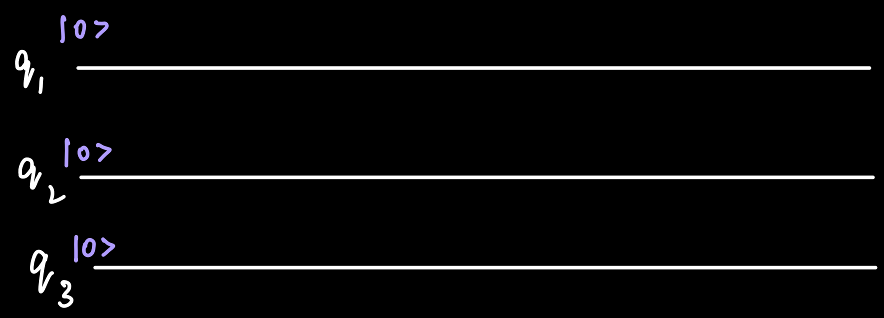
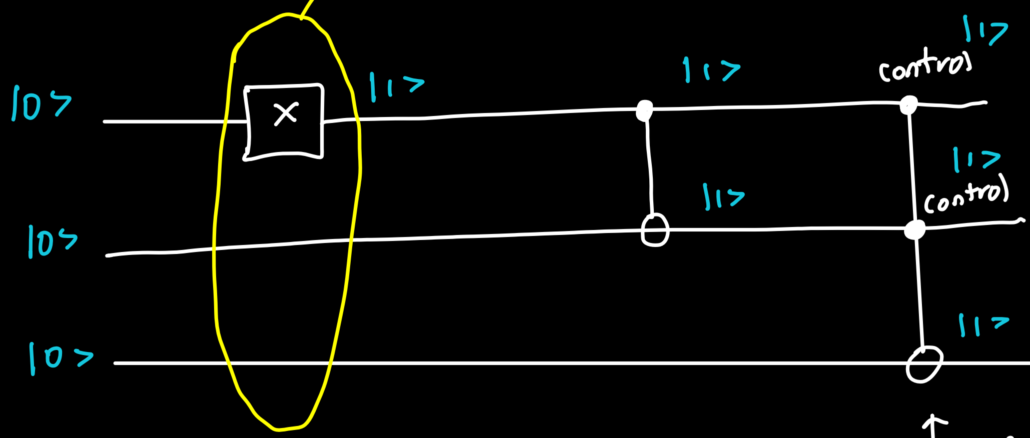
MATRIX PRODUCT - TENSOR PRODUCT



$\longrightarrow$ time

$q_1, q_2, q_3 \rightarrow$ QUBIT REGISTER

MOMENT $(X \otimes I \otimes I)$

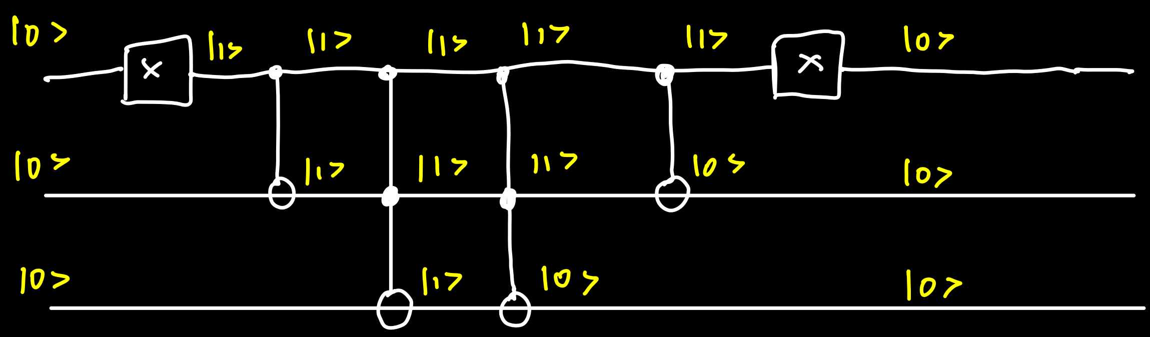


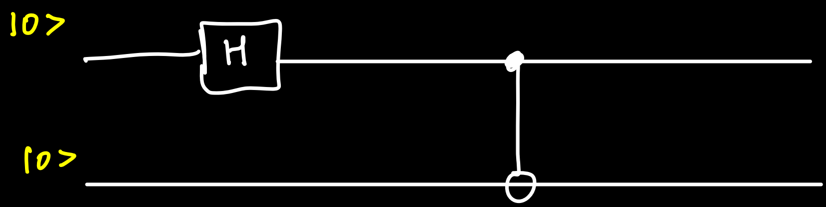
flip if both controls are 1.

CNOT $\rightarrow$ flip if control $= 1$

CNOT :

$$\begin{array}{cc} & \text{INPUT} \\ \text{OUTPUT} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$





$$\text{CNOT} \circ (H \otimes I)\ |00\rangle$$

$$= \text{CNOT} \circ (H \otimes I)\ (|0\rangle \otimes |0\rangle)$$

$$= \text{CNOT} \circ (H|0\rangle \otimes I|0\rangle)$$

$$= \text{CNOT}\ (|+\rangle \otimes |0\rangle)$$

$$= \text{CNOT}\ \left(\tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle\right)$$

$$= \text{CNOT}\ \left(\underbrace{\tfrac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle)}_{\text{no flip}} + \underbrace{\tfrac{1}{\sqrt{2}}(|1\rangle \otimes |0\rangle)}_{\text{flip}}\right)$$

$$= \tfrac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle) + \tfrac{1}{\sqrt{2}}(|1\rangle \otimes |1\rangle)$$

SUPERDENSE CODING:

n qubits → how many bits can we extract from them?

$$n \text{ qubits} \xrightarrow{\text{theorem}} n \text{ bits}$$

Alice, Bob exchange information.

2bits (ab)

in other words, the qubits cannot be written as a separate tensor product.

IDEA: Share a pair of entangled qubits ahead of time

both 0 or 1

$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

To exchange data now, we need only 1 qubit instead of the 2 earlier.

$\frac{1}{\sqrt{2}}(|\overset{A}{0}\overset{B}{0}\rangle + |11\rangle)$

↓

share earlier

qubit A is with Alice and B is with bob

Alice :

* if $a = 1$, then apply Z to A .
* if $b = 1$, then apply X to A .
* sends A to Bob

Bob :

* CNOT (A, B)
* $H(A)$
* Measures both A and B

How does it work :

ab	STEP 1	Alice STEP 2
00	$\frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 11\rangle$	$\frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 11\rangle)$
01	$\frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 11\rangle)$	$\frac{1}{\sqrt{2}}(\lvert 10\rangle + \lvert 01\rangle)$
10	$\frac{1}{\sqrt{2}}(\lvert 00\rangle - \lvert 11\rangle)$	$\frac{1}{\sqrt{2}}(\lvert 00\rangle - \lvert 11\rangle)$
11	$\frac{1}{\sqrt{2}}(\lvert 00\rangle - \lvert 11\rangle)$	$\frac{1}{\sqrt{2}}(\lvert 10\rangle - \lvert 01\rangle)$

Initially : $\frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 11\rangle)$

$Z|0\rangle = |0\rangle$

$Z|1\rangle = -|1\rangle$

SEND A TO BOB

BOB

ab	STEP 1	STEP 2
00	$\frac{1}{\sqrt{2}}(\|00\rangle + \|10\rangle) = \frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)\|0\rangle$	$\|00\rangle$
01	$\frac{1}{\sqrt{2}}(\|11\rangle + \|01\rangle) = \frac{1}{\sqrt{2}}(\|1\rangle + \|0\rangle)\|1\rangle$	$\|01\rangle$
10	$\frac{1}{\sqrt{2}}(\|00\rangle - \|10\rangle) = \frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)\|0\rangle$	$\|10\rangle$
10	$\frac{1}{\sqrt{2}}(\|11\rangle - \|01\rangle) = \frac{1}{\sqrt{2}}(\|1\rangle - \|0\rangle)\|1\rangle$	$-\|11\rangle$

When we measure, we take the norm to get 11.

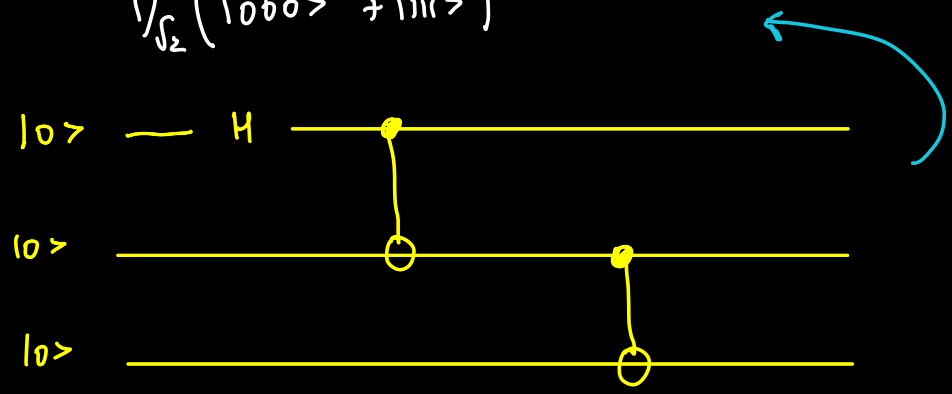
So alice and bob have one of the entangled qubits A, B each (A with Alice and B with Bob). Then to send 2 bits from Alice to Bob, we don't need 2 qubits to be sent across the channel. We need only to send 1, A from

Alice to Bob. Total is still 2 qubits as per the theorem.

ENTANGLED QUBITS:

3 qubits

$\frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$



QUANTUM TELEPORTATION :

Alice , Bob : Ahead of time , they share

$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

1 qubit

$\alpha|0\rangle + \beta|1\rangle$

So Alice wants to share 1 qubit to Bob. Alice sends 2 bits to share the qubit. (Opposite to the former).

QUBITS :

TO BE SENT, C $(\alpha|0\rangle + \beta|1\rangle)$

ENTANGLED : $\frac{1}{\sqrt{2}}(\overset{AB}{|00\rangle} + |11\rangle)$

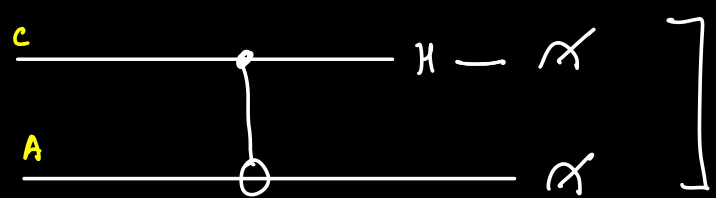
$$(\alpha|0\rangle + \beta|1\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$\frac{1}{\sqrt{2}}(\alpha\overset{CAB}{|000\rangle} + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$$

Alice has A, C

Bob has B

Alice :



→ send the 2bit ab to BOB

ORDER OF QUBIT: CAB

Bob:

* if $b=1$, then apply x to his qubit.
* if $a=1$, then apply z to his qubit.

Alice:

Step 1:

$$\frac{1}{\sqrt{2}}\left(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle\right)$$

Step 2:

$$\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\left(\overbrace{\alpha|000\rangle + \alpha|100\rangle} + \alpha|011\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle\right)$$

$$\frac{1}{2}\left(|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle)\right)$$

↳ on measure
25% chance of getting 00, 01, 10, 11 for CA (ones Alice have)

Bob:

Step 1: if $b=1$ apply X

$$\alpha|0\rangle + \beta|1\rangle$$

$$\frac{1}{2}\Big(|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle)$$

$$+ |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle)\Big)$$

$$\alpha|0\rangle - \beta|1\rangle$$

Step 2: if $a=1$ change sign of $|1\rangle$

$$\frac{1}{2}\Big(|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|0\rangle + \beta|1\rangle)$$

$$+ |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|0\rangle - \beta|1\rangle)\Big)$$

$$\alpha|0\rangle + \beta|1\rangle \qquad \alpha|0\rangle + \beta|1\rangle$$

ALL 4 COMBINATIONS RESULT IN $\alpha|0\rangle + \beta|1\rangle$

$ab = 00 \longrightarrow \alpha|0\rangle + \beta|1\rangle$

$ab = 01 \longrightarrow X(\alpha|1\rangle + \beta|0\rangle)$

$ab = 10 \longrightarrow Z(\alpha|0\rangle - \beta|1\rangle)$

$ab = 11 \longrightarrow ZX(\alpha|1\rangle - \beta|0\rangle)$

NO - CLONING THEOREM:

No quantum operation maps $|\psi\rangle|0\rangle$ to $|\psi\rangle|\psi\rangle$.

Quantum teleportation allows the qubit to reach Bob, but alice has to destroy hers.

PROOF:

Suppose $U|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle$ — Assumption

Assume $|\psi_1\rangle, |\psi_2\rangle$ not orthogonal, not proportional

$$\langle \psi_1|\psi_2\rangle \neq 0 \ , \ \langle \psi_1|\psi_2\rangle \neq 1$$

note:

$$\langle v_1 \otimes v_2 | w_1 \otimes w_2\rangle = \langle v_1|w_1\rangle \cdot \langle v_2|w_2\rangle$$ Theorem 1

$$\langle \psi_1|\psi_2\rangle = \langle \psi_1|\psi_2\rangle\langle 0|0\rangle$$ inner product of 0, 0 = 1

$$= \langle \psi_1 \otimes |0\rangle \,|\, \psi_2 \otimes |0\rangle\rangle$$ Reverse Theorem 1

$$= \langle U(\psi_1 \otimes |0\rangle) \,|\, U(\psi_2 \otimes |0\rangle)\rangle$$ Unitary matrices preserve inner product

$$= \langle \psi_1 \otimes |\psi_1\rangle \,|\, \psi_2 \otimes |\psi_2\rangle\rangle$$ Assumption

$$= \langle \psi_1|\psi_2\rangle \cdot \langle \psi_1|\psi_2\rangle$$ Theorem 1

CONTRADICTION

UNIVERSAL GATE SETS:

Classical Computers $\rightarrow$ {NAND}

{AND, OR, NOT}

$$OR(x_1, x_2) = NOT(AND(NOT(x_1), NOT(x_2)))$$

$$x_1 \cup x_2 = (\overline{\bar{x}_1 \cap \bar{x}_2})$$

$$AND(x_1, x_2) = NOT(NAND(x_1, x_2))$$

$$NOT(x) = NAND(x, \overset{\text{helper bit}}{1})$$

So ONLY NAND IS ENOUGH.

Universal $\rightarrow$ NAND GATE + helper bits

Quantum :

NAND is not reversible. So we need another universal gate.

TAKE CCNOT

$$CCNOT(000) = 000$$

$$CCNOT(110) = 111$$

$$\text{CCNOT} = \begin{array}{c} \\ 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{array} \begin{array}{c} \begin{array}{cccccccc} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \end{array} \\ \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & 0 & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & 0 & & & 1 & & \\ & & & & & & 0 & 1 \\ & & & & & & 1 & 0 \end{bmatrix} \end{array}$$

$$\text{NAND}(x_1, x_2) = \text{CCNOT}(x_1, x_2, 1)$$

↳ 0 only when

$x_1 = 1$

$x_2 = 1$ like

NAND

output.

Quantum Universal Gate Set:

$$\{ \underbrace{\text{CCNOT}}_{\text{For all real matrices}}, H, S \}$$

$S \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ covers complex.

This is not great for programming.

PROGRAMMERS:

$\{ CNOT, H, T \}$

$S = TT$.

ACTUAL QUANTUM COMPUTERS:

IBM: $\{ CNOT, \underbrace{U_1, U_2, U_3}_{\text{single qubit gates}} \}$

ION TRAP: $\{ XX, \underbrace{R_x, R_y, R_z}_{\text{1-QUBIT GATE.}} \}$