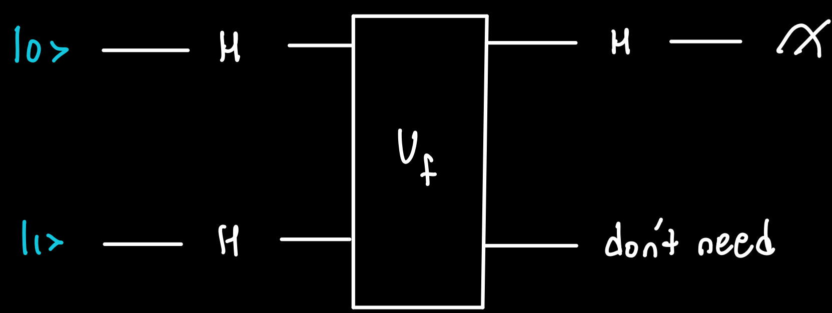
1) $f(0) = 0 \qquad f(1) = 1$



$(H \otimes I)\, U_f\, (H \otimes H)\, |01\rangle \qquad (0+1)(0-1)$

$= (H \otimes I)\, U_f\, \frac{1}{2} \left(|00\rangle - |01\rangle + |10\rangle - |11\rangle \right)$

$U_f |x\rangle |b\rangle = |x\rangle |b \oplus f(x)\rangle$

$= (H \otimes I)\, \frac{1}{2} \left(|0\rangle |0 \oplus f(0)\rangle - |0\rangle |1 \oplus f(0)\rangle + |1\rangle |0 \oplus f(1)\rangle - |1\rangle |1 \oplus f(1)\rangle \right)$

$= (H \otimes I)\, \frac{1}{2} \left(|00\rangle - |01\rangle + |11\rangle - |10\rangle \right)$

$= (H \otimes I)\, |-\rangle |-\rangle$

$= |1\rangle$

$\downarrow$

measure, we will get 1 => BALANCED

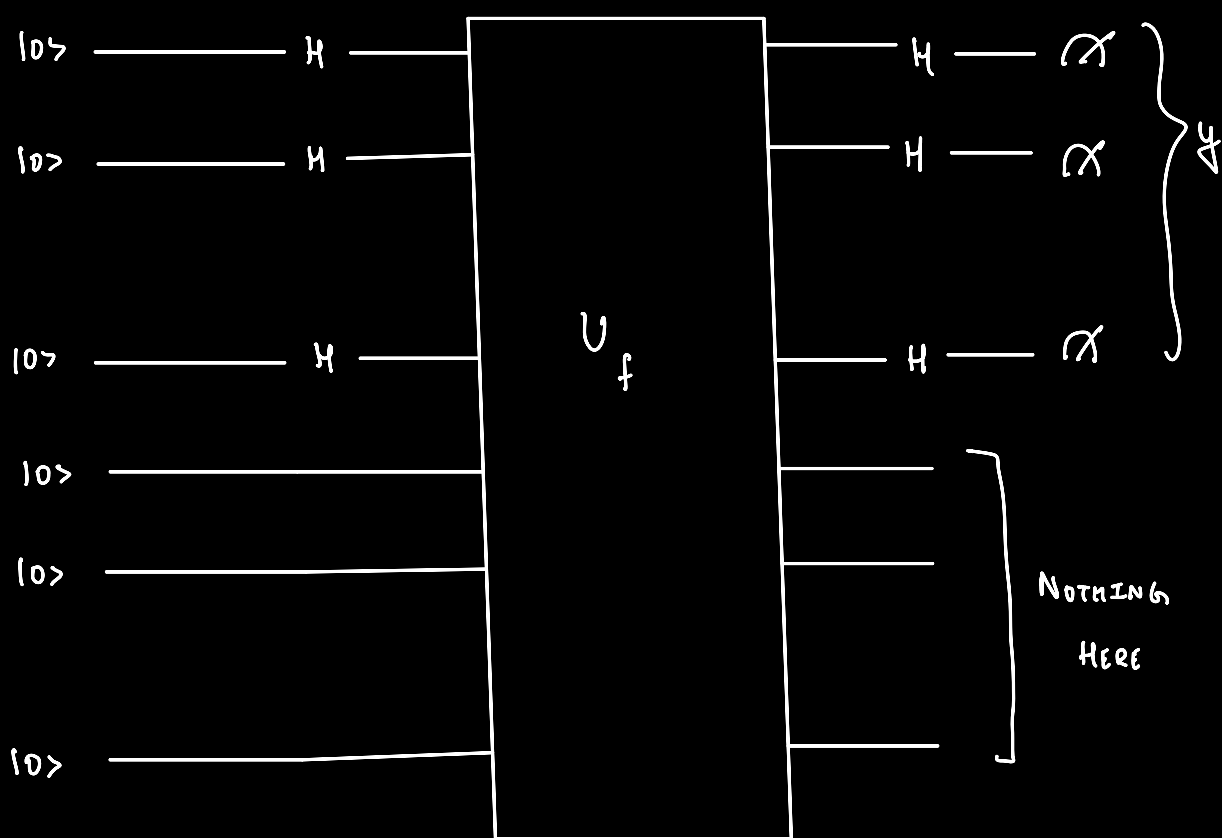
2) $n = 3$

$f(000) = f(011) = 010$

$f(001) = f(010) = 101$

$f(100) = f(111) = 110$

$f(101) = f(110) = 001$



STEP 1:

$$(H \otimes H \otimes H \otimes I \otimes I \times I)|0000000\rangle$$

$$= \left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right)$$
$$\otimes |000\rangle$$

$$= \tfrac{1}{2\sqrt{2}}\left(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle\right.$$
$$\left. + |110\rangle + |111\rangle\right) \otimes |000\rangle$$

STEP 2:

APPLY U_f

$$U_f(x, b) = (x, \overset{|000\rangle}{b} \oplus f(x)) = (x, f(x))$$

$$\tfrac{1}{2\sqrt{2}}\left(|000\,\underline{010}\rangle + |001\,\underline{101}\rangle + |010\,\underline{101}\rangle + |011\,\underline{010}\rangle\right.$$
$$\left. + |100\,\underline{110}\rangle + |101\,\underline{001}\rangle + |110\,\underline{001}\rangle + |111\,\underline{110}\rangle\right)$$

STEP 3:

APPLY $H^{\otimes 3}$

$$H^{\otimes 3}|000\rangle = \frac{1}{2\sqrt{2}}\left(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle\right)$$

$$H^{\otimes 3}|001\rangle = \frac{1}{2\sqrt{2}}\left(|000\rangle - |001\rangle + |010\rangle - |011\rangle + |100\rangle - |101\rangle + |110\rangle - |111\rangle\right)$$

$$H^{\otimes 3}|010\rangle = \frac{1}{2\sqrt{2}}\left(|000\rangle + |001\rangle - |010\rangle - |011\rangle + |100\rangle + |101\rangle - |110\rangle - |111\rangle\right)$$

$$H^{\otimes}|011\rangle = \frac{1}{2\sqrt{2}}\left(|000\rangle - |001\rangle - |010\rangle + |011\rangle + |100\rangle - |101\rangle - |110\rangle + |111\rangle\right)$$

$$H^{\otimes 3}|100\rangle = \frac{1}{2\sqrt{2}}\left(|000\rangle + |001\rangle + |010\rangle + |011\rangle - |100\rangle - |101\rangle - |110\rangle - |111\rangle\right)$$

$$H^{\otimes 3}|101\rangle = \frac{1}{2\sqrt{2}}\left(|000\rangle - |001\rangle + |010\rangle - |011\rangle - |100\rangle + |101\rangle - |110\rangle + |111\rangle\right)$$

$$H^{\otimes 3}|110\rangle = \frac{1}{2\sqrt{2}}\left(|000\rangle + |001\rangle - |010\rangle - |011\rangle - |100\rangle - |101\rangle + |110\rangle + |111\rangle\right)$$

$$H^{\otimes 3}|111\rangle = \frac{1}{2\sqrt{2}}\left(|000\rangle - |001\rangle - |010\rangle + |011\rangle - |100\rangle + |101\rangle + |110\rangle - |111\rangle\right)$$

$$\frac{1}{2\sqrt{2}}\Big[\left(H^{\otimes 3}|000\rangle + H^{\otimes 3}|011\rangle\right)|010\rangle +$$

$$\left(H^{\otimes 3}|001\rangle + H^{\otimes 3}|010\rangle\right)|101\rangle +$$

$$\left(H^{\otimes 3}|100\rangle + H^{\otimes 3}|111\rangle\right)|110\rangle +$$

$$\left(H^{\otimes 3}|101\rangle + H^{\otimes 3}|110\rangle\right)|001\rangle\Big]$$

$$H^{\otimes 3}|000\rangle = \frac{1}{2\sqrt{2}}\left(|000\rangle + \cancel{|001\rangle} + \cancel{|010\rangle} + |011\rangle + |100\rangle + \cancel{|101\rangle} + \cancel{|110\rangle} + |111\rangle\right)$$

$$H^{\otimes 3}|001\rangle = \frac{1}{2\sqrt{2}}\left(|000\rangle - \cancel{|001\rangle} + \cancel{|010\rangle} - |011\rangle + |100\rangle - \cancel{|101\rangle} + \cancel{|110\rangle} - |111\rangle\right)$$

$$H^{\otimes 3}|010\rangle = \frac{1}{2\sqrt{2}}\left(|000\rangle + \cancel{|001\rangle} - \cancel{|010\rangle} - |011\rangle + |100\rangle + \cancel{|101\rangle} - \cancel{|110\rangle} - |111\rangle\right)$$

$$H^{\otimes}|011\rangle = \frac{1}{2\sqrt{2}}\left(|000\rangle - \cancel{|001\rangle} - \cancel{|010\rangle} + |011\rangle + |100\rangle - \cancel{|101\rangle} - \cancel{|110\rangle} + |111\rangle\right)$$

$$H^{\otimes 3}|100\rangle = \frac{1}{2\sqrt{2}}\left(|000\rangle + \cancel{|001\rangle} + \cancel{|010\rangle} + |011\rangle - |100\rangle - \cancel{|101\rangle} - \cancel{|110\rangle} - |111\rangle\right)$$

$$H^{\otimes 3}|101\rangle = \frac{1}{2\sqrt{2}}\left(|000\rangle - \cancel{|001\rangle} + \cancel{|010\rangle} - |011\rangle - |100\rangle + \cancel{|101\rangle} - \cancel{|110\rangle} + |111\rangle\right)$$

$$H^{\otimes 3}|110\rangle = \frac{1}{2\sqrt{2}}\left(|000\rangle + \cancel{|001\rangle} - \cancel{|010\rangle} - |011\rangle - |100\rangle - \cancel{|101\rangle} + \cancel{|110\rangle} + |111\rangle\right)$$

$$H^{\otimes 3}|111\rangle = \frac{1}{2\sqrt{2}}\left(|000\rangle - \cancel{|001\rangle} - \cancel{|010\rangle} + |011\rangle - |100\rangle + \cancel{|101\rangle} + \cancel{|110\rangle} - |111\rangle\right)$$

$$\frac{1}{8}\Big[2\left(|000\rangle + |011\rangle + |100\rangle + |111\rangle\right) \otimes |010\rangle +$$

$$2\left(|000\rangle - |011\rangle + |100\rangle - |111\rangle\right) \otimes |101\rangle +$$

$$2\left(|000\rangle + |011\rangle - |100\rangle - |111\rangle\right) \otimes |110\rangle +$$

$$2\left(|000\rangle - |011\rangle - |100\rangle + |111\rangle\right) \otimes |001\rangle\Big]$$

$\frac{1}{4}$ Probability

$y = 000$

$y = 011$

$y = 100$

$y = 111$

$$\boxed{\begin{array}{l} 011 \cdot s = 0 \\ 111 \cdot s = 0 \\ 011 \cdot s = 0 \end{array}}$$

$s = s_1 s_2 s_3$

EQUATIONS : $\rightarrow$ $s_1 \cdot 0 \oplus s_2 \cdot 0 \oplus s_3 \cdot 0 = 0$ NOT AN EQUATION

$\rightarrow$ $s_1 \cdot 0 \oplus s_2 \cdot 1 \oplus s_3 \cdot 1 = 0$

$$s_2 \oplus s_3 = 0$$

$\rightarrow$ $s_1 \cdot 1 \oplus s_2 \cdot 0 \oplus s_3 \cdot 0 = 0$

$$s_1 = 0$$

$\rightarrow$ $s_1 \cdot 1 \oplus s_2 \cdot 1 \oplus s_3 \cdot 1 = 0$

$$s_1 \oplus s_2 \oplus s_3 = 0$$

The algorithm obtain y that are orthogonal to s. This generates equations of the form $y \cdot s = 0$.

Using many such equations we can find s //

$$
\begin{aligned}
& \Sigma_{y\in\{0,1\}^3}\ |y\rangle\ \left(\frac{1}{2^3}\ \Sigma_{x\in\{0,1\}^3}(-1)^{x\cdot y}\ |f(x)\rangle\right) \\
= \ & \Sigma_{y\in\{0,1\}^3}\ |y\rangle\ (\frac{1}{8}\ (((-1)^{010\cdot y} + (-1)^{100\cdot y})\ |000\rangle + \\
& \qquad ((-1)^{001\cdot y} + (-1)^{111\cdot y})\ |010\rangle + \\
& \qquad ((-1)^{000\cdot y} + (-1)^{110\cdot y})\ |101\rangle + \\
& \qquad ((-1)^{011\cdot y} + (-1)^{101\cdot y})\ |110\rangle)) \\
= \ & \Sigma_{y\in\{0,1\}^3}\ |y\rangle\ (\frac{1}{8}\ (((-1)^{010\cdot y} + (-1)^{(010\oplus 110)\cdot y})\ |000\rangle + \\
& \qquad ((-1)^{001\cdot y} + (-1)^{(001\oplus 110)\cdot y})\ |010\rangle + \\
& \qquad ((-1)^{000\cdot y} + (-1)^{(000\oplus 110)\cdot y})\ |101\rangle + \\
& \qquad ((-1)^{011\cdot y} + (-1)^{(011\oplus 110)\cdot y})\ |110\rangle)) \\
= \ & \Sigma_{y\in\{0,1\}^3}\ |y\rangle\ (\frac{1}{8}\ ((-1)^{010\cdot y}(1 + (-1)^{110\cdot y})\ |000\rangle + \\
& \qquad (-1)^{001\cdot y}(1 + (-1)^{110\cdot y})\ |010\rangle + \\
& \qquad (-1)^{000\cdot y}(1 + (-1)^{110\cdot y})\ |101\rangle + \\
& \qquad (-1)^{011\cdot y}(1 + (-1)^{110\cdot y})\ |110\rangle))
\end{aligned}
$$

3) $f(10)=1$ $\quad f(00)=f(01)=f(11)=0$

GROVER'S ALGORITHM :

$$G = -H^{\otimes n} Z_0 H^{\otimes n} Z_f$$

$\boxed{\begin{array}{l} x = |0^n\rangle \quad (n\ |0\rangle \text{ qubits}) \\ H^{\otimes n} x \\ \text{repeat } \{ \text{apply } G \text{ to } x \} \ O(\sqrt{2^n}) \text{ times} \\ \text{measure } x \text{ and output the result} \end{array}}$

$x = |00\rangle$

1. $H^{\otimes n} x$

$$\tfrac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

2. Apply G

$$-H^{\otimes n} Z_0 H^{\otimes n} Z_f \left[\tfrac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)\right]$$

$$= -H^{\otimes n} Z_0 H^{\otimes n} \left[\tfrac{1}{2}\left(|00\rangle + |01\rangle - |10\rangle + |11\rangle\right)\right]$$

$$= -H^{\otimes n} Z_0 \Big[\tfrac{1}{4} \Big[\left(|00\rangle + \underline{|01\rangle} + \underline{|10\rangle} + \underline{|11\rangle}\right)$$

$$+ \left(\underline{|00\rangle} - |01\rangle + |10\rangle - \underline{|11\rangle}\right)$$

$$- \left(|00\rangle + |01\rangle - |10\rangle - |11\rangle \right)$$

$$+ \left(|00\rangle - |01\rangle - |10\rangle + |11\rangle \right) \Big]$$

$$= -H^{\otimes} Z_0 \left[\tfrac{1}{2} \left(|00\rangle - |01\rangle + |10\rangle + |11\rangle \right) \right]$$

$$= -H^{\otimes} \left[\tfrac{1}{2} \left(-|00\rangle - |01\rangle + |10\rangle + |11\rangle \right) \right]$$

$$= -\tfrac{1}{4} \Big[- \left(|00\rangle + |01\rangle + |10\rangle + |11\rangle \right)$$

$$- \left(|00\rangle - |01\rangle + |10\rangle - |11\rangle \right)$$

$$+ \left(|00\rangle + |01\rangle - |10\rangle - |11\rangle \right)$$

$$+ \left(|00\rangle - |01\rangle - |10\rangle + |11\rangle \right) \Big]$$

$$= -\tfrac{1}{4} \left(-4|10\rangle \right) = |10\rangle \;//$$

↳ Measure. We get 10 //

$N = 4$

$$k = \frac{\pi}{4\theta} - \frac{1}{2} = \frac{\pi\sqrt{N}}{4} - \frac{1}{2}$$

$$k = \frac{\pi \times 2}{4} - \frac{1}{2} = \frac{1}{2}(\pi - 1) = \frac{2.14}{2} \approx 1$$

note:

$\Rightarrow H^{\otimes 2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$

$(|0\rangle + |1\rangle)(|0\rangle + |1\rangle)$

$\rightarrow |00\rangle$

$\Rightarrow |00\rangle + |01\rangle + |10\rangle - |11\rangle$ same

$\Rightarrow |00\rangle + |01\rangle - |10\rangle + |11\rangle \Leftrightarrow |00\rangle - |01\rangle + |10\rangle + |11\rangle$