \frac{z}{0} \qquad \therefore X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

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Modal matrix, $P = [X_1 \quad X_2 \quad X_3]$

$$P = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & -1 \\ 0 & -2 & -2 \end{bmatrix}$$

$$P = IP$$

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & -1 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} P$$

Apply $R_2 \to R_2 + R_1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} P$$

Apply $R_1 \to R_1 - 2R_2$
$R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 0 \\ 1 & 1 & 0 \end{bmatrix} P$$

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Que $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ find eigen values & eigen vectors of this matrix.

Sol Characteristic eqn. of A is given by

$$\det[A - \lambda I] = 0$$

$$\left| \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} = 0$$

$$\Rightarrow \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

Using hit & trial $\lambda = -3$

i.e. $\Rightarrow$ $\lambda + 3 = 0$ is a factor.

$$\lambda + 3 \,\big)\, \lambda^3 + \lambda^2 - 21\lambda - 45 \,\big(\, \lambda^2 - 2\lambda - 15$$

$$\lambda^3 + 3\lambda^2$$

$$-2\lambda^2 - 21\lambda$$

$$-2\lambda^2 - 6\lambda$$

$$+ \quad + \quad +$$

$$-15\lambda - 45$$

$$-15\lambda - 45$$

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$\lambda^2 - 2\lambda - 15 = 0$

$\Rightarrow \lambda = -3, 5$

$\therefore \lambda = -3, -3, 5$ are eigen values

When $\lambda = -3$, $X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigen vector corresponding to eigen value $\lambda = -3$

as $[A - \lambda I] X_1 = 0$

$[A + 3I] X_1 = 0$

$$\begin{bmatrix} -2+3 & 2 & -3 \\ 2 & 1+3 & -6 \\ -1 & -2 & 0+3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + 2y - 3z = 0$

$2x + 4y - 6z = 0$

$-x - 2y + 3z = 0$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 2 & 4 & -6 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right]$$

$R_3 \to R_3 + R_1$

$R_2 \to R_2 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore \rho = 1 <$ no of unknowns $= 3$

$\therefore$ infinitely many soln.

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From ① $x + 2y - 3z = 0$

Let $y = 1$

$z = 1$

$\therefore \; x + 2 - 3 = 0$

$x = 1$

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $x = 2$, $z = 0$

From R_1, $2 + 2y - 0 = 0$

$2y = -2$

$y = -1$

$$X_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$ is the eigen vector corresponding to eigen value $\lambda = -3$

When $\lambda = 5$, $X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigen vector corresponding to eigen value $\lambda = 5$

$$[A - \lambda I] X_3 = 0$$

$$[A - 5I] X_3 = 0$$

$$\begin{bmatrix} -2-5 & 2 & -3 \\ 2 & 1-5 & -6 \\ -1 & -2 & 0-5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-7x + 2y - 3z = 0$$
$$2x - 4y - 6z = 0$$
$$-1x - 2y - 5z = 0$$

From 2nd & 3rd eqn.

$$\begin{array}{cccc} 2 & 3 & 1 & 2 \\ \hline -4 & -6 & 2 & -4 \\ -2 & -5 & -1 & -2 \end{array}$$

$$\frac{x}{\begin{matrix} -4 & -6 \\ -2 & -5 \end{matrix}} = \frac{y}{\begin{matrix} -6 & 2 \\ -5 & -1 \end{matrix}} = \frac{z}{\begin{matrix} 2 & -4 \\ -1 & -2 \end{matrix}}$$

$$\frac{x}{8} = \frac{y}{16} = \frac{z}{-8}$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$$

$$\therefore X_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

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Que- $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ Cal. eigen values of this matrix. & eigen vectors also.

Sol: Characteristic eqn. of A is given by

$$\det.[A-\lambda I] = 0$$

$$\left| \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \right| = 0$$

$$(6-\lambda)[(3-\lambda)(3-\lambda)-1] + 2(-6+2\lambda+2)$$
$$+2(2-6+2\lambda) = 0$$

$$(6-\lambda)(9-3\lambda-3\lambda+\lambda^2-1) + 2(-4+2\lambda) + 2(-4+2\lambda) = 0$$

$$(6-\lambda)(\lambda^2-6\lambda+8) + (-8+4\lambda) + (-8+4\lambda) = 0$$

$$6\lambda^2 - 36\lambda + 48 - \lambda^3 + 6\lambda^2 - 8\lambda - 16 + 8\lambda = 0$$

$$-\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$$

Using hit & trial method

$$\lambda = 2$$

i.e. $(\lambda - 2)$ is a factor

$-8+48-72+32$

$+1+12+28+32$

$-1+12-28+32$

$-8+48-56+32$

$40-24$

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$$
\begin{array}{r|l}
 & -\lambda^2 + 10\lambda - 16 \\
\hline
\lambda - 2 & -\lambda^3 + 12\lambda^2 - 36\lambda + 32 \\
 & -\lambda^3 + 2\lambda^2 \\
 & + \quad - \\
\hline
 & 10\lambda^2 - 36\lambda \\
 & 10\lambda^2 - 20\lambda \\
 & - \quad + \\
\hline
 & -16\lambda + 32 \\
 & -16\lambda + 32 \\
 & + \quad - \\
\hline
 & 0
\end{array}
$$

$$\lambda^2 - 10\lambda + 16 = 0$$
$$\lambda^2 - 8\lambda - 2\lambda + 16 = 0$$
$$\lambda(\lambda - 8) - 2(\lambda - 8) = 0$$
$$(\lambda - 2)(\lambda - 8) = 0$$
$$\therefore \lambda = 2, 8$$

$$\therefore \lambda = 2, 2, 8$$

When $\lambda = 2$, $X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigen vector corresponding to eigen value $\lambda = 2$

as $[A - \lambda I]X_1 = 0$

$$\begin{bmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$4x - 2y + 2z = 0$$
$$-2x + 1y - 1z = 0$$
$$2x - y + z = 0$$

$$[A:B] = \left[\begin{array}{ccc:c} 4 & -2 & 2 & 0 \\ -2 & 1 & -1 & 0 \\ 2 & -1 & 1 & 0 \end{array}\right]$$

Apply $R_1 \to R_1/2$

$$\left[\begin{array}{ccc:c} 2 & -1 & 1 & 0 \\ -2 & 1 & -1 & 0 \\ 2 & -1 & 1 & 0 \end{array}\right]$$

$R_2 \to R_2 + R_1$
$R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc:c} 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho = 1 <$ no of unknowns

Thus the system has infinitely many solns.

Let $y = 1$, $z = 1$

from R_1 $\quad 2x - y + z = 0$

$$2x - 1 + 1 = 0$$
$$2x = 0$$
$$x = 0$$

$$X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

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Let $y = 3$, $z = 1$

$$2x - y + z = 0$$
$$2x - 3 + 1 = 0$$
$$2x - 2 = 0$$
$$x = 1$$

$$X_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

When $\lambda = 8$, $X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigen vector corresponding to eigen value $\lambda = 8$

as $[A - \lambda I] X_3 = 0$

$$\begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x - 2y + 2z = 0$$
$$-2x - 5y - z = 0$$
$$2x - y - 5z = 0$$

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from 2nd & 3rd eqn.

$$\begin{array}{cccc} 2 & 3 & 1 & 2 \\ \hline -5 & -1 & -2 & -5 \\ -1 & -5 & 2 & -1 \end{array}$$

$$\frac{x}{\begin{matrix}-5 & -1\\ -1 & -5\end{matrix}} = \frac{y}{\begin{matrix}-1 & -2\\ -5 & 2\end{matrix}} = \frac{z}{\begin{matrix}-2 & -5\\ 2 & -1\end{matrix}}$$

$$\frac{x}{24} = \frac{y}{-12} = \frac{z}{12}$$

$$\frac{x}{2} = \frac{y}{-2} = \frac{z}{1}$$

$$X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Modal matrix.

$$P = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

~~Apply~~ $P = IP$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & -2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} P$$

~~Apply~~ $R_1 \leftrightarrow R_2$

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Result → If A is a diagonalizable matrix & P is its modal matrix then

$$f(A) = P[f(D)]P^{-1}$$

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Que For matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$, find $A^{10} + A^5 + A^2 + 5A + 3I$

Sol (i) Find eigen values of A, $\lambda = 1, 2, 3$

(ii) Find eigen vectors, $X_1 = (1, -1, 1)$

$X_2 = (1, 0, 1)$

$X_3 = (0, 1, 1)$

(iii) Find Modal matrix P. $P = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(iv) Diagonalise A, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Case I Here $f(A) = A^2 + 5A + 3I$

$f(D) = D^2 + 5D + 3I$

Hence $(A^2 + 5A + I) = P[f(D)]P^{-1}$

$= P[D^2 + 5D + 3I]P^{-1}$

$= P\left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 15 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right]P^{-1}$

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$$= P\begin{bmatrix} 9 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 27 \end{bmatrix} P^{-1}$$

Case II $f(A) = A^{10} + A^5$

$$f(D) = D^{10} + D^5 = \begin{bmatrix} 1^{10} & 0 & 0 \\ 0 & 2^{10} & 0 \\ 0 & 0 & 3^{10} \end{bmatrix} + \begin{bmatrix} 1^5 & 0 & 0 \\ 0 & 2^5 & 0 \\ 0 & 0 & 3^5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1024 & 0 \\ 0 & 0 & 59049 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 243 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1056 & 0 \\ 0 & 0 & 59292 \end{bmatrix}$$

$$\therefore A^{10} + A^5 = P[f(D)]P^{-1} = P\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1056 & 0 \\ 0 & 0 & 59292 \end{bmatrix} P^{-1}$$

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Properties of Eigen values & vectors

① If λ is an eigen value of matrix A, then $\frac{1}{\lambda}$ is eigen value of A^{-1}.

eigen vector remain same.

Proof ⑧ Let X be the corresponding eigen vector of A

for eigen value λ

$\therefore AX = \lambda X$

Premultiplying both sides by A^{-1}

$$\Rightarrow AA^{-1}X = \lambda A^{-1}X$$

$$\Rightarrow IX = \lambda A^{-1}X$$

$$\Rightarrow \frac{1}{\lambda}X = A^{-1}X$$

$\therefore \frac{1}{\lambda}$ is eigen value of A^{-1}, eigen vector remaining the same.

② If λ is eigen value of A, X being the corresponding eigen vector, then λ^m is eigen value of A^m, eigen vector remaing the same.

Proof:- Since λ is eigen value & X is corresponding eigen vector of A, $\therefore AX = \lambda X$ ----(i)

Premultiplying both sides by A

$$A^2X = \lambda(AX)$$

$$\Rightarrow A(AX) = \lambda(\lambda X) \text{ using (i)}$$

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$\Rightarrow A^2X = \lambda^2 X$ ---(ii)

$\therefore \lambda^2$ is eigen value of A^2, eigen vector remaining the same.

Premultiplying (ii) by A

$$A(A^2X) = \lambda^2(AX)$$

$\Rightarrow A^3X = \lambda^2(\lambda X)$ ----- using (i)

$\Rightarrow A^3X = \lambda^3 X$

$\therefore \lambda^3$ is eigen value of A^3, eigen vector remaining same.

Continuing the process, we get

— — — — — — — — —

— — — — — — — —

$$A^m X = \lambda^m X$$

$\therefore \lambda^m$ is eigen value of A^m, eigen vector remaining the same.

(3) If λ is eigen value of A, X is corresponding eigen vector, then $g(\lambda)$ is eigen value of $g(A)$ where $g(x)$ is a polynomial, eigen vector remaining the same.

Proof:- Let $g(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$.

$\therefore g(A) = a_0 + a_1 A + a_2 A^2 + \dots + a_n A^n$

$g(\lambda) = a_0 + a_2 \lambda + a_2 \lambda^2 + \dots + a_n \lambda^n$

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Since λ is eigen value of A, X is the corresponding eigen vector.

$\therefore AX = \lambda X$ -----(i)

We have to prove $g(A)X = g(\lambda)X$

Now $g(A)X = (a_0 + a_1 A + \dots + a_n A^n)X$

$= a_0 X + a_1 A X + a_2 A^2 X + \dots + a_n A^n X$

$= a_0 X + a_1 \lambda X + a_2 \lambda^2 X + \dots + a_n \lambda^n X$

$\because$ $AX = \lambda X$, $A^2 X = \lambda^2 X$, ---- $A^m X = \lambda^m X$ as if λ is eigen value of A, then λ^m is eigen value of A^m, eigen vector remaining the same)

If this property comes to solve in 8 marks then the applied & used property should also be proved.

$= (a_0 + a_1 \lambda + a_2 \lambda^2 + \dots + a_n \lambda^n) \cdot X$

$= [g(\lambda)]X \Rightarrow [g(A)]X = [g(\lambda)]X$

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SIMILAR MATRICES

A square matrix A is said to be similar to a square matrix B if there exists an invertible matrix P such that.

$$PB = AP$$

i.e. $B = P^{-1}AP$

Result:- If A is similar to B then B is similar to A, i.e. property of similarity is symmetric.

Proof:- Since A is similar to B.

$\therefore$ there exists an invertible matrix P such that $B = P^{-1}AP$ ... (i)

Let $P^{-1} = Q$

It is an invertible matrix.

$\therefore (P^{-1})^{-1} = Q^{-1}$

$\Rightarrow P = Q^{-1}$

$\therefore$ (i) becomes

$B = QAQ^{-1}$

$$\boxed{Q^{-1}BQ = A}$$

Premultiplying both sides by Q^{-1}

$Q^{-1}B = Q^{-1}QAQ^{-1}$

$\Rightarrow Q^{-1}B = AQ^{-1}$

Post multiplying both sides by Q

$Q^{-1}BQ = AQ^{-1}Q$

$\Rightarrow Q^{-1}BQ = A$

Hence B is similar to A

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Que- Examine whether A is similar to B where

$$A = \begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

Sol Let there exists a matrix $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$$PB = AP$$

$$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a-3b & 2a+4b \\ c-3d & 2c+4d \end{bmatrix} = \begin{bmatrix} 5a+5c & 5b+5d \\ -2a & -2b \end{bmatrix}$$

$$\Rightarrow \quad a-3b = 5a+5c$$
$$2a+4b = 5b+5d$$
$$c-3d = -2a$$
$$2c+4d = -2b$$

$\Rightarrow$ ~~4a+5c~~

$$\Rightarrow \quad 4a+3b+5c+0.d=0$$
$$2a-b+0.c-5d=0$$
$$2a+0.b+c-3d=0$$
$$0.a+2b+2c+4d=0$$

$$[A:B] = \left[\begin{array}{cccc|c} 4 & 3 & 5 & 0 & 0 \\ 2 & -1 & 0 & -5 & 0 \\ 2 & 0 & 1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \end{array}\right]$$

$$R_1 \leftrightarrow R_2 \left[\begin{array}{cccc|c} 2 & -1 & 0 & -5 & 0 \\ 4 & 3 & 5 & 0 & 0 \\ 2 & 0 & 1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \end{array}\right]$$

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$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & -5 & 0 \\ 0 & 5 & 5 & 10 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 2 & 2 & 4 & 0 \end{array}\right]$$

Apply $R_2 \leftrightarrow R_3$
$R_4 \to R_4/2$

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & -5 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 5 & 5 & 10 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{array}\right]$$

$R_3 \to R_3 - 5R_2$
$R_4 \to R_4 - R_2$

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & -5 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = 2 <$ no of unknowns $= 4$

thus system has infinitely many solns.

Let $c = 1$, $d = 1$

From R_2, $b + c + 2d = 0$
$\Rightarrow b + 1 + 2 = 0$
$\Rightarrow b = -3$

From R_1, $2a - b - 5d = 0$
$\Rightarrow 2a + 3 - 5 = 0$
$\Rightarrow a = 1$

$$P = \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}$$

$|P| = 1 + 3 = 4 \neq 0$

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$\therefore$ P is an invertible matrix satisfying the eqn.
$$PB = AP$$

Hence A is similar to B.

Que- (Dec. 2018) Similar matrices have same characteristic eqn. & hence the same eigen values.

Sol If X is eigen vector of A corresponding to eigen value λ & P is the similarity matrix, then $P^{-1}X$ is eigen vector of B corresponding to eigen value λ.

Sol Proof :- Let A be similar to B $\therefore$ There exists an invertible matrix P such that $B = P^{-1}AP$

~~Now characteristic polynomial of R is given by~~

Now $B - \lambda I = P^{-1}AP - \lambda I$

$$B - \lambda I = P^{-1}AP - \lambda P^{-1}P$$

$$B - \lambda I = P^{-1}(A - \lambda I)P$$

taking determinant on both sides

$$|B - \lambda I| = |P^{-1}(A - \lambda I)P|$$

$$|B - \lambda I| = |P^{-1}||A - \lambda I||P|$$

$$= |A - \lambda I||P^{-1}||P| = |A - \lambda I||P^{-1}P|$$

$$(\because |M||N| = |MN|)$$

$$\Rightarrow |B - \lambda I| = |A - \lambda I||I|$$

$$\Rightarrow |B - \lambda I| = |A - \lambda I|$$

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Hence characteristic polynomials of similar matrices are equal.

Hence similar matrices have same eigen values.

Let λ be eigen value of A & X be the corresponding eigen vector of A

$$\therefore \quad AX = \lambda X \text{ ------ (i)}$$

Now Premultiplying both sides by P^{-1}

$$P^{-1}AX = \lambda(P^{-1}X) \text{ ------ (ii)}$$

Let $X = PY$ i.e. $Y = P^{-1}X$

$\therefore$ (ii) becomes $P^{-1}APY = \lambda(P^{-1}PY)$

$$\Rightarrow BY = \lambda Y \qquad (\because P^{-1}P = I)$$

$$\Rightarrow B(P^{-1}X) = \lambda(P^{-1}X)$$

$\Rightarrow \lambda$ is eigen value of B & Y i.e. $P^{-1}X$ is eigen vector of B.

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Conjugate matrix – The matrix obtained by replacing elements of a matrix A by their conjugates is called conjugate matrix of A.

Eg. If $A = \begin{bmatrix} 2-i & 3+4i \\ 6 & 7-5i \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} 2+i & 3-4i \\ 6 & 7+5i \end{bmatrix}$$

Tranjugate matrix – Transpose of conjugate matrix is called Tranjugate matrix.

$$(\bar{A})^T = A^* = A^\theta = \begin{bmatrix} 2+i & 6 \\ 3-4i & 7+5i \end{bmatrix}$$

Result → ① $(\bar{A})^T = \overline{(A^T)}$

Hermitian matrix – A matrix A is called Hermitian matrix if $A^* = A$. It is basically a square matrix.

i.e. $a_{ij} = \overline{a_{ji}}$

Result → The diagonal element of a Hermitian matrix are real.

Proof:- In a Hermitian matrix, $a_{ij} = \overline{a_{ji}}$

∴ For a diagonal element, $a_{ii} = \overline{a_{ii}}$

⇒ a_{ii} is real

∴ Hence diagonal elements of a Hermitian matrix are real.

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E.g., $A = \begin{bmatrix} 2 & 6+7i & 9-10i \\ 6-7i & 3 & 1+i \\ 9+10i & 1-i & 5 \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} 2 & 6-7i & 9+10i \\ 6-7i & 3 & 1-i \\ 9-10i & 1+i & 5 \end{bmatrix}$$

$$(\bar{A})^T = A^* = \begin{bmatrix} 2 & 6+7i & 9-10i \\ 6-7i & 3 & 1+i \\ 9+10i & 1-i & 5 \end{bmatrix}$$

$\therefore$ A is a Hermitian matrix

$\rightarrow$ Result $\rightarrow (A^*)^* = A$

Proof:

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Tutorial

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Que $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ Check whether the matrices are similar.

Sol $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

let there exists a matrix $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $PB = AP$

$$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \begin{aligned} a &= a & &\Rightarrow 0 = 0 \\ a+b &= b & &\Rightarrow a = 0 \\ c &= c & &\Rightarrow 0 = 0 \\ c+d &= d & &\Rightarrow c = 0 \end{aligned}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{aligned} \cancel{0.a + 0.b + 0.c + 0.d = 0} \\ \cancel{a + 0.b + 0.c + 0.d = 0} \\ \cancel{0.a + 0.b + c + 0.d = 0} \\ \cancel{0.a + 0.b + c + 0.d = 0} \end{aligned}$$

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

$\rho = 2 <$ no of unknowns $= 4$

$a = 0$, $c = 0$, let $b = 1$, $d = 2$

$P = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ $|P| = 0$

P is not invertible

Matrices are not similar

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Que Diagonalise if possible $A = \begin{bmatrix} 6 & 0 & 9 \\ 0 & 7 & 1 \\ 0 & -4 & 3 \end{bmatrix}$

Sol. characteristic eqn. of A is given by

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 6-\lambda & 0 & 9 \\ 0 & 7-\lambda & 1 \\ 0 & -4 & 3-\lambda \end{bmatrix} \right| = 0$$

$$(6-\lambda)[(7-\lambda)(3-\lambda)+4] = 0$$

$$(6-\lambda)[21 - 7\lambda - 3\lambda + \lambda^2 + 4] = 0$$

$$(6-\lambda)[\lambda^2 - 10\lambda + 25] = 0$$

$$6\lambda^2 - 60\lambda + 150 - \lambda^3 + 10\lambda^2 - 25\lambda = 0$$

$$-\lambda^3 + 16\lambda^2 - 85\lambda + 150 = 0$$

$$\lambda^3 - 16\lambda^2 + 85\lambda - 150 = 0$$

~~$\lambda = 6, 7, 3$~~ By hit & trial method

$$\lambda = 5$$

i.e. $(\lambda - 5)$ is a factor

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$$
\begin{array}{r|l}
 & \lambda^2 - 11\lambda + 30 \\
\hline
\lambda - 5 & \lambda^3 - 16\lambda^2 + 85\lambda - 150 \\
 & \lambda^3 - 5\lambda^2 \\
 & - \quad + \\
\hline
 & -11\lambda^2 + 85\lambda \\
 & -11\lambda^2 + 55\lambda \\
 & + \quad - \\
\hline
 & 30\lambda - 150 \\
 & 30\lambda - 150 \\
 & - \quad + \\
\hline
 & 0
\end{array}
$$

$\lambda^2 - 11\lambda + 30 = 0$

$\lambda = 6, 5$

$\lambda = 5, 5, 6$

when $\lambda = 5$, $X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the corresponding eigen vector to value $\lambda = 5$

$[A - \lambda I] X_1 = 0$

$$\begin{bmatrix} 1 & 0 & 9 \\ 0 & 2 & 1 \\ 0 & -4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$1.x + 0.y + 9.z = 0$

$0.x + 2.y + 1.z = 0$

$0.x - 4y - 2z = 0$

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$$\left[\begin{array}{ccc|c} 1 & 0 & 9 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & -4 & -2 & 0 \end{array}\right] \quad R_3 \to R_3 + 2R_2 \quad \left[\begin{array}{ccc|c} 1 & 0 & 9 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Infinitely many solns.

Let $z = k$

$\therefore \quad y = \frac{-k}{2}, \quad x = -9k$

$k = 2 \quad \begin{bmatrix} -18 \\ -1 \\ 2 \end{bmatrix}$

When $\lambda = 6$. $\quad X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$[A - 6I]X_3 = 0$$

$$\begin{bmatrix} 0 & 0 & 9 \\ 0 & 1 & 1 \\ 0 & -4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$\begin{cases} 9z = 0 \\ y + z = 0 \\ -4y - 3z = 0 \end{cases} \qquad \left[\begin{array}{ccc|c} 0 & 0 & 9 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -4 & -3 & 0 \end{array}\right]$$

$$R_3 \to R_3 + 4R_2 \quad \left[\begin{array}{ccc|c} 0 & 0 & 9 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{array}{l} R_1 \to R_1 - 9R_3 \\ R_2 \to R_2 - R_3 \end{array} \quad \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

~~Infinitely many soln.~~

$$R_3 \leftrightarrow R_1 \quad \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$C_1 \leftrightarrow C_3 \quad \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ z & y & x \end{array} \end{array}$$

Infinitely many soln.

$x = 0, \; y = 0, \; z = 1$

$$X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

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Hermitian matrix

Result → ① $(A^*)^* = A$

Proof $(A^*)^* = \overline{[(\overline{A})^T]}^T = [(\overline{\overline{A}})^T]^T = A$

② $(AB)^* = B^* A^*$

Proof $(AB)^* = \overline{[(AB)^T]} = \overline{[B^T A^T]}$

$= \overline{(B^T)}\,\overline{(A^T)}$

$= B^* A^*$

③ $(A+B)^* = A^* + B^*$

Que- Prove that eigen values of a Hermitian matrix are real.

Proof Let A be a Hermitian matrix, λ be its eigen value & X be the corresponding eigen vector.

$\therefore \quad A^* = A$ ---(i)

$AX = \lambda X$ ---(ii)

$\Rightarrow (AX)^* = (\lambda X)^*$

$\Rightarrow X^* A^* = \bar{\lambda} X^*$ $\quad (\because A^* = A)$

$\Rightarrow X^* A = \bar{\lambda} X^*$

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Postmultiplying both sides by X

$$\Rightarrow X^* A X = \bar{\lambda} X^* X \quad (\because AX = \lambda X)$$

$$\Rightarrow \lambda (X^* X) = \bar{\lambda} X^* X$$ ~~Post mu~~

Post multiplying both sides by $(X^* X)^{-1}$

$$\lambda = \bar{\lambda}$$

$\therefore \lambda$ is real. Hence eigen values of Hermitian matrix are real.

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Skew-Hermitian Matrix – A square matrix A is said to be skew-hermitian if $A^* = -A$
i.e. $a_{ij} = -\overline{a_{ji}}$

Result → The diagonal elements of a skew Hermitian matrix are either 0 or purely imaginary

e.g. $A = \begin{bmatrix} 2i & 6+7i & 3-4i \\ -6+7i & 0 & 7i \\ -3-4i & 7i & 5i \end{bmatrix}$

$\bar{A} = \begin{bmatrix} -2i & 6-7i & 3+4i \\ -6-7i & 0 & -7i \\ -3+4i & -7i & -5i \end{bmatrix}$

$(\bar{A})^T = A^* = \begin{bmatrix} -2i & -6-7i & -3+4i \\ 6-7i & 0 & -7i \\ 3+4i & -7i & -5i \end{bmatrix}$

$-(A^*) = \begin{bmatrix} 2i & 6+7i & 3-4i \\ -6+7i & 0 & 7i \\ -3-4i & 7i & 5i \end{bmatrix} = A$

$\therefore$ A is a skew Hermitian matrix.

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PROBLEMS :-

Que-1 For any sq. matrix A, A can be represented as sum of a Hermitian matrix & a skew-hermitian matrix.

Ans Proof :-

$$A = \frac{1}{2}(A+A^*) + \frac{1}{2}(A-A^*)$$

Claim I

$\frac{1}{2}(A+A^*)$ is a Hermitian matrix.

Proof of Claim I :-

$$\left[\frac{1}{2}(A+A^*)\right]^* = \frac{1}{2}(A+A^*)^*$$

$$= \frac{1}{2}\left(A^* + (A^*)^*\right) = \frac{1}{2}(A^* + A)$$

$$= \frac{1}{2}(A + A^*)$$

Hence $\frac{1}{2}(A+A^*)$ is a Hermitian matrix.

Claim II

$\frac{1}{2}(A-A^*)$ is a Skew-Hermitian matrix.

Proof of Claim II :-

$$\left[\frac{1}{2}(A-A^*)\right]^* = \frac{1}{2}\left[A^* - (A^*)^*\right]$$

$$= \frac{1}{2}\left[A^* - A\right]$$

$$= -\frac{1}{2}\left[A - A^*\right]$$

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$\therefore \frac{1}{2}(A - A^*)$ is a skew Hermitian matrix.

Hence A can be represented as sum of a Hermitian matrix & a skew Hermitian matrix.

Que-2 verify Cayley Hamilton theorem for

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Hence find $A^8 - 11A^7 - 4A^6 + A^5 + A^4 - 11A^3 - 3A^2 + 2A + I$

Sol 2 Characteristic eqn. of A is given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 5 \\ 3 & 5 & 6-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(4-\lambda)(6-\lambda) - 25] - 2[12 - 2\lambda - 15] + 3[10 - 12 + 3\lambda] = 0$$

$$(1-\lambda)[24 - 4\lambda - 6\lambda + \lambda^2 - 25] - 2[-2\lambda - 3] + 3[-2 + 3\lambda] = 0$$

$$(1-\lambda)[\lambda^2 - 10\lambda - 1] + 4\lambda + \cancel{6} - \cancel{6} + 9\lambda = 0$$

$$\lambda^2 - 10\lambda - 1 - \lambda^3 + 10\lambda^2 + \lambda + 4\lambda + 9\lambda = 0$$

$$-\lambda^3 + 11\lambda^2 + 4\lambda + 1 = 0$$

$$\Rightarrow \lambda^3 - 11\lambda^2 - 4\lambda + I = 0$$

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Verification

By Cayley Hamilton Thm.

$$A^3 - 11A^2 - 4A + I = 0$$

$$L.H.S. = A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}$$

$$A^3 = AA^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix}$$

$$A^3 - 11A^2 - 4A + I$$

$$= \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix} - 11 \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}$$

$$- 4 \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Hence Cayley Hamilton Thm. is verified for given Ques.

Now $A^8 - 11A^7 - 4A^6 + A^5 + A^4 - 11A^3 - 3A^2 + 2A + I$

$\cancel{A^5[A^3 - 11A^2 - 4A + I] + A[A^3 - 11A^2 - 3A + A + 2A + I]}$

$A^5[A^3 - 11A^2 - 4A + I] + A[A^3 - 11A^2 - 4A + I] + A^2 + A + I$

$= 0 + 0 + A^2 + A + I$

$$= \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 27 & 34 \\ 27 & 50 & 56 \\ 34 & 61 & 77 \end{bmatrix}$$

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