Q1 : PROVE THE MATRICES ARE UNITARY

MATRIX U IS UNITARY IFF $UU^\dagger = U^\dagger U = I$

(i) $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$X^\dagger X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

(ii) $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$Y^\dagger Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

(iii) $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$Z^\dagger Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

(iv) $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$I^{+} I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

(v) $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$H^{+} H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \times \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= I$$

(vi) $R_x(\theta) = \begin{bmatrix} \cos\theta/2 & -i\sin\theta/2 \\ -i\sin\theta/2 & \cos\theta/2 \end{bmatrix}$

$$R_x^{+}(\theta)\, R_x(\theta) = \begin{bmatrix} \cos\theta/2 & i\sin\theta/2 \\ i\sin\theta/2 & \cos\theta/2 \end{bmatrix} \begin{bmatrix} \cos\theta/2 & -i\sin\theta/2 \\ -i\sin\theta/2 & \cos\theta/2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta/_2 - i^2 \sin^2\theta/_2 & -i\sin\theta/_2 \cos\theta/_2 + i\sin\theta/_2 \cos\theta/_2 \\ i\sin\theta/_2 \cos\theta/_2 - i\sin\theta/_2 \cos\theta/_2 & -i^2 \sin^2\theta/_2 + \cos^2\theta/_2 \end{bmatrix}$$

$$\Rightarrow \sin^2\theta + \cos^2\theta = 1$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

(vii) $R_y(\theta) = \begin{bmatrix} \cos\theta/2 & -\sin\theta/_2 \\ \sin\theta/_2 & \cos\theta/_2 \end{bmatrix}$

$$R_y(\theta)^{+} R_y(\theta) = \begin{bmatrix} \cos\theta/2 & \sin\theta/_2 \\ -\sin\theta/_2 & \cos\theta/_2 \end{bmatrix} \begin{bmatrix} \cos\theta/_2 & -\sin\theta/_2 \\ \sin\theta/_2 & \cos\theta/_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta/_2 + \sin^2\theta/_2 & -\cos\theta/_2 \sin\theta/_2 + \sin\theta/_2 \cos\theta/_2 \\ -\sin\theta/_2 \cos\theta/_2 + \sin\theta/_2 \cos\theta/_2 & \sin^2\theta/_2 + \cos^2\theta/_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

(viii) $R_z(\theta) = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$

$$R_z(\theta)^{+} R_z(\theta) = \begin{bmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{bmatrix} \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$

$$= \begin{bmatrix} e^{0} & 0 \\ 0 & e^{0} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

(ix) $R_\varphi = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{bmatrix}$

$$R_\varphi^{+} R_\varphi = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\varphi} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{0} \end{bmatrix} = I$$

(x) $S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$

$$S^{+}S = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

(xi) $T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$

$$T^{+}T = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{0} \end{bmatrix}$$

$$= I$$

Q2: GIVEN: U IS UNITARY

$$UU^{+} = U^{+}U = I$$

TO PROVE: U^{+} is UNITARY

TO PROVE: $(U^{+})^{+}U^{+} = I$

$$U^{+} = \bar{U}^{T}$$

$$(U^{+})^{+} = \left(\overline{\bar{U}^{T}}\right)^{T}$$

WE KNOW THAT $\bar{(\bar{U})} = U$

AND $(U^{T})^{T} = U$

SO $(U^{+})^{+} = U$

SO $(U^{+})^{+}U^{+} = UU^{+} = I$

$$U^{+}(U^{+})^{+} = U^{+}U = I$$

THEREFORE, U^{+} IS UNITARY.

Q3: U_1, U_2 ARE UNITARY

$$U_1^+ U_1 = U_1 U_1^+ = I$$

$$U_2^+ U_2 = U_2 U_2^+ = I$$

So $U_1 U_2$ be the product

To PROVE : $(U_1 U_2)(U_1 U_2)^+ = I$

$$(U_1 U_2)(U_1 U_2)^+ = (U_1 U_2)(U_2^+ U_1^+)$$

$$= U_1 (U_2 U_2^+) U_1^+$$

$$= U_1 I U_1^+$$

$$= U_1 U_1^+$$

$$= I \quad //$$

THEREFORE, THE PRODUCT OF TWO UNITARY MATRICES IS UNITARY.

Q4: $U = R + iQ$

$$U^{+} = (R+iQ)^{+} = (\bar{R} - i\bar{Q})^{T} = R^{T} - iQ^{T}$$

$$[\bar{R} = R, \bar{Q} = Q]$$

U IS UNITARY:

$$UU^{+} = I_N \Rightarrow (R+iQ)(R^{T} - iQ^{T}) = I_N$$

$$\Rightarrow RR^{T} + QQ^{T} + i(QR^{T} - RQ^{T}) = I_N$$

$$\left.\begin{array}{l} RR^{T} + QQ^{T} = I_N \\ QR^{T} - RQ^{T} = 0_N \qquad QR^{T} = RQ^{T} \end{array}\right] - (1)$$

$$U' = \begin{bmatrix} R & Q \\ -Q & R \end{bmatrix} \qquad (U')^{+} = \begin{bmatrix} R^{T} & -Q^{T} \\ Q^{T} & R^{T} \end{bmatrix}$$

$$(U')(U')^{+} = \begin{bmatrix} RR^{T} + QQ^{T} & -RQ^{T} + QR^{T} \\ -QR^{T} + RQ^{T} & QQ^{T} + RR^{T} \end{bmatrix}$$

USING (1)

$$(U')(U')^{+} = \begin{bmatrix} I_N & 0_N \\ 0_N & I_N \end{bmatrix} = I_{2N}$$ //

HENCE U' IS UNITARY //

PAULI MATRIX:

$$y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$y = R + iQ = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + i \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$y' = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$(y')^{+} (y') = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

HENCE y' IS UNITARY FOR PAULI MATRIX.

Q5: X, Y, Z, I

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

TO PROVE: X, Y, Z AND I FORM AN ORTHONORMAL BASIS FOR THE SPACE OF 2×2 MATRIX.

WE NEED TO PROVE THESE:

1. X, Y, Z AND I ARE ORTHONORMAL
2. X, Y, Z AND I FORM A BASIS
   - 2i. X, Y, Z AND I ARE LINEARLY INDEPENDENT
   - 2ii. EVERY 2×2 MATRIX CAN BE WRITTEN AS A COMBINATION OF X, Y, Z AND I.

PROOF :

1. X, Y, Z AND I ARE ORTHONORMAL

U and V are orthonormal if

$$\langle U | U \rangle = 1 \quad , \langle V | V \rangle = 1$$

$$\langle U | V \rangle = 0$$

$$\langle U | V \rangle = \frac{1}{2} \mathrm{Tr}(U^{+} V)$$

$$X^{T} X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$Y^{+} Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$Z^{+} Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$I^\dagger I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\langle X | X \rangle = \langle Y | Y \rangle = \langle Z | Z \rangle = \langle I | I \rangle = 1 \ //$$

TRACE

$$X^\dagger Y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \longrightarrow 0$$

$$Y^T X = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \longrightarrow 0$$

$$X^\dagger Z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \longrightarrow 0$$

$$Z^\dagger X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \longrightarrow 0$$

$$X^\dagger I = X^\dagger = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \longrightarrow 0$$

$$I^\dagger X = I X = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \longrightarrow 0$$

$$Y^\dagger Z = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \longrightarrow 0$$

$$Z^{+}Y = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \rightarrow 0$$

$$Y^{+}I = Y^{+} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \rightarrow 0$$

$$I^{+}Y = IY = Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \rightarrow 0$$

$$Z^{+}I = Z^{+} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow 0$$

$$I^{+}Z = IZ = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow 0$$

$$\langle X|Y\rangle = \langle Y|X\rangle = \langle X|Z\rangle = \langle Z|X\rangle = \langle X|I\rangle = \langle I|X\rangle$$

$$= \langle Y|Z\rangle = \langle Z|Y\rangle = \langle Y|I\rangle = \langle I|Y\rangle = \langle Z|I\rangle$$

$$= \langle I|Z\rangle = 0 \;//$$

HENCE X, Y, Z, I ARE ORTHONORMAL //

2i. X, Y, Z AND I ARE LINEARLY INDEPENDENT

They are linearly independent if

$$aX + bY + cZ + dI = 0$$

gives $a = b = c = d = 0$, $a, b, c, d \in \mathbb{C}$

$$aX + bY + cZ + dI = a\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} +$$

$$c\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + d\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c+d & a-bi \\ a+bi & d-c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $c + d = 0$

$a - bi = 0$ $\quad$ $a = 0$ $\quad$ $c = 0$

$a + bi = 0$ $\quad$ $b = 0$ $\quad$ $d = 0$

$d - c = 0$

$$a = b = c = d = 0$$

THEREFORE X, Y, Z AND I ARE LINEARLY INDEPENDENT //

2ii. EVERY 2×2 MATRIX CAN BE WRITTEN AS A COMBINATION OF X, Y, Z AND I.

LET M BE ANY 2×2 MATRIX

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$M = \alpha X + \beta Y + \gamma Z + \delta I$$

$$\alpha, \beta, \gamma, \delta \in \mathbb{C}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \alpha \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} +$$

$$\gamma \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \delta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \gamma + \delta & \alpha - i\beta \\ \alpha + \beta i & -\gamma + \delta \end{bmatrix}$$

$$\gamma + \delta = a \qquad \alpha - i\beta = b$$

$$-\gamma + \delta = d \qquad \alpha + i\beta = c$$

$$\delta = \left(\frac{a+d}{2} \right) \qquad \gamma = \left(\frac{a-d}{2} \right)$$

$$\alpha = \left(\frac{b+c}{2}\right) \quad \beta = \left(\frac{c-b}{2i}\right) = \left(\frac{b-c}{2}\right) i$$

SO ANY GIVEN 2x2 MATRIX CAN BE EXPRESSED AS THE LINEAR COMBINATION OF X, Y, Z AND I.

COMBINING THE THREE PROOFS, WE CAN PROVE THAT THE FOUR PAULI MATRICES X, Y, Z AND I FORM AN ORTHONORMAL BASIS FOR THE SPACE OF 2x2 MATRICES.