1. $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

2. $\langle \psi | = (|\psi\rangle)^\dagger$

$$\langle \psi | \phi \rangle = \underbrace{|\psi\rangle . |\phi\rangle}_{\text{inner-product.}}$$

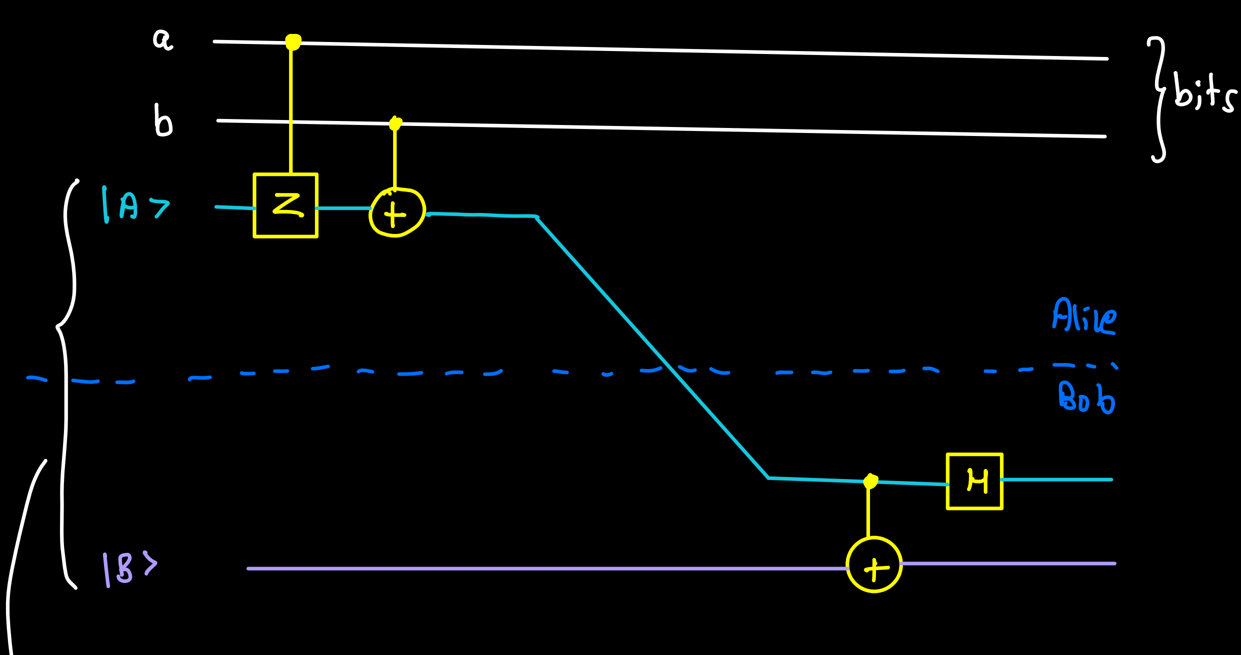
3. SUPER DENSE ENCODING:

Alice $\longrightarrow$ Bob

(a,b)

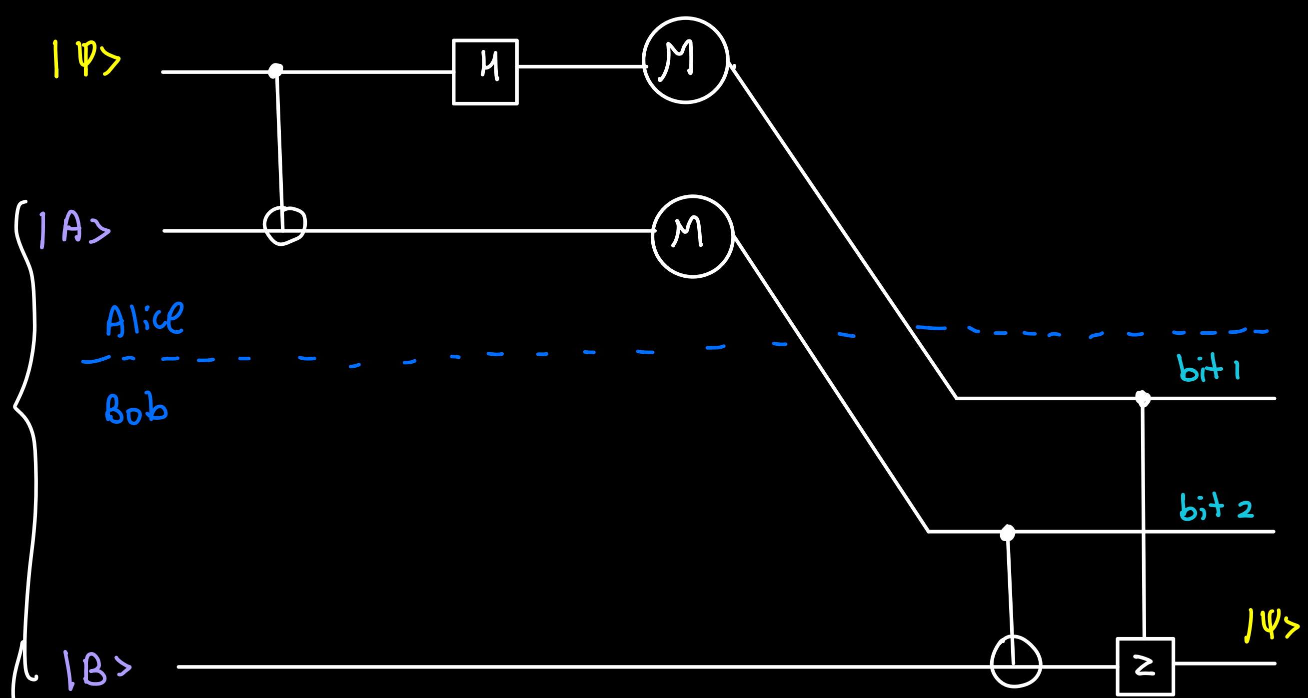
A <- - - - - - -> B

Send A



$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

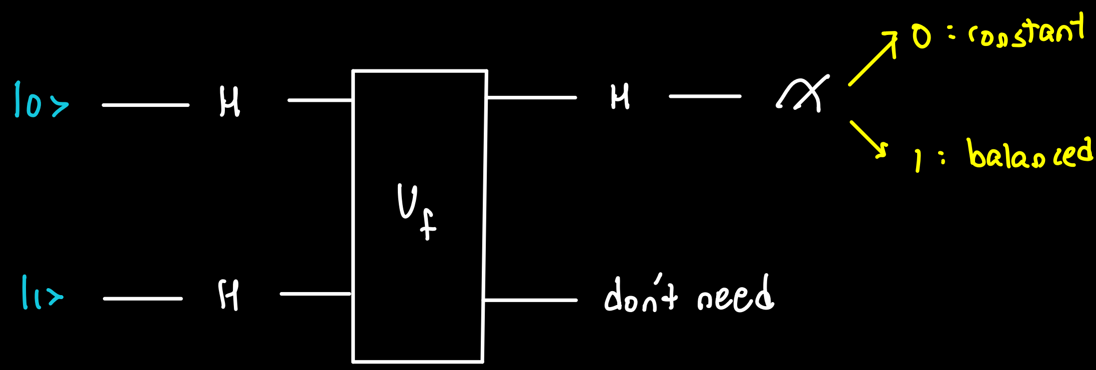
4. Quantum Teleportation :



$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

5. Deutsch's Algorithm:

$$f : \{0,1\}^n \rightarrow \{0,1\}$$



$$U_f |x\rangle |b\rangle = |x\rangle |b \oplus f(x)\rangle$$

Lemma 1:

$$\forall a \in \{0,1\} : |0 \oplus a\rangle - |1 \oplus a\rangle = (-1)^a (|0\rangle - |1\rangle)$$

Lemma 2:

$$\forall x \in \{0,1\}^n :$$

$$U_f |x\rangle |-\rangle = (-1)^{f(x)} |x\rangle |-\rangle$$

Lemma 3:

$$\forall a \in \{0,1\} : H\left(\tfrac{1}{\sqrt{2}} |0\rangle + \tfrac{1}{\sqrt{2}} (-1)^a |1\rangle \right) = |a\rangle$$

$$(H \otimes I)\, U_f\, (H^{\otimes 2})\, |01\rangle$$

$$= (H \otimes I)\, U_f \left[\tfrac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \otimes \tfrac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \right]$$

$$= (H \otimes I)\, U_f\, \tfrac{1}{\sqrt{2}} \left[(\underbrace{|0\rangle}_{x} + \underbrace{|1\rangle}_{x}) \otimes |-\rangle \right]$$

$$= (H \otimes I) \left(\tfrac{1}{\sqrt{2}} (-1)^{f(|0\rangle)} |0\rangle - \tfrac{1}{\sqrt{2}} (-1)^{f(|1\rangle)} |1\rangle \right) |-\rangle$$

$$= (H \otimes I)\, (-1)^{f(|0\rangle)} \left(\tfrac{1}{\sqrt{2}} |0\rangle - \tfrac{1}{\sqrt{2}} (-1)^{f(|1\rangle) \oplus f(|0\rangle)} |1\rangle \right) |-\rangle$$

$$= \underbrace{(-1)^{f|0\rangle} \left(f(|1\rangle) \oplus f(|0\rangle) \right)}_{\text{Measure}} |-\rangle$$

With $P = \left| (-1)^{f|0\rangle} \right|^2 = 1$,

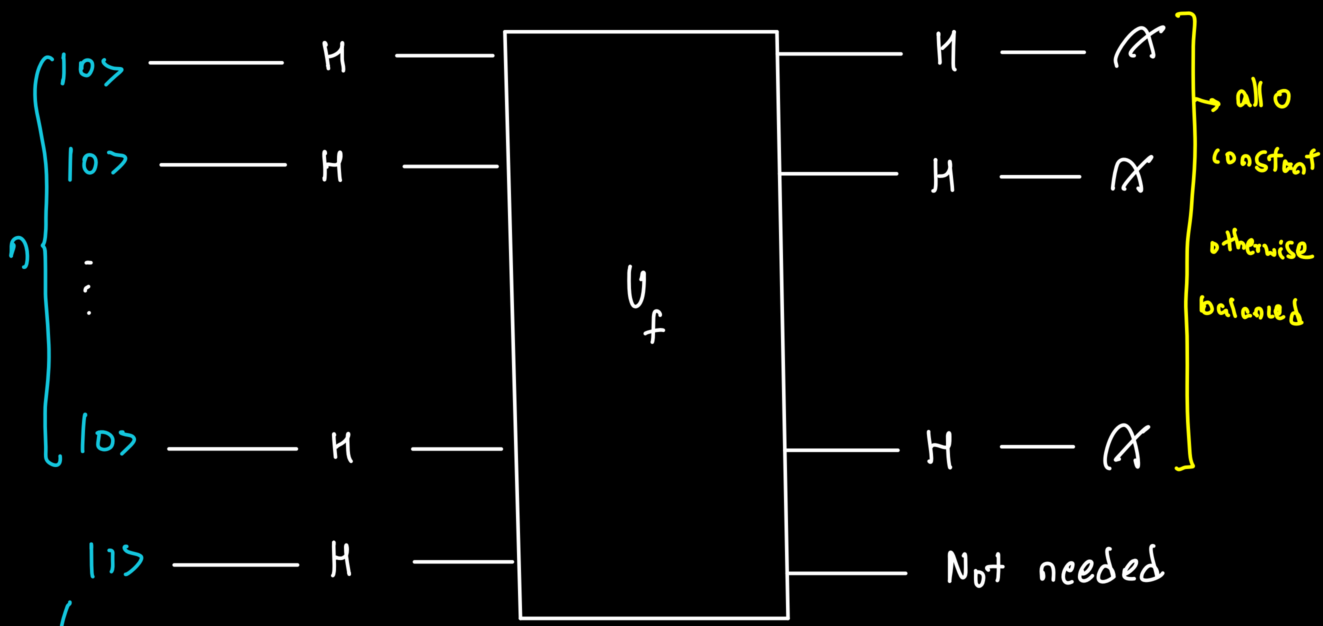
we observe $f(|0\rangle) \oplus f(|1\rangle) \rightarrow 0$ CONSTANT

$\hookrightarrow 1$ BALANCED //

6. DEUTSCH - JOZSA ALGORITHM:

$$f : \{0,1\}^n \to \{0,1\}$$

$$U_f |x\rangle |b\rangle = |x\rangle |b \oplus f(x)\rangle$$



helper qubit : b.

LEMMA 5 : $\forall x \in \{0,1\}$:

$$H|x\rangle = \frac{1}{\sqrt{2}} \sum_{y \in \{0,1\}} (-1)^{x \cdot y} |y\rangle$$

LEMMA 6 :

$$H^{\otimes n} |x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle$$

$x \cdot y$ → dot product

$|x\rangle$ → n bits

$$x \cdot y = \sum_{i=1}^{n} x_i y_i \pmod 2$$

$$\left(H^{\otimes n} \otimes I\right) U_f \left(H^{\otimes n+1}\right) |00\ldots 01\rangle$$

$$= \left(H^{\otimes n} \otimes I\right) \underline{U_f} \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} \underline{|x\rangle} \, \underline{|-\rangle}$$

$$= \left(H^{\otimes n} \otimes I\right) \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle |-\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle |-\rangle$$

$$= \sum_{y \in \{0,1\}^n} \left(\frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x) + x \cdot y} \right) |y\rangle \, |-\rangle$$

$|-\rangle$ → discarded

if $y = |0^n\rangle$

$x \cdot y = 0$

$$\alpha = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x)}$$

CONSTANT:

All $f(x) = 0$

$$\alpha^2 = \left(\frac{1}{2^n} \sum_{x \in \{0,1\}^n} 1 \right)^2$$

$$= \left(\frac{1}{2^n} \times 2^n \right)^2 = 1$$

All $f(x) = 1$

$$\alpha^2 = \left(\frac{1}{2^n} \times -2^n \right)^2 = 1$$

So ALWAYS 0^n FOR CONSTANT

BALANCED:

$f(x) = 0$ and $f(x) = 1$

for 2^{n-1} for 2^{n-1}

$$\sum_{x \in \{0,1\}^n} (-1)^{f(x)} = 0$$

$$\alpha^2 = 0$$

NEVER //

7. BERNSTEIN VAZIRANI ALGORITHM:

$$f : \{0,1\}^n \rightarrow \{0,1\}$$

$$f(x) = (a \cdot x) \oplus b$$

find a, b

note:

if $a = 0^n$

$f(x) = b$ CONSTANT

if $a \neq 0^n$

$f(x) = (a \cdot x) \oplus b$

say $a = 0100$ $b = 0$

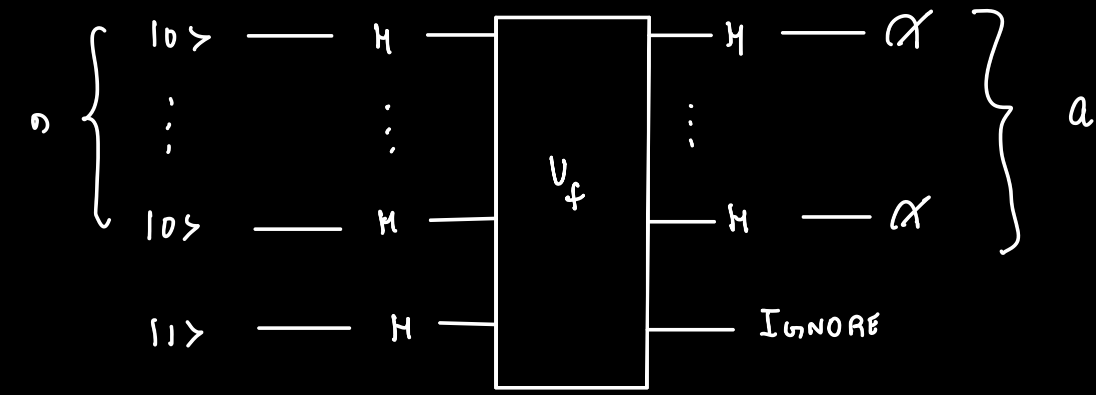
$a \cdot x = 0$ for half the inputs

BALANCED //

USE CLASSICAL COMPUTER TO FIND b:

$f(0^n) = b$

SAME CIRCUIT AS DJ:



$$\left(H^{\otimes n} \otimes I\right) U_f \left(H^{\otimes n+1}\right) |0^n\rangle |1\rangle$$

$$= \left(H^{\otimes n} \otimes I\right) U_f \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle \, |-\rangle$$

$$= \left(H^{\otimes n} \otimes I\right) \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle \, |-\rangle$$

$$= \underbrace{\frac{1}{2^n} \sum_{x \in \{0,1\}^n} \sum_{y \in \{0,1\}^n} (-1)^{f(x)} (-1)^{x \cdot y} |y\rangle}_{\frac{1}{2^n} \sum_{x \in \{0,1\}^n} \sum_{y \in \{0,1\}^n} (-1)^{(x \cdot y) \oplus ((a \cdot x) \oplus b)} |y\rangle} \, |-\rangle$$

dropped

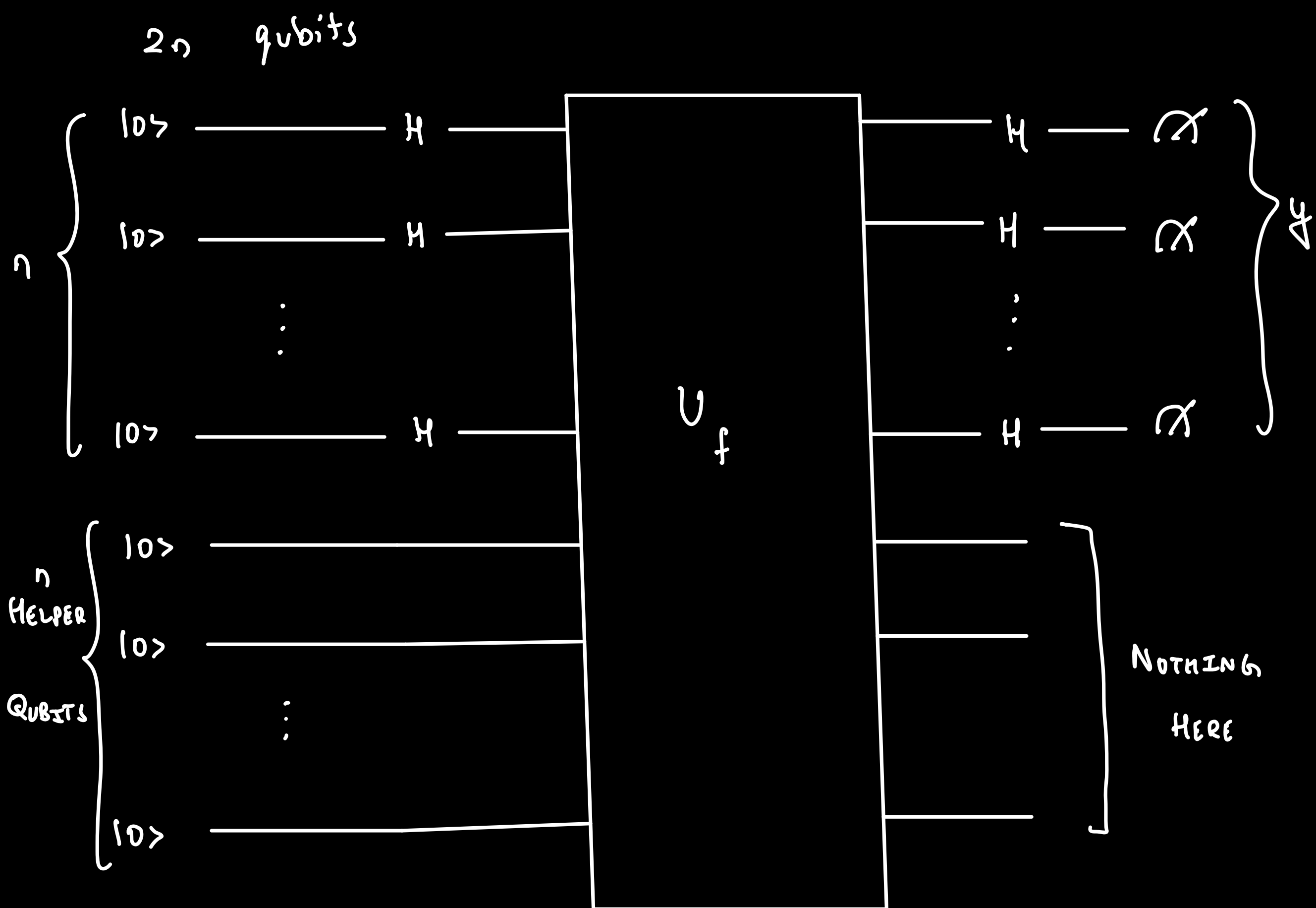
PROBABILITY TO OBSERVE a $= \left\| \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{(x \cdot a) \oplus ((a \cdot x) \oplus b)} \right\|^2$

$= \left\| \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^b \right\|^2$

$= \left\| \frac{1}{2^n} (-1)^b \times 2^n \right\|^2$

$= 1$ //

8. SIMON'S ALGORITHM:



$$\left(H^{\otimes n} \otimes I^{\otimes n}\right) U_f \left(H^{\otimes n} \otimes I^{\otimes n}\right) |0^n\rangle |0^n\rangle$$

$$= \left(H^{\otimes n} \otimes I^{\otimes n}\right) U_f \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle |0^n\rangle$$

$$= \left(H^{\otimes n} \otimes I^{\otimes n}\right) \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle |f(x)\rangle$$

$$= \frac{1}{2^n} \sum_{x \in \{0,1\}^n} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle |f(x)\rangle$$

PROBABILITY OF EACH $y = \left\| \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot y} |f(x)\rangle \right\|^2$

CASE 1:

FOR $s = 0^n$

$$\sum_{x \in \{0,1\}^n} |f(x)\rangle = \sum_{x \in \{0,1\}^n} |x\rangle$$

$$P = \left\| \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot y} |x\rangle \right\|^2$$

$x = 100\ldots0\rangle$

$$= \left\| \frac{1}{2^n} \left((-1)^{x \cdot y} |100\ldots0\rangle + (-1)^{x \cdot y} |0\ldots1\rangle \ldots \right.\right.$$

$$\left.\left. (-1)^{x \cdot y} |11\ldots1\rangle \right) \right\|^2$$

$$= \left\| \frac{(-1)^{x \cdot y}}{2^n} |100\ldots0\rangle + \frac{(-1)^{x \cdot y}}{2^n} |0\ldots1\rangle \ldots \right.$$

$$\left. + \frac{(-1)^{x \cdot y}}{2^n} |11\ldots1\rangle \right\|^2$$

$$= \left[\frac{(-1)^{x\cdot y}}{2^n}\right]^2 + \left[\frac{(-1)^{x\cdot y}}{2^n}\right]^2 + \cdots$$

$$+ \left[\frac{(-1)^{x\cdot y}}{2^n}\right]^2$$

$$= \frac{1}{2^{2n}} \times 2^n = 1/2^n$$

UNIFORMLY DISTRIBUTED

n- BIT STRING

CASE 2:

FOR $s \neq 0^n$

PROBABILITY OF EACH $y = \left\| \frac{1}{2^n} \sum_{x\in\{0,1\}^n} (-1)^{x\cdot y} |f(x)\rangle \right\|^2$

$$\sum_{x\in\{0,1\}^n} |f(x)\rangle = \sum_{z\in A} 2|z\rangle$$

$$f(x) = f(x') = z$$

$$x \oplus x' = s \Rightarrow x' = x \oplus s$$

$$\sum_{x \in \{0,1\}^n} (-1)^{x \cdot y} |f(x)\rangle = \sum_{z \in A} \left[(-1)^{x \cdot y} + (-1)^{x' \cdot y} \right] |z\rangle$$

$$= \sum_{z \in A} \left[(-1)^{xy} + (-1)^{(x \oplus s) \cdot y} \right] |z\rangle$$

$$= \sum_{z \in A} (-1)^{xy} \left[1 + (-1)^{sy} \right] |z\rangle$$

$$P = \left\| \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot y} |f(x)\rangle \right\|^2$$

$$= \left\| \frac{1}{2^n} \sum_{z \in A} (-1)^{xy} \left[1 + (-1)^{sy} \right] |z\rangle \right\|^2$$

if $s \cdot y = 1$

$$P = 0$$

if $s \cdot y = 0$

$$P = \left\| \frac{1}{2^n} \sum_{z \in A} (-1)^{sy} \cdot 2 \, |z\rangle \right\|^2$$

$$= \left\| \frac{(-1)^{sy}}{2^{n-1}} |z_1\rangle + \frac{(-1)^{sy}}{2^{n-1}} |z_2\rangle \ldots \right.$$

$$+ \frac{(-1)^{s y}}{2^{n-1}} |2_{2^{n-1}}\rangle \|^2$$

$$= \left(\frac{1}{2^{n-1}}\right)^2 + \left(\frac{1}{2^{n-1}}\right)^2 + \cdots \left(\frac{1}{2^{n-1}}\right)^2$$

$$= \left(\frac{1}{2^{n-1}}\right)^2 \times 2^{n-1}$$

$$= \frac{1}{2^{n-1}}$$

UNIFORMLY DISTRIBUTED OVER ALL

$y \ \forall \ y \cdot s = 0$ //

RUN $(n-1) \times 4m$ times. 　　　CLASSICAL $\sqrt{2^n}$

If we need 99% probability

$$e^{-m} < 1/100$$

Each iteration guaranteed produce $y \ \forall \ y \cdot s = 0$, but may not be independent.

If $s = 101 \rightarrow y = \underbrace{000/101/010/111}_{2^{3-1} \text{ possibilities}}$

9. GROVER'S ALGORITHM:

$\Rightarrow$ Find x $\ni$ $f(x) = 1$

$$Z_f|x\rangle = (-1)^{f(x)}|x\rangle$$

$$Z_0|x\rangle = \begin{cases} -|x\rangle & \text{if } x = 0^n \\ |x\rangle & \text{if } x \neq 0^n \end{cases}$$

$$G = -H^{\otimes n} Z_0 H^{\otimes n} Z_f$$

$X = |0^n\rangle$

$\rightarrow$ $H^{\otimes n} X$

$\rightarrow$ Apply G to X $O(\sqrt{2^n})$ times

$\rightarrow$ Measure X

$$A = \{x \in \{0,1\}^n \mid f(x) = 1\}$$

$$B = \{x \in \{0,1\}^n \mid f(x) = 0\}$$

$$a = |A|$$

$$b = |B|$$

$$N = a + b = 2^n$$

$$|A\rangle = \frac{1}{\sqrt{a}} \sum_{x \in A} |x\rangle$$

$$|B\rangle = \frac{1}{\sqrt{b}} \sum_{x \in B} |x\rangle$$

STEP 1 : $H^{\otimes n} |0^n\rangle$

$$|h\rangle = H^{\otimes n} |0^n\rangle = \frac{1}{\sqrt{N}} \sum_{x \in \{0,1\}^n} |x\rangle$$

$$= \sqrt{\frac{a}{N}} |A\rangle + \sqrt{\frac{b}{N}} |B\rangle$$

LEMMA 7 :

$$Z_0 = I - 2|0^n\rangle\langle 0^n|$$

$$|0^n\rangle = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \langle 0^n| = \left(|0^n\rangle\right)^{+} = [1 \; 0 \; \dots \; 0]$$

$$|0^n\rangle\langle 0^n| = \begin{bmatrix} 1 & & \\ & 0 & & 0 \\ & & \ddots & \\ & 0 & & \\ & & & 0 \end{bmatrix}$$

$$Z_0 = \begin{bmatrix} -1 & & \\ & 0 & & 0 \\ & & \ddots & \\ & 0 & & \\ & & & 0 \end{bmatrix}$$

LEMMA 8:

$$H^{\otimes n} Z_0 H^{\otimes n} = I - 2|h\rangle\langle h|$$

$$H^{\otimes n} (I - 2|0\rangle\langle 0|) H^{\otimes n} = I - 2|h\rangle\langle h|$$

LEMMA 9:

$$G|A\rangle = \left(1 - \frac{2a}{N}\right)|A\rangle - \frac{2\sqrt{ab}}{N}|B\rangle$$

$$G|B\rangle = \frac{2\sqrt{ab}}{N}|A\rangle - \left(1 - \frac{2b}{N}\right)|B\rangle$$

$$G|A\rangle = -H^{\otimes n} Z_0 H^{\otimes n} Z_f |A\rangle$$

$$= (I - 2|h\rangle\langle h|)(\underbrace{-Z_f |A\rangle}) \rightarrow$$

all have -1 as all are $f(x)=1$ cases

$$= (I - 2|h\rangle\langle h|)|A\rangle$$

$$= |A\rangle - 2\underbrace{\langle h|A\rangle}_{\text{Dot product of } \left(\sqrt{\frac{a}{N}}|A\rangle + \sqrt{\frac{b}{N}}|B\rangle\right) \text{ and } |A\rangle \,=\, \sqrt{\frac{a}{N}}} |h\rangle$$

$$= |A\rangle - 2\sqrt{\frac{a}{N}}\left(\sqrt{\frac{a}{N}}|A\rangle + \sqrt{\frac{b}{N}}|B\rangle\right)$$

$$= \left(1 - \frac{2a}{N}\right)|A\rangle - \frac{2\sqrt{ab}}{N}|B\rangle$$

$$G|B\rangle = -H^{\otimes n} Z_0 H^{\otimes n} Z_f |B\rangle$$

$$= (I - 2|h\rangle\langle h|)(\underbrace{-Z_f|B\rangle}) \rightarrow \text{all have } +1 \text{ as all are } f(x) \text{ cases}$$

$$= (I - 2|h\rangle\langle h|) - |B\rangle$$

$$= -|B\rangle + 2\underbrace{\langle h|B\rangle}_{\text{Dot product of } \left(\sqrt{\frac{a}{N}}|A\rangle + \sqrt{\frac{b}{N}}|B\rangle\right) \text{ and } |B\rangle \,=\, \sqrt{\frac{b}{N}}} |h\rangle$$

$$= -|B\rangle + 2\sqrt{\frac{b}{N}}\left(\sqrt{\frac{a}{N}}\,|A\rangle + \sqrt{\frac{b}{N}}\,|B\rangle\right)$$

$$= \frac{2\sqrt{ab}}{N}\,|A\rangle - \left(1 - \frac{2b}{N}\right)|B\rangle$$

FROM

TO

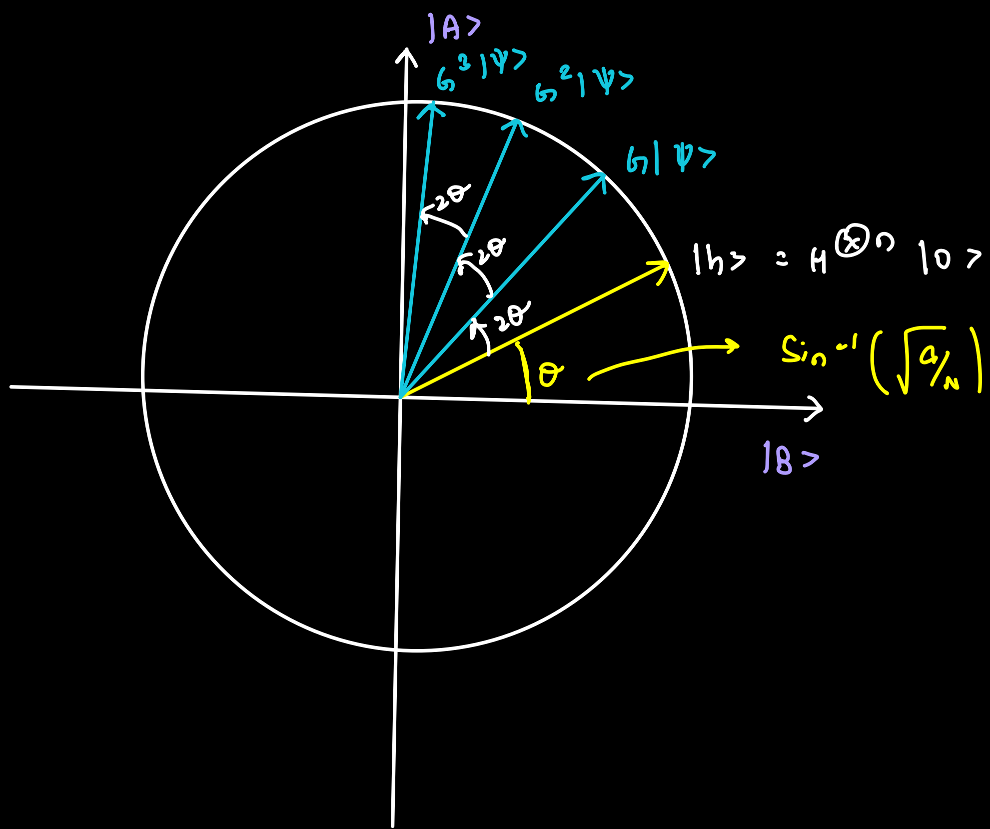
$$M = \begin{array}{c c} & \begin{array}{cc} |B\rangle & |A\rangle \end{array} \\ \begin{array}{c} |B\rangle \\ |A\rangle \end{array} & \begin{bmatrix} -(1 - 2b/N) & -2\sqrt{ab}/N \\ 2\sqrt{ab}/N & 1 - 2a/N \end{bmatrix} \end{array}$$

$$\sin\theta = \sqrt{\frac{a}{N}} \qquad \cos\theta = \sqrt{\frac{b}{N}}$$

$$M = \begin{bmatrix} -(1 - 2\cos^2\theta) & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & 1 - 2\sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$$

Rotates by 2θ



We stop when

$$\sin\left((2k+1)\theta\right) \approx 1$$

$$(2k+1)\theta \approx \pi/2$$

$$k = \pi/4\theta - 1/2$$

$a = 1$

For small θ

$$\sin\theta \approx \theta$$

$$\theta \approx \sqrt{\frac{a}{N}} = \sqrt{\frac{1}{N}}$$

$$k = \frac{\pi}{4}\sqrt{N} - \frac{1}{2}$$

Probability of find $x \ \forall \ f(x) = 1$ for this k

$$|\sin(2k+1)\theta|^2 \simeq \left|\sin\left(2\left(\frac{\pi\sqrt{N}}{4} - \frac{1}{2}\right)+1\right)\left(\frac{1}{\sqrt{N}}\right)\right|^2$$

$$\simeq \left|\sin\left(\frac{\pi}{2}\sqrt{N} - 1 + 1\right)\left(\frac{1}{\sqrt{N}}\right)\right|^2$$

$$= 1 //$$

$$\theta = \sin^{-1}\left(\sqrt{\frac{a}{N}}\right) \qquad N = n^2$$

Each Iteration G does $R_\theta^2 \rightarrow$ Shift by 2θ.

After k iterations

$$\cos((2k+1)\theta)\,|B\rangle + \sin((2k+1)\theta)\,|A\rangle$$

$$P = \sin^2((2k+1)\theta)$$

1. TENSOR PRODUCT:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & 0\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

2. $\left(\frac{3}{5}|0\rangle - \frac{4}{5}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$

$$= \frac{3}{5\sqrt{2}}|00\rangle + \frac{3}{5\sqrt{2}}|01\rangle - \frac{4}{5\sqrt{2}}|10\rangle - \frac{4}{5\sqrt{2}}|11\rangle$$

3. $(H \otimes I)$ CNOT

$$\sqrt{\frac{1}{3}}|00\rangle + \sqrt{\frac{2}{3}}|11\rangle$$

CNOT

$$\sqrt{\frac{1}{3}}|00\rangle + \sqrt{\frac{2}{3}}|10\rangle$$

$$\sqrt{\frac{1}{3}}\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)|0\rangle$$

$$+\sqrt{\frac{2}{3}}\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)|0\rangle$$

$$\left(\frac{1}{\sqrt{6}}|0\rangle + \frac{1}{\sqrt{6}}|1\rangle + \frac{\sqrt{2}}{\sqrt{6}}|0\rangle - \frac{\sqrt{2}}{\sqrt{6}}|1\rangle\right)|0\rangle$$

$$\frac{(\sqrt{2}+1)}{\sqrt{6}}|00\rangle + \frac{(1-\sqrt{2})}{\sqrt{6}}|10\rangle$$

4. $|10\rangle$

5. $H^{\otimes 3}\ |111\rangle$

$$(0-1)(0-1)(0-1)$$

$$(00-01-10+11)(0-1)$$

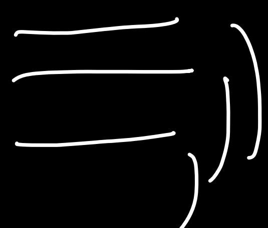
$$\frac{1}{2\sqrt{2}}(000 - 001 - 010 + 011 - 100 + 101$$
$$+ \underline{110} - \underline{111})$$

$1/4$ //

$_8C_4$

$$\frac{\cancel{8}\times 7\times \overset{2}{\cancel{6}}\times 5}{1\times \cancel{2}\times \cancel{3}\times \cancel{4}} \quad 70$$

1 2 3 4 5 6 7 8



$3 + {}^3C_2$

$6 + 6 = 12$

CNOT (1, 4)

CNOT (2, 4)

CNOT (3, 4)