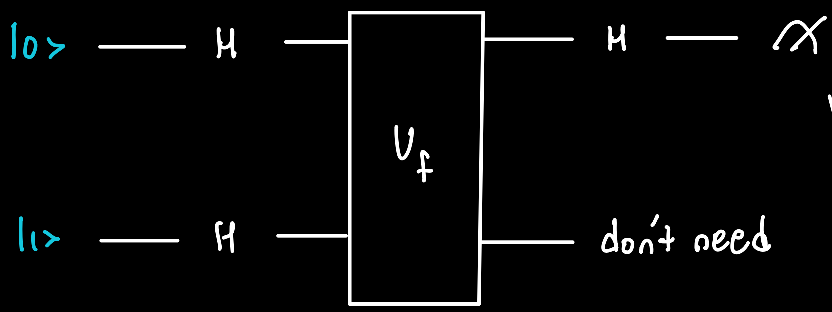
$n = 1$

INPUT	f_0 (CONSTANT-0)	f_1 (CONSTANT-1)	f_2 (BALANCED)	f_3 (BALANCED)
0	0	1	0	1
1	0	1	1	0

$$U_f |x\rangle |b\rangle = |x\rangle |b \oplus f(x)\rangle$$



0, if constant

1, if balanced.

WE NEED TO FIND THE U_f!

CONSTANT - 0 : f_0

$$f(x) = 0 \qquad b \oplus f(x) = b$$

$$U_f |x\rangle |b\rangle = |x\rangle |b\rangle$$

IDENTITY //

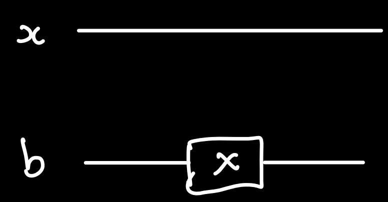
$\Rightarrow$ So, U_f IS JUST IDENTITY //

CONSTANT - 1 : f_1

$f(x) = 1$ $\qquad$ $b \oplus f(x) = !b$

$U_f |x\rangle |b\rangle = |x\rangle |!b\rangle$

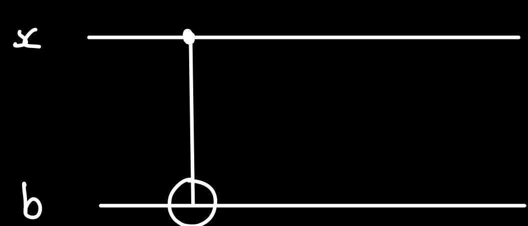
$U_f =$



BALANCED : f_2

INPUT		$f(x)$	OUTPUT	
x	b		x	$b \oplus f(x)$
0	0	0	0	0
0	1	0	0	1
1	0	1	1	1
1	1	1	1	0

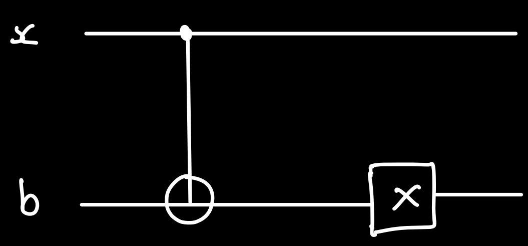
$U_f =$



BALANCED : f_3

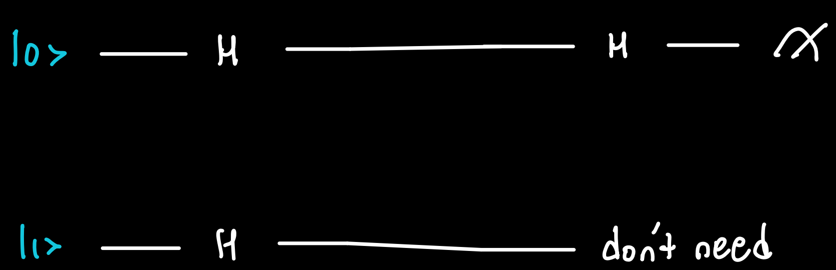
INPUT		$f(x)$	OUTPUT	
x	b		x	$b \oplus f(x)$
0	0	1	0	1
0	1	1	0	0
1	0	0	1	0
1	1	0	1	1

$U_f =$



LET US VERIFY IF THESE U_f WORK :

CONSTANT - 0 :



$(H \otimes I)(H \otimes H)|01\rangle$

$= (H \otimes I)(|+\rangle \otimes |-\rangle)$

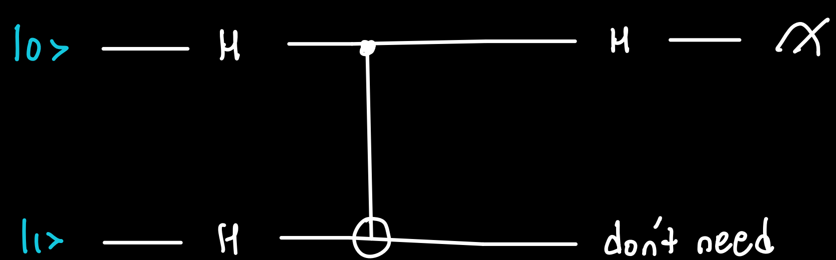
$= H|+\rangle \otimes |-\rangle$

$= |0\rangle \otimes |-\rangle$

$\uparrow$

measure, we will get 0 => CONSTANT

BALANCED: (f_2):



$(H \otimes I)\, C(X)[1,2]\, (H \otimes H)|01\rangle$

$= (H \otimes I)\, C(X)[1,2]\, \underbrace{(|+\rangle \otimes |-\rangle)}$

$\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$

$\frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$

$= (H \otimes I)\, \frac{1}{2}\left(|00\rangle - |01\rangle + |11\rangle - |10\rangle\right)$

$$= \frac{1}{2}\Big[\left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right)|0\rangle - \left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right)|1\rangle$$

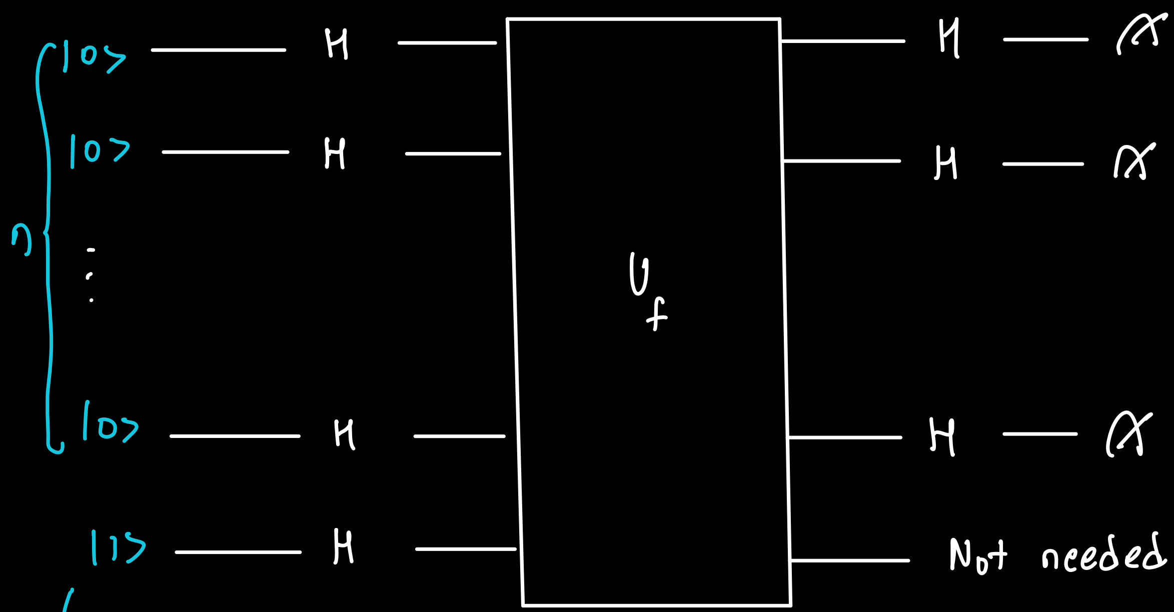
$$+ \left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)|1\rangle - \left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)|0\rangle\Big]$$

$$= \frac{1}{2\sqrt{2}}\left(\underline{|00\rangle} + |10\rangle - \underline{|01\rangle} - |11\rangle + \underline{|01\rangle} - |11\rangle - \underline{|00\rangle} + |10\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|10\rangle - |11\rangle\right)$$

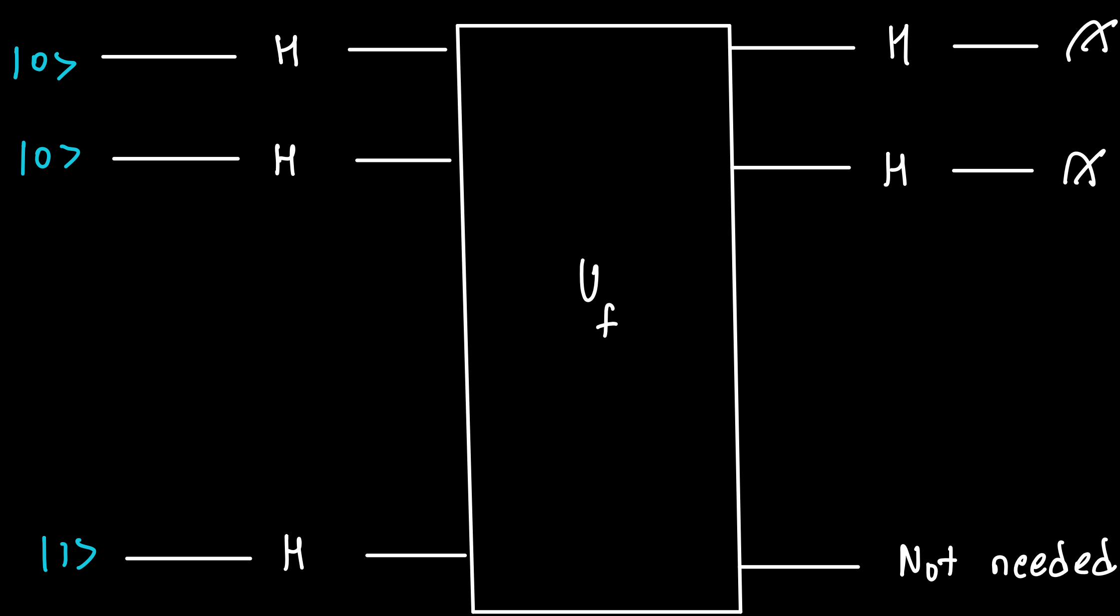
MEASURE QUBIT 1 $\rightarrow$ $|1\rangle$ $\Rightarrow$ BALANCED//

GENERIC DEUTSCH JOZSA CIRCUIT:



helper qubit = b.

$n = 2$



WE NEED TO FIND U_f //

$n = 2$

CONSTANT - 0 : f_0

$f(x) = 0$

$b \oplus f(x) = b$

$U_f |x\rangle |b\rangle = |x\rangle |b\rangle$

IDENTITY //

U_f IS AGAIN IDENTITY //

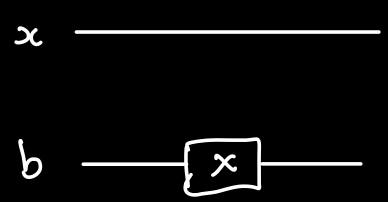
CONSTANT - 1 : f_1

$f(x) = 1$

$b \oplus f(x) = !b$

$U_f |x\rangle |b\rangle = |x\rangle |!b\rangle$

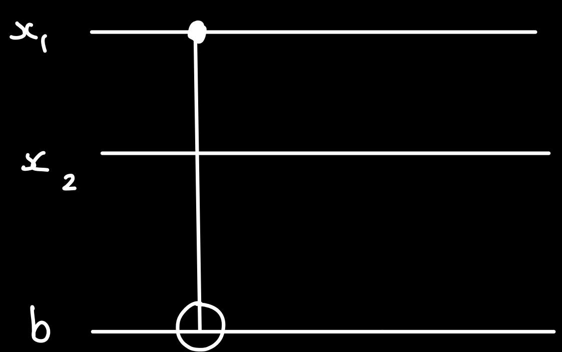
U_f :



BALANCED1:

INPUT		$f(x)$	OUTPUT	
x_1x_2	b		x_1x_2	$b \oplus f(x)$
00	0	0	00	0
00	1	0	00	1
01	0	0	01	0
01	1	0	01	1
10	0	1	10	1
10	1	1	10	0
11	0	1	11	1
11	1	1	11	0

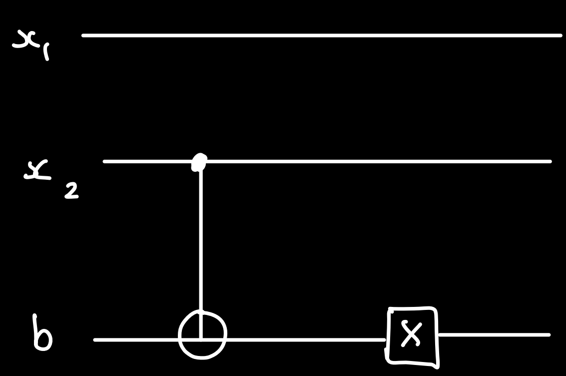
U_f =



BALANCED2:

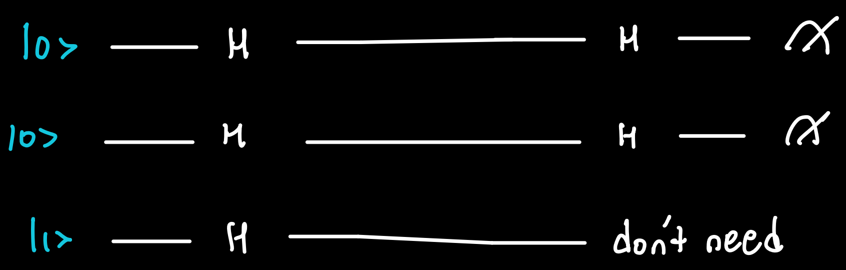
INPUT $x_1 x_2$	INPUT b	$f(x)$	OUTPUT $x_1 x_2$	OUTPUT $b \oplus f(x)$
00	0	1	00	1
00	1	1	00	0
01	0	0	01	0
01	1	0	01	1
10	0	1	10	1
10	1	1	10	0
11	0	0	11	0
11	1	0	11	1

U_f



LET US VERIFY IF THESE U_f WORK :

CONSTANT -0 :



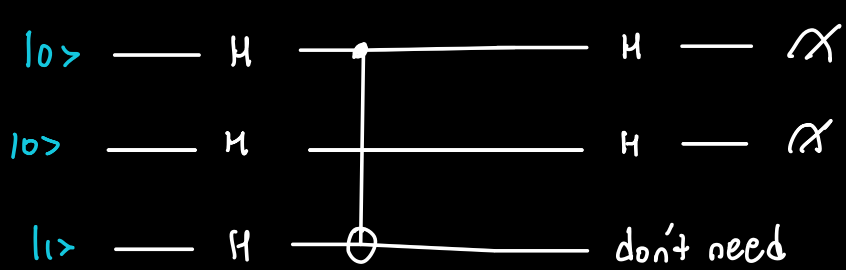
$$(H \otimes H \otimes I)(H \otimes H \otimes H)\,|001\rangle$$

$$= (H \otimes H \otimes I)(|+\rangle \otimes |+\rangle \oplus |-\rangle)$$

$$= \underbrace{|0\rangle \otimes |0\rangle}\otimes |-\rangle$$

MEASURE $|00\rangle$ => CONSTANT

BALANCED 1 :



$$(H \otimes H \otimes I)\,CNOT(1,3)\,(H \otimes H \otimes H)\,|001\rangle$$

$$= (H \otimes H \otimes I)\,CNOT(1,3)\,\underbrace{(|+\rangle \otimes |+\rangle \otimes |-\rangle)}$$

$$\left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)$$

$$= \tfrac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right) \otimes \tfrac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$= \tfrac{1}{2\sqrt{2}}\left(|000\rangle + |010\rangle + |100\rangle + |110\rangle - |001\rangle - |011\rangle - |101\rangle - |111\rangle\right)$$

$$= (H \otimes H \otimes I)\ \tfrac{1}{2\sqrt{2}}\left(|000\rangle + |010\rangle + |101\rangle + |111\rangle - |001\rangle - |011\rangle - |100\rangle - |110\rangle\right)$$

$$= \tfrac{1}{2\sqrt{2}}\Big[\left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right)\left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right)|0\rangle$$

$$+ \left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right)\left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)|0\rangle$$

$$+ \left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)\left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right)|1\rangle$$

$$+ \left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)\left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)|1\rangle$$

$$- \left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right)\left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right)|1\rangle$$

$$- \left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right)\left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)|1\rangle$$

$$- \left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)\left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right)|0\rangle$$

$$- \left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)\left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)|0\rangle$$

$$= \tfrac{1}{4\sqrt{2}}\Big(|000\rangle + |010\rangle + |100\rangle + |110\rangle +$$

$$|000\rangle - |010\rangle + |100\rangle - |110\rangle +$$

$$|001\rangle + |011\rangle - |101\rangle - |111\rangle +$$

$$|001\rangle - |011\rangle - |101\rangle + |111\rangle$$

$$- |001\rangle - |011\rangle - |101\rangle - |111\rangle$$

$$- |001\rangle + |011\rangle - |101\rangle + |111\rangle$$

$$- |000\rangle - |010\rangle + |100\rangle + |110\rangle$$

$$- |000\rangle + |010\rangle + |100\rangle - |110\rangle \Big)$$

$$= \tfrac{1}{\sqrt{2}}\left(|100\rangle - |101\rangle\right)$$

MEASURE QUBITS 1, 2 => $|10\rangle$ WITH 100% PROBABILITY

NOT $|00\rangle$ => SO BALANCED //