EXERCISE 6 (*b*)

1. If $\beta(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$, then what is value of $\beta\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$.

2. (*a*) Show that $\Gamma(n) = k^n \displaystyle\int_0^\infty x^{n-1} e^{-kx}\, dx$

   (*b*) Show that $\Gamma(n) = \displaystyle\int_0^\infty e^{-x^{\frac{1}{n-1}}}\, dx$

3. Show that $\Gamma(n) = 2 \displaystyle\int_0^\infty x^{2n-1} e^{-x^2}\, dx$

4. Show that

   (*i*) $\displaystyle\int_{-\infty}^0 e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$ (*ii*) $\displaystyle\int_{-\infty}^\infty e^{-x^2}\, dx = \sqrt{\pi}$

5. Evaluate

   (*i*) $\dfrac{\Gamma(6)}{2\Gamma(3)}$ (*ii*) $\dfrac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$ (*iii*) $\dfrac{\Gamma(3)\,\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{11}{2}\right)}$ (*iv*) $\dfrac{6\,\Gamma\left(\frac{8}{3}\right)}{5\Gamma\left(\frac{2}{3}\right)}$

6. Show $\displaystyle\int_0^\infty x^6\, e^{-2x}\, dx = \frac{45}{8}$.

7. Show (*i*) $\displaystyle\int_0^\infty \sqrt{y}\, e^{-y^3}\, dy = \frac{\sqrt{\pi}}{3}$ (*ii*) $\displaystyle\int_0^\infty (3)^{-4x^2}\, dx = \frac{\sqrt{\pi}}{4\sqrt{\log 3}}$

8. Evaluate

(i) $\int_0^\infty e^{-x^3}\,dx$ (ii) $\int_0^\infty x^{\frac{1}{4}} e^{-\sqrt{x}}\,dx$ (iii) $\int_0^1 x^2(1-x)^3\,dx$

(iv) $\int_0^1 \sqrt{\frac{1-x}{x}}\,dx$ (v) $\int_0^1 x^5(1-x^3)^{10}\,dx$ (vi) $\int_0^\infty e^{-x^4}\,dx$ (P.T.U. 20..)

9. Evaluate

(i) $\int_0^\infty e^{-4x} x^{\frac{3}{2}}\,dx$ (ii) $\int_0^2 \frac{x^2}{\sqrt{2-x}}\,dx$ (iii) $\int_0^1 x^4(1-x)^3\,dx$

(iv) $\int_0^\infty \frac{1}{x^{\frac{7}{4}}} e^{-\sqrt{x}}\,dx$ (P.T.U. 20..)

10. (a) Evaluate $\int_0^{\frac{\pi}{2}} \sin^5 x\,dx$

(b) Using Beta and Gamma functions, evaluate $\int_{-1}^1 (1-x^2)^n\,dx$, where n is a positive integer.

(P.T.U. 20..)

11. Show $\int_0^\infty x^m e^{-ax^n}\,dx = \frac{1}{n(a)^{\frac{m+1}{n}}}\Gamma\left(\frac{m+1}{n}\right)$

where m, n, a are positive constants

12. Show that $\int_0^a y^4 \sqrt{a^2-y^2}\,dy = \frac{\pi a^6}{32}$ by using Beta, Gamma function.

13. Show $\int_0^\infty \frac{x}{1+x^6}\,dx = \frac{1}{3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{\pi}{3\sqrt{3}}$.

14. (a) Show $\int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx = \frac{1}{8}\sqrt{\frac{2}{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$

(b) Show that $\int_0^1 y^{q-1}\left(\log\frac{1}{y}\right)^{p-1} dy = \frac{\Gamma(p)}{q^p}$, where $p > 0$, $q > 0$.