Report On Generative Model and Objective Function

The purpose of our generative model here, is to remove the bias field in the brain MRI image and segment the image into 3 clusters according to the intensities. The bias field can be generated due to non-static magnetic field or RadioFrequency(RF) Coil. Vignetting is another effect which might change the actual intensity of a voxel. Considering the image has N voxels,
At voxel i , intensity be x_i , bias field be b_i and observed intensity be y_i.
Also we consider the noise model(n) to be I.I.D additive zero-mean Gaussian.i.e. $n \sim G(0, \sigma^2)$

$$y_i = x_i b_i + n_i$$

Also since there are 3 tissues in a brain , thus each tissue has its (uncorrupted) intensity represented by c_k.
Thus, At voxel i, if its neighbor j belongs to class k, then,

$$x_j = c_k$$
$$b_j \approx b_i$$
$$\Rightarrow x_j b_j \approx c_k b_i$$

Our strategy is to minimize the difference between the observed neighborhood intensities(y_j) and intensities predicted by model($x_j b_j$).
For a particular voxel i, and it’s neighbor voxel j :
Since we don’t know what class j lies in (due to bias field which alters the intensity of a voxel) we find difference for all class weighted by membership $(u_{jk})^q$
in each class with the constraint that $\forall j$: $\sum_{k=1}^{3} u_{jk} = 1$ and q is the fuzziness parameter i.e. $(u_{jk})^q \times (y_j - c_k b_i)$.

Also since bias field varies smoothly , it is nearly constant for a short range in neighborhood of a voxel i and thus $x_j b_j \approx c_k b_i$ holds only for a short range. And thus we define weight (w_{ij}) which is 0 for distance between voxels > threshold. And for distance between voxels < threshold , weights are chosen such that they sum to 1. Weights can be chosen as Gaussian and its width is determined by smoothness of bias field i.e. if bias field is smooth it will be nearly equal for a greater range around a voxel while if bias field is irregular(say around a edge) then since bias field is irregular $x_j b_j \approx c_k b_i$ holds only for small range till bias field is regular. Hence, our objective function is :

$$J = \sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij} \sum_{k=1}^{K} ((u_{jk})^q (y_j - c_k b_i)^2)$$

$$J = \sum_{j=1}^{N}\sum_{k=1}^{K} (u_{jk})^q (\sum_{i=1}^{N} w_{ij} (y_j - c_k b_i)^2)$$

$$J = \sum_{j=1}^{N}\sum_{k=1}^{K} (u_{jk})^q d_{jk}$$

where d_{jk} is the distance between mean c_k and y_j .

We optimize our objective function using Lagrange multipliers

$$L(\{u_{jk}\}, \{b_i\}, \{c_k\}, \{\lambda_j\}) = \sum_{j=1}^{N}\sum_{k=1}^{K} (u_{jk})^q d_{jk} + \sum_{j=1}^{N} \lambda_j (1 - \sum_k u_{jk})$$

Differentiating L w.r.t $\{u_{jk}\}, \{b_i\}, \{c_k\}, \{\lambda_j\}$ and setting them to 0 gives us:

$$u_{jk} = \frac{(\frac{1}{d_{kj}})^{\frac{1}{q-1}}}{\sum_k (\frac{1}{d_{kj}})^{\frac{1}{q-1}}}$$

$$b_i = \frac{\sum_{j=1}^{N} w_{ij} y_j \sum_{k=1}^{K} (u_{jk})^q c_k}{\sum_{j=1}^{N} w_{ij} \sum_{k=1}^{K} (u_{jk})^q c_k^{\ 2}}$$

$$c_k = \frac{\sum_j (u_{jk})^q y_j \sum_i w_{ij} b_i}{\sum_j (u_{jk})^q \sum_i w_{ij} b_i^2}$$

If memberships are crisp i.e. one voxel belongs to only one class then:

$$b_i \approx \frac{\sum_j w_{ij} y_j c_{k(j)}}{\sum_j w_{ij} c_{k(j)}^2}$$

Also, if bias field varies slowly over neighborhood then:

$$c_k \approx \frac{\sum_{j \in C^k} y_j b_j}{\sum_{j \in C^k} b_j^2}$$