ALGORITHM : PHASE ESTIMATION

INPUT: Unitary U, eigenvector $|\psi\rangle$ of U :

$$U|\psi\rangle = e^{2\pi i\theta}|\psi\rangle$$

OUTPUT : θ

METHOD :

1. Iteration : $\underbrace{U \dots U}_{k\in\{0,1,\dots 2^m-1\}} (\dots)$: $\Lambda_m(U)|k\rangle|\phi\rangle = |k\rangle U^k|\phi\rangle$

2. QFT^+ (inverse QFT)

$$\Lambda_m(U)\underbrace{\left(H^{\otimes m}\otimes I^{\otimes n}\right)}_{\text{working on } (m+n) \text{ qubits}}|0^m\rangle|\psi\rangle = \Lambda_m(U)\left(\frac{1}{2^{m/2}}\sum_{k=0}^{2^m-1}|k\rangle\right)|\psi\rangle$$

$$= \frac{1}{2^{m/2}}\sum_{k=0}^{2^m-1}|k\rangle\left(U^k|\psi\rangle\right)$$

$$\boxed{= \frac{1}{2^{m/2}}\sum_{k=0}^{2^m-1}|k\rangle e^{2\pi ik\theta}|\psi\rangle}$$

$$\omega = e^{2\pi i/2^m}$$

$$QFT_{2^m}|j\rangle = \boxed{\frac{1}{2^{m/2}} \sum_{k=0}^{2^m-1} \omega^{jk} |k\rangle}$$

$$\theta = j/2^m$$

$$QFT^{\dagger}_{2^m}|k\rangle = \frac{1}{2^{m/2}} \sum_{j=0}^{2^m-1} \omega^{-kj} |j\rangle$$

$$QFT_{2^m} = \frac{1}{2^{m/2}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{2^m-1} \\ 1 & \omega^2 & \omega^4 & & \omega^{2(2^m-1)} \\ \vdots & & & & \vdots \\ 1 & \omega^{2^m-1} & \omega^{2(2^m-1)} & \cdots & \omega^{(2^m-1)^2} \end{pmatrix}$$

LEMMA 1 :

$$\sum_{k=0}^{2^m-1} \omega^{k(j-l)} = \begin{cases} 2^m, & \text{if } j = l \\ 0, & \text{if } j \neq l \end{cases}$$

$$= \sum_{k=0}^{2^m-1} \left(\omega^{(j-l)}\right)^k$$

$$= \frac{1 - \left(\omega^{(j-l)}\right)^{2^m}}{1 - \omega^{(j-l)}} = \frac{1-1}{1-\omega^{(j-l)}} = 0$$

$j \neq l$

$$QFT_{2^m}^{\dagger} \, QFT_{2^m} |j\rangle = QFT_{2^m}^{\dagger} \left(\frac{1}{2^{m/2}} \sum_{k=0}^{2^m-1} \omega^{jk} |k\rangle \right)$$

$$= \frac{1}{2^{m/2}} \sum_{k=0}^{2^m-1} \omega^{jk} \left(1/_{2^{m/2}} \sum_{l=0}^{2^m-1} \omega^{-kl} \cdot |l\rangle \right)$$

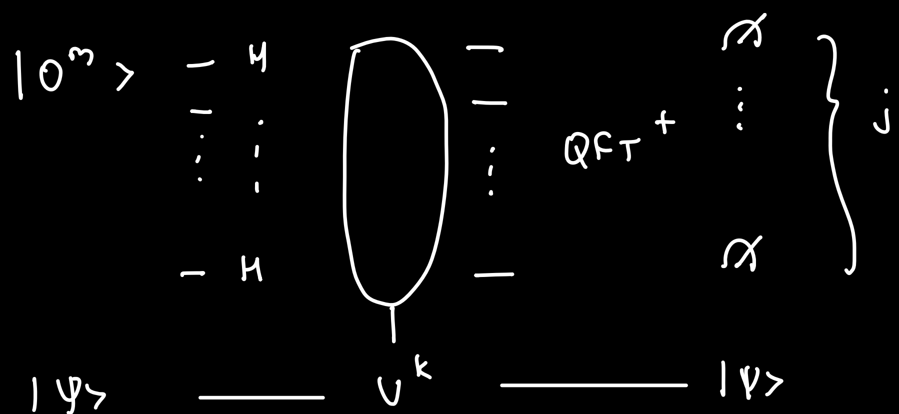
$$= \frac{1}{2^m} \underbrace{\sum_{l=0}^{2^m-1} \left(\sum_{k=0}^{2^m-1} \omega^{k(j-l)} \right)}_{0 \text{ unless } l=j} |l\rangle$$

$$= \frac{1}{2^m} \cdot 2^m |j\rangle$$

$$= |j\rangle .$$

Phase Estimation

$$U|\psi\rangle = \left(QFT^{\dagger}_{2^m} \otimes I^{\otimes n}\right) \Lambda_m(U) \left(H^{\otimes m} \otimes I^{\otimes n}\right)|0^m\rangle|\psi\rangle$$



IF : $\theta = j/2^m$

$$\text{PhEst } U|\psi\rangle = \left(QFT^{\dagger}_{2^m} \otimes I^{\otimes n}\right) \underbrace{\ldots |0^m\rangle|\psi\rangle}$$

$$= QFT^{\dagger}_{2^m}\left(\frac{1}{2^{m/2}} \sum_{k=0}^{2^m-1} e^{2\pi i k\theta} |k\rangle\right)|\psi\rangle$$

$j/2^m$

$$= QFT^{\dagger}_{2^m}\left(\frac{1}{2^{m/2}} \sum_{k=0}^{2^m-1} \omega^{kj} |k\rangle\right)|\psi\rangle$$

$$= |j\rangle|\psi\rangle$$

ω^{kj} generally is $\omega^{k 2^m \theta}$

$$\text{PhEst}_m \, U|\Psi\rangle = (QFT^{\dagger} \ldots) \frac{1}{2^{m/2}} \sum_{k=0}^{2^m-1} \omega^{k 2^m \theta} |k\rangle |\Psi\rangle$$

$$= \frac{1}{2^{m/2}} \left(\sum_{k=0}^{2^m-1} \omega^{k 2^m \theta} \frac{1}{2^{m/2}} \sum_{j=0}^{2^m-1} \omega^{-kj} |j\rangle \right) |\Psi\rangle$$

$$= \sum_{j=0}^{2^m-1} \underbrace{\left(\frac{1}{2^m} \sum_{k=0}^{2^m-1} \omega^{k(2^m\theta - j)} |j\rangle \right)}_{\text{probability of measuring a } j.} |\psi\rangle$$

$$P_j = \left| \frac{1}{2^m} \sum_{k=0}^{2^m-1} \omega^{k(2^m\theta - j)} \right|^2$$

$$= \frac{1}{2^{2m}} \left| \sum_{k=0}^{2^m-1} \left(\omega^{(2^m\theta - j)} \right)^k \right|^2$$

$$P_j = \frac{1}{2^{2m}} \left| \frac{1 - (\omega^{2^m\theta - j})^{2^m}}{1 - \omega^{2^m\theta - j}} \right|^2$$

$$= \frac{1}{2^{2m}} \frac{\left|\left(\omega^{2^m\theta - j}\right)^{2^m} - 1\right|^2}{\left|\left(\omega^{2^m\theta - j}\right) - 1\right|^2}$$

$$|\varepsilon| \leq \frac{1}{2^{m+1}}$$

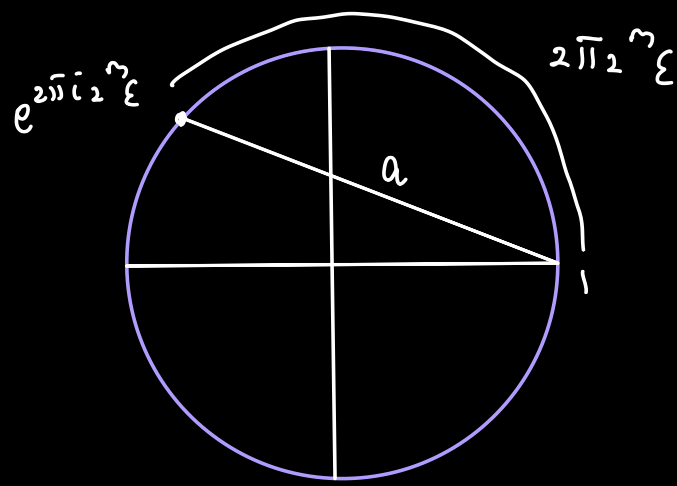
$$\theta = \left(\frac{j}{2^m} + \varepsilon\right) (\bmod 1)$$

$$2^m\theta - j = 2^m\varepsilon$$

$$p_j = \frac{1}{2^{2m}} \frac{\left|\left(\omega^{2^m\varepsilon}\right)^{2^m} - 1\right|^2}{\left|\omega^{2^m\varepsilon} - 1\right|^2}$$

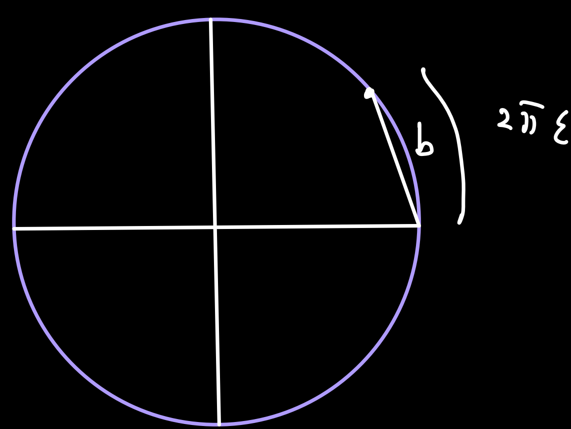
$$= \frac{1}{2^{2m}} \frac{\left|\left(\omega^{2^{2m}\varepsilon}\right) - 1\right|^2}{\left|\left(\omega^{2^m\varepsilon}\right) - 1\right|^2} \qquad \omega = e^{2\pi i/2^m}$$

$$= \frac{1}{2^{2m}} \frac{\left|\overbrace{e^{2\pi i 2^m\varepsilon} - 1}^{a}\right|^2}{\left|\underbrace{e^{2\pi i\varepsilon} - 1}_{b}\right|^2}$$



$$\frac{2\pi 2^m \varepsilon}{a} \leq \frac{\pi}{2}$$

$$4 \varepsilon 2^m \leq a$$



$$b \leq 2\pi\varepsilon$$

$$p_j = \frac{1}{2^{2m}} \frac{a^2}{b^2} \geq \frac{1}{2^{2m}} \frac{(4\varepsilon 2^m)^2}{(2\pi \varepsilon)^2}$$

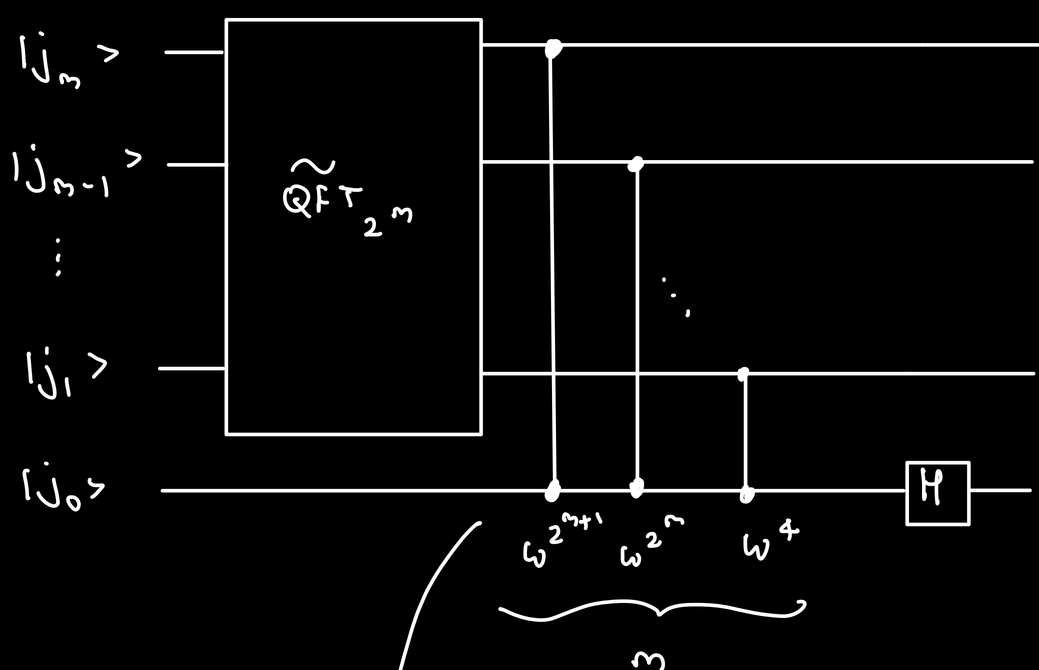
$$= \frac{4}{\pi^2} > 0.4.$$

Really good with $p = 0.4$.

$$\omega_N = e^{2\pi i/N}$$

$$\widetilde{QFT}_{2^1} |j\rangle = H|j\rangle$$

$\widetilde{QFT}_{2^{m+1}}$:



$$\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & \omega^{2^{m+1}} \end{bmatrix}$$ like cz.

$$g(1) = 1$$

$$g(n+1) = g(n) + n + 1$$

For 2^n, we need n^2 gates.