EIGEN VALUES & VECTORS

Eigen Values :—

Let $A = [a_{ij}]_{n\times n}$ be any square matrix of order n and λ be an indeterminate.
$[A - \lambda I]$ is called characteristic matrix and $|A - \lambda I| = 0$ is called characteristic equation and roots of this equation is called the characteristic roots or characteristic values or eigen values or latent roots or proper values of the mtx A.

Note:- The set of the eigen values of A is called the spectrum of A.

Eigen Vectors :— If λ is a characteristic root of an $n\times n$ matrix A, then a non-zero vector $\underset{\sim}{x}$ such that

$$A\underset{\sim}{x} = \lambda \underset{\sim}{x}$$

is called a characteristic vector or eigen vector of A corresponding to the characteristic root λ.

Remark:— 1. λ is a characteristic root of a matrix A if and only if there exists a non-zero vector $\underset{\sim}{x}$ such that $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$.

2. If $\underset{\sim}{x}$ is a characteristic vector of a matrix A corresponding to the characteristic value λ. Here k is any non-zero scalar, then $k\underset{\sim}{x}$ is also a characteristic vector of A corresponding to the same characteristic value λ.

3. The characteristic vectors corresponding to distinct characteristic roots of a matrix are linearly independent.

Nature of an eigen value of the special types of matrices :—

1. The eigen value of a Hermitian matrix are all real.
2. The eigen value of a real symmetric mtx are all real.
3. The eigen value of a skew-Hermitian mtx are either pure imaginary or zero.
4. The eigen value of a skew symmetric mtx are either pure imaginary or zero.
5. The eigen values of a unitary matrices & an orthogonal matrix are of unit modulus.

Ex. S.T. the eigen value of a triangular matrix are just the diagonal elements of the matrix.

Sol.
Let, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$ be a triangular mtx of order 3.

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) = 0$$

$$\Rightarrow \lambda = a_{11}, a_{22}, a_{33}.$$

EX.2. Determine the eigen values and eigen vectors of the matrix
$$A=\begin{bmatrix}5 & 4\\ 1 & 2\end{bmatrix}.$$

Sol.

$|A-\lambda I|=0$

$\Rightarrow \begin{vmatrix}5-\lambda & 4\\ 1 & 2-\lambda\end{vmatrix}=0$

$\Rightarrow (5-\lambda)(2-\lambda)-4=0$

$\Rightarrow (\lambda-6)(\lambda-1)=0$

$\therefore \lambda_1=6,\ \lambda_2=1$

Eigen vectors $\underset{\sim}{X}_1=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ of A corresponding to the eigen value 6.

$(A-6I)\underset{\sim}{x}=\underset{\sim}{0}$

$\Rightarrow \begin{bmatrix}5-6 & 4\\ 1 & 2-6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$

$\Rightarrow \begin{bmatrix}-1 & 4\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$ by $R_2 \leftrightarrow R_2+R_1$

rank of the coefficient matrix $=1$, there are $(2-1)=1$ L.I. solution.

$4x_2=x_1$

$\therefore x_2=4,\ x_1=1$

$\therefore \begin{bmatrix}4\\ 1\end{bmatrix}$ is an eigen vector of A corresponding to the eigen value 6.

The eigen vectors $\underset{\sim}{X}_2=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ corresponding to the eigen value 1.

$(A-1I)\underset{\sim}{x}=\underset{\sim}{0}$

$\Rightarrow \begin{bmatrix}4 & 4\\ 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$

$\Rightarrow \begin{cases}4x_1+4x_2=0\\ x_1+x_2=0\end{cases}$

$\therefore x_1=-x_2$

if $x_2=1,\ x_1=-1$.

$\therefore \underset{\sim}{X}_2=\begin{bmatrix}1\\ -1\end{bmatrix}$ is the eigen vector of A corresponding to the eigen value 1.

So, $\underset{\sim}{X}_1=k\begin{bmatrix}4\\ 1\end{bmatrix},\ \underset{\sim}{X}_2=h\begin{bmatrix}1\\ -1\end{bmatrix}$ are the eigenvectors corresponding to the eigen values 6, 1, respectively, where k, h are any non-zero scalars.

The Cayley-Hamilton theorem:-

Every square matrix satisfies its characteristic equation.
i.e. if for a square matrix A of order n,

$$|A-\lambda I| = (-1)^n\left[\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots\dots + a_n\right]$$

then the matrix equation

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n I = 0$$

is satisfied by $X = A$,

i.e. $A^n + a_1 A^{n-1} + a_2 A^{n-2} + \dots + a_n I = 0$.

Corollary 1:- If A be a non-singular matrix, $|A| \neq 0$.

Premultiplying by A^{-1}

$$A^{n-1} + a_1 A^{n-2} + a_2 A^{n-3} + \dots + a_{n-1} I + a_n A^{-1} = 0$$

or, $A^{-1} = -\left(\frac{1}{a_n}\right)\left(A^{n-1} + a_1 A^{n-2} + \dots + a_{n-1} I\right)$

Corollary 2:- If m be a positive integer such that $m \geq n$, then multiplying the results by A^{m-n}.

$$A^m + a_1 A^{m-1} + \dots + a_n A^{m-n} = 0.$$

Eigen values and Eigen vectors:-

If V is a vector space over the field F and T is a linear operator on V. An eigen value of T is a scalar c in F such that there is a non-zero vector $\alpha \in V$ with $T\alpha = c\alpha$.

If c is an eigen value of T, then

(a) Any α such that $T\alpha = c\alpha$ is called eigen vector of T associated with the eigen value c;

(b) The collection of all α such that $T\alpha = c\alpha$ is called the eigen space associated with c.

Eigen value of matrix A over F:- If A is an $n\times n$ matrix over the field F, an eigen value of A over F is a scalar c in F such that the matrix $(A - cI)$ is singular (not invertible).

Diagonalisable:- If T is a linear operator on the finite dimensional space V. Then T is diagonalizable if there is a basis for V each vector of which is an eigen vector of T.

Eigen polynomial:-

$$f(c) = |A - cI|.$$

Some Important Theorem:—

1. If T is a linear operation on a finite dimensional space V and c is any scalar. then followings are equivalent :
   (a) c is an eigen value of T
   (b) The operator $(T-cI)$ is singular (not invertible)
   (c) $\det(T-cI)=0$.
2. Similar matrices have the same eigen ~~...~~ polynomial.
3. If $T\alpha = c\alpha$ and F is any polynomial, then $F(T)\alpha) = F(c)\alpha$
4. Suppose T is a linear operator on the finite dimensional space V; $c_1,\dots,c_k$ are k-distinct eigen values of T and W is the space of the eigen vectors associated with the eigen value c_i. If $W = W_1 + W_2 + \dots + W_k$, then
$$\dim(W) = \dim(W_1) + \dim(W_2) + \dots + \dim(W_k).$$
In fact, if B_i is an ordered basis for W_i, then $B = (B_1,\dots,B_k)$ is an ordered basis for W.
5. If T is a linear operator on a finite dimensional values of T and W_i is a null space of $(T-c_iI)$. The followings are equivalent:
   (i) T is diagonalizable
   (ii) The eigen polynomial for T is $F = (x-c_1)^{d_1}(x-c_2)^{d_2}\dots(x-c_k)^{d_k}$; with $\dim(W_i) = d_i$; $i = 1(1)k$.
   (iii) $\dim V = \dim(W_1) + \dim(W_2) + \dots + \dim(W_k)$.

Theorem:— If $\underset{\sim}{\alpha}$ is a characteristic vector of T corresponding to the eigen value λ, then $k\underset{\sim}{\alpha}$ is also a ch. vector of T corresponding to the same ch. value λ. Here k is any non-zero scalar.

Proof:- Since $\underset{\sim}{\alpha}$ is a characteristic vector of T corresponding to the ch. value λ, therefore, $\underset{\sim}{\alpha} \neq \underset{\sim}{0}$ and

$$T(\underset{\sim}{\alpha}) = \lambda\underset{\sim}{\alpha}$$; let, k be any non-zero scalar.

$$T(k\underset{\sim}{\alpha}) = kT(\underset{\sim}{\alpha}) = k(\lambda\underset{\sim}{\alpha})$$
$$= \lambda(k\underset{\sim}{\alpha}).$$

$\therefore$ $k\underset{\sim}{\alpha}$ is a characteristic vector of T corresponding to the characteristic value λ.

Thus, corresponding to a ch. value λ, there may correspond more than one characteristic vectors.

Theorem:— If $\underset{\sim}{\alpha}$ is a ch. vector of T, then $\underset{\sim}{\alpha}$ can't correspond to more than one ch. values of T.

Proof:-

$T\underset{\sim}{\alpha} = c_1\underset{\sim}{\alpha}$ and $T\underset{\sim}{\alpha} = c_2\underset{\sim}{\alpha}$; where c_1, c_2 are two distinct ch. values of T.

$$c_1\underset{\sim}{\alpha} = c_2\underset{\sim}{\alpha}$$
$$\Rightarrow (c_1-c_2)\underset{\sim}{\alpha} = \underset{\sim}{0} \quad [\because \underset{\sim}{\alpha} \neq \underset{\sim}{0}]$$
$$\Rightarrow c_1 - c_2 = 0$$
$$\Rightarrow c_1 = c_2.$$

So, a ch. vector of a matrix, can't correspond to more than one characteristic value of that mtx.

Theorem:— Let T be a linear operator on an n-dimensional vector space V and A be the matrix of T relative to any ordered basis B. Then a vector α in V is an eigenvector of T corresponding to its eigenvalue c if and only if its coordinate vector $\underset{\sim}{X}$ relative to the basis B is an eigen-vector of A corresponding to its eigenvalue c.

Proof:— We have $[T-cI]_B = [T]_B - c[I]_B$

$$= A - cI.$$

If $\alpha \neq 0$, then the coordinate vector $\underset{\sim}{X}$ of $\underset{\sim}{\alpha}$ is also non-zero.

Now, $[(T-cI)(\alpha)]_B = [T-cI]_B[\alpha]_B$

$$= (A-cI)\underset{\sim}{X}$$

$\therefore (T-cI)(\alpha) = \underset{\sim}{0}$ iff $(A-cI)\underset{\sim}{X} = 0$

or, $T(\alpha) = c\underset{\sim}{\alpha}$ iff $A\underset{\sim}{X} = c\underset{\sim}{X}$.

or, α is an eigenvector of T iff $\underset{\sim}{X}$ is an eigen vector of A.

Ex. 1. Let V be an n-dimensional vector space over F. What is the ch. polynomial of (i) the identity operator on V, (ii) the zero operator on V.

Sol. Let B be any ordered basis for V.

(i) If I is the identity operator on V, then $[I]_B = I$.

The characteristic polynomial of I is $\det(I - xI)$

$$= \begin{vmatrix} 1-x & 0 & \cdots & 0 \\ 0 & 1-x & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1-x \end{vmatrix} = (1-x)^n.$$

(ii) If $\hat{0}$ is the zero operator on V, then $[\hat{0}]_B = \hat{0}$, i.e. the null matrix of order n, then the ch. polynomial of $\hat{0}$ is

$$\det(\underset{\sim}{0} - \underset{\sim}{x}I) = \begin{vmatrix} -x & 0 & \cdots & 0 \\ 0 & -x & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -x \end{vmatrix} = (-1)^n x^n.$$

Ex. 2. If $c \in F$ is a ch. value of a linear operator T on a vector space $V(F)$, then for any polynomial $p(x)$ over F, $p(c)$ is a ch. value of $p(T)$.

Sol. Since c is a ch. value of T, therefore $\exists$ a non-zero vector $\underset{\sim}{\alpha}$ in $V \ni$

$$T\underset{\sim}{\alpha} = c\underset{\sim}{\alpha}$$

$$\Rightarrow T(T\underset{\sim}{\alpha}) = T(c\underset{\sim}{\alpha})$$

$$\Rightarrow T^2\underset{\sim}{\alpha} = cT(\underset{\sim}{\alpha}) = c^2\underset{\sim}{\alpha}$$

$\therefore c^2$ is a ch. value of T^2.

Repeating this process k times, we get

$$T^k\underset{\sim}{\alpha} = c^k\underset{\sim}{\alpha}.$$

$\therefore c^k$ is a ch. values of T^k, where k be any positive integer

Let $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$, where $a_i \in F$.

Then $p(T) = a_0 I + a_1 T + a_2 T^2 + \cdots + a_m T^m$.

We have $[p(T)](\alpha) = (a_0 I + a_1 T + \cdots + a_m T^m)(\alpha)$

$$= a_0 I \underset{\sim}{\alpha} + a_1 T(\underset{\sim}{\alpha}) + \cdots + a_m T^m(\underset{\sim}{\alpha})$$

$$= a_0 \underset{\sim}{\alpha} + a_1 (c\underset{\sim}{\alpha}) + \cdots + a_m (c^m \underset{\sim}{\alpha})$$

$$= (a_0 + c a_1 + \cdots + c^m a_m)\, \underset{\sim}{\alpha}.$$

$\therefore P(c) = a_0 + c a_1 + \cdots + a_m c^m$ is a ch. value of $P(T)$.

Ex. 3. Find all (complex) ch. values and ch. vectors of the following matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Sol.

$$|A - xI| = \begin{vmatrix} -x & 1 \\ 0 & -x \end{vmatrix} = 0$$

$$\Rightarrow x^2 = 0 \Rightarrow x = 0$$

$\therefore 0$ is the only ch. value of A.

Let x_1, x_2 be the components of the ch. vector corresponding to this ch. value 0.

Let $\underset{\sim}{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Now, $[A - 0.I]\underset{\sim}{X} = \underset{\sim}{0}$

$$\Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

i.e. $x_2 = 0$

let $x_1 = k$; where k is any non-zero complex number.

$\therefore \underset{\sim}{X} = \begin{bmatrix} k \\ 0 \end{bmatrix}$ is the ch. vector corresponding to the eigen value $\lambda = 0$, where k is any non-zero complex no.

WORKED EXAMPLES:-

1. The eigen values of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ is

(A) 2, 3, 6 (B) 2, 6, 7 (C) −2, 3, 6 (D) None.

Sol. (C) $|A-\lambda I| = 0$

$$\Leftrightarrow \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 5-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 5-\lambda \\ 3 & 1 \end{vmatrix}$$

$$= (\lambda+2)(\lambda-3)(\lambda-6) = 0$$

$$\Rightarrow \lambda = -2, 3, 6.$$

2. Find the eigen values of $(A^4+3A-2I)$, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ are

(A) 2, 3, 20 (B) 2, 2, 2 (C) 2, 2, 20 (D) 20, 20, 2

Sol. (D)

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 3 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 15 & 15 \\ 0 & 16 & 15 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, $B = A^4+3A-2I$

$$= \begin{bmatrix} 1 & 15 & 15 \\ 0 & 16 & 15 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 3 & 3 \\ 0 & 6 & 3 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 18 & 18 \\ 0 & 20 & 18 \\ 0 & 0 & 2 \end{bmatrix}$$

Then the eigen values of $|B-\lambda I| = 0$

$$\Leftrightarrow \begin{vmatrix} 2-\lambda & 18 & 18 \\ 0 & 20-\lambda & 18 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = 2, 20, 2.$$

3. Find the eigen values of A^4, where $A = \begin{pmatrix} 1 & 0 & -1 \\ 9 & 4 & 1 \\ 3 & 1 & 1 \end{pmatrix}$.

Sol.

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 9 & 4 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Then the eigen values of the matrix can be determined from the ch. equation

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & -1 \\ 9 & 4-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(\lambda-2)(\lambda-3) = 0$$

$$\Rightarrow \lambda = 1, 2, 3.$$

$\therefore$ The eigen values of A are: 1, 2, 3.

So, the eigen values of A^4 are: $1^4, 2^4, 3^4$, i.e. 1, 16, 81.

4. Use Cayley Hamilton theorem, find the inverse of $A = \begin{bmatrix} -1 & 1 & 1 \\ 3 & 1 & -1 \\ 2 & 2 & 1 \end{bmatrix}$

Sol.

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 1 & 1 \\ 3 & 1-\lambda & -1 \\ 2 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda^3 + \lambda^2 - 4\lambda - 4 = 0$$

$$\Rightarrow A^3 - A^2 + 4A + 4 = 0$$

$$\Rightarrow A^{-1} = \frac{-A^2 + A + 4I}{4}$$

$$= \frac{1}{4}\left[- \begin{bmatrix} 6 & 2 & -1 \\ -2 & 2 & 1 \\ 6 & 6 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 & 1 \\ 3 & 1 & -1 \\ 2 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \right]$$

$$= \frac{1}{4} \begin{bmatrix} -3 & -1 & 2 \\ 5 & 3 & -2 \\ 4 & -4 & 4 \end{bmatrix}$$

5. If λ_1 and λ_2 are the values of λ for which $\begin{vmatrix} 1 & \lambda & 0 \\ \lambda & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0$, then $\lambda_1 + \lambda_2$ equals (A) -1 (B) 0 (C) 1 (D) 2.

Sol. (B) $\begin{vmatrix} 1 & \lambda & 0 \\ \lambda & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0$

$$\Rightarrow 1(2-1) - \lambda(\lambda) = 0$$

$$\Rightarrow \lambda^2 = 1 \qquad \text{i.e. } \lambda_1 + \lambda_2 = 1 - 1 = 0.$$

$$\Rightarrow \lambda = \pm 1$$

6. Find the characteristic roots and corresponding characteristic vectors for each of the following matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Sol. The characteristic equation is

$$|A - \lambda I| = \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = -\lambda^3 + 18\lambda^2 - 45\lambda = 0$$

$\Rightarrow \lambda(\lambda-3)(\lambda-15) = 0$

$\Rightarrow \lambda = 0, 3, 15$ are the 3 ch. roots of the mtx A.

If $\underset{\sim}{X}$ is a characteristic vector corresponding to the ch. root 0, then we have

$$A\underset{\sim}{X} = \lambda \underset{\sim}{X}$$

$$\Rightarrow [A - \lambda I]\underset{\sim}{X} = \underset{\sim}{0}$$

$$\Rightarrow \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 8x_1 - 6x_2 + 2x_3 = 0, \; -6x_1 + 7x_2 - 4x_3 = 0, \; 2x_1 - 4x_2 + 3x_3 = 0$

let $x_1 = 1$, then $x_2 = 2$, $x_3 = 2$.

$\therefore \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ is a ch. vector corresponding to the

It may similarly be shown by considering the equation

$(A - 3I)\underset{\sim}{X} = \underset{\sim}{0}$,

$(A - 15I)\underset{\sim}{X} = \underset{\sim}{0}$,

that the ch. vectors corresponding to the ch. roots 3 and 15 are arbitrary non-zero multiples of the vectors

$$\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

7. Find the eigen value of the mtx $\begin{bmatrix} 1 & 0 & 0 & -a/2 \\ 0 & 1 & 0 & -a/2 \\ 0 & 0 & 1 & -a/2 \\ 0 & 0 & 0 & a \end{bmatrix}$.

Sol.

$\det[A - \lambda I] = 0$

$\Rightarrow (1-\lambda)(1-\lambda)(1-\lambda)(a-\lambda) = 0$

$\Rightarrow \lambda = 1, 1, 1, a$ are the eigen values.

8. Let $M = \begin{pmatrix} 1 & 1+i & 2-i \\ 1-i & 2 & 3+i \\ 2+i & 3-i & 3 \end{pmatrix}$. If $B = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$, where $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ and $\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$ are LIN eigen vectors of M, then the main diagonal of the matrix $B^{-1}MB$ has

(A) exactly one real entry (B) exactly two real entry

(C) exactly 3 real entry (C) no real entry.

Sol.

$M^T = M^*$

$\Rightarrow$ M is a Hermitian matrix

Since B is invertible mtx,

$\Rightarrow B^{-1}MB$ is a diagonal mtx whose diagonal entries are eigenvalues of M.

We know the eigen values of Hermitian mtx are real.

$\Rightarrow$ all ~~three~~ eigen values are real.

9. Let P be a 3X3 mtx $\ni$ for some c, the linear system $P = c$ has infinite number of solutions. Which one of the following is TRUE?

(A) The linear system $Px = b$ has infinite no. of solutions $\forall b$.

(B) Rank $(P) = 3$ (C) Rank $(P) \neq 1$ (D) Rank $(P) \leq 2$.

Sol. (D) rank $(P) < n \Rightarrow Px = b$ has infinite no. of solutions

$\therefore$ Rank $(P) \leq 2$.

10. Let P be a 2X2 mtx $\ni P^{102} = 0$. Then

(A) $P^2 = 0$ (B) $(1-P)^2 = 0$ (C) $(1+P)^2 = 0$ (D) $P = 0$

Sol. (A) Since P is mtx of order 2 so its ch. equation is of order 2.

So P^{102} is equal to 0 iff $P^2 = 0$.

11. Let A be an nxn matrix $\ni P^{-1}AP > 0$ for every non-zero invertible mtx P where P is also an nxn mtx. Which of the following is TRUE?

(A) All eigen values of A are negative (B) All eigen values of A are positive.

Sol. (B) Since given that for mtx A, $P^{-1}AP > 0$.

Here $P^{-1}AP$ is a diagonal mtx whose diagonal elements are eigen values of matrix A. But $P^{-1}AP > 0$ shows the eigen values of A are all positive.

12) Let $P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, The eigenvectors corresponding to the eigenvalues i and $-i$ are respectively

(A) $\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\begin{pmatrix} -1 \\ i \end{pmatrix}$ (B) $\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ (C) $\begin{pmatrix} -1 \\ i \end{pmatrix}\begin{pmatrix} i \\ -i \end{pmatrix}$ (D) $\begin{pmatrix} i \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ i \end{pmatrix}$

Sol.

$\lambda_1 = i, \lambda_2 = -i$

$(P - \lambda_1 I)\underset{\sim}{X} = 0$

$\Rightarrow \begin{bmatrix} 0-i & 1 \\ -1 & -i \end{bmatrix} \underset{\sim}{X} = \underset{\sim}{0}$

$\Rightarrow \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{cases} -x_1 i + x_2 = 0 \\ -x_1 - ix_2 = 0 \end{cases}$

$x_1 = 1,\ x_2 = i$

$\therefore \begin{bmatrix} 1 \\ i \end{bmatrix}$ is the eigen vector corresponding to the eigen value i.

$(P - \lambda_2 I)\underset{\sim}{X}' = \underset{\sim}{0}$

$\Rightarrow \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{cases} ix_1 + x_2 = 0 \\ -x_1 + ix_2 = 0 \end{cases}$

$\therefore x_1 = -1,\ x_2 = i$

$\therefore \begin{bmatrix} -1 \\ i \end{bmatrix}$ is the eigenvector corresponding to the eigen value $-i$.

13) Let P, M, B be $n \times m$ matrices $\ni$ M and N are non-singular. If $\underset{\sim}{x}$ is an eigen vector P corresponding to the eigen value λ, then an eigen vector of $N^{-1}MPM^{-1}N$ corresponding to the eigenvalue λ is

(A) $MN^{-1}\underset{\sim}{x}$ (B) $M^{-1}N\underset{\sim}{x}$ (C) $NM^{-1}\underset{\sim}{x}$ (D) $N^{-1}M\underset{\sim}{x}$

Sol. (C) Since λ is eigenvalue of P and $\underset{\sim}{x}$ be the eigen vector corresponding to it.

$\Rightarrow P\underset{\sim}{x} = \lambda\underset{\sim}{x}$

$$\begin{aligned} N^{-1}MPM^{-1}N(N^{-1}M\underset{\sim}{x}) &= N^{-1}MPM^{-1}(NN^{-1})M\underset{\sim}{x} \\ &= N^{-1}MPM^{-1}M\underset{\sim}{x} \\ &= N^{-1}MP(M^{-1}M)\underset{\sim}{x} \\ &= N^{-1}MP\underset{\sim}{x} \\ &= N^{-1}M\lambda\underset{\sim}{x} \\ &= N^{-1}M\ldots \end{aligned}$$

$\therefore N^{-1}M\underset{\sim}{x}$ is eigenvector corresponding to λ.

14) Let $P=\begin{bmatrix}1 & 1\\ i & -1\end{bmatrix}$. Then

(A) P has two linearly independent eigenvectors (B) P has an eigen vectors
(C) P is non-singular (D) $\nexists$ a non-singular S $\ni$ $S^{-1}PS$ is a diagonal mtx

sol.

(A)

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14) Let P be an $n\times n$ idempotent mtx, that is $P^2=P$, which of the following is FALSE?

(A) P^T is idempotent
(B) The possible eigenvalues of P are 0 and 1.
(C) The non-diagonal entries of P can be zero.
(D) There are infinite no. of $n\times n$ non-singular matrices that are idempoten

Solution:-

(D) Since P is idempotent matrix, i.e. $P^2=P$, then P^T is also idempotent, as $(P^2)^T=P^T \Rightarrow (P^T)^2=P^T$.

$P^2=P \Rightarrow P(P-I)=0 \Rightarrow$ eigen values of P are 0 and 1.

The non-diagonal entries of an idempotent mtx can be zero,

e.g. $P=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$, $P^2=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}=P$.

So, (D) is FALSE.

15) Let A and B are any arbitrary square matrices of order 2. Then show that AB and BA have same eigen values but may have different eigen vectors.

Solution:- Let us ~~take~~ an example.

$$A=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix},\quad B=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

$$AB=\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}$$

$$|AB-\lambda I|=0 \Rightarrow (1-\lambda)(2-\lambda)=2$$
$$\Rightarrow \lambda^2-3\lambda=0$$
$$\Rightarrow \lambda(\lambda-3)=0$$
$$\Rightarrow \lambda=0,\ \lambda=3.$$

For $\lambda=0$,

$$[AB-0.I]\underset{\sim}{X}=\underset{\sim}{0}$$
$$\Rightarrow \begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$\therefore x_1+x_2=0$ $\quad\therefore \begin{bmatrix}1\\ -1\end{bmatrix}$ be an eigen vector corresponding to $\lambda=0$.

$\therefore x_1=1,\ x_2=-1$

For $\lambda=3$,

$$[AB-3.I]\underset{\sim}{X}=\underset{\sim}{0}$$
$$\Rightarrow \begin{bmatrix}1-3 & 1\\ 2 & 2-3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$\therefore x_2=2,\ x_1=1$

$-2x_1+x_2=0$ $\quad 2x_1-x_2=0$ $\quad\therefore \begin{bmatrix}1\\ 2\end{bmatrix}$ is the eigen

Here $BA = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

$\therefore |BA - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda = 0, 3.$

for $\lambda = 0$, $[BA - 0.I]\underset{\sim}{X} = \underset{\sim}{0}$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 + 2x_2 = 0$

$\Rightarrow x_1 = -2x_2$

$\therefore x_2 = -1/2, \quad x_1 = 1$

$\therefore \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$ be the eigenvector corresponding to $\lambda = 0$.

for $\lambda = 3$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ be the eigenvector corresponding to $\lambda = 3$.

$\therefore$ AB & BA have same eigen values but not same eigen vectors.

16) A real 3×3 mtx M has eigen values ±1 and 2. S.T.
(i) M is invertible (ii) $M^3 - 2M^2$ is singular (iii) M is diagonalisable.

Sol. We have a mtx M whose 3 eigenvalues are $+1, -1$ and 2.
it's ch. equation can be given as

$(M+I)(M-I)(M-2I) = 0$

$\Rightarrow (M^2 - I)(M-2I) = 0$

$\Rightarrow M^3 - 2M^2 = M - 2I$

(i) By the properties of eigenvalues:
determinant of mtx = multiplication of eigenvalues

$|M| = (-1)\times(1)\times(2)$

$|M| = -2$

here $|M| \neq 0 \Rightarrow$ M is invertible.

(ii) $|M^3 - 2M^2| = |M - 2I|$

Since $|M| = -2$

$|M - 2I| = 0$

$\Rightarrow |M^3 - 2M^2| = 0$

$\Rightarrow M^3 - 2M^2$ is singular.

(iii) Since M having 3 distinct eigenvalues, then the ch. vectors corresponding to destinct characteristics roots of a mtx are LIN.
And an $n\times n$ mtx is diagonalisable if and only if it possesses n linearly independent eigenvectors.

$\Rightarrow$ M is diagonalisable.