

MATRICES

Definition:- A collection of numbers arranged in rows and columns is said to be an array. A matrix is a rectangular array of numbers closed in addition, subtraction, multiplication, and division.

We represent a matrix by $A = [a_{ij}]_{m \times n}$, $i=1(1)m$, $j=1(1)n$.

$$\text{i.e., } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

Diagonal matrix:-

$A = [a_{ij}]$ is such that $a_{ij} = 0 \forall i \neq j$

$$\text{i.e., } A = \text{diag}[d_1, d_2, \dots, d_n]$$

Eg. $A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$ is a 3×3 diagonal mtx.

Scalar matrix:-

$$A = [a_{ij}] \Rightarrow a_{ij} = 0 \forall i \neq j$$

$$a_{ij} = k \forall i=j, k \in \mathbb{N}.$$

$$\text{Eg} \quad A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$$

Triangular mtx:- If every element above or below the leading diagonal of a square matrix is zero, then the matrix is called a triangular matrix.

Upper triangular mtx:-

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

Lower Triangular mtx:-

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Equality of matrices:- Two matrices A and B of the same order are said to be equal, if and only if the corresponding elements are equal.

Multiplication of matrix by a scalar:- Matrix multiplication is associative and distributive but not commutative.

$$A = [a_{ij}]_{m \times n}$$

$$KA = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix} = [ka_{ij}]_{m \times n}$$

Addition of Matrices:-

$$A = [a_{ij}]_{m \times n} \quad B = [b_{ij}]_{m \times n}$$

$$A + B = [a_{ij} + b_{ij}]$$

Matrix addition is commutative, associative and distributive.

Note:- Only a square matrix can have a determinant.

Transpose of a matrix:-

$$A = [a_{ij}]_{m \times n}$$

$$A^T = [a_{ji}]_{n \times m}$$

$$(1) (A')' = A.$$

$$(2) (A+B)' = A' + B'.$$

$$(3) (AB)' = B'A'.$$

Symmetric and Skew-symmetric matrices:-

For a symmetric mtx. A , $a_{ij} = a_{ji} \forall i, j$.

For a skew-symmetric mtx A , $a_{ij} = -a_{ji} \forall i \neq j$
 $= 0 \quad \forall i = j$.

Ex.1. If A be any matrix, S.T. AA' and $A'A$ are symmetric.

$$(AA')' = (A')'(A)' = AA'$$

$$(A'A)' = (A)'(A')' = A'A$$

Hence AA' and $A'A$ both are symmetric.

Ex.2. If A and B are both symmetric, then AB is symmetric iff A and B commute.

Sol.

$$A' = A, \quad B' = B$$

$$(AB)' = B'A'$$

$$= BA$$

$= AB$ iff A and B commute.

This shows $(AB)'$ is symmetric.

Ex.3. S.T. A^2 is symmetric, if A is either symmetric or skew-symmetric.

Sol.

A^2 exists only if A is ^{square} mtx.

Let $A = [a_{ij}]$, $i, j = 1, 2, \dots, n$. Then

$$A^2 = [c_{ij}]$$

$$c_{ij} = \sum_{k=1}^n a_{ik}a_{kj}$$

Case I:- When A is symmetric, then $a_{kj} = a_{jk}$.

$$\therefore c_{ij} = \sum_{k=1}^n a_{ik} a_{jk}$$

$\therefore c_{ji} = \sum a_{jk} a_{ik}$; on interchanging i and j .

\therefore Clearly, $c_{ij} = c_{ji}$, $\therefore A^2$ is symmetric.

Case II:- When A is skew-symmetric, then $a_{kj} = -a_{jk}$.

$$\therefore c_{ij} = \sum_{k=1}^n a_{ik} (-a_{jk}) = -\sum_{k=1}^n a_{ik} a_{jk}$$

so that $c_{ji} = \sum a_{jk} a_{ik}$, on interchanging i and j .

Clearly, $c_{ij} = c_{ji}$

Hence A^2 is again symmetric.

$\therefore A^2$ is symmetric if A is either symmetric or skew-symmetric.

Ex. 4. S.T. all positive integral powers of a symmetric matrix are symmetric.

Sol.

$$A' = A$$

$$(A^n)' = (AA \dots n \text{ times})'$$
, n be a positive integer

$$= A'A' \dots n \text{ times}$$

$$= AA \dots n \text{ times} \text{ as } A' = A$$

$$= A^n; \text{ hence } A^n \text{ is symmetric.}$$

Ex. 5. S.T. all positive odd (even) integral powers of a skew-symmetric matrix are skew-symmetric (symmetric).

Sol.

$$A' = -A$$

$$(A^n)' = (AA \dots n \text{ times})'$$

$$= A'A' \dots n \text{ times}$$

$$= (-A)(-A) \dots n \text{ times} \text{ as } A' = -A.$$

$$= (-1)^n A^n$$

$$= \begin{cases} A^n, & n = \text{even} \quad \therefore A^n \text{ is symmetric.} \\ -A^n, & n = \text{odd} \quad \therefore A^n \text{ is skew-symmetric.} \end{cases}$$

Ex. 6. If A is symmetric (skew-symmetric), show that $B'AB$ is symmetric (skew-symmetric).

Sol.

Case I:- $A^T = A$ $(B'AB)' = (B)'A'(B')'$
 $= B'A'B$

Case II:- $A^T = -A$, $(B'AB)' = -B'AB$

Ex.7. If A and B are symmetric (skew-symmetric), s.t. A+B is symmetric (skew-symmetric).

Ex.8. If A be any ^{sq.} matrix, s.t. A+A' is symmetric and A-A' is skew-symmetric.

Sol. $(A+A')' = (A') + (A')' = A' + A = (A+A')$

$$(A-A')' = (A') - (A')' = A' - A = -(A-A').$$

Ex.9. If A, B are symmetric, s.t. AB+BA is symmetric and AB-BA is skew-symmetric.

Ex.10. Show that every square matrix can be uniquely expressed as the sum of a symmetric and a skew-symmetric matrix.

Sol.

$$A = \frac{1}{2}(A+A') + \frac{1}{2}(A-A')$$

$$(A+A')' = A' + (A')' = A' + A$$

$$(A-A')' = A' - A = -(A-A')$$

$\therefore A+A'$ is symmetric and $A-A'$ is skew-symmetric.

Conjugate and Triangulated of matrix:-

A matrix obtained by replacing each element of a mtx A by its complex conjugate is called the conjugate mtx of A and is denoted by \bar{A} .

A matrix is said to be real iff $\bar{A} = A$.

$(\bar{A})'$ is called triangulated matrix of A.

Hermitian and Skew-Hermitian matrices:-

$A^* = [a_{ij}]$ is Hermitian iff $a_{ij} = \bar{a}_{ij} \forall i, j$
 $= \bar{a}_{ji} \forall i = j$

i.e. every diagonal element of a Hermitian mtx is real.

e.g. $\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \begin{bmatrix} 4 & 1-i \\ 1+i & 2 \end{bmatrix}$ are the examples of Hermitian mtx.

$A = [a_{ij}]$ is skew-Hermitian iff $a_{ij} = -\bar{a}_{ji} \forall i, j$.

i.e. Every diagonal element of a skew-Hermitian mtx is either purely imaginary or zero.

e.g. $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1-i \\ -1-i & 0 \end{bmatrix}$

Q. Show that $A = \begin{bmatrix} 3 & 1+2i \\ 1-2i & 2 \end{bmatrix}$ is Hermitian.

Sol. \bar{A} = conjugate of A = $\begin{bmatrix} 3 & 1-2i \\ 1+2i & 2 \end{bmatrix}$

and $A^* = (\bar{A})'$ transpose of $\bar{A} = \begin{bmatrix} 3 & 1+2i \\ 1-2i & 2 \end{bmatrix}$

Clearly, $A^* = A$; hence A is Hermitian.

Note:- The ~~above~~ all results of symmetric and skew-symmetric

Hermitian matrix and skew-symmetric mtx by skew-Hermitian mtx.

Note:- Every square mtx. can be uniquely represented as the sum of a hermitian and a skew-hermitian matrix.

2. Every square mtx. can be uniquely expressed as $P + iQ$, where P and Q are hermitian.

3. If A is Hermitian (skew-hermitian) mtx, then iA is a skew-hermitian (Hermitian) mtx.

Polynomials in square matrix with scalar coefficients:

The algebra of polynomials in one square mtx A with scalar coefficients is the same as the algebra of ordinary polynomials.

For example, two parallel identities are:

$$x^2 - (\alpha + \beta)x + \alpha\beta \equiv (x - \alpha)(x - \beta)$$

$$A^2 - (\alpha + \beta)A + \alpha\beta I \equiv (A - \alpha I)(A - \beta I)$$

More generally, if

$$x^n + p_1 x^{n-1} + \dots + p_{n-1} x + p_n \equiv (x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n)$$

$$\text{then } A^n + p_1 A^{n-1} + \dots + p_{n-1} A + p_n I \equiv (A - \alpha_1 I)(A - \alpha_2 I) \dots (A - \alpha_n I).$$

Ex.1. Find the scalar solutions of the mtx equation $A^2 - 5A + 7I = 0$, and show that $\begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$ is a non-scalar solution.

Sol. Consider the algebraic equation

$$x^2 - 5x + 7 = 0;$$

$$x = \frac{5 \pm i\sqrt{3}}{2}$$

Hence the scalar solution are $A = \frac{1}{2}(5 \pm i\sqrt{3})I$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\text{i.e. } A = \begin{bmatrix} \frac{1}{2}(5+i\sqrt{3}) & 0 \\ 0 & \frac{1}{2}(5+i\sqrt{3}) \end{bmatrix}$$

2nd part:-

$$\text{If } A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}, \text{ then } A^2 = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$A^2 - 5A = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5 \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= -7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= -7I$$

Hence $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ is a non-scalar solution.

Idempotent Matrix:-

Definition:- A square matrix A such that $A^2 = A$ is called idempotent. If $A^T = A$ and $A^2 = A$, then the square matrix A is called symmetric idempotent.

Theorem:- Every non-singular idempotent mtx is an identity mtx.

Proof:- A matrix A is said to be non-singular or regular if $|A| \neq 0$.

As A is idempotent so $A^2 = A$ or $AA = A$.

Now, a square mtx B of the same order as A , such that $AB = BA = I$, is called the inverse mtx of A and

is denoted by A^{-1} and exists as A is non-singular.

$$A^{-1}(AA) = A^{-1}A$$

$$\Rightarrow IA = I$$

$$\Rightarrow A = I.$$

Theorem:- If A and B are idempotent matrices, then AB is idempotent

Proof:- if A and B commute.

$$A^2 = A, B^2 = B$$

$$(AB)^2 = (AB)(AB)$$

$$= A(BA)B$$

$$= A(AB)B$$

$$= (AA)(BB)$$

$$= A^2B^2 = AB \text{ as } A \text{ and } B \text{ commute, i.e., } AB = BA.$$

Theorem:- $\therefore AB$ is idempotent.

If A is idempotent and $A+B = I$, then B is idempotent and $AB = BA = 0$.

Proof:- since $A+B = I$, $\therefore B = (I-A)$

$$B^2 = (I-A)(I-A)$$

$$= I - A + A - A^2$$

$$= I - A \quad [\because A^2 = A]$$

$$= B.$$

$\therefore B$ is an idempotent matrix.

2nd Part:-

$$A+B = I$$

$$A(A+B) = AI$$

$$A^2 + AB = A$$

$$A + AB = A$$

$$\therefore AB = 0$$

similarly $BA = 0$.

Theorem:- Show that the mtx A defined as

$$A = I_n - X(X'X)^{-1}X'$$
 is a symmetric and idempotent mtx.

$$\text{Sol.} \quad A = I_n - X(X^{-1}(X')^{-1})X'$$

$$= I_n - \{(XX^{-1})\} \{(X')^{-1}X'\}$$

$$= I_n - (I_n)(I_n)$$

$$= I_n - I_n = 0$$

Therefore $A' = A$ and $A^2 = A$.

Nilpotent Matrix:-

Definition:- If A be a nilpotent mtx such that $A^m=0$, where m is positive integer, then A is called a nilpotent matrix.

If m be the least positive integer for which $A^m=0$, then A is said to be a nilpotent matrix of index m . Thus a square mtx $A \ni A^m=0$, but $A^{m-1} \neq 0$ is a nilpotent mtx of order m , m being a positive integer.

Involutory Matrix:-

Definition:- A square mtx A is such that

$$A^2 = I \text{ or, } (I+A)(I-A) = 0$$

is called involutory. Clearly I is involutory.

Ex. Show that $A = \begin{bmatrix} ab & b \\ -a & -ab \end{bmatrix}$ is nilpotent.

Sol.

$$A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Orthogonal matrix:-

Definition:- A square mtx A is said to be orthogonal if

$$A'A = I = AA'$$

Note:- When A is orthogonal $|A|^2 = 1$, $|A| = \pm 1$. If $|A|$ is equal to 1, then A is called a proper mtx.

Theorem:- If A and B are n -square orthogonal matrices, then AB and BA are orthogonal matrices.

Proof:- Since A and B are orthogonal matrices, we have

$$AA' = I \text{ and } BB' = I$$

$$\begin{aligned} (AB)(AB') &= (AB)(B'A') \\ &= A(BB')A' \\ &= AA' \\ &= I \end{aligned}$$

$\therefore AB$ is orthogonal, similarly BA is also orthogonal.

Ex.1. S.T. the mtx $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

Sol.

$$AA' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \text{ hence } A \text{ is orthogonal.}$$

Ex.2. If A is real skew-symmetric matrix such that $A^2 + I = 0$. Show that A is orthogonal.

Sol.

$$A = -A'$$

$$AA = -AA'$$

$$A^2 = -AA'$$

$$-I = -AA'$$

$$\therefore AA' = I$$

$\therefore A$ is orthogonal.

Unitary matrix:-

Definition:- A square matrix A is called unitary if $A^*A = I = AA^*$.
 If A is real then $A^* = A'$, so that A is unitary if $A'A = AA' = I$.

Ex. Show that $A = \begin{bmatrix} \frac{1}{2}(1+i) & -\frac{1}{2}(1-i) \\ \frac{1}{2}(1+i) & \frac{1}{2}(1-i) \end{bmatrix}$ is unitary mtx.

Sol. $A = \frac{1}{2} \begin{bmatrix} 1+i & 1-i \\ 1+i & 1-i \end{bmatrix}$

$$A^* = \frac{1}{2} \begin{bmatrix} 1+i & 1-i \\ -1-i & 1+i \end{bmatrix}$$

$$AA^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore A$ is unitary.

Trace (or spur) of a square matrix:-

Definition:- The sum of the elements of the principal diagonal of a square matrix is called the trace (or, spur, a german word) of the matrix.

Thus if $A = [a_{ij}]$ be a square matrix of order n , then

$$\text{trace } A = a_{11} + a_{22} + a_{33} + \dots + a_{nn} = \sum_{i=1}^n a_{ii}$$

Remark:- For the identity mtx I_n , $\text{trace } I_n = n$.

Ex. 1. Show that for any sq. mtx A, B :-

- (i) $\text{trace } (kA) = k \text{trace } (A)$, k being a scalar.
- (ii) $\text{trace } (KA + B) = k\text{trace } (A) + \text{trace } (B)$.

Sol. (i) $A = [a_{ij}]_{n \times n}$, Then $kA = [ka_{ij}]_{n \times n}$

$$\begin{aligned} \text{trace } (kA) &= \sum_{i=1}^n ka_{ii} \\ &= k \text{trace } (A). \end{aligned}$$

(ii) $A = [a_{ij}]_{n \times n}, B = [b_{ij}]_{n \times n}$

$$KA + B = [ka_{ij} + b_{ij}]_{n \times n}$$

$$\begin{aligned} \text{trace } (KA + B) &= \sum_{i=1}^n (ka_{ii} + b_{ii}) \\ &= k \sum_{i=1}^n a_{ii} + \sum_{i=1}^n b_{ii} \\ &= k \text{trace } (A) + \text{trace } (B) \end{aligned}$$

e.g. If $\text{trace } (A) = 20$, $\text{trace } (B) = -8$, then $\text{trace } (A + B) = 12$.

Ex.2. Show that $\text{trace } A' = \text{trace } A$.

Sol. Let $A = [a_{ij}]_{n \times n}$, $A' = [a'_{ij}]_{n \times n} = [a_{ji}]_{n \times n}$

therefore when $i=j$, we have $\text{trace}(A) = \text{trace}(A')$.

Ex.3. Show that $\text{trace}(AB) = \text{trace}(BA)$, if AB and BA co-exist.

Sol.

$$A = [a_{ij}]_{m \times n} \quad B = [b_{ij}]_{n \times m}$$

$$AB = [c_{ij}]_{m \times m} \quad ; \quad c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

$$\therefore \text{trace}(AB) = \sum_{i=1}^m c_{ii}$$

$$= \sum_{i=1}^m \sum_{k=1}^n a_{ik} b_{ki}$$

$$BA = [d_{ij}]_{n \times n}$$

$$d_{ij} = \sum_{k=1}^m b_{ik} a_{kj}$$

$$\therefore \text{trace}(BA) = \sum_{i=1}^n d_{ii}$$

$$= \sum_{i=1}^n \sum_{k=1}^m b_{ik} a_{ki}$$

$$= \sum_{k=1}^n \sum_{i=1}^m b_{ki} a_{ik}, \text{ interchanging } i \text{ and } k.$$

$$= \sum_{i=1}^n \sum_{k=1}^m a_{ik} b_{ki}$$

$$\therefore \text{trace}(AB) = \text{trace}(BA).$$

Ex.4. S.T. $\text{trace}(AA') \geq 0$.

Sol. $A = [a_{ij}]_{m \times n}$, $A' = [a'_{ij}]_{n \times m}$

$$AA' = [c_{ij}]_{m \times m} \quad ; \quad c_{ij} = \sum_{k=1}^n a_{ik} a'_{kj}$$

$$\therefore \text{trace}(AA') = \sum_{i=1}^m c_{ii}$$

$$= \sum_{i=1}^m \sum_{k=1}^n a_{ik} a'_{ki}$$

$$= \sum_{i=1}^m \sum_{k=1}^n (a_{ik})^2 \text{ as } a_{ik} = a'_{ki}$$

Ex.5. S.T. $\text{trace}(C'Ac) > 0$, if C is an orthogonal mtx.

Sol. $\text{trace}(C'Ac) = \text{trace}\{C'(Ac)\}$

$$= \text{trace}\{(Ac)c'\}, \text{ as } \text{trace } AB = \text{trace } BA.$$

$$= \text{trace}\{A(cc')\}, \text{ as } c \text{ is orthogonal mtx.}$$

$$= \text{trace}(A).$$

The adjoint of a square matrix:-

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \text{ Adj } A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix}$$

Theorem:- If A is a square matrix, then

$$A(\text{adj } A) = (\text{adj } A)A = |A| \cdot I.$$

Sol. We have

The (i, j) th element of the product $A \cdot (\text{adj } A)$

= Product of the i th row of A and j th column of $\text{adj } A$

$$= a_{11}A_{j1} + a_{12}A_{j2} + \dots + a_{1n}A_{jn}$$

$$= \begin{cases} 0 & \text{if } i \neq j \\ |A| & \text{if } i=j \end{cases}$$

Thus in the product, only the diagonal elements exist and each is equal to $|A|$ while all other elements are zero, so that

$$A \cdot (\text{adj } A) = \begin{bmatrix} |A| & 0 & 0 & \dots & 0 \\ 0 & |A| & 0 & \dots & 0 \\ 0 & 0 & |A| & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & |A| \end{bmatrix}$$

$$= |A| \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

$$= |A| \cdot I.$$

similarly, $(\text{adj } A) \cdot A = |A| \cdot I.$

Corollary 1:- If $|A| \neq 0$, we have

$$|A| \cdot |\text{adj } A| = \begin{bmatrix} |A| & 0 & \dots & 0 \\ 0 & |A| & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & |A| \end{bmatrix} = |A|^n$$

$$\therefore |\text{adj } A| = |A|^{n-1}.$$

Corollary 2:- If $|A| \neq 0$, then

$$A \left(\frac{1}{|A|} \text{adj } A \right) = I = \left(\frac{1}{|A|} \text{adj } A \right) A.$$

Singular matrix:- A sq. matrix is said to be singular if its determinant is zero.

Theorem:- If A and B are n -square matrices, then

$$\text{Adj } AB = \text{Adj } A \cdot \text{Adj } B$$

Sol. We have $AB \cdot (\text{Adj } AB) = |AB| \cdot I = (\text{Adj } AB) \cdot AB$

$$\begin{aligned} \text{Now, } AB \cdot (\text{Adj } B) \cdot (\text{Adj } A) &= A \cdot (B \cdot \text{Adj } B) \cdot (\text{Adj } A) \\ &= A \cdot (|B| \cdot I) \cdot (\text{Adj } A) \\ &= |B| \cdot (A \cdot \text{Adj } A) \\ &= |B| \cdot |A| \cdot I \\ &= |AB| \cdot I \end{aligned}$$

$$\begin{aligned} \text{Also, } (\text{Adj } B) \cdot (\text{Adj } A) \cdot AB &= (\text{Adj } B) [\text{Adj}(A) \cdot A] B \\ &= (\text{Adj } B) |A| \cdot I B \\ &= |A| [(\text{Adj } B) \cdot B] \\ &= |A| \cdot |B| \cdot I \\ &= |AB| \cdot I. \end{aligned}$$

$$\therefore \text{Adj}(AB) = \text{Adj}(A) \text{Adj}(B).$$

Ex.1. Show that $\text{Adj } A' = (\text{Adj } A)'$.

Sol. Obviously, the matrices $\text{Adj } A'$ and $(\text{Adj } A)'$ both will be of the same order as A . Now,

$$\begin{aligned} \text{The } (i,j)\text{th element of } \text{Adj } A' &= \text{the co-factor of } (j,i)\text{th element of } A' \text{ in} \\ &\quad \text{the determinant } |A'|. \\ &= \text{the co-factor of } (i,j)\text{th element of } A \text{ in} \\ &\quad \text{the determinant } |A| \\ &= (j,i)\text{th element of } \text{Adj } A. \\ \therefore \text{Adj } A' &= (\text{Adj } A)'. \end{aligned}$$

Ex.2. Show that every skew-symmetric matrix of odd order is singular.

Solution:- Let A be a skew-symmetric matrix of order n , where n is odd. Since A is skew-symmetric, we have

$$A' = -A.$$

$$\begin{aligned} |A'| &= |-A| \\ &= (-1)^n |A| \\ &= -|A|, \text{ where } n \text{ is odd and } |A'| = |A| \end{aligned}$$

$$\therefore |A| = -|A|$$

$$\therefore 2|A| = 0$$

$$\Rightarrow |A| = 0.$$

Hence a skew-symmetric matrix of odd order is singular.

WORKED EXAMPLES:-

1. Let A be a non-singular sq. mtx of order 3. If B is the mtx obtained from A by adding 3-multiple of its first row to its second row, then the value of $\det(2A^{-1}B)$ is

- (A) 8 (B) 3 (C) 6 (D) 2

Sol. (D) $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$|A| \neq 0,$$

$$A \sim B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 3a_{11} + a_{21} & 3a_{12} + a_{22} & 3a_{13} + a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Since A and B are equivalent so $|A|=|B|$

$$\begin{aligned} \therefore \det(2A^{-1}B) &= 2|A^{-1}| |B| \\ &= 2 \cdot \frac{1}{|A|} \cdot |B| \\ &= 2. \end{aligned}$$

2. Let u be a unit column vector and $A = I - 2uu^T$. Then A^{-1} is

- (A) $I - 2uu^T$ (B) $I + 2uu^T$ (C) $2u^Tu - I$ (D) $4u^Tu$.

Sol.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \cdot [1 \ 1 \ 1] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = I - 2uu^T.$$

3. Let P, Q, R be matrices of order $3 \times 5, 5 \times 7, 7 \times 3$ respectively. The number of scalar additions required to compute $P(QR)$ is

- (A) 114 (B) 126 (C) 128 (D) 138.

Sol. We have to calculate $P(QR)$.

Q and R is of order 5×7 and 7×3 respectively. When we multiply it we have to make 6 scalar addition for each entry so 5 column in Q and 3 rows in R then we multiply Q and R .

We do $6 \times 5 \times 3 = 90$ addition further QR is of 5×3 matrix.

Similarly when we multiply P to QR we perform $4 \times 3 \times 3$ addition

$$\therefore \text{Total no. of additions} = 90 + 36 = 126.$$

4. If A is a square matrix and $A - \frac{1}{2}I$ and $A + \frac{1}{2}I$ are orthogonal, prove that A is skew-symmetric & $A^2 = -\frac{3}{4}I$.

Sol. Since $A - \frac{1}{2}I$ is orthogonal,

$$\text{so, } (A - \frac{1}{2}I)(A - \frac{1}{2}I)^T = I$$

$$\text{or, } (A - \frac{1}{2}I)(A' - \frac{1}{2}I) = I$$

$$\text{or, } AA' - \frac{1}{2}IA' - \frac{1}{2}AI + \frac{1}{4}I^2 = I$$

$$\text{or, } AA' - \frac{1}{2}A' - \frac{1}{2}A = \frac{3}{4}I \quad \dots \textcircled{1}$$

Similarly since $A + \frac{1}{2}I$ is orthogonal, we have

$$AA' + \frac{1}{2}A' + \frac{1}{2}A = \frac{3}{4}I. \quad \dots \textcircled{2}$$

$\therefore \textcircled{1} \& \textcircled{2}$ gives $A' + A = 0$

$$\textcircled{1} - \textcircled{2} \Rightarrow A' = -A$$

$\therefore A$ is skew-symmetric.

$$\textcircled{1} + \textcircled{2} \text{ gives } 2AA' = \frac{3}{2}I$$

$$\therefore 2A(-A) = \frac{3}{2}I$$

$$\therefore A^2 = -\frac{3}{4}I.$$

5. If $A = \begin{bmatrix} 1+i & 2-3i & 2 \\ 3-4i & 4+5i & 1 \\ 5 & 3 & 3-i \end{bmatrix}$, find the conjugate of matrix A ?

Sol. $\bar{A} = \begin{bmatrix} 1-i & 2+3i & 2 \\ 3+4i & 4-5i & 1 \\ 5 & 3 & 3+i \end{bmatrix}$

6. Let P and Q be two $n \times n$ non-zero matrices $\exists P+Q=0$, then show that, (i) P is non-singular

(ii) $P = Q^T$

(iii) $P = Q^{-1}$

(iv) $\text{Rank}(P) = \text{Rank}(Q)$.

Sol. Given that P and Q are two $n \times n$ matrices such that $P+Q=0$.

$\therefore P+Q=0 \Rightarrow P=-Q$

$$\Rightarrow |P|=|Q|$$

$$\Rightarrow |P| \neq 0 \text{ iff } |Q| \neq 0$$

i.e. P is non-singular iff Q is non-singular.

If Q is non-singular then Q^{-1} exists. $|Q^{-1}| = |Q^T| = |Q|$

$$\therefore P = Q^T = Q^{-1}$$

But if $P+Q=0$ then $\text{Rank}(P) = \text{Rank}(Q)$.

Example - $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Then $P+Q=0$ and $\text{Rank}(P) = \text{Rank}(Q)$.

7. Show that all $n \times n$ symmetric matrices over \mathbb{F} form a vector space of dimension $\frac{n(n+1)}{2}$ over \mathbb{F} . What is the dimension of the space of skew-symmetric matrices?

Sol.

Let $A = (a_{ij})$ be an $n \times n$ symmetric matrix.
 Then $a_{ij} = a_{ji} \forall i \neq j$. Thus the number of independent entries are a_{ij} ($i < j$) and a_{ij} ($1 \leq i \leq n$) and there are $\frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$ in number.

Hence the space has dimension $\frac{n(n+1)}{2}$.

For a skew-symmetric matrix, $a_{ij} = 0, 1 \leq i \leq n, \forall i = j$ and $a_{ij} = -a_{ji}$.
 Then the number of LIN entries are $\frac{n(n-1)}{2}$ which is also the dimension of the space of skew-symmetric matrices.

8. (a) If A is a real skew-symmetric matrix and $A^2 + I = 0$, then S.T. A is orthogonal.

(b) If H is Hermitian matrix, what kind of matrix is e^{iH} ?

Sol. (a) Let A be the real skew-symmetric matrix.

then $A^T = -A$.

Also we have $A^2 + I = 0$.

We have $A^T A = (-A)(A) = I$.

$\therefore A$ is orthogonal.

(b) Let H is Hermitian matrix. $H^* = H$.

for any matrix M , $(e^M)^* = e^{M^*}$

Let $e^{iH} = A$

Then $A^* A = (e^{iH})^* \cdot e^{iH}$
 $= I$.

Hence e^{iH} is unitary matrix.

Inverse or Reciprocal of a matrix:-

Let A be a square matrix of order n . Then the mtx B of order n , if it exists, such that

$$AB = BA = I_n,$$

is called the inverse or reciprocal of A and is denoted by A^{-1} .

$$A^{-1} = \left(\frac{\text{Adj } A}{|A|} \right) * I, \text{ provided } |A| \neq 0.$$

Corollary! We have $|A| |A^{-1}| = |AA^{-1}| = |I| = 1$

$$\therefore |A^{-1}| = |A|^{-1}.$$

Theorem! - the inverse of a matrix is unique.

Sol. If possible let B and C be two inverses of the same mtx A , then by definition

$$AB = BA = I$$

$$AC = CA = I$$

$$1. C(AB) = CI = C$$

$$\therefore (CA)B = IB = B$$

$\therefore C = B$, \therefore the inverse is unique.

Theorem! - A square matrix A has an inverse if and only if $|A| \neq 0$, i.e. only a non-singular matrix has an inverse.

Proof! - the condition is necessary. Let B be the inverse of the mtx A .

$$\text{then } AB = I$$

$$\text{so, that } |A||B| = |I| = 1$$

$$\therefore |A| \neq 0.$$

2. the condition is sufficient.

Let $|A| \neq 0$, we assume that $B = \frac{\text{Adj } A}{|A|}$

$$\therefore AB = A \left(\frac{\text{Adj } A}{|A|} \right)$$

$$= \frac{1}{|A|} \cdot (A \cdot \text{Adj } A)$$

$$= \frac{|A| \cdot I}{|A|} = I$$

Similarly $BA = I$.

$\therefore AB = BA = I$, \therefore Hence A has an inverse.

Ex. 1 find A^{-1} when $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & 6 & -7 \end{bmatrix}$.

Sol. $|A| = 20$; $\text{Adj } A = \begin{bmatrix} 6 & 4 \\ -7 & -8 \\ -18 & 6 \end{bmatrix}$

Hence $A^{-1} = \frac{\text{Adj } A}{|A|} = \begin{bmatrix} \frac{6}{20} & \frac{4}{20} & \frac{-1}{20} \\ \frac{-7}{20} & \frac{-8}{20} & \frac{-8}{20} \\ \frac{-18}{20} & \frac{6}{20} & \frac{4}{20} \end{bmatrix}$

Ex. 2. Show that the inverse of a regular symmetric (hermitian) matrix is symmetric (hermitian).

Sol. Let A be symmetric, so that $A' = A$

$$\& AA^{-1} = I = A^{-1}A$$

$$\text{since } I = I', \text{ so, } A^{-1}A = (A^{-1}A)'$$

$$\begin{aligned} AA^{-1} &= (A^{-1}A)' \\ &= A' \cdot (A^{-1})' \\ &= A(A^{-1})' \end{aligned}$$

$$\therefore A^{-1} = (A^{-1})'$$

$\therefore A^{-1}$ is symmetric.

2nd Part:- Let A be hermitian, so that

$$A^* = A$$

$$A^{-1}A = AA^{-1} = I$$

$$\therefore (A^{-1}A)^* = I^* = I = (AA^{-1})$$

$$\& A^*(A^{-1})^* = AA^{-1}$$

$$\therefore A(A^{-1})^* = AA^{-1}$$

$$\therefore (A^{-1})^* = A^{-1}$$

$\therefore A^{-1}$ is hermitian.

Ex. 3. find the inverse of $A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ -1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix}$.

Sol.

$$\boxed{A = I \times A}$$

$$\begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 1 & 3 \\ 1 & 3 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \quad \left[\begin{array}{l} c_1' \leftrightarrow c_2 - c_1 \\ c_3' \leftrightarrow c_4 - c_3 \end{array} \right]$$

$$\left[\begin{array}{cccc} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & 0 \end{array} \right] = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] A \quad \left[\begin{array}{l} R_2' \leftrightarrow R_3 - R_2 \\ R_4' \leftrightarrow R_4 - R_3 \end{array} \right]$$

$$\left[\begin{array}{cccc} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -1 & -2 \end{array} \right] = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 2 & 2 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{array} \right] A \quad \left[\begin{array}{l} R_4' \leftrightarrow R_4 - R_1 \\ R_3' \leftrightarrow R_3 - 2R_2 \end{array} \right]$$

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{array} \right] = \left[\begin{array}{cccc} 2 & 1 & -1 & 0 \\ -1 & -1 & 1 & 0 \\ 2 & 2 & -1 & 0 \\ 1 & 2 & -1 & 1 \end{array} \right] A \quad \left[\begin{array}{l} R_4' = R_4 + R_3 \\ R_1' = R_1 - R_2 \end{array} \right]$$

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] = \left[\begin{array}{cccc} 0 & -3 & 1 & -2 \\ -1 & -1 & 1 & 0 \\ -1 & -4 & 2 & -3 \\ 1 & 2 & -1 & 1 \end{array} \right] A \quad \left[\begin{array}{l} R_1' = R_1 - 2R_4 \\ R_3' = R_3 - 3R_4 \end{array} \right]$$

$$I = A^{-1} A$$

$$\therefore A^{-1} = \begin{bmatrix} 0 & -3 & 1 & -2 \\ -1 & -1 & 1 & 0 \\ -1 & -4 & 2 & -3 \\ 1 & 2 & -1 & 1 \end{bmatrix}$$

Rank and Nullity of a matrix:-

The maximum order of the non-singular square sub-matrix of A is called the rank of A. The mtx A is said to be of rank n, if and only if it has at least one nonsingular square sub-mtx of order n and all square submatrices of (n+1) and higher orders are singular. The rank of a mtx A is denoted by rank(A).

If A is a square mtx of order n, then n - rank(A) is called the nullity of the mtx A and denoted by N(A).

Remarks:-

1. If I is a unit mtx of order n, rank(I)=n.

2. If A is of order mxn, then rank(A) ≤ m and ≤ n.

3. If A' is the transpose of A, rank(A')=rank(A).

Ex. S.T. the rank of a skew-symmetric mtx can't be 1.

Sol. For a skew-symmetric mtx, the elements of the leading diagonal are all zero. If one element is non-zero, say x, then we have a minor of the form $\begin{vmatrix} 0 & x \\ -x & 0 \end{vmatrix} \neq 0$

Therefore rank of the mtx is ≥ 2. If all entries are zero then rank is 0.

Ex. 2. Find the rank of the mtrix A, where $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{bmatrix}$.

Sol.

$$|A| = 2(-9+8) + 2(-3+4) = 0$$

But $\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} \neq 0$, so $\text{rank}(A) = 2$.

Normal form of a matrix:-

Theorem:- By means of elementary transformations every mtrix A of order $m \times n$ and rank $r (> 0)$ can be reduced to the following form:

- (i) $\left[\begin{array}{c|c} I_n & 0 \\ \hline 0 & 0 \end{array} \right]$
- (ii) $\left[\begin{array}{c|c} I_n & 0 \\ \hline 0 & 0 \end{array} \right]$
- (iii) $\left[\begin{array}{c|c} I_n & 0 \\ \hline 0 & 0 \end{array} \right]$
- (iv) $\left[\begin{array}{c|c} I_n & 0 \\ \hline 0 & 0 \end{array} \right]$

Ex. Reduce the mtrix A to its normal form, where $A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 2 \end{bmatrix}$ and hence determine its rank.

Solution:-

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix}$$

$$R_2' \leftrightarrow R_2 - R_1$$

$$R_3' \leftrightarrow R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_3' \leftrightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & -6 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1' \leftrightarrow R_1 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_3' \leftrightarrow C_3 + C_1$$

$$C_3' \leftrightarrow C_3 - 2C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_4' \leftrightarrow C_4 + 6C_1$$

$$C_4' \leftrightarrow C_4 - 5C_2$$

Hence the normal form of A is
and $\text{rank}(A) = 2$.

$$\left[\begin{array}{c|c} I_2 & 0 \\ \hline 0 & 0 \end{array} \right]$$

WORKED EXAMPLES:-

1. If $r(A)$ denotes rank of a mtx A, then $r(AB)$ is
 (a) $r(A)$ (b) $r(B)$ (c) $\leq \min[r(A), r(B)]$ (d) $> \min[r(A), r(B)]$

Sol: $\text{rank}(AB) \leq \min[r(A), r(B)]$

e.g. $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
 $AB = \begin{bmatrix} 7 & 10 \\ 7 & 10 \end{bmatrix} \quad \text{rank}(A) = 1, \text{rank}(B) = 2,$
 $\text{rank}(AB) = 1 \leq \min[r(A), r(B)]$.

2. Show that $\text{rank}(A') = \text{rank}(A^*) = \text{rank}(A)$.

Sol. The transpose of a mtx is obtained by interchanging rows into columns into rows, and clearly this change does not alter the values of the determinants of minors. Hence

$$\text{rank}(A') = \text{rank}(A).$$

Now assume that the value of a minor of A with complex elements be $a+ib$. Then the value of the corresponding minor of the conjugate mtx of A be $a-ib$.

In case $a+ib=0$ then $a-ib=0$.

In case $a+ib \neq 0$ then $a-ib \neq 0$.

$$\text{Hence } \text{rank}(A^*) = \text{rank}(A) = \text{rank}(A').$$

3. If A is an $(n \times 1)$ non-zero mtx and B is a $(1 \times n)$ non-zero mtx, then show that $\text{rank}(AB)=1$.

Sol. Let $A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}$ and $B = [b_{11} \ b_{12} \ \dots \ b_{1n}]$.

$$\text{Then } AB = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} \dots a_{11}b_{1n} \\ a_{21}b_{11} & a_{21}b_{12} \dots a_{21}b_{1n} \\ \vdots & \vdots \\ a_{n1}b_{11} & a_{n1}b_{12} \dots a_{n1}b_{1n} \end{bmatrix}$$

AB is non-zero and there will be at least one non-zero minor of order 1.

Hence, $\text{rank}(AB)=1$ because all 2nd and higher order minors are all zero here.

4. Find the rank of the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix}$

Solution:-

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 3 & -2 & 0 \\ 2 & 0 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{bmatrix} \quad c_4' \leftrightarrow c_4 - c_1$$

$\therefore \text{rank}(A) = \text{No. of LIN columns in } A$

$\hat{=} 3.$

5. Reduce the mtx/A to its normal form, where
and hence find the rank of the mtx.

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

Solution:-

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4' \leftrightarrow R_4 - (R_1 + R_2)$$

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{E.C.} \leftrightarrow c_1 - (c_2 + c_4)$$

$$\sim \begin{bmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C_2' = c_1 + c_2$$

5. Reduce the matrix A to its normal form, where

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 2 \end{bmatrix}$$

and hence determine its rank.

Sol. To reduce a mtx into its normal form, let us use congruent row and column operations.

$$A \sim \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & 2 & 5 \\ 3 & 4 & 5 & 2 \end{bmatrix} \quad \text{By } R_{21}(-1)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix} \quad \text{By } R_{31}(-3)$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix} \quad \text{By } C_{21}(-1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix} \quad \text{By } C_{31}(-1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix} \quad \text{By } C_{41}(1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{By } R_{32}(-1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{By } C_{32}(-1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{By } C_{42}(5)$$

Hence the normal form of A is $\left[\begin{array}{c|c} I_2 & 0 \\ \hline 0 & 0 \end{array} \right]$ and $\text{rank}(A) = 2$.

6. Reduce the matrix A to its normal form, where

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

and hence find the rank of the matrix.

Solution:- By performing the operation R_{12} , we have

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

By $R_{31}(-3)$, $R_{41}(-1)$ $A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$

By $C_{31}(-1)$, $C_{41}(-1)$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$

By $R_{32}(-1)$, $\cancel{R_{42}(-1)}$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

By $C_{32}(3)$, $C_{42}(1)$, $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

which is of the form $\left[\begin{array}{c|cc} I_2 & 0 \\ \hline 0 & 0 \end{array} \right]$; hence $\text{rank}(A) = 2$.

— . —

System of Linear Equations

1. The equations

$$x - y + 2z = 4$$

$$3x + y + 4z = 6$$

$$x + y + z = 1 \quad \text{have}$$

- (A) Unique solution (B) Infinite solutions (C) No solution (D) None.

Sol. →

$$(B) [A|b] = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 3 & 1 & 4 & 6 \\ 1 & 1 & 1 & 1 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 4 & -2 & -6 \\ 0 & 2 & -1 & -3 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 4 & -2 & -6 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\therefore \text{rank } [A|b] = \text{rank } (A) = 2 < 3$$

∴ it has infinite solutions.

2. The system of linear equations $x + y + z = 2$, $2x + y - z = 3$, $3x + 2y + kz = 4$ has a unique solution if

- (A) $k=0$ (B) $k \neq 1$ (C) $k=1$ (D) $k \neq 0$.

Sol.

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 3 \\ 3 & 2 & k & 4 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -3 & -1 \\ 0 & -1 & k-3 & -2 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -3 & -1 \\ 0 & 0 & k-3 & -1 \end{array} \right]$$

$$\therefore k \neq 0$$

AH. method:-

Given system has a unique solution if

$$\left| \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{array} \right| \neq 0$$

$$\Rightarrow k \neq 0$$

3. The system of equations

$$2x + y = 5$$

$$x - 3y = -1$$

$$3x + 4y = k$$

is consistent, when k is

$$(A) 1$$

$$(B) 2$$

$$(C) 5$$

$$(D) 10$$

Sol.

$$(D) [A|B] = \left[\begin{array}{cc|c} 2 & 1 & 5 \\ 1 & -3 & -1 \\ 3 & 4 & k \end{array} \right]$$

$$A = \left[\begin{array}{cc} 2 & 1 \\ 1 & -3 \\ 3 & 4 \end{array} \right] \sim \left[\begin{array}{cc} 2 & 1 \\ 1 & -3 \\ 3 & 4 \end{array} \right]$$

$$\text{rank}(A) = 2$$

$$\text{Det}[A|B] = -7k + 70$$

$$\therefore \text{the system is consistent if } \text{Det}[A|B] = 0$$

$$\Rightarrow -7k = 70$$

$$\Rightarrow k = 10.$$

4. S.T. the S.O.E.s $\left. \begin{array}{l} x_1 - 2x_2 + x_3 - x_4 + 1 = 0 \\ 3x_1 - 2x_3 + 3x_4 + 4 = 0 \\ 5x_1 - 4x_2 + x_4 + 3 = 0 \end{array} \right\}$ is inconsistent.

Sol. Here, we have

$$A = \left[\begin{array}{cccc} 1 & -2 & 1 & -1 \\ 3 & 0 & -2 & 3 \\ 5 & -4 & 0 & 1 \end{array} \right], [A:b] = \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 1 \\ 3 & 0 & -2 & 2 & -1 \\ 5 & -4 & 0 & 1 & -3 \end{array} \right]$$

By elementary transformations, A and $[A:b]$ can be reduced in the following forms:

$$A \sim \left[\begin{array}{cccc} 1 & 0 & \frac{1}{6} & 1 \\ 0 & 1 & -\frac{5}{6} & 1 \\ 0 & 0 & 0 & 0 \end{array} \right], [A:b] \sim \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{6} & 1 & -4/3 \\ 0 & 1 & -\frac{5}{6} & 1 & -1/6 \\ 0 & 0 & 0 & 0 & 3 \end{array} \right]$$

$$\therefore \text{rank}(A) = 2 \text{ but } \text{rank}(A:b) = 3$$

so, the equations are inconsistent.

5. Find the rank of the matrix $A = \left[\begin{array}{cccc} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 1 & a & b & 0 \\ 0 & c & d & 1 \end{array} \right].$

Sol.

$$A \sim \left[\begin{array}{cccc} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\therefore \text{rank}(A) = 2.$$

6. Find the inverse of the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$$

Sol. $A = I_3 A$

Applying E-norm transformation to the matrix applying $R_2 \rightarrow R_2 - 3R_1$,
 $R_3 \rightarrow R_3 - R_1$.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

$$R_2 \rightarrow -\frac{1}{4}R_2$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{4} & \frac{1}{4} & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

$$R_1 \rightarrow R_1 - 2R_2, R_3 \rightarrow R_3 + R_2$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 1 \end{bmatrix} A$$

$$R_1 \rightarrow R_1 - R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & -1 \\ \frac{3}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 1 \end{bmatrix} A$$

$$\therefore I_3 = BA$$

$$\therefore A^{-1} = B.$$

$$\therefore A^{-1} = B = \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & -1 \\ \frac{3}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 1 \end{bmatrix}$$

$$\begin{aligned}
 7. \quad & \text{Solve: } x_1 - x_2 + x_3 = 2 \\
 & 3x_1 - x_2 + 2x_3 = -6 \\
 & 3x_1 + x_2 + x_3 = -18
 \end{aligned}$$

Sol. Here, we have $A = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 3 & -1 & 2 & -6 \\ 3 & 1 & 1 & -18 \end{bmatrix}$

$$\text{Now, } |A| = 1(-1-2) + 1(3-6) + 1(3+3) = 0.$$

$$\text{Also, } \begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix} \neq 0; \text{ hence rank}(A) = 2.$$

We now proceed to reduce matrix B to its Echelon form

$$\text{by } R_{21}(-3), R_{31}(-3),$$

$$B \sim \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & -1 & -12 \\ 0 & 4 & -2 & -24 \end{bmatrix}$$

$$\text{By } R_{32}(-2),$$

$$B \sim \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & -1 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{By } R_1(1/2)$$

$$B \sim \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & -1/2 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \text{rank}(B) = 2 = \text{rank}(A).$$

Hence the given equations are consistent.

Also the mtix equation becomes

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 0 \end{bmatrix}$$

$$\begin{cases} x_1 - x_2 + x_3 = 2 \\ x_2 - 1/2 x_3 = -6 \end{cases} \quad \begin{cases} x_1 = -1/2 x_3 - 4, \\ x_2 = 1/2 x_3 - 6. \end{cases}$$

and x_1, x_2 can be expressed in terms of x_3 which is arbitrary. Since the rank is 2, two unknowns x_1, x_2 are expressed in terms of x_3 , the system has ~~finite~~ infinite number of solutions.

8. Find two non-singular matrices P and Q such that PAQ is in the normal form where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}. \text{ Also find the rank of the matrix A.}$$

Solution:

$$A = I_3 A I_3$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

E-operations on the matrix A (left hand equation) until it is reduced to the normal form.
 Every E-type operation will also be applied to the pre-factors I_3 of the product on the right hand member of the above equation and every column operation to the factors I_3

Applying $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing $C_2 \rightarrow C_2 - C_1, C_3 \rightarrow C_3 - C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing $R_2 \rightarrow -\frac{1}{2}R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing $R_3 \rightarrow R_3 + 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing $C_3 \rightarrow C_3 - C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_2 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = P A Q, \text{ where }$$

$$P = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -2 & -1 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Rank(A) = 2