$Z_f |x\rangle = (-1)^{f(x)} |x\rangle$

$x \in \{0,1\}^n$

Helper function

$\hookrightarrow$ $f(x)$'s encoder (like $U_f$)

$$Z_0 |x\rangle = \begin{cases} -|x\rangle, & \text{if } x = 0^n \\ |x\rangle, & \text{otherwise} \end{cases}$$

$f(0^n) = 1$

$f(x) = 0$, for all other $x$

**GROVER'S ALGORITHM :**

$$G = -H^{\otimes n} Z_0 H^{\otimes n} Z_f$$

$X = |0^n\rangle$ ($n$ $|0\rangle$ qubits)

$H^{\otimes n} X$

repeat { apply $G$ to $X$ } $O(\sqrt{2^n})$ times

measure $X$ and output the result

LETS IMPLEMENT $Z_0$ FOR 2 QUBITS:

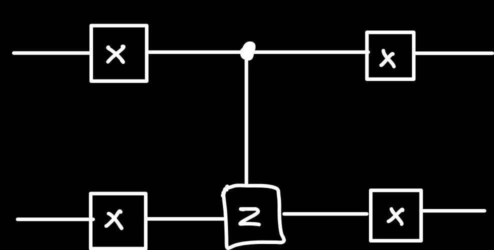
$n = 2$

$f(00) = 1$

| $x$ | $f(x)$ | $(-1)^{f(x)} x$ |
|---|---|---|
| 00 | 1 | $-00$ |
| 00 | 1 | $-00$ |
| 01 | 0 | 01 |
| 01 | 0 | 01 |
| 10 | 0 | 10 |
| 10 | 0 | 10 |
| 11 | 0 | 11 |
| 11 | 0 | 11 |

So WE APPLY $Z$, when both qubits are 0

$Z_0 =$



Q1: $f(00) = 1$

$Z_f = Z_0$

STEP 1: $H^{\otimes 2}|00\rangle = |++\rangle$

$= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$

STEP 2: $N = 4$

$$k = \frac{\pi}{4\theta} - \frac{1}{2} = \frac{\pi\sqrt{N}}{4} - \frac{1}{2}$$

$$k = \frac{\pi \times 2}{4} - \frac{1}{2} = \frac{(\pi - 1)}{2} \approx 1$$

RUN ONCE

$$G = -H^{\otimes n} Z_0 H^{\otimes n} Z_f$$

$$H^{\otimes 2} Z_0 H^{\otimes 2} Z_f \underbrace{\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)}$$

$$X^{\otimes 2} CZ X^{\otimes n} \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

$$= X^{\otimes 2} CZ \frac{1}{2}(|11\rangle + |10\rangle + |01\rangle + |00\rangle)$$

$$= X^{\otimes 2} \tfrac{1}{2}\left(-|11\rangle + |10\rangle + |01\rangle + |00\rangle\right)$$

$$= \tfrac{1}{2}\left(-|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

$$H^{\otimes 2} Z_0 \underbrace{H^{\otimes 2} \tfrac{1}{2}\left(-|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)}_{}$$

$$\underbrace{H^{\otimes 2} \tfrac{1}{2}\left(-2|00\rangle\right)}_{} + \tfrac{1}{2} \times 2|00\rangle$$

$$= -\tfrac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right) + |00\rangle$$

$$= -\tfrac{1}{2}\left(-|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

$$Z_0\left(-\tfrac{1}{2}\left(-|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)\right)$$

$$X^{\otimes 2} CZ X^{\otimes n}\left(-\tfrac{1}{2}\left(-|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)\right)$$

$$= X^{\otimes 2} CZ\left(-\tfrac{1}{2}\left(-|11\rangle + |10\rangle + |01\rangle + |00\rangle\right)\right)$$

$$= X^{\otimes 2}\left(-\tfrac{1}{2}\left(|11\rangle + |10\rangle + |01\rangle + |00\rangle\right)\right)$$

$= -\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$

$H^{\otimes 2}\left(-\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)\right)$

$= -|00\rangle$

STEP 3:

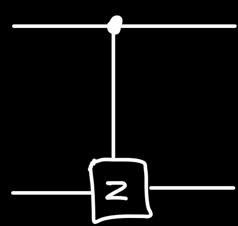
MEASURE $\Rightarrow$ 00 //

Q2 : $f(11) = 1$

$Z_0$ is same

$Z_f$ change slighty

$Z_f =$



$Z_f = CZ$

STEP 1 : $H^{\otimes 2}|00\rangle = |{+}{+}\rangle$

$= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$

STEP 2 : $N = 4$

$$k = \frac{\pi}{4\theta} - \frac{1}{2} = \frac{\pi\sqrt{N}}{4} - \frac{1}{2}$$

$$k = \frac{\pi \times 2}{4} - \frac{1}{2} = \frac{(\pi - 1)}{2} \approx 1$$

RUN ONCE

$G = -H^{\otimes n} Z_0 H^{\otimes n} Z_f$

$$H^{\otimes 2} Z_0 H^{\otimes 2} \underbrace{Z_f \tfrac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)}$$

$$CZ \tfrac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

$$= \tfrac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle - |11\rangle\right)$$

$$H^{\otimes 2} Z_0 \underbrace{H^{\otimes 2} \tfrac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle - |11\rangle\right)}$$

$$H^{\otimes 2} Z_0 \tfrac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle - |11\rangle\right)$$

$$Z_0 \left(\tfrac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle - |11\rangle\right)\right)$$

$$X^{\otimes 2} CZ X^{\otimes n}\left(\tfrac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle - |11\rangle\right)\right)$$

$$X^{\otimes 2} CZ \tfrac{1}{2}\left(|11\rangle + |10\rangle + |01\rangle - |00\rangle\right)$$

$$X^{\otimes 2} \tfrac{1}{2}\left(-|11\rangle + |10\rangle + |01\rangle - |00\rangle\right)$$

$$\tfrac{1}{2}\left(-|00\rangle + |01\rangle + |10\rangle - |11\rangle\right)$$

$H^{\otimes 2} \ \frac{1}{2} \left( -|00\rangle + |01\rangle + |10\rangle - |11\rangle \right)$

$H^{\otimes 2} \ -\frac{1}{2} \left( |00\rangle - |01\rangle - |10\rangle - |11\rangle \right)$

$= H^{\otimes 2} \quad - \left( |--\rangle \right)$

$= -|11\rangle$

STEP 3 :

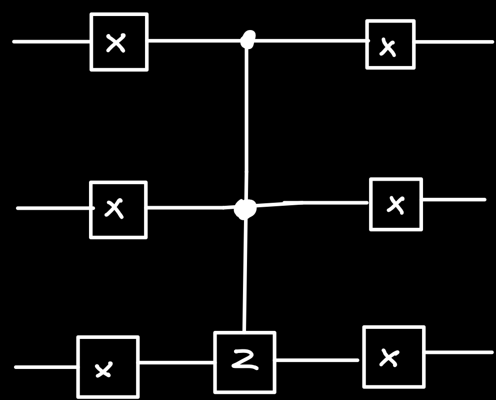
MEASURE => 11 //

Q3: $n = 3$

$f(101) = 1$

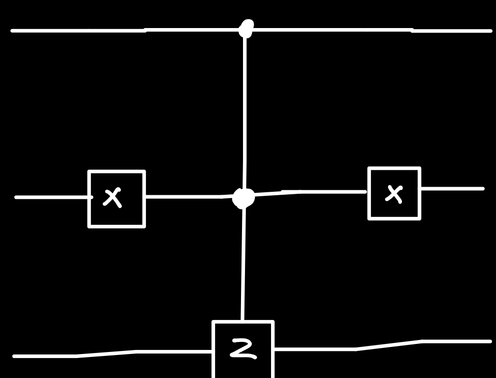
We can extend $Z_0, Z_f$ from before to get:

$Z_0$ :





$Z_f$ :





$$Z_f = (I \times I)\ CIZ\ (I \times I)$$

$$Z_f \left(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle\right)$$

$$(I \times I) \rightarrow \left(|010\rangle + |011\rangle + |000\rangle + |001\rangle + |110\rangle + |111\rangle + |100\rangle + |101\rangle\right)$$

$$CCZ \rightarrow \left(|010\rangle + |011\rangle + |000\rangle + |001\rangle + |110\rangle - |111\rangle + |100\rangle + |101\rangle\right)$$

$(I \times I) \rightarrow (|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle - |101\rangle + |110\rangle + |111\rangle)$

WORKS AS EXPECTED

STEP 1: $H^{\otimes 3}|000\rangle = |+++\rangle$

$$= \frac{1}{2\sqrt{2}}\big(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110 + |111\rangle\big)$$

STEP 2: $N = 8$

$$k = \frac{\pi}{4\theta} - \frac{1}{2} = \frac{\pi\sqrt{N}}{4} - \frac{1}{2}$$

$$k = \frac{\pi \times 2\sqrt{2}}{4} - \frac{1}{2} = \frac{1}{2}(\pi\sqrt{2} - 1) = 1.72 \approx 2$$

RUN TWICE

$$G = -H^{\otimes n} Z_0 H^{\otimes n} Z_f$$

$$Z_f \; \frac{1}{2\sqrt{2}}\big(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle\big)$$

$$H^{\otimes 3}\ \frac{1}{2\sqrt{2}}\big(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle - |101\rangle + |110\rangle + |111\rangle\big)$$

$$-2|101\rangle + |101\rangle$$

$$H^{\otimes 3} \times \frac{1}{2\sqrt{2}}\big(-2|101\rangle + |000\rangle$$

$$-\frac{1}{\sqrt{2}} \times \Big( \underbrace{\underbrace{(|0\rangle - |1\rangle)(|0\rangle + |1\rangle)}_{|00\rangle + |01\rangle - |10\rangle - |11\rangle}(|0\rangle - |1\rangle)}_{} \Big) \times \frac{1}{2\sqrt{2}}$$

$$\big(|000\rangle + |010\rangle - |100\rangle - |110\rangle - |001\rangle - |011\rangle + |101\rangle + |111\rangle\big)$$

$$\frac{-1}{4}\big(|000\rangle + |010\rangle - |100\rangle - |110\rangle - |001\rangle - |011\rangle + |101\rangle + |111\rangle - 4|000\rangle\big)$$

$$\frac{-1}{4}\big(-3|000\rangle + |010\rangle - |100\rangle - |110\rangle - |001\rangle - |011\rangle + |101\rangle + |111\rangle\big)$$

APPLY $Z_0$

$$\frac{-1}{4}\Big(3|000\rangle + |010\rangle - |100\rangle - |110\rangle$$
$$- |001\rangle - |011\rangle + |101\rangle + |111\rangle\Big)$$

APPLY $H^{\otimes 3}$

$$H^{\otimes 3} \quad \frac{-1}{4}(2|000\rangle) + \left(\frac{-1}{4}\right) |101\rangle \times 2\sqrt{2}$$

$$\frac{1}{2\sqrt{2}} \times \frac{-1}{2}\Big(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle$$
$$+ |111\rangle\Big)$$
$$+ \frac{-1}{\sqrt{2}}(|101\rangle)$$

$$\frac{-1}{4\sqrt{2}}\Big(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + 5|101\rangle$$
$$+ |110\rangle + |111\rangle\Big)$$

ROUND 2:

APPLY $G = H^{\otimes 3} Z_0 H^{\otimes 3} Z_f$

APPLY $Z_f$

$$\frac{-1}{4\sqrt{2}}\Big(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle - 5|101\rangle$$
$$+ |110\rangle + |111\rangle\Big)$$

APPLY $H^{\otimes 3}$

$$H^{\otimes 3} \frac{-1}{4\sqrt{2}}\left(-6\,|1101\rangle\right) - \frac{1}{4\sqrt{2}}\left(|1000\rangle\right) \times 2\sqrt{2}$$

$$\frac{3}{2\sqrt{2}}\Big(|1000\rangle + |1010\rangle - |1100\rangle - |1110\rangle - |1001\rangle - |1011\rangle + |1101\rangle + |1111\rangle\Big) \times \frac{1}{2\sqrt{2}}$$

$$-\frac{4}{8}\,|1000\rangle$$

$$\frac{1}{8}\Big(-|1000\rangle + 3\,|1010\rangle - 3\,|1100\rangle - 3\,|1110\rangle - 3\,|1001\rangle - 3\,|1011\rangle + 3\,|1101\rangle + 3\,|1111\rangle\Big)$$

Apply $Z_0$

$$\frac{1}{8}\Big(|1000\rangle + 3\,|1010\rangle - 3\,|1100\rangle - 3\,|1110\rangle - 3\,|1001\rangle - 3\,|1011\rangle + 3\,|1101\rangle + 3\,|1111\rangle\Big)$$

Apply $H^{\otimes 3}$

$$H^{\otimes 3}\ \frac{3}{8}\Big(|1000\rangle + |1010\rangle - |1100\rangle - |1110\rangle - |1001\rangle - |1011\rangle + |1101\rangle + |1111\rangle$$

$$-\frac{2}{8}\,|1000\rangle$$

$$\frac{3}{8}|101\rangle \times 2\sqrt{2} \quad - \frac{2}{8} \times \frac{1}{2\sqrt{2}} \Big(|000\rangle + |001\rangle + |010\rangle + |011\rangle$$
$$+ |100\rangle + |101\rangle + |110\rangle + |111\rangle\Big)$$

$$\frac{3\sqrt{2}}{4}|101\rangle - \frac{1}{8\sqrt{2}}\Big(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle$$
$$+ |101\rangle + |110\rangle + |111\rangle\Big)$$

$$\frac{-1}{8\sqrt{2}}\Big(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle - 11|101\rangle$$
$$+ |110\rangle + |111\rangle\Big)$$

STEP 3 :

Measure

We observe 101 with $p = \left(\frac{11}{8\sqrt{2}}\right)^2 = \frac{121}{128}$

$= 94.5\%$