ONLINE LEARNING

→ Do not see all the data at once.

example:

→ Online advertising

→ Weather prediction

→ Resource allocation.

→ Games

→ Stock Market Predictions.

HOW TO FORMALIZE ONLINE LEARNING:

IDEA 1 : Mistake bound model

IDEA 2 : Regret Minimization

MISTAKE BOUNDED MODEL:

"true" label for the day $\in \{0,1\}$

Day 1 : $(x_1, \hat{y}_1, y_1)$

↳ your prediction $\in \{0,1\}$

Is $\hat{y}_1 = y_1$?

Day 2 : $(x_2, \hat{y}_2, y_2)$

⋮

YOU ARE ONLY ALLOWED TO MAKE A BOUNDED NUMBER OF MISTAKES.

Assume: The true labels are generated using some function f_* from a hypothesis class H.

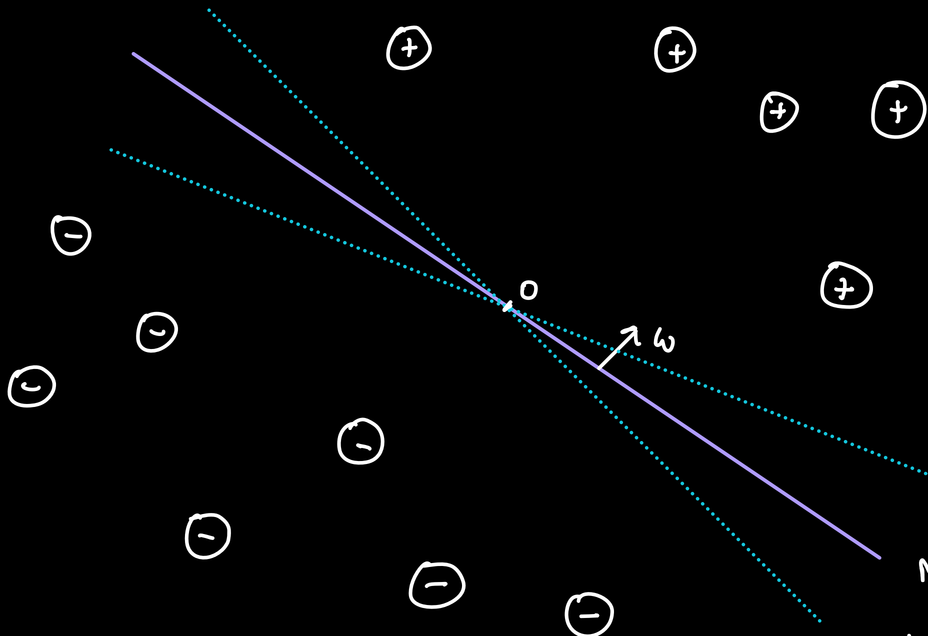
$$y_i = f_*(x_i)$$

ONLINE LEARNING OF HALFSPACES:

$f_*(x) = \text{sign}(\langle w_*, x \rangle)$ for some w_*

Domain $x \in \mathbb{R}^d$, $L = \{1, -1\}$

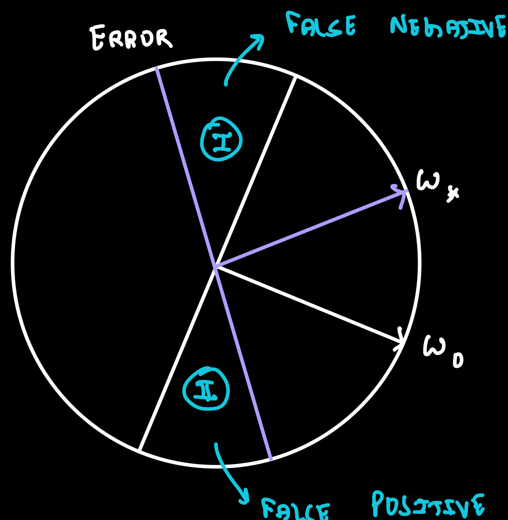
$$h(x) = \begin{cases} 1 & \text{if } \langle w, x \rangle > 0 \\ -1 & \text{if } \langle w, x \rangle < 0 \end{cases}$$



Margin of the halfspace. (Assume buffer exists).

Margin of halfspace $\boxed{\gamma = \min_{x \to \infty} \frac{|\langle w_*, x \rangle|}{\|w_*\| \, \|x\|_2}}$

Assume that $\|w_*\| = 1$ and let us normalize all examples $\|x\| = 1$.



PERCEPTRON (1958, ROSENBLATT):

THEOREM: Perceptron makes at most $\frac{1}{\gamma^2} + 1$ mistakes

($\gamma \equiv$ margin of w_*)

→ $w^0 \equiv$ random vector

→ On Day i

$$w_i = \begin{cases} w_{i-1} & \text{if no mistake} \\ w_{i-1} + y_i \cdot x_i & \text{if mistake.} \end{cases}$$

Day 1: 1. Start with random w_0.

Predict $\hat{y}_1 = \text{Sign}(\langle w_0, x \rangle)$

$y_1 \equiv$ true label

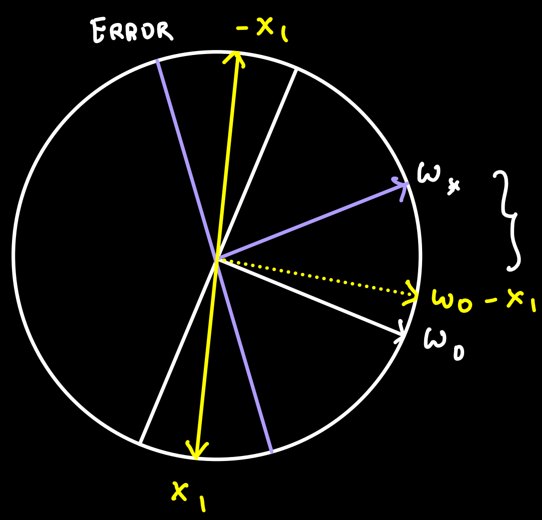
If correct: Do nothing

If wrong: If true label is '−' but we say "+".

$w_1 = w_0 - x_1$

If true label is '+' but we say "−".

$w_1 = w_0 + x_1$



always gets closer

PROOF:

When mistake $w_i = w_{i-1} + y_i \cdot x_i$

$$\langle w_i , w_* \rangle = \langle w_{i-1} , w_* \rangle + \underbrace{y_i \langle x_i , w_* \rangle}_{\text{always positive}}$$

$$= \langle w_{i-1} , w_* \rangle + |\langle w_* , x_i \rangle|$$

$$\geq \langle w_{i-1} , w_* \rangle + \gamma$$

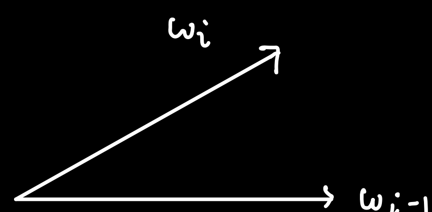
(because we have margin γ!)

So inner-product to w_* increases. This can increase by increasing the amplitube.

CLAIM: $\|w_i\|^2 \leq \|w_{i-1}\|^2 + 1$ (for each iteration)

PROOF: When we make a mistake

$$w_i = w_{i-1} + y_i \cdot x_i$$



$$\|w_i\|^2 = \|w_{i-1}\|^2 + \underbrace{y_i^2 \cdot \|x_i\|^2}_{\text{unit norm}} + 2\langle w_{i-1}, y_i \cdot x_i \rangle$$

$$= \|w_{i-1}\|^2 + 1 + 2y_i \cdot \underbrace{\langle w_{i-1}, x_i \rangle}$$

$\hat{y}_i = \text{sign}(\langle w_{i-1}, x_i \rangle)$

we know mistake

so $y_i \neq \hat{y}_i$

is ≤ 0

$$\leq \|w_{i-1}\|^2 + 1 \quad //$$

So we have whenever we make a mistake

$$\langle w_{i-1}, w_* \rangle + \gamma \le \langle w_i, w_* \rangle \quad \text{and}$$

$$\|w_i\|^2 \le \|w_{i-1}\|^2 + 1$$

At any point in time

$$\|w_n\|^2 \le 1 + M_n$$

mistakes made until then.

$$\langle w_0 + w_* \rangle + \gamma M_n \le \langle w_n, w_* \rangle \le \|w_n\| \cdot \|w_*\|$$

$$= \|w_n\|$$

$$\le \sqrt{1 + M_n}$$

$$\langle u, v \rangle \le \|u\| \cdot \|v\|$$

$$\gamma M_n \le \sqrt{1 + M_n} - \langle w_0, w_* \rangle$$

$$\to \ \ge -1$$

$$\gamma M_n \le \sqrt{1 + M_n} + 1$$

$$\boxed{M_n \le 1/\gamma^2 + 1}$$

→ Only depends on margin.

$$\gamma M_n - 1 \leq \sqrt{1 + M_n}$$

$$\gamma^2 M_n^2 + 1 - 2\gamma M_n \leq 1 + M_n$$

$$\gamma^2 M_n^2 \leq 2\gamma M_n + M_n$$

$$\gamma^2 M_n \leq 2\gamma + 1$$

$$M_n \leq 2/\gamma + 1/\gamma^2$$

PERCEPTRON AS SGD WITH "HINGE LOSS":

$$\text{SGD}: \quad w_i = w_{i-1} - \eta \underbrace{G(w_{i-1})}_{\text{estimator for gradient.}}$$

SGD for ERM:

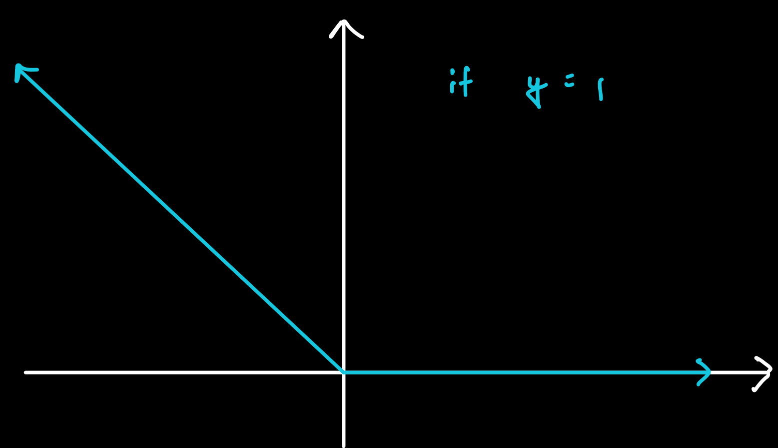
$$w_i = w_{i-1} - \eta \nabla_w \left(\ell(h_w(x_i), y_i) \right)$$

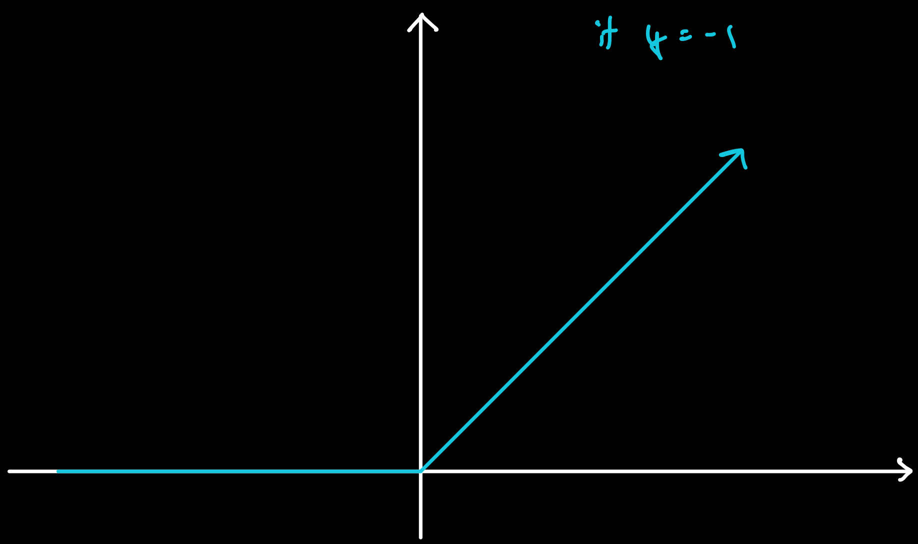
$$(x_1, y_1), (x_2, y_2) \cdots, (x_n, y_n)$$

$$L(w) = \frac{1}{n} \sum_{i=1}^{n} \ell(h_w(x_i), y_i)$$

$$h_w(x) = \langle w, x \rangle$$

Loss function:





$$\nabla_w \ell(h_w(x_i), y_i) = \begin{cases} 0 & \text{if } \operatorname{sign}(\langle w, x_i \rangle) = y_i \\ -y_i x_i & \text{if not.} \end{cases}$$

"SGD" can be used for online learning problems.

PROBLEM WITH MISTAKE BOUNDED?

→ What if $f_\star$ changes?

LEARNING WITH EXPERTS:

- "Predict" based on prediction of experts.

	E_1	E_2	$\dots$		E_d	OUR PREDICTION	TRUTH
Day 1:	Up	Down	Up	Down	Down	Up	Down
Day 2:							

"Loss" of	E_1	E_2	$\dots$	E_d	Our Loss

$L(i,t)$ = Loss of expert i on day t $\left(\begin{array}{ll} 1 & \text{if wrong} \\ 0 & \text{otherwise} \end{array}\right)$

GOAL:

Do as well as the best expert in hindsight.

$$\boxed{\text{Regret}(n) = \sum_{t=1}^{n} L(t) - \min_{i=1\dots d} \sum_{t=1}^{n} L(i,t).}$$