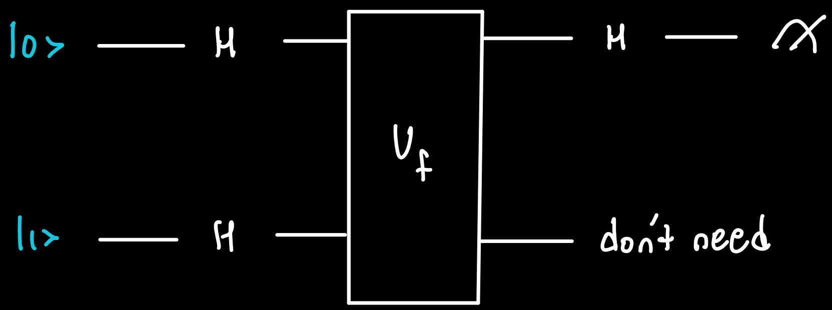
1) $f(0) = f(1) = 1$

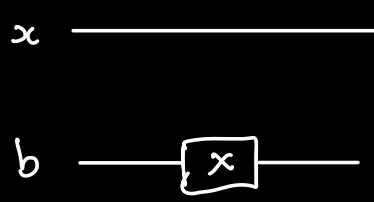


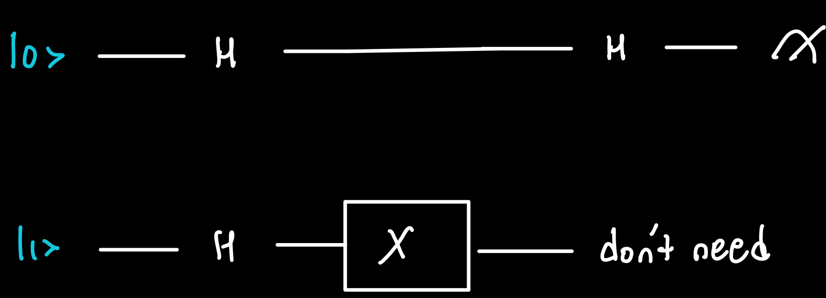
CONSTANT -1 : f_1

$f(x) = 1$ $\qquad b \oplus f(x) = !b$

$U_f|x\rangle|b\rangle = |x\rangle|!b\rangle$

$U_f =$





$(H \otimes I)(I \otimes X)(H \otimes H)\,|01\rangle$

$= (H \otimes I)(I \otimes X)(|+\rangle \otimes |-\rangle)$

$= (H \otimes I)(I \otimes X)\left(\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)\right)$

$= (H \otimes I)\left(\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|0\rangle\right)\right)$

$= |0\rangle \otimes (-|-\rangle)$

$\downarrow$

measure, we will get 0 => CONSTANT

(OR)

WITHOUT FINDING U_f

$U_f\,|x\rangle|b\rangle = U_f\,|x\rangle|b \oplus f(x)\rangle$

$(H \otimes I)\,U_f\,(H \otimes H)\,|0\rangle$

$= (H \otimes I)\,U_f\left(\frac{1}{2}\left(|00\rangle - |01\rangle + |10\rangle - |11\rangle\right)\right)$

$= (H \otimes I)\,\frac{1}{2}\left[|01\rangle - |00\rangle + |11\rangle - |10\rangle\right]$

$= (H \otimes I)\left(\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|0\rangle\right)\right)$

$= |0\rangle \otimes (-|-\rangle)$ $\rightarrow$ Measure $= 0$ => CONSTANT

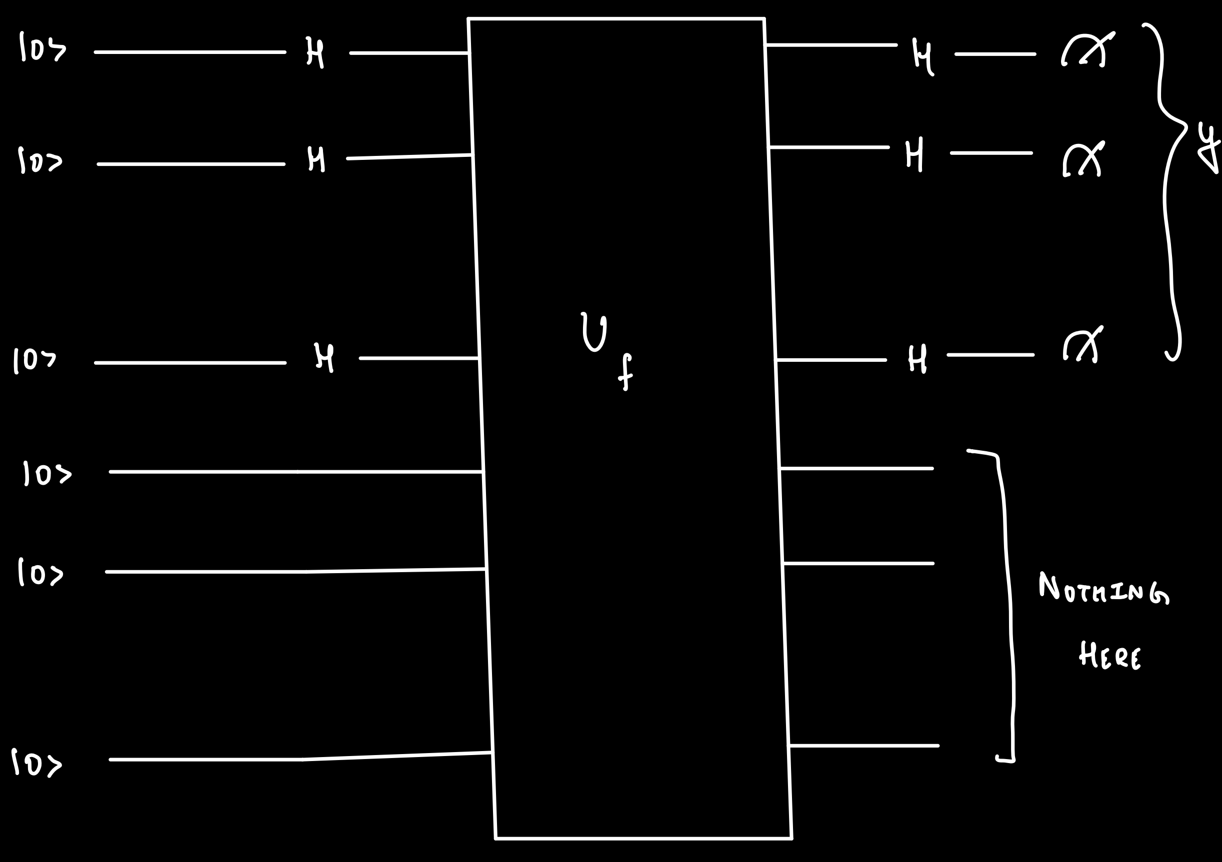
2) $n = 3$

$f(000) = f(010) = 110$

$f(001) = f(011) = 101$

$f(100) = f(110) = 011$

$f(111) = f(101) = 111$



STEP 1:

$$(H \otimes H \otimes H \otimes I \otimes I \times I)\,|000000\rangle$$

$$= \left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right)$$

$$\otimes\, |000\rangle$$

$$= \tfrac{1}{2\sqrt{2}}\big(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle$$

$$+ |110\rangle + |111\rangle\big) \otimes |000\rangle$$

STEP 2:

APPLY U_f

$$U_f(x, b) = (x, \overset{|000\rangle}{b} \oplus f(x)) = (x, f(x))$$

$$\tfrac{1}{2\sqrt{2}}\big(|000110\rangle + |001101\rangle + |010110\rangle + |011101\rangle$$

$$+ |100011\rangle + |101111\rangle + |110011\rangle + |111111\rangle\big)$$

STEP 3:

APPLY $H^{\otimes 3}$

$\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \quad \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

$\frac{1}{2}(00 - 01 - 10 + 11)$

$$\frac{1}{2\sqrt{2}}\Big[\frac{1}{2\sqrt{2}}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle) \otimes |110\rangle$$

$$+ \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) \otimes |101\rangle$$

$$+ \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes |110\rangle$$

$$+ \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) \otimes |101\rangle$$

$$+ \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle - |11\rangle) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes |011\rangle$$

$$+ \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle - |11\rangle) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) \otimes |111\rangle$$

$$+\frac{1}{2}\left(|00\rangle - |01\rangle - |10\rangle + |11\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$\otimes |011\rangle$$

$$+\frac{1}{2}\left(|00\rangle - |01\rangle - |10\rangle + |11\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$\otimes |111\rangle \Big]$$

$$= \frac{1}{8}\Big[\big(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle$$

$$+|000\rangle + |001\rangle - |010\rangle - |011\rangle + |100\rangle + |101\rangle - |110\rangle - |111\rangle\big)$$

$$\otimes |110\rangle$$

$$+\big(|000\rangle - |001\rangle + |010\rangle - |011\rangle + |100\rangle - |101\rangle + |110\rangle - |111\rangle$$

$$+|000\rangle - |001\rangle - |010\rangle + |011\rangle + |100\rangle - |101\rangle - |110\rangle + |111\rangle\big)$$

$$\otimes |101\rangle$$

$$+\big(|000\rangle + |001\rangle + |010\rangle + |011\rangle - |100\rangle - |101\rangle - |110\rangle - |111\rangle$$

$$+|000\rangle + |001\rangle - |010\rangle - |011\rangle - |100\rangle - |101\rangle + |110\rangle + |111\rangle\big)$$

$$\otimes |011\rangle$$

$$+\big(|000\rangle - |001\rangle + |010\rangle - |011\rangle - |100\rangle + |101\rangle - |110\rangle + |111\rangle$$

$$+|000\rangle - |001\rangle - |010\rangle + |011\rangle - |100\rangle + |101\rangle + |110\rangle - |111\rangle\big)$$

$\otimes |111\rangle$

$$= \frac{1}{4}\Big[\left(|000\rangle + |001\rangle + |100\rangle + |101\rangle\right) \otimes |110\rangle$$

$$+ \left(|000\rangle - |001\rangle + |100\rangle - |101\rangle\right) \otimes |101\rangle$$

$$+ \left(|000\rangle + |001\rangle - |100\rangle - |101\rangle\right) \otimes |011\rangle$$

$$+ \left(|000\rangle - |001\rangle - |100\rangle + |101\rangle\right) \otimes |111\rangle \Big]$$

$\frac{1}{4}$ Probability

$y = 000$

$y = 001$

$y = 100$

$y = 101$

$$S = s_1 s_2 s_3$$

EQUATIONS : $\rightarrow s_1 \cdot 0 \oplus s_2 \cdot 0 \oplus s_3 \cdot 0 = 0$ NOT AN EQUATION

$\rightarrow s_1 \cdot 0 \oplus s_2 \cdot 0 \oplus s_3 \cdot 1 = 0$

$s_3 = 0$

$\rightarrow \quad s_1 \cdot 1 \oplus s_2 \cdot 0 \oplus s_3 \cdot 0 = 0$

$$s_1 = 0$$

$\rightarrow \quad s_1 \cdot 1 \oplus s_2 \cdot 0 \oplus s_3 \cdot 1 = 0$

$$s_1 \oplus s_3 = 0$$

The algorithm obtain y that are orthogonal to s. This generates equations of the form $y \cdot s = 0$. Using many such equations we can find s //

3) $f(01) = 1$ $\quad f(00) = f(10) = f(11) = 0$

GROVER'S ALGORITHM :

$$G = -H^{\otimes n} Z_0 H^{\otimes n} Z_f$$

$X = |0^n\rangle$ (n $|0\rangle$ qubits)

$H^{\otimes n} X$

repeat { apply G to X } $O(\sqrt{2^n})$ times

measure X and output the result

$X = |00\rangle$

1. $H^{\otimes n} X$

$$\tfrac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

2. Apply G

$$-H^{\otimes n} Z_0 H^{\otimes n} Z_f \left[\tfrac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \right]$$

$$= -H^{\otimes n} Z_0 H^{\otimes n} \left[\tfrac{1}{2}(|00\rangle - |01\rangle + |10\rangle + |11\rangle) \right]$$

$$= -H^{\otimes n} Z_0 \Big[\tfrac{1}{4} \big[(\underline{|00\rangle} + |01\rangle + \underline{|10\rangle} + \underline{|11\rangle})$$

$$- (\underline{|00\rangle} - |01\rangle + |10\rangle - |11\rangle)$$

$$+\left(|00\rangle + |01\rangle - |10\rangle - |11\rangle\right)$$

$$+\left(|00\rangle - |01\rangle - |10\rangle + |11\rangle\right)\Big]$$

$$= -H^{\otimes} Z_0 \left[\tfrac{1}{2} \left(|00\rangle + |01\rangle - |10\rangle + |11\rangle \right)\right]$$

$$= -H^{\otimes} \left[\tfrac{1}{2} \left(-|00\rangle + |01\rangle - |10\rangle + |11\rangle \right)\right]$$

$$= -\tfrac{1}{4}\Big[-\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

$$+\left(|00\rangle - |01\rangle + |10\rangle - |11\rangle\right)$$

$$-\left(|00\rangle + |01\rangle - |10\rangle - |11\rangle\right)$$

$$+\left(|00\rangle - |01\rangle - |10\rangle + |11\rangle\right)\Big]$$

$$= -\tfrac{1}{4}\left(-4|01\rangle\right) = |01\rangle \,//$$

↳ Measure. We get 01//