1) a) i) $AA^T = I$

$$A^T = A^{-1}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$a^2 + b^2 = 1$$

$$c^2 + d^2 = 1$$

$$ac + bd = 0$$

$$a = 1/\sqrt{2}$$

$$b = -1/\sqrt{2}$$

$$c = 1/\sqrt{2}$$

$$d = 1/\sqrt{2}$$

one has to be -ve

$$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

EIGENDECOMPOSITION:

$$Av = \lambda v$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1/\sqrt{2} - \lambda & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} - \lambda \end{vmatrix} = 0$$

$$(1/\sqrt{2} - \lambda)^2 + 1/2 = 0$$

$$1/2 + \lambda^2 - 2\lambda/\sqrt{2} + 1/2 = 0$$

$$\lambda^2 - \sqrt{2}\lambda + 1 = 0$$

$$\lambda = \frac{\sqrt{2} \pm \sqrt{2-4}}{2} = \frac{\sqrt{2} \pm \sqrt{2}\, i}{2}$$

$$= 1/\sqrt{2} \pm i/\sqrt{2}$$

EIGENVALUES:

$$\lambda_1 = 1/\sqrt{2} + i/\sqrt{2} \qquad \lambda_2 = 1/\sqrt{2} - i/\sqrt{2}$$

$$A - \lambda_1 I = \begin{bmatrix} -i/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -i/\sqrt{2} \end{bmatrix} x = 0$$

$$-i/\sqrt{2}\, x_1 - 1/\sqrt{2}\, x_2 = 0$$

$$1/\sqrt{2}\, x_1 - i/\sqrt{2}\, x_2 = 0$$

$$\left.\begin{array}{l} -ix_1 = x_2 \\ x_1 = ix_2 \end{array}\right]$$ SAME EQUATION

SO A PLANE OF POSSIBLE EIGENVECTORS.

ASSUME $x_2 = 1$

$\Rightarrow x_1 = i$

$$X_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$A - \lambda_2 I = \begin{bmatrix} i/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & i/\sqrt{2} \end{bmatrix}$$

$$(A - \lambda_2 I)x = 0$$

$$ix_1 - x_2 = 0$$

$$x_1 + ix_2 = 0$$

$$\left.\begin{array}{l} x_2 = ix_1 \\ x_1 = -ix_2 \end{array}\right]$$ SAME EQUATION

SO, A PLANE OF POSSIBLE EIGENVECTORS.

ASSUME $x_2 = 1$

$x_1 = -i$

$$X_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

EIGENVECTORS:

$$x_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} \qquad x_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

OBSERVATIONS:

→ EIGENVALUES :

* Can be complex. When complex, they exist in complex conjugates.
* They have norm = 1.

→ EIGENVECTORS :

* They are orthogonal to each other if the eigenvalues are distinct.

$$x_1^H x_2 = \begin{bmatrix} -i & 1 \end{bmatrix} \begin{bmatrix} -i \\ 1 \end{bmatrix} = i^2 + 1 = 0 \,//$$

ii) $AA^T = I$ $\quad$ A is real

$$Ax = \lambda x \qquad -(1)$$

$$(Ax)^H = (\lambda x)^H \longrightarrow$$ TAKE CONJUGATE TRANSPOSE

$$x^H A^T = \bar{\lambda} x^H \qquad -(2)$$ [$A^H = A^T$ as A is real]

[$\lambda^H = \bar{\lambda}$ as it is scalar]

$$AA^T = I \Rightarrow A^T = A^{-1} \Rightarrow A^T A = I$$

(left and right inverse of square matrices are same)

$$(2) \times (1)$$

$$x^H \underbrace{A^T A}_{I} x = \bar{\lambda} x^H \lambda x$$

$$x^H x = (\bar{\lambda}\lambda) x^H x$$

$$x^H x (1 - \bar{\lambda}\lambda) = 0$$

$$\|x\|^2 (1 - |\lambda|^2) = 0$$

WE ASSUME $\|x\|$ cannot be 0.

(non-zero eigenvectors).

Then $|\lambda|^2 = 1$

$|\lambda| = 1$ as it is always positive.

THEREFORE,

EIGENVALUES HAVE NORM = 1.

WE CAN ALSO ASSUME ALL ARE REAL:

$$Ax = \lambda x \quad - (1)$$

$$(Ax)^T = (\lambda x)^T$$

$$x^T A^T = \lambda x^T \quad - (2)$$

$$(2) \times (1)$$

$$x^T \underline{A^T} A x = |\lambda|^2 x^T x$$

$$\|x\|^2 = |\lambda|^2 \|x\|^2$$

↳ non-zero eigenvectors

$$|\lambda|^2 = 1$$

iii) $Ax_1 = \lambda_1 x_1$

$Ax_2 = \lambda_2 x_2$

$x_1^H x_2 = x_1^H \cdot I \cdot x_2$ $\quad [A^T A = A A^T = I]$

$= x_1^H \cdot A^T A \cdot x_2$ $\quad [A^H = A^T]$

$= (Ax_1)^H (Ax_2)$

$= (\lambda_1 x_1)^H (\lambda_2 x_2)$

$= \bar{\lambda}_1 x_1^H \cdot \lambda_2 x_2$

$= \bar{\lambda}_1 \lambda_2 (x_1^H x_2)$

$(x_1^H x_2)(\bar{\lambda}_1 \lambda_2 - 1) = 0$

EITHER $x_1^H x_2 = 0$

OR

$\bar{\lambda}_1 \lambda_2 = 1$

$$\left.\begin{aligned} \lambda_1 &= e^{i\theta} \\ \lambda_2 &= e^{i\phi} \end{aligned}\right\} |\lambda_1| = |\lambda_2| = 1 \text{ , from (1 ii).}$$

$\bar{\lambda}_1 \lambda_2 = (e^{-i\theta} \cdot e^{i\phi}) = 1$

$e^{i(\phi - \theta)} = 1$

$= e^0$

$$\phi - \theta = 0$$

$$\phi = \theta$$

$$\Rightarrow \lambda_1 = e^{i\theta}$$

$$\lambda_2 = e^{i\theta}$$

$$\Rightarrow \lambda_1 = \lambda_2$$

BUT QUESTION SAYS DISTINCT EIGENVALUES

CONTRADICTION

$$\therefore \bar{\lambda}_1 \lambda_2 \neq 1$$

HENCE, $x_1^H x_2 = 0$ $\Rightarrow$ EIGENVECTORS CORRESPONDING TO DISTINCT EIGENVALUES ARE ORTHOGONAL //

WE CAN ALSO ASSUME ALL ARE REAL:

$$A x_1 = \lambda_1 x_1$$

$$A x_2 = \lambda_2 x_2$$

$$x_1^T x_2 = x_1^T \cdot I \cdot x_2 = x_1^T A^T A x_2 = (Ax_1)^T (Ax_2)$$

$$= (\lambda_1 x_1)^T (\lambda_2 x_2)$$

$$= \lambda_1 x_1^T \lambda_2 x_2$$

$$= \lambda_1 \lambda_2 x_1^T x_2$$

$$(\lambda_1 \lambda_2 - 1) x_1^T x_2 = 0$$

Since $\lambda_1 \neq \lambda_2$ and we know from (iii) that $|\lambda_1| = |\lambda_2| = 1$,

$$x_1^T x_2 = 0 //$$

iv) Ax

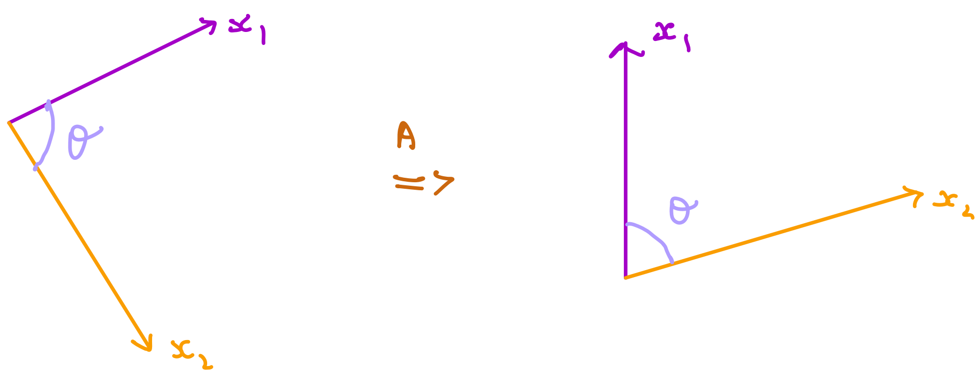
THE ORTHOGONAL MATRIX PRESERVES THE INNER PRODUCT OF VECTORS AND HENCE IT DOES A ROTATION/ REFLECTION OF ANY VECTOR, WHEN APPLIED.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Any rotation transformation by angle θ, can be shown to be an orthogonal matrix.

It preserves length and angles between vectors.

↳ magnitude of x is maintained.



So for vector x,

→ magnitude / length is maintained

→ rotated / reflected by an angle //

b) i) $A = U \Sigma V^T$

$$AA^T = U \Sigma V^T (U \Sigma V^T)^T$$

$$= U \Sigma \underbrace{V^T V}_{\text{ORTHONORMAL } [V^T V = I]} \Sigma^T U^T$$

$$= U (\Sigma \Sigma^T) U^T$$

COMPARE THIS WITH EIGENDECOMPOSITION OF AA^T

$$AA^T = S \Lambda S^{-1}$$

So S here is U and $\underbrace{S^{-1} = U^T}$

As X here is AA^T, which is always a symmetric positive definite matrix, $S^{-1} = S^T$

Symmetric matrices have orthogonal eigenvectors. So if we normalise the eigenvectors, $S^{-1} = S^T$ and $X = S \Lambda S^T$.

Therefore U are the eigenvectors of AA^T.

$\Rightarrow$ Left Singular vectors of A are the eigenvectors of AA^T.

$$A = U \Sigma V^T$$

$$A^T A = V \Sigma^T \underbrace{U^T U}_{\text{ORTHONORMAL}} \Sigma V^T$$

$$= V (\Sigma^T \Sigma) V^T$$

EIGENDECOMPOSITION:

$$A^T A = S \Lambda S^{-1} = S \Lambda S^T$$

$A^T A \rightarrow$ Symmetric positive definite.

$$S = V$$

$$\Lambda = \Sigma^T \Sigma$$

Therefore V are the eigenvectors of $A^T A$.

$\Rightarrow$ Right Singular vectors of A are the eigenvectors of $A^T A$.

ii)

$$A = U \Sigma V^T$$

$$AA^T = U \Sigma V^T (U \Sigma V^T)^T$$

$$= U \Sigma \underbrace{V^T V}_{\text{ORTHONORMAL}} \Sigma^T U^T$$

ORTHONORMAL $[V^T V = I]$

$$= U (\Sigma \Sigma^T) U^T$$

COMPARE THIS WITH EIGENDECOMPOSITION OF AA^T

$$AA^T = S \Lambda S^{-1} = S \Lambda S^T$$ (SYMMETRIC POSITIVE DEFINITE)

$$\Lambda = \Sigma \Sigma^T = \begin{bmatrix} \sigma_1^2 & & & 0 \\ & \sigma_2^2 & & \\ & & \ddots & \\ 0 & & & \sigma_n^2 \end{bmatrix}$$

Therefore the square of singular values of A are the eigenvalues of AA^T.

$$A = U \Sigma V^T$$

$$A^T A = V \Sigma^T \underbrace{U^T U}_{\text{ORTHONORMAL}} \Sigma V^T$$

$$= V (\Sigma^T \Sigma) V^T$$

EIGENDECOMPOSITION :

$$A^T A = S \Lambda S^{-1} = S \Lambda S^T$$

$A^T A \rightarrow$ Symmetric positive definite.

$$\Lambda = \Sigma^T \Sigma = \begin{bmatrix} \sigma_1^2 & & & 0 \\ & \sigma_2^2 & & \\ & & \ddots & \\ 0 & & & \sigma_n^2 \end{bmatrix}$$

Therefore the square of singular values of A are the eigenvalues of $A^T A$ as well. So $A^T A$ and $A A^T$ have the same eigenvalues //

or

The singular values of A are the square roots of the eigenvalues of $A A^T$ and $A^T A$.

c) i) FALSE

Every linear operator in an n-dimensional vector space has at most n distinct eigenvalues.

eg: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a linear operator in 2-D vector space.

$n=2$

EIGENVALUES ARE (1, 1)

DISTINCT EIGENVALUES = 1 < 2 //

ii) FALSE

$Ax_1 = \lambda_1 x_1$ — 1

$Ax_2 = \lambda_2 x_2$ — 2

if $\lambda_1 = \lambda_2$

$A(x_1+x_2) = \lambda(x_1+x_2)$

x_1+x_2 is an eigenvector

BUT IF $\lambda_1 \neq \lambda_2$ AND ASSUME x_1+x_2 IS AN EIGENVECTOR, WITH EIGENVALUE, μ.

$A(x_1+x_2) = \mu(x_1+x_2)$

$Ax_1 + Ax_2 = \mu x_1 + \mu x_2$

$\lambda_1 x_1 + \lambda_2 x_2 = \mu x_1 + \mu x_2$ [FROM 1, 2]

$$x_1(\lambda_1-\mu)+x_2(\lambda_2-\mu)=0 \quad - (3)$$

$$A\times(3) \Rightarrow Ax_1(\lambda_1-\mu)+Ax_2(\lambda_2-\mu)=0$$

$$\lambda_1 x_1(\lambda_1-\mu)+\lambda_2 x_2(\lambda_2-\mu)=0 \quad - (4)$$

$$\lambda_1\times(3) \Rightarrow \lambda_1 x_1(\lambda_1-\mu)+\lambda_1 x_2(\lambda_2-\mu)=0 \quad - (5)$$

$(4)-(5)$

$$(\lambda_2-\lambda_1)x_2(\lambda_2-\mu)=0.$$

$\lambda_1\neq\lambda_2$ (ASSUMPTION)

$x_2\neq 0$ (NON ZERO EIGENVECTORS)

NON-TRIVIAL.

So $\lambda_2-\mu=0$

$\lambda_2=\mu$

$$\lambda_2\times(3) \Rightarrow \lambda_2 x_1(\lambda_1-\mu)+\lambda_2 x_2(\lambda_2-\mu)=0 \quad - (6)$$

$(4)-(6)$

$$\underbrace{(\lambda_1-\lambda_2)}_{\neq 0}\overbrace{x_1}^{\neq 0}(\lambda_1-\mu)=0$$

So $\lambda_1=\mu$

So $\lambda_1=\lambda_2=\mu$ [CONTRADICTION]

THEREFORE, A NON ZERO SUM OF 2 EIGENVECTORS OF A MATRIX A IS NOT ALWAYS AN EIGENVECTOR.

eg: $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 = 4$$

$1-\lambda = 2$ $\qquad$ $1-\lambda = 2$

$\lambda_1 = -1$ $\qquad$ $\lambda_2 = 3$

$(A - \lambda_1 I)\, x = 0$ $\qquad$ $(A - \lambda_2)\, x = 0$

$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$ $\qquad$ $\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$x_1 + x_2 = 0$ $\qquad$ $2x_1 = 2x_2$

$x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\qquad$ $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$x_3 = x_1 + x_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$Ax_3 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

If $Ax_3 = \lambda_3 x_3$

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2\lambda_3 \end{bmatrix}$$

$$4 \neq 0$$

HENCE PROVED

$x_1 + x_2 = x_3$ IS NOT AN EIGENVECTOR //

iii) TRUE

$x^T A x \geq 0 \quad \forall x$, including eigenvectors

Let x_i be an eigenvector

$$x_i^T A x_i \geq 0$$

$$A x_i = \lambda_i x_i$$

$$x_i^T \lambda_i x_i \geq 0$$

$$\lambda_i \, x_i^T x_i \geq 0$$

$$\lambda_i \|x_i\|^2 \geq 0$$

$\hookrightarrow$ NORM ≥ 0

$$\Rightarrow \lambda_i \geq 0 \quad //$$

iv) TRUE

The rank can exceed the number of distinct non-zero eigenvalues. eg:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

RANK = 2

EIGENVALUES = (1, 1)

NUMBER OF DISTINCT EIGENVALUES, $n = 1$.

RANK > n //

v) TRUE

$$Ax_1 = \lambda x_1$$

$$Ax_2 = \lambda x_2$$

PROVE $x_1 + x_2$ is an eigenvector

$$A(x_1 + x_2) = Ax_1 + Ax_2 = \lambda x_1 + \lambda x_2 = \lambda(x_1 + x_2) \text{ //}$$

2) a) i) $P(H \mid M50) = 0.5$ $\quad P(T \mid M50) = 0.5$

$P(H \mid M60) = 0.6$ $\quad P(T \mid M60) = 0.4$

$P(M50) = P(M60) = 1/2$

$$P(M50 \mid T) = \frac{P(T \mid M50) \times P(M50)}{P(T)}$$

$$= \frac{P(T \mid M50) \times P(M50)}{P(T \mid M50) \times P(M50) + P(T \mid M60) \times P(M60)}$$

$$= \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.4 \times 0.5}$$

$$= 5/9$$

$$= 0.55 //$$

ii) $P(H50) = P(H60) = 1/2$

$$P(H50 \mid THHH) = \frac{P(THHH \mid H50) \cdot P(H50)}{P(THHH \mid H50) \cdot P(H50) + P(THHH \mid H60) \cdot P(H60)}$$

$$= \frac{(1/2)^4}{(1/2)^4 + (0.4)(0.6)^3}$$

$$= \frac{0.0625}{0.0625 + 0.0864}$$

$$= \frac{0.0625}{0.1489}$$

$$= 0.4197$$

iii) $P(H50 \mid 9H1T) = \dfrac{P(9H1T \mid H50) \cdot P(H50)}{P(9H1T \mid H50) \cdot P(H50) + P(9H1T \mid H55) \cdot P(H55) + P(9H1T \mid H60) \cdot P(H60)}$

$$= \frac{(0.5)^{10}}{(0.5)^{10} + (0.55)^9 (0.45) + (0.6)^9 (0.4)}$$

$$= 0.137931 //$$

$$P(H55 \mid 9H1T) = \frac{P(9H1T \mid H55) \cdot P(H55)}{P(9H1T)}$$

$$= \frac{(0.55)^9 (0.45)}{(0.5)^{10} + (0.55)^9 (0.45) + (0.6)^9 (0.4)}$$

$$= \frac{0.00207}{0.0070800055862}$$

$$= 0.292711$$

$$P(H60 \mid 9H1T) = \frac{P(9H1T \mid H60) \cdot P(H60)}{P(9H1T)}$$

$$= \frac{(0.6)^9 (0.4)}{0.0070800055862}$$

$$= 0.569356 //$$

b) $P(+ \mid Pg) = 99/100$

$P(- \mid Pg) = 1/100$

$P(+ / NP) = 10/100$

$P(- / NP) = 9/10$

$P(NP) = 99/100$

$P(Pg) = 1/100$

$$P(Pg \mid +) = \frac{P(+ \mid Pg) \cdot P(Pg)}{P(+ \mid Pg) \cdot P(Pg) + P(+ \mid NP) \cdot P(NP)}$$

$$= \frac{99/100 \times 1/100}{99/100 \times 1/100 + 10/100 \times 99/100}$$

$$= \frac{99}{99 + 990}$$

$$= \frac{99}{1089}$$

$$= 0.0909 \,//$$

The test is not that good with 10% failure when not pregnant. Moreover, the populalation has quite a huge majority (99%) of not pregnant women. Therefore the positive test results from non-pregnant women, though just 10% is still 10% of the 99% population and hence far outnumbers the total number of pregnant women.

Moreover the test has high false positive rate of 10%.

So assume 100,000 people. 99000 are not pregnant and 1000 are pregnant. 10% of 99,000 → 9900 are tested positive.

99% of 1000 => 990 are tested positive.

We can see that false positives far outnumber the true positive cases.

c) $E(Ax+b)$

$$A: m\times n,\quad x: n\times 1,\quad b: m\times 1$$

$$E(x) = \begin{bmatrix} E(x_1) \\ \vdots \\ E(x_n) \end{bmatrix}$$

$$C = Ax+b = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \vdots \\ \vdots & & & \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + b_m \end{bmatrix}$$

$$C_i = \sum_{j=1}^{n} (a_{ij}x_j) + b_i$$

$$E[C_i] = E(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + b_i)$$

$$= a_{i1}E(x_1) + a_{i2}E[x_2] + \cdots + a_{in}E(x_n) + b_i$$

$$E[C] = \begin{bmatrix} a_{11}E(x_1) + a_{12}E(x_2) \cdots + a_{1n}E(x_n) + b_1 \\ \vdots \\ a_{m1}E(x_1) + a_{m2}E(x_2) \cdots + a_{mn}E(x_n) + b_m \end{bmatrix}$$

$$E[c] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \vdots \\ \vdots & & & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix} \begin{bmatrix} E(x_1) \\ E(x_2) \\ \vdots \\ E(x_n) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$E[c] = A\,E(x) + b$$

$$E(Ax+b) = AE(x) + b \,//$$

OR

Without expanding the matrix, we can solve it

$$E(Ax_i + b_i) = E(Ax_i) + E(b_i)$$

$$= E\left(\sum_{j=1}^{n} A_{ij}\, x_j\right) + b_i$$

$$= \sum_{j=1}^{n} A_{ij}\, E(x_j) + b_i$$

$$= \sum_{j=1}^{n} A_{ij}\, E(x)_j + b_i$$

$$= [A\,E(x)]_i + b_i$$

$$E(Ax+b) = A\,E(x) + b \,//$$

d) $\operatorname{cov}(x) = E[(x - Ex)(x - Ex)^T]$ — 1

$$\operatorname{cov}(Ax+b) = E[(Ax+b - E(Ax+b))(Ax+b - E(Ax+b))^T]$$

$$= E[(Ax+b - AE(x) - b)(Ax+b - AE(x) - b)^T]$$

$$= E[A\underbrace{(x - E(x))(x - E(x))^T}A^T]$$

As A and A^T are deterministic, they can be taken out of the expectation like in question (c)

$$\operatorname{cov}(Ax+b) = A \cdot E(x - E(x))(x - E(x))^T \cdot A^T$$

$$\operatorname{cov}(Ax+b) = A \cdot \operatorname{cov}(x) \cdot A^T \quad //$$

3) a) $x \in \mathbb{R}^n, y \in \mathbb{R}^m, A \in \mathbb{R}^{n \times m}$

$\nabla_x x^T A y$

Let $z = x^T A y$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11}y_1 + a_{12}y_2 + \cdots + a_{1m}y_m \\ \vdots \\ a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nm}y_m \end{bmatrix}$$

$$= (a_{11}y_1 + a_{12}y_2 + \cdots + a_{1m}y_m)x_1 +$$
$$(a_{21}y_1 + a_{22}y_2 + \cdots + a_{2m}y_m)x_2 +$$
$$\vdots$$
$$(a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nm}y_m)x_n$$

$$z = \sum_{i=1}^{n} \sum_{j=1}^{m} (a_{ij} y_j) x_i$$

$$\frac{\partial z}{\partial x_i} = \sum_{j=1}^{m} a_{ij} y_j$$

$$\frac{\partial z}{\partial x} = \begin{bmatrix} \sum_{j=1}^{m} a_{1j} y_j \\ \sum_{j=1}^{m} a_{2j} y_j \\ \vdots \\ \sum_{j=1}^{m} a_{nj} y_j \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & & & \vdots \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nm} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$= Ay$$

$$\nabla_x x^T A y = Ay$$

b) $x \in \mathbb{R}^n, y \in \mathbb{R}^m, A \in \mathbb{R}^{n\times m}$

$\nabla_y x^T A y$

LET $z = x^T A y$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11}y_1 + a_{12}y_2 + \cdots + a_{1m}y_m \\ \vdots \\ a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nm}y_m \end{bmatrix}$$

$$= (a_{11}y_1 + a_{12}y_2 + \dots + a_{1m}y_m)x_1 +$$

$$(a_{21}y_1 + a_{22}y_2 + \dots + a_{2m}y_m)x_2 +$$

$$\vdots$$

$$(a_{n1}y_1 + a_{n2}y_2 + \dots + a_{nm}y_m)x_n$$

$$z = \sum_{i=1}^{n} \sum_{j=1}^{m} (a_{ij}\, y_j) x_i$$

$$\frac{\partial z}{\partial y_j} = \sum_{i=1}^{n} (a_{ij}\, x_i)$$

$$\frac{\partial z}{\partial y} = \begin{bmatrix} a_{11}x_1 + a_{21}x_2 + \cdots + a_{n1}x_n \\ a_{12}x_1 + a_{22}x_2 + \cdots + a_{n2}x_2 \\ \vdots \\ a_{1m}x_1 + a_{2m}x_2 + \cdots + a_{nm}x_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & & & \vdots \\ \vdots & & & \\ a_{1m} & \cdots & & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= A^T x$$

$$\nabla_y \, x^T A y = A^T x$$

c) $x \in \mathbb{R}^n, y \in \mathbb{R}^m, A \in \mathbb{R}^{n \times m}$

$\nabla_A x^T A y$

LET $z = x^T A y$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11}y_1 + a_{12}y_2 + \cdots + a_{1m}y_m \\ \vdots \\ a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nm}y_m \end{bmatrix}$$

$$= (a_{11}y_1 + a_{12}y_2 + \cdots + a_{1m}y_m)x_1 +$$
$$(a_{21}y_1 + a_{22}y_2 + \cdots + a_{2m}y_m)x_2 +$$
$$\vdots$$
$$(a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nm}y_m)x_n$$

$$z = \sum_{i=1}^{n} \sum_{j=1}^{m} (a_{ij} y_j) x_i$$

$$\frac{\partial z}{\partial a_{ij}} = x_i y_j$$

$$\frac{\partial z}{\partial A} = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_m \\ x_2 y_1 & & & \vdots \\ \vdots & & & \\ x_n y_1 & & \cdots & x_n y_m \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix}$$

$$= x y^T$$

$$\nabla_A x^T A y = x y^T \;//$$

d) $f = x^T A x + b^T x$

Let $g = x^T A x$ and $h = b^T x$

$$g = [x_1 \quad x_2 \quad \dots \quad x_n] \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & \dots & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

if $i = j$

$$\frac{\partial g}{\partial x_i} = 2 a_{ii} x_i$$

if $i \neq j$

$$\frac{\partial g}{\partial x_i} = \sum_{\substack{j=1, \\ i \neq j}}^{n} a_{ij} x_j + \sum_{\substack{j=1 \\ i \neq j}}^{n} a_{ji} x_j$$

$$\frac{\partial g}{\partial x_i} = 2 a_{ii} x_i + \sum_{\substack{j=1, \\ i \neq j}}^{n} (a_{ij} + a_{ji}) x_j$$

$$= \sum_{j=1}^{n} (a_{ij} + a_{ji}) x_j$$

$$= \sum_{j=1}^{n} a_{ij} x_j + \sum_{j=1}^{n} a_{ji} x_j$$

$$\frac{\partial g}{\partial x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} & & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$+ \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ \vdots & & & \vdots \\ a_{1n} & & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= Ax + A^T x$$

$$h = b^T x$$

$$h = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= b_1 x_1 + \cdots + b_n x_n$$

$$= \sum_{i=1}^{n} b_i x_i$$

$$\frac{\partial h}{\partial x_i} = b_i$$

$$\frac{\partial h}{\partial x} = b$$

$$\nabla_x f = \frac{\partial f}{\partial x} = \frac{\partial g}{\partial x} + \frac{\partial h}{\partial x} = (A + A^T)x + b \,//$$

e) $f = tr(AB)$

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & & & \\ a_{n1} & & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \\ b_{m1} & \cdots & b_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1m}b_{m1} & \cdots & a_{11}b_{1n} + a_{12}b_{2n} + \dots + a_{1m}b_{mn} \\ & a_{21}b_{12} + a_{22}b_{21} + \dots + a_{2n}b_{m2} & \\ & \ddots & \\ & & a_{n1}b_{1n} + a_{n2}b_{2n} + \dots + a_{nm}b_{mn} \end{bmatrix}$$

$$tr(AB) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} b_{ji}$$

$$\frac{\partial f}{\partial a_{ij}} = b_{ji}$$

$$\frac{\partial f}{\partial A} = B^T \quad //$$

4) $\hat{y} = Wx$

$$L = \frac{1}{2} \sum_{i=1}^{n} \| y^{(i)} - Wx^{(i)} \|^2$$

$$= \frac{1}{2} \sum_{i=1}^{n} (y_i - Wx_i)^T (y_i - Wx_i)$$

Let us take

$$y - Wx = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} - \begin{bmatrix} W \end{bmatrix} \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

$$(y - Wx)^T (y - Wx) = \begin{bmatrix} (y_1 - Wx_1)^T (y_1 - Wx_1) & (y_1 - Wx_1)^T (y_2 - Wx_2) & \cdots \\ & (y_2 - Wx_2)^T (y_2 - Wx_2) & \\ & & \ddots \\ & & & (y_n - Wx_n)^T (y_n - Wx_n) \end{bmatrix}$$

All the elements of L are along the diagonals of $(y - Wx)^T (y - Wx)$

$$L = \frac{1}{2} \operatorname{tr} \left[(y - Wx)^T (y - Wx) \right]$$

$$= \frac{1}{2} \operatorname{tr} \left[(y^T - x^T W^T)(y - Wx) \right]$$

$$= \frac{1}{2} \operatorname{tr} \left(y^T y - x^T W^T y - y^T W x + x^T W^T W x \right)$$

Find W to minimize L, $\frac{\partial L}{\partial W} = 0$

$$\frac{\partial L}{\partial W} = \frac{\partial}{\partial W} tr(y^T y) + \frac{\partial}{\partial W} tr(x^T W^T W x) - \frac{\partial}{\partial W} tr(x^T W^T y)$$
$$- \frac{\partial}{\partial W} tr(y^T W x)$$

WE KNOW THAT

$$tr(A^T) = tr(A)$$

$$tr(ABC) = tr(CAB) = tr(BCA)$$

$$\frac{\partial}{\partial W} tr(y^T y) = 0 \quad [\text{independent of } W]$$

$$\frac{\partial}{\partial W} tr(x^T W^T W x) = \frac{\partial}{\partial W} tr(x x^T W^T W) = \frac{\partial}{\partial W} tr(W x x^T W^T)$$

We know $\frac{\partial}{\partial W} tr(W A W^T) = W A^T + W A$

$$\text{So } \frac{\partial}{\partial W} tr(x^T W^T W x) = \frac{\partial}{\partial W} tr(W \underline{x x^T} W^T) = W(x x^T)^T + W x x^T$$

$$= W x x^T + W x x^T$$

$$= 2 W x x^T$$

$$\frac{\partial}{\partial W} tr(x^T W^T y) = \frac{\partial}{\partial W} tr(x^T W^T y)^T = \frac{\partial}{\partial W} tr(y^T W x)$$

$$= \frac{\partial}{\partial W} tr(x y^T W)$$

$$= \frac{\partial}{\partial W} tr(W x y^T)$$

$$= y x^T$$

$$\left[\text{FROM } \frac{\partial}{\partial W} tr(WA) = A^T \right]$$

$$\frac{\partial L}{\partial W} = 0 + 2 W x x^T - y x^T - y x^T$$

$$= 2 W x x^T - 2 y x^T$$

$$\frac{\partial L}{\partial W} = 0$$

$$W x x^T = y x^T$$

$$W = y x^T (x x^T)^{-1} \text{ //}$$