VECTORS

DIRAC NOTATION:

$$\left.\begin{array}{lll} |\psi\rangle & \rightarrow & \text{ket} \\ \langle\psi| & \rightarrow & \text{Bra} \end{array}\right] \text{ DIRAC NOTATION}$$

$$|\psi\rangle \rightarrow \text{COLUMN VECTORS}$$

$$\langle\psi| \rightarrow \text{ROW VECTORS}$$

L^P NORM:

$$\|v\|_p = (v_1^p + v_2^p + \dots + v_n^p)^{1/p}.$$

L2 NORM OF COMPLEX NUMBER:

$$\|z\|_2^2 = \bar{z}z.$$

$$|v|^2 = \bar{v}_1 v_1 + \bar{v}_2 v_2 + \dots + \bar{v}_n v_n.$$

INNER PRODUCT

$$\langle u, v\rangle = \bar{u}_1 v_1 + \bar{u}_2 v_2 + \dots + \bar{u}_n v_n$$

$$= \bar{u}^T v = u^\dagger v$$

$$\langle u|v\rangle = \langle u, v\rangle \Rightarrow \langle u| = u^\dagger$$

POLAR COORDINATES:

$$a+bi \Rightarrow r = \sqrt{a^2+b^2}$$

$$\theta = \tan^{-1}\left(b/a\right).$$

$$z = a+bi = re^{i\theta} = r\cos\theta + ir\sin\theta$$

MATRICES

SIMILAR MATRICES:

- → Same trace
- → Differ only by a change in basis

TAYLOR SERIES - EXPONENTIATION:

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n \quad , A = \text{Matrix}.$$

OUTER PRODUCT:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|0\rangle\langle 1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

OUTER PRODUCT

$$|u\rangle\langle v| = uv^{+}$$ //

TENSOR PRODUCT:

1 - TENSOR : SCALAR

2 - TENSOR : VECTOR

3 - TENSOR : MATRIX

$$|0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \otimes \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{bmatrix} xa \\ xb \\ xc \\ ya \end{bmatrix}$$

$$\begin{bmatrix} yb \\ yc \end{bmatrix}$$

$$u \otimes v \longrightarrow w$$

$$\begin{array}{ccc} u & v & w \\ \downarrow & \downarrow & \downarrow \\ a \times b & c \times d & (ac) \times (bd) \end{array}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \otimes \begin{pmatrix} a & b & c \end{pmatrix} = \begin{bmatrix} xa & xb & xc \\ ya & yb & yc \end{bmatrix}$$

2×1 $\quad$ 1×3 $\quad$ $(2 \times 1) \times (1 \times 3)$

SET THEORY:

FUNCTION $f(X \to Y)$

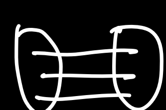
$X \downarrow$ Domain

$Y \to$ Codomain

if (x_1, y_1) AND (x_1, y_2) exists,

$y_1 = y_2$ //

INJUCTIVE $\to$ (ONE TO ONE)



SURJECTIVE $\to$ ONTO

$f: X \to Y, \ \forall y \in Y \Rightarrow f(x) = y$

$x \in X$

BIJECTIVE: BOTH ONE TO ONE + ONTO

$f: X \to Y$ = BIJECTIVE $\Rightarrow f^{-1}: Y \to X$

exists.

eg: $f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$

It is a function

$f^{-1}(x) = \sqrt{x}$ $\quad f^{-1}: \mathbb{R} \to \mathbb{R}$

is not a function

eg: $x = 4$

$f^{-1}(x) = 2$ and -2

$\Rightarrow$ LETS MAKE IT ONE TO ONE

$f(x) = x^2 \quad f: [0, \infty) \to \mathbb{R}$

$f^{-1}(x) = \sqrt{x} \quad f^{-1}: \mathbb{R} \to [0, \infty)$

WHAT ABOUT $x = -1$

f^{-1} doesn't exist

$\Rightarrow$ ONTO

$f(x) = x^2 \quad f: [0,\infty) \to [0,\infty)$

$f^{-1}(x) = \sqrt{x} \quad f^{-1}: [0,\infty) \to [0,\infty)$

EXISTS //

LINEAR TRANSFORMATION:

TRANSFORMATION $T: \mathbb{R} \to \mathbb{R}$ is linear iff

* $x, y \in \mathbb{R} \quad T(x+y) = T(x) + T(y)$

AND

* $a \in \mathbb{R} \quad T(ax) = aT(x)$

eg: $T(x) = x+1$ is not linear

$T(x+y) = (x+y)+1$

$T(x) + T(y) = x+1 + y+1 = x+y+2$

NOT SAME

$T(3x) = 3x+1$

$3T(x) = 3x+3$

NOT SAME

VECTOR SPACE:

Vector Space, V over a Field F is an abelian group V equipped with an action of the field F on V.

GROUP:

GROUP $(G, *)$ satisfies

→ CLOSURE

$$*: G \times G \to G$$

→ ASSOCIATIVITY:

$$(x * y) * z = x * (y * z)$$

→ IDENTITY:

$$\forall x \in G$$

$$x * e = e * x = x$$

→ INVERSE:

$$\forall x \in G$$

$$x * x^{-1} = x^{-1} * x = e$$

ABELIAN GROUP:

$(G, *) \rightarrow$ ABELIAN IFF

GROUP + COMMUTATIVE //

So $\mathbb{C}^2$ VECTORS IS AN ABELIAN GROUP

WITH RESPECT TO ADDITION.

FIELDS:

F is

* Abelian Group over $+$
* Satisfies these ore $\times$
  * Closure
  * Associativity
  * Identity
  * Inverse
  * Commutativity
  * Distributivity

$\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are all fields

ACTION OF A FIELD ON AN ABELIAN GROUP:

An action of a field F on an abelian group V is a function $\cdot : F \times V \to V$ satisfying

→ Distributivity I : For all $a \in F$ and all $u, v \in V$

$$a \cdot (u + v) = a \cdot u + a \cdot v$$

→ Distributivity II : For all $a, b \in F$ and all $v \in V$

$$(a+b) \cdot v = a \cdot v + b \cdot v$$

→ Compatibility of action with multiplication in F

$$(ab) \cdot v = a \cdot (b \cdot v)$$

→ IDENTITY:

$$1 \cdot v = v.$$

SPAN:

We say set of vectors $v_1 \cdots v_m$ span vector space V over field F iff $\forall v \in V$, $\exists\ a_1 \cdots a_m \in F$

$$a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = v$$

BASIS:

Basis of vector space V over F is a linearly independent spanning set of vectors.

DIMENSION:

Dimension of vector space V is the number of vectors for any basis for V.

$$\dim_{\mathbb{C}}(\mathbb{C}^2) = 2$$

$$\dim_{\mathbb{R}}(\mathbb{C}^2) = 4$$

V

F

ORTHOGONAL VECTORS:

$$\langle u, v \rangle = \langle u | v \rangle = u^{+} v = 0.$$

ORTHONORMAL BASIS:

$\rightarrow$ BASIS

$$\rightarrow \langle v_i, v_j \rangle = \begin{cases} 1, & i = j \end{cases}$$

[redacted]