Example 1. *Using the equation,* $\frac{P \sin \alpha a}{\alpha a} + \cos \alpha a = \cos ka$, *for the Kronig Penney model, where the letters have their usual meanings, show that if* $P << 1$, *the energy of lowest band is given by* $E = \frac{\hbar^2 P}{ma^2}$.

Solution. The given equation is $\frac{P \sin \alpha a}{\alpha a} + \cos \alpha a = \cos ka$.

The energy of the lowest band corresponds to

$$ka = \pm \frac{\pi}{a} \quad i.e., \quad \cos ka = \pm 1$$

$$\therefore \quad \frac{P \sin \alpha a}{\alpha a} + \cos \alpha a = 1 \qquad \text{[taking only +1 for simplicity]}$$

$$\therefore \quad \frac{P\left(2 \sin \frac{\alpha a}{2} \cos \frac{\alpha a}{2}\right)}{\alpha a} = 1 - \cos \alpha a = 2 \sin^2 \frac{\alpha a}{2}$$

$$\Rightarrow \quad \frac{\sin \frac{\alpha a}{2}}{\cos \frac{\alpha a}{2}} = \frac{P}{\alpha a} \quad \Rightarrow \quad \tan \frac{\alpha a}{2} = \frac{P}{\alpha a}$$

For $P << 1$, $\frac{P}{\alpha a} \simeq \tan \frac{P}{\alpha a}$

$$\therefore \text{ from eqn. (1),} \quad \tan \frac{\alpha a}{2} = \tan \frac{P}{\alpha a} \quad \Rightarrow \quad \frac{\alpha a}{2} = \frac{P}{\alpha a}$$

$$\therefore \quad \alpha^2 = \frac{2P}{a^2} \qquad \text{...(2)}$$

$$\text{But} \quad \alpha^2 = \frac{2mE}{\hbar^2} \qquad \text{...(3) (see Kronig Penney model)}$$

Comparing eqns. (2) and (3)

$$\frac{2mE}{\hbar^2} = \frac{2P}{a^2} \quad \Rightarrow \quad E = \frac{\hbar^2 P}{ma^2}$$

Example 2. *Using the Kronig Penney model, show that the energy of electron becomes* $E = \frac{\hbar^2 k^2}{2m}$ *if the electron is a free electron.*

Solution. For a free electron, $V_0 = 0$.

Therefore $$P = \frac{mV_0ab}{\hbar^2} = 0$$

∴ from equation

$$\frac{P\sin\alpha a}{\alpha a} + \cos\alpha a = \cos ka$$

We have $$\cos\alpha a = \cos ka \quad i.e., \quad \alpha a = ka$$

or $$\alpha = k \qquad \text{...(}$$

But $$\alpha^2 = \frac{2mE}{\hbar^2} \quad \text{or} \quad E = \frac{\alpha^2\hbar^2}{2m}.$$

Therefore, form eqn. (1).

$$E = \frac{k^2\hbar^2}{2m}.$$