DEUTSH - JOZSA :

INPUT : $f : \{0,1\}^n \to \{0,1\}$

Assumption : f is either constant or balanced.

$\forall x : f(x) = 0$

$\forall x : f(x) = 1$

$f(x) = 0$ on half of all x.

OUTPUT : 0 , if f is constant

1 , if f is balanced .

CLASSICAL SOLUTION:

$f(000\ldots0) = 1$

$f(000\ldots1) = 1$

$f(000\ldots10) = 1$

Try more than half of the inputs.

$2^{n-1} + 1$ tries //

QUANTUM:

$n = 1$

INPUT	f_0	f_1	f_2	f_3
0	0	0	1	1
1	0	1	0	1

Try on:

$$2^{n-1} - 1 = 2^0 + 1 = 2 \text{ inputs.}$$

$$f : \{0,1\} \rightarrow \{0,1\}$$

$$U_f : \text{Qubit}^{\otimes 2} \rightarrow \text{Qubit}^{\otimes 2}$$

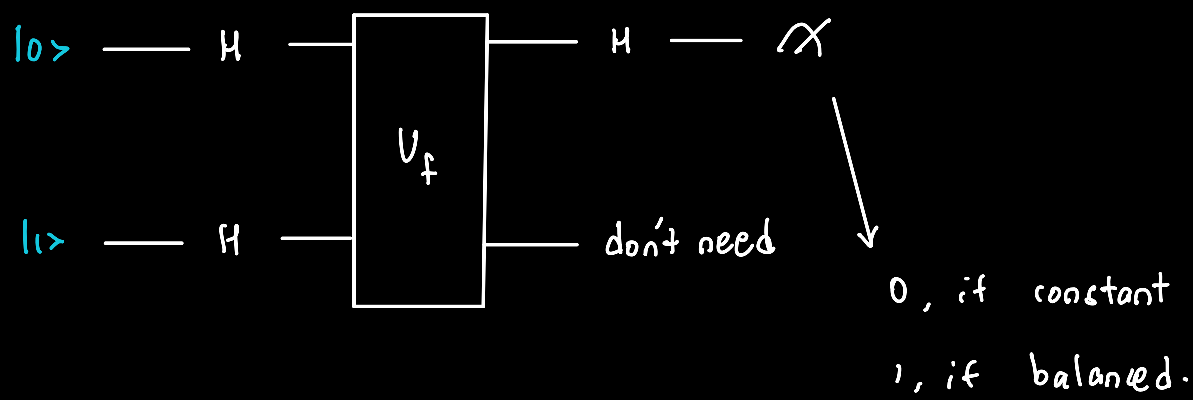
$$U_f |x\rangle |b\rangle = |x\rangle |b \oplus f(x)\rangle$$

$U_{f_0} = I$ $\quad$ $U_{f_1} = \text{CNOT}$ $\rightarrow$ for f_1, when $x = 0$, b becomes $b \oplus 0 = b$

when $x = 1$, b becomes $b \oplus 1 = !b$

Oracles : encoding of f

So, the user has f and gives us the U_f of the f to us and asks us to guess f.



LEMMA 1:

$$\forall a \in \{0,1\} : |0 \oplus a\rangle - |1 \oplus a\rangle = (-1)^a (|0\rangle - |1\rangle)$$

PROOF:

$a = 0$: LHS $|0 \oplus 0\rangle - |1 \oplus 0\rangle = |0\rangle - |1\rangle$

RHS $(-1)^0 (|0\rangle - |1\rangle) = |0\rangle - |1\rangle$

$a = 1$: LHS $|0 \oplus 1\rangle - |1 \oplus 1\rangle = |1\rangle - |0\rangle$

RHS $(-1)^1 (|0\rangle - |1\rangle) = |1\rangle - |0\rangle$

LEMMA 2:

$$\forall x \in \{0,1\}^n : U_f |x\rangle |-\rangle = (-1)^{f(x)} |x\rangle |-\rangle$$

$$|-\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$$

$$U_f|x\rangle|-\rangle = U_f|x\rangle\left(\tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right) = \tfrac{1}{\sqrt{2}}\left(U_f|x\rangle|0\rangle - U_f|x\rangle|1\rangle\right)$$

$$= \tfrac{1}{\sqrt{2}}\left(|x\rangle|0 \oplus f(x)\rangle - |x\rangle|1 \oplus f(x)\rangle\right)$$

$$= \tfrac{1}{\sqrt{2}}|x\rangle\left(|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle\right)$$

$$= \tfrac{1}{\sqrt{2}}|x\rangle(-1)^{f(x)}\left(|0\rangle - |1\rangle\right)$$

$$= (-1)^{f(x)}|x\rangle\tfrac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$= (-1)^{f(x)}|x\rangle|-\rangle \quad //$$

LEMMA 3:

$$\forall a \in \{0,1\} : \quad H\left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}(-1)^a|1\rangle\right) = |a\rangle$$

PROOF:

$a = 0$

$$H\left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right)$$

$$= H|+\rangle$$

$$= |0\rangle = |a\rangle$$

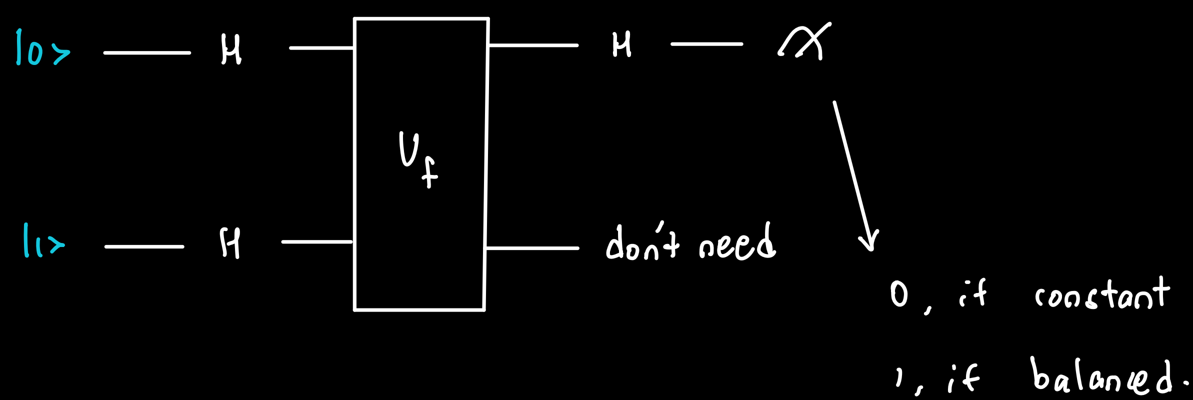
$a = 1$

$$H\left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)$$

$$= H|-\rangle$$

$$= |1\rangle = |a\rangle$$

Proof:



$$(H \otimes I)\, U_f\, (H \otimes H)\, |01\rangle$$

$$= (H \otimes I)\, U_f\, (H|0\rangle \otimes H|1\rangle)$$

$$= (H \otimes I)\, U_f\, (|+\rangle \otimes |-\rangle)$$

$$= (H \otimes I)\, U_f\, \left(\tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |-\rangle\right)$$

$$= (H \otimes I)\, U_f \left(\tfrac{1}{\sqrt{2}} \sum_{x \in \{0,1\}} |x\rangle \otimes |-\rangle \right)$$

$$= (H \otimes I)\, \tfrac{1}{\sqrt{2}} \left(\sum_{x \in \{0,1\}} (-1)^{f(x)} |x\rangle |-\rangle \right) \quad \text{[LEMMA 2]}$$

$$= (H \otimes I)\, \tfrac{1}{\sqrt{2}} \left((-1)^{f(0)} |0\rangle + (-1)^{f(1)} |1\rangle \right) |-\rangle$$

$$= (H \otimes I)\, \tfrac{1}{\sqrt{2}} (-1)^{f(0)} \left(|0\rangle + \frac{(-1)^{f(1)}}{(-1)^{f(0)}} |1\rangle \right) |-\rangle$$

$\downarrow$

$f(0), f(1) \in \{0,1\}$

	$(-1)^{f(1)-f(0)}$	$(-1)^{f(0) \oplus f(1)}$
$f(0) = f(1) = 0$	$(-1)^0 = 1$	$(-1)^0 = 1$
$f(0) = 0, f(1) = 1$	$(-1)^1 = -1$	$(-1)^1 = -1$
$f(0) = 1, f(1) = 0$	$(-1)^{-1} = 1/_{-1} = -1$	$(-1)^1 = -1$
$f(0) = f(1) = 1$	$(-1)^0 = 1$	$(-1)^0 = 1$

$$= (H \otimes I)(-1)^{f(0)} \tfrac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{f(0) \oplus f(1)} |1\rangle\right)|-\rangle$$

[LEMMA 3]

$$= (-1)^{f(0)} \cdot |f(0) \oplus f(1)\rangle |-\rangle$$

b not used

MEASURE

with prob: α^2

$$\alpha^2 = |(-1)^{f(0)}|^2$$

$$= 1^2 = 1$$

prob = 1, we observe

$$|f(0) \oplus f(1)\rangle$$

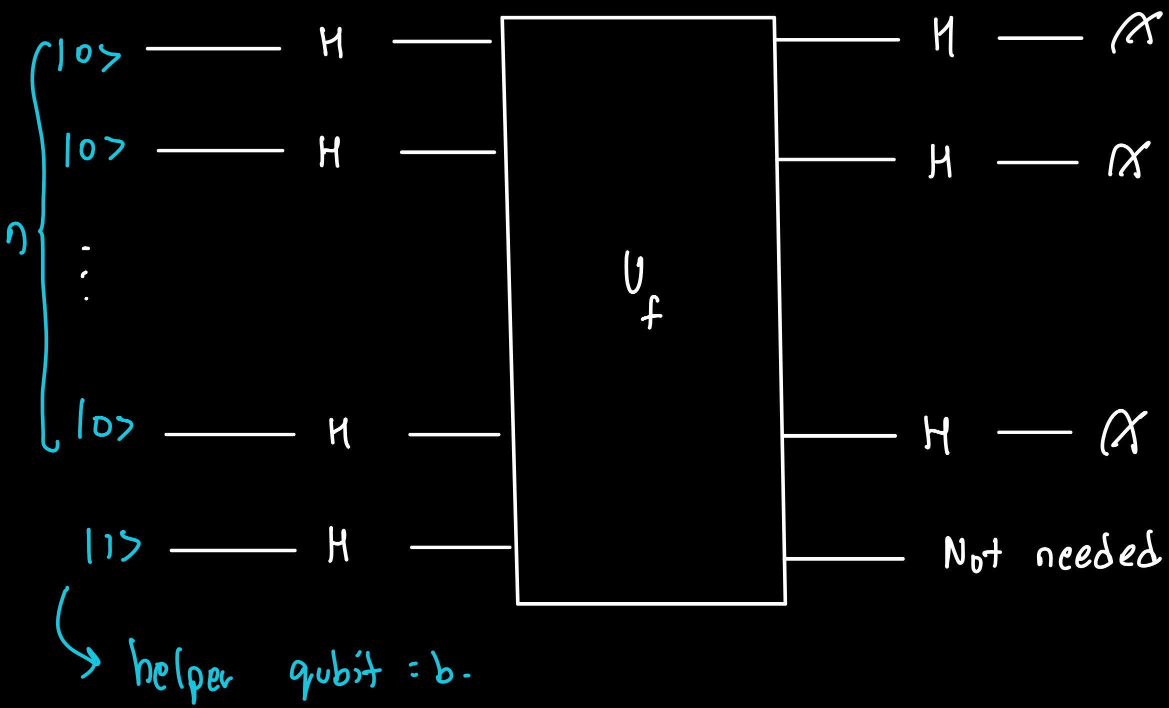
So if CONSTANT

$$|f(0) \oplus f(1)\rangle = 0$$

else

it is 1

DEUTSCH - JOZSA:



LEMMA 5: $\forall x \in \{0,1\}$;

$$H|x\rangle = \frac{1}{\sqrt{2}} \sum_{y \in \{0,1\}} (-1)^{x \cdot y} |y\rangle$$

PROOF:

$$H|x\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}(-1)^{x}|1\rangle \quad // \quad [\text{FROM LEMMA 3}]$$

LEMMA 6 :

$$H^{\otimes n} |x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle$$

dot product

n bits

PROOF :

$x : x_1 \dots x_n$

$$H^{\otimes n} |x\rangle : H|x_1\rangle \otimes \dots \otimes H|x_n\rangle$$

Apply LEMM 5

$$= \left(\frac{1}{\sqrt{2}}\right)^n \left(\sum_{y_1 \in \{0,1\}} (-1)^{x_1 \cdot y_1} |y_1\rangle \right) \otimes \dots \otimes$$

$$\left(\sum_{y_n \in \{0,1\}} (-1)^{x_n \cdot y_n} |y_n\rangle \right)$$

$$= \left(\frac{1}{\sqrt{2^n}}\right) \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle .$$

PROOF :

$$(H^{\otimes n} \otimes I) \cdot U_f \cdot H^{\otimes n+1} \, |0 \dots 0 1\rangle$$

$H|1\rangle$

$$= (H^{\otimes n} \otimes I) \cdot U_f \cdot \frac{1}{\sqrt{2^n}} \left(\sum_{x \in \{0,1\}^n} |x\rangle \right) |-\rangle$$

(LEMMA 6)

$$= (H^{\otimes n} \otimes I) \cdot \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle \, |-\rangle$$

(LEMMA 2)

$$= \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} \left(\frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle \right) |-\rangle$$

[LEMMA 6]

$$= \frac{1}{2^n} \sum_{x} \sum_{y} (-1)^{f(x) \oplus x \cdot y} |y\rangle |-\rangle$$

To SEE $|0 \dots 0 -\rangle$,

f constant $f(x) = 0$ always

$f(x) = 1$ always

if $f(x) = 0$ and $y = |0 \cdots 0\rangle$

then $\frac{1}{2^n} \sum_x (-1)^{0 \oplus 0} = \frac{1}{2^n} \times 2^n = 1$

if $f(x) = 1$ and $y = |0 \cdots 0\rangle$

then $\frac{1}{2^n} \sum_x (-1)^{1 \oplus 0} = \frac{1}{2^n} (-1) \times 2^n = -1$

$f(x) = 0$, always $\rightarrow$ amplitude $= 1$

$f(x) = 1$ always $\rightarrow$ amplitude $= -1$

$|\text{amplitude}|^2 = 1$ //

if f is balanced, amplitude $= 0$ to observe $|0 \cdots 0\rangle$

$|0, \cdots, 0\rangle$

So if we measure all 0's $\rightarrow$ constant

$00 \cdots 0$

else $\rightarrow$ balanced //.