Module-IV

Linear Algebra

Question Bank

2 Marks

Q. Define symmetric matrix. (IKGPTU Dec. 2019 and Nov. 2018)

Sol. **Symmetric matrix:** A square matrix is called symmetric matrix if $A = A^T$

i.e.$a_{ij} = a_{ji}$

e.g.$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 7 \end{bmatrix}$

Q. Define Skew Symmetric matrix

Sol. **Skew-Symmetric matrix:** A square matrix is called symmetric matrix if $A = -A^T$

i.e. $a_{ij} = -a_{ji}$. The diagonal elements of a skew-symmetric matrix are zero because $a_{ii} = -a_{ii}$ if and only if $a_{ii} = 0$

e.g. $\begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}$

Q. Define Orthogonal matrix

Sol. **Orthogonal matrix:** A square matrix A is said to be orthogonal if

$$AA^T = A^TA = I.$$

$$\text{e.g. } A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Q. Define Conjugate matrix

Sol. **Conjugate Matrix:**

A matrix $\bar{A}$ obtained by replacing all the elements of matrix A by their

conjugate numbers is called conjugate matrix of A.

$$\text{e.g. } A = \begin{bmatrix} 1+3i & 2 & 2-5i \\ 2 & 1 & -2+4i \\ 2+6i & -2 & 1+i \end{bmatrix}$$

Then

$$\bar{A} = \begin{bmatrix} 1-3i & 2 & 2+5i \\ 2 & 1 & -2-4i \\ 2-6i & -2 & 1-i \end{bmatrix}$$

Q. Define Conjugate matrix

Sol. **Tranjugate Matrix:** Transpose of conjugate matrix is called tranjugate matrix.

$$A^{\theta} \quad or \quad A^{*} = (\bar{A})' = \begin{bmatrix} 1-3i & 2 & 2-6i \\ 2 & 1 & -2 \\ 2+5i & -2-4i & 1-i \end{bmatrix}$$

Q. Define Unitary matrix

Sol. **Unitary matrix:** A square matrix A is said to be Unitary if

$$A^{\theta}A = AA^{\theta} = I$$

where $A^{\theta} = \left(\overline{A}\right)^{T}$

e.g. $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

Note: Every orthogonal matrix is unitary.

Q. Define Hermitian matrix

Sol.

Hermitian matrix: A square matrix A is said to be Hermitian matrix if

$A^{\theta} = A$ i.e. $a_{ij} = \overline{a_{ji}}$

Diagonal elements of a Hermitian matrix are real numbers.

e.g. $A = \begin{bmatrix} 1 & 2+3i & 5-6i \\ 2-3i & 2 & 9-6i \\ 5+6i & 9+6i & -11 \end{bmatrix}$

Q. Define Skew-Hermitian matrix

Sol.

Skew Hermitian matrix: A square matrix A is said to be skew Hermitian matrix if

$A^{\theta} = -A$ i.e. $a_{ij} = -\overline{a_{ji}}$

Diagonal elements of a skew Hermitian matrix are either zero or purely imaginary numbers.

$$\text{e.g. } A = \begin{bmatrix} 1 & 2+3i & -5-6i \\ -2+3i & 2 & -9+6i \\ 5-6i & 9+6i & -11 \end{bmatrix}$$

Q. Prove $\left(A^{\theta}\right)^{\theta} = A$

Proof: $\left(A^{\theta}\right)^{\theta} = \overline{(\bar{A}')'} = \overline{\left((A')'\right)} = \overline{(\bar{A})} = A$

Q. Prove that $A + A^{\theta}$ is a Hermitian matrix.

Proof: Now

$$\left(A + A^{\theta}\right)^{\theta} = A^{\theta} + \left(A^{\theta}\right)^{\theta} = A^{\theta} + A = A + A^{\theta}$$

Hence $A + A^{\theta}$ is a Hermitian matrix.

Q. Prove that $A - A^{\theta}$ is a Skew-Hermitian matrix.

Proof: Now

$$\left(A - A^{\theta}\right)^{\theta} = A^{\theta} - \left(A^{\theta}\right)^{\theta} = A^{\theta} - A = -\left(A - A^{\theta}\right)$$

Hence $A - A^{\theta}$ is a Skew-Hermitian matrix.

Q. Define Similar matrices

Sol. **Similar matrices:** A square matrix A is said to be similar to a square matrix B if there exists an invertible matrix P such that $\mathrm{A} = \mathrm{P}^{-1}\mathrm{BP}$. P is called similarity matrix. This relation of similarity is a symmetric relation.

Q. State Cayley Hamilton theorem

Sol. **Cayley Hamilton theorem:** Every square matrix satisfies its own characteristic equation.

Q. Define characteristic equation.

Sol. Let A be a square matrix. Then the equation

determinant $(\mathrm{A} - \alpha\mathrm{I})$ =0 is called characteristic equation of A.

Q. What are Eigen Values and Eigen Vectors?

Sol. **Eigen values and Eigen Vectors:** The roots of characteristic equation of A are called Eigen values or latent roots of matrix A.

A column vector X satisfying the equation $\mathrm{AX} = \alpha\mathrm{X}$ i.e.

$(A - \alpha I)X = 0$ is called Eigen vector or latent vector of matrix A corresponding to eigen value α.

Q. Write a short note on diagonalizable matrix.

Sol.

Diagonalizable matrix: A square matrix A is said to be diagonalizable if there exists an invertible matrix P such that

$$P^{-1}BP = D$$

Where D is a diagonal matrix and the diagonal elements of D are Eigen values of A.

Q. Write two properties of Eigen Values.

Sol. . Two properties of Eigen Values are:

(i) Atleast one Eigen Value of every singular matrix is zero.
(ii) A square matrix 'A' and its transpose have the same set of Eigen values.

Q. Show that if zero is an Eigen Value of a matrix then it is singular. (IKGPTU Nov. 2018)

Sol. Let A be a square matrix whose one of the Eigen value is zero.

Now Characteristic equation of A is $|A - \alpha I| = 0$

Now if Eigen Value i.e. $\alpha = 0$, then we get

$$|A - (0)I| = 0 \xrightarrow{yields} |A| = 0$$

Hence A is a singular matrix.

Q. What is product of Eigen values of a matrix A equal to?

Sol. The product of Eigen values of a matrix A is equal to determinant of A

Q. What is sum of Eigen values of a matrix A equal to?

Sol. The sum of Eigen values of a matrix A is equal to trace of A i.e. equal to sum of diagonal elements of A.

Q. Find sum and product of latent roots of $\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ (IKGPTU May 2019)

Sol. Let $A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$

Trace of given matrix A is = sum of diagonal matrix = 2+3 = 5

Since sum of latent roots of a matrix is equal to trace of that matrix, therefore

sum of latent roots given matrix A = 5

Now $|A| = 6 - 2 = 4$

Since product of latent roots of a matrix is equal to determinant of that matrix, therefore

product of latent roots given matrix A = 4

Q. The characteristics equation of a matrix A is $t^2 - t - 1 = 0$, then determine A^{-1}.

Sol. By Cayley Hamilton theorem, every square matrix satisfies its characteristic equation.

Therefore $A^2 - A - 1 = 0$

or $A^2 - A = 1$

Premultipying both sides by A

$$A-I=A^{-1}$$

Q. Prove that product of two orthogonal matrices is orthogonal matrix

Sol. Let A and B be two orthogonal matrices. Therefore

$AA^T = A^TA = I$ and $BB^T = B^TB = I$

Now $(AB)(AB)^T = ABB^TA^T = AIA^T = AA^T = I$ and

$(AB)^T(AB) = B^TA^T\,AB = BIB^T = BB^T = I$

Hence AB is an orthogonal matrix. Therefore product of two orthogonal matrices is orthogonal

matrix.

Q. Prove that transpose of an orthogonal matrix is orthogonal matrix.

Sol. Let A be orthogonal matrix. Therefore

$AA^T = A^TA = I$

Now $A^T(A^T)^T = A^TA = I$ and

$(A^T)^TA^T = AA^T = I$

Hence A^T is an orthogonal matrix

Therefore transpose of an orthogonal matrix is orthogonal matrix.

Q. Prove that inverse of an orthogonal matrix is an orthogonal matrix.

Sol. Let A be orthogonal matrix. Therefore

$AA^T = A^TA = I$

Now $A^{-1}(A^{-1})^T = A^{-1}(A^T)^{-1} = (A^T A)^{-1} = I^{-1} = I$
and

$(A^{-1})^T A^{-1} = (A^T)^{-1} A^{-1} = (AA^T)^{-1} = I^{-1} = I$

Hence A^{-1} is an orthogonal matrix

Therefore inverse of an orthogonal matrix is orthogonal matrix.

Q. Prove that determinant of an orthogonal matrix is ± 1.
(IKGPTU Nov. 2018)

Sol. Let A be orthogonal matrix. Therefore

$AA^T = A^T A = I$

Taking determinant on both sides

$|AA^T| = |I| \xrightarrow{yields} |A||A^T| = 1 \xrightarrow{yields} |A||A| = 1 \xrightarrow{yields} |A|^2 = 1 \xrightarrow{yields} |A| = \pm 1$

(Because $|CD| = |C||D|, \quad |I| = 1, \quad |A| = |A^T|$)

Q. Find characteristic equation of $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ (IKGPTU Dec. 2019)

Sol. $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$

Characteristic equation of A is $|A - \alpha I| = 0 \xrightarrow{yields} \begin{vmatrix} 1-\alpha & 2 \\ 3 & 1-\alpha \end{vmatrix} = 0$

$\xrightarrow{yields} \alpha^2 - 2\alpha - 5 = 0 \xrightarrow{yields} \alpha = \frac{2 \pm \sqrt{4+20}}{2} = \frac{2 \pm \sqrt{24}}{2} = \frac{2 \pm 2\sqrt{6}}{2}$

$\xrightarrow{yields} \alpha = 1 \pm \sqrt{6}$ *are the Eigen values of given matrix.*

Q. Find characteristic equation of $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Sol. $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Characteristic equation of A is $|A - \alpha I| = 0$

$$\xrightarrow{yields} \begin{vmatrix} 1-\alpha & 0 & -1 \\ 1 & 2-\alpha & 1 \\ 2 & 2 & 3-\alpha \end{vmatrix} = 0$$

$$\xrightarrow{yields} \alpha^3 - 6\alpha^2 + 11\alpha - 6 = 0$$

Q. Is $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ diagonalizable?

Sol. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

Characteristic equation of A is $|A - \alpha I| = 0$

$$\xrightarrow{yields} \begin{vmatrix} 1-\alpha & 0 & 0 \\ 0 & 3-\alpha & -1 \\ 0 & -1 & 3-\alpha \end{vmatrix} = 0$$

$$\xrightarrow{yields} \alpha^3 - 7\alpha^2 + 14\alpha - 8 = 0 \xrightarrow{yields} \alpha = 1,2,4$$

Since A has three distinct Eigen values, $\therefore$ it has three linearly independent Eigen vectors. Hence A

A is diagonalizable.

4 Marks Questions

1. Define orthogonal matrix. Is the matrix $\begin{bmatrix} 4 & 2 & 1 \\ 6 & 3 & 4 \\ 2 & 1 & 0 \end{bmatrix}$ orthogonal?

Sol. **Orthogonal matrix:** A square matrix A is said to be orthogonal if

$$AA^T = A^T A = I.$$

Let $A = \begin{bmatrix} 4 & 2 & 1 \\ 6 & 3 & 4 \\ 2 & 1 & 0 \end{bmatrix}$

Then $A^T = \begin{bmatrix} 4 & 6 & 2 \\ 2 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}$

Now $AA^T = \begin{bmatrix} 4 & 2 & 1 \\ 6 & 3 & 4 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 4 & 6 & 2 \\ 2 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix} =$

$$\begin{bmatrix} 16+4+1 & 24+6+4 & 8+2 \\ 24+6+4 & 36+9+16 & 12+3 \\ 8+2 & 12+3 & 4+1 \end{bmatrix} = \begin{bmatrix} 21 & 34 & 10 \\ 34 & 61 & 15 \\ 10 & 15 & 5 \end{bmatrix} \neq I$$

Hence given matrix A is not an orthogonal matrix.

2. Prove eigen value of a Hermitian matrix is real.

Sol. Let A be a Hermitian matrix. Therefore $A^{\theta} = A - - - - - -(1)$

Let α be eigen value of A and X be the corresponding non-zero eigen vector. Then

$$AX = \alpha X \xrightarrow{yields} (AX)^{\theta} = (\alpha X)^{\theta} \xrightarrow{yields} X^{\theta} A^{\theta} = \bar{\alpha} X^{\theta}$$
$$\xrightarrow{yields} X^{\theta} A = \bar{\alpha} X^{\theta} \quad \text{(using (1))}$$

Post multiplying both sides by X, we get

$$X^{\theta}(AX) = \bar{\alpha}\left(X^{\theta} X\right) \xrightarrow{yields} X^{\theta}\ \alpha X = \bar{\alpha}\left(X^{\theta} X\right)$$
$$\xrightarrow{yields} \alpha\left(X^{\theta} X\right) = \bar{\alpha}\left(X^{\theta} X\right) \xrightarrow{yields} \alpha = \bar{\alpha}$$

Hence α is a real number. Therefore Eigen value of a Hermitian matrix is real.

3. Prove eigen value of a Skew-Hermitian matrix is purely imaginary.

Sol. Let A be a Skew-Hermitian matrix.

Therefore $A^{\theta} = -A - - - - - - -(1)$

Let α be eigen value of A and X be the corresponding non-zero eigen vector. Then

$$AX = \alpha X \xrightarrow{yields} (AX)^{\theta} = (\alpha X)^{\theta} \xrightarrow{yields} X^{\theta} A^{\theta} = \bar{\alpha} X^{\theta}$$

$\xrightarrow{yields} X^\theta(-A) = \bar{\alpha}X^\theta$ (using (1))

Post multiplying both sides by X, we get

$$X^\theta(-AX) = \bar{\alpha}\left(X^\theta X\right) \xrightarrow{yields} X^\theta\ (-\alpha X) = \bar{\alpha}\left(X^\theta X\right)$$

$\xrightarrow{yields} -\alpha\left(X^\theta X\right) = \bar{\alpha}\left(X^\theta X\right) \xrightarrow{yields} -\alpha = \bar{\alpha}$

Hence α is purely imaginary or zero.

(Because if $\alpha = x + iy \xrightarrow{yields} \bar{\alpha} = x - iy$

Now $-\alpha = \bar{\alpha} \xrightarrow{yields} -x - iy = x - iy$

Comparing real and imaginary parts, we get

$$-x = x \xrightarrow{yields} x = 0$$

And $y = y$

Hence α is either zero or purely imaginary)

Therefore Eigen value of a Hermitian matrix is purely imaginary or zero.

4. Prove $\frac{|A|}{\alpha}$ is an eigen value of $adj\ (A)$eigen vector remaining the same if α is an eigen value of A and X is corresponding Eigen vector.

Sol. Let A be a square matrix$------$ (1)

Let α be eigen value of A and X be the corresponding non-zero eigen vector. Then

$AX = \alpha X$ (using (1))

Pre- multiplying both sides by adj (A), we get

$adj\ (A)(AX) = adj\ (A)\alpha X \xrightarrow{yields} (adj\ (A)A)X = \alpha(adj\ (A)X) \xrightarrow{yields} |A|X = \alpha(adj\ (A)X)$

$$adj\ (A)X = \frac{|A|}{\alpha}X$$

Hence $\frac{|A|}{\alpha}$ is an eigen value of $adj\ (A)$ and X is corresponding Eigen vector.

5. Prove that inverse of a unitary matrix is a unitary matrix.

Sol. Let A be unitary matrix. Therefore

$A^{\theta}A = AA^{\theta} = I$ where $A^{\theta} = \left(\overline{A}\right)^{T}$

Now $A^{-1}(A^{-1})^{\theta} = A^{-1}\left(A^{\theta}\right)^{-1} = \left(A^{\theta}A\right)^{-1} = I^{-1} = I$

and

$(A^{-1})^{\theta}A^{-1}$=$\left(A^{\theta}\right)^{-1}A^{-1} = \left(AA^{\theta}\right)^{-1} = I^{-1} = I$

Hence A^{-1} is an orthogonal matrix

Therefore inverse of an orthogonal matrix is orthogonal matrix.

6. Prove that the relation of similarity is symmetric.

Proof: Let A and B be two square matrices such that A is to similar to B. Therefore there exists an invertible matrix P

such that $A = P^{-1}BP$ or $PA = BP$

Let $P^{-1} = Q \xrightarrow{yields} P = Q^{-1}$

Hence $A = P^{-1}BP \xrightarrow{yields} A = QBQ^{-1}$

Pre-multiplying both sides by Q^{-1}, we get

$$Q^{-1}A = Q^{-1}QBQ^{-1} \xrightarrow{yields} Q^{-1}A = BQ^{-1}$$

Post-multiplying both sides by Q, we get

$$Q^{-1}AQ = BQ^{-1}Q \xrightarrow{yields} Q^{-1}AQ = B$$

Hence B is similar to A.

Therefore relation of similarity is symmetric.

7. Prove that every square matrix A can be expressed as the sum of a Hermitian matrix and a Skew-Hermitian matrix.

Proof: A can be written as

$$A = \frac{1}{2}\left(A + A^{\theta}\right) + \frac{1}{2}\left(A - A^{\theta}\right) \text{ ----------- (I)}$$

Claim 1: $\frac{1}{2}\left(A + A^{\theta}\right)$ is a Hermitian matrix

Pf: Now

$$\frac{1}{2}\left(A + A^{\theta}\right)^{\theta} = \frac{1}{2}\left(A^{\theta} + \left(A^{\theta}\right)^{\theta}\right) = \frac{1}{2}\left(A^{\theta} + A\right) = \frac{1}{2}\left(A + A^{\theta}\right)$$

Hence $\frac{1}{2}\left(A + A^{\theta}\right)$ is a Hermitian matrix.

Claim 2: $\frac{1}{2}\left(A - A^{\theta}\right)$ is a Skew-Hermitian matrix

Pf: Now

$$\frac{1}{2}\left(A - A^{\theta}\right)^{\theta} = \frac{1}{2}\left(A^{\theta} - \left(A^{\theta}\right)^{\theta}\right) = \frac{1}{2}\left(A^{\theta} - A\right) = -\frac{1}{2}\left(A - A^{\theta}\right)$$

Hence $\frac{1}{2}\left(A + A^{\theta}\right)$ is a Skew-Hermitian matrix.

From (I), Claim 1 and Claim 2 we get that A has been expressed as the sum of a Hermitian matrix and a Skew-Hermitian matrix.

8. Prove that $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$ can be expressed as the sum of a symmetric and a skew symmetric matrix.

Sol. $Let\ A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$

$$\therefore A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & -8 \\ 3 & 6 & 9 \end{bmatrix}$$

$$\frac{1}{2}(A + A^T) = \frac{1}{2}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & -8 \\ 3 & 6 & 9 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 6 & 10 \\ 6 & 10 & -2 \\ 10 & -2 & 18 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & -1 \\ 5 & -1 & 9 \end{bmatrix}$$

$Now\ \left[\frac{1}{2}(A + A^T)\right]^T = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & -1 \\ 5 & -1 & 9 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & -1 \\ 5 & -1 & 9 \end{bmatrix} =$
$\frac{1}{2}(A + A^T)$

$Hence\ \frac{1}{2}(A + A^T)\ is\ a\ symmetric\ matrix.$

$$\frac{1}{2}(A - A^T) = \frac{1}{2}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & -8 \\ 3 & 6 & 9 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & 14 \\ 4 & -14 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 7 \\ 2 & -7 & 0 \end{bmatrix}$$

Now $\left[\frac{1}{2}(A - A^T)\right]^T = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 7 \\ 2 & -7 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -7 \\ -2 & 7 & 0 \end{bmatrix} =$
$-\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 7 \\ 2 & -7 & 0 \end{bmatrix} = -\frac{1}{2}(A - A^T)$

Hence $\frac{1}{2}(A - A^T)$ *is a skew-symmetric matrix.*

Since $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

$$\therefore \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & -1 \\ 5 & -1 & 9 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 7 \\ 2 & -7 & 0 \end{bmatrix}$$

Is representation of given matrix as the sum of a symmetric matrix and a skew-symmetric matrix.

8 **Marks Questions**

1. State and prove Cayley Hamilton theorem.

   Sol. Statement: Every square matrix satisfies its own characteristic equation.

   Proof: Let A be a square matrix of order n and its characteristic equation be $|A - \lambda I| = 0$

   i.e. $(-1)^n \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots \ldots \ldots + a_n = 0$

   Required to be proved: $(-1)^n A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots \ldots \ldots + a_n I = 0$

   Here λ is an eigen value of A.

   $[A - \lambda I]$ is a matrix of order n $\xrightarrow{yields}$ $adj.(A - \lambda I)$ is a matrix of order (n-1).

   Therefore we can write $adj.(A - \lambda I) = P_1 \lambda^{n-1} + P_2 \lambda^{n-2} + \cdots \ldots \ldots + P_n$ where

   $P_1, P_2, \ldots \ldots \ldots\ P_n$ are square matrices.

   Also $A(adj.A) = |A|I \xrightarrow{yields} (A - \lambda I)adj.(A - \lambda I) = |A - \lambda I|I$

$\xrightarrow{yields} (A - \lambda I)[P_1 \lambda^{n-1} + P_2 \lambda^{n-2} + \cdots \ldots \ldots + P_n]= [(-1)^n \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots \ldots \ldots + a_n]I$

Comparing coefficients of like powers of A, we get

$$-P_1 = (-1)^n I$$

$$AP_1 - P_2 = a_1 I$$

$$AP_2 - P_3 = a_2 I$$

$AP_3 - P_4 = \mathrm{a}_3 I$

…………………….. (and so on)

$AP_{n-1} - P_n = a_{n-1} I$

$AP_n = a_n I$

Pre-multiplying these equations by $A^n,\ A^{n-1},\ A^{n-2}, \dots \dots \dots, A,\ I$ respectively on both sides and

adding, we get $0 = (-1)^n A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots \dots \dots + a_n I$

$\xrightarrow{yields} (-1)^n A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots \dots \dots + a_n I = 0$

(Hence proved).

2. Verify Cayley Hamilton theorem for $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$. Hence find A^{-1}. Also find Eigen values and vectors of A

Sol. $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

Characteristic equation of A is $|A - \alpha I| = 0$

$\xrightarrow{yields} \begin{vmatrix} 1-\alpha & 4 \\ 3 & 2-\alpha \end{vmatrix} = 0$

$\xrightarrow{yields} \alpha^2 - 3\alpha - 10 = 0 \xrightarrow{yields} \alpha = -2,5$

By Cayley Hamilton theorem $A^2 - 3A - 10I = 0$ ………………………(*)

Now $A^2 = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 12 \\ 9 & 16 \end{bmatrix}$,

$$\therefore A^2 - 3A - 10I = \begin{bmatrix} 13 & 12 \\ 9 & 16 \end{bmatrix} + \begin{bmatrix} -3 & -12 \\ -9 & -6 \end{bmatrix} + \begin{bmatrix} -10 & 0 \\ 0 & -10 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ Cayley Hamilton theorem is verified for given matrix A.

Multiplying both sides of (*) by A^{-1}, we get $A - 3I = 10A^{-1}$
$\xrightarrow{yields} A^{-1} = \frac{1}{10}\begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$

Let $X_1 = \begin{bmatrix} x \\ y \end{bmatrix}$be the Eigen vector of A corresponding to Eigen value $\alpha = -2$.

$$\therefore \ [A - \alpha I]X_1 = 0 \xrightarrow{yields} [A - (-2)I]X_1 = 0 \xrightarrow{yields} \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{yields} 3x + 4y = 0 \, , 3x + 4y = 0 \xrightarrow{yields} \frac{x}{-4} = \frac{y}{3}$$

$\therefore \ X_1 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ is the Eigen vector of A corresponding to Eigen value $\alpha = -2$.

Let $X_2 = \begin{bmatrix} x \\ y \end{bmatrix}$be the Eigen vector of A corresponding to Eigen value $\alpha = 5$.

$$\therefore \ [A - \alpha I]X_2 = 0 \xrightarrow{yields} [A - (5)I]X_2 = 0 \xrightarrow{yields} \begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{yields} -4x + 4y = 0 \, , 3x - 3y = 0 \xrightarrow{yields} x = y \xrightarrow{yields} \frac{x}{1} = \frac{y}{1}$$

$\therefore \ X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the Eigen vector of A corresponding to Eigen value $\alpha = 5$.

3. Find Eigen Values and Eigen Vectors of $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

(IKGPTU Dec. 2019)

Sol. $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Characteristic equation of A is $|A - \alpha I| = 0 \xrightarrow{yields} \begin{vmatrix} 5-\alpha & 4 \\ 1 & 2-\alpha \end{vmatrix} = 0$

$\xrightarrow{yields} \alpha^2 - 7\alpha + 10 - 4 = 0 \xrightarrow{yields} \alpha^2 - 7\alpha + 6 = 0 \xrightarrow{yields} \alpha = 1 \; or \; 6$

Are the Eigen Values of given matrix A.

Let $X_1 = \begin{bmatrix} x \\ y \end{bmatrix}$ *be the Eigen vector of A corresponding to Eigen value* $\alpha = 1$.

$\therefore \; [A - \alpha I]X_1 = 0 \xrightarrow{yields} [A - (1)I]X_1 = 0 \xrightarrow{yields} \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\xrightarrow{yields} 4x + 4y = 0 \;\; and \; x + y = 0 \xrightarrow{yields} x + y = 0 \xrightarrow{yields} \frac{x}{1} = \frac{y}{-1}$

$\therefore \; X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ *is the Eigen vector of A corresponding to Eigen value* $\alpha = 1$.

Let $X_2 = \begin{bmatrix} x \\ y \end{bmatrix}$ *be the Eigen vector of A corresponding to Eigen value* $\alpha = 6$.

$\therefore \; [A - \alpha I]X_2 = 0 \xrightarrow{yields} [A - (6)I]X_2 = 0 \xrightarrow{yields} \begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\xrightarrow{yields} -x + 4y = 0 \; and \; x - 4y = 0 \xrightarrow{yields} x = 4y \; \xrightarrow{yields} \frac{x}{4} = \frac{y}{1}$

$\therefore \; X_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ *is the Eigen vector of A corresponding to Eigen value* $\alpha = 6$.

4. Verify Cayley Hamilton theorem for $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

(IKGPTU Dec. 2019)

Sol. $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

Characteristic equation of A is $|A - \alpha I| = 0$

$$\xrightarrow{yields} \begin{vmatrix} 1-\alpha & 2 & 0 \\ -1 & 1-\alpha & 2 \\ 1 & 2 & 1-\alpha \end{vmatrix} = 0$$

$$\xrightarrow{yields} (1-\alpha)[(1-\alpha)^2 - 4] - 2[(\alpha - 1) - 2] = 0$$

$$\xrightarrow{yields} (1-\alpha)[\alpha^2 - 2\alpha - 3] - 2\alpha + 6 = 0$$

$$\xrightarrow{yields} -\alpha^3 + 2\alpha^2 + 3\alpha + \alpha^2 - 2\alpha - 3 - 2\alpha + 6 = 0$$

$\xrightarrow{yields} \alpha^3 - 3\alpha^2 + \alpha - 3 = 0$ *is the characteristic equation of matrix A.*

By Cayley Hamilton theorem $A^3 - 3A^2 + A - 3I = 0$ *……………………..(i)*

$$L.H.S. A^2 = A.A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{bmatrix}$$

Hence $A^3 - 3A^2 + A - 3I$

$$= \begin{bmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{bmatrix} - 3\begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1+3+1-3 & 10-12+2 & 12-12 \\ 1-0-1 & 11-9+1-3 & 10-12+2 \\ -1-0+1 & 16-18+2 & 17-15+1-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence Cayley Hamilton theorem is verified for the given matrix A

5. Verify Cayley Hamilton theorem for $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

Hence find A^{-1}.

Sol. $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Characteristic equation of A is $|A - \alpha I| = 0$

$$\xrightarrow{yields} \begin{vmatrix} 2-\alpha & -1 & 1 \\ -1 & 2-\alpha & -1 \\ 1 & -1 & 2-\alpha \end{vmatrix} = 0$$

$$\xrightarrow{yields} \alpha^3 - 6\alpha^2 + 9\alpha - 4 = 0$$

By Cayley Hamilton theorem $A^3 - 6A^2 + 9A - 4I = 0$ ……………………..(i)

L.H.S.$A^2 = A.A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$

$$A^3 = AAA = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Hence $A^3 - 6A^2 + 9A - 4I$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22-36+18-4 & -21+30-9 & 21-30+9 \\ -21+30-9 & 22-36+18-4 & -21+30-9 \\ 21-30+9 & -21+30-9 & 22-36+18-4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence Cayley Hamilton theorem is verified for the given matrix A

From(i), $4I = A^3 - 6A^2 + 9A$

Multiplying both sides by A^{-1}, we get

$$A^{-1} = \frac{1}{4}[A^2 - 6A + 9I]$$

$$= \frac{1}{4}\left[\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}\right]$$

$$= \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

6. Find Eigen values and vectors of $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

Sol. $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

Characteristic equation of A is $|A - \alpha I| = 0$

$$\xrightarrow{yields} \begin{vmatrix} 3-\alpha & 1 & -1 \\ -2 & 1-\alpha & 2 \\ 0 & 1 & 2-\alpha \end{vmatrix} = 0$$

$\xrightarrow{yields} \alpha^3 - 6\alpha^2 + 11\alpha - 6 = 0 \xrightarrow{yields} \alpha = 1,2,3$ are Eigen values of given matrix.

Let $X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the Eigen vector of A corresponding to Eigen value $\alpha = 1$.

$$\therefore\ [A - \alpha I]X_1 = 0 \xrightarrow{yields} [A - (1)I]X_1 = 0$$

$$\xrightarrow{yields} \begin{bmatrix} 2 & 1 & -1 \\ -2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{yields}\ 2x + y - z = 0\,, -2x + 2z = 0, y + z = 0$$

From first two equations, $\frac{x}{\begin{matrix} 1 & -1 \\ 0 & 2 \end{matrix}} = \frac{y}{\begin{matrix} -1 & 2 \\ 2 & -2 \end{matrix}} = \frac{z}{\begin{matrix} 2 & 1 \\ -2 & 0 \end{matrix}} \xrightarrow{yields} \frac{x}{2} = \frac{y}{-2} = \frac{z}{2} \xrightarrow{yields} \frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$

$\therefore\ X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is the Eigen vector of A corresponding to Eigen value $\alpha = 1$.

Let $X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the Eigen vector of A corresponding to Eigen value $\alpha = 2$.

$$\therefore\ [A-\alpha I]X_2 = 0 \xrightarrow{yields} [A-(2)I]X_2$$

$$= 0 \xrightarrow{yields} \begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{yields} x+y-z=0\,, -2x-y+2z=0, y=0$$

From first two equations, $\frac{x}{\begin{matrix} 1 & -1 \\ -1 & 2 \end{matrix}} = \frac{y}{\begin{matrix} -1 & 1 \\ 2 & -2 \end{matrix}} = \frac{z}{\begin{matrix} 1 & 1 \\ -2 & -1 \end{matrix}}$

$\xrightarrow{yields} \frac{x}{1} = \frac{y}{0} = \frac{z}{1}$

$\therefore\ X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is the Eigen vector of A corresponding to Eigen value $\alpha = 2$.

Let $X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the Eigen vector of A corresponding to Eigen value $\alpha = 3$.

$$\therefore\ [A-\alpha I]X_3 = 0 \xrightarrow{yields} [A-(3)I]X_3$$

$$= 0 \xrightarrow{yields} \begin{bmatrix} 0 & 1 & -1 \\ -2 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\xrightarrow{yields} y-z=0\,, -2x-2y+2z=0, y-z=0 \xrightarrow{yields} y-z=0\,, -2x-2y+2z=0$

$\therefore$ we get , $\frac{x}{\begin{matrix} 1 & -1 \\ -2 & 2 \end{matrix}} = \frac{y}{\begin{matrix} -1 & 0 \\ 2 & -2 \end{matrix}} = \frac{z}{\begin{matrix} 0 & 1 \\ -2 & -2 \end{matrix}} \xrightarrow{yields} \frac{x}{0} = \frac{y}{2} = \frac{z}{2}$

$\xrightarrow{yields} \frac{x}{0} = \frac{y}{1} = \frac{z}{1}$

$\therefore\ X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is the Eigen vector of A corresponding to Eigen value $\alpha = 2$.

7. Find Eigen values and vectors of $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

   (IKGPTU Nov. 2018)

Sol. $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

Characteristic equation of A is $|A - \alpha I| = 0$

$$\xrightarrow{yields} \begin{vmatrix} 2-\alpha & -2 & 3 \\ 1 & 1-\alpha & 1 \\ 1 & 3 & -1-\alpha \end{vmatrix} = 0$$

(Operating $R_2 \to R_2 - R_3$)

$$\xrightarrow{yields} \begin{vmatrix} 2-\alpha & -2 & 3 \\ 0 & -2-\alpha & \alpha+2 \\ 1 & 3 & -1-\alpha \end{vmatrix} = 0$$

$$\xrightarrow{yields} (\alpha+2)\begin{vmatrix} 2-\alpha & -2 & 3 \\ 0 & -1 & 1 \\ 1 & 3 & -1-\alpha \end{vmatrix} = 0$$

$$\xrightarrow{yields} (\alpha+2)[(2-\alpha)(1+\alpha-3) + 2(0-1) + 3(0+1)] = 0$$

$$\xrightarrow{yields} (\alpha+2)(-\alpha^2 + 4\alpha - 3) = 0$$

$\xrightarrow{yields}$ $\alpha = -2,1,3$ are Eigen values of given matrix.

Let $X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the Eigen vector of A corresponding to Eigen value $\alpha = 1$.

$$\therefore \ [A - \alpha I]X_1 = 0 \xrightarrow{yields} [A - (1)I]X_1 = 0$$

$$\xrightarrow{yields} \begin{bmatrix} 1 & -2 & 3 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(Operating $R_2 \to R_2 - R_1$ *and* $R_3 \to R_3 - R_1$)

$$\xrightarrow{yields} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -2 \\ 0 & 5 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(Operating $R_3 \to R_3 - \frac{5}{2}R_2$)

$$\xrightarrow{yields} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $z = k$

Therefore from R_2 we get $2y - 2z = 0 \xrightarrow{yields} y = k$

From R_1 we get $x - 2y + 3z = 0 \xrightarrow{yields} x = -k$

$\therefore \ X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ is the Eigen vector of A corresponding to Eigen value $\alpha = 1$.

Let $X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the Eigen vector of A corresponding to Eigen value $\alpha = -2$.

$$\therefore\ [A-\alpha I]X_1 = 0 \xrightarrow{yields} [A-(-2)I]X_1 = 0$$

$$\xrightarrow{yields} \begin{bmatrix} 4 & -2 & 3 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(Operating $R_3 \to R_3 - R_2$)

$$\xrightarrow{yields} \begin{bmatrix} 4 & -2 & 3 \\ 1 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(Operating $R_2 \to R_2 - \frac{1}{4}R_1$)

$$\xrightarrow{yields} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 7/2 & 1/4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $z = k$

Therefore from R_2 we get $\frac{7}{2}y + \frac{1}{4}z = 0 \xrightarrow{yields} y = \frac{-1}{14}k$

From R_1 we get $x - 2y + 3z = 0 \xrightarrow{yields} x = \frac{-22}{28}k$

$$\therefore\ X_2 = \begin{bmatrix} \frac{-22}{28} \\ \frac{-1}{14} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-11}{14} \\ \frac{-1}{14} \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \\ 14 \end{bmatrix}$$ is the Eigen vector of A corresponding to Eigen value $\alpha = -2$.

Let $X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the Eigen vector of A corresponding to Eigen value $\alpha = 3$.

$$\therefore\ [A-\alpha I]X_3 = 0 \xrightarrow{yields} [A-(3)I]X_3 = 0$$

$$\xrightarrow{yields} \begin{bmatrix} -1 & -2 & 3 \\ 1 & -2 & 1 \\ 1 & 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(Operating $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + R_1$)

$$\xrightarrow{yields} \begin{bmatrix} -1 & -2 & 3 \\ 0 & -4 & 4 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(Operating $R_2 \to \frac{-1}{4} R_2$)

$$\xrightarrow{yields} \begin{bmatrix} -1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(Operating $R_3 \to R_3 - R_2$)

$$\xrightarrow{yields} \begin{bmatrix} -1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $z = k$

Therefore from R_2 we get $y - z = 0 \xrightarrow{yields} y = k$

From R_1 we get $-x - 2y + 3z = 0 \xrightarrow{yields} x = k$

$\therefore\ X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is the Eigen vector of A corresponding to

Eigen value $\alpha = 3$.

8. Find Eigen values and vectors of $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Sol. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Characteristic equation of A is $|A - \alpha I| = 0$

$$\xrightarrow{yields} \begin{vmatrix} 1-\alpha & 1 & 0 \\ 0 & 1-\alpha & 1 \\ 0 & 0 & 1-\alpha \end{vmatrix} = 0$$

$\xrightarrow{yields} (1-\alpha)^3 \xrightarrow{yields} \alpha = 1,1,1$ are Eigen values of given matrix.

Let $X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the Eigen vector of A corresponding to Eigen value $\alpha = 1$.

$$\therefore\ [A - \alpha I]X_1 = 0 \xrightarrow{yields} [A - (1)I]X_1 = 0 \xrightarrow{yields} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\xrightarrow{yields} y = 0, z = 0.\ Take\ x = 1$

$\therefore\ X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the Eigen vector of A corresponding to Eigen value $\alpha = 1$.

9. Examine whether the following matrix is diagonalizable. If so, obtain the matrix P such that $P^{-1}AP$ is a diagonal matrix. $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Sol. $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Characteristic equation of A is $|A - \alpha I| = 0$

$$\xrightarrow{yields} \begin{vmatrix} -2-\alpha & 2 & -3 \\ 2 & 1-\alpha & -6 \\ -1 & -2 & 0-\alpha \end{vmatrix} = 0$$

$\xrightarrow{yields} -(\alpha+3)(\alpha+3)(\alpha-5) = 0 \xrightarrow{yields} \alpha = -3, -3, 5$ are Eigen values of given matrix.

Let $X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the Eigen vector of A corresponding to Eigen value $\alpha = -3$.

$$\therefore\ [A - \alpha I]X_1 = 0 \xrightarrow{yields} [A - (-3)I]X_1$$

$$= 0 \xrightarrow{yields} \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(Operating $R_2 \to R_2 - 2R_1, R_3 \to R_3 + R_1$)

$$\xrightarrow{yields} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{yields} x + 2y - 3z = 0$$

Choose $y = 0 \xrightarrow{yields} x - 3z = 0 \xrightarrow{yields} \frac{x}{3} = \frac{z}{1}$

$\therefore\ X_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ is the first Eigen vector of A corresponding to Eigen value $\alpha = -3$.

Choose $z = 0 \xrightarrow{yields} x + 2y = 0 \xrightarrow{yields} \frac{x}{-2} = \frac{y}{1}$

$\therefore\ X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ is another Eigen vector of A corresponding to Eigen value $\alpha = -3$.

Let $X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the Eigen vector of A corresponding to Eigen value $\alpha = 5$.

$$\therefore\ [A - \alpha I]X_3 = 0 \xrightarrow{yields} [A - (5)I]X_3$$

$$= 0 \xrightarrow{yields} \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\xrightarrow{yields} -7x + 2\,y - 3z = 0\,, 2x - 4y - 6z = 0, -x - 2y - 5z = 0$

$\therefore$ from first two equations we get , $\dfrac{x}{\begin{matrix} 2 & -3 \\ -4 & -6 \end{matrix}} = \dfrac{y}{\begin{matrix} -3 & -7 \\ -6 & 2 \end{matrix}} = \dfrac{z}{\begin{matrix} -7 & 2 \\ 2 & -4 \end{matrix}}$

$\xrightarrow{yields} \dfrac{x}{-24} = \dfrac{y}{-48} = \dfrac{z}{24} \xrightarrow{yields} \dfrac{x}{1} = \dfrac{y}{12} = \dfrac{z}{-1}$

$\therefore\ X_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is the Eigen vector of A corresponding to Eigen value $\alpha = 5$.

$\therefore$ Modal Matrix P $= \begin{bmatrix} -2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$

$|P| = \begin{vmatrix} -2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{vmatrix} = 8 \neq 0$. Hence vectors are linearly independent and the given matrix is

Diagonalizable.

$$P^{-1} = \frac{Adj.P}{|P|} = \frac{1}{8}\begin{bmatrix} -2 & 4 & 6 \\ 1 & 2 & 5 \\ 1 & 2 & -3 \end{bmatrix}$$

Diagonal Matrix = D = $P^{-1}AP =$

$$\frac{1}{8}\begin{bmatrix} -2 & 4 & 6 \\ 1 & 2 & 5 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} -2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} -24 & 0 & 0 \\ 0 & -24 & 0 \\ 0 & 0 & 40 \end{bmatrix}\begin{bmatrix} -2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

10. Diagonalize the matrix $\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ (IKGPTU May 2019)

Sol. $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

Characteristic equation of A is $|A - \alpha I| = 0$

$$\xrightarrow{yields} \begin{vmatrix} -1-\alpha & 2 & -2 \\ 1 & 2-\alpha & 1 \\ -1 & -1 & -\alpha \end{vmatrix} = 0$$

$$\xrightarrow{yields} (-1-\alpha)[-2\alpha + \alpha^2 + 1] - 2(-\alpha + 1) - 2[-1 + 2 - \alpha] = 0$$

$$\xrightarrow{yields} 2\alpha - \alpha^2 - 1 + 2\alpha^2 - \alpha^3 - \alpha + 2\alpha - 2 - 2 + 2\alpha = 0$$

$$\xrightarrow{yields} -\alpha^3 + \alpha^2 + 5\alpha - 5 = 0$$

$$\xrightarrow{yields} \alpha^3 - \alpha^2 - 5\alpha + 5 = 0 \xrightarrow{yields} \alpha = 1, \sqrt{5}, -\sqrt{5}$$ are Eigen values of given matrix.

Let $X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the Eigen vector of A corresponding to Eigen value $\alpha = 1$.

$$\therefore\ [A - \alpha I]X_1 = 0 \xrightarrow{yields} [A - (1)I]X_1 = 0$$

$$\xrightarrow{yields} \begin{bmatrix} -2 & 2 & -2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{yields} -2x + 2y - 2z = 0\ ,\ \ x + y + z = 0,\ \ -x - y - z = 0$$

$$\xrightarrow{yields} x - y + z = 0\ , \qquad x + y + z = 0$$

From first two equations,

$$\frac{x}{\begin{matrix} -1 & 1 \\ 1 & 1 \end{matrix}} = \frac{y}{\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}} = \frac{z}{\begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix}} \xrightarrow{yields} \frac{x}{-2} = \frac{y}{0} = \frac{z}{2} \xrightarrow{yields} \frac{x}{-1} = \frac{y}{0} = \frac{z}{1}$$

$\therefore\ X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is the Eigen vector of A corresponding to Eigen value $\alpha = 1$.

Let $X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the Eigen vector of A corresponding to Eigen value $\alpha = \sqrt{5}$.

$$\therefore\ [A - \alpha I]X_2 = 0 \xrightarrow{yields} \left[A - \left(\sqrt{5}\right)I\right]X_2$$

$$= 0 \xrightarrow{yields} \begin{bmatrix} -1-\sqrt{5} & 2 & -2 \\ 1 & 2-\sqrt{5} & 1 \\ -1 & -1 & -\sqrt{5} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{yields} \left(-1-\sqrt{5}\right)x + 2y - 2z = 0\,,\;\; x + \left(2-\sqrt{5}\right)y + z = 0,$$
$$-x - y - \sqrt{5}z = 0$$

From last two equations i.e.

$$x + \left(2-\sqrt{5}\right)y + z = 0,\;\; x + y + \sqrt{5}z = 0,$$

we get

$$,\;\; \frac{x}{\begin{matrix} \left(2-\sqrt{5}\right) & 1 \\ 1 & \sqrt{5} \end{matrix}} = \frac{y}{\begin{matrix} 1 & 1 \\ \sqrt{5} & 1 \end{matrix}} = \frac{z}{\begin{matrix} 1 & \left(2-\sqrt{5}\right) \\ 1 & 1 \end{matrix}}$$

$$\xrightarrow{yields} \frac{x}{2\sqrt{5}-5-1} = \frac{y}{1-\sqrt{5}} = \frac{z}{1-2+\sqrt{5}}$$

$$\xrightarrow{yields} \frac{x}{2\sqrt{5}-6} = \frac{y}{1-\sqrt{5}} = \frac{z}{-1+\sqrt{5}}$$

$$\xrightarrow{yields} \frac{x}{-2\sqrt{5}+6} = \frac{y}{-1+\sqrt{5}} = \frac{z}{1-\sqrt{5}}$$

$$\xrightarrow{yields} \frac{x}{\left(\sqrt{5}-1\right)^2} = \frac{y}{\sqrt{5}-1} = \frac{z}{1-\sqrt{5}}$$

$$\xrightarrow{yields} \frac{x}{\sqrt{5}-1} = \frac{y}{1} = \frac{z}{-1}$$

$$\therefore\ X_2 = \begin{bmatrix} \sqrt{5}-1 \\ 1 \\ -1 \end{bmatrix}$$ is the Eigen vector of A corresponding to Eigen value $\alpha = \sqrt{5}$.

Let $X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the Eigen vector of A corresponding to Eigen value $\alpha = -\sqrt{5}$.

$$\therefore\ [A-\alpha I]X_3 = 0 \xrightarrow{yields} \left[A-\left(-\sqrt{5}\right)I\right]X_3$$

$$= 0 \xrightarrow{yields} \begin{bmatrix} -1+\sqrt{5} & 2 & -2 \\ 1 & 2+\sqrt{5} & 1 \\ -1 & -1 & \sqrt{5} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{yields} \left(-1+\sqrt{5}\right)x + 2y - 2z = 0\,,\ x + \left(2+\sqrt{5}\right)y + z = 0,$$
$$-x - y + \sqrt{5}z = 0$$

From first two equations,

$$\frac{x}{\begin{matrix} 2 & -2 \\ \left(2+\sqrt{5}\right) & 1 \end{matrix}} = \frac{y}{\begin{matrix} -2 & \left(-1+\sqrt{5}\right) \\ 1 & 1 \end{matrix}} = \frac{z}{\begin{matrix} \left(-1+\sqrt{5}\right) & 2 \\ 1 & \left(2+\sqrt{5}\right) \end{matrix}}$$

$$\xrightarrow{yields} \frac{x}{6+2\sqrt{5}} = \frac{y}{-1-\sqrt{5}} = \frac{z}{1+\sqrt{5}}$$

$$\xrightarrow{yields} \frac{x}{\left(1+\sqrt{5}\right)^2} = \frac{y}{-\left(1+\sqrt{5}\right)} = \frac{z}{1+\sqrt{5}}$$

$$\xrightarrow{yields} \frac{x}{\left(1+\sqrt{5}\right)} = \frac{y}{-1} = \frac{z}{1}$$

$\therefore\ X_3 = \begin{bmatrix} 1+\sqrt{5} \\ -1 \\ 1 \end{bmatrix}$ is the Eigen vector of A corresponding to Eigen value $\alpha = -\sqrt{5}$.

$\therefore$ Modal Matrix P = $\begin{bmatrix} 1 & \sqrt{5}-1 & 1+\sqrt{5} \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$

$$|P| = \begin{vmatrix} 1 & \sqrt{5}-1 & 1+\sqrt{5} \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix}$$

$$= 1(1-1) - 0\left(\sqrt{5}-1+1+\sqrt{5}\right) - 1\left(-\sqrt{5}+1-1-\sqrt{5}\right)$$

$= 0 - 0 + 2\sqrt{5} = 2\sqrt{5} \neq 0.$

Hence vectors are linearly independent and the given matrix is Diagonalizable.

$$c_{11} = (-1)^{1+1}\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 0$$

$$c_{12} = (-1)^{1+2}\begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} = 1$$

$$c_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix} = 1$$

$$c_{21} = (-1)^{2+1}\begin{vmatrix} \sqrt{5}-1 & 1+\sqrt{5} \\ -1 & 1 \end{vmatrix} = -2\sqrt{5}$$

$$c_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1+\sqrt{5} \\ -1 & 1 \end{vmatrix} = 2+\sqrt{5}$$

$$c_{23} = (-1)^{2+3}\begin{vmatrix} 1 & \sqrt{5}-1 \\ -1 & -1 \end{vmatrix} = 2-\sqrt{5}$$

$$c_{31} = (-1)^{3+1}\begin{vmatrix} \sqrt{5}-1 & 1+\sqrt{5} \\ 1 & -1 \end{vmatrix} = -2\sqrt{5}$$

$$c_{32} = (-1)^{3+2}\begin{vmatrix}1 & 1+\sqrt{5}\\0 & -1\end{vmatrix} = 1$$

$$c_{33} = (-1)^{3+3}\begin{vmatrix}1 & \sqrt{5}-1\\0 & 1\end{vmatrix} = 1$$

$$\therefore adj.P = \begin{bmatrix}0 & 1 & 1\\-2\sqrt{5} & 2+\sqrt{5} & 2-\sqrt{5}\\-2\sqrt{5} & 1 & 1\end{bmatrix}^T$$

$$= \begin{bmatrix}0 & -2\sqrt{5} & -2\sqrt{5}\\1 & 2+\sqrt{5} & 1\\1 & 2-\sqrt{5} & 1\end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|}adj.P = \frac{1}{2\sqrt{5}}\begin{bmatrix}0 & -2\sqrt{5} & -2\sqrt{5}\\1 & 2+\sqrt{5} & 1\\1 & 2-\sqrt{5} & 1\end{bmatrix}$$

Diagonal Matrix = D = $P^{-1}AP$

$$= \frac{1}{2\sqrt{5}}\begin{bmatrix}0 & -2\sqrt{5} & -2\sqrt{5}\\1 & 2+\sqrt{5} & 1\\1 & 2-\sqrt{5} & 1\end{bmatrix}\begin{bmatrix}-1 & 2 & -2\\1 & 2 & 1\\-1 & -1 & 0\end{bmatrix}\begin{bmatrix}1 & \sqrt{5}-1 & 1+\sqrt{5}\\0 & 1 & -1\\-1 & -1 & 1\end{bmatrix}$$

$$= \frac{1}{2\sqrt{5}}\begin{bmatrix}0 & -2\sqrt{5} & -2\sqrt{5}\\1 & 2+\sqrt{5} & 1\\1 & 2-\sqrt{5} & 1\end{bmatrix}\begin{bmatrix}1 & (1+2+2-\sqrt{5}) & (-1-2-2-\sqrt{5})\\0 & (\sqrt{5}-1+2-1) & (\sqrt{5}+1-2+1)\\-1 & (-\sqrt{5}+1-1) & (-\sqrt{5}-1+1)\end{bmatrix}$$

$$= \frac{1}{2\sqrt{5}}\begin{bmatrix}0 & -2\sqrt{5} & -2\sqrt{5}\\1 & 2+\sqrt{5} & 1\\1 & 2-\sqrt{5} & 1\end{bmatrix}\begin{bmatrix}1 & 5-\sqrt{5} & -5-\sqrt{5}\\0 & \sqrt{5} & \sqrt{5}\\1 & -\sqrt{5} & -\sqrt{5}\end{bmatrix}$$

$$= \frac{1}{2\sqrt{5}}\begin{bmatrix}2\sqrt{5} & -10+10 & -10+10\\0 & (5-\sqrt{5}+2\sqrt{5}+5-\sqrt{5}) & (-5-\sqrt{5}+2\sqrt{5}+5-\sqrt{5})\\0 & (5-\sqrt{5}+2\sqrt{5}-5-\sqrt{5}) & (-5-\sqrt{5}+2\sqrt{5}-5-\sqrt{5})\end{bmatrix}$$

$$= \frac{1}{2\sqrt{5}} \begin{bmatrix} 2\sqrt{5} & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}$$

11. Prove that Similar matrices have same characteristic equation and hence same Eigen values. Also if X is an eigen vector of A corresponding to eigen value λ then $P^{-1}X$ is an eigen vector of B corresponding to eigen value λ, where P is Similarity matrix. (IKGPTU Nov. 2018)

Proof: Let matrix B be similar matrix A and P is similarity matrix, therefore

$AP = PB$ or $B = P^{-1}AP$

Let λ be eigen value and X be corresponding eigen vector of A

$\therefore \ AX = \lambda X$

Now $B - \lambda I = P^{-1}AP - \lambda I = P^{-1}AP - \lambda P^{-1}P$

$= P^{-1}AP - P^{-1}\lambda IP$

$$= P^{-1}(A - \lambda I)P$$

$\therefore\ |B-\lambda I| = |P^{-1}(A-\lambda I)P|$

$= |P^{-1}||A-\lambda I||P|$

$$= |A-\lambda I||P^{-1}||P|$$

$$= |A-\lambda I||P^{-1}P|$$

$$= |A-\lambda I||I|$$

$$= |A-\lambda I|$$

Hence similar matrices have same characteristic polynomial and hence same Eigen values.

Since X is an eigen vector of A corresponding to eigen value λ

$$\therefore AX = \lambda X$$

Pre-multiplying both sides by P^{-1}, we get

$$P^{-1}AX = P^{-1}\lambda X$$

$\xrightarrow{yields} P^{-1}AX = \lambda P^{-1}X$ --------- (1)

Let $X = PY$

Therefore (1) becomes

$$P^{-1}APY = \lambda P^{-1}PY$$

$$\xrightarrow{yields} BY = \lambda IY$$

$$\xrightarrow{yields} BY = \lambda Y$$

Therefore Y i.e. $P^{-1}X$ is the Eigen vector of matrix B corresponding to Eigen value λ

12. Show that the matrix $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ satisfies the equation

$A^3 - 6A^2 + 11A - I = 0$ (IKGPTU May 2019)

Sol. Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

$$\textit{Characteristic equation of A is } |A - \alpha I| = 0$$

$$\xrightarrow{yields} \begin{vmatrix} 2-\alpha & 0 & -1 \\ 5 & 1-\alpha & 0 \\ 0 & 1 & 3-\alpha \end{vmatrix} = 0$$

$$\xrightarrow{yields} (2-\alpha)\begin{vmatrix} 1-\alpha & 0 \\ 1 & 3-\alpha \end{vmatrix} - 0 - 1\begin{vmatrix} 5 & 1-\alpha \\ 0 & 1 \end{vmatrix} = 0$$

$$\xrightarrow{yields} (2-\alpha)(1-\alpha)(3-\alpha) - 5 = 0$$

$$\xrightarrow{yields} (\alpha^2 - 3\alpha + 2)(3-\alpha) - 5 = 0$$

$$\xrightarrow{yields} 3\alpha^2 - 9\alpha + 6 - \alpha^3 + 3\alpha^2 - 2\alpha - 5 = 0$$

$$\xrightarrow{yields} -\alpha^3 + 6\alpha^2 - 11\alpha + 1 = 0$$

$\xrightarrow{yields} \alpha^3 - 6\alpha^2 + 11\alpha - 1 = 0$ *is the characteristic equation of matrix A.*

Since Cayley Hamilton theorem states that every square matrix satisfies its own characteristic equation, therefore A satisfies the equation

$$A^3 - 6A^2 + 11A - I = 0$$

Alternate Method

$$L.H.S. = A^2 = A.A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 5 & 4 & 9 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 5 & 4 & 9 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -6 & -19 \\ 35 & -4 & -30 \\ 30 & 13 & 22 \end{bmatrix}$$

Hence $A^3 - 6A^2 + 11A - I$

$$= \begin{bmatrix} 3 & -6 & -19 \\ 35 & -4 & -30 \\ 30 & 13 & 22 \end{bmatrix} - 6 \begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 5 & 4 & 9 \end{bmatrix} + 11 \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-24+22-1 & -6+6+0-0 & -19+30-11-0 \\ 35-90+55-0 & -4-6+11-1 & -30+30+0-0 \\ 30-30+0-0 & 13-24+11-0 & 22-54+33-1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R.H.S.$$

Therefore A satisfies the equation

$$A^3 - 6A^2 + 11A - I = 0$$