1) $SWAP[q_1, q_2] = C(X)[q_1, q_2]\, C(X)[q_2, q_1]\, C(X)[q_1, q_2]$

ASSUME:

$|k_{q_1}\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$

$|k_{q_2}\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$

THE REGISTERS:

$|k_1\rangle \ldots (\alpha_1|0\rangle + \beta_1|1\rangle) \ldots (\alpha_2|0\rangle + \beta_2|1\rangle) \ldots |k_n\rangle$

$|k_1\rangle \ldots (\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle) \ldots |k_n\rangle$

APPLY GATES RIGHT TO LEFT

RHS :

* APPLY $C(X)[q_1, q_2]$

we get

$|k_1\rangle \ldots (\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|11\rangle + \beta_1\beta_2|10\rangle) \ldots |k_n\rangle$

* APPLY $C(X)[q_2 q_1]$

we get

$|k_1\rangle \ldots (\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|11\rangle + \beta_1\alpha_2|01\rangle + \beta_1\beta_2|10\rangle) \ldots$
$|k_n\rangle$

* Apply $C(X)[q_1, q_2]$

we get

$$|k_1\rangle \dots \left(\alpha_1\alpha_2 |00\rangle + \alpha_1\beta_2 |10\rangle + \beta_1\alpha_2 |01\rangle + \beta_1\beta_2 |11\rangle \right) \dots |k_n\rangle$$

$$|k_1\rangle \dots (\alpha_2|0\rangle + \beta_2|1\rangle) \dots (\alpha_1|0\rangle + \beta_1|1\rangle) \dots |k_n\rangle$$

$$= |k_1\rangle \dots |k_{q_2}\rangle \dots |k_{q_1}\rangle \dots |k_n\rangle$$

$$= \text{SWAP}[q_1, q_2]$$

$$= \text{LHS} //$$

2) $C(X)[p,q] = H[q]\, C(Z)[p,q]\, H[q]$

ASSUME:

$$|k_p\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$$

$$|k_q\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$$

THE REGISTERS:

$$|k_1\rangle \ldots (\alpha_1|0\rangle + \beta_1|1\rangle) \ldots (\alpha_2|0\rangle + \beta_2|1\rangle) \ldots |k_n\rangle$$

$$|k_1\rangle \ldots (\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle) \ldots |k_n\rangle$$

RHS:

* APPLY $H[q]$

$$|k_1\rangle \ldots (\alpha_1\alpha_2|0\rangle\left(\tfrac{1}{\sqrt2}|0\rangle + \tfrac{1}{\sqrt2}|1\rangle\right) + \alpha_1\beta_2|0\rangle\left(\tfrac{1}{\sqrt2}|0\rangle - \tfrac{1}{\sqrt2}|1\rangle\right)$$

$$+ \beta_1\alpha_2|1\rangle\left(\tfrac{1}{\sqrt2}|0\rangle + \tfrac{1}{\sqrt2}|1\rangle\right) + \beta_1\beta_2|1\rangle\left(\tfrac{1}{\sqrt2}|0\rangle - \tfrac{1}{\sqrt2}|1\rangle\right))$$

$$\ldots |k_n\rangle$$

$$= |k_1\rangle \ldots \tfrac{1}{\sqrt2}\Big[(\alpha_1\alpha_2 + \alpha_1\beta_2)|00\rangle + (\alpha_1\alpha_2 - \alpha_1\beta_2)|01\rangle +$$

$$(\beta_1\alpha_2 + \beta_1\beta_2)|10\rangle - (\beta_1\alpha_2 - \beta_1\beta_2)|11\rangle)\Big] \ldots |k_n\rangle$$

* APPLY $C(Z)[p, q]$

$\hookrightarrow$ if p is 0 do nothing

if p is 1 apply Z to q

$[|0\rangle$ is $|0\rangle$, $|1\rangle$ is $-|1\rangle]$

$$|k_1\rangle \ldots \frac{1}{\sqrt{2}}\Big[(\alpha_1\alpha_2 + \alpha_1\beta_2)|00\rangle + (\alpha_1\alpha_2 - \alpha_1\beta_2)|01\rangle +$$

$$(\beta_1\alpha_2 + \beta_1\beta_2)|10\rangle + (\beta_1\alpha_2 - \beta_1\beta_2)|11\rangle\Big] \ldots |k_n\rangle$$

* APPLY $H[q]$

$$|k_1\rangle \ldots \frac{1}{\sqrt{2}}\Big[(\alpha_1\alpha_2 + \alpha_1\beta_2)|0\rangle\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) +$$

$$(\alpha_1\alpha_2 - \alpha_1\beta_2)|0\rangle\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) +$$

$$(\beta_1\alpha_2 + \beta_1\beta_2)|1\rangle\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) +$$

$$(\beta_1\alpha_2 - \beta_1\beta_2)|1\rangle\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)\Big] \ldots |k_n\rangle$$

$$|k_1\rangle \ldots \ \tfrac{1}{2}\Big[(\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_1\alpha_2 - \alpha_1\beta_2)|00\rangle +$$

$$(\alpha_1\alpha_2 + \alpha_1\beta_2 - \alpha_1\alpha_2 + \alpha_1\beta_2)|01\rangle +$$

$$(\beta_1\alpha_2 + \beta_1\beta_2 + \beta_1\alpha_2 - \beta_1\beta_2)|10\rangle +$$

$$(\beta_1\alpha_2 + \beta_1\beta_2 - \beta_1\alpha_2 + \beta_1\beta_2)|11\rangle)\Big] \ldots \ |k_n\rangle$$

$$= |k_1\rangle \ldots \ (\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle) \ldots |k_n\rangle \quad - (1)$$

LHS

$C(x)[p, q]$

$$|k_1\rangle \ldots \ (\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|11\rangle + \beta_1\beta_2|10\rangle) \ldots |k_n\rangle \quad - (2)$$

(1) = (2)

LHS = RHS

HENCE PROVED//

3) $C(2)[p, q] = C(2)[q, p]$

ASSUME:

$|k_p\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$

$|k_q\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$

THE REGISTERS:

$|k_1\rangle \ldots (\alpha_1|0\rangle + \beta_1|1\rangle) \ldots (\alpha_2|0\rangle + \beta_2|1\rangle) \ldots |k_n\rangle$

$|k_1\rangle \ldots (\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle) \ldots |k_n\rangle$

LHS:

APPLY $C(2)[p, q]$

$|k_1\rangle \ldots (\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta\alpha_2|10\rangle - \beta_1\beta_2|11\rangle) \ldots |k_n\rangle$ - (1)

RHS

APPLY $C(2)[q, p]$

$|k_1\rangle \ldots (\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta\alpha_2|10\rangle - \beta_1\beta_2|11\rangle) \ldots |k_n\rangle$ - (2)

(1) = (2)

LHS = RHS

HENCE PROVED//

4) $H[p]\, H[q]\, C(X)[p,q]\, H[p]\, H[q] = C(X)[q,p]$

ASSUME:

$$|k_p\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$$

$$|k_q\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$$

THE REGISTERS:

$$|k_1\rangle \ldots (\alpha_1|0\rangle + \beta_1|1\rangle) \ldots (\alpha_2|0\rangle + \beta_2|1\rangle) \ldots |k_n\rangle$$

$$|k_1\rangle \ldots (\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle) \ldots |k_n\rangle$$

LHS:

* APPLY $H[q]$

$$|k_1\rangle \ldots \left(\alpha_1\alpha_2|0\rangle\left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right) + \alpha_1\beta_2|0\rangle\left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)\right.$$

$$\left. + \beta_1\alpha_2|1\rangle\left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right) + \beta_1\beta_2|1\rangle\left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)\right) \ldots |k_n\rangle$$

$$= |k_1\rangle \ldots \tfrac{1}{\sqrt{2}}\left[(\alpha_1\alpha_2 + \alpha_1\beta_2)|00\rangle + (\alpha_1\alpha_2 - \alpha_1\beta_2)|01\rangle + \right.$$

$$\left. (\beta_1\alpha_2 + \beta_1\beta_2)|10\rangle + (\beta_1\alpha_2 - \beta_1\beta_2)|11\rangle\right] \ldots |k_n\rangle$$

* APPLY $H[p]$

$$|k_1\rangle \ldots \tfrac{1}{\sqrt{2}}\left[(\alpha_1\alpha_2 + \alpha_1\beta_2)\left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right)|0\rangle + \right.$$

$$(\alpha_1\alpha_2 - \alpha_1\beta_2)\left(1/\sqrt{2}\,|0\rangle + 1/\sqrt{2}\,|1\rangle\right)|1\rangle +$$

$$(\beta_1\alpha_2 + \beta_1\beta_2)\left(1/\sqrt{2}\,|0\rangle - 1/\sqrt{2}\,|1\rangle\right)|0\rangle +$$

$$(\beta_1\alpha_2 - \beta_1\beta_2)\left(1/\sqrt{2}\,|0\rangle - 1/\sqrt{2}\,|1\rangle\right)|1\rangle\Big] \ldots |k_n\rangle$$

$$= |k_1\rangle \ldots \; 1/2\Big[(\alpha_1\alpha_2 + \alpha_1\beta_2 + \beta_1\alpha_2 + \beta_1\beta_2)|00\rangle +$$

$$(\alpha_1\alpha_2 - \alpha_1\beta_2 + \beta_1\alpha_2 - \beta_1\beta_2)|01\rangle +$$

$$(\alpha_1\alpha_2 + \alpha_1\beta_2 - \beta_1\alpha_2 - \beta_1\beta_2)|10\rangle +$$

$$(\alpha_1\alpha_2 - \alpha_1\beta_2 - \beta_1\alpha_2 + \beta_1\beta_2)|11\rangle\Big] \ldots |k_n\rangle$$

* Apply $c(x)[p, q]$

$$= |k_1\rangle \ldots \; 1/2\Big[(\alpha_1\alpha_2 + \alpha_1\beta_2 + \beta_1\alpha_2 + \beta_1\beta_2)|00\rangle +$$

$$(\alpha_1\alpha_2 - \alpha_1\beta_2 + \beta_1\alpha_2 - \beta_1\beta_2)|01\rangle +$$

$$(\alpha_1\alpha_2 + \alpha_1\beta_2 - \beta_1\alpha_2 - \beta_1\beta_2)|11\rangle +$$

$$(\alpha_1\alpha_2 - \alpha_1\beta_2 - \beta_1\alpha_2 + \beta_1\beta_2)|10\rangle\Big] \ldots |k_n\rangle$$

* APPLY $H[q]$

$$|k_1\rangle \ldots \tfrac{1}{2}\Big[(\alpha_1\alpha_2 + \alpha_1\beta_2 + \beta_1\alpha_2 + \beta_1\beta_2)|0\rangle \left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right) +$$

$$(\alpha_1\alpha_2 - \alpha_1\beta_2 + \beta_1\alpha_2 - \beta_1\beta_2)|0\rangle \left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right) +$$

$$(\alpha_1\alpha_2 + \alpha_1\beta_2 - \beta_1\alpha_2 - \beta_1\beta_2)|1\rangle \left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right) +$$

$$(\alpha_1\alpha_2 - \alpha_1\beta_2 - \beta_1\alpha_2 + \beta_1\beta_2)|1\rangle \left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right)\Big] \ldots |k_n\rangle$$

$$= |k_1\rangle \ldots \tfrac{1}{2\sqrt{2}}\Big[(\alpha_1\alpha_2 + \alpha_1\beta_2 + \beta_1\alpha_2 + \beta_1\beta_2 + \alpha_1\alpha_2 - \alpha_1\beta_2 + \beta_1\alpha_2 - \beta_1\beta_2)|00\rangle +$$

$$(\alpha_1\alpha_2 + \alpha_1\beta_2 + \beta_1\alpha_2 + \beta_1\beta_2 - \alpha_1\alpha_2 + \alpha_1\beta_2 - \beta_1\alpha_2 + \beta_1\beta_2)|01\rangle +$$

$$(\alpha_1\alpha_2 + \alpha_1\beta_2 - \beta_1\alpha_2 - \beta_1\beta_2 + \alpha_1\alpha_2 - \alpha_1\beta_2 - \beta_1\alpha_2 + \beta_1\beta_2)|10\rangle +$$

$$(-\alpha_1\alpha_2 - \alpha_1\beta_2 + \beta_1\alpha_2 + \beta_1\beta_2 + \alpha_1\alpha_2 - \alpha_1\beta_2 - \beta_1\alpha_2 + \beta_1\beta_2)|11\rangle\Big] \ldots |k_n\rangle$$

$$= |k_1\rangle \ldots \tfrac{1}{\sqrt{2}}\Big[(\alpha_1\alpha_2 + \beta_1\alpha_2)|00\rangle + (\alpha_1\beta_2 + \beta_1\beta_2)|01\rangle +$$

$$(\alpha_1\alpha_2 - \beta_1\alpha_2)|10\rangle + (-\alpha_1\beta_2 + \beta_1\beta_2)|11\rangle\Big] \ldots |k_n\rangle$$

* APPLY $H[p]$

$$|k_1\rangle \ldots \tfrac{1}{\sqrt{2}}\Big[(\alpha_1\alpha_2 + \beta_1\alpha_2)\left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right)|0\rangle +$$

$$(\alpha_1\beta_2 + \beta_1\beta_2)\left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right)|1\rangle +$$

$$(\alpha_1\alpha_2 - \beta_1\alpha_2)\left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)|0\rangle +$$

$$(-\alpha_1\beta_2 + \beta_1\beta_2)\left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)|1\rangle \Big] \ldots |k_n\rangle$$

$$= |k_1\rangle \ldots \tfrac{1}{2}\Big[(\alpha_1\alpha_2 + \beta_1\alpha_2 + \alpha_1\alpha_2 - \beta_1\alpha_2)|00\rangle +$$

$$(\alpha_1\beta_2 + \beta_1\beta_2 - \alpha_1\beta_2 + \beta_1\beta_2)|01\rangle +$$

$$(\alpha_1\alpha_2 + \beta_1\alpha_2 - \alpha_1\alpha_2 + \beta_1\alpha_2)|10\rangle +$$

$$(\alpha_1\beta_2 + \beta_1\beta_2 + \alpha_1\beta_2 - \beta_1\beta_2)|11\rangle \Big] \ldots |k_n\rangle$$

$$= |k_1\rangle \ldots (\alpha_1\alpha_2|00\rangle + \beta_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \alpha_1\beta_2|11\rangle) \ldots |k_n\rangle \quad -①$$

RHS

APPLY $c(x)[q,p]$

$$|k_1\rangle \ldots (\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|11\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|01\rangle) \ldots |k_n\rangle \quad -②$$

1 = 2

LHS = RHS

HENCE PROVED //

5) $c(x)[p,q]\ x[p]\ c(x)[p,q] = x[p]\ x[q]$

ASSUME:

$$|k_p\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$$

$$|k_q\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$$

THE REGISTERS:

$$|k_1\rangle \ldots (\alpha_1|0\rangle + \beta_1|1\rangle) \ldots (\alpha_2|0\rangle + \beta_2|1\rangle) \ldots |k_n\rangle$$

$$|k_1\rangle \ldots (\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle) \ldots |k_n\rangle$$

LHS:

* APPLY $c(x)[p,q]$

we get

$$|k_1\rangle \ldots (\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|11\rangle + \beta_1\beta_2|10\rangle) \ldots |k_n\rangle$$

* APPLY $x[p]$

we get

$$|k_1\rangle \ldots (\alpha_1\alpha_2|10\rangle + \alpha_1\beta_2|11\rangle + \beta_1\alpha_2|01\rangle + \beta_1\beta_2|00\rangle) \ldots |k_n\rangle$$

* APPLY $c(x)[p,q]$

we get

$$|k_1\rangle \ldots (\alpha_1\alpha_2|11\rangle + \alpha_1\beta_2|10\rangle + \beta_1\alpha_2|01\rangle + \beta_1\beta_2|00\rangle) \ldots |k_n\rangle \quad -1$$

RHS :

* APPLY $X[q]$

$$|k_1\rangle \ldots (\alpha_1\alpha_2|01\rangle + \alpha_1\beta_2|00\rangle + \beta_1\alpha_2|11\rangle + \beta_1\beta_2|10\rangle) \ldots |k_n\rangle$$

* APPLY $X[p]$

$$|k_1\rangle \ldots (\alpha_1\alpha_2|11\rangle + \alpha_1\beta_2|10\rangle + \beta_1\alpha_2|01\rangle + \beta_1\beta_2|00\rangle) \ldots |k_n\rangle \quad -(2)$$

$$(1) = (2)$$

LHS = RHS

HENCE PROVED //

b) $C(x)[p,q]\ Y[p]\ C(x)[p,q] = Y[p]\ X[q]$

Assume:

$|k_p\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$

$|k_q\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$

The registers:

$|k_1\rangle \ldots (\alpha_1|0\rangle + \beta_1|1\rangle) \ldots (\alpha_2|0\rangle + \beta_2|1\rangle) \ldots |k_n\rangle$

$|k_1\rangle \ldots (\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle) \ldots |k_n\rangle$

LHS:

* Apply $C(x)[p,q]$

we get

$|k_1\rangle \ldots (\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|11\rangle + \beta_1\beta_2|10\rangle) \ldots |k_n\rangle$

* Apply $Y[p]$ $\quad |0\rangle \rightarrow i|1\rangle \qquad |1\rangle \rightarrow -i|0\rangle$

we get

$|k_1\rangle \ldots (\alpha_1\alpha_2 i|10\rangle + \alpha_1\beta_2 i|11\rangle - i\beta_1\alpha_2|01\rangle - i\beta_1\beta_2|00\rangle) \ldots |k_n\rangle$

* Apply $C(x)[p,q]$

we get

$|k_1\rangle \ldots (\alpha_1\alpha_2 i|11\rangle + \alpha_1\beta_2 i|10\rangle - i\beta_1\alpha_2|01\rangle - i\beta_1\beta_2|00\rangle) \ldots |k_n\rangle$

$-(1)$

RHS:

* APPLY $x[q]$

$$|k_1\rangle \dots (\alpha_1\alpha_2|01\rangle + \alpha_1\beta_2|00\rangle + \beta_1\alpha_2|11\rangle + \beta_1\beta_2|10\rangle) \dots |k_n\rangle$$

* APPLY $y[p]$

$$|k_1\rangle \dots (\alpha_1\alpha_2 i|11\rangle + i\alpha_1\beta_2|10\rangle - i\beta_1\alpha_2|01\rangle - i\beta_1\beta_2|00\rangle) \dots |k_n\rangle \quad -②$$

$$① = ②$$

LHS = RHS

HENCE PROVED//

7) $C(X)[p,q]\, Z[p]\, C(X)[p,q] = Z[p]$

ASSUME:

$|k_p\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$

$|k_q\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$

THE REGISTERS:

$|k_1\rangle \ldots (\alpha_1|0\rangle + \beta_1|1\rangle) \ldots (\alpha_2|0\rangle + \beta_2|1\rangle) \ldots |k_n\rangle$

$|k_1\rangle \ldots (\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle) \ldots |k_n\rangle$

LHS:

* APPLY $C(X)[p,q]$

we get

$|k_1\rangle \ldots (\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|11\rangle + \beta_1\beta_2|10\rangle) \ldots |k_n\rangle$

* APPLY $Z[p]$ $\quad |0\rangle \rightarrow |0\rangle \quad |1\rangle \rightarrow -|1\rangle$

we get

$|k_1\rangle \ldots (\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle - \beta_1\alpha_2|11\rangle - \beta_1\beta_2|10\rangle) \ldots |k_n\rangle$

* APPLY $C(X)[p,q]$

we get

$|k_1\rangle \ldots (\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle - \beta_1\alpha_2|10\rangle - \beta_1\beta_2|11\rangle) \ldots |k_n\rangle$ - 1

RHS:

* APPLY $Z[p]$

$$|k_1\rangle \ldots (\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle - \beta_1\alpha_2|10\rangle - \beta_1\beta_2|11\rangle) \ldots |k_n\rangle$$

-2

1 = 2

LHS = RHS

HENCE PROVED//

8) $C(X)[p,q]\, X[q]\, C(X)[p,q] = X[q]$

ASSUME:

$|k_p\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$

$|k_q\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$

THE REGISTERS:

$|k_1\rangle \ldots (\alpha_1|0\rangle + \beta_1|1\rangle) \ldots (\alpha_2|0\rangle + \beta_2|1\rangle) \ldots |k_n\rangle$

$|k_1\rangle \ldots (\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle) \ldots |k_n\rangle$

LHS :

* APPLY $C(X)[p,q]$

we get

$|k_1\rangle \ldots (\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|11\rangle + \beta_1\beta_2|10\rangle)$

* APPLY $X[q]$

we get

$|k_1\rangle \ldots (\alpha_1\alpha_2|01\rangle + \alpha_1\beta_2|00\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle) \ldots |k_n\rangle$

* APPLY $C(X)[p,q]$

we get

$|k_1\rangle \ldots (\alpha_1\alpha_2|01\rangle + \alpha_1\beta_2|00\rangle + \beta_1\alpha_2|11\rangle + \beta_1\beta_2|10\rangle) \ldots |k_n\rangle$ - 1

RHS :

* APPLY $x[q]$

$$|k_1\rangle \ \ldots \ (\alpha_1\alpha_2|01\rangle + \alpha_1\beta_2|00\rangle + \beta_1\alpha_2|11\rangle + \beta_1\beta_2|10\rangle) \ldots |k_n\rangle \quad -2$$

$$1 = 2$$

LHS = RHS

HENCE PROVED //

9) $c(x)[p,q]\ y[q]\ c(x)[p,q] = z[p]\ x[q]$

Assume:

$|k_p\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$

$|k_q\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$

The registers:

$|k_1\rangle \ldots (\alpha_1|0\rangle + \beta_1|1\rangle) \ldots (\alpha_2|0\rangle + \beta_2|1\rangle) \ldots |k_n\rangle$

$|k_1\rangle \ldots (\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle) \ldots |k_n\rangle$

LHS:

* Apply $c(x)[p,q]$

we get

$|k_1\rangle \ldots (\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|11\rangle + \beta_1\beta_2|10\rangle)$

* Apply $y[q]$ $\quad |0\rangle \rightarrow i|1\rangle \quad |1\rangle \rightarrow -i|0\rangle$

we get

$|k_1\rangle \ldots i(\alpha_1\alpha_2|01\rangle - \alpha_1\beta_2|00\rangle - \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle) \ldots |k_n\rangle$

* Apply $c(x)[p,q]$

we get

$|k_1\rangle \ldots i(\alpha_1\alpha_2|01\rangle - \alpha_1\beta_2|00\rangle - \beta_1\alpha_2|11\rangle + \beta_1\beta_2|10\rangle) \ldots |k_n\rangle$

-1

RHS:

* APPLY $x[q]$

$|k_1\rangle \ldots (\alpha_1\alpha_2|01\rangle + \alpha_1\beta_2|00\rangle + \beta_1\alpha_2|11\rangle + \beta_1\beta_2|10\rangle) \ldots |k_n\rangle$

* APPLY $z[p]$ $|0\rangle \rightarrow |0\rangle$ $|1\rangle \rightarrow -|1\rangle$

$|k_1\rangle \ldots (\alpha_1\alpha_2|01\rangle + \alpha_1\beta_2|00\rangle - \beta_1\alpha_2|11\rangle - \beta_1\beta_2|10\rangle) \ldots |k_n\rangle$ - (2)

(1) $\neq$ (2)

LHS $\neq$ RHS

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10) $C(X)[p,q]\; Z[q]\; C(X)[p,q] = Z[p]\; Z[q]$

Assume:

$|k_p\rangle : \alpha_1|0\rangle + \beta_1|1\rangle$

$|k_q\rangle : \alpha_2|0\rangle + \beta_2|1\rangle$

The Registers:

$|k_1\rangle \ldots (\alpha_1|0\rangle + \beta_1|1\rangle) \ldots (\alpha_2|0\rangle + \beta_2|1\rangle) \ldots |k_n\rangle$

$|k_1\rangle \ldots (\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle) \ldots |k_n\rangle$

LHS:

* Apply $C(X)[p,q]$

we get

$|k_1\rangle \ldots (\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|11\rangle + \beta_1\beta_2|10\rangle)$

* Apply $Z[q]$ $\quad |0\rangle \to |0\rangle \quad |1\rangle \to -|1\rangle$

we get

$|k_1\rangle \ldots (\alpha_1\alpha_2|00\rangle - \alpha_1\beta_2|01\rangle - \beta_1\alpha_2|11\rangle + \beta_1\beta_2|10\rangle) \ldots |k_n\rangle$

* Apply $C(X)[p,q]$

we get

$|k_1\rangle \ldots (\alpha_1\alpha_2|00\rangle - \alpha_1\beta_2|01\rangle - \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle) \ldots |k_n\rangle$ — 1

RHS :

- APPLY $Z[q]$

We get

$$|k_1\rangle \ldots \left(\alpha_1\alpha_2|00\rangle - \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle - \beta_1\beta_2|11\rangle\right) \ldots |k_n\rangle$$

- APPLY $Z[p]$

We get

$$|k_1\rangle \ldots \left(\alpha_1\alpha_2|00\rangle - \alpha_1\beta_2|01\rangle - \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle\right) \ldots |k_n\rangle \quad - 2$$

$1 = 2$

LHS = RHS

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