TENSORS:

More than 2 axes.

KEY NOTES:

1. $C = A + b \rightarrow$ ADDED TO EACH ROW

   $C_{i,j} = A_{i,j} + b_j$

2. ELEMENT-WISE PRODUCT: $A \odot B$.

3. $Ax = b$ $\quad b = m \times 1$ $\quad A = m \times n$

   $x = A^{-1} b$ $\quad x = n \times 1$

   A^{-1} EXISTS ONLY IF

   $\Rightarrow m = n$

   $\Rightarrow$ COLUMNS - LINEARLY INDEPENDENT

FOR A^{-1} TO EXIST

↳ NOT 0 SOLUTIONS

↳ NOT ∞ SOLUTIONS

] OF x FOR GIVEN b

TO GET A SOLUTION FOR ALL B,

Ax MUST SPAN ALL $\mathbb{R}^m$.

FOR THIS THE COLUMNS OF A MUST BE ATLEAST $m \Rightarrow n \geq m$ (n should have m independent columns)

IF $n > m$, THEN EACH SUBSET OF m INDEPENDENT COLUMNS $\Rightarrow$ 1 SOLUTION

SO NOT UNIQUE.

UNIQUE $\Rightarrow \boxed{n = m}$

SINGULAR MATRIX:

SQUARE MATRIX WITH LINEARLY DEPENDENT COLUMNS.

NORMS:

L^p : $\|x\|_p = \left(\sum_i |x_i|^p\right)^{1/p}$

EUCLIDEAN NORM:

L^2 NORM $\|x\|$

$$\left(\sum_i |x_i|^2\right)^{1/2}$$

squared L^2 NORM is preferred.

L^1 NORM:

If difference between 0 and non-zero is of atmost importance, use L1 NORM.

$$\|x_1\| = \sum_i |x_i|$$

MAX NORM:

L^∞ NORM: $\|x\|_\infty = \max_i |x_i|$

FROBENIUS NORM:

$$\|A\|_F = \sqrt{\sum_{i,j} A_{i,j}^2}$$

MATRICES:

DIAGONAL MATRIX:

$$D_{ij} = 0 \quad \forall \; i \neq j$$

* Can be non-square

$$D = \begin{bmatrix} a & & & 0 \\ & b & & \\ & & \ddots & \\ 0 & & & z \end{bmatrix}$$

$$v = [a \; b \ldots z]^T$$

$$D = \mathrm{diag}(v)$$

$$D^{-1} = \begin{bmatrix} 1/a & & & 0 \\ & 1/b & & \\ & & \ddots & \\ 0 & & & 1/z \end{bmatrix}$$

SYMMETRIC MATRIX:

$$A = A^T$$

ORTHOGONAL MATRIX:

→ Square

→ Rows are mutually Orthonormal

→ Columns are mutually orthonormal

$$A^T A = A A^T = I$$

$$\Rightarrow A^{-1} = A^T.$$

MATRIX DECOMPOSITION:

EIGENDECOMPOSITION:

$$A v = \lambda v$$

$\lambda \rightarrow$ eigenvalues

$v \rightarrow$ eigenvectors

$$A = V \operatorname{diag}(\lambda) V^{-1}$$

$$A = V \Delta V^{-1}$$

If $A \rightarrow$ real symmetric matrix

$$A = Q \Delta Q^T$$

Q: orthogonal

Δ: diagonal

PROPERTIES:

* A is singular iff $\lambda_i = 0$ for some i.

* $f(x) = x^T A x$, A : symmetric , $\|x\|_2 = 1$

  x : eigenvectors

$$A = A^T \qquad Ax = \lambda x$$

$$f(x) = x^T \lambda x$$

$$= \lambda x^T x$$

$$= \lambda \|x\|_2$$

$$= \lambda$$

$$v_i \xrightarrow{I/P} f(x) \xrightarrow{O/P} \lambda_i$$

POSITIVE DEFINITE:

$$\lambda_i > 0$$

POSITIVE SEMIDEFINITE:

$$\lambda_i \geq 0 .$$

Singular Value Decomposition:

$$A = U D V^T$$

$$\underset{m \times n}{A} = \underset{m \times m}{U}\ \underset{m \times n}{D}\ \underset{n \times n}{V^T}$$

U, V : orthogonal

D : Diagonal

U columns : left-singular vectors = EV of AA^T

V columns : right-singular vectors = EV of A^TA

D : singular values = $\sqrt{\text{Evalues}}$ of A^TA / AA^T.

Moore - Penrose Pseudoinverse:

$$Ax = y$$

$$x = A^+ y$$

[A^{-1} doesn't exist]

$$A^+ = V \underset{\text{(Reciprocal of non-zero diagonals of } D)^T}{D^+} U^T$$

if $\quad cols(A) > rows(A)$:

$x = A^{+} y$ gives one of the ∞ solutions, such that

$\|x\|_2$ is least.

if $\quad rows(A) > cols(A)$:

no solution

we get x such that $\|Ax - y\|_2$ is least.

TRACE:

$Tr(AB) = Tr(BA)$

$Tr(ABC) = Tr(CAB) = Tr(BCA) \neq Tr(BAC)$

Only cyclic permutation

DETERMINANT :

$det(x) = 0 \implies$ contracts space to 0 volume

$det(x) = 1 \implies$ transformation preserves volume.

EIGENDECOMPOSITION :

EIGENVALUES :

$$\sum \lambda_i = \text{trace}(A) \quad || \quad \prod \lambda_i = \det(A)$$

$$Ax = \lambda x$$

$$\underbrace{(A - \lambda I)}_{B} x = 0$$

$$B_{11}x_1 + B_{12}x_2 + \dots + B_{1n}x_n = 0$$

$$B_{21}x_1 + B_{22}x_2 + \dots + B_{2n}x_n = 0$$

$$\vdots$$

$$\underline{B_{n1}}x_1 + B_{n2}x_2 + \dots + B_{nn}x_n = 0$$

$$B_1^T x + B_2^T x + \dots + B_n^T x = 0$$

column

We know x is not 0.

So $B_1, B_2 \dots B_n$ are linearly dependent

As B is a square matrix

linear dependence $\Rightarrow \det = 0$

$$\det(B) = 0$$

$$\det(A - \lambda I) = 0$$

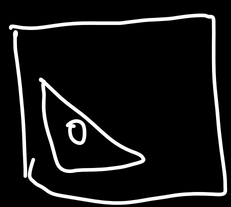
IT IS A SINGULAR MATRIX

ORTHOGONAL MATRICES:

Rotates space

→ complex E Values

TRIANGULAR MATRICES :



→ Repeated E values

→ $< n$ independent Evectors.

SYMMETRIC MATRICES:

Always real E Values.

E. Vectors orthogonal

$A = A^T$

USUAL $A = S \Lambda S^{-1}$

SYMMETRIC:

$$A = Q \Lambda Q^{-1} = Q \Lambda Q^T$$

Q ↓ Orthogonal

Why real eigenvalues?

$$Ax = \lambda x \overset{\text{always}}{\Longrightarrow} A\bar{x} = \bar{\lambda}\bar{x}$$

We assume real matrices

$$\bar{x}^T A^T = \bar{x}^T \bar{\lambda}^T$$

$$A^T = A$$

$$\bar{x}^T A = \bar{x}^T \bar{\lambda}^T$$

$$\bar{x}^T A x = \lambda \bar{x}^T x \qquad \bar{x}^T A x = \bar{\lambda} \bar{x}^T x$$

COMPARE

$$\lambda \bar{x}^T x = \bar{\lambda} \bar{x}^T x$$

$\lambda = \bar{\lambda}$ $\qquad$ λ is real

$$\bar{x}^T x = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$\downarrow$

> 0

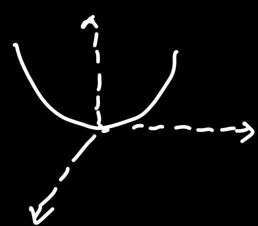
↳ its 0 only if all x is 0.

POSITIVE DEFINITE : (MATRIX S)

$\lambda_i > 0$

$x^T S x > 0$ for all $x \neq 0$

Convex function :



QUADRATIC : Convex can be

$x^T S x \geq 0$

$\lambda_i \geq 0$

S, T → positive definite

→ $S + T$

$$x^T(S+T)x = x^T S x + x^T T x > 0$$

$(S+T)$ is positive definite //

$\rightarrow S^{-1}$?

S^{-1} has eigenvalues $1/\lambda$ (Symmetric).

So S^{-1} is PD //

$\rightarrow$ $A = S^{-1} B S$ SIMILAR MATRICES

S : nonsingular

A and B are Similar matrices

$\Rightarrow$ Same eigenvalues.

EIGENDECOMPOSITION:

n Independent eigenvectors

Put them in columns of S

$$AS = A\begin{bmatrix} \vdots & \vdots & & \vdots \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

$$= \begin{bmatrix} \vdots & & \vdots \\ x_1 & \cdots & x_n \\ \vdots & & \vdots \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & 0 \\ 0 & \vdots & \cdots & 0 \\ 0 & & & \vdots \\ 0 & 0 & & \lambda_n \end{bmatrix}$$

$$= S\Lambda$$

Diagonal eigenvalue matrix

$$S^{-1}AS = S^{-1}S\Lambda = \Lambda$$

possible only if eigenvectors are independent

$$\underline{\underline{S\Lambda S^{-1} = A}}$$

DECOMPOSISTION

If $Ax = \lambda x$

$A^2 x = \lambda \underline{Ax} = \lambda^2 x$

Eigenvalues of $A^2 = \lambda^2$

Eigenvectors of A^2: Same as A.

or

$A = S \Lambda S^{-1}$

$A^2 = S \Lambda \underline{S^{-1} S} \Lambda S^{-1} = S \Lambda^2 S^{-1}$

$A^k = S \Lambda^k S^{-1}$ //

THEOREM:

$A^k \to 0$ as $k \to \infty$

if all $|\lambda_i| < 1$.

A is sure to have n different eigenvectors,

if λ_i s are different.

if A has repeated eigenvalues, then we may

or may not have n independent eigenvectors.

eg: I has n evalues $\rightarrow (1, 1, \dots 1)$

all vectors are evectors

↳ choose n independent.

→ Not for triangle matrices.

SINGULAR VALUE DECOMPOSITION:

We have a set of orthogonal Basis

↓ Apply A

Generate a set of orthogonal Basis again

new orthogonal Basis

$$v_1 \xrightarrow{A} Av_1 \qquad u_1$$

$$v_2 \xrightarrow{A} Av_2 \qquad u_2$$

$Av_1 \rightarrow$ same direction as u_1

$$Av_1 = \sigma_1 u_1$$

$$Av_2 = \sigma_2 u_2$$

$$A \begin{bmatrix} \vdots & \vdots & & \vdots \\ v_1 & v_2 & \dots & v_n \\ \vdots & \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ u_1 & u_2 & \dots & u_n \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \ddots \sigma_n \; 0 \end{bmatrix}$$

$$AV = U\Sigma$$

$$A = \underset{\substack{m \times r \\ \text{left} \\ \text{singular} \\ \text{vectors}}}{U}\ \underset{\substack{r \times r \\ \text{singular values}}}{\Sigma}\ \underset{\substack{r \times n \\ \text{right singular vectors}}}{V^T}$$

r = rank

OR

$U \to m \times m$

$\Sigma \to m \times n$

$V \to n \times n$

$$\begin{bmatrix} \sigma_1 & \cdots & 0 \\ & \sigma_r & \\ 0 & & 0 \end{bmatrix}$$

others 0

$A \to m \times n$

$U \to m \times n$

$V \to n \times m$

Σ

$A^T A$

$n \times m \quad m \times n \qquad n \times n$

$\to$ symmetric

$\to$ positive definite

$$A^T A = V \Lambda V^T$$

Λ: ≥ 0

V: orthogonal

EIGENDECOMPOSITION

MATRIC IS POSITIVE DEFINITE IF

$$S^T A S > 0$$

$$S^T (A^T A) S$$

$$(AS)^T (AS)$$

$$\|AS\|^2 > 0 \quad //$$

$$\underset{m\times n}{A} \; \underset{n\times m}{A^T} \rightarrow m\times m$$

$$AA^T = U \Lambda U^T$$

$\downarrow$

same eigenvalues as $A^T A$

(but only m instead of n)

if $m<n$, remaining are 0's.

$$(A^T A)\vec{v} = \lambda \vec{v}$$

$$A(A^T A)\vec{v} = \lambda A \vec{v}$$

$$(AA^T)(A\vec{v}) = \lambda (A\vec{v})$$

$\hookrightarrow$ same

Eigenvectors are $\vec{u} = A\vec{v}$.

$\downarrow$

Find these vector combinations

such that

$$A v_1 = \sigma_1 u_1$$

$$\vdots$$

$$A v_r = \sigma_r u_r$$

r = rank

$$A \begin{bmatrix} v_1 & \cdots & v_r \end{bmatrix} = \begin{bmatrix} u_1 & \cdots & u_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}$$

$$AV = U\Sigma$$

$$A = U\Sigma V^{-1}$$

$$= U\Sigma V^T$$

$$A^T A = V\Sigma^T U^T U \Sigma V^T = V(\Sigma^T \Sigma) V^T$$

eigenvectors of $A^T A$

eigenvalues $\rightarrow \sigma_i^2$

$$AA^T = U(\Sigma\Sigma^T) U^T$$

WHAT IF $\sigma_i = \sigma_j$

and v_i and v_j can be 2 orthogonal from a plane of possible vectors.

Will u_i and u_j be orthogonal always?

$$Av_1 = \sigma_1 u_1$$

$$u_1 = \frac{Av_1}{\sigma_1}$$

$$u_2 = \frac{Av_2}{\sigma_2}$$

$$u_1^T u_2 = \left(\frac{Av_1}{\sigma_1}\right)^T \left(\frac{Av_2}{\sigma_2}\right) = \frac{v_1^T A^T A v_2}{\sigma_1 \sigma_2}$$

$$= \frac{v_1^T \sigma_2^2 v_2}{\sigma_1 \sigma_2}$$

$$= 0 \;//$$

(Computation)

NEVER USE A^TA IN REAL LIFE TO FIND u or v

as A^TA has a lot of rounding off.

So SVD says

$$Ax = U \Sigma V^T x$$

A matrix transformation is a

rotation / reflection $\rightarrow$ stretch $\rightarrow$ rotation / reflection

$V^T x$ $\quad$ $\Sigma V^T x$ $\quad$ $U \Sigma V^T x$

of all vectors.

PRODUCT OF σ_i $\rightarrow$ DETERMINANT OF A.

$$\sigma_1 \leq \lambda_1 \leq \lambda_2 \leq \sigma_2$$

$$\lambda_1 \lambda_2 = \sigma_1 \sigma_2$$