- Define matrix.

Sol. A matrix is a rectangular array of numbers, algebraic symbols, or mathematical functions, provided that such arrays are added and multiplied according to certain rules.

- Define order of a matrix.

Sol. Order or size of Matrix $= no.of\ rows \times no.of\ columns = m \times n$

Where m is number of rows and n is number of columns.

- Define row matrix.

Sol. A matrix with one row is called a row matrix.

- Define column matrix

Sol. A matrix with one column is called a row matrix.

- Define square matrix.

Sol. A matrix is called a square matrix if the number of its rows equals the number of columns.

- Define rectangular matrix.

Sol. A matrix is called a rectangular matrix if the number of its rows is not equal to the number of columns.

- Define diagonal matrix.

Sol. A matrix is called a diagonal matrix if all its off-diagonal elements are equal to zero, but at least one of the diagonal elements is nonzero:

$$\mathrm{a_{ij}} = 0 \text{ if } \mathrm{i} \neq \mathrm{j}$$

- Define scalar matrix.

Sol. A matrix is called a scalar matrix if all its off-diagonal elements are equal to zero, but its diagonal elements are non-zero and equal.

- Define identity matrix.

Sol. An identity matrix is a diagonal matrix whose diagonal elements are equal to unity.

- Define zero matrix.

Sol. A matrix is called a zero-matrix (or 0-matrix) if all its elements are equal to zero.

- Define transpose of a matrix.

Sol. A matrix obtained from matrix A by interchanging its rows and columns is called transpose of A and is denoted by $\mathrm{A^T}$ or $\mathrm{A'}$.

e.g. $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & -3 \\ 1 & 3 & 0 \end{bmatrix}$

- Write two properties of transpose of a matrix.

Sol. Two properties of transpose of matrix are: (1) $(\mathrm{A^T})^T = A$ and (2) $(AB)^T = \mathrm{B^T A^T}$

- Define upper triangular matrix.

Sol. A square matrix A is called upper triangular matrix if all the elements below the principal diagonal are zero.

e.g. $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & 4 \end{bmatrix}$

- Define lower triangular matrix.

Sol. A square matrix A is called lower triangular matrix if all the elements above the principal diagonal are zero.

e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 2 & -3 & 6 \end{bmatrix}$

- Define symmetric matrix.

Sol. A square matrix is called symmetric matrix if $A = A^T$

i.e.$a_{ij} = a_{ji}$

e.g.$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 7 \end{bmatrix}$

- Define skew-symmetric matrix.

Sol. A square matrix is called symmetric matrix if $A = -A^T$

i.e. $a_{ij} = -a_{ji}$. The diagonal elements of a skew-symmetric matrix are zero because $a_{ii} = -a_{ii}$ if and only if $a_{ii} = 0$

e.g. $\begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}$

- Explain Nilpotent matrix.

Sol. A square matrix 'A' is called nilpotent matrix if there exists a positive integer n such that $A^n = 0$.

If n is the least positive integer such that $A^n = 0$, then n is called index of nilpotent matrix A.

- What is the condition for addition of two matrices?

Sol. Only matrices of same order can be added. In case of matrix addition, the corresponding elements of two similar matrices are added to each other.

e.g. $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix} + \begin{bmatrix} 5 & -7 & 1 \\ 6 & -8 & -1 \\ 3 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+(-7) & 1+1 \\ 1+6 & 0+(-8) & 3+(-1) \\ 2+3 & -3+2 & 0+(-6) \end{bmatrix} = \begin{bmatrix} 6 & -5 & 2 \\ 7 & -8 & 2 \\ 5 & -1 & -6 \end{bmatrix}$

Matrix addition is commutative and associative.

- What is the condition of product of two matrices?

Sol. Two matrices A and B can be multiplied i.e. AB is possible if number of columns in A is equal to number of rows in B.

If order of A is $m \times n$ and order of B is $n \times p$, then order of AB is $m \times p$.

Matrix multiplication is not commutative i.e. $AB \neq BA$.

- Define a singular matrix.

Sol. A matrix 'A' is said to be singular if $|A| = 0$

- Define a non-singular matrix.

Sol. A matrix 'A' is said to be non-singular if $|A| \neq 0$

- Define rank of a matrix.

Sol._A matrix **A** is said to be of rank r if

(i) All the minors, in **A**, of order greater than r are zero.

(ii) There exists atleast one minor of order r in **A** which is non zero.

- What is the rank of a non-singular matrix of order n?

Sol. The rank of a non-singular matrix of order n is n because the determinant of non-singular matrix A is non-zero.

- If A is a non- zero row and B is a non- zero column matrix, show that rank AB =1.

Sol. Let A = $[y_1 \quad y_2 \quad \cdots \cdots \quad y_n]$ and B = $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$

Then AB = $[x_1y_1 + x_2y_2 + \ldots \ldots . + x_ny_n]$ which is a singleton matrix. Hence rank AB = 1.

- Define linearly dependent vectors.

Sol. A set of vectors $X_1, X_2, \ldots \ldots .., X_n$ is said to be linearly dependent if there exist scalars $\alpha_1, \alpha_2, \ldots \ldots ., \alpha_n$, atleast one α_i non- zero, such that

$$\alpha_1X_1 + \alpha_2X_2 + \cdots + \alpha_nX_n = 0$$

e.g. $X_1 = (2,4)$and $X_2 = (1,2)$, then $X_1 + (-2)X_2 = 0$. Therefore $X_1$and X_1 are linearly dependent vectors.

- Define linearly independent vectors.

Sol. A set of vectors $X_1, X_2, \ldots \ldots .., X_n$is said to be linearly

independent If for scalars $\alpha_1, \alpha_2, \ldots \ldots ., \alpha_n$,

$$\alpha_1X_1 + \alpha_2X_2 + \cdots + \alpha_nX_n = 0$$

Implies all α_i are zero.

e.g. $X_1 = (2,0)$and $X_2 = (0,4)$,

then $\alpha_1X_1 + \alpha_2X_2 = 0 \xrightarrow{yields} (2\alpha_1, 4\alpha_2) = 0 \xrightarrow{yields} \alpha_1 = 0 \; and \; \alpha_2 = 0$. Therefore $X_1$and X_2 are linearly independent vectors.

- Determine whether the set $\{(3,2,4), \; (1,0,2), \; (1,-1,-1)\}$ of vectors linearly independent.

Sol. $\begin{vmatrix} 3 & 2 & 4 \\ 1 & 0 & 2 \\ 1 & -1 & -1 \end{vmatrix} = 3(0+2) - 2(-1-2) + 4(-1) = 4 \neq 0$

$\therefore$ Vectors linearly independent.

- Determine whether the set $\{(2,2,1), \; (1,-1,1), \; (1,0,1)\}$ of vectors linearly independent.

Sol. $\begin{vmatrix} 2 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2(-1-0) - 2(1-1) + 1(0+1) = -1 \neq 0$

$\therefore$ Vectors linearly independent.

- Write conditions for consistency of a non-homogenous system of equations.

Sol. **<u>Conditions for consistency of Non-homogenous system of linear equations (AX=B)</u>**

i. If $\rho(A{:}B) = \rho(A) = number\ of\ unknowns$, the system has unique solution.
ii. If $\rho(A{:}B) = \rho(A) < number\ of\ unknowns$, the system has an infinite number of solutions.
iii. If $\rho(A{:}B) \neq \rho(A)$, the system is inconsistent i.e. it has no solution.

- Write conditions for consistency of a non-homogenous system of equations.

Sol. **<u>Conditions for consistency of homogenous system of linear equations (AX=B)</u>**

i. This system always has a solution X=0 called the null or trivial solution.
ii. If $\rho(A) = number\ of\ unknowns$, the system has unique solution i.e. trivial solution.
iii. If $\rho(A) < number\ of\ unknowns$, the system has an infinite number of solutions.

- Write conditions for consistency of a system of equations in case of Cramer's rule.

Sol. Let the system of equations be

$$a_{11}x + a_{12}y + a_{13}z = b_1, a_{21}x + a_{22}y + a_{23}z = b_2, \qquad a_{31}x + a_{32}y + a_{33}z = b_3$$

Then $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$,

$$\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

i. If $\Delta \neq 0$, then $x = \frac{\Delta_1}{\Delta},\ \ y = \frac{\Delta_2}{\Delta},\ \ z = \frac{\Delta_3}{\Delta}$
ii. If $\Delta = 0$ and at least one of $\Delta_1, \Delta_2, \Delta_3 \neq 0$ then system is inconsistent i.e. has no solution.
iii. If $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$, then system has infinitely many solutions.

- Write conditions for consistency of a system of equations in case of matrix inversion method.

Sol. Let the system of equations be

$$a_{11}x + a_{12}y + a_{13}z = b_1, a_{21}x + a_{22}y + a_{23}z = b_2, \qquad a_{31}x + a_{32}y + a_{33}z = b_3$$

Then $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

i. If $|A| \neq 0$, then system has unique solution.

The system can be written as $AX = B \xrightarrow{yields} X = A^{-1}B$

ii. If $|A| = 0\ and\ (adj.A)B = 0$, then system is consistent and has infinitely many solutions.
iii. If $|A| = 0\ and\ (adj.A)B \neq 0$, then system is inconsistent.

- Define normal form of a matrix.

Sol. Any non-zero matrix of order $m \times n$ can be reduced to one of the following forms by performing E-operations :

a) I_r
b) $\begin{bmatrix} I_r & 0 \end{bmatrix}$
c) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$
d) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

Where I_r is an identity matrix of order r. All these forms are known as normal forms of the matrix.

- Define minor of a matrix.

Sol. Let $\left[a_{ij}\right]$ be a square matrix of order n. Then the minor M_{ij} of a_{ij} in A is the determinant of the square sub-matrix of order (n – 1) obtained by leaving i^{th} row and j^{th} column of A.

e.g. $A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 7 & 4 \\ 6 & 5 & 1 \end{bmatrix}$

Then $M_{23} = \begin{vmatrix} 2 & 3 \\ 6 & 5 \end{vmatrix}$

- Define cofactor of a matrix.

Sol. Let $\left[a_{ij}\right]$ be a square matrix of order n. Then the minor C_{ij} of a_{ij} in A is equal to $(-1)^{i+j}$ times the determinant of the square sub-matrix of order (n – 1) obtained by leaving i^{th} row and j^{th} column of A.

e.g. $A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 7 & 4 \\ 6 & 5 & 1 \end{bmatrix}$

Then $C_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 3 \\ 6 & 5 \end{vmatrix}$

- For what value of ‘a’ the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & a \end{bmatrix}$ is a singular matrix?

Sol. A is singular if $|A| = 0 \xrightarrow{yields} a - 8 = 0 \xrightarrow{yields} a = 8$

- Find adjoint of $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$

Sol. $adj.(A) = \begin{bmatrix} s & -q \\ -r & p \end{bmatrix}$ (because adjoint of a square matrix of order 2 can be obtained by interchanging the diagonal elements and changing of signs of off-diagonal elements.

- If A is non-singular matrix, prove that $|A^{-1}| = \frac{1}{|A|}$

Proof: Since $|A| \neq 0 \xrightarrow{yields} A^{-1}$ exists such that $AA^{-1} = I = A^{-1}A \xrightarrow{yields} |AA^{-1}| = |I|$

$\xrightarrow{yields} |A||A^{-1}| = 1 \xrightarrow{yields} |A^{-1}| = \frac{1}{|A|}$

- If A is non-singular matrix of order n, then what is $|adj\ A|$?

Sol. $|adj\ A| = |A|^{n-1}$

Proof: $A(adj\ A) = |A|I_n \xrightarrow{yields} A(adj\ A) = \begin{bmatrix} |A| & 0 & \dots.. & 0 \\ 0 & |A| & \dots.. & 0 \\ \dots.. & . & \dots.. & \dots.. \\ 0 & 0 & \dots\dots & |A| \end{bmatrix}_{n\times n}$

$\xrightarrow{yields} |A(adj\ A)| = \begin{vmatrix} |A| & 0 & \dots.. & 0 \\ 0 & |A| & \dots.. & 0 \\ \dots.. & . & \dots.. & \dots.. \\ 0 & 0 & \dots\dots & |A| \end{vmatrix} = |A|^n \xrightarrow{yields} |A||adj\ A| = |A|^n$

$\xrightarrow{yields} |adj\ A| = |A|^{n-1}$

- If A and B are two non-singular matrices of same order, then what is $j\ AB$?

Sol. $adj\ AB = (adj\ B)(adj\ A)$

- If $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$, find $adj\ AB$

Sol. Since adjoint of matrix is $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$ is $\begin{bmatrix} s & -q \\ -r & p \end{bmatrix}$

$\therefore adjA = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$ and $adjB = \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}$

Also $adj\ AB = (adj\ B)(adj\ A) = \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$

OR

$AB = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} 67 & 87 \\ 47 & 61 \end{bmatrix}$

$\therefore adj\ AB = \begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$

- If A is an invertible square matrix, then what is $j\ A^T$?

Sol. $adj\ A^T = (adj\ A)^T$

- If A is a non-singular square matrix, then $adj(adjA)\ is$?

Sol. $adj(adjA) = |A|^{n-2}A$

- Find the inverse of $A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$

Sol. $|A| = 2 + 15 = 17$

$$adj.(A) = \begin{bmatrix} 1 & -5 \\ 3 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adj.(A) = \frac{1}{17}\begin{bmatrix} 1 & -5 \\ 3 & 2 \end{bmatrix}$$

- If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$, find K so that $A^2 = KA - 2I$

Sol. **L.H.S.** $A^2 = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix}$

R.H.S. $KA - 2I = \begin{bmatrix} 3K - 2 & -2K \\ 4K & -2K - 2 \end{bmatrix}$

$$A^2 = KA - 2I \xrightarrow{yields} \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} 3K - 2 & -2K \\ 4K & -2K - 2 \end{bmatrix}$$

$$\xrightarrow{yields} 3K - 2 = 1, \quad -2 = -2K, \quad 4 = 4K, \quad -4 = -2K - 2 \quad \xrightarrow{yields} K = 1$$

- If A and B are two square matrices of order 3 such that $|A| = -1\ \ and\ \ |B| = 2$, then what will be the value of $|4AB|$?

Sol. We know that $|AB| = |A||B|$ and $|KA| = K^n|A|$, if A is matrix of order $n \times n$ and AB

Is a matrix of order 3×3

$$\therefore |4AB| = 4^3|A||B| = 64(-1)(2) = -128$$

- Find rank of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Sol. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$\therefore \rho(A) \neq 3$$

All the minors of order 2×2 are zero, $\therefore \rho(A) \neq 2$

$M_{11} = 1$

$\therefore \rho(A) = 1$

- Construct a 2×3 matrix with elements $a_{ij} = \frac{(i-2j)^2}{2}$

Sol. Since $a_{ij} = \frac{(i-2j)^2}{2}$

$\therefore\ a_{11} = \frac{(1-2)^2}{2} = \frac{1}{2}$, $a_{12} = \frac{(1-4)^2}{2} = \frac{9}{2}$, $a_{13} = \frac{(1-6)^2}{2} = \frac{25}{2}$

$a_{21} = \frac{(2-2)^2}{2} = 0$, $a_{22} = \frac{(2-4)^2}{2} = 2$, $a_{23} = \frac{(2-6)^2}{2} = 8$

$\therefore$ Required matrix is $\begin{bmatrix} \frac{1}{2} & \frac{9}{2} & \frac{25}{2} \\ 0 & 2 & 8 \end{bmatrix}$

- Find rank of $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -3 & 0 \\ 3 & -3 & 1 \end{bmatrix}$

Sol. $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -3 & 0 \\ 3 & -3 & 1 \end{bmatrix}$

$\begin{vmatrix} 1 & 1 & 0 \\ 2 & -3 & 0 \\ 3 & -3 & 1 \end{vmatrix} = 1(-3) - 1(2) + 0 = -5 \neq 0 \therefore \rho(A) = 3$

- Define rank of $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -9 & 2 \\ 5 & -3 & 4 \end{bmatrix}$

Sol. $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -9 & 2 \\ 5 & -3 & 4 \end{bmatrix}$ (Operating $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 5R_1$)

$\xrightarrow{yields} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -12 & -1 \\ 0 & -8 & -1 \end{bmatrix}$ (Operating $R_2 \to \frac{R_2}{-12}$)

$\xrightarrow{yields} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{1}{12} \\ 0 & -8 & -1 \end{bmatrix}$ (Operating , $R_3 + 8R_2$) $\xrightarrow{yields} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{1}{12} \\ 0 & 0 & \frac{-4}{12} \end{bmatrix}$

Since number of non-zero rows in Echelon form is 3, $\therefore \rho(A) = 3$

- Find the value of x such that $[1 \quad 1 \quad x] \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = [0]$

Sol. $\begin{bmatrix} 1 & 1 & x \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = [0] \xrightarrow{yields} \begin{bmatrix} 1+2x & 2+x & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = [0]$

$\xrightarrow{yields} [1 + 2x + 2 + x + 3] = [0] \xrightarrow{yields} 3x + 6 = 0 \xrightarrow{yields} x = -2$