2
VECTOR SPACES

2.0. INTRODUCTION

First of all, we shall deal with an algebraic structure, called as **vector space** (or **linear vector space**) because the whole study of Linear Algebra is based upon it. The motivation for the study of vector spaces comes from the study of ordinary notion of vectors. Here we shall be concerned with two sets and two compositions at a time, The former deals with vectors and later with scalars.

Let R be the set of all real numbers, then the set $R \times R$

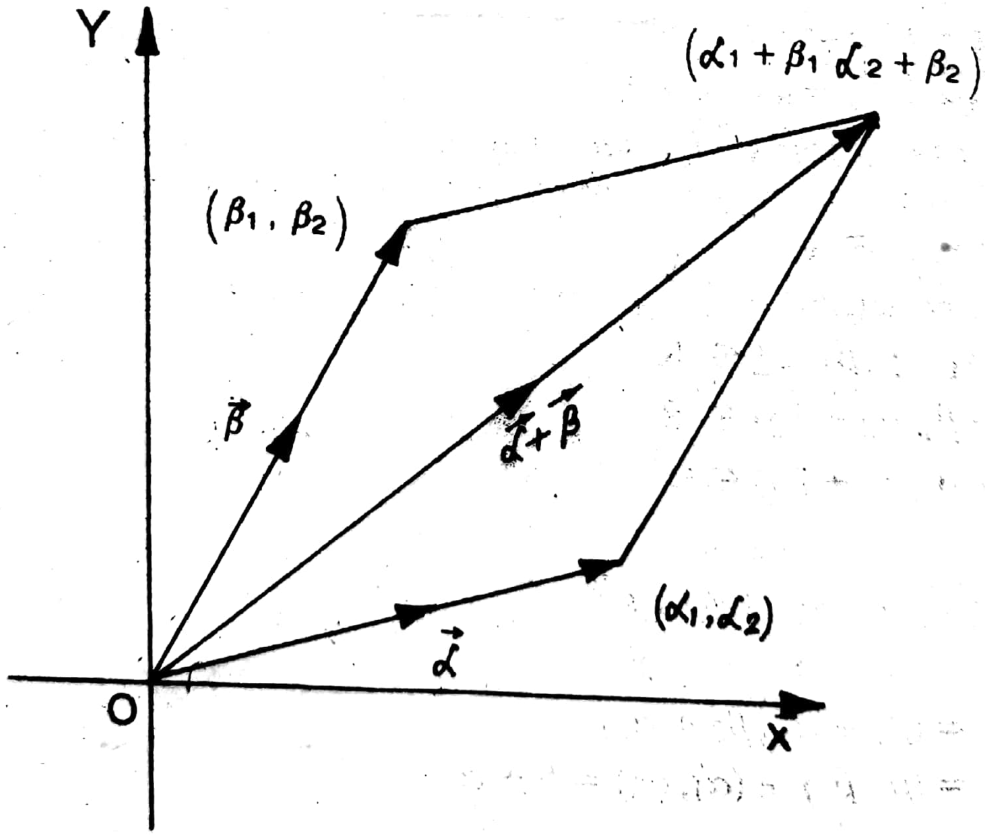
i.e. $R^2 = \{ (\alpha_1, \alpha_2) : \alpha_1, \alpha_2 \in R\}$.

The set R^2 is concerned with an algebraic structure. **Vector spaces** are the sets with algebraic structure similar to those on R^2. The algebraic structure on R^2 refers that the elements of R^2 refer to vectors rather than points. An element of R^2 is known both as a point and a vector in the plane R^2. The length of the vector $(\alpha_1, \alpha_2) = \sqrt{{\alpha_1}^2 + {\alpha_1}^2}$.

A vector of length 1 is called a **unit vector.**

(*a*) **Addition of vectors.** This is given by **Parallelogram Law.**

Let (α_1, α_2) and (β_1, β_2) be the end points of the vectors $\vec{\alpha}$ and $\vec{\beta}$ emanating from the point O (0,0).



Then $(\alpha_1 + \beta_1, \alpha_2 + \beta_2)$ is the end point of the vector $\vec{\alpha} + \vec{\beta}$, as shown in the above diagram.

Thus the sum of the vectors (α_1, α_2) and (β_1, β_2) is the vector $(\alpha_1 + \beta_1, \alpha_2 + \beta_2)$ *i.e.*

$$(\alpha_1, \alpha_2) + (\beta_1, \beta_2) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2).$$

For Ex. $(2, -3) + (4, 8) = (2 + 4, -3 + 8) = (6, 5)$.

(*h*) **Scalar Multiplication.** The product $a\vec{\alpha}$ of a real number a by the vector $\vec{\alpha}$ is obtained by multiplying the magnitude of $\vec{\alpha}$ by a and retains the same direction if $a \geq 0$ or the opposite direction if $a < 0$.

If (α_1, α_2) be the end point of the vector $\vec{\alpha}$, then $(a\,\alpha_1, a\,\alpha_2)$ is the point of $a\vec{\alpha}$.

Here the scalar multiplication $a\,(\alpha_1, \alpha_2)$ of the vector (α_1, α_2) by the scalar $a \in \mathbf{R}$ is the vector $(a\,\alpha_1, a\,\alpha_2)$.

For Ex. $2(-1, 2) + 3(0, 2) = (-2, 4) + (0, 6) = (-2 + 0, 4 + 6) = (-2, 10)$.

Properties : *For any $\alpha, \beta, \gamma \in R^2$ and scalars $a, b \in R$, the following properties hold :*

(*i*) $\alpha + \beta \in R^2$ (*ii*) $\alpha + \beta = \beta + \alpha$

(*iii*) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

(*iv*) $\exists\, 0 \in R^2$ s.t. $\alpha + 0 = \alpha\ \forall\, \alpha \in R^2$

(*v*) $\forall\, \alpha \in R^2, \exists\, \alpha' \in R^2$ s.t. $\alpha + \alpha' = 0$

(*vi*) $a\,\alpha \in R^2$

(*vii*) $(ab)\,\alpha = a\,(b\,\alpha)$

(*viii*) $(a + b)\,\alpha = a\,\alpha + b\,\alpha$

(*ix*) $a(\alpha + \beta) = a\,\alpha + a\,\beta$

(*x*) $1\,\alpha = \alpha$, *where 1 is the unity in R.*

Proof : Let $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2)$ and $\gamma = (\gamma_1, \gamma_2)$, where $\alpha_1, \alpha_2; \beta_1, \beta_2; \gamma_1, \gamma_2 \in \mathbf{R}$.

(*i*) $\alpha + \beta = (\alpha_1, \alpha_2) + (\beta_1, \beta_2) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2)$.

Since $\alpha_1, \alpha_2; \beta_1, \beta_2 \in \mathbf{R}$,

$\therefore\ (\alpha_1 + \beta_1, \alpha_2 + \beta_2) \in \mathbf{R}$

$\therefore\ (\alpha_1 + \beta_1, \alpha_2 + \beta_2) \in \mathbf{R}^2$

$\Rightarrow\ \alpha + \beta \in \mathbf{R}^2$.

(*ii*) $\alpha + \beta = (\alpha_1, \alpha_2) + (\beta_1, \beta_2)$

$= (\alpha_1 + \beta_1, \alpha_2 + \beta_2)$

$= (\beta_1 + \alpha_1, \beta_2 + \alpha_2)$

$= (\beta_1, \beta_2) + (\alpha_1, \alpha_2) = \beta + \alpha$.

(*iii*) $(\alpha+\beta)+\gamma=((\alpha_1,\alpha_2)+(\beta_1,\beta_2))+(\gamma_1,\gamma_2)$

$= (\alpha_1+\beta_1,\alpha_2+\beta_2)+(\gamma_1,\gamma_2)$

$=((\alpha_1+\beta_1)+\gamma_1,(\alpha_2+\beta_2)+\gamma_2)$

$= ((\alpha_1+(\beta_1+\gamma_1),\alpha_2+(\beta_2+\gamma_2))$

$=(\alpha_1,\alpha_2)+(\beta_1+\gamma_1,\beta_2+\gamma_2)$

$=(\alpha_1,\alpha_2)+((\beta_1,\beta_2)+(\gamma_1,\gamma_2))$

$=\alpha+(\beta+\gamma).$

(*iv*) **Let 0** $=(0,0)\in \mathbf{R}^2$

Now $\alpha+\mathbf{0}=(\alpha_1,\alpha_2)+(0,0)$

$=(\alpha_1+0,\alpha_2+0)=(\alpha_1,\alpha_2)=\alpha \;\forall\, \alpha\in \mathbf{R}^2.$

(*v*) $\forall\, \alpha=(\alpha_1,\alpha_2)\in \mathbf{R}^2,\ \exists\ \alpha'=(-\alpha_1,-\alpha_2)\in \mathbf{R}^2$ s.t.

$\alpha+\alpha'=(\alpha_1,\alpha_2)+(-\alpha_1,-\alpha_2)$

$=(\alpha_1-\alpha_1,\alpha_2-\alpha_2)=(0,0)=\mathbf{0}.$

(*vi*) $a\alpha=a(\alpha_1,\alpha_2)=(a\alpha_1,a\alpha_2).$

Since α_1,α_2 and $a\in \mathbf{R}$, $\therefore\ a\alpha_1,\ a\alpha_2\in \mathbf{R}$

$\Rightarrow\ (a\alpha_1,a\alpha_2)\in \mathbf{R}^2$.

$\therefore\quad a\alpha\in \mathbf{R}^2.$

(*vii*) $(ab)\alpha=(ab)(\alpha_1,\alpha_2)=((ab)\alpha_1,(ab)\alpha_2)$

$=(a(b\alpha_1),a(b\alpha_2))=a(b\alpha_1,b\alpha_2)$

$=a(b(\alpha_1,\alpha_2))=a(b\alpha).$

(*viii*) $(a+b)\alpha=(a+b)(\alpha_1,\alpha_2)$

$=((a+b)\alpha_1,(a+b)\alpha_2)$

$=(a\alpha_1+b\alpha_1,a\alpha_2+b\alpha_2)$

$=(a\alpha_1,a\alpha_2)+(b\alpha_1,b\alpha_2)$

$=a(\alpha_1,\alpha_2)+b(\alpha_1,\alpha_2)=a\alpha+b\alpha.$

(*ix*) $a(\alpha+\beta)=a((\alpha_1,\alpha_2)+(\beta_1,\beta_2))$

$=a(\alpha_1+\beta_1,\alpha_2+\beta_2)$

$=(a(\alpha_1+\beta_1),a(\alpha_2+\beta_2))$

$=(a\alpha_1+a\beta_1,a\alpha_2+a\beta_2)$

$=(a\alpha_1,a\alpha_2)+(a\beta_1\,a\beta_2)$

$=a(\alpha_1,\alpha_2)+a(\beta_1,\beta_2)=a\alpha+a\beta.$

(*x*) $1\alpha=1(\alpha_1,\alpha_2)=(1\alpha_1,1\alpha_2)$

$=(\alpha_1,\alpha_2)=\alpha.$

2.1. BINARY COMPOSITIONS

There are two types of compositions viz.

(*i*) Internal Composition

(*ii*) External Composition.

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(*i*) **Internal Composition. Def.** *Let A be a set. Then the mapping $f : A \times A \to A$ is said to be internal composition on it.*

This is also called binary composition.

The mapping associates to each ordered pair $(a, b) \in A \times A$ a unique number of $f((a, b))$ of A, where $a, b \in A$.

For Ex. Let R be the set of all real numbers.

If $f : R \times R \to R$ is defined as

$f((a, b)) = ab \ \forall (a, b) \in R \times R$; $a, b \in R$, then f is a composition in R.

Another Definition. *Let A be a set. If there exists a rule, denoted by $\oplus$, which associates to each ordered pair (a, b), $a, b \in A$, a unique element $a \oplus b$ of A, then $\oplus$ is said to be binary composition in A.*

We use addition composition by '+' and multiplication composition by '.' in scalars.

(*ii*) **External Composition. Def.** Let V and F be two non-empty sets. Then the mapping $f : V \times F \to V$ is said to be an external composition in V over F.

2.2. VECTOR SPACES *(G.N.D.U. 1993)*

Def. Let $(F, +, \cdot)$ be a given field when elements of F are scalars. Let V be a non-empty set, whose elements are vectors.

Then V is said to be a vector space (denoted as V.S.) over the field F if it satisfies the following axioms :

(*a*) **Under Addition :** Addition of vectors, denoted by '+' is defined as internal composition in V so as to satisfy the following :

(*i*) **Closure.** $\forall \alpha, \beta \in V, \alpha + \beta \in V$.

(*ii*) **Associativity.** $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma) \ \forall \alpha, \beta, \gamma \in V$.

(*iii*) **Existence of Identity.** $\exists \ 0 \in V$ s.t.

$$\alpha + 0 = \alpha = 0 + \alpha \ \forall \alpha \in V.$$

Note : 0 is said to be zero vector in V.

(*iv*) **Existence of Inverse.** $\forall \alpha \in V, \exists \ -\alpha \in V$ s.t.

$$(-\alpha) + \alpha = 0 = \alpha + (-\alpha).$$

(*v*) **Commutativity.** $\alpha + \beta = \beta + \alpha \ \forall \alpha, \beta \in V$.

(*b*) **Under Scalar Multiplication.** Scalar multiplication is defined as external composition in V over F so as to satisfy the following :

(*vi*) $a\alpha \in V \ \forall a \in F \ \alpha \in V$

(*vii*) $a(\alpha + \beta) = a\alpha + a\beta \ \forall a \in F$ and $\forall \alpha, \beta \in V$

(*viii*) $(a + b)\alpha = a\alpha + b\alpha \ \forall a, b \in F$ and $\forall \alpha \in V$

(*ix*) $(ab)\alpha = a(b\alpha) \ \forall a, b \in F$ and $\forall \alpha \in V$

(*x*) $1\alpha = \alpha \ \forall \alpha \in V$.

Note. 1 is said to be unity element in F.

Caution. **0** is the zero element of V and 0 is the zero element of the field F.

Notation. The vector space V over the field F is denoted by V (F).

V (F) is a (*i*) *real vector space* if F is the field R of all real numbers

(*ii*) *rational vector space* if F is the field Q of all rational numbers ; etc.

2.2.1. Properties : *Let V be a vector space over a given field F, then the following properties hold :*

(*i*) $\alpha \cdot \mathbf{0} = \mathbf{0}, \mathbf{0} \in V, \forall \alpha \in F$.

Proof. We know that $\mathbf{0}+\mathbf{0}=\mathbf{0}$, where **0** is the zero vector of V

$\Rightarrow \alpha(\mathbf{0}+\mathbf{0})=\alpha\mathbf{0} \quad \forall \alpha \in F$

$\Rightarrow \alpha \mathbf{0} + \alpha\mathbf{0} = \alpha \mathbf{0}$

$\Rightarrow \alpha \mathbf{0} + \alpha \mathbf{0} = \alpha \mathbf{0} + \mathbf{0}$

$\Rightarrow \alpha \mathbf{0}=\mathbf{0}$. *[By Cancellation Law]*

(*ii*) $0 . x=\mathbf{0} \ \forall x \in V, 0 \in F, \mathbf{0} \in V$.

Proof. We know that $0+0=0$, where 0 is the zero element of F

$\Rightarrow (0 + 0)x = 0x \ \forall x \in V$

$\Rightarrow 0x + 0x = 0x$

$\Rightarrow 0x + 0x = 0x + \mathbf{0}$

$\Rightarrow \quad 0x = \mathbf{0}$ *[By Cancellation Law]*

(*iii*) $(-\alpha)x=-(\alpha x) \ \forall \alpha \in F$ and $x \in V$.

Proof. Since $\alpha \in F$,

$\therefore \ -\alpha \in F$. *[$\because$ F is a field]*

$\Rightarrow \alpha + (-\alpha) = 0 \in F$.

Now $\alpha x + (-\alpha)x = [\alpha + (-\alpha)]x \ \forall x \in V$

$\Rightarrow \alpha x + (-\alpha)x = 0x \ \forall x \in V$

$\Rightarrow \alpha x + (-\alpha)x = \mathbf{0} \ \forall x \in V$ *[Using (ii)]*

$\Rightarrow (-\alpha)x$ is the additive inverse of αx in V.

Hence $(-\alpha)x=-(\alpha x)$.

(*iv*) $\alpha(-x)=-(\alpha x) \ \forall \alpha \in F$ and $x \in V$.

Proof. Since $x \in V$, $\therefore \ -x \in V$

$\Rightarrow x + (-x) = \mathbf{0} \in V$.

Now $\alpha x + \alpha(-x)=\alpha[x + (-x)] \ \forall x \in V$

$\Rightarrow \alpha x + \alpha(-x) = \alpha\mathbf{0} \ \forall x \in V$

$\Rightarrow \alpha x + \alpha(-x) = \mathbf{0} \ \forall x \in V$. *[Using (i)]*

$\Rightarrow \alpha(-x)$ is the additive inverse of αx in V.

Hence $\alpha(-x) = -(\alpha x)$.

(v) $\alpha(x-y) = \alpha x - \alpha y \; \forall \; \alpha \in F$ and $x, y \in V$.

Proof. $\alpha\,(x-y) = \alpha\,[x + (-y)]$

$$= \alpha x + \alpha\,(-y)$$
$$= \alpha x - (\alpha y). \qquad \textit{[Using (iv)]}$$

(vi) $\alpha x = 0,\; x \neq 0 \Rightarrow \alpha = 0$, where $x \in V, \alpha \in F$.

Proof. We have : $\alpha x = 0,\; x \neq 0$.

If possible, let $\alpha \neq 0$. Then α^{-1} exists, where $\alpha^{-1} \in F$.

$\therefore\; \alpha^{-1}(\alpha x) = \alpha^{-1}(0)$

$\Rightarrow\; (\alpha^{-1}\alpha)\,x = 0 \;\Rightarrow\; 1x = 0 \qquad [\because \alpha^{-1}\alpha = 1,\text{ where } \alpha \in F]$

$\Rightarrow\; x = 0$, which is a contradiction.

Hence $\alpha = 0$.

2.3. CANCELLATION LAWS IN V UNDER ADDITION

If V be a vector space over a field F, then

(i) $x + y = x + z \Rightarrow y = z$

(ii) $y + x = z + x \Rightarrow y = z$

$\forall\; x, y, z \in V$.

Proof. (i) $x + y = x + z$

$\Rightarrow\; (-x) + (x + y) = (-x) + (x + z) \qquad$ [*Adding $-x$ to both sides*]

$\Rightarrow\; ((-x) + x) + y = ((-x) + x) + z \qquad$ [*By Associative law*]

$\Rightarrow\; 0 + y = 0 + z$

$\Rightarrow\; y = z.$

(ii) Left as an exercise for the readers.

ILLUSTRATIVE EXAMPLES

Example 1. *Which of the following sets form vector spaces over reals ? If not, give reasons.*

(i) *The set of all rationals over R* (G.N.D.U. 1987)

(ii) $V = \{a + ib\,;\; \forall\; a, b \in Z\}$ (G.N.D.U. 1987)

(iii) *All polynomials over R with constant term zero*

(iv) *All polynomials over R with constant term 1*

(v) *All polynomials with positive real coefficients*

(vi) *All polynomials $f(x)$ over R such that $f(1) = 0$*

(vii) *All polynomials $f(x)$ over R such that $f(1) = 5$*

(viii) *All lower (upper) triangular matrices of order n over R*

(ix) *All n-rowed symmetric (skew-symmetric) matrices over C.*

(G.N.D.U. 1987)

Solution.

(*i*) No. [$\because$ *It is not closed under scalar multiplication*]

$$\frac{1}{2}(3+4i) = \frac{3}{2} + 2i \notin V$$

$$a = \frac{1}{2} \in R \quad 3 + 4i \in V$$

[$\because$ $a \in R, \alpha \in Q, a\alpha \in Q$ *is not necessarily true*]

(*ii*) No. [$\because$ *V is not closed under scalar multiplication*]

(*iii*) Yes.

(*iv*) No. $2(2x^2 + 3x + 1) = 4x^2 + 6x + 2 \notin V$

[$\because$ *Polynomials under consideration are not closed under addition*]

(*v*) No. [$\because$ *Zero element does not exist*]

(*vi*) Yes.

(*vii*) No. [$\because$ $f(x)$ *is not closed under addition*]

(*viii*) Yes.

(*ix*) Yes

Example 2. *Let V be the set of all pairs (a, b) of real numbers. Examine the following, whether V is a vector space over R or not :*

(*i*) $(a, b) + (a', b') = (0, b + b')$; $\alpha(a, b) = (\alpha a, \alpha b)$

(*ii*) $(a, b) + (a', b') = (a + a', b + b')$; $\alpha(a, b) = (\alpha^2 a, \alpha^2 b)$.

where $a, b, a', b' ; \alpha \in R$.

Solution. In order to show that V is not a vector space, we have only to show that one of the postulates of a vector space does not hold.

(*i*) Here there exists no additive identity *i.e.* there exists no ordered pair $(c, d) \in V$ such that

$$\therefore \quad (c, d) + (a, b) = (a, b) \ \forall \quad (a, b) \in V$$

[$\because$ $(c, d) + (a, b) = (0, d + b) \neq (a, b)$] (*by def.*)

Thus V(R) is not satisfied.

Hence V(R) is not a vector space.

(*ii*) By def., $(\alpha + \beta)(a, b), ((\alpha + \beta)^2 a, (\alpha + \beta)^2 b)$ for $\alpha, \beta \in R$

and $\alpha(a, b) + \beta(a, b) = (\alpha^2 a, \alpha^2 b) + (\beta^2 a, \beta^2 b)$

$$= (\alpha^2 a + \beta^2 a, \alpha^2 b + \beta^2 b) \qquad [Def.]$$

$$= ((\alpha^2 + \beta^2) a, (\alpha^2 + \beta^2) b).$$

[*By Distributive Law in R*]

Since $(\alpha+\beta)^2 \neq \alpha^2+\beta^2 \;\forall\, \alpha,\beta \in \mathbf{R}$,

$\therefore\ (\alpha+\beta)(a,b) \neq \alpha(a,b)+\beta(a,b)$.

Thus V(R) is not satisfied.

Hence V(R) is not a vector space.

Example 3. *If (F, +, ·) is a field, then show that* **F(F)** *is a vector space and deduce that*

(*a*) ***C** is vector space over field **C***

(*b*) ***R** is vector space over field **R***

(*c*) ***C** is vector space over field **R***

(*d*) ***R is not a vector space over field C***

Solution. To prove that F(F) is a vector space, we are to verify all the **axioms** for the vector space.

Since (F, +, .) is a field, [*Given*]

$\therefore$ (F, +) is an abelian group

$\therefore$ (*i*) and (*v*) axioms are satisfied.

The scalar multiplication is the same as multiplication of field because the set F and field are same, so

(*vi*) $\alpha(x+y)=\alpha x+\alpha y \;\forall\, \alpha \in \mathbf{F}$ and $\forall\, x,y \in \mathbf{F}$

[$\because$ *Elements of the field are distributive*]

(*vii*) $(\alpha+\beta)x=\alpha x+\beta x \;\forall\, \alpha,\beta \in \mathbf{F}$ and $\forall\, x \in \mathbf{F}$

(*viii*) $\alpha(\beta x)=(\alpha\beta)x \;\forall\, \alpha,\beta \in \mathbf{F}$ and $\forall\, x \in \mathbf{F}$

[$\because$ *Elements of the field are associative for multiplication*]

(*ix*) $1(x)=x$ for $1 \in \mathbf{F}, \forall\, x \in \mathbf{F}$

[$\because$ *1 is the multiplicative identity of F*]

Hence **F(F)** is a vector space.

(*a*) Since (C, +, ·) is a field,

$\therefore$ as proved above C(C) is a vector space.

(*b*) Since (R, +, ·) is a field,

$\therefore$ as proved above **R(R)** is a vector space.

(*c*) In order to prove that C(R) is a vector space, we have to verify all the **axioms** for the vector space.

Since (C, +, ·) is a field,

$\therefore$ (C, +) is an abelian group.

$\therefore$ (*i*) – (*v*) are satisfied.

(*vi*) Since the product of a real number by a complex number is again a complex number,

$\therefore \alpha \in \mathbf{R}, x \in \mathbf{C} \Rightarrow \alpha x \in \mathbf{C}$.

Thus closure property is verified.

Since the elements of R are also elements of C ($\because \mathbf{R} \subset \mathbf{C}$) and elements of C are distributive w.r.t. addition and multiplication, so

(*vii*) $\alpha(x+y) = \alpha x + \alpha y \;\forall\, \alpha \in \mathbf{R}$ and $\forall\, x, y \in \mathbf{C}$

(*viii*) $(\alpha + \beta)x = \alpha x + \beta x \;\forall\, \alpha, \beta \in \mathbf{R}$ and $\forall\, x \in \mathbf{C}$

(*ix*) $\alpha(\beta x) = (\alpha\beta)x \;\forall\, \alpha, \beta \in \mathbf{R}$ and $\forall\, x \in \mathbf{C}$.

i.e. the elements of C are associative for multiplication.

(*x*) $1.(x) = x$ for $1 \in R$ and $\forall\, x \in \mathbf{C}$.

i.e. $1 \in \mathbf{R}$ is the multiplicative identity of C.

Hence C(R) is a vector space.

(*d*) Since product of complex number by a real number is a complex number and not a real number *i.e.*

$$\alpha \in \mathbf{C}, x \in \mathbf{R} \Rightarrow \alpha x \in \mathbf{C} \text{ but } \alpha x \notin \mathbf{R},$$

$\therefore$ R(C) is not closed for scalar multiplication.

Hence R(C) is not a vector space.

Example 4. *If R is the field of real numbers and V is the set of vectors in a plane. Further if addition of vectors is the internal binary composition in V and the multiplication of the elements of R with those of V as the external compostition, prove that V(R) is a vector space.*

Solution. Given

$V = \{ (x, y) \} \mid x, y \in \mathbf{R}\}$

[$\because$ *V is a set of vectors in a plane,* $\therefore$ *elements are ordered pairs*]

Let us define the *addition of vectors* in V as

$$(x, y) + (x', y') = (x + x', y + y')$$

and the *scalar multiplication of* $\alpha \in \mathbf{R}$ and $(x, y) \in \mathbf{V}$ *as*

$$\alpha(x, y) = (\alpha x, \alpha y).$$

1. Under Addition :

(*i*) **Closure.**

$$(x_1 + x_2, y_1 + y_2) \in \mathbf{V} \;\forall\, (x_1, y_1), (x_2, y_2) \in \mathbf{V}$$

[$\because x_1, y_1 ; x_2, y_2 \in \mathbf{R} \Rightarrow x_1 + x_2, y_1 + y_2 \in \mathbf{R}$

and $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$]

(*ii*) **Associativity.**

$((x_1, y_1) + (x_1, y_2)) + (x_3, y_3)$

$= (x_1, y_1) + (x_2, y_2), + (x_3, y_3) \ \forall \ (x_1, y_1), (x_2, y_2), (x_3, y_3) \in V.$

Because $((x_1, y_1) + (x_2, y_2)) + (x_3, y_3)$

$=(x_1 + x_2, y_1 + y_2) + (x_3, y_3)$ *[By def.]*

$=(x_1 + x_2 + x_3, \ y_1 + y_2 + y_3)$ *[By def.]*

$=(x_1, y_1) + (x_2 + x_3, y_2 + y_3)$ [$\because$ *Associativity in* R]

$=(x_1, y_1) + ((x_2, y_2) + (x_3, y_3)).$

(*iii*) **Existence of Identity.** $\forall (x, y) \in V, \exists (0, 0) \in V$ s.t.

$(x, y) + (0, 0)=(x, y)=(0, 0) + (x, y).$

Because $(x, y) + (0, 0))=(x + 0, y + 0)$

$=(x, y).$ *[By def.]*

Similarly $(0, 0) + (x, y) = (x, y).$

Thus $(x, y) + (0, 0)=(x, y) = (0, 0) + (x, y).$

Here $(0, 0)$ is the **zero element** of V.

(*iv*) **Existence of Inverse.** $\forall (x, y) \in V, \exists (-x, -y) \in V$ *such that*

$(-x, -y) + (x, y) = (0, 0)= (x, y) + (-x, -y).$

Because $(-x, -y) + (x, y) = ((-x) + x, (-y) + y))$ *[By def.]*

$=(0, 0)$

Similarly $(x, y) + (-x, -y)=(0, 0).$

$\therefore \ (-x, -y) + (x, y) = (0, 0) = (x, y) + (-x, -y).$

Here $(-x, -y)$ is **the inverse element** of (x, y) in V.

(*v*) **Commutativity.** $(x_1, y_1) + (x_2, y_2) = (x_2, y_2) + (x_1, y_1)$

$\forall (x_1, y_1), (x_2, y_2) \in V$

Because $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ *[By def.]*

$=(x_2 + x_1, y_2 + y_1)$

[$\because$ $x_1, x_2 ; y_1, y_2 \in R$ *and commutative law holds in* R]

$=(x_2 , y_2) + (x_1, y_1)$ *[By def.]*

II. Under Scalar Multiplication :

(*vi*) $\forall \alpha \in$ R and $\forall (x, y) \in V, x, y \in R$ [$\because$ $\alpha x, \alpha y \in R$]

$\alpha (x, y) = (\alpha x, \alpha y) \in V.$

(*vii*) $\alpha ((x_1, y_1) + (x_2, y_2)) = \alpha (x_1, y_1) + \alpha (x_1, y_2)$

$\forall \ \alpha \in R$ and $\forall \ (x_1, y_1), \ (x_2, y_2) \in V.$

Because $\alpha((x_1, y_1) + (x_2, y_2)) = \alpha (x_1 + x_2, y_1 + y_2)$ *[By def.]*

$=(\alpha (x_1 + x_2), \alpha (y_1 + y_2))$ [$\because$ *of (vi)*]

$=(\alpha x_1 + \alpha x_2, \alpha y_1 + \alpha y_2)$ [By def.]

$=(\alpha x_1, \alpha y_1) + (\alpha x_2, \alpha y_2)$ [$\because$ of (vi)]

$=\alpha\,(x_1, y_1) + \alpha\,(x_2, y_2)$.

(viii) $(\alpha + \beta)\,(x, y) = \alpha\,(x, y) + \beta\,(x, y) \;\forall\, \alpha, \beta \in R$ and $\forall\, (x, y) \in V$.

Because $(\alpha + \beta)\,(x, y) = ((\alpha + \beta)x, (\alpha + \beta)\,y)$ [$\because$ of (vi)]

$=(\alpha x + \beta x, \alpha y + \beta y)$ [$\because$ of (vi)]

$=(\alpha x, \alpha y) + (\beta x, \beta y)$ [By def.]

$=\alpha\,(x, y) + \beta\,(x, y)$. [$\because$ of (vi)]

(ix) $(\alpha, \beta)\,(x, y) = \alpha\,(\beta\,(x, y)) \;\forall\, \alpha, \beta \in R$ and $\forall\, (x, y) \in V$.

Because [$\because$ of (vi)]

$(\alpha\,.\,\beta)\,(x, y) = ((\alpha\,.\,\beta)\,x, (\alpha\,.\,\beta)\,y)$

$=(\alpha\,(\beta x), \alpha(\beta y))$ [$\because$ of (vi)]

$=\alpha\,(\beta x, \beta y)$

$=\alpha\,(\beta\,(x, y))$. [$\because$ of(vi)]

(x) $1\,(x, y) = (x, y) \;\forall\; (x, y) \in V$ and $1 \in R$.

Because. [By def.]

$1(x, y) = (1\,.\,x\; 1\,.\,y)$

$=(x, y)$

Hence V(R) is a vector space.

Example 5. *Let F be a field and V the set of all m×n matrices over the field F. The addition of matrices is defined as internal composition and multiplication of any scalar with a matrix as the external composition. Prove that V(F) is a vector space.* **(G.N.D.U. 1986, 85)**

Sol. Let $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$, $C = [c_{ij}]_{m \times n} \in V(F)$, where $a_{ij}, b_{ij}, c_{ij} \in F$.

I. Under Addition :

(i) **Closure.** $\forall\, A, B \in V(F)$, $A + B \in V(F)$.

Because $A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [a_{ij} + b_{ij}]_{m \times n} \in V(F)$.

(ii) **Associativity.**

$(A + B) + C = A + (B + C)$

$\forall\, A, B, C \in V(F)$.

Because

$(A+B)+C=([a_{ij}]_{m\times n}+[b_{ij}]_{m\times n})+[c_{ij}]_{m\times n}$

$=[a_{ij}+b_{ij}]_{m\times n}+[c_{ij}]_{m\times n}$

$=[(a_{ij}+b_{ij})+c_{ij}]_{m\times n}$

$=[a_{ij}+b_{ij}+c_{ij}]_{m\times n}$ [$\because$ *Associativity under addition in F*]

$=[a_{ij}]_{m\times n}+[b_{ij}+c_{ij}]_{m\times n}$

$=[a_{ij}]_{m\times n}+([b_{ij}]_{m\times n}+[c_{ij}]_{m\times n})$

$=A+(B+C)$.

(*iii*) **Existence of Identity.** $A+O=A=A+O$ $\forall A \in V(F), O \in V(F)$.

Because $A+O=[a_{ij}]_{m\times n}+[0]_{m\times n}$

$=[a_{ij}+0]_{m\times n}=[a_{ij}]_{m\times n}=A$.

Similarly $O+A=A$

$\therefore \quad A+O=A=O+A$

Here $\quad O=O_{m\times n}=[0]_{m\times n}$ is the **identity.**

(*iv*) **Existence of Inverse.** $\forall A \in V(F)$,

$\exists A \in V(F)$ such that

$A+(-A)=O=(-A)+A$.

Because $A+(-A)=[a_{ij}]_{m\times n}+[-a_{ij}]_{m\times n}$

$=[a_{ij}+(-a_{ij})]_{m\times n}=[0]_{m\times n}=O$.

Similarly $(-A)+A=O$.

$\therefore \quad A+(-A)=O=(-A)+A$.

Here $-A$ is the **inverse** of A.

(*v*) **Commutativity.** $A+B=B+A \ \forall A, B \in V(F)$.

Because $A+B=[a_{ij}]_{m\times n}+[b_{ij}]_{m\times n}=[a_{ij}+b_{ij}]_{m\times n}$

$=[b_{ij}+a_{ij}]_{m\times n}$ [*Commutativity under addition in F*]

$=[b_{ij}]_{m\times n}+[a_{ij}]_{m\times n}=B+A$.

II. Under Scalar Multiplication :

(*vi*) $\forall \alpha \in F, \forall A \in V(F)$

$\alpha A=\alpha\,[a_{ij}]_{m\times n}=[\alpha a_{ij}]_{m\times n} \in V(F)$.

(*vii*) $(\alpha+\beta)A=\alpha A+\beta A \ \forall \alpha, \beta \in F, \forall A \in V(F)$.

Because

$(\alpha+\beta)A=(\alpha+\beta)\,[a_{ij}]_{m\times n}$

$=[(\alpha+\beta)\,a_{ij}]_{m\times n}=[\alpha a_{ij}+\beta a_{ij}]_{m\times n}$

$=[\alpha a_{ij}]_{m\times n}+[\beta a_{ij}]_{m\times n}=\alpha[a_{ij}]_{m\times n}+\beta[a_{ij}]_{m\times n}$

$=\alpha A+\beta A$.

(*viii*) $\alpha(A+B)=\alpha A+\alpha B \ \forall\ \alpha \in F, \forall\ A, B \in V(F)$.

Because

$\alpha(A+B)=\alpha([a_{ij}]_{m\times n}+[b_{ij}]_{m\times n})$

$=\alpha[a_{ij}+b_{ij}]_{m\times n}=[\alpha(a_{ij}+b_{ij})]_{m\times n}$

$=[\alpha a_{ij}+\alpha b_{ij}]_{m\times n}=[\alpha a_{ij}]_{m\times n}+[\alpha b_{ij}]_{m\times n}$

$=\alpha[a_{ij}]_{m\times n}+\alpha[b_{ij}]_{m\times n}$

$=\alpha A+\alpha B$.

(*ix*) $(\alpha\beta)A=\alpha(\beta A) \ \forall\ \alpha,\beta \in F, \forall\ A \in V(F)$.

Because $(\alpha\beta)A=(\alpha\beta)[a_{ij}]_{m\times n}=[(\alpha\beta)a_{ij}]_{m\times n}$

$=[\alpha(\beta a_{ij})]_{m\times n}=\alpha[\beta a_{ij}]_{m\times n}$

$=\alpha(\beta[a_{ij}]_{m\times n})=\alpha(\beta A)$.

(*x*) $1.A=A \ \forall\ A \in V(F), 1 \in F$.

Because $1.A=1[a_{ij}]_{m\times n}$

$=[1.a_{ij}]_{m\times n}=[a_{ij}]_{m\times n}=A$.

Hence V(F) is a vector space.

EXERCISE 2 (a)

1. Examine the truth or otherwise of the following statements :

(*i*) A vector space must have atleast two elements. *(G.N.D.U. 1985 S, 85)*

(*ii*) A vector space has always an infinite number of elements. *(G.N.D.U. 1985 S)*

(*iii*) In the definition of a vector space V(F), the axiom $1.v=v$, for all $v \in V$ can be dropped. *(G.N.D.U. 1993, 86)*

2. What is the zero vector in (*i*) $V_2(R)$ (*ii*) $V_3(R)$ (*iii*) $V_4(R)$?

3. Show that the set of all elements of the type $a+b\sqrt{2}+c\sqrt[3]{3}$, where $a, b, c \in Q$ for a vector space over the field Q under usual addition and scalar multiplication of real numbers.

4. Prove that the set of all $m \times n$ diagonal matrices over the reals is a vector space.

5. Does the set V of all ordered pairs of integers for a vector space over the field R of real numbers with addition and scalar multiplication defined as below :

$(a_1, a_2)+(b_1, b_2)=(a_1+b_1, a_2+b_2)$ for $(a_1, a_2), (b_1, b_2) \in V$

$\alpha(a_1, a_2)=(\alpha a_1, \alpha a_2)$ for $\alpha \in R, (a_1, a_2) \in V$?

6. Let V be the set of all pairs (a, b) of real numbers. Examine the following, whether V is a vector space over R or not.

(i) $(a,b)+(a',b')=(a+a',b+b'),\ \alpha(a,b)=(0,\alpha b)$

(ii) $(a,b)+(a',b')=(a,b);\ \alpha(a,b)=(\alpha a,\alpha b)$,

where $a, b, a', b', \alpha \in R$.

7. Show that the set Q^4 of 4-tuples of rational numbers is a vector space over the field Q under usual addition and scalar multiplication of 4-tuples.

8. Prove that the set $V_n(F)$ of all ordered n tuples of all elements of any field F is a vector space with vector addition and scalar multiplication defined as below :

$$\forall x=(x_1,x_2,\ldots,x_n) \text{ and } y=(y_1,y_2,\ldots,y_n)\in V_n(F)$$

$$x+y=(x_1+y_1,x_2+y_2,\ldots,x_n+y_n),$$

where $x_1,x_2,\ldots,x_n;y_1,y_2,\ldots,y_n \in F$ and $\forall \alpha \in F$

we define $\alpha x=(\alpha x_1,\alpha x_2,\ldots,\alpha x_n)$.

9. If V is the set of all real-valued continuous (differentiable or integrable) functions defined in some interval [0,1]. Then show that V(R) is a vector space with addition and scalar multiplication defined as follows :

$$(f+g)x=f(x)+g(x)\quad \forall f,g\in V$$

and $$(\alpha f)x=\alpha f(x)\quad \forall \alpha\in R, f\in V.$$

10. If **P** (n) denotes the set of all polynomials V in one indeterminate x over a field F, then show that **P**(x) is a vector space over F with addition defined as addition of polynomials and scalar multiplication defined as the product of polynomials by an element of F.

11. If $v_1=(1,-2,3)$, $v_2=(-3,1,-1)$ be in the vector space Q^3 over the field Q, solve the equation

$$3v_1+2x-v_2=0$$

in the vector space Q^3.

12. Let V_1, V_2 be two vector spaces over the same field F, show that

$$V_1\times V_2=\{(u_1,u_2);\text{ for all } u_1\in V_1, u_2\in V_2\}$$

is a vector space over the field F under addition and scalar multiplication defined by

$$(u_1,u_2)+(v_1,v_2)=(u_1+v_1,u_2+v_2) \text{ and}$$

$$\alpha(u_1,u_2)=(\alpha u_1,\alpha u_2)$$

for all $u_1, v_1, \in V_1, u_2, v_2 \in V_2$ and $\alpha \in F$.

ANSWERS

1. (*i*) False (*ii*) False (*iii*) False.
2. (*i*) (0, 0) (*ii*) (0, 0, 0) (*iii*) (0, 0, 0, 0).
5. No.
6. (*i*) No (*ii*) No.
11. $x = \left(-3, \frac{7}{2}, -5\right)$.

2.3. VECTOR SUB-SPACES

Def. *A non-empty subset W of a vector space V over a field F is said to be a sub-space of V iff W is a vector space over the field F in itself under the addition of V and the scalar multiplication of V by F.*

Each vector space V over a field F has two **trivial** sub-spaces $\{0\}$ and V itself.

[$\because$ *$V \subset V$ and V is a vector space & $\{0\} \subset V$* and *$\{0\}$ is a vector space*]

Any other sub-space of V is called **non-trivial** sub-space.

Illustrations :

(I) The field C of complex numbers is a vector space over the field Q of rational numbers.

The field **R** of reals is a proper sub-space of C.

(II) Let $V = V_3(R) = \{(a_1, a_2, a_3) : a_1, a_2, a_3 \in R\}$.

This is a vector space under addition and scalar multiplication of vectors.

Let $W = \{(a_1, a_2, 0) : a_1, a_2 \in R\}$. **[Check !]**

It can be checked that W is also a vector space under addition and scalar multiplication of vectors.

Also $W \subset V$, therefore, W is a vector sub-space of V.

2.3.1. Theorems :

Theorem I. *A non-empty subset W of a vector space V (F) is a sub-space of V iff W is closed under vector addition and scalar multiplication.* **(G.N.D.U. 1985 S)**

Proof. Since W is a subspace of V (F),

$\therefore$ W is closed under vector addition and scalar multiplication.

Hence the result.

Conversely : W is closed under vector addition and scalar multiplication in V.

W is closed under scalar multiplication.

Bacause $\forall x \in W,\ -1 \in F \Rightarrow (-1)x \in W$

$\Rightarrow\ -x \in W$ $[\because x \in W \Rightarrow x \in V$ *and* $(-1)(x) = -x \in V]$

(I) *Thus additive inverse of each element of W exists.*

Now W is closed under vector addition.

$\therefore \forall x \in W,\ -x \in W \Rightarrow x + (-x) \in W$

$[\because x \in W \Rightarrow -x \in W \Rightarrow x + (-x) = \mathbf{0} \in V]$

$\Rightarrow \mathbf{0} \in W.$

(II) *Thus* $\mathbf{0}$ *is the additive identity of W.*

(III) Since the elements of W are elements of V,

$\therefore$ vector addition is commutative and associative in W.

Thus W is an abelian group under vector addition.

Further, W is closed under multiplication and thus the remaining properties of vector space also hold in W because they hold in V.

Hence W is a vector space and is a vector subspace of V.

Theorem II. *A sub-set W of a vector space V (F) is a sub-space of V iff*

(*i*) *W is non-empty*

(*ii*) $\forall x, y \in W \Rightarrow x - y \in W$

(*iii*) $\forall \alpha \in F, x \in W \Rightarrow \alpha x \in W.$

Proof. (*i*) Since W is a sub-sapce of V (F),

$\therefore$ (W , +) is an abelian group and is a sub-group of (V , +).

$\therefore$ W contains the additive identity **0**.

Hence W is non empty.

(*ii*) $\forall x, y \in W \Rightarrow x, -y \in W$ $[\because y \in W \Rightarrow -y \in W]$

$\Rightarrow x + (-y) \in W.$ [$\because$ *W is closed under addition*]

$\Rightarrow x - y \in W.$

(*iii*) $\forall \alpha \in F, x \in W$, where W is a vector space

$\therefore \alpha x \in W.$ [*By def.*

Conversely. Given. Conditions (*i*), (*ii*) and (*iii*) hold true.

To prove. W is a vector sub-space of V (F).

For this, all the properties for a vector space must hold true.

$\forall x \in W \Rightarrow x \in V$

Also $1 \in F \Rightarrow -1 \in F$ $[\because W \subset V$

$\therefore \forall\ y \in$ W and $-1 \in$ F $\Rightarrow$ $(-1)y \in$ W. [*By above (iii)*]

$\Rightarrow$ $-y \in$ W.

Thus inverse of each element of W exists.

$\forall\ x \in$ W and $y \in$ W

$\Rightarrow$ $x \in$ W and $-y \in$ W $\Rightarrow$ $x - (-y) \in$ W [*By above (ii)*]

$\Rightarrow$ $x + y \in$ W.

Thus W is closed under addition.

$\forall x \in$ W and $-x \in$ W $\Rightarrow$ $x + (-x) \in$ W.

$\Rightarrow$ $0 \in$ W.

Thus identity exists in W under addition.

Since W $\subset$ V,

$\therefore$ the remaining properties also hold true for W.

Hence W is a vector sub-space of V (F).

Theorem III. *A sub-set W of a vector space V (F) is a sub-space of V (F) iff $\forall\ \alpha, \beta \in F$ and $\forall\ x, y \in W \Rightarrow \alpha x + \beta y \in W$.* **(P.U. 1985)**

Proof. **Given.** W is a sub-space of V (F).

To prove. $\forall\ \alpha, \beta \in$ F and $\forall\ x, y \in$ W $\Rightarrow$ $\alpha x + \beta y \in$ W.

$$\left.\begin{array}{l} \alpha \in F, x \in W \Rightarrow \alpha x \in W \\ \text{and } \beta \in F,\ y \in W \Rightarrow \beta y \in W. \end{array}\right\}$$

[$\because$ *W is closed under scalar multiplication*]

Now $\alpha x \in$ W, $\beta y \in$ W $\Rightarrow$ $\alpha x + \beta y \in$ W.

[$\because$ *W is closed under addition*]

Hence $\forall\ \alpha, \beta \in$ F and $\forall\ x, y \in$ W $\Rightarrow$ $\alpha x + \beta y \in$ W.

Conversely. **Given.** $\forall\ \alpha, \beta \in$ F and $\forall x, y \in$ W

$\Rightarrow$ $\alpha x + \beta y \in$ W.

To prove. W is a sub-space of V (F).

Taking $\alpha = 1, \beta = -1$

$1.x + (-1)y \in$ W $\Rightarrow$ $x - y \in$ W.

Taking $\beta = 0$,

$\alpha x + 0.y \in$ W $\Rightarrow$ $\alpha x + 0 \in$ W [$\because$ $0y = 0$]

$\Rightarrow$ $\alpha x \in$ W.

Hence W is a sub-space of V (F).

Theorem IV. *The intersection of the sub-spaces W_1 and W_2 of a vector space V (F) is also a sub-space.* **(G.N.D.U. 1989)**

Proof. Let $x, y \in W_1 \cap W_2$ and $\alpha, \beta \in$ F.

Now $x \in W_1 \cap W_2 \Rightarrow x \in W_1$ and $x \in W_2$

and $y \in W_1 \cap W_2 \Rightarrow y \in W_1$ and $y \in W_2$.

Since $x, y \in W_1$ and W_1 is a sub-space of V,

$\therefore \alpha x + \beta y \in W_1$ ...(1) [Th. III]

Similarly $\alpha x + \beta y \in W_2$ ...(2)

(1) and (2) $\Rightarrow \alpha x + \beta y \in W_1 \cap W_2$.

Hence $W_1 \cap W_2$ is a sub-space of V(F).

Remark. The union of the sub-spaces may not be a sub-space.

For Ex. *Consider V_3 (F) to be a vector-space and W_1, W_2 be its two sub-spaces with elements of the type (a, 0, 0) and (0, b, 0) respectively.*

Let $x = (a, 0, 0) \in W_1$ and $y = (0, b, 0) \in W_2$.

If α, β are two non-zero scalars, then

$\alpha x + \beta y = \alpha(a, 0, 0) + \beta(0, b, 0)$

$= (\alpha a, \beta b, 0)$

$\Rightarrow \alpha x + \beta y \notin W_1$ and $\alpha x + \beta y \notin W_2$

$\Rightarrow \alpha x + \beta y \notin W_1 \cup W_2$

Hence $W_1 \cup W_2$ is not a sub-space.

Extension : *The intersection of an arbitrary family $\{W_\alpha \mid \alpha \in N\}$ of sub-spaces of a vector space V is a sub-space of V.* **(G.N.D.U. 1989)**

Proof. Let $W_1, W_2, \ldots\ldots, W_n$ be n sub-spaces of a vector space W.

Let $W = \cap W_\alpha ; \alpha = 1, 2, \ldots n$

Since $0 \in W_\alpha$, $\therefore 0 \in \cap W_\alpha$

$\Rightarrow 0 \in W$

$\Rightarrow$ W is non-empty.

Let $\alpha, \beta \in F$ and $x, y \in W_\alpha$

$\Rightarrow \alpha x + \beta y \in$ each W_α, which is a sub-space

$\Rightarrow \alpha x + \beta y \in \cap W_\alpha$

Hence $W = \cap W_\alpha$ is a sub-space.

Remark. The above result does not hold for union.

Theorem V. *The union of two sub-spaces of a vector space V over a field F is a sub-space of V iff they are comparable.* **(P.U. 1989 ; G.N.D.U. 1985)**

Proof. Let W_1, W_2 be two sub-spaces of the vector space W over the field F.

Let W_1, W_2 be comparable. [*Given*

$\therefore$ either $W_1 \subset W_2$ or $W_2 \subset W_1$.

If $W_1 \subset W_2$, then $W_1 \cup W_2 = W_2$.

If $W_2 \subset W_1$, then $W_1 \cup W_2 = W_1$.

Thus $W_1 \cup W_2$ is either W_1 or W_2.

But W_1 and W_2 are both sub-spaces of V,

$\therefore$ $W_1 \cup W_2$ is a sub-space of V.

Conversely : Let $W_1 \cup W_2$ be a sub-space.

To prove : W_1 and W_2 are comparable

i.e. either $W_1 \subset W_2$ or $W_2 \subset W_1$.

Let us assume that W_1 is not a sub-set of W_2

and W_2 is not a sub-set of W_1.

Then $W_1 \not\subset W_2 \therefore \exists x \in W_1, x \notin W_2$ ...(1)

and $W_2 \not\subset W_1 \therefore \exists y \in W_2, y \notin W_1$ ...(2)

From (1), $x \in (W_1 \cup W_2)$ $[\because x \in W_1]$

From (2), $y \in (W_1 \cup W_2)$ $[\because y \in W_2]$

But $W_1 \cup W_2$ is a sub- space [*Given*]

$\therefore$ $x + y \in W_1 \cup W_2$

$\Rightarrow$ $x + y \in W_1$ or W_2

Let $x + y \in W_1$.

Let $x \in W_1$ and W_1 is a sub-space.

$\therefore$ $(x + y) - x \in W_1 \Rightarrow y \in W_1$, which contradicts (2).

Thus either $W_1 \subset W_2$ or $W_2 \subset W_1$.

Hence W_1 and W_2 are comparable.

ILLUSTRATIVE EXAMPLES

Example 1. *Let V be a vector space of polynomials of degree* ≤ 6. *Which of the following are sub-spaces ? Justify your answers. In each case f (x) belongs to V.*

(*i*) $W = \{f(x) : f(0) = 1\}$

(*ii*) $W = \{f(x) : \deg f(x) \leq 4\}$

(*iii*) $W = \{f(x) : f(1) = 0, f(3) = 0\}$

(*iv*) $W = \{f(x) : \text{coeff. of } x^2 \text{ is } 1 \text{ or } -1\}$

(*v*) $W = \{f(x) : \text{having +ve coefficients}\}$. **(G.N.D.U. 1988 S)**

Solution. (*i*) W is not a sub-space of V, because zero polynomial does not belong to W.

(*ii*) W is a sub-space of V.

Clearly W is a non-empty sub-set of V.

Consider $f(x), g(x) \in W$, then $\deg f(x) \leq 4$, $\deg g(x) \leq 4$.

For all $\alpha, \beta \in F$,

$\alpha f(x) + \beta g(x)$ is a polynomial of degree ≤ 4

$\Rightarrow \alpha f(x) + \beta g(x) \in W \Rightarrow W$ is a sub-space of V.

(*iii*) W is a sub-space of V.

Let $f(x) = x^2 - 4x + 3$, which is a polynomial in x of degree 2 and $f(1) = f(3) = 0$ so $f(x) \in W$ and thus W is non-empty subset of V.

As in part (*ii*), we can prove that W is a sub-space of V.

(*iv*) W is not a sub-space of V, because zero polynomial does not belong to W.

(*v*) W is not a sub-space of V, because zero polynomial does not belong to W.

Example 2. *Let $V = R^3 = \{ (a, b, c) : a, b, c \in R \}$ and if W be the set of all triples (a,b,c) such that $a - 3b + 4c = 0$, then prove that W is a sub-space of R^3 (R).*

Solution. Since $a - 3b + 4c = 0$, $\therefore a = 3b - 4c$.

Let us choose two elements of W so as to satisfy the above condition as :

$$x = (a_1, b_1, c_1) \text{ and } y = (a_2, b_2, c_2)$$

i.e. $x = (3b_1 - 4c_1, b_1, c_1)$ and $y = (3b_2 - 4c_2, b_2, c_2)$.

To prove : W is a sub-space.

$\alpha x + \beta y \in W$, where $\alpha, \beta \in R$.

Because $\alpha x + \beta y = \alpha (3b_1 - 4c_1, b_1, c_1) + \beta(3b_2 - 4c_2, b_2, c_2)$

$$= (3(\alpha b_1 + \beta b_2) - 4(\alpha c_1 + \beta c_2), \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2)$$

$$= (3B - 4C, B, C)$$

$$= (A, B, C) \in W,$$

where $A = 3B - 4C$ or $A - 3B + 4C = 0$,

where $A, B, C \in R$ because $\alpha b_1 + \beta b_2 \in R$, $\alpha c_1 + \beta c_2 \in R$.

Hence W is a sub-space.

Example 3. *If a vector space is the set of real valued continuous functions over R, then show that set W of solutions of differential equation*

$$2\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 2y = 0 \text{ is a sub-space of } V.$$

Solution. We have : $W = \left\{ y : 2\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 2y = 0 \right\}$, where $y = f(x)$.

Since y satisfies the differential equation, therefore it $\in$ W.

Now select $y_1, y_2 \in W$ so that

$$2\frac{d^2y_1}{dx^2} - 9\frac{dy_1}{dx} + 2y_1 = 0 \text{ and } 2\frac{d^2y_2}{dx^2} - 2\frac{dy_2}{dx} + 2y_2 = 0.$$

In order to prove that W is a sub-space we are to show that $\alpha y_1 + \beta y_2 \in W$, where $\alpha, \beta \in F$.

$$\text{Now } 2\frac{d^2}{dx^2}(\alpha y_1 + \beta y_2) - 9\frac{d}{dx}(\alpha y_1 + \beta y_2) + 2(\alpha y_1 + \beta y_2) = 0$$

$$\Rightarrow 2\alpha\frac{d^2y_1}{dx^2} + 2\beta\frac{d^2y_2}{dx^2} - 9\left(\alpha\frac{dy_1}{dx} + \beta\frac{dy_2}{dx}\right) + 2\alpha y_1 + 2\beta y_2 = 0$$

$$\Rightarrow \alpha\left(2\frac{d^2y_1}{dx^2} - 9\frac{dy_1}{dx} + 2y_1\right) + \beta\left(2\frac{d^2y_2}{dx^2} - 9\frac{dy_2}{dx} + 2y_2\right) = 0$$

$\Rightarrow \alpha(0) + \beta(0) = 0$, which is true.

Since $\alpha y_2 + \beta y_2$ satisfies the given differential equation as and when y_1, y_2 satisfy it.

Hence W is a sub-space.

Example 4. *Let V be a vector space of all 2 × 2 matrices over reals. Determine whether W is a sub-space of V or not, where :*

(a) *W consists of all matrices with non-zero determinant*

(b) *W consists of all matrices A s.t. $A^2 = A$.*

Solution. (i) Let $W = \left\{\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}; x, y \in R \text{ and } xy \neq 0\right\}$

Since $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \in W$, therefore W is a non-empty subset of V.

We shall show with the help of an example, that W is not a sub-space of V.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ be any two members of W

($\because$ det A $\neq 0$, det. B $\neq 0$)

Taking $\alpha = 2, \beta = -1$, we have

$$\alpha A + \beta B = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 1\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin W$$

[$\because$ det. $\alpha A + \beta B = 0$]

$\therefore$ W is not a sub-space of V.

(ii) W is not a sub-space of V because W is not closed under addition.

To verify. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, so that

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

$\therefore A \in W$.

But $A + A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

and $(A + A)^2 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \neq A + A$

Thus $A + A \notin W$.

EXERCISE 2 (b)

1. Discuss whether or not R^2 is a sub-space of R^3.

2. Let R be the field of real numbers. Which of the following are sub-spaces of the vector space R^3 over R ? Justify your answer :

(*a*) (*i*) $W = \{ (a, b, c) ; a, b, c \in R ; a \geq 0 \}$

(*ii*) $W = \{ (a, b, c) ; a, b, c \in R ; a \leq b \leq c \}$

(*iii*) $W = \{ (a, b, c) ; a, b, c \in R ; a = b = c \}$

(*iv*) $W = \{ (a, b, c) ; a, b, c \in R ; ab = 0 \}$

(*v*) $W = \{ (a, b, c) : a, b, c \in R ; a = 2b \}$

(*vi*) $W = \{ (a, b, c) : a, b, c \in R ; a = b^2 \}$

(*vii*) $W = \{ (a, b, c) : a, b, c \in R ; a^2 + b^2 + c^2 \leq 1 \}$

(*viii*) $W = \{ (a, b, c) : a, b, c \in R ; k_1 a + k_2 b + k_3 c = 0 ; k$'s being scalars $\}$. (*G.N.D.U. 1985*)

(*b*) (*i*) $W = \{ (a, b, c) ; a, b, c \in R$ *s.t.* $2a + b - 5c = 0 \}$

(*ii*) $W = \{ (a, b, c) ; a, b, c \in R$ *s.t.* $2a - b - 3c = 0 \}$

(*iii*) $W = \{ (a, b, c) ; a, b, c \in R$ *s.t.* $b = 2a, c = a + b \}$

(*iv*) $W = \{ (a, b, c) ; a, b, c \in R$ *s.t.* $a - b + c = 0, 2a + 3b - c_3 = 0 \}$

(*v*) $W = \{ (a, 2a, a + 1) ; a \in R \}$

(*vi*) $W = \{ (a, b, c) ; a, b, c \in Q \}$.

3. Let R be the field of real numbers. Show that

(*i*) $V_1 = \{ (a, b, c, d) ; b - 2c + d = 0 \}$

(*ii*) $V_2 = \{ (a, b, c, d) ; a = d \text{ and } b = 2c \}$

are proper sub-spaces of the vector space R^4 over R.

4. Which of the following sets of vectors

$x = (a_1, a_2, \ldots, a_n)$

in R^n are sub-spaces of R^n ? $(n \geq 3)$

(i) all x such that $a_1 \geq 0$

(ii) all x such that $a_1 + 3a_2 = a_3$

(iii) all x such that $a_2 = a_1^2$

(iv) all x such that $a_1 a_2 = 0$

(v) all x such that a_2 is rational.

5. Let V be the vector space of all polynomials in x of degree ≤ 5 (including the zero polynomial) over the field Q. Which of the following subsets are sub-spaces of V ? Explain why ?

Here $f(x) \in$ V.

(i) $W = \{ f(x) ; f(3) = 0 \}$

(ii) $W = \{ f(x) ; f(1) = 0 \text{ and } f(5) = 0 \}$

(iii) $W = \{ f(x) ; f(0) = 1 \}$

(iv) $W = \{ f(x) ; \text{coefficient of } x^2 \text{ is zero} \}$

(v) $W = \{ f(x) ; \text{coefficient of } x^3 \text{ is } 1 \text{ or } -1 \}$

(vi) $W = \{ f(x) ; \text{either } f(x) \text{ is zero polynomial or } \deg f(x) \leq 3 \}$.

6. Let V be a vector space of polynomials of degree ≤ 6. Which of the following are sub-spaces ? Justify your answers. In each case $f(x) \in$ V.

(i) $W = \{ f(x) ; f(0) = 1 \}$

(ii) $W = \{ f(x) ; \text{coefficient of } x^3 = 0 \}$.

7. Let V be a vector space of a function F : V → R. Which of the following are sub-spaces ? In each case $f(x) \in$ V.

(i) All f such that $f(x^2) = [f(x)]^2$

(ii) all f such that $f(0) = f(1)$

(iii) all f such that $f(-1) = 0$

(iv) all f such that $f(3) = 1 + f(-5)$

(v) all f which are continuous.

8. Let V be a vector space of all $n \times n$ matrices over reals. Examine whether the following are sub-spaces of V or not :

(i) collection of all scalar matrices

(ii) collection of all diagonal matrices

(iii) collection of all symmetric matrices

(iv) collection of all singular matrices

(v) collection of all lower (upper) triangular matrices

(vi) collection of all skew-symmetric matrices.

9. Let V be a vector space of all 2×2 matrices over reals. Examine whether the following are sub-spaces of V or not :

(i) collection of all matrices A *s.t.* $A^2 = A$

(ii) collection of matrices with zero determinant.

10. Let V be a vector space of all 2×2 matrices over the field of reals. Determine which of the following are sub-spaces of V ? Explain why ?

(*i*) $W = \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & 0 \end{bmatrix} ; \alpha, \beta, \gamma \in R \right\}$

(*ii*) $W = \{ A ; A \in V, A \text{ is singular} \}$.

11. Let V be a vector space of all $n \times n$ matrices over the field F. Let $W = \{ A \in V ;\ AT = TA \}$, where T is a given $n \times n$ matrix over F, then show that W is a sub-space of V.

12. If W_1 and W_2 are sub-spaces V, prove that

$W_1 + W_2 = \{ w_1 + w_2 : w_1 \in W_1, w_2 \in W_2 \}$

is a sub-space of V. (*P.U. 1985 S*)

13. Let { F, + , . } be any field. Show that $W_1 = \{(a, 0, 0) ; \forall a \in F\}$ and $W_2 = \{ (0, b, c) ; \forall b, c \in F \}$ are proper sub-spaces of the vector space F^3 over the field F.

14. Prove that the set of all polynomials in one indeterminate x over a field F of degree less than or equal to n is a sub-space of the vector space of all polynomials over F

$\{ p(x) : p(x) = a_0 + a_1 x + a_2 x^2 + ... \} ; a_i \in F.$

15. Let V be a vector space of all polynomials over R. Show that collection of all polynomials of degree $\leq n$ with zero polynomial is a sub-space of V.

ANSWERS

1. No.

2. (*a*) (*i*) No (*ii*) No (*iii*) Yes (*iv*) No (*v*) Yes (*vi*) No (*vii*) No (*viii*) No.

(*b*) (*i*) Yes (*ii*) Yes (*iii*) Yes (*iv*) Yes (*v*) No (*vi*) No.

4. (*i*) No (*ii*) Yes (*iii*) No (*iv*) No (*v*) No.

5. (*i*) Yes (*ii*) Yes (*iii*) No (*iv*) Yes (*v*) No (*vi*) Yes.

6. (*i*) No (*ii*) Yes.

7. (*i*) No (*ii*) Yes (*iii*) Yes (*iv*) No (*v*) Yes.

8. (*i*) Yes (*ii*) Yes (*iii*) Yes (*iv*) No (*v*) Yes (*vi*) Yes.

9. (*i*) No (*ii*) No.

10. (*i*) Yes (*ii*) No.

2.4. LINEAR COMBINATION

Def. *Let V be a vector space over a field F. Let $x_1, x_2 ... x_n \in V$, then any element x, which can be written in the form*

$$x = a_1 x_1 + a_2 x_2 + ... + a_n x_n$$

$$= \sum_{i=1}^{n} a_i x_i \text{ for } a_i \in F, \text{ where } i \le i \le n$$ *is said to be* **linear combination** *of the vectors $x_1, x_2, ..., x_n$ over the field F.*

As V is a vector space, then by addition of vectors and scalar multiplication in V, $x \in V$.

Remark. For $x_1, x_2, ...x_n$, we get different linear combinations by selecting different sets of scalars.

ILLUSTRATIVE EXAMPLES

Example 1. *Write the vector $x = (1, -2, 5)$ as a linear combination of the vectors :*

$x_1 = (1, 1, 1), x_2 = (1, 2, 3), x_3 = (2, -1, 1)$ *in the vector space $V_3(R)$.*

Solution. Let $x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$; α's $\in$ R

i.e. $(1, -2, 5) = \alpha_1(1, 1, 1) + \alpha_2(1, 2, 3) + \alpha_3(2, -1, 1)$

$\Rightarrow (1, -2, 5) = (\alpha_1, \alpha_1, \alpha_1) + (\alpha_2, 2\alpha_2, 3\alpha_2) + (2\alpha_3, -\alpha_3, \alpha_3)$

$\Rightarrow (1, -2, 5) = (\alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 - \alpha_3, \alpha_1 + 3\alpha_2 + \alpha_3)$

Comparing, $1 = \alpha_1 + \alpha_2 + 2\alpha_3$ ...(1)

$-2 = \alpha_1 + 2\alpha_2 - \alpha_3$ ...(2)

and $5 = \alpha_1 + 3\alpha_2 + \alpha_3$ ...(3)

Solving (1), (2) and (3) :

Adding (2) and (3), $3 = 2\alpha_1 + 5\alpha_2 \Rightarrow 2\alpha_1 + 5\alpha_2 = 3$ ...(4)

(1) + 2 (2) gives $-3 = 3\alpha_1 + 5\alpha_2 \Rightarrow 3\alpha_1 + 5\alpha_2 = -3$ ...(5)

Subtracting (4) from (5), $\alpha_1 = -6$.

Putting in (4), $2(-6) + 5\alpha_2 = 3 \Rightarrow 5\alpha_2 = 3 + 12$

$\Rightarrow 5\alpha_2 = 15 \Rightarrow \alpha_2 = 3$.

Putting in (1), $-6 + 3 + 2\alpha_3 = 1 \Rightarrow 2\alpha_3 = 1 + 6 - 3$

$\Rightarrow 2\alpha_3 = 4 \Rightarrow \alpha_3 = 2$.

Hence $x = -6x_1 + 3x_2 + 2x_3$, which is the reqd. linear combination.

Example 2. *Write the polynomial $v = t^2 + 4t - 3$ over R as a linear combination of polynomials :*

$e_1 = t^2 - 2t + 5, e_2 = 2t^2 - 3t$ and $e_3 = t + 3$. **(Pbi. U. 1995, 1997)**

Solution. Let $v = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$; α's $\in$ R

i.e. $t^2 + 4t - 3 = \alpha_1(t^2 - 2t + 5) + \alpha_2(2t^2 - 3t) + \alpha_3(t + 3)$

$= (\alpha_1 + 2\alpha_2)t^2 + (-2\alpha_1 - 3\alpha_2 + \alpha_3)t + (5\alpha_1 + 3\alpha_3)$

Comparing coeffs. of t^2, $1 = \alpha_1 + 2\alpha_2 \Rightarrow \alpha_1 + 2\alpha_2 = 1$ ...(1)

Comparing coeffs. of t, $4 = -2\alpha_1 - 3\alpha_2 + \alpha_3$

$\Rightarrow 2\alpha_1 + 3\alpha_2 - \alpha_3 = -4$ ...(2)

Comparing constant terms, $-3 = 5\alpha_1 + 3\alpha_3 \Rightarrow 5\alpha_1 + 3\alpha_3 = -3$ ...(3)

Solving (1), (2) and (3) :

3(2) – (3) gives : $15 = -11\alpha_1 - 9\alpha_2 \Rightarrow 11\alpha_1 + 9\alpha_2 = -15$ ...(4)

Multiplying (1) by 11, $11\alpha_1 + 22\alpha_2 = 11$ ...(5)

Subtracting (4) from (5), $13\alpha_2 = 26 \Rightarrow \alpha_2 = 2$.

Putting in (1), $\alpha_1 + 2(2) = 1 \Rightarrow \alpha_1 = 1 - 4 = -3$.

Putting in (3), $5(-3) + 3\alpha_3 = -3 \Rightarrow 3\alpha_3 = 15 - 3$

$\Rightarrow 3\alpha_3 = 12 \Rightarrow \alpha_3 = 4$.

Hence $v = -3e_1 + 2e_2 + 4e_3$.

Example 3. *Write the vector* $x = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$, *in vector space of* 2×2 *matrices, as a linear combination of*

$x_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, x_2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, x_3 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$.

Solution. Let $x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$, ; α's $\in$ F

i.e. $\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} \alpha_1 & \alpha_1 \\ 0 & -\alpha_1 \end{bmatrix} + \begin{bmatrix} \alpha_2 & \alpha_2 \\ -\alpha_2 & 0 \end{bmatrix} + \begin{bmatrix} \alpha_3 & -\alpha_3 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} \alpha_1 + \alpha_2 + \alpha_3 & \alpha_1 + \alpha_2 - \alpha_3 \\ -\alpha_2 & -\alpha_1 \end{bmatrix}$

Comparing, we have :-

$3 = \alpha_1 + \alpha_2 + \alpha_3 \Rightarrow \alpha_1 + \alpha_2 + \alpha_2 = 3$ ...(1)

$-1 = \alpha_1 + \alpha_2 - \alpha_3 \Rightarrow \alpha_1 + \alpha_2 - \alpha_3 = -1$ ...(2)

$1 = -\alpha_2 \Rightarrow \alpha_2 = -1$ ...(3)

$-2 = -\alpha_1 \Rightarrow \alpha_1 = 2$ ...(4)

From (4) and (3), $\alpha_1 = 2, \alpha_2 = -1$.

Putting in (1), $2 - 1 + \alpha_3 = 3 \Rightarrow \alpha_3 = 2$.

These satisfy (2) because $2 - 1 - 2 = -1$ *i.e.* $-1 - 1$, which is true.

Thus the system (A) of equations is a consistent solution.

$\therefore x = 2x_1 - x_2 + 2x_3.$

Hence x is a linear combination of x_1, x_2 and x_3.

EXERCISE 2 (c)

1. (*a*) Write the vector $x = (1, 7, -4)$ as linear combination of the vectors $x_1 = (1, -3, 2)$ and $x_2 = (2, -1, 1)$ in the vector space V_3 (R).

(*b*) Write the vector $x = (2, -5, 4)$ as linear combination of the vectors $x_1 = (1, -3, 2)$ and $x_2 = (2, -1, 1)$ in the vector space V_3 (R).

(*c*) Write the vector $x = (1, -2, 5)$ as linear combination of the vectors $x = (1, 1, 1), x_2 = (2, -1, 1), x_3 = (2, -1, 1)$ in the vector space V_3 (R).

2. Is the vector $(2, -5, 3)$ in V_3 (R) a linear combination of the vectors

$x_1 = (1, -3, 2), x_2 = (2, -4, -1), x_3 = (1, -5, 7)$?

3. For what value of k will the vector $x = (1, k, 5)$ in V_3 (R) be a linear combination of vectors

$x_1 = (1, -3, 2)$ and $x_2 = (2, -1, 1)$?

4. Show that the vectors

$x_1 = (1, 2, 3), x_2 = (0, 1, 2), x_3 = (0, 0, 1)$

generate V_3 (R).

5. Write the vectors $x = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$ in vector space of 2×2 matrices, as linear combination of

$$x_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, x_2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, x_3 = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}.$$

ANSWERS

1. (*a*) $x = -3x_1 + 2x_2$ (*b*) Not possible

(*c*) $x = -6x_1 + 2x_2 + 3x_3$

2. No. 3. $k = -8$.

5. $x = \quad - 2x_2 - x_3.$

2.5. LINEAR SPAN

Def. *If S is any non-empty subset of a vector space V(F), then the set of all linear combinations of finite sets of elements of S (S itself may not be finite) is said to be the* **linear span** *of S and is denoted by L(S).*

In Symbols :

$$L(S) = \left\{ \sum_{i=1}^{n} \alpha_i x_i : x_i \in S,\ \alpha_i \in F,\ 1 \le i \le n \right\}.$$

2.5.1. THEOREMS

Theorem I. *The linear span L(S) of any subset S of a vector space V(F), is a sub-space of V(F).* **(P.U. 1992)**

Proof. Let $x, y \in L(S)$.

Then by def.,

$x = \Sigma \alpha_i x_i$, where $\alpha_i \in F$, $x_i \in S$ for $i = 1, 2, \ldots\ldots, n$

and $y = \Sigma \beta_j y_j$, where $\beta_j \in F, y_j \in S$ for $j = 1, 2, \ldots, m$.

To prove : *L(S) is a sub-space.*

For this, we are to prove that

For $\alpha, \beta \in F$ and $x, y \in L(S) \Rightarrow \alpha x + \beta y \in L(S)$.

$$\text{Now } \alpha x + \beta y = \alpha \left(\sum_{i=1}^{n} \alpha_i x_i \right) + \beta \left(\sum_{j=1}^{m} \beta_j y_j \right)$$

$$= \sum_{i=1}^{n} \alpha(\alpha_i x_i) + \sum_{j=1}^{m} \beta(\beta_j y_j)$$

$$= \sum_{i=1}^{n} (\alpha\alpha_i) x_i + \sum_{j=1}^{m} (\beta\beta_j) y_j \qquad [\textit{Associativity}]$$

$$= (\alpha\alpha_1) x_1 + (\alpha\alpha_2) x_2 + \ldots\ldots + (\alpha\alpha_n) x_n + (\beta\beta_1) y_1 + (\beta\beta_2) y_2 + \ldots\ldots + (\beta\beta_m) y_m$$

$[\because \alpha, \alpha_i \in F \Rightarrow \alpha\alpha_i \in F; \beta, \beta_j \in F$

$\Rightarrow \beta\beta_j \in F$ *for* $i = 1, 2, \ldots\ldots, n$ *and* $j = 1, 2, \ldots\ldots, m]$

Hence $\alpha x + \beta y$ has been expressed as a linear combination of finite number of vectors :

$x_1, x_2, \ldots\ldots, x_n ; y_1, y_2, \ldots\ldots, y_m$

of S and consequently it belongs to L(S).

Thus $\alpha x + \beta y \in L(S)$, where $\alpha, \beta \in F$ and $x, y \in L(S)$.

Hence L(S) is a sub-space of V(F).

Remark. If W is any other sub-space of V(F) containing S, then $L(W) \subset W$.

L(S) is the smallest sub-space of V(F) containing S and is called sub-space **spanned** or **generated** by S.

Question. If S is a subset of V(F), then L(S) is a sub-space of V(F) and also prove that L(S) is the smallest sub-space.

(G.N.D.U. 1991 S, 90)

Theorem II. *If S and T are any two subsets of a vector space V(F), then*

(i) $S \subset L(T) \Rightarrow L(S) \subset L(T)$

(ii) $S \subset T \Rightarrow L(S) \subset L(T)$

(iii) *S is a sub-space of V(F)* $\Leftrightarrow L(S) = S$. **(P.U. 1992)**

Proof. (i) *Given* : $S \subset L(T)$.

Let $x \in L(S) \Rightarrow \exists\ x_1, x_2, \ldots, x_n \in S\ ; \alpha_1, \alpha_2, \ldots, \alpha_n \in F$

s.t. $x = \Sigma\ \alpha_i x_i$ for $i = 1, 2 \ldots, n$

$\Rightarrow x_i \in L(T)$ for $i = 1, 2, \ldots, n$ $[\because S \subset L(T)]$

$\Rightarrow \Sigma\ \alpha_i x_i \in L(T)$ for $i = 1, 2, \ldots, n$

[$\because$ *L(T) is a sub-space of V(F)*]

$\Rightarrow x \in L(T)$.

Thus $L(S) \subset L(T)$.

Hence $S \subset L(T) \Rightarrow L(S) \subset L(T)$.

(ii) Given : $S \subset T$.

Let $x \in L(S) \Rightarrow \exists\ x_1, x_2, \ldots, x_n \in S\ ; \alpha_1, \alpha_2, \ldots, \alpha_n \in F$

s.t. $x = \Sigma\ \alpha_i x_i$ for $i = 1, 2, \ldots, n$

$\Rightarrow x = \Sigma\ \alpha_i x_i \in L(T)$ [$\because S \subset T$ *so* $x_1, x_2, \ldots, x_n \in T$]

Thus $L(S) \subset L(T)$.

Hence $S \subset T \Rightarrow L(S) \subset L(T)$.

(iii) Given : S is a sub-space of V(F).

Let $x \in L(S) \Rightarrow \exists\ x_1, x_2, \ldots, x_n \in S\ ; \alpha_1, \alpha_2, \ldots, \alpha_n \in F$

s.t. $x = \Sigma\ \alpha_i x_i$ for $i = 1, 2, \ldots, n$

$\Rightarrow x = \Sigma\ \alpha_i x_i \in S$

[$\because$ *S is a sub-space of V(F), hence it is closed for addition and scalar multiplication*]

$\therefore L(S) \subset S$ ...(1)

Also $S \subset L(S)$ ...(2)

From (1) and (2), $L(S) = S$

Conversely: *Given:* $L(S) = S$.

Since $L(S)$ is a sub-space of (F),

$\therefore$ S is also a sub-space of V(F).

Theorem III. *If S and T are any two subsets of V(F), then*

(i) $L(S \cup T) = L(S) + L(T)$ **(Pbi. U. 1995, 1997)**

(ii) $L(L(S)) = L(S)$.

Proof. (i) Let $x \in L(S \cup T)$ $\Rightarrow \exists x_1, x_2, \ldots\ldots x_n \in S \cup T$

and $\alpha_1, \alpha_2, \ldots\ldots \alpha_n \in F$

s.t. $x = \Sigma \alpha_i x_i$ for $i = 1, 2, \ldots, n$

$\Rightarrow$ $x = \Sigma \alpha_j x_j + \Sigma \alpha_k x_k$, where x_j's $\in$ S and remaining x_k's $\in$ T

[$\because$ *Each x_i is either an element of S or an element of T or an element of both, so dividing the elements x_i into elements belonging to S and belonging to T*]

$\Rightarrow$ $x \in L(S) + L(T)$.

Thus $L(S \cup T) \subset L(S) + L(T)$, ...(1)

Let $z \in L(S) + L(T)$

$\Rightarrow$ $z = x + y$, where $x \in L(S)$, $y \in L(T)$

$\Rightarrow$ $z = \Sigma \alpha_j x_j + \Sigma \alpha_k x_k$, where x_i's $\in$ S, x_k's $\in$ T;

α_j's, α_k's $\in$ F

$\Rightarrow$ $z = \Sigma \alpha_i x_i$, where x_i's, $\in S \cup T$ [$\because \{x_i\} = \{x_j\} \cup \{x_k\}$]

$\Rightarrow$ $z \in L(S \cup T)$

$\Rightarrow$ $L(S) + L(T) \subset L(S \cup T)$ ...(2)

From (1) and (2), $L(S \cup T) = L(S) + L(T)$.

(ii) Since L(S) is a sub-space of V(F),

$\because$ $L(L(S)) = L(S)$. [*Th. II, (iii)*]

2.6. LINEAR-DEPENDENCE AND LINEAR-INDEPENDENCE

(a) **Linear Dependent (L.D.) Def.** *If V is a vector space over field F, then the vectors $x_1, x_2, \ldots . x_n \in V$ are said to be linearly dependent over F if there exist elements $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n \in F$,* **not all zero,** *such that*

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots + \alpha_n x_n = 0.$$

(b) **Linear Independent (L.I.) Def.** *If V is a vector space over field F, then the vectors $x_1, x_2, \ldots\ldots, x_n \in V$ are said to be linearly independent if there exist elements $\alpha_1, \alpha_2, \ldots\ldots\ldots, \alpha_n \in F$,* **all zero,** *such that*

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots . + \alpha_n x_n = 0.$$

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ILLUSTRATIONS :

(I) *Consider the vectors :*

$x_1 = (1, 2, 3)$, $x_2 = (1, 0, 0)$, $x_3 = (0, 1, 0)$ and $x_4 = (0, 0, 1)$.

Since $1.x_1 + (-1)x_2 + (-2)x_3 + (-3)x_4$

$= (1, 2, 3) + (-1)(1,0,0) + (-2)(0, 1, 0) + (-3)(0, 0, 1)$

$= (1,2, 3) + (-1, 0, 0) + (0, -2, 0) + (0, 0, -3)$

$= (1-1+0+0,\ 2+0-2+0, 3+0+0-3) = (0, 0, 0) = 0$

$\Rightarrow \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = 0,$

where $\alpha_1 = 1 \neq 0, \alpha_2 = -1 \neq 0,\ \alpha_3 = -2 \neq 0, \alpha_4 = -3 \neq 0.$

Hence the given vectors are L.D.

(II) *Consider the vectors :*

$x_1 = (1, 0, 0), x_2 = (0, 1, 0)$ and $x_3 = (0, 0, 1)$.

If $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$, s.t.

$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$

$\Rightarrow \alpha_1 (1, 0, 0) + \alpha_2 (0, 1, 0) + \alpha_3 (0, 0, 1) = (0, 0, 0)$

$\Rightarrow (\alpha_1, 0, 0) + (0, \alpha_2, 0) + (0, 0, \alpha_3) = (0, 0, 0)$

$\Rightarrow (\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$

$\Rightarrow \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0.$

Hence the given vectors are L.I.

2.6.1. CRITERIA FOR LINEAR DEPENDENCE/INDEPENDENCE

(i) *An infinite sub-space S of a vector space is said to be L.I. if each subset of S is L.I.*

This is deduced from the definition of L.I. of a finite set.

(ii) *Any superset of L.D. set is L.D.*

Let $S = \{x_1, x_2, \ldots, x_m\}$ be a L.D. set

s.t. $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_m x_m = 0$ ...(1),

where α_i's are not all zero.

Consider $T = \{x_1, x_2, \ldots, x_m, x\}$, which is a superset of S.

Then $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_m x_m + 0x = 0,$ [*Using (1)*]

where all α_i's are not zero.

Hence T is L.D.

(iii) *Any subset of a L.I. set is L.I.*

Let $S = \{x_1, x_2, \ldots, x_m\}$ be a L.I. set

s.t. $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_m x_m = 0$ ...(1),

where $\alpha_1 = 0, \alpha_2 = 0 \ldots, \alpha_m = 0.$

Consider $T = \{x_1, x_2, \ldots, x_n\}$, where $1 \leq n \leq m$.

This is a subset of S.

From (1), $\alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots + \alpha_n x_n = 0$

$\Rightarrow \alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots + \alpha_n x_n + 0x_{n+1} + \ldots\ldots + 0x_m = 0$

$\Rightarrow \alpha_1 = 0, \alpha_2 = 0, \ldots\ldots, \alpha_n = 0$ [$\because$ *S is L.I.*]

$\Rightarrow$ T is L.I.

(iv) Any subset of a vector space is either L.D. or L.I.

(v) A set containing only **0** *vector i.e.* $\{\mathbf{0}\}$ *is L.D.*

Let $S = \{\mathbf{0}\}$.

Since $\alpha.\mathbf{0} = \mathbf{0}$ $\forall$ α, α being a scalar.

Hence S is L.D.

(vi) A set containing the single non-zero vector is L.D.

Let $S = \{x\}$, where $x \neq \mathbf{0}$.

Since $0x = \mathbf{0}$ $\Rightarrow$ $\alpha = 0$ [$\because x \neq 0$]

$\therefore$ S is L.I.

(vii) A set having one of the vectors as zero vector is L.D.

Let $S = \{x_1, x_2, \ldots, x_n\}$ have one of the vectors ; say $x_i = \mathbf{0}$.

Then $\{0x_1 + 0x_2 + \ldots\ldots\ldots + 1x_i + \ldots\ldots + 0x_n = \mathbf{0}$,

which shows that the coeff. of $x_i = 1$, which $\neq 0$.

Hence S is L.D.

2.6.2. THEOREMS

Theorem I. *Let V be a vector space. Then*

(a) the set $\{v\}$ *is L.D. iff.* $v = \mathbf{0}$,

(b) the set $\{v_1, v_2\}$ *is L.D.*

iff v_1 and v_2 *are collinear (i.e. one of them is a scalar multiple of the other)*

(c) the set $\{v_1, v_2, v_3\}$ *is L.D.*

iff v_1, v_2 and v_3 *are coplanar (i.e. one of them is a linear combination of the other two).* **(G.N.D.U. 1985)**

Proof. (*a*) Let $v = \mathbf{0}$.

$\therefore$ $\alpha v = \alpha\,\mathbf{0} = \mathbf{0}$ $\forall$ $\alpha \neq 0$, α being a scalar

Hence the set $\{v\}$ is L.D.

Conversely : Let $\{v\}$ be L.D.

$\therefore$ $\exists$ a scalar α $(\neq 0)$ s.t. $\alpha v = \mathbf{0}$

$\Rightarrow$ $v = \mathbf{0}$.

(*b*) Let $\{v_1, v_2\}$ be L.D.

Then there exist scalars α_1, α_2 (with at least one of them ; say $\alpha_1 \neq 0$) s.t.

$$\alpha_1 v_1 + \alpha_2 v_2 = \mathbf{0}$$

$\Rightarrow v_1 = -\left(\frac{\alpha_2}{\alpha_1}\right) v_2$ [$\because \alpha_1 \neq 0$]

$\Rightarrow v_1$ is a scalar multiple of v_2

$\Rightarrow v_1$ and v_2 are collinear.

Conversely. Let v_1 and v_2 be collinear.

$\therefore v_1$ is a scalar multitple of v_2

i.e. $v_1 = \alpha v_2$, where α is a scalar

$\Rightarrow 1.v_1 - \alpha v_2 = 0$ [$\because -1, \alpha$ *are not both zero*]

$\Rightarrow \{ v_1, v_2 \}$ is L.D.

(c) Let $\{ v_1, v_2, v_3 \}$ be L.D.

Then there exist scalars α_1, α_2 and α_3 (with at least one of them ; say $\alpha_1 \neq 0$) s.t.

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

$\Rightarrow v_1 = \left(\frac{-\alpha_2}{\alpha_1}\right) v_2 + \left(\frac{-\alpha_3}{\alpha_1}\right) v_3$ [$\because \alpha_1 \neq 0$]

$\Rightarrow v_1$ is a linear combination of v_2 and v_3

$\Rightarrow v_1 \in [v_2, v_3]$

$\Rightarrow v_1, v_2$ and v_3 are coplanar.

Conversely. Let v_1, v_2 and v_3 be coplanar.

$\therefore v_1 \in [v_2, v_3]$

$\therefore$ there exist some scalars α_2, α_3 s.t.

$$v_1 = \alpha_2 v_2 + \alpha_3 v_3$$

$\Rightarrow 1.v_1 - \alpha_2 v_2 - \alpha_3 v_3 = 0$

$\Rightarrow \{ v_1, v_2, v_3 \}$ is L.D. [$\because 1, -\alpha_2, -\alpha_3$ *are not zero*]

Theorem II. *If V(F) is a vector space, then the set S of non-zero vectors* $x_1, x_2 \ldots\ldots, x_n \in V$ *(i.e.* $S = \{x_1, x_2, \ldots\ldots, x_n\} \subset V$ *) is L.D. iff some elements of S is a linear combination of others.*

Proof. *Given :* $S = \{ x_1, x_2, \ldots., x_n \}$ is L.D. Then there exist $\alpha_1, \alpha_2, \ldots.. \alpha_n \in F$, (not all zero), s.t.

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots.. + \alpha_n x_n = 0$$

$$\Rightarrow \sum_i \alpha_i x_i = 0$$

[$\because$ *set S is L.D. so at least one* α_i*'s is not equal to zero, let* $\alpha_k \neq 0$]

$$\Rightarrow \sum_{i \neq k} \alpha_i x_i + \alpha_k x_k = 0$$

$$\Rightarrow \alpha_k x_k = -\sum_{i \neq k} \alpha_i x_i$$ [*By def. of vector space*]

$$\Rightarrow \quad x_k = \sum_{i \neq k} -\frac{\alpha_i}{\alpha_k} x_i,$$

[$\because$ $\alpha_k \neq 0 \Rightarrow \alpha_k^{-1} \in F$, *thus by left inverse of* α_k]

which shows that some $x_k \in S$ is expressed as a linear combination of others.

Conversely : Let some $x_j \in S$ be expressible as a linear combination of vectors $x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots x_n \in S$, *i.e.*

$$x_j = \sum_{i \neq j} \beta_i x_i \text{ for } \beta_i\text{'s} \in F$$

$$= \sum_{i \neq j} \beta_i x_i + (-1) x_j = 0$$ [*By def. of vector space*]

Hence at least one coefficient $-1 \neq 0$ showing that S is L.D.

Theorem III. *If V(F) is a vector space, then the set S of non-zero vectors* $x_1, x_2, \ldots, x_n \in V$ *(i.e.* $S = \{x_1, x_2, \ldots, x_n\} \subset V$*) is linearly dependent if and only if some vector* $x_k \in S$, $2 \leq k \leq n$, *can be expressed as a linear combination of its preceding vectors.* **(P. U. 1995)**

Proof. *Given* $S = \{x_1, x_2, \ldots, x_n\}$ is L.D. Then there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$, not all zero, such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0$$

or $\quad \Sigma \alpha_i x_i = 0$ for $i = 1, 2, \ldots, n$.

Let k be the largest suffix of α (*i.e.* the largest value of i) for which $\alpha_k \neq 0$. Then

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_k x_k + 0x_{k+1} + \ldots 0x_n = 0$$

$$\Rightarrow \quad \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_k x_k = 0 \quad \ldots(1)$$

Suppose $k=1$, then $\alpha_1 x_1 = 0$, but $\alpha_1 \neq 0$, so $x_1 = 0$, which is a contradiction because each x_i is a non-zero vector.

Hence $k > 1$, *i.e.* $2 \leq k \leq n$, $\therefore$ from (1), we have

$$\alpha_k x_k = -\alpha_1 x_1 - \alpha_2 x_2 \ldots - \alpha_{k-1} x_{k-1}.$$

As $\alpha_k \neq 0$, $\alpha_k^{-1} \in F$. Thus by left inverse of α_k, we have

$$x_k = -\alpha_k^{-1} \alpha_1 x_1 - \alpha_k^{-1} \alpha_2 x_2 \ldots - \alpha_k^{-1} \alpha_{k-1} x_{k-1}$$

$$= \beta_1 x_1 + \beta_2 x_2 \ldots + \beta_{k-1} x_{k-1},$$

where $\beta_1 = -\alpha_k^{-1} \alpha_1, \beta_2 = -\alpha_k^{-1} \alpha_2, \ldots, \beta_{k-1} = -\alpha_k^{-1} \alpha_{k-1} \in F.$

Thus x_k is expressed as a linear combination of its preceding vectors.

Conversely : Let some $x_i \in S$ be expressible as a linear combination of its preceding vectors *i.e.*

$$x_i = \beta_1 x_1 + \beta_2 x_2 + \text{.......} + \beta_{i-1} x_{i-1} \text{ for } \beta_i\text{'s} \in F$$

$$\Rightarrow \quad \beta_1 x_1 + \beta_2 x_2 + \text{.......} + \beta_{i-1} x_{i-1} + (-1) x_i = 0$$

[*By def. of vector space*]

$$\Rightarrow \beta_1 x_1 + \beta_2 x_2 + \text{........} + \beta_{i-1} x_{i-1} + (-1) x_i + 0 x_{i+1} + \text{.......} 0 x_n = 0.$$

Thus atleast one coefficient $-1 \neq 0$ showing that S is L.D.

Note. If x is a linear combination of the set of vectors

$x_1, x_2, \text{.......} x_n,$

then the set of vectors $\{x, x_1, x_2, \text{.....} x_n\}$ is L.D.

ILLUSTRATIVE EXAMPLES

Example 1. *Examine whether (1, –3, 5) belongs to the linear space generated by S, where*

S = {(1, 2, 1), (1, 1, –1), (4, 5, –2)}, *or not.* **(G.N.D.U. 1998)**

Solution. The vector $(1, -3, 5) \in L(S)$

iff it is a linear combination of the elements of S

iff $\exists$ scalars α, β, γ s.t.

$$(1, -3, 5) = \alpha(1, 2, 1) + \beta(1, 1, -1) + \gamma(4, 5, -2)$$

iff $(1, -3, 5) = (\alpha, 2\alpha, \alpha) + (\beta, \beta, -\beta) + (4\gamma, 5\gamma, -2\gamma)$

iff $(1, -3, 5) = (\alpha + \beta + 4\gamma, 2\alpha + \beta + 5\gamma, \alpha - \beta - 2\gamma)$

iff
$$1 = \alpha + \beta + 4\gamma \Rightarrow \alpha + \beta + 4\gamma = 1 \quad \text{...(1)}$$
$$-3 = 2\alpha + \beta + 5\gamma \Rightarrow 2\alpha + \beta + 5\gamma = -3 \quad \text{...(2)}$$
$$5 = \alpha - \beta - 2\gamma \Rightarrow \alpha - \beta - 2\gamma = 5 \quad \text{...(3)}$$

are consistent

(2) – (1) gives : $\alpha + \gamma = -4$ ...(4)

Adding (1) and (3), $2\alpha + 2\gamma = 6 \Rightarrow \alpha + \gamma = 3$ ...(5)

(4) and (5) $\Rightarrow$ $-4 = 3$, which is false.

Hence (1, –3, 5) does not belong to L(S).

Example 2. *Let S* = {(1, 2, – 1), (2, – 3, 2)} *and T* = {(4, 1, 3), (– 3, 1, 2)}. *Show that* $L(S) \neq L(T)$.

(P.U. 1992)

Solution. We have :

$$S=\{v_1, v_2\}, \text{ where } v_1=(1,2,-1) \text{ and } v_2=(2,-3,2)$$

and $T=\{v_3, v_4\}$, where $v_3=(4,1,3)$ and $v_4=(-3,1,2)$.

To Prove : $L(S) \neq L(T)$.

If possible, let $= L(S) = L(T)$.

Consider any $v \neq 0 \in L(S)$, then

$$v = \alpha_1 v_1 + \alpha_2 v_2 \text{ for } \alpha_1, \alpha_2 \in R \quad \text{(not both zero)}$$

Since $L(S)=L(T)$, $\therefore v \in L(T)$.

$\therefore v = \alpha_3 v_3 + \alpha_4 v_4$ for $\alpha_3, \alpha_4 \in R$ (not both zero)

Thus $\alpha_1 v_1 + \alpha_2 v_2 = \alpha_3 v_3 + \alpha_4 v_4$

$\Rightarrow \alpha_1(1,2,-1) + \alpha_2(2,-3,2) = \alpha_3(4,1,3) + \alpha_4(-3,1,2)$

$\Rightarrow (\alpha_1, 2\alpha_1, -\alpha_1) + (2\alpha_2, -3\alpha_2, 2\alpha_2) = (4\alpha_3, \alpha_3, 3\alpha_3) + (-3\alpha_4, \alpha_4, 2\alpha_4)$

$\Rightarrow (\alpha_1 + 2\alpha_2, 2\alpha_1 - 3\alpha_2, -\alpha_1 + 2\alpha_2) = (4\alpha_3 - 3\alpha_4, \alpha_3 + \alpha_4, 3\alpha_3 + 2\alpha_4)$

$$\Rightarrow \alpha_1 + 2\alpha_2 = 4\alpha_3 - 3\alpha_4 \qquad ...(1)$$

$$2\alpha_1 - 3\alpha_2 = \alpha_3 + \alpha_4 \qquad ...(2)$$

$$\text{and } -\alpha_1 + 2\alpha_2 = 3\alpha_3 + 2\alpha_4 \qquad ...(3)$$

Solving (1) and (3):

Adding (1) and (3), $4\alpha_2 = 7\alpha_3 - \alpha_4 \Rightarrow \alpha_2 = \frac{1}{4}(7\alpha_3 - \alpha_4)$

(1) – (3) gives : $2\alpha_1 = \alpha_3 - 5\alpha_4$

$\Rightarrow \alpha_1 = \frac{1}{2}(\alpha_3 - 5\alpha_4)$.

Putting these in (2), we get :-

$$2\alpha_1 - 3\alpha_2 = \alpha_3 - 5\alpha_4 - \frac{3}{4}(7\alpha_3 - \alpha_4)$$

$$= -\frac{17}{4}(\alpha_3 + \alpha_4) \neq \alpha_3 + \alpha_4.$$

Thus (2) is not verified.

Hence $L(S) \neq L(T)$.

Example 3. *Prove that the system of vectors*

$$x=(1,2,3), y=(4,1,5), z=(-4,6,2)$$

of $V_3(R)$ is L.D.

Solution. Let $\alpha_1 x + \alpha_2 y + \alpha_3 z = 0$, where $\alpha_1, \alpha_2, \alpha_3 \in R$

$$\therefore \quad \alpha_1(1,2,3) + \alpha_2(4,1,5) + \alpha_3(-4,6,2) = (0,0,0)$$

$\Rightarrow (\alpha_1, 2\alpha_1, 3\alpha_1) + (4\alpha_2, \alpha_2, 5\alpha_2) + (-4\alpha_3, 6\alpha_3, 2\alpha_3) = (0,0,0)$

$\Rightarrow (\alpha_1 + 4\alpha_2 - 4\alpha_3, 2\alpha_1 + \alpha_2 + 6\alpha_3, 3\alpha_1 + 5\alpha_2 + 2\alpha_3) = (0, 0, 0)$;

$\Rightarrow \quad \alpha_1 + 4\alpha_2 - 4\alpha_3 = 0 \quad \ldots(1)$

$2\alpha_1 + \alpha_2 + 6\alpha_3 = 0 \quad \ldots(2)$

and $3\alpha_1 + 5\alpha_2 + 2\alpha_3 = 0 \quad \ldots(3)$

Solving (1), (2) and (3) :

3(1) + 2(2) gives : $7\alpha_1 + 14\alpha_2 = 0 \Rightarrow \alpha_1 = -2\alpha_2$

Putting in (1), $-2\alpha_2 + 4\alpha_2 - 4\alpha_3 = 0$

$$\Rightarrow 2\alpha_2 = 4\alpha_3 \Rightarrow \alpha_3 = \frac{1}{2}\alpha_2$$

$\therefore$ (1) and (2) $\Rightarrow \alpha_1 = -2\alpha_2, \alpha_3 = \frac{1}{2}\alpha_2.$

These satisfy (3). $[\because -6\alpha_2 + 5\alpha_2 + \alpha_2 = 0]$

Hence the given system is L.D.

ALITER. Do upto equation (3) as above.

The equations (1), (2) and (3) can be written in the form AX = O

i.e.
$$\begin{bmatrix} 1 & 4 & -4 \\ 2 & 1 & 6 \\ 3 & 5 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

where
$$A = \begin{bmatrix} 1 & 4 & -4 \\ 2 & 1 & 6 \\ 3 & 5 & 2 \end{bmatrix}, \ X = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}.$$

Now
$$\det A = \det \begin{bmatrix} 1 & 4 & -4 \\ 2 & 1 & 6 \\ 3 & 5 & 2 \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 4 & -4 \\ 0 & -7 & 14 \\ 0 & -7 & 14 \end{bmatrix}$$

[Operating $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$]

$= 0$ *[$\because$ R_2 and R_3 are identical]*

$\Rightarrow \alpha_1, \alpha_2, \alpha_3$ are not all zero*

Hence the given system is L.D.

Example 4. *Prove that the system of vectors $x=(1, 2, -3), y=(1, -3, 2), z=(2, -1, 5)$ of $V_3(R)$ is L.I.*

(P.U. 1985 S)

Solution. Let $\alpha_1 x + \alpha_2 y + \alpha_3 z = 0$, where $\alpha_1, \alpha_2, \alpha_3 \in R$

$\therefore\ \alpha_1 (1, 2, -3) + \alpha_2 (1, -3, 2) + \alpha_3 (2, -1, 5) = (0, 0, 0)$

$\Rightarrow (\alpha_1, 2\alpha_1, -3\alpha_1) + (\alpha_2, -3\alpha_2, 2\alpha_2)$

$+ (2\alpha_3, -\alpha_3, 5\alpha_3) = (0, 0, 0)$

$\Rightarrow\ (\alpha_1 + \alpha_2 + 2\alpha_3, 2\alpha_1 - 3\alpha_2 - \alpha_3, -3\alpha_1 + 2\alpha_2 + 5\alpha_3) = (0, 0, 0)$

$\Rightarrow\ \alpha_1 + \alpha_2 + 2\alpha_3 = 0$ ...(1)

$2\alpha_1 - 3\alpha_2 - \alpha_3 = 0$ ...(2)

and $-3\alpha_1 + 2\alpha_2 + 5\alpha_3 = 0$ ...(3)

Solving (1), (2) and (3) :

(1) + 2 (2) gives : $5\alpha_1 - 5\alpha_2 = 0 \Rightarrow \alpha_1 = \alpha_2$ ...(4)

Putting in (1), $\alpha_2 + \alpha_2 + 2\alpha_3 = 0 \Rightarrow \alpha_3 = -\alpha_2$ ...(5)

Putting these in (3),

$-3\alpha_2 + 2\alpha_2 - 5\alpha_2 = 0 \Rightarrow -6\alpha_2 = 0 \Rightarrow \alpha_2 = 0$

(4) $\Rightarrow \alpha_1 = 0$, and (5) $\Rightarrow \alpha_3 = 0$.

Thus $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$.

Hence the given system is L.I.

ALITER. Do upto equation (3) as above.

The equations (1), (2) and (3) can be put in the form AX = O

i.e.
$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & -3 & -1 \\ -3 & 2 & 5 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

where
$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -3 & -1 \\ -3 & 2 & 5 \end{bmatrix}, \quad X = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}.$$

Now
$$\det A = \det \begin{bmatrix} 1 & 1 & 2 \\ 2 & -3 & -1 \\ -3 & 2 & 5 \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 1 & 2 \\ 0 & -5 & -5 \\ 0 & 5 & 11 \end{bmatrix}$$

[Operating $R_2 \to R_2 - 2R_1, R_3 \to R_3 + 3R_1$]

$$= 1 . \det \begin{bmatrix} -5 & -5 \\ 5 & 11 \end{bmatrix}$$ *[Expanding by C_1]*

$$=1.(-55+25)=-30 \neq 0.$$

$\therefore\ \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0.$

Hence the given system is L.I.

Example 5. *Prove that the four vectors* $x=(1, 0, 0), y=(0, 1, 0), z=(0, 0, 1), u=(1, 1, 1)$ *in* $V_3(C)$ *form a linearly dependent set, but any three of them are linearly independent.*

Solution. **To prove :** *x, y, z, u are L.D.*

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in C$, such that

$$\alpha_1 x + \alpha_2 y + \alpha_3 z + \alpha_4 u = 0 \qquad ...(1)$$

i.e., $\alpha_1 (1, 0, 0) + \alpha_2 (0, 1, 0) + \alpha_3 (0, 0, 1) + \alpha_4 (1, 1, 1) = (0, 0, 0)$

$$\Rightarrow \qquad (\alpha_1 + \alpha_4, \alpha_2 + \alpha_4, \alpha_3 + \alpha_4) = (0, 0, 0)$$

$$\therefore \quad \alpha_1 + \alpha_4 = 0 \quad \Rightarrow \quad \alpha_1 = -\alpha_4$$

$$\alpha_2 + \alpha_4 = 0 \quad \Rightarrow \quad \alpha_2 = -\alpha_4$$

$$\alpha_3 + \alpha_4 = 0 \quad \Rightarrow \quad \alpha_3 = -\alpha_4$$

Thus if $\alpha_4 = -k$, then $\alpha_1 = k, \alpha_2 = k, \alpha_3 = k$, showing that

$$x + y + z - u = 0. \qquad [By\ (1)]$$

Hence the vectors are L.D.

To Prove : *Any three of the four vectors are L.I.*

Let $\alpha_1, \alpha_2, \alpha_3 \in C$, such that

$$\alpha_1 x + \alpha_2 y + \alpha_3 x = 0$$

i.e., $\alpha_1 (1, 0, 0) + \alpha_2 (0, 1, 0) + \alpha_3 (1, 1, 1) = (0, 0, 0)$

$$\Rightarrow \qquad (\alpha_1 + \alpha_3, \alpha_2 + \alpha_3, \alpha_3) = (0, 0, 0).$$

$$\therefore \qquad \alpha_1 + \alpha_3 = 0 \qquad ...(2)$$

$$\alpha_2 + \alpha_3 = 0 \qquad ...(3)$$

$$\alpha_3 = 0 \qquad ...(4)$$

Thus we have $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$, which show that vectors x, y, u are L.I.

Similarly we can show for other collection of three vectors.

Example 6. *If V(R) be a vector space of* 2×3 *matrices over R, then show that the matrices*

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix}; B = \begin{bmatrix} 1 & 1 & -3 \\ -2 & 0 & 5 \end{bmatrix}; C = \begin{bmatrix} 4 & -1 & 2 \\ 1 & -2 & 3 \end{bmatrix}$$

in V(R) are linearly independent.

Solution. Let $\alpha_1, \alpha_2, \alpha_3$ be the scalars in R such that

$$\alpha_1 \begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 1 & -3 \\ -2 & 0 & 5 \end{bmatrix} + \alpha_3 \begin{bmatrix} 4 & -1 & 2 \\ 1 & -2 & 3 \end{bmatrix} = O$$

$$\Rightarrow \begin{bmatrix} 2\alpha_1 & \alpha_1 & -\alpha_1 \\ 3\alpha_1 & -2\alpha_1 & 4\alpha_1 \end{bmatrix} + \begin{bmatrix} \alpha_2 & \alpha_2 & -3\alpha_2 \\ -2\alpha_2 & 0 & 5\alpha_2 \end{bmatrix} + \begin{bmatrix} 4\alpha_3 & -\alpha_3 & 2\alpha_3 \\ \alpha_3 & -2\alpha_3 & 3\alpha_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2\alpha_1 + \alpha_2 + 4\alpha_3 & \alpha_1 + \alpha_2 - \alpha_3 & -\alpha_1 - 3\alpha_2 + 2\alpha_3 \\ 3\alpha_1 - 2\alpha_2 + \alpha_3 & -2\alpha_1 - 2\alpha_3 & 4\alpha_1 + 5\alpha_2 + 3\alpha_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad ...(1)$$

Comparing,

$2\alpha_1 + \alpha_2 + 4\alpha_3 = 0, \alpha_1 + \alpha_2 - \alpha_3 = 0, -\alpha_1 - 3\alpha_2 + 2\alpha_3 = 0$

$3\alpha_1 - 2\alpha_2 + \alpha_3 = 0, -2\alpha_1 - 2\alpha_3 = 0, 4\alpha_1 + 5\alpha_2 + 3\alpha_3 = 0$

From 2nd and 5th, $2\alpha_1 + \alpha_2 = 0$

From 1st, $4\alpha_3 = 0 \Rightarrow \alpha_3 = 0$

From 5th, $\alpha_1 = 0 \Rightarrow \alpha_2 = 0$

Thus (1) is true only if

$\alpha_1 = \alpha_2 = \alpha_3 = 0.$

Hence A, B, C is V(R) are L.I.

Example 7. *Find the value of k so that the vectors*

$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ *and* $\begin{bmatrix} k \\ 0 \\ 1 \end{bmatrix}$ *are L.D.* **(P.U. 1992 ; G.N.D.U. 1986)**

Solution. Let $\alpha_1, \alpha_2, \alpha_3$ be scalars in R (not all zero) s.t.

$$\alpha_1 \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + \alpha_3 \begin{bmatrix} k \\ 0 \\ 1 \end{bmatrix} = O$$

$$\Rightarrow \begin{bmatrix} \alpha_1 \\ -\alpha_1 \\ 3\alpha_1 \end{bmatrix} + \begin{bmatrix} \alpha_2 \\ 2\alpha_2 \\ -2\alpha_2 \end{bmatrix} + \begin{bmatrix} k\alpha_3 \\ 0 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha_1 + \alpha_2 + k\alpha_3 \\ -\alpha_1 \cancel{+2\alpha_2} \\ 3\alpha_1 - 2\alpha_2 + \alpha_3 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Comparing, $\alpha_1 + \alpha_2 + k\alpha_3 = 0$ ...(1)

$-\alpha_1 + 2\alpha_2 = 0$ ...(2)

and $3\alpha_1 - 2\alpha_2 + \alpha_3 = 0$ ...(3)

From (2), $\alpha_1 = 2\alpha_2 \Rightarrow \alpha_2 = \frac{1}{2}\alpha_1.$

Putting in (3), $3\alpha_1 - \alpha_1 + \alpha_3 = 0$

$\Rightarrow \alpha_3 = -2\alpha_1.$

Putting in (1), $\alpha_1 + \frac{1}{2}\alpha_1 - 2\alpha_1 k = 0$

$\Rightarrow \frac{3}{2}\alpha_1 - 2\alpha_1 k = 0$

$\Rightarrow 2\alpha_1 \left(\frac{3}{4} - k\right) = 0$

But $\alpha_1 \neq 0$

[$\because$ *If* $\alpha_1 = 0$, *then* $\alpha_2 = 0$ *and* $\alpha_3 = 0 \Rightarrow$ *given vectors are L.I.*]

$\therefore \frac{3}{4} - k = 0 \Rightarrow k = \frac{3}{4}.$

EXERCISE 2 (d)

1. State whether the following statements are true or false :

(*i*) If A and B are subsets of a vector space, then

$A \neq B \Rightarrow L(A) \neq L(B).$

(*ii*) Union of two linearly independent subsets of a vector space is linearly independent.

(*iii*) Intersection of two sub-spaces of a vector space V is the sub-space of V.

(*iv*) A set containing a linearly independent set of vectors is itself linearly independent.

2. Fill in the blanks :

(*i*) Any set of vectors containing the zero vector as a member is linearly....

(*ii*) A system consisting of a simple non-zero vector is always linearly....

(iii) Intersection of two linearly independent subsets of a vector space will be linearly...

(iv) In the vector space V_3 (R), the vectors

(1, 0, 1), (2, 5, 0) and (−1, 0, −1)

are linearly.......

3. In the vector space V_3 (R),

$$x = (1, 2, 1), y = (3, 1, 5), z = (3, -4, 7),$$

prove that the sub-space spanned by

$$S = \{x, y\} \text{ and } T = \{x, y, z\} \text{ are same.}$$

4. Find the condition on a, b, c so that $(a, b, c) \in R^3$ belongs to the space generated by :

$u = (2, 1, 0), v = (1, -1, 2)$ and $w = (0, 3, -4)$ *(H.P. 1998)*

5. Show that any system of vectors containing zero vector is L.D.

6. Prove that two vectors comprise a L.D. set iff one is a scalar multiple of the other.

7. Prove that the set of vectors

(i) $\{(2, -3), (8, -12)\}$ of V_2 (R) is L.D.

(ii) $\{(4, 3, -2), (2, -6, 7)\}$ of V_3 (R) is L.I.

8. Prove that the following systems of vectors of V_3 (R) are L.D.

(i) $x = (1, 3, 2), y = (1, -7, -8), z = (2, 1, -1)$

(ii) $x = (1 -2, 1), y = (2, 1, -1), z = (7, -4, 1)$

(iii) $x = (1, 2, 3), y = (1, 0, 0), z = (0, 1, 0), u = (0, 0, 1)$

(iv) $x = (0, 2, -4), y = (1, -2, -1), z = (1, -4, 3)$.

9. (a) Prove that the following systems of vectors of V_3 (R) are L.I.

(i) $x = (1, 0, 0), y = (0, 1, 0), z = (0, 0, 1)$

(ii) $x = (0, 1, -2), y = (1, -1, 1), z = (1, 2, 1)$

(iii) $x = (1, 1, 1), y = (0, 4, 1), z = (3, 0, 1)$.

(b) Prove that the system of vectors

$x_1 = (1, 2, 0), x_2 = (0, 3, 1), x_3 = (-1, 0, 1)$ of V_3(Q) is L.I. when Q is the field of rational numbers.

10. If $v_1 = (2, -1, 0), v_2 = (1, 2, 1)$ and $v_3 = (0, 2, -1)$,

show that v_1, v_2, v_3 are L.I.

Express (3, 2, 1) as a linear combination of v_1, v_2, v_3. *(Pbi. U. 1986)*

11. Which of the following systems of vectors in vector space of polynomials of degree ≤ 4 are linearly independent :

(i) $x^4 - x, x^3 + 1, x^3 - 1, x$

(ii) $x^3 - x + 1, x^3 + 2x + 1, x + 1$

(iii) $1, 1 + x, (1 + x)^2, (1 + x)^3, (1 + x)^4$

(iv) $1 + x, x + x^2, x^2 + x^3, x^3 + x^4, x^4 - 1.$ (P.U 1995)

12. Under what condition on scalar $a_1 \in C$ are the vectors $(1 + a_1, 1 - a_1)$ and $(1 - a_1, 1 + a_1)$ in V_3 (C) are linearly dependent ?

13. Show that three row (column) vectors comprising the matrix

$$\begin{bmatrix} 2 & 7 & 5 \\ 3 & -6 & 2 \\ 1 & 17 & 7 \end{bmatrix} \text{ are L.I.}$$

14. Find the condition on a, b, c such that the matrix

$$\begin{bmatrix} a & -b \\ b & c \end{bmatrix} \text{ is a linear combination of}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$.(G.N.D.U. 1995)

15. Prove the set of vectors $\{A, B\}$, where

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 1 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -6 & 12 \\ 9 & 3 & -3 \end{bmatrix}$$

in the vector space of all 2×3 matrices over R is L.D.

16. Find 'a' if the vectors

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} a \\ 0 \\ 1 \end{bmatrix} \text{ are L.D.}$$

17. Show that

(i) row vectors (ii) column vectors

comprising the matrix

$$\begin{bmatrix} 2 & 7 & 5 \\ 3 & -6 & 2 \\ 1 & 17 & 7 \end{bmatrix}$$

are linearly independent.

18. If $v_1, v_2, \ldots\ldots, v_n$ are L.I. and $v_1, v_2, \ldots v_n, v$ are L.D., prove that v is a linear combination of $v_1, v_2, \ldots\ldots, v_n$.

19. The set of vectors $\{x_1, x_2, ..., x_n\}$ forms a linearly dependent set if either (*i*) at least one of the vectors is zero or (*ii*) at least one vector is a linear combination of others.

20. (*a*) Let V be a vector space of real valued derivable functions on $(0, \infty)$, show that the set

(*i*) $S = \{\sin x, \cos x, \sin (x + 1)\}$ is L.D. (*G.N.D.U. 1992*)

(*ii*) $S = \{x^2 e^x, xe^x, (x^2 + x - 1) e^x\}$ is L.I. (*G.N.D.U. 1992 S*)

21. Let v be a vector space over a field F. Let the vectors v_1, v_2, v_3 be L.I. over F, show that

$v_1 + v_2, v_1 - v_2$ and $v_1 - 2v_2 + v_3$ are also L.I. over F.

ANSWERS

1. (*i*) False (*ii*) False (*iii*) True (*iv*) False.
2. (*i*) dependent (*ii*) independent
   (*iii*) independent (*iv*) independent.
4. $2a - 4b - 3c = 0$
10. $\frac{8}{9}v_1 + \frac{11}{9}v_2 + \frac{2}{9}v_3$.
11. (*i*) L.I. (*ii*) L.I. (*iii*) L.I. (*iv*) L.D.
12. $a_1 = 0$. 14. $a + b + 2c = 0$. 16. $a = 1$.

2.7. BASIS AND DIMENSIONS

(*a*) **Basis. Def.** *Let V (F) be a vector space. A subset S of V is said to be basis of B iff*

(*i*) *S is linearly independent set*

(*ii*) $L(S) = V$

i.e. S generates (spans) V

i.e. each element of V is a linear combination of the elements of S.

The basis set of V is generally denoted by B.

Note. A set of vectors having zero vector is always L.D. and as such it cannot be the basis of a vector space.

Thus *zero vector cannot be an element of a basis set of a vector space.*

(*b*) **Finite Dimensional. Def.** *A vector space V(F) is said to be finite dimensional (or finitely generated) if there exists a finite subset S of V such that L(S) = V i.e. linear span of S is equal to V and the set S is linearly independent.*

Note. if there exists no finite linear independent subset which generates V, then V is said to be an **infinite dimensional** vector space.

2.7.1. THEOREMS

Theorem I. *In a finite dimensional vector space V(F) basis set is* $B = \{x_1, x_2, ..., x_n\}$, *iff each vector* $x \in V$ *is uniquely expressible as linear combination of the vectors in B.* **(P.U. 1996)**

Proof. Since B is a basis set of V, [*Given*]

$\therefore$ any vector $x \in V$ can be expressed as a linear combination of vectors in B

i.e. $x = \Sigma \alpha_i x_i$ for α_i's $\in F$ ...(1)

Uniqueness :

If possible, let $x = \Sigma \beta_i x_i$ for β_i's $\in F$ ...(2)

be another representation.

Subtracting (2) from (1), we get :-

$$0 = \Sigma \alpha_i x_i - \Sigma \beta_i x_i$$

$\Rightarrow \Sigma (\alpha_i - \beta_i) x_i = 0$

$\Rightarrow \alpha_i - \beta_i = 0$ for $i = 1, 2, ..., n$ [$\because$ *B, being basis set of V, is L.I.*]

$\Rightarrow \quad \alpha_i = \beta_i$ for $i = 1, 2, ..., n.$

Thus the expressions (1) and (2) are same.

Hence each vector $x \in V$ can be uniquely expressible as linear combination of the vectors in B.

Conversely : Let each $x \in V$ be uniquely expressed as

$$x = \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n ; \alpha_i \in F, 1 \leq i \leq n.$$

To Prove : $B = \{x_1, x_2, ..., x_n\}$ is the basis of V(F).

(*i*) **B is L.I.**

Consider $a_1 x_1 + a_2 x_2 + ... + a_n x_n = 0$ for a's $\in F$ ...(3)

Also $0 x_1 + 0 . x_2 + ... + 0 . x_n = 0$ ...(4)

Since representation of $0 \in V$ is unique,

$\therefore$ from (3) and (4), $a_1 = 0, a_2 = 0, ..., a_n = 0$

$\Rightarrow$ the vectors $x_1, x_2, ..., x_n$ are **L.I.**

Thus $B = \{x_1, x_2, ..., x_n\}$ is L.I.

(*ii*) **L(B) = V.**

We know that

$L(B) \subset V$ ...(5)

Let $x \in V$

$\Rightarrow \exists \alpha_i \ (1 \leq i \leq n) \in F$,

s.t. $x = \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n$

$\Rightarrow x \in L(B)$

$\Rightarrow V \subset L(B)$ ...(6)

From (5) and (6), $L(B) = V$

Hence B is the basis of V(F).