HILBERT SPACES:

Complex number $z = a + b \cdot i$

$$z^* = a - b \cdot i$$

conjugation

$$|z|^2 = z \cdot z^* = (a+bi)(a-bi)$$

$$= a^2 + b^2$$

norm

length

$\hookrightarrow$ $|z| = \sqrt{a^2+b^2}$

EULER'S FORMULA:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

HILBERT SPACE:

Vector space + inner product + (some conditions)

Scalar = complex number

$$\alpha = \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{k-1} \end{bmatrix} \quad \beta = \begin{bmatrix} \beta_0 \\ \vdots \\ \beta_{k-1} \end{bmatrix}$$

INNER PRODUCT:

$$\left\langle \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{k-1} \end{pmatrix} \middle| \begin{pmatrix} \beta_0 \\ \vdots \\ \beta_{k-1} \end{pmatrix} \right\rangle = \sum_i \alpha_i^* \cdot \beta_i$$

DIRAC NOTATION

$|v\rangle$, $|w\rangle$

$$\langle v| \cdot |w\rangle = \langle v|w\rangle$$

$\underline{}$

MATRIX PRODUCT

linearly independent vectors that span the space.

BASIS

$|v\rangle = a_0|0\rangle + a_1|1\rangle$

$|w\rangle = b_0|0\rangle + b_1|1\rangle$

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$^{*T} \longleftrightarrow \dagger$ "dagger"

$$\langle v|w\rangle = \left\langle \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}^{*} \middle| \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} \right\rangle$$

$$= \left\langle \begin{pmatrix} a_0^* & a_1^* \end{pmatrix} \middle| \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} \right\rangle$$

$$= a_0^* \cdot b_0 + a_1^* \cdot b_1$$

$$\langle 0|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{+} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= 0 \,//$$

OUTER PRODUCT:

$$|\alpha\rangle\langle\omega| = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}^{+}$$

$$\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} \begin{bmatrix} \alpha_2^* & \beta_2^* \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 \alpha_2^* & \alpha_1 \beta_2^* \\ \beta_1 \alpha_2^* & \beta_1 \beta_2^* \end{bmatrix}$$

$|v\rangle \quad |w\rangle$

INNER: $v^{\dagger} \cdot w$

OUTER $v \cdot w^{\dagger}$

UNITARY MATRICES:

U, V

$UU^{\dagger} = U^{\dagger}U = I$

↳ the inverse of U = the conjugate transpose of U.

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$HH^{\dagger} = HH = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I \,//$$

NOT

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

PAULI MATRICES

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

KET 1

$$\text{NOT}|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$$

KET 0

$$\text{NOT}|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

$$\text{NOT}\left(\alpha|0\rangle + \beta|1\rangle\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \beta|0\rangle + \alpha|1\rangle //$$

ORTHONORMAL BASIS:

$$\{ |0\rangle, |1\rangle \}$$

$$\{ |+\rangle, |-\rangle \}$$

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

H : Change of basis.

$$H|0\rangle = |+\rangle$$

$$H|1\rangle = |-\rangle$$

$$H|+\rangle = |0\rangle$$

$$H|-\rangle = |1\rangle$$

QUBIT AND MEASUREMENTS:

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha|0\rangle + \beta|1\rangle$$

$\downarrow$ MEASURE

GET	: 0	1
PROBABILITY	: $\lvert\alpha\rvert^2$	$\lvert\beta\rvert^2$
AFTER	$\lvert 0\rangle$	$\lvert 1\rangle$

GENERALLY WE USE ONLY 2 BASIS

$|0\rangle, |1\rangle$ or

$|+\rangle, |-\rangle$ //

$|01011 \cdots 0\rangle$ is generally how it is.

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

got 00 with $P = |\alpha_{00}|^2$ and it becomes 00.

Measure n qubits, we get n bits.

3 qubits → can be any of 2^3 states

↓ once measured

we get 3bits.

MEASURE ONLY A FEW QUBITS:

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

Measure the first qubit and we get 0

with probability $|\alpha_{00}|^2 + |\alpha_{01}|^2$.

After measurement the qubit is:

$$\frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}}$$

$$= |0\rangle \otimes \left(\frac{\alpha_{00}|0\rangle + \alpha_{01}|1\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}} \right)$$

$\downarrow$

Probability of getting 0

$$\left(\frac{|\alpha_{00}|^2}{|\alpha_{00}|^2 + |\alpha_{01}|^2} \right)$$

$HH|0\rangle = |0\rangle$

$|0\rangle$

$|0\rangle - H - H$

$|+\rangle$ or $|-\rangle$

$|0\rangle - H - \not\!\!\alpha - H$

0 or 1

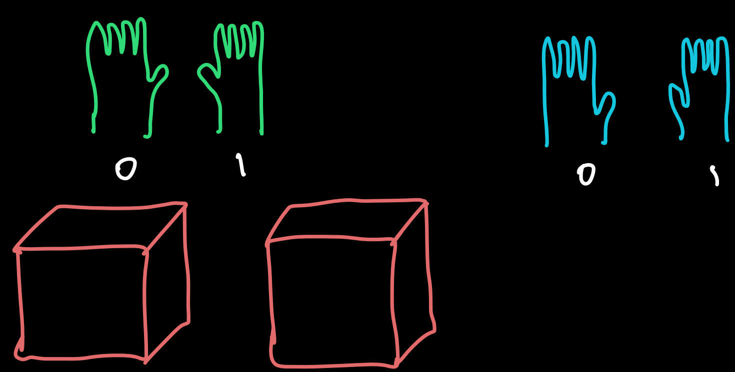
Order of measurement doesn't matter if there are no gates between them.

ENTANGLEMENT:

$$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

Alice and Bob are given 2 boxes. Each box has a glove.

Glove set 1: Glove set 2:



Take 2 boxes, take either both left or both right gloves and odd one to each.

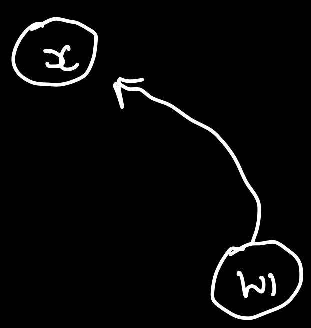
Alice moves to MARS and opens the box. If she has a left gloves, she knows Bob has left too.

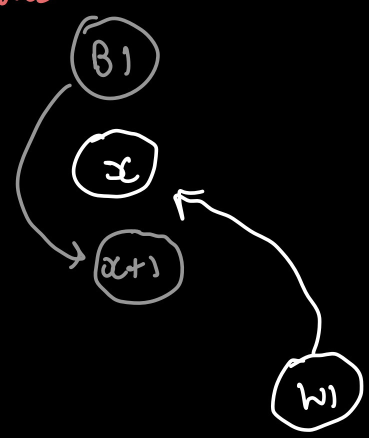
Assume quantum chess, white moves to position x with 50% or doesn't move with 50%.

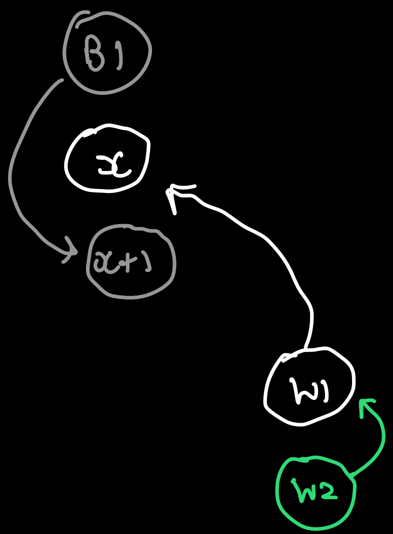
Now black moves to $x+1$, if white didn't move or if doesn't move.

Now white wants to move to previous white's position if it had moved the first time. Now we need to know if white had moved earlier.

We measure, 2 states possible. W move + B didn't move or W didn't move + B moved.







measure W1 move

2 States:

B1

W1 at x

W2 at prev W1 position

OR

B1 at x+1

W1

W2 stays

TENSOR PRODUCT:

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \otimes \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$$

a copy of the second vector for each element of the first vector

$$= \begin{bmatrix} \alpha_1 \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} \\ \beta_1 \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 \alpha_2 \\ \alpha_1 \beta_2 \\ \beta_1 \alpha_2 \\ \beta_1 \beta_2 \end{bmatrix}$$

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \\ A_{n1} & \cdots & A_{nm} \end{pmatrix}$$

$$\underbrace{\underset{n\times m}{A} \otimes \underset{p\times q}{B}}_{(n\cdot p)\times(m\cdot q)} = \begin{pmatrix} A_{11}B & \cdots & A_{1m}B \\ \vdots & & \vdots \\ A_{n1}B & \cdots & A_{nm}B \end{pmatrix}$$

DIRAC NOTATION:

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} 1\begin{pmatrix}1\\0\end{pmatrix} \\ 0\begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix}$$

$$= |0\rangle|0\rangle$$

$$= \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$|10\rangle = |1\rangle \otimes |0\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} 0\begin{pmatrix}1\\0\end{pmatrix} \\ 1\begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\left(|1000000\rangle + |111111\rangle\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \left.\vphantom{\begin{matrix}0\\\vdots\\0\end{matrix}}\right] \text{62 zeroes}$$

$$H \otimes I = \frac{1}{\sqrt{2}}\begin{bmatrix} 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & -1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$(H \otimes I)\,\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

IN DIRAC NOTATION

$$(H \otimes I)\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right)$$

$$= (H \otimes I)\,\frac{1}{\sqrt{2}}\left(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle\right)$$

$$= \frac{1}{\sqrt{2}} \left(H|0\rangle \otimes I|0\rangle + H|1\rangle \otimes I|1\rangle \right)$$

IMPORTANT

RULES:

* $(A \otimes B).(C \otimes D) = (A.C) \otimes (B.D)$

* $(\underbrace{\alpha}_{\text{scalar}} \cdot A) \otimes B = A \otimes (\alpha B) = \alpha (A \otimes B)$

* $|\psi\rangle . \underbrace{\langle\theta|}_{\text{bra}} . \underbrace{|\gamma\rangle}_{\text{ket}} = |\psi\rangle \underbrace{\langle\theta|\gamma\rangle}_{\text{scalar}}$

$= \langle\theta|\gamma\rangle . |\psi\rangle$

$$M = \begin{array}{cc} & \quad\; j \\ i & \begin{bmatrix} & \vdots & \\ \cdots & \boxed{M_{ij}} & \\ & & \end{bmatrix} \end{array}$$

$\underbrace{|i\rangle \langle j\rangle}_{\text{binary}}$

$$M = \sum_{ij} M_{ij} . |i\rangle\langle j|$$

$$A = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix} = A_{00} \cdot |0\rangle\langle 0| + A_{01} \cdot |0\rangle\langle 1| + A_{10} |1\rangle\langle 0| + A_{11} |1\rangle\langle 1|$$

$$|0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\underset{\substack{\downarrow \\ 57}}{|0111001\rangle} = \left.\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right] \text{Size} = 128$$

$1 \rightarrow 58^{TH}$ position