1) $\frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{4}|10\rangle - \frac{\sqrt{7}}{4}|11\rangle$

MEASURE QUBIT 2 $\rightarrow$ 0

ONLY $\frac{1}{2}|00\rangle$ AND $\frac{1}{4}|10\rangle$ ARE POSSIBLE NOW

NORMALIZE THE AMPLITUDES

$$\alpha_1^2 + \alpha_2^2 = 1$$

$$\text{FINAL STATE} = \frac{1/2}{\sqrt{(1/2)^2 + (1/4)^2}}|00\rangle + \frac{1/4}{\sqrt{(1/2)^2 + (1/4)^2}}|10\rangle$$

$$1/4 + 1/16 = 5/16$$

$$\sqrt{5/16} = \sqrt{5}/4$$

$$= \frac{2}{\sqrt{5}}|00\rangle + \frac{1}{\sqrt{5}}|10\rangle \;//$$

Once we measure qubit 2 and observe a 0, qubit 2 gets converted to state 0. And hence all states with qubit 2 state as 1 are not possible anymore. Now only 2 states are possible and we need to normalize the amplitudes. So, initially $|00\rangle$ had twice the amplitude as $|10\rangle$ and this ratio remains in the final state.

2) $H^{\otimes 3}|101\rangle$

$$\left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)$$

$$\left(\tfrac{1}{2}|00\rangle - \tfrac{1}{2}|10\rangle + \tfrac{1}{2}|01\rangle - \tfrac{1}{2}|11\rangle\right) \otimes \left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)$$

$$\Big(\tfrac{1}{2\sqrt{2}}|000\rangle - \tfrac{1}{2\sqrt{2}}|100\rangle + \tfrac{1}{2\sqrt{2}}|010\rangle - \tfrac{1}{2\sqrt{2}}|\underline{11}0\rangle - \tfrac{1}{2\sqrt{2}}|001\rangle$$
$$+ \tfrac{1}{2\sqrt{2}}|101\rangle - \tfrac{1}{2\sqrt{2}}|011\rangle + \tfrac{1}{2\sqrt{2}}|\underline{11}1\rangle\Big)$$

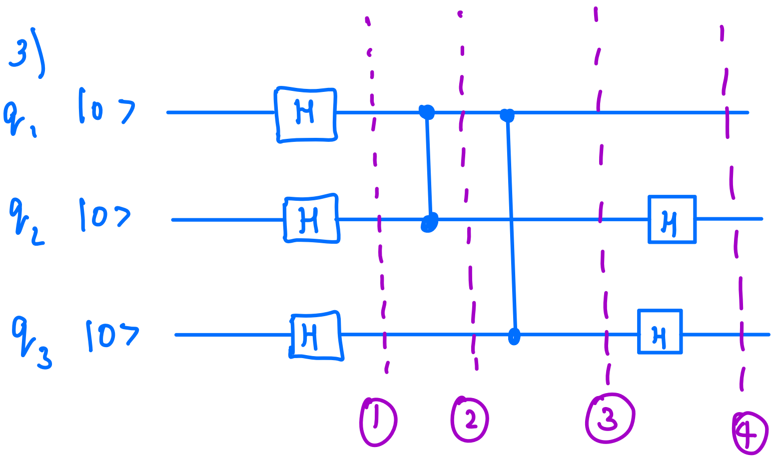
OBSERVE 11 WHEN STATE $\in \{110, 111\}$

PROBABILITY $= \left(\tfrac{1}{2\sqrt{2}}\right)^2 + \left(\tfrac{1}{2\sqrt{2}}\right)^2$

$= \tfrac{1}{8} + \tfrac{1}{8}$

$= \tfrac{1}{4}$ //

3)



Stage 1:

$$H^{\otimes 3}|000\rangle = \frac{1}{2\sqrt{2}}\big(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle\big)$$

Stage 2:

$C(Z)$ with q_1 as control bit

$$\frac{1}{2\sqrt{2}}\big(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle - |110\rangle - |111\rangle\big)$$

Stage 3:

$C(Z)$ with q_1 as control bit

$$\frac{1}{2\sqrt{2}}\big(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle - |101\rangle - |110\rangle + |111\rangle\big)$$

Stage 4:

$I\,H^{\otimes 2}$ applied

$$H^{\otimes 2}(|00\rangle) = \frac{1}{2}\big(|00\rangle + |01\rangle + |10\rangle + |11\rangle\big)$$

$$H^{\otimes 2}(|01\rangle) = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$= \frac{1}{2}\left(|00\rangle - |01\rangle + |10\rangle - |11\rangle\right)$$

$$H^{\otimes 2}(|10\rangle) = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$= \frac{1}{2}\left(|00\rangle + |01\rangle - |10\rangle - |11\rangle\right)$$

$$H^{\otimes 2}(|11\rangle) = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$= \frac{1}{2}\left(|00\rangle - |01\rangle - |10\rangle + |11\rangle\right)$$

STATE :

$$\frac{1}{2\sqrt{2}}\Big(\frac{1}{2}\left(|000\rangle + |001\rangle + |010\rangle + |011\rangle\right) + \frac{1}{2}\left(|000\rangle - |001\rangle + |010\rangle - |011\rangle\right)$$

$$+\frac{1}{2}\left(|000\rangle + |001\rangle - |010\rangle - |011\rangle\right) + \frac{1}{2}\left(|000\rangle - |001\rangle - |010\rangle + |011\rangle\right)$$

$$+\frac{1}{2}\left(|100\rangle + |101\rangle + |110\rangle + |111\rangle\right) - \frac{1}{2}\left(|100\rangle - |101\rangle + |110\rangle - |111\rangle\right)$$

$$-\frac{1}{2}\left(|100\rangle + |101\rangle - |110\rangle - |111\rangle\right) + \frac{1}{2}\left(|100\rangle - |101\rangle - |110\rangle + |111\rangle\right)\Big)$$

$$\frac{1}{4\sqrt{2}}\left(4|000\rangle + 4|111\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

PROBABILITY OF GETTING 1000 $= \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$ //

4) $\frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$

→ MEASURE QUBIT 1

$P(0) = 1/4$

STATE = $|01\rangle$

STATE OF QUBIT 2 → $|1\rangle$

→ MEASURE QUBIT 2

$P(1) = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2$

$= \frac{1}{4} + \frac{1}{2}$

$= 3/4$

STATE $= \frac{1/2}{\sqrt{3/4}}|01\rangle + \frac{1/\sqrt{2}}{\sqrt{3/4}}|11\rangle$

$= \frac{1}{\sqrt{3}}|01\rangle + \frac{\sqrt{2}}{\sqrt{3}}|11\rangle$

STATE OF QUBIT 1 : $\begin{cases} |0\rangle, & \text{with probability} = 1/3 \\ |1\rangle, & \text{with probability} = 2/3 \end{cases}$ //