LEARNING WITH EXPERTS:

Loss

Day	E_1	E_2		E_d	Our
1	0	1	...		1
2					
3					
4					
⋮					

$L(t,i)$: Loss of expert i on day t

$L(t)$: Loss of our algorithm on day t

$$\text{Regret}(T) = \sum_{t=1}^{T} L(t) - \underbrace{\min_{i} \sum_{t=1}^{T} L(t,i)}_{\text{Loss of best expert.}}$$

Strategy 1: Follow the leader (FTL)

↳ Ties, pick smaller ID.

WORST CASE

Day	E_1	E_2	E_3	…	E_d	Our
1	Up	Down	Down	…	Down	Up
	1	0	0	…	0	1
2	Up	Up	Down		Down	Up
	1	1	0	…	0	1
3						
4						
⋮						
d	1	1	1	… 1	0	1

PREDICTION

LOSSES

Our loss = d

Best loss = 0

After d days

$$\text{Regret}(d) = d$$

Can generalize this so that

$L_*(T) \equiv$ loss of best expert $= T/d$

$L(T) \equiv T$.

"Toy Case": Suppose there is infact an infallible expert.

(i.e. $\exists$ some $i^* \in \{1, \dots d\}$ such that $L(t, i^*) = 0$ $\forall t$).

Q: What regret can we achieve now?

Suggestion 1: - Keep a set of "eligible experts"

- Whenever someone makes a mistake, throw them out.

Strategy 2: Follow the majority (FTM)

Algorithm

→ $\mathcal{E}[0] = \{1, 2, \dots d\}$

→ On each day $t = 1, \dots, d$:

→ Predict according to the majority prediction of experts in $\mathcal{E}[t-1]$

→ $\mathcal{E}[t] = \mathcal{E}[t-1] \setminus$ experts who made mistake on day t.

THEOREM: $\text{Regret}(T) \leq \log_2 d$

PROOF: Whenever we make a mistake, the set of eligible experts shrinks by atleast a factor of 2.

$w(t)$: # Eligible experts

$= |\varepsilon(t)|$

a. $w(0) = d$

b. Every mistake, $w(t) \leq \dfrac{w(t-1)}{2}$

$L(t)$ = # mistakes our algorithm makes.

$\Rightarrow \quad w(t) \leq w(0) \cdot \left(\frac{1}{2}\right)^{L(t)}$

Because there is an infallible exper

$w(t) \geq 1$

$1 \leq d \cdot \left(\frac{1}{2}\right)^{L(t)}$

$2^{L(t)} \leq d$

$L(t) \leq \log_2 d$

Assumption of infallible expert is very strong!

IN GENERAL :

Let us maintain a weight for each expert

→ $w(t,i)$ for expert i after round t

→ We will make prediction based on "weighted majority".

But

→ How do we update weights?

Every mistake halves your weight.

STRATEGY 3 : Weighted Majority Algorithm (WMA)

→ $w(0,i) = 1 \quad \forall \quad i = 1, \dots d$

→ On each day $t = 1, \dots, T$:

→ Predict according to weighted majority

If sum of weights of experts predicting UP > Sum of weights of experts prediting DOWN → predict UP.

→ For each expert i :

→ If mistake then $w(t,i) = \dfrac{w(t-1,i)}{2}$

THEOREM: $L(T) \leq 2.4\left(L_*(T) + \log_2 d\right)$

$L(T)$ ↓ Our loss

$L_*(T)$ ↓ Loss of best expert

PROOF:

$$w(t) = \sum_{i=1}^{d} w(t, i) = \text{Total weight (of all experts)}$$

CLAIM: $w(t) \leq w(t-1)$

CLAIM: If we make a mistake on day t,

$$w(t) \leq \left(\frac{3}{4}\right) w(t-1)$$

PROOF:

$$w(t) = \sum_{t=1}^{d} w(t, i)$$

$$= \sum_{\substack{i:\ \text{expert } i \\ \text{is correct}}} w(t,i) + \sum_{\substack{i:\ \text{expert } i \\ \text{was wrong}}} w(t,i)$$

$$= \sum_{\substack{i:\ \text{expert } i \\ \text{is correct}}} w(t-1,i) + \underbrace{\frac{1}{2} \sum_{\substack{i:\ \text{expert } i \\ \text{was wrong}}} w(t-1,i)}_{\geq w(t-1)/2}$$

$$= \sum_{\substack{i:\text{ expert } i \\ \text{is correct}}} w(t-1,i) + \sum_{\substack{i:\text{ expert } i \\ \text{was wrong}}} w(t-1,i) - \frac{1}{2} \sum_{\substack{i:\text{ expert } i \\ \text{was wrong}}} w(t-1,i)$$

$$\leq w(t-1) - \frac{1}{2} \cdot \frac{w(t-1)}{2} = \frac{3}{4}\, w(t-1)$$

If $L(t)$ = our loss after t rounds, then

$$w(t) \leq w(0) \cdot \left(\frac{3}{4}\right)^{L(t)}$$

$$= d \cdot \left(\frac{3}{4}\right)^{L(t)}$$

CLAIM: $\left(\frac{1}{2}\right)^{L_*(t)} \leq w(t)$

$L_*(t)$: # mistakes of best expert

PROOF: $w(t,i) = \left(\frac{1}{2}\right)^{\sum_{j \leq t} L(j,i)}$

So $\left(\frac{1}{2}\right)^{L_*(t)} \leq w(t) \leq d \cdot \left(\frac{3}{4}\right)^{L(t)}$

$$\Rightarrow \left(\frac{4}{3}\right)^{L(t)} \leq 2^{L_*(t)} \cdot d$$

$$L(t) \cdot \log_2 (4/3) \leq L_*(t) + \log_2 d$$

$$L(t) \leq \left(\frac{1}{\log_2 4/3} \right) (L_*(t) + \log_2 d)$$

$$\approx 2.4 (L_*(t) + \log_2 d)$$

$$\text{Regret}(T) = L(t) - L_*(T)$$

$$\leq 1.4 L_*(T) + 2.4 \log_2 d$$

ALGORITHMS WITH $\frac{\text{Regret}(T)}{T} \to 0$ as $T \to \infty$ ARE CALLED

"NO - REGRET ALGORITHMS".

Remark: WMA is not a "No- Regret" Algorithm!

CLAIM: There is no deterministic "No - Regret" Algorithm.

There is no deterministic algorithm that can do better than a factor of 2!

THEOREM: Multiplicative Weights Method (MWM) satisfies

$$E[L(T)] \leq L_*(T) + 2\sqrt{T\ln d}$$

$$\Rightarrow \text{Regret}(T) \leq 2\sqrt{T\ln d}$$

$$\Rightarrow \frac{\text{Regret}(T)}{T} \leq 2\sqrt{\frac{\ln d}{T}}$$

STRATEGY 4: Multiplicative Weights Update Method

1. $w(0,i) = 1 \quad , i = 1, \dots, d$

2. On each day $t = 1, \dots, T$:

→ * Pick an expert i with probability $\propto w(t-1, i)$

$$\Pr[\text{Expert } i \text{ is picked}] = \frac{w(t-1,i)}{\sum_{i=1}^{d} w(t-1,i)}$$

* Then follow expert i.

→ Update weight of every expert:

* if correct, $w(t,i) = w(t-1,i)$

* if wrong, $w(t,i) = (1-\varepsilon)\, w(t-1,i)$

↳ some parameter

THEOREM: $E[L(T)] \leq (1+\varepsilon) L_*(T) + \frac{\ln d}{\varepsilon}$, $\varepsilon < 1/2$

COROLLARY: Set $\varepsilon = \sqrt{\frac{\ln d}{T}}$. Then

$$E[L(T)] \leq L_*(T) + 2\sqrt{T \ln d}$$

PROOF: $E[L(T)] \leq L_*(T) + \varepsilon L_*(T) + \frac{\ln d}{\varepsilon}$

$$= L_*(T) + \sqrt{\frac{\ln d}{T}}\, \underbrace{L_*(T)}_{\leq T} + \sqrt{T(\ln d)}$$

Best expert can make upto T losses

$$\leq L_*(T) + 2\sqrt{T \ln d}$$