MEAN VALUE THEOREMS

1.) Rolle's Theorem
2.) Langrange mean value theorem.
3.) Tayler's theorem.
4.) Cauchy's mean value theorem

Rolle's Theorem

Rolle's theorem states that if a function 'f' is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b) such that $f(a) = f(b)$, then $f'(x) = 0$ for some x with $a \leq x \leq b$.

f,
(i) cts, $x \in [a,b]$
(2) diff, $x (a,b)$
(3) $f(a) = f(b)$
$\exists \ c \in (a,b)$
$f'(c) = 0$

(1) f continuous for all $x \in [a,b]$
(2) f differentiable for all $x \in (a,b)$
(3) $f(a) = f(b)$

There exists atleast one point $c \in (a,b)$

such that $f'(c) = 0$

Geometrical meaning of Rolle's Theorem.

Let AB be the graph of f^n $y = f(x)$ such that the point A and B of the graph correspond to the numbers ~~to the~~ a and b of the interval $[a,b]$.

$\because$ $f(x)$ is continuous in the interval $[a,b]$

$\therefore$ its graph is a cts curve b/w A and B.

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Again as $f(x)$ is derivable in the open interval (a,b), therefore graph of $f(x)$ has a unique tangent at every point b/w A and B.

$\because f(a) = f(b)$

$\therefore AM = BN$ i.e, ordinates A and B are equal.

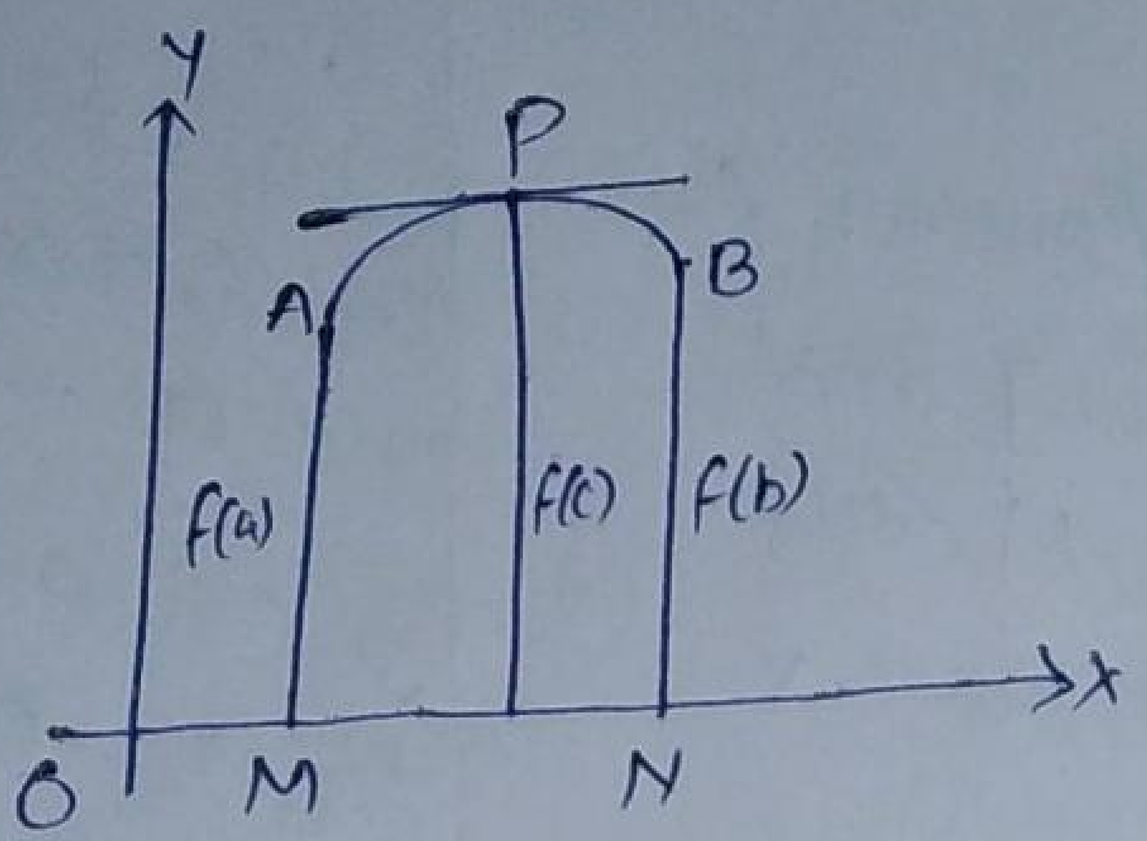

Fig (i)

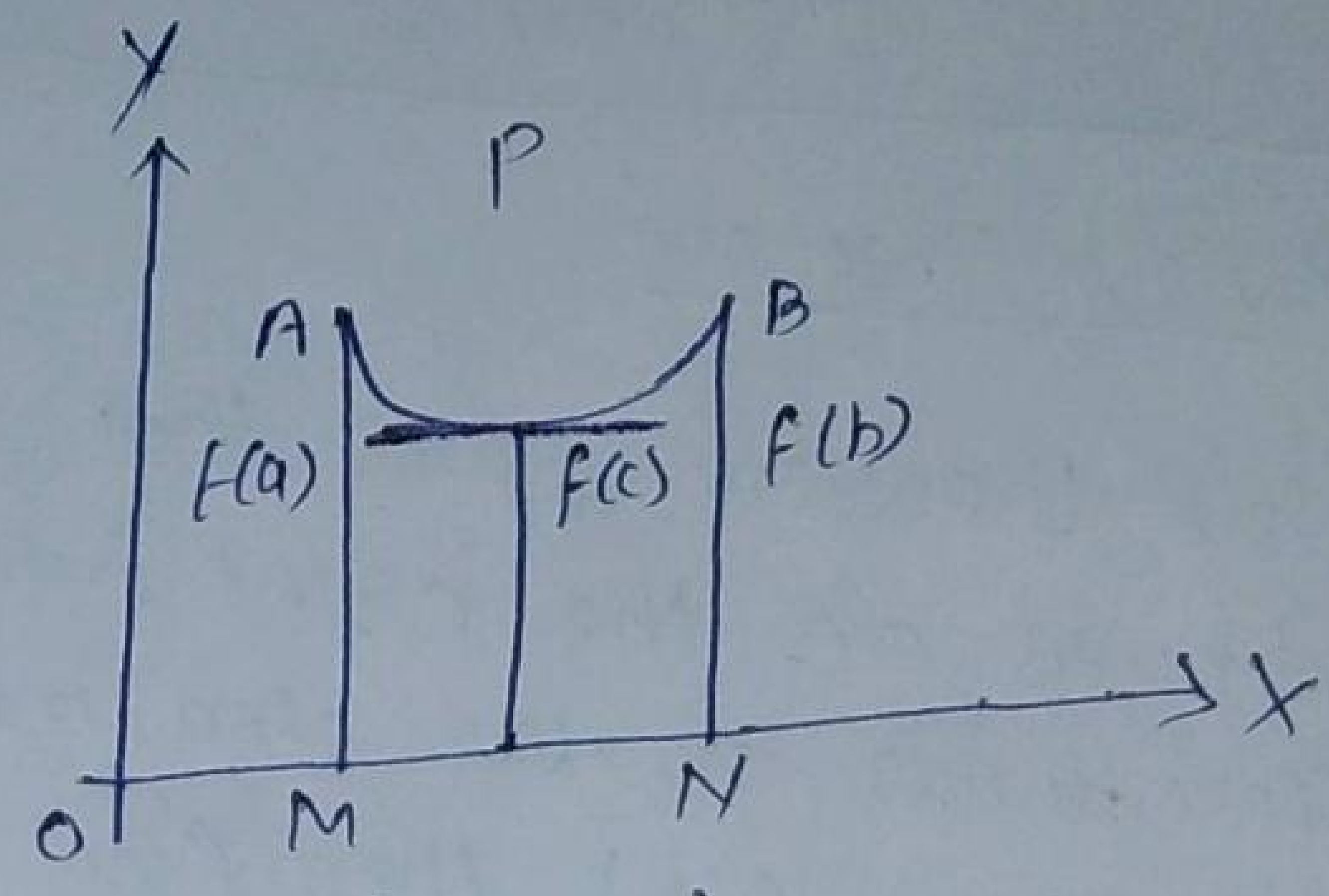

Fig (ii)

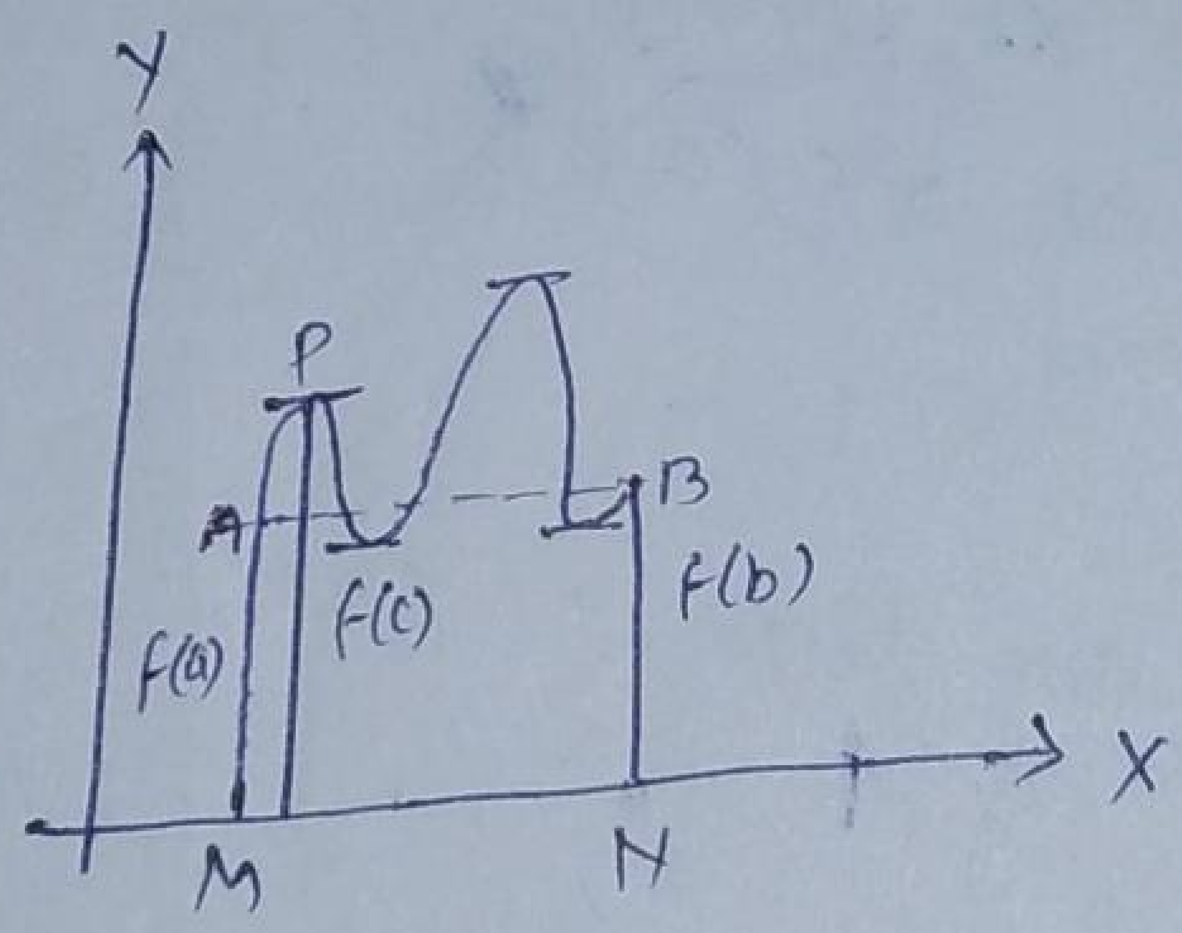

Fig (iii)

From the three figures drawn above, it is clear that there is at least one point P on the curve between A and B, the tangent at which parallel to x-axis.

$\therefore$ slope of tangent at $P = 0$

$\therefore f'(c) = 0$ where c is abscissa of P

Proof

Case I. If f is a constant function in $[a,b]$,

then $f'(x) = 0 \; \forall x \in [a,b]$

$f'(c) = 0 \; \forall c \in (a,b)$.

$\therefore$ the result is true in this case.

Case II When f is not a constant function throughout $[a,b]$, then as f is continuous in $[a,b]$, it is bounded and attains its l.u.b. and g.l.b. in $[a,b]$.

$\because$ f is not a constant f^n.

~~$\therefore$ let $f(c)$~~

$\therefore$ At least one of the these bounds (l.u.b and g.l.b) must be different from the equal values $f(a)$, $f(b)$ and must be attained at $x = c$ (say)

let this bound be attained at $x=c$ i.e., $c \in (a,b)$ where c is different from a and b

If $h > 0$, then $\dfrac{f(c+h)-f(c)}{h} \leq 0$

$$\Rightarrow \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \leq 0$$

$\Rightarrow Rf'(c) \leq 0$

Again if $h<0$, then $\dfrac{f(c+h)-f(c)}{h} \geq 0$

$$\Rightarrow \lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \geq 0$$

$\Rightarrow Lf'(c) \geq 0$

$\because$ $f(x)$ is derivable in (a,b)

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$\therefore f'(x)$ exists $\forall x \in (a,b)$

$\Rightarrow f'(c)$ exists.

$\Rightarrow Rf'(c) = Lf'(c) = f'(c)$

$\Rightarrow f'(c) = 0$ [$\because$ 0 is the common value of $Lf'(c)$ and $Rf'(c)$]

$\Rightarrow f'(c) = 0$ for some $c \in (a,b)$

$\therefore$ The theorem is proved.

Note:- A Polynomial, e^x, $\sin x$, $\cos x$ are always continuous.

Q. Prove Rolle's theorem for $f(x) = (x-a)^m (x-b)^n$ where a & b are +ve integers and $x \in [a,b]$

Soln:- (1) Continuous in $[a,b]$

(2.) $f'(x) = m(x-a)^{m-1}(x-b)^n + n(x-a)^m (x-b)^{n-1}$

It is a polynomial and it exists at each point $\forall x \in (a,b)$

$\therefore f(x)$ is diff for all $x \in (a,b)$

(3.) $f(a) = (a-a)^m (a-b)^n = 0$

$f(b) = (b-a)^m (b-b)^n = 0$

$\therefore f(a) = f(b)$

$\therefore$ All the conditions of Rolle's theorem have been satisfied.

$\therefore$ By Rolle's theorem $\exists$ a point $c \in (a,b)$ such that

$f'(c) = 0$

Now $f'(c) = 0$

$\Rightarrow m(c-a)^{m-1}(c-b)^n + n(c-a)^m(c-b)^{n-1} = 0$

$\Rightarrow (c-a)^{m-1}(c-b)^{n-1}\left[m(c-b) + n(c-a)\right] = 0$

$\Rightarrow mc - mb + nc - na = 0$

$$\Rightarrow \boxed{c = \frac{mb+na}{m+n}}$$

$\therefore c \in (a,b)$ and c divides line ab internally in the ratio $m:n$.

Hence Rolle's theorem has been verified.

Q. Verify Rolle's theorem for $f(x) = x(x+3)e^{-x/2}$ for $-3 \le x \le 0$

Soln:- (1). $x(x+3)$ being a polynomial is continuous. Also $e^{-x/2}$ is also continuous at all x.

$\therefore$ Their product $f(x)$ is cts $\forall x \in [-3,0]$

(2) $f'(x) = (2x+3)e^{-x/2} + (x^2+3x)\left(-\frac{1}{2}e^{-x/2}\right)$

$= e^{-x/2}\left[2x+3 - \frac{1}{2}x^2 - \frac{3}{2}x\right]$

$= e^{-x/2}\left[-\frac{1}{2}x^2 + \frac{x}{2} + 3\right]$

It exists at all points $x \in (-3,0)$

$\therefore f(x)$ is diff $\forall x \in (-3,0)$

3. $f(-3) = (9-9)e^{3/2} = 0$

$f(0) = (0+0)e^{0} = 0$

$\therefore f(-3) = f(0)$

$\therefore f(x)$ satisfies all condition of Rolle's theorem.

Now $f'(c) = 0$

$\Rightarrow e^{-c/2}\left[-\frac{1}{2}c^2 + \frac{c}{2} + 3\right] = 0$

$\Rightarrow c^2 - c - 6 = 0$

$\Rightarrow c = 3, -2$

$\boxed{(-3, 0)}$

$c = -2 \in (-3, 0)$ & $f'(-2) = 0$.

$\therefore$ Rolle's theorem is verified.

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~~LAGRT~~ LAGRANGE'S MEAN VALUE THEOREM

Let 'f' be a function such that

① 'f' is continuous in $[a,b]$

② 'f' is differentiable in (a,b)

then there exists at least one real number $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Geometrical Meaning

When $x=a$, $f(x)=f(a)$

When $x=b$, $f(x)=f(b)$

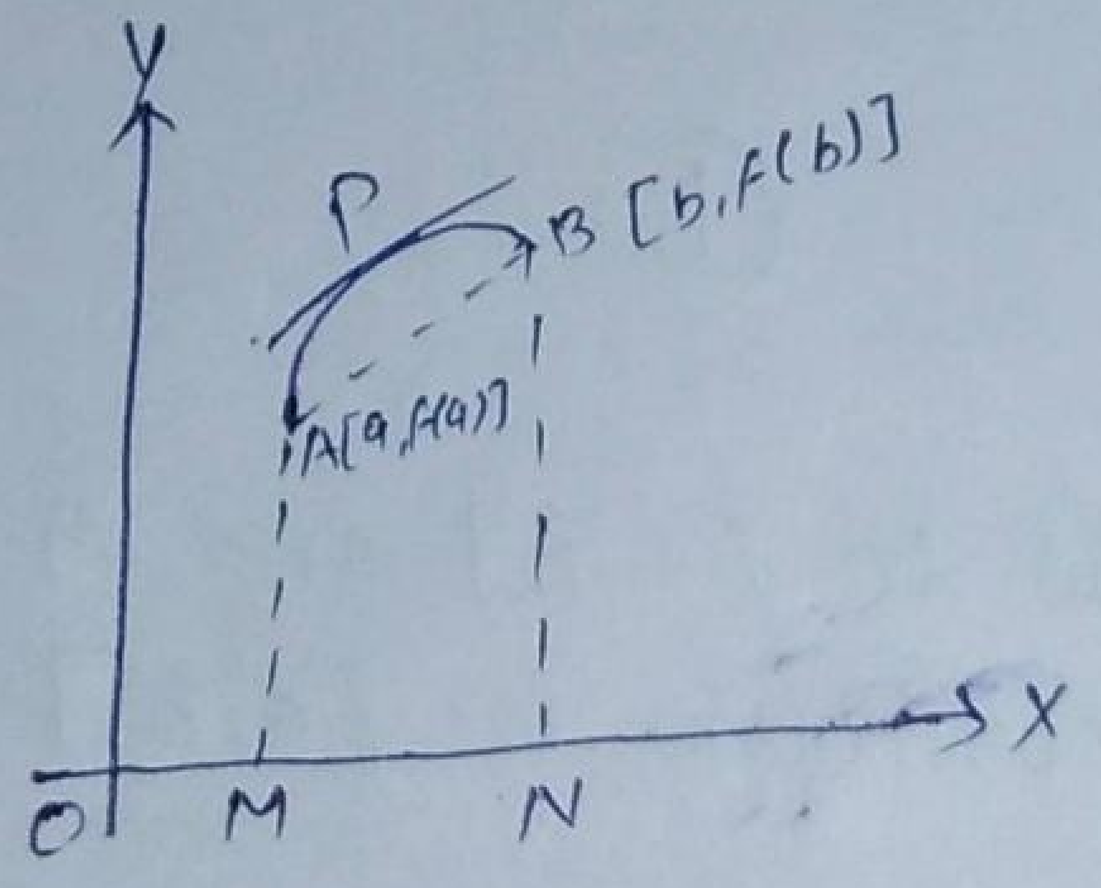
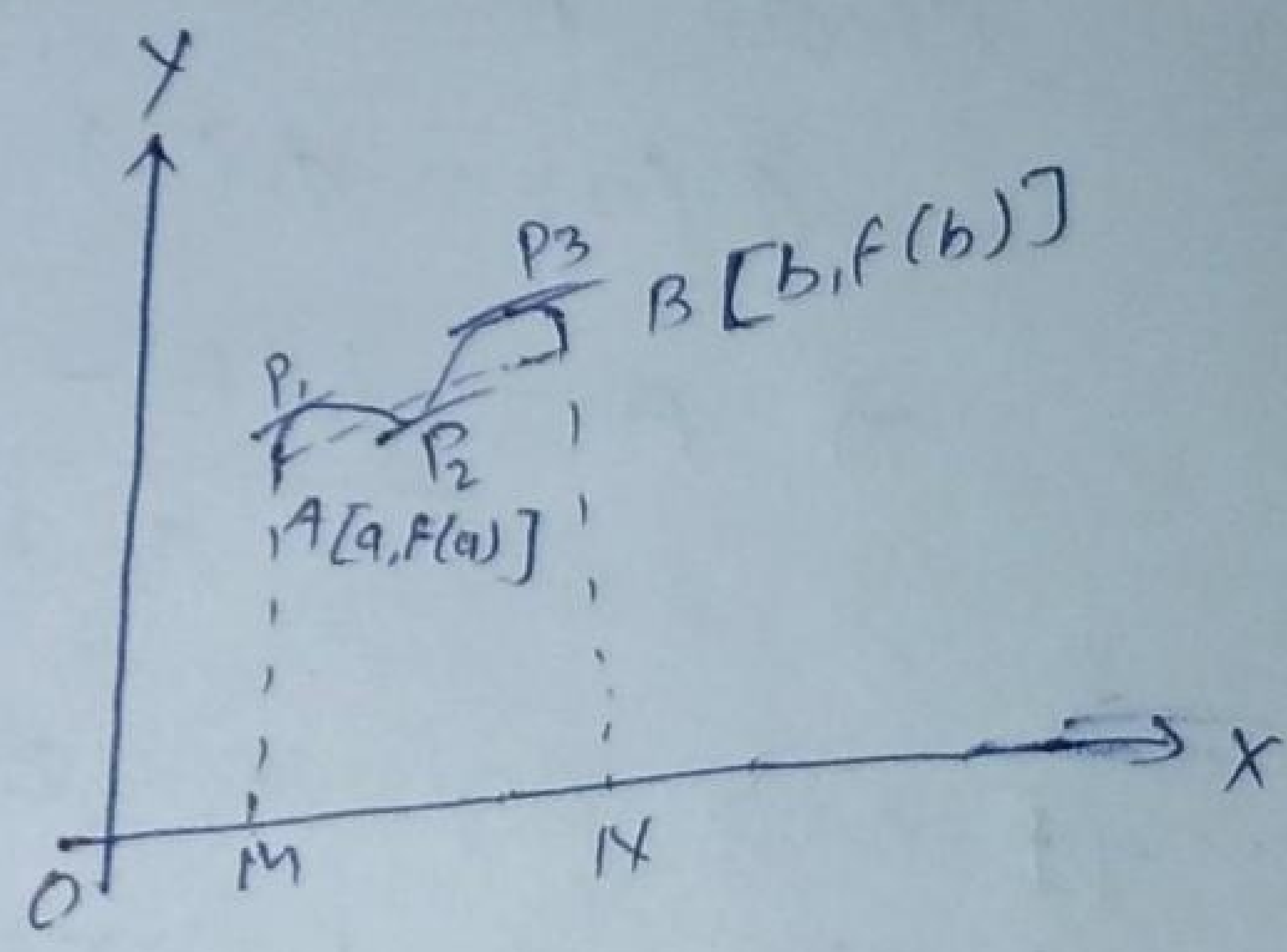


Fig (i)



$\therefore$ $(a, f(a))$, $(b, f(b))$ are two points on the graph of $y=f(x)$. Let these points be A and B.

$\because$ $f(x)$ is cts in $a \le x \le b$

$\because$ $f(x)$ is cts for $a \le x \le b$.

$\therefore$ Graph of $f(x)$ is cts for $a \le x \le b$.

$\because$ $f(x)$ is derivable for $a < x < b$.

$\therefore$ Tangent exists at each point of the graph for $a < x < b$.

$\therefore$ There exists at least one point P on graph between the points A and B such that the tangent P is parallel to the chord AB. If C is abscissa of P. then

$$f'(c) = \text{slope of chord } AB = \frac{f(b)-f(a)}{b-a}$$

Note :- Lagrange's Mean Value theorem is also known as first mean value theorem or mean value theorem or law of Mean.

Proof: Let $\phi(x) = f(x) + Ax$

Choose A such that

$$\phi(a) = \phi(b)$$

$$\Rightarrow A = \frac{f(b)-f(a)}{a-b} \quad \text{---} \quad (1)$$

(1) Since $f(x)$ is continuous in $[a,b]$ and Ax being a polynomial is continuous in $[a,b]$

$\therefore$ Sum of these two i.e.

$\phi(x)$ is also continuous for all $x \in [a,b]$

(2) ~~Since f(x) is differe~~

(2.) $\phi'(x) = f'(x) + A$

$f'(x)$ exists $\forall x \in (a,b)$

$\therefore \phi'(x)$ also exists $\forall x \in (a,b)$

Hence $\phi(x)$ is differentiable for all $x \in (a,b)$

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$\therefore$ ϕ satisfies all properties of Rolle's theorem

Hence there exists atleast one $c \in (a,b)$ such that

$$\phi'(c) = 0$$

$$\Rightarrow f'(c) + A = 0$$

$$\Rightarrow f'(c) = -A$$

$$\Rightarrow f'(c) = -\left[\frac{f(b)-f(a)}{a-b}\right]$$

$$= \frac{f(b)-f(a)}{b-a}$$

CAUCHY'S MEAN VALUE THEOREM

Let f & g be two functions such thats.

(1) f & g are continuous for all $x \in [a,b]$

(2) f & g are differentiable $\forall\, x \in (a,b)$

(3) $g'(x) \neq 0$ for any $x \in (a,b)$

then there exists at least one real number $c \in (a,b)$ such that.

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Proof: Let $\phi(x) = f(x) + Ag(x)$ where A is to be chosen in such a way that $\phi(a) = \phi(b)$

$\Rightarrow f(a) + Ag(a) = f(b) + Ag(b)$

$$\Rightarrow A = \frac{f(b) - f(a)}{g(a) - g(b)}$$

① Since $f(x)$ & $g(x)$ are continuous $\forall\ x \in [a,b]$

$\therefore$ Their sum i.e. $\phi(x)$ is also continuous $\forall\ x \in [a,b]$

(2) $\phi'(x) = f'(x) + Ag'(x)$ exists $\forall\ x \in (a,b)$ because $f'(x)$ & $g'(x)$ exist $\forall\ x \in (a,b)$

$\therefore$ ϕ is differentiable $\forall\ x \in (a,b)$

$\therefore$ ϕ satisfies all properties of Rolle's theorem

$\therefore$ $\exists$ atleast one $c \in (a,b)$ such that

$\phi'(c) = 0$ i.e $f'(c) + Ag'(c) = 0$

$$\Rightarrow \frac{f'(c)}{g'(c)} = -A = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Q. Verify Lagrange's theorem for $f(x) = \log x$ in $[1, e]$

Solⁿ:- ① $\log x$ is continuous for all $x \in [1, e]$

② $f'(x) = \frac{1}{x}$ exists for all $x \in (1, e)$

$\therefore$ 'f' satisfies all conditions of Lagrange's mean value theorem

Now $f'(c) = \dfrac{f(e) - f(1)}{(e-1)}$

$\Rightarrow \dfrac{1}{c} = \dfrac{\log e - \log 1}{(e-1)}$

$\boxed{\begin{aligned} &2.7 - 1 \\ &= 1.7 \end{aligned}}$

$\Rightarrow \dfrac{1}{c} = \dfrac{1-0}{e-1} \Rightarrow c = (e-1)$
$\in (1, e)$

$\therefore$ we have found such a point $c \in (1, e)$ such that

$$f'(c) = \frac{f(e) - f(1)}{(e-1)}$$

Hence Lagrange's theorem is verified.

Q. Check the applicability of Lagrange's mean value theorem for $x^{2/3}$ in $[-8, 8]$

Solⁿ:- ① $f(x) = x^{2/3}$ is an algebraic function and continuous in $[-8, 8]$

② $f'(x) = \dfrac{2}{3x^{1/3}}$ doesnot exist at $x = 0 \in (-8, 8)$

$\therefore$ Lagrange's mean value theorem is not applicable to given function.

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Q. Verify Cauchy's mean value theorem for the functions $f(x) = \log x$ and $g(x) = 1/x$ in the interval $[1, e]$.

Soln: $f(x) = \log x$ and $g(x) = 1/x$

$\therefore f'(x) = \frac{1}{x}$ and $g'(x) = -1/x^2$

Now f and g are two functions such that

(i) Both are continuous in $[1, e]$

(ii) Both are derivable in $(1, e)$

(iii) $g'(x) \neq 0$ for any x in $(1, e)$

$\therefore$ $f(x)$ and $g(x)$ satisfy all the conditions of Cauchy's mean value theorem in $[1, e]$

$\therefore$ by Cauchy's mean value theorem.

$$\frac{f(e) - f(1)}{g(e) - g(1)} = \frac{f'(c)}{g'(c)} \quad \text{where } 1 < c < e.$$

$$\frac{\log e - \log 1}{\frac{1}{e} - \frac{1}{1}} = \frac{1/c}{-1/c^2}$$

$$\frac{1 - 0}{\frac{1 - e}{e}} = -c$$

$$\therefore c = \frac{e}{e-1}$$

$$\left[\because c - 1 = \frac{e}{e-1} - 1 = \frac{1}{e-1} > 0 \qquad e - c = e - \frac{e}{e-1} = \frac{e(e-2)}{e-1} > 0 \right]$$

Clearly $c \in (1, e)$

Hence the theorem is verified.

TAYLOR'S THEOREM

Let f be a function such that

① $f, f', f'', \dots, f^{(n-1)}$ are continuous in $[a,b]$

② f^n exists $\forall x \in (a,b)$

Then there exists a point $c \in (a,b)$ such that

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \dots + \frac{(b-a)^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{(b-a)^n}{n!}f^n(c)$$

Lagrange's remainder after n terms.

Another form

$$f(b) = f(a) + hf'(a) + \frac{(h)^2}{2!}f''(a) + \dots + \frac{(h)}{(n-1)!}f^{n-1}(a) + \frac{(h)^n}{n!}f^n(a+\theta h)$$

Lagrange's remainder after n terms

Q. Expand $\tan x$ in the power of $(x - \pi/4)$

Soln: Here $f(x) = \tan x$

Here $a = \pi/4$, $b = x$

$f(a) = f(\pi/4) = \tan \pi/4 = 1$

$f'(x) = \sec^2 x$

$f'(a) = f'(\pi/4) = \sec^2 \pi/4$

$= 1 + \tan^2 \pi/4 = 2$

$f''(x) = 2\sec x \cdot \sec x \tan x$

$= 2\sec^2 x \tan x$

$f''(a) = 2\sec^2 \pi/4 \tan \pi/4$

$= 2(2)(1) = 4$

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By Taylor series.

$$f(b) = f(a) + (b-a)\, f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots$$

$$\Rightarrow \tan x = 1 + (x - \pi/4)\,(2) + \frac{(x-\pi/4)^2}{2!}\,(4) + \cdots$$

Q. $f(x) = x^3 - 2x + 5$. find approx value of $f(2.001)$

Soln: Here $f(x) = x^3 - 2x + 5$

let $a = 2$, $b = 2.001$; $h = b - a$
$= 0.001$

$f'(x) = 3x^2 - 2$
$f''(x) = 6x$
$f'''(x) = 6$
$f^{iv}(x) = 0$

$f(2) = (2)^3 - 2(2) + 5$
$= 9$

$f'(2) = 3(2)^2 - 2$
$= 10$

$f''(2) = 6(2) = 12$

$f'''(2) = 6$

By Taylor series

$$f(b) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots$$

$$\Rightarrow f(2.001) = 9 + (0.001)(10) + \frac{(0.001)^2}{2!}(12)$$

$$= \frac{(0.001)^2}{3!}(6)$$

$$\boxed{= 9.0100600!}$$

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V. IMP

Q.1. Verify Rolle's theorem for $f(x) = x(x-2)e^{3x/4}$ in $[0,2]$

Soln:- (1) continuous.

(2) Differentiable

$$f'(x) = (x-2)e^{3x/4} + xe^{3x/4} + x(x-2)\frac{3}{4}e^{3x/4}$$

(3.) $f(0) = 0, \quad f(2) = 0$

$c \in (0,2)$ such that $f'(c) = 0$

$\Rightarrow c = -2, 1.3$

$c = 1.3 \in (0,2)$

There for Rolle's theorem is verified.

Q. Suppose f is differentiable in $[0,1]$ & its derivative is never zero in $[0,1]$ using mean value theorem show that

$f(0) \neq f(1)$

(1) Continuity in $[a,b]$

(2) Diff (a,b)

Soln:- (1) Since it is given that f is differentiable ~~in for all x [0,1]~~ in $[0,1]$

$\Rightarrow f$ is continuous for all $x \in [0,1]$

(2) Given f is differentiable in $(0,1)$

$\therefore f$ satisfies all the conditions of Lagrange's mean Value Theorem

$\therefore \exists$ a point $c \in (0,1)$ such that

$$f'(c) = \frac{f(1) - f(0)}{(1-c)}$$

$\Rightarrow f'(c) = f(1) - f(0)$

$\Rightarrow f'(c) + f(0) = f(1)$

$\Rightarrow f(0) \neq f(1)$

($\because$ given that derivative of f is never zero in (0,1))

Q.2. Discuss the applicability of Rolle's theorem to the function $f(x) = 1 + |x-2|$ in $[0,2]$

Soln:- $f'(x) = 0 + \frac{(x-2)}{|x-2|}$

doesnot exist at $x = 2 \in (0,2)$ doesnot applicable.

$\therefore$ Rolle's theorem is not applicable.

Q. Prove that between two roots of $e^x \sin x = 1$, there exists a root of $e^x \cos x + 1 = 0$

Soln:- Let $f(x) = e^x \sin x - 1$

It is a composite function of e^x & $\sin x$

i.e two continuous function

$\therefore$ f is continuous for all x. & hence in $[\alpha, \beta]$

let α and β be two roots of $e^x \sin x = 1$

$f'(x) = e^x \sin x + e^x \cos x$.

$= 1 + e^x \cos x$.

It exists $\forall x \in (\alpha, \beta)$

$\therefore f(x)$ is differentiable $\forall x \in (\alpha, \beta)$

Since α & β are roots of $f(x)$

$\therefore f(\alpha) = 0$ & $f(\beta) = 0$

Hence $f(\alpha) = f(\beta)$

$\therefore f$ satisfies all conditions of Rolle's theorem.

$\therefore \exists$ atleast one real number $c \in (\alpha, \beta)$ such that

$f'(c) = 0$

i.e. $1 + e^c \cos c = 0$

$\therefore c$ is a root of

$1 + e^x \cos x = 0$ lying in (α, β)

$\therefore \exists$ a root of $1 + e^x \cos x = 0$ laying between roots of $e^x \sin x = 1$

Q. Discuss the applicability of Rolle's theorem, L.M.V.T for the function

$$f(x) = \begin{cases} x^2+1, & 0 \le x \le 1 \\ 3-x, & 1 < x \le 2 \end{cases}$$

Solⁿ:- Since in $[0,1]$ $f(x)$ is a Polynomial.

It is continuous in $[0,1]$ as well as diff in $(0,1)$

Same is the case for $[1,2]$

Continuity at 1 L.HL $\lim_{x\to 1^-} f(x)$

$= \lim_{x\to 1^-} (x^2+1) = 2.$

R.HL $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 3-x = 2$

$f(1) = (1)^2+1 = 2$

$\therefore$ f is continuous at 1

$$\boxed{\begin{aligned} &\operatorname{Lt}_{x\to a^-} f(x) \\ &= \operatorname{Lt}_{h\to 0} f(a-h) \end{aligned}}$$

Differentiable at 1

LHL $\lim_{x\to 1^-} \dfrac{f(x)-f(1)}{(x-1)}$

$= \lim_{x\to 1^-} \dfrac{x^2+1-2}{x-1} = \lim_{x\to 1^-} \dfrac{(x^2-1)}{(x-1)}$

$= \lim_{h\to 0} \dfrac{(1-h)^2-1}{1-h-1} = \lim_{h\to 0} \dfrac{h^2-2h}{(-h)}$

$= 2$

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RHL $\lim_{x\to 1^+} \frac{f(x)-f(1)}{(x-1)}$

$= \lim_{x\to 1^+} \frac{(3-x)-2}{x-1} = \lim_{x\to 1^+} \frac{(1-x)}{(x-1)}$

$= \lim_{h\to 0} \frac{1-(1+h)}{(1+h)-1} = -1$

$\therefore$ L.H.L $\neq$ R.H.L. Hence f is not differentiable at 1.

$\therefore$ Rolle's theorem and L.M.V.T are not applicable to the given question.

Q. If $f(x), g(x), h(x)$ have derivatives when $a \le x \le b$ such that show that $\exists\, c \in (a,b)$

$$\begin{vmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f'(c) & g'(c) & h'(c) \end{vmatrix} = 0$$

Soln:- Let $F(x) = \begin{vmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f'(x) & g'(x) & h(x) \end{vmatrix}$

$= Af(x) + Bf(x) + C\,h(x)$

F is continuous in $[a,b]$

Given that f, g, h are differentiable in (a,b)

$\therefore$ There sum F is differentiable in (a,b) and $\therefore$ continuous.

$F(a) = 0$
$F(b) = 0$

$\therefore$ F satisfies all condition of Rolle's theorem

Q. Calculate the approximate value of $\sqrt{17}$ to four decimal places by taking first there terms of a Taylor's Expansion.

Soln:- Let $f(x+h) = \sqrt{x+h}$

Putting $h=0$, we get $f(x) = \sqrt{x}$

$\therefore f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2} x^{-1/2}, \quad f''(x) = -\frac{1}{4} x^{-3/2} \ldots$

By Taylor's Expansions.

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \cdots$$

$$\therefore \sqrt{x+h} = \sqrt{x} + \frac{h}{2\sqrt{x}} - \frac{h^2}{8x\sqrt{x}} + \cdots$$

Put $x = 16$, $h = 1$

$$\therefore \sqrt{17} = 4 + \frac{1}{2 \times 4} - \frac{1}{8 \times 16 \times 4} + \cdots = 4 + \frac{1}{8} - \frac{1}{512} + \cdots$$

$$= 4 + 0.125 - 0.00195 + \cdots$$

$$= 4.12305 \text{ nearly}$$

$$= 4.1231 \text{ nearly}$$

Ans.

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Maclaurin's Theorem

If $f(x)$ be a function of the variable x such that it can be expanded in ascending power of x and this expansion be differentiable any number of times then

$$\boxed{f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \cdots \frac{x^n}{n!} f^n(0) + \cdots}$$

Proof $f(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4$

$\boxed{f(0) = A_0} \Rightarrow A_0 = f(0)$

$f'(x) = A_1 + 2A_2 x + 3A_3 x^2 + 4A_4 x^3$

$\boxed{f'(0) = A_1}$ $\boxed{A_1 = f'(0)}$

$f''(x) = 2A_2 + 6A_3 x + 12 A_4 x^2$

$\boxed{f''(0) = 2A_2} \Rightarrow \boxed{A_2 = \frac{f''(0)}{2!}}$

$f'''(x) = 6A_3 + 24 A_4 x + \cdots$

$\boxed{f'''(0) = 6A_3} \Rightarrow \boxed{A_3 = \frac{f'''(0)}{3!}}$

$$f(x) = f(0) + \frac{x f'(0)}{1!} + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

Q. Expand $\log(1+x)$ in terms of infinite series.

Soln:- Let $f(x) = \log(1+x)$

Expand it with respect two Maclaurin's series.

	$f(0) = \log(1+0) = 0$
$f'(x) = \frac{1}{1+x}$	$f'(0) = \frac{1}{1+0} = 1$
$f''(x) = \frac{-1}{(1+x)^2}$	$f''(0) = \frac{-1}{(1+0)^2} = -1$

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$f'''(x) = \frac{2}{(1+x)^3}$

$f^{IV} = \frac{-6}{(1+x)^4}$

$f'''(0) = 2!$

$f^{IV}(0) = -3!$

$\log(1+x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$

$= 0 + x + (-1)\frac{x^2}{2} + \frac{(2!)x^3}{3!} \dots$

$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots$

Another forms to solve.

Soln $y = \log(1+x)$ $\quad y(0) = 0$

$y_1 = \frac{1}{1+x}$ $\quad y_1(0) = 1$

$y_2 = -\frac{1}{(1+x)^2}$ $\quad y_2(0) = -1$

$y_3 = \frac{2}{(1+x)^3}$ $\quad y_3(0) = 2$

$y_4 = \frac{-6}{(1+x)^4}$ $\quad y_4(0) = -6$

$y = y(0) + \frac{x}{1!} y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0) + \dots$

$\log(1+x) = \frac{x}{1!} - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{6x^4}{4!} \dots$

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Q. using Taylor's theorem, prove $\cos x \geq 1 - \frac{x^2}{2!}$ $\forall x \in R$

Soln:- Case I - When $x = 0$

L.H.S. $\cos 0 = 1$

R.H.S $1 - \frac{(0)^2}{2!} = 1$

LHS $\geq$ RHS

$\therefore$ Result is true for $x = 0$

Case-II $x > 0$ By maclaurin's theorem upto second derivative

$$f(x) = f(0) + x f'(x) + \frac{x^2}{2!} f''(\theta x) \text{ ——— } (*)$$

Take $f(x) = \cos x$

$f(0) = \cos 0 = 1$

$f'(x) = -\sin x$ | $f'(0) = -\sin 0 = 0$

$f''(x) = -\cos x$.

put in (*)

$$\cos x = 1 + 0 + \frac{x^2}{2!}(-\cos \theta x)$$

$$\Rightarrow \cos x = 1 - \frac{x^2}{2!} \cos \theta x \text{ - - - } (1)$$

$$\cos \theta x \leq 1$$

$\therefore$ For $x > 0$ ~~$\Rightarrow \left(1 - \frac{x^2}{2!} \cos \theta x\right) \geq$~~

$$\frac{x^2}{2!} \cos \theta x \leq \frac{x^2}{2!}$$

$$\Rightarrow -\frac{x^2}{2!} \cos \theta x \geq -\frac{x^2}{2!}$$

$\therefore$ (1) becomes

$$\cos x = \left(1 - \frac{x^2}{2!} \cos \theta x\right) \geq \left(1 - \frac{x^2}{2!}\right)$$

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$\Rightarrow \cos x \geq \left(1-\frac{x^2}{2!}\right)$

$\therefore$ Result has been proved for $x>0$

Case III - where $x<0$

let $x=-y$, where y is positive.

since y is a +ve no; $\therefore$ from case 2

$$\cos y \geq 1-\frac{y^2}{2!}$$

$$\Rightarrow \cos(-x) \geq 1-\frac{(-x)^2}{2!}$$

$$\Rightarrow \cos x > 1-\frac{x^2}{2!} \quad ; \quad x<0$$

$\therefore$ Result is true for $x<0$

Hence result is true for $\forall\, x \in R$.

Q: Prove $|\sin x - \sin y| \leq |x-y| \;\forall\, x, y \in R$

Soln: Let $f(t) = \sin t$

Take $a=y$, $b=x$

Case I - when $x \neq y$.

$$\frac{|\sin x - \sin y|}{|x-y|}$$

$$\frac{f(b)-f(a)}{b-a}$$

(1) $f(t) = \sin t$ is continuous $\forall\, t \in R$ &
hence $\forall\, t \in [y, x]$

(2) $f'(t) = \cos t$ exists $\forall\, t \in (y, x)$

$\therefore f$ is differentiable $\forall\, t \in (y, x)$

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Hence f satisfies all the conditions of L.M.V.T

$\therefore \exists$ atleast one $c \in (y, x)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow \cos c = \frac{\sin x - \sin y}{x - y}$$

$$\Rightarrow |\cos c| = \left| \frac{\sin x - \sin y}{x - y} \right|$$

$$\Rightarrow 1 \geq \frac{|\sin x - \sin y|}{|x - y|}$$

$$\Rightarrow |\sin x - \sin y| \leq |x - y| \; \forall \; x, y \in R$$

Case II - When $x = y$

L.H.S $\quad |\sin x - \sin y|$

$= |\sin x - \sin x| = 0$

R.H.S : $\quad |x - y|$

$= |x - x| = 0$

$\therefore$ L.H.S $\leq$ RHS

Hence result is also true $x = y$

$\therefore$ From case I and II ; result is true for all $x, y \in R$

Q. Use Lagrange's mean value theorem the function $f(x) = \log(1+x)$ to show that $0 < [\log(1+x)]^{-1} - x^{-1} < 1 \ \forall x > 0$

Soln:- Here $f(t) = \log(1+t)$

Let $[a, b] = [0, x]$

(1) $\log(1+t)$ is continuous $\forall\ t \in [0, x]$

(2) $f'(t) = \dfrac{1}{(1+t)}$ exists $\forall\ t \in (0, x)$

$\therefore$ f is differentiable $\forall\ t \in (0, x)$

$\therefore$ By L.M.V.T, $\exists\ \theta x \in (0, x)$

such that.

$$f'(\theta x) = \frac{f(x) - f(0)}{x - 0}$$

$$\Rightarrow \frac{1}{(1+\theta x)} = \frac{\log(1+x) - \log(1+0)}{x - 0}$$

$$\Rightarrow \frac{1}{(1+\theta x)} = \frac{\log(1+x)}{x}$$

$$\Rightarrow \frac{x}{(1+\theta x)} = \log(1+x) \ \text{---} \ (1)$$

Given $x > 0$ & $0 < \theta < 1$

$\Rightarrow 0 < \theta x < x$

$\Rightarrow 1 < 1 + \theta x < 1 + x$

$\Rightarrow 1 > \dfrac{1}{1+\theta x} > \dfrac{1}{1+x}$

$\Rightarrow x > \frac{x}{1+\theta x} > \frac{x}{1+x}$

$\Rightarrow x > \log(1+x) > \frac{x}{1+x}$ (using ①)

$\Rightarrow \frac{1}{x} < \frac{1}{\log(1+x)} < \frac{1+x}{x}$

$\Rightarrow \frac{1}{x} < \frac{1}{\log(1+x)} < \frac{1}{x} + 1$

$\Rightarrow 0 < \frac{1}{\log(1+x)} - \frac{1}{x} < 1$ (by subtracting $\frac{1}{x}$)

<u>Q.</u> Prove $\frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta$ where $0 < \alpha < \theta < \beta < \pi/2$

Solⁿ:- Let $f(x) = \sin x$.
$g(x) = \cos x$.

$f(x)$ & $g(x)$ are continuous everywhere & hence $\forall x \in [\alpha, \beta]$

$f'(x) = \cos x$ and $g'(x) = -\sin x$ exist $\forall x \in (\alpha, \beta)$

$\therefore f$ & g are differentiable in (α, β)

Now $g'(x) = -\sin x \neq 0 \ \forall x \in (\alpha, \beta)$ where $(\alpha, \beta) \subset [0, \pi/2]$

Hence cauch's Mean Value theorem is applicable to the given two functions

$\therefore \exists$ a real numbes $\theta \in (\alpha, \beta)$ such that

$$\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} = \frac{f'(\theta)}{g'(\theta)}$$

$$\Rightarrow \frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} = \frac{\cos\theta}{-\sin\theta}$$

$$\Rightarrow \frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta$$