DISCRETE MATHEMATICS

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Mathematical Logic

- Propositional logic,
- propositional Equivalences,
- Predicates and Quantifiers,
- Nested Quantifiers,
- Rules of Interference,
- Introduction to proofs,
- Normal forms,
- proof methods and strategy.

What is Proposition ?

A declarative sentence to which can assign one and only one of the truth value true or false is called proposition.

T = True

F = False

Example :

India's Capital is new Delhi. (True)

2 x 3 = 5 (False)

Law of Excluded middle :

If a proposition is not true then it is false.

Not True = False

Not False = True

Similarly, If a proposition is not false then it is true.

Law of Contradiction :

A proposition can not be simulteniously true & false, then it is called low of contradiction.

Types of propositions :

Atomic Proposition

- A proposition which can not be divided further into two or more proposition is said to be atomicproposition.
- It is denoted by p, q, r, s, ...

Compound Proposition :

- Two or more atomic propositions can be combine using connectives is called compoundproposition.
- There are five connectives

Connective Name	Symbol
And	$\cap$
Or	$\cup$
Iff (if and only if)	$\leftrightarrow$
Implies	$\rightarrow$
Not	$\sim$

Exp :

India's capital is New Delhi **and** 2x3=5

$P \longrightarrow (Q \vee R)$

Negation :

If P is a proposition then "not P" is written as symbolically $\sim P$ is a proposition whose truth value is false only when if P is true.

Exp :

(P) India's capital is New Delhi.

($\sim$P) India's capital is not New Delhi

Truth Table

P	**~P**
T	F
F	T

Disjunction(or):

If P & Q are any two propositions then "P or Q" are symbolically written as ($P \cup Q$) is a proposition whose value is false only when both P and Q are false.

Exp : ($P \cup Q$) is true if atleast one of the two proposition is true.

Truth Table

P	**Q**	**P∪Q**
F	F	F
F	T	T
T	F	T
T	T	T

Disjunctive Syllolism

IF (P∪Q) is true and P is false then Q is true.

Conjunction(and):

If P & Q are any two propositions then "P and Q" are symbolically written as (P∩Q) is a proposition whose value is true only when both P and Q are true.

Exp : (P∩Q) is FALSE if atleast one of the two proposition is false.

Truth Table

P	**Q**	**P∩Q**
F	F	F
F	T	F
T	F	F
T	T	T

Conjunctive Syllolism

IF (P∩Q) is FALSE and P is TRUE then Q is FALSE.

Implication(Conditional)

If P & Q are any two propositions, then "P implies Q" or "P then Q" symbolically written as P →Q is a propositions whose truth value is FALSE only when P is true and Q is false.

P →Q Read as "P implies Q"

Anticedent Consequent

Truth Table

P	Q	$P \rightarrow Q$
F	F	T
F	T	T
T	F	F
T	T	T

NOTE :

1) $(Q \rightarrow P)$ is **converse** of $(P \rightarrow Q)$
2) $(\sim P \rightarrow \sim Q)$ is **inverse** of $(P \rightarrow Q)$
3) $(\sim Q \rightarrow \sim P)$ is **contrapositive** of $(P \rightarrow Q)$
4) $(P \rightarrow Q)$ is **equivalance** of $(\sim Q \rightarrow \sim P)$

Verify with truth table :

P	Q	$\sim P$	$\sim Q$	$(P \rightarrow Q)$	$(Q \rightarrow P)$	$\sim P \rightarrow \sim Q$	$\sim Q \rightarrow \sim P$
F	F	T	T	T	T	T	T
F	T	T	F	T	F	F	T
T	F	F	T	F	T	T	F
T	T	F	F	T	T	T	T

Biconditional(iff)

If P & Q are any two propositions, then "P iff Q" symbolically written as $P \leftrightarrow Q$ is a proposition whose truth value is TRUE only when both P and Q have same truth value.

OR,

Biconditional is false if P & Q have different truth values.

Truth Table

P	Q	$P \leftrightarrow Q$
F	F	T
F	T	F
T	F	F
T	T	T

If Biconditional is equivalent

$$(P \leftrightarrow Q) \equiv \{(P \rightarrow Q) \cap (Q \rightarrow P)\}$$

Tautology

A propositional function which is always TRUE is called a tautology.

Truth Table

P	Q	f(P,Q)
F	F	T
F	T	T
T	F	T
T	T	T

Exp : $(P \cup \sim P)$

$(P \cup (P \rightarrow Q))$

P	Q	(P→Q)	P ∪ (P→Q)
F	F	T	T
F	T	T	T
T	F	F	T
T	T	T	T

Contradiction

A propositional function which is always FALSE is called a contradiction.

Truth Table

P	Q	f(P,Q)
F	F	F
F	T	F
T	F	F
T	T	F

Exp : **$(P \cap \sim P)$**

P	~P	(P ∩ ~P)
F	T	F
F	T	F
T	F	F
T	F	F

Propositional Equivalence

Two logical expressions are said to be equivalent if they have the same truth value in all cases.

Types of propositions based on Truth values

There are three types of propositions when classified according to their truth values

1. **Tautology –** A proposition which is always true, is called a tautology.
2. **Contradiction –** A proposition which is always false, is called a contradiction.
3. **Contingency –** A proposition that is neither a tautology nor a contradiction is called a contingency.

Example

- $(P \cup \sim P)$ is a tautology.
- $(P \cap \sim P)$ is a contradiction.
- $(P \cup Q)$ is a contingency.

Logical Equivalence

Two propositions P and Q are said to be logically equivalent if **P⟷Q** is a **Tautology**. The notation $P \equiv Q$ is used to denote that P and Q are logically equivalent.

Exp :

If P & Q are a compound proposition then, P is equivalent to Q is written as $P \equiv Q$ or $P \Leftrightarrow Q$.

$$(p \rightarrow q) \equiv (\sim p \cup q)$$

p	q	(p→q)	~p	q	(~p ∪ q)
F	F	T	T	F	T
F	T	T	T	T	T
T	F	F	F	F	F
T	T	T	F	T	T

Predicate Logic

- Predicate logic is an extension of Propositional logic.
- It adds the concept of predicates and quantifiers to better capture the meaning of statements that cannot be adequately expressed by propositional logic.

What is a predicate?

- Consider the statement, " x is greater than 5″. It has two parts.
- The first part, the variable x, is the subject of the statement.
- The second part, "is greater than 5", is the **predicate**.
- It refers to a property that the subject of the statement can have.

The statement "x is greater than 5″ can be denoted by P(x) where P denotes the predicate "is greater than 5" and x is the variable.

Exp1 : Concider the statement

John is a politician.

Here,

John – Subject

is a politician – predicate

Therefore,

j : John

P : Politician

Then

P(j) : John is a politician.

Exp2 : Kalam is a Scientist.

k : Kalam

S : is a scientist

S(k) : Kalam is a Scientist.

Example 1: Let P(x) denote the statement “x > 20″. What are the truth values of P(21) and P(15) ?

Solution:

P(21) is equivalent to the statementx > 20,
21 > 20 which is True.

P(15) is equivalent to the statement

x > 20,

15 > 20, which is False.

Example 2: Let R(x,y) denote the statement “x = y + 1“. What is the truth value of the propositions R(1,3) and R(2,1)?

Solution:

R(1,3) is the statement

x = y + 1

1 = 3 + 1, which is False.

R(2,1) is the statement

x = y + 1

2 = 1 + 1, which is True.

What are quantifiers?

- Quantifiers are words that refer to quantities such as “some”or “all”.
- It tells that how many elements a given predicate is true.
- In predicate logic, predicates are used alongside quantifiers to express the extent to which a predicate is true over a range of elements.

There are two types of quantification-

1. Universal Quantifiers(∀)
2. Existential Quantifiers(∃)

Universal Quantifiers(∀)

∀ : For All

$\forall_x$: For all x in the universe of discourse

Exp : The universal quantification of P(x) is the statement " P(x) for all values of x in the domain"

The notation ∀P(x) denotes the universal quantification of P(x) .

Here ∀ is called the universal quantifier.

∀P(x) is read as "for all x P(x) ".

Ques :

Let P(x) be the statement " $x + 3 > x$ ". What is the truth value of the statement $\forall_x P(x)$?

Solution:

As $x + 3$ is greater than x for +ve integer, so $P(x) \equiv T$ for all x.

Existential Quantifiers(∃)

∃ : There Exist

$\exists_x$: For some x in the universe of discourse

Exp : The existential quantification of P(x) is the statement

"There exists an element x in the domain such that P(x)"

The notation ∃P(x) denotes the existential quantification of P(x).

Here ∃ is called the existential quantifier.

∃P(x) is read as "There is atleast one such x such that P(x) ".

Ques:

Let P(x) be the statement "$x > 5$". What is the truth value of the statement $\exists_x P(x)$?

Solution:

P(x) is true for all real numbers greater than 5 and false for all real numbers less than 5.

So $\exists_x P(x) \equiv T$

Quantifiers

- Quantifiers are expressions that indicate the scope of the term to which they are attached, here predicates.
- A predicate is a property the subject of the statement can have.

Nested Quantifiers

Nested quantifiers are quantifiers that occur one quantifier within the scope of other quantifiers.

OR,

Two quantifiers are nested if one is within the scope of the other.

Example:

$$\boxed{\forall_x \exists_y Q(x, y)}$$

Different combination of Nested Quantifiers

$\forall_x \forall_y Q(x, y)$

$\forall_x \exists_y Q(x, y)$

$\exists_y \forall_x Q(x, y)$

$\exists_x \exists_y Q(x, y)$

$$\boxed{\begin{array}{l} \forall_x \forall_y Q(x, y) = \forall_y \forall_x Q(x, y) \\ \exists_x \exists_y Q(x, y) = \exists_y \exists_x Q(x, y) \end{array}}$$

$$\boxed{\forall_x \exists_y Q(x, y) \neq \exists_y \forall_x Q(x, y)}$$

Q. Let x and y be the real numbers and P(x,y) denotes "x + y = 0".

a) $\forall_x \forall_y Q(x, y)$

b) $\forall_x \exists_y Q(x, y)$

c) $\exists_y \forall_x Q(x, y)$

d) $\exists_x \exists_y Q(x, y)$

Arguments :

- An arguments is a collection of statements.
- The last statement is the conclusion and all its preceding statements are called premises (or hypothesis).
- The symbol "∴", (read therefore) is placed before the conclusion.
- A valid argument is the conclusion follows from the truth values of the premises.

Example :

Statements

If a man is a bachelor, he is unhappy.

A man is he is unhappy, he is an Engineer.

Conclusion

∴ Bachelor is an Engineer.

Here, decide this conclusion is valid or not.

p : man is a bachelor

q : he is unhappy

r : he is an Engineer

$$\begin{array}{c} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

How to Check conclusion is valid or not.

1. Use truth table
2. Critical rows are always true.

P	Q	R	P →q	P →r	q→r
			T	T	T
			T	T	T
			T	T	T
			T	T	T

Rules of Inference/Lows of Syllogism/Hypothetical Syllogism

- Rules of Inference provide the guidelines for constructing valid arguments from the statements.
- Inference means driving conclusion from evidence.
- It is used to deduce new statements from the statements whose truth that we already know.

Inference – अनुमान

Syllogism – न्याय

Hypothetical – काल्पनिक

Types of Inference rules :

1. Modus Ponens
2. Modus Tollens
3. Hypothetical Syllogism
4. Disjunctive Sullogism
5. Addition
6. Simplification

1. Modus Ponens(Low of detachment)

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

$$\boxed{p \mathrm{A} (p \rightarrow q) \rightarrow q}$$

p	**q**	$p \rightarrow q$	$p \mathrm{A} (p \rightarrow q)$	$p \mathrm{A} (p \rightarrow q) \rightarrow q$
T	T			
T	F			
F	T			
F	F			

1. **Modus Tollens(Law of denial)**

$p \rightarrow q$

$\sim q$

$\therefore \sim p$

$\boxed{(p \rightarrow q)\ A\ (\sim \mathbf{q}\ ) \rightarrow (\sim \mathbf{p}}$

p	**q**	$p \rightarrow q$	**~q**	$(p \rightarrow q)$ A $(\sim \mathbf{q})$	**~p**	$(p \rightarrow q)$ A $(\sim \mathbf{q}) \rightarrow (\sim \mathbf{p})$
T	T	T	F	T	F	F
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

2. **ypothetical Syllogism**

$p \rightarrow q$

$q \rightarrow r$

$\therefore\ p \rightarrow r$

$\boxed{(\mathbf{p} \rightarrow \mathbf{q})\ A\ (\ \mathbf{q} \rightarrow \mathbf{r}) \rightarrow (\ \mathbf{p} \rightarrow \mathbf{r})}$

3. **Disjunctive Syllogism**

$p \vee q$

$\sim p$

$\therefore\ q$

$\boxed{(\mathbf{p} \vee \mathbf{q})\ A\ (\sim \mathbf{p}) \rightarrow \mathbf{q}}$

4. **Addition**

P

$\therefore$ P V q

$\boxed{\mathbf{p} \rightarrow (\mathbf{p} \vee \mathbf{q})}$

5. **Simlification**

p A q

$\therefore$ P

OR

p A q

$\therefore$ q

$\boxed{(p\ A\ q) \rightarrow \mathbf{p}}$

$\boxed{(p\ A\ q) \rightarrow \mathbf{q}}$

Introduction to Proofs

- Proof is an argument we give logically to validate a mathematical statement.
- To validate a statement, we consider two things: A statement and Logical operators.
- A statement is either true or false but not both.
- Logical operators are AND, OR, NOT, If then, and If and only if.

Types of mathematical proofs:

1. Direct Proof
2. Indirect Proof
3. Proof by contradiction

Normal Forms

- Suppose, **A (P1, P2, … , Pn)** is a statements formula where **P1, P2, …, P6** are the atomic variables.
- If we consider all possible assignments of the truth value to **P1, P2, …, Pn** and obtain the resulting truth values of the formula A then we get the truth table for **A**, such a truth table contains **2^6** rows.
- The problem of finding whether a given statement is tautology or contradiction or satisfiable in a finite number of steps is called the Decision Problem.
- For Decision Problem, construction of truth table may not be practical always.
- We consider an alternate procedure known as the reduction to normal forms.

There are two such forms:

1. Disjunctive Normal Form (DNF)
2. Conjunctive Normal Form

Disjunctive Normal Form (DNF):

- If p, q are two statements, then "p or q" is a compound statement, denoted by $p \vee q$ and referred as the disjunction of p and q.
- The disjunction of p and q is true whenever at least one of the two statements is true, and it is false only when both p and q are false

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example: - if p is "4 is a positive integer" and q is "$\sqrt{3}$ is a rational number", then $p \vee q$ is true as statement p is true, although statement q is false.

Conjunctive Normal Form:

- If p, q are two statements, then "p and q" is a compound statement, denoted by $p \wedge q$ and referred as the conjunction of p and q.
- The conjunction of p and q is true only when both p and q are true, otherwise, it is false

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example: if statement p is "6<7" and statement q is "-3>-4" then the conjunction of p and q is true as both p and q are true statements.

UNIT-02 : Introduction to SET THEORY

❑Concept of sets:

❖Notation –

❖subset,

❖superset,

❖Empty set,

❖Universal set. Examples –

❖Definition of power set

❖cardinality of a set,

❖Finite and Infinite sets.

❑Operation on sets:

❖Union –

❖Intersection –

❖Complementation –

❖Difference –

❖Symmetric difference – problems relating simple set identities,,

❖Cartesian product of finite number of sets, simple problems –

❑Concept of sets:

❑Repersentation of Set

❑Types of Set

Set :

- A well defined unordered collection of distinct objects is called set.
- A set is a collection of objects or groups of objects.
- These objects are called elements or members of a set.

Well Defined :

Exp1 :

The collection of vowels in English alphabets.

This set contains five elements, namely, a, e, i, o, u.

Exp2 :

A group of "Singers with ages between 18 years and 30 years" is a set.

Exp4 :

The collection of past presidents of the India is a set.

Exp5 :

A group of “Young dancers” is not a set, as the range of the ages of young dancers is not given and so it can’t be decided that which dancer is to be considered young i.e., the objects are not well-defined.

Unordered :

A = {1, 2, 3, 4, 5}

B = {3, 4, 5, 2, 1}

C = {1, 1, 3, 2, 2, 5, 3, 4}

Distinct :

A = {1, 2, 3, 4}

B = {3, 4, 2, 1, 3, 2}

C = {1, 1, 3, 2, 2, 5, 3, 4}

Representation of Sets

1. Roster Form or Tabular form
2. Set Builder Form

Roster Form

In roster form, all the elements of the set are listed, separated by commas and enclosed between curly braces { }.

Exp : A ={1, 2, 4, 8}

Set Builder Form

- In set builder form, all the elements have a common property.

Example:

A={ x : x is an integer}

‘:’ means ‘such that’

‘{}’ means ‘the set of all’

Types of Sets

1) **Finite set:** The number of elements is finite
2) **Infinite set:** The number of elements are infinite
3) **Empty set:** It has no elements
4) **Singleton set:** It has one only element
5) **Equal set:** Two sets are equal if they have same elements
6) **Equivalent set:** Two sets are equivalent if they have same number of elements
7) **Power set:** A set of every possible subset.
8) **Universal set:** Any set that contains all the sets under consideration.
9) **Subset:** When all the elements of set A belong to set B, then A is subset of B.

1) Empty Set
2) Properties of Empty Sets
3) Zero set Vs Empty set
4) Cardinality of a Set
5) Subset :
6) Proper Subset

Empty Set :

- A set with no element is called empty set.
- A set can be defined as an empty set or a null set if it doesn't contain any elements.
- Its size & cardinality is zero.
- The empty set is denoted by $\emptyset = \{\ \}$. (symbol '$\emptyset$' - read as 'phi')

Example :

$\emptyset$ = A whole number between 6 and 7.

$\emptyset = \{x : x \text{ is a prime number } 8 \le x \le 10\}$

Properties of Empty Sets

1) Subset of any Set
2) Union with an Empty Set
3) Intersection with an Empty Set
4) Cardinality of Empty Set is zero
5) Cartesian Product of Empty Set

Cardinality of a Set

- Cardinality refers to the size of the set.
- It is the total number of elements in the given set.
- An empty set contains no elements.
- Thus, it has cardinality equal to zero.

For example :

Consider a set A = {0, 1, 2, 3}.
There are 4 elements in set A Therefore, the cardinality of set A = 4.

Zero Set	Empty Set or Null Set
A zero set can be defined as a set that contains zero as the only element.	An empty set is a set that does not contain any elements.
It is denoted as {0}.	An empty set can be denoted as {}.

Subset :

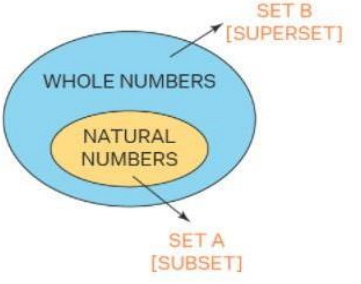
- If every elements of a set A is also an element of set B, then set A is called a subset of set B.
- Set A is a subset of set B is written as $A \subseteq B$.
- If all elements of set A are in another set B, then set A is said to be a subset of set B.

Example :

If set 1 = {A,B,C} and

set 2 =

{A,B,C,D,E,F}



- A = {1, 2, 3} is a subset of B = {1, 2, 3, 4, 10}
- A = {p, q, r} is a subset of B = set of all alphabets

Note : Every set is a subset of itself and also the empty set (Φ) is also a subset of every set.

Number of Subsets of a Set

- The number of subsets of a set with n elements is 2^n.
- **Example :**

if A = {1, 2, 3}, then the number of elements of A = 3.

The subsets of A are { }, {1}, {2}, {3}, {1, 2}, {2, 3}, {3, 1}, and {1, 2, 3}.

Proper Subset

- A **proper subset** is any subset of the set except itself.
- We know that every set is a subset of itself but it is NOT a proper subset of itself.
- **Example :**

if A = {1, 2, 3},

Its proper subsets are {}, {1}, {2}, {3}, {1, 2}, {2, 3}, and {3, 1}, but the set itself {1, 2, 3} is NOT a proper subset of A.

- The proper subset symbol is $\subset$.
  - $A \subset B$ and
  - $A \neq B$

Proper Subset Formula

The number of proper subsets of a set with 'n' elements is $2^n - 1$.

Example:

- The number of proper subsets of A = {1, 2, 3} is, $2^3 - 1 = 7$.
- The number of proper subsets of A = {a, b} is, $2^2 - 1 = 3$.

❖Finite Set

❖Infinite Sets

❖Finite set Vs Infinite set

❖Superset

❖Power Set

❖Universal Set

Finite Set

- Finite Sets are defined as sets with a finite number of elements.
- Elements of finite sets can be counted.
- All finite sets are countable but not all countable sets are finite.

Example :

A = {2, 4, 6, 8, 10}.

B = {x : x is the prime number between 1 and 20}

Infinite Set

❖Infinite sets in set theory are defined as sets that are not finite.

❖The number of elements in an infinite set cannot determine.

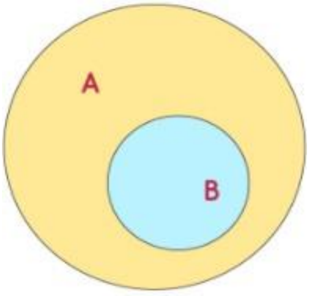
Example :

The set of integers, Z = {……… -2, -1, 0, 1, 2, ……….}

Finite Sets	Infinite Sets
All finite sets are countable.	Infinite sets can be countable or uncountable.
The union of two finite sets is finite.	The union of two infinite sets is infinite.
A subset of a finite set is finite.	A subset of an infinite set may be finite or infinite.
The power set of a finite set is finite.	The power set of an infinite set is infinite.

What is a Superset?

- Set A is called the **superset** of another set B if all elements of set B are elements of set A.
- This superset relationship is denoted as $A \supset B$.



A is the superset of B

Example

If A = {a, b, c, d, e, f, g}
B = {a, b, c, d},
then $A \supset B$.

Example

Let's consider the two sets to be,

set A = {11, 12, 13, 14, 15, 16} and

set B = {11, 13, 15}.

Power Set of a Set

- The power set of a set is a set of all the subsets (along with the empty set and the original set).
- The power set of a set A is denoted by P(A).
- If A has 'n' elements then P(A) has 2^n.

Examples :

If A = {1, 2},

then P(A) = { { }, {1}, {2}, {1, 2} }

Examples :

If A = {a, b, c},

then P(A) = { { }, {a}, {b}, {c}, {a, b}, {b, c}, {c, a}, {a, b, c} }

Important Formulas

If a set A has **'n'** elements:

- The number of subsets of A = 2^n.
- Hence, the number of elements of the power set of A (P(A)) is 2^n.
- The number of proper subsets of A is $2^n - 1$.
- The number of improper subsets of A is 1.

Universal Set

- The universal set is the set of all elements of all related sets.
- It is usually denoted by the symbol U.
- A universal set can be either a finite or infinite set.

Example :

Let's consider three sets, A, B, and C,

A = {2, 4, 6},

B = {1, 3, 7, 9, 11}, and

C = {4, 8, 11}.

U = {1, 2, 3, 4, 6, 7, 8, 9, 11}

- ❖ **Singleton Set**
- ❖ **Equivalent set**
- ❖ **Equal sets**

Singleton Set :

A set that has only one element is called singleton set.

Example :

A = {0}

Equivalent set :

- Any two sets are said to be equivalent set if their cardinality is same.
- In other words, if two sets having equal number of elements then it is called equivalent sets.
- Example :

A = {2, 4, 6},

B = {1, 7, 3},

Here, A & B are equivalent sets.

Equal sets :

- Any two sets are said to be equal if they are equivalent & their elements are identical.
- The cardinality of an equal sets is same.

Example

set A = {1, 2, 3, 4, 5}

B = {1, 2, 3, 4, 5}.

❑ Operation on sets:

❖ Union –

❖ Intersection –

❖ Complementation –

❖ Set Difference –

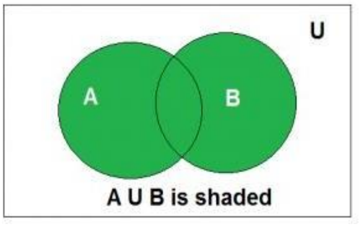
❖ Symmetric difference – problems relating simple set identities

What are Set Operations?

- Set operations are the operations that are applied on two or more sets to develop a relationship between them.
- There are four main kinds of set operations which are as follows.
  1) Union of sets
  2) Intersection of sets
  3) Complement of a set
  4) Difference between sets

Union of Sets(All)

- For two given sets A and B, AUB (read as A union B) is the set of distinct elements that belong to set A and set B or both.
- The number of elements in A U B is given by



Example: $$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

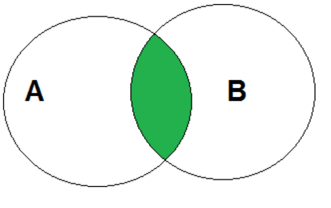
If A = {1, 2, 3, 4} and

B = {4, 5, 6, 7},

A ∪ B = {1, 2, 3, 4, 5, 6, 7}.

Intersection of Sets(Common)

- For two given sets A and B, A∩B (read as A intersection B) is the set of common elements that belong to set A and B.
- The number of elements in A∩B is given by



$$n(A \cap B) = n(A) + n(B) - n(A \cup B)$$

Example:

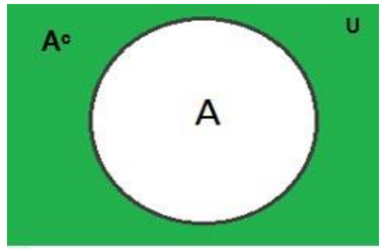
If A = {1, 2, 3, 4} and

B = {3, 4, 5, 7},

Then, **A ∩ B = {3, 4}.**

Complement of Sets(Absent)

- The complement of a set A denoted as A′ or A^c (read as A complement).
- It is defined as the set of all the elements in the given universal set(U) that are not present in set A.



Example:

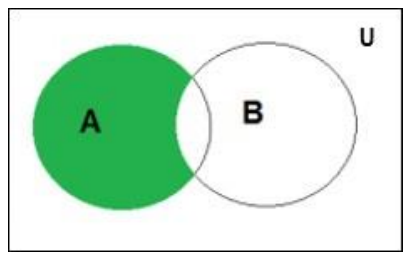
If U = {1, 2, 3, 4, 5, 6, 7, 8, 9} and

A = {1, 2, 3, 4},

A' = {5, 6, 7, 8, 9}.

Set Difference

- The difference between sets A and set B is denoted as A – B.
- It has list of all the elements that are present in set A but not in set B.



Example:

If A = {1, 2, 3, 4} and

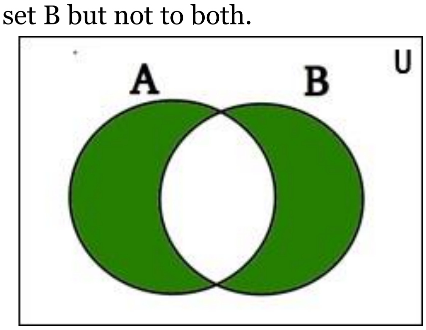
B = {3, 4, 5, 7},

Then,

A - B = {1, 2}.

Symmetric difference

- For two sets A and B, symmetric difference is expressed as A o B.
- A o B is the set of all those elements which belongs either set A or set B but not to both.



A o B = (A ∪ B) - (B ∩ A).

Or,

A o B = (A – B) ∪ (B – A)

Q1. If A = {1, 2, 4, 7, 9} and B = {2, 3, 7, 8, 9} then find A o B ?

Q2. If P = {a, c, f, m, n} and Q = {b, c, m, n, j, k} then find P o Q.

Q3. If A = {1, 2, 3, 4, 5, 6, 7, 8} and B = {1, 3, 5, 6, 7, 8, 9}, then find :

a) A – B
b) B – A and
c) A o B

Properties of Set Operations

- Commutative Law
- Associative Law
- Distributive Law :
- Absorption Law
- De-Morgan's Law
- Idempotents
- Identity
- Complements

Commutative Law

- $A \cup B = B \cup A$
- $A \cap B = B \cap A$

Absorption Law

- $A \cup (A \cap B) = A$
- $A \cap (A \cup B) = A$

Associative Law

- $(A \cup B) \cup C = A \cup (B \cup C)$
- $(A \cap B) \cap C = A \cap (B \cap C)$

Distributive Law :

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

De-Morgan's Law

- $(A \cup B)' = A' \cap B'$
- $(A \cap B)' = A' \cup B'$

Identity

- $A \cap \emptyset = \emptyset$
- $A \cup \emptyset = A$

Complements

- $A \cap A^c = \emptyset$
- $A \cup A^c = U$
- $(A^c)^c = A$
- $\emptyset^c = U$
- $U^c = \emptyset$

Idempotents

- $A \cap A = A$
- $A \cup A = A$

Cartesian Product :

❖ Let A & B are any two sets then the Cartesian product of set A and set B denoted as (A x B).

$$\mathbf{A \times B = \{(a, b) : a \in A \text{ and } b \in B\}}$$

❖The **Cartesian Product** of sets A and B is defined as the set of all ordered pairs (x, y) such that x belongs to A and y belongs to B.

Example :

if A = {1, 2} and B = {3, 4, 5},

A x B = {(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5)}.

Multiset :

- An unordered collection of objects, in which a object can appear more than once is called multiset.
- Example :

A = {a, a, b, b, c, d, d, d, k, k, k, k}

A = {2a, 2b, 1c, 3d, 4k}

Disjoint Sets

- Two sets are said to be disjoint if there are no common elements.
- When the intersection of the two sets is empty, then those sets are said to be disjoint sets.

Let's consider two distinct sets

A = {a, b} and B = {c, d}.

Relation Between two sets

Syllabus

- Binary relation as a subset of Cartesian product,
- Reflexive,
- Symmetric
- transitive relations – Examples,
- Equivalence relation – Examples.

Types of Relations

- Identity/Diagonal Relation
- Inverse Relation
- Reflexive Relation
- Symmetric Relation
- Transitive Relation
- Equivalence Relation

Relations :

- In mathematics a Relations is a subset of the Cartesian/cross product of two sets.
- If A and B are two sets, then every subset of (A x B) is called a relation from A to B.
- **Example :**

Let A = {a, b} and B = {1, 2, 3}. Find the number of relations from A to B.

A x B = {(a, 1),(a, 2),(a, 3),(b, 1),(b, 2),(b, 3)}

n(A) = 2 and n(B) = 3 n(A x B) = 2 x 3 = 6

Therefore,

The number of relation from set A to B = $2^{2x3} = 2^6 = 64$

No. of Relations :

If A and B are two sets having element m & n respectively then,

No. of relation possible from A to B = $2^{n(A)xn(B)} = 2^{m.n}$

Q. Let the number of elements of the sets A and B be p and q respectively. Then, the number of relations from the set A to the set B is

a) 2^{p+q}

b) 2^{pq}

c) p+q

d) pq

Q. Let A = {1,2} and B = {3,4}. Find the number of relations from A to B.

Q. Let n(A)=n. Then the number of all relations on A is

a) None

b) 2^n

c) $2^{(n)!}$

d) 2^{n2}

Relation Between two sets

Q. A and B are two sets having 3 and 4 elements respectively and having 2 elements in common. The number of relations which can be defined from A to B is

a) 2^5

b) $2^{10}-1$

C) $2^{12}-1$

d) None of these

Q7. If A={a,b,c,d}, B={1,2,3} find whether or not the following sets of ordered pairs are relations from A to B or not.

R1={(a,1),(a,3)}
R2={(a,1),(c,2),(d,1)}

R3={(a,1),(b,2),(3,c)}.

a) R1 R2 are relations but R3 is not a relation.

b) R1 R3 are relations but R2 is not a relation.

c) All are relations

d) None of these

Types of Relations

- Identity/Diagonal Relation
- Inverse Relation
- Reflexive Relation
- Symmetric Relation
- Transitive Relation
- Equivalence Relation

Identity/Diagonal Relation

- If all elements in a set are related to itself then it becomes an identity relation.
- It is written as I = {(x, x) : for all x ∈ A}.
- Example

A = {1, 2, 3}

A x A = {(1, 1), (1, 2), (1, 3),
(2, 1), (2, 2), (2, 3),
(3, 1), (3, 2), (3, 3),**}**

Q. A = {3, 7, 9} then
I = {(3, 3), (7, 7), (9, 9)}

Inverse Relation

- An inverse relation is obtained by interchanging the elements of each ordered pair of the given relation.
- Let R be a relation from a set A to set B.
- Then R is of the form $\{(x, y) : x \in A \text{ and } y \in B\}$.
- The inverse relationship of R is denoted by R^{-1} and its formula is $R^{-1} = \{(y, x): y \in B \text{ and } x \in A\}$.
- The inverse of a relation R is denoted as R^{-1}.

i.e., $\mathbf{R^{-1} = \{(y, x) : (x, y) \in R\}}$

$\mathbf{= (A \times B) - R}$

$\mathbf{A \times A = \{}(1, 1), (1, 2), (1, 3),$
$(2, 1), (2, 2), (2, 3),$
$(3, 1), (3, 2), (3, 3),\mathbf{\}}$

Examples :

A = {a, b, c, d, e} and B = {1, 2, 3, 4, 5}.

- If $R = \{(a, 2), (b, 4), (c, 1)\} \Leftrightarrow$ $R^{-1} = \{(2, a), (4, b), (1, c)\}$
- If $R = \{(c, 1), (b, 2), (a, 3)\} \Leftrightarrow$ $R^{-1} = \{(1, c), (2, b), (3, a)\}$
- If $R = \{(b, 3), (c, 2), (e, 1)\} \Leftrightarrow$ $R^{-1} = \{(3, b), (2, c), (1, e)\}$

Example 1: Find the inverse of the following relations:

a) $R = \{(2, 7), (8, 3), (5, 5), (4, 3\})$ and

b) $R = \{(x, x^2): x \text{ is a prime number less than } 15\}$. Find the domain and range in each of these cases.

Solution:

a) $R^{-1} = \{(7, 2), (3, 8), (5, 5), (3, 4)\}$.

b) Let us write the given relation in roster form.

The list of prime numbers less than 15 are 2, 3, 5, 7, 11, and 13. Thus,

$R = \{(2, 2^2), (3, 3^2), (5, 5^2), (7, 7^2), (11, 11^2), (13, 13^2)\} = \{(2, 4), (3, 9), (5, 25), (7, 49), (11, 121), (13, 169)\}$

Now,

$R^{-1} = \{(x^2, x): x \text{ is a prime number less than } 15\}$

$= \{(4, 2), (9, 3), (25, 5), (49, 7), (121, 11), (169, 13)\}$.

Reflexive Relation

- A binary relation R defined on a set A is said to be reflexive if, for every element a ∈ A, we have aRa, that is, (a, a) ∈ R.
- This implies that a relation defined on a set is a reflexive relation if and only if every element of the set is related to itself.
- Example :

  Let A = {a, b, c, d, e} and R is a relation defined on A as R = {(a, a), (a, b), (b, b), (c, c), (d, d), (e, e),

  (c, e)}.

Example :

Set A = {1, 2, 3} then relation

R = {(1, 1),(2, 2),(3, 3),(1, 2),(2, 3)} is a reflexive.

Symmetric Relation

- A relation is said to be a symmetric relation if one set, A, contains ordered pairs, (x, y) as well as the reverse of these pairs, (y, x).
- In other words, if (x, y) ∈ R then (y, x) ∈ R for the relation to be symmetric.
- Example :

  A = {3, 4},

  then a symmetric relation can be R = {(3, 4), (4, 3)}.

Example :

Set A = {1, 2, 3} then relation

R1 = {(1, 1),(2, 2),(1, 2),(2, 1)}

R1 = {(1, 1),(1, 3),(3, 1)}

R1 = {(1, 1),(2, 2),(3, 2),(2, 3)}

Transitive Relation

- Suppose (x, y) ∈ R and (y, z) ∈ R then R is a transitive relation if and only if (x, z) ∈ R.
- Example,

  P = {p, q, r},

  then a transitive relationcan be R = {(p, q), (q, r), (p, r)}

Equivalence Relation

- An equivalence relation is a type of relation that is reflexive, symmetric and transitive.

Example :

$A = \{1, 2, 3\}$ then

$R_1 = \{(1, 1),(2, 2),(3, 3)\}$

$R_2 = \{(1, 1),(2, 2),(3, 3),(2, 1),(1, 2)\}$

$R_3 = \{(1, 1),(2, 2),(3, 3)\}$

Note :

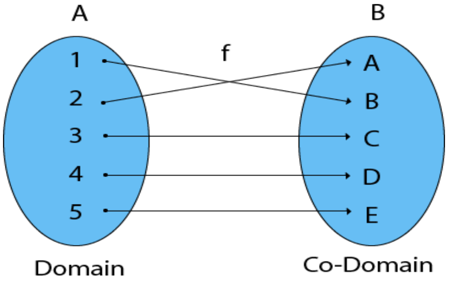
❖ Smallest equivalence relation on A = Diagonal relation

❖ Largest equivalence relation on A = A x A

Unit : 04
Function

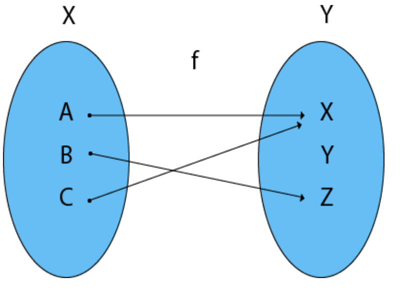
- ➢ Definition of function
- ➢ Domain,
- ➢ Co-domain &
- ➢ Range of a function –Related problems.

Function :



- A relation from a set A to set B is called a function if each element of a set A assign a unique element in set B.
- Every function is relation but every relation is not a function.
- The set of A is called Domain and set of B is called Co-domain of a function.

Representation of a Function



- The two sets A and B are represented by two circles.
- The function f: X → Y is represented by a collection of arrows joining the points which represent the elements of X and corresponds elements of Y.

Example :

Let X = {a, b, c} and Y = {x, y, z}

and f: X → Y such that

f = {(a, x), (b, z), (c, x)}

Number of function possible :

- If n(A) = m and n(B) = n

Number of function possible from A to B = $\mathbf{n^m}$

Example: If a set A has n elements, how many functions are there from A to A?
Solution: If a set A has n elements, then there are n^n functions from A to A.

Domain, Co-Domain, and Range of a Function: Domain of a Function:

- Let f be a function from A to B.
- Then the every elements of a set A is called the domain of the function f.

Co-Domain of a Function:

- Let f be a function from A to B.
- Then the every elements of a set B is called Co-domain of the function f.

Range of a Function:

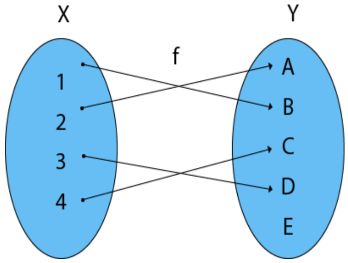
- The range of a function is the set of picture of its domain.
- In other words, we can say it is a subset of its co-domain.

Example: Find the Domain, Co-Domain, and Range of function.

Let x = {1, 2, 3, 4}

y = {a, b, c, d, e}

f = {(1, b), (2, a), (3, d), (4, c)}



Solution:

Domain of function: {1, 2,

3, 4} Range of function:

{a, b, c, d}

Co-Domain of function: {a, b, c, d, e}

Example2: Let X = {x, y, z, k} and Y = {1, 2, 3, 4}. Determine which of the following functions.

Give reasons if it is not. Find range if it is a function.

f = {(x, 1), (y, 2), (z, 3), (k, 4)

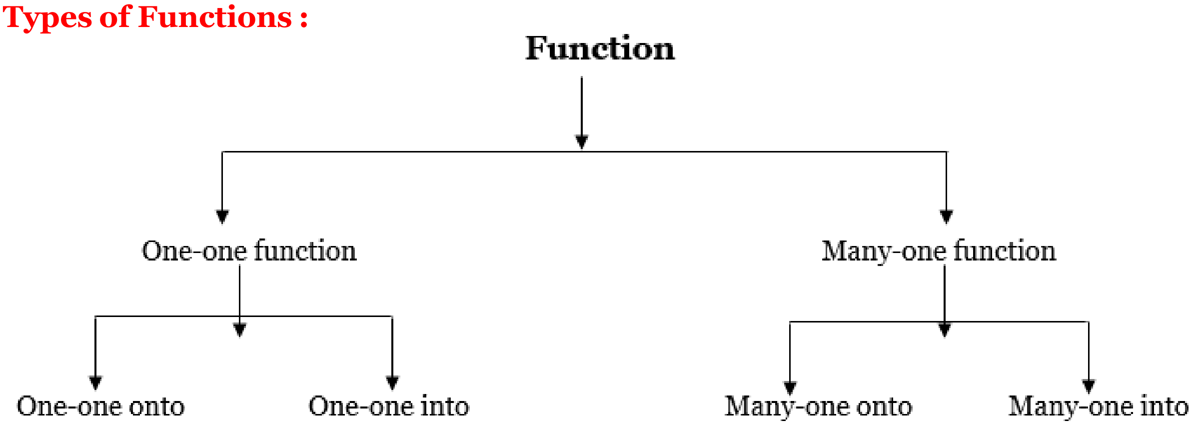
g = {(x, 1), (y, 1), (k, 4)

h = {(x, 1), (x, 2), (x, 3), (x, 4)

l = {(x, 1), (y, 1), (z, 1), (k, 1)}

d = {(x, 1), (y, 2), (y, 3), (z, 4), (k, 4)}.

Types of Functions :



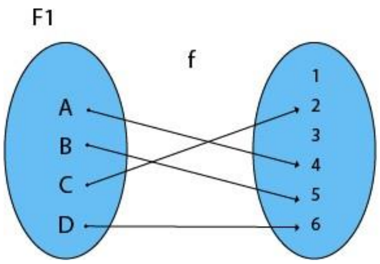
One-one function :

❖ A function in which one element of Domain is connected to one element of Co-Domain is called one-one.

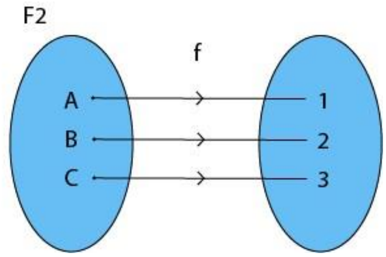
❖ The one-to-one function is also called an injective function.

Example :

Unit : 04
Function



One-one into function



One-one onto function

Example 1: Let D = {3, 4, 8, 10} and C = {w, x, y, z}. Which of the following relations represent a one to one function?

{(3, w), (3, x), (3, y), (3,z)}

{(4,w), (3,x), (10,z), (8, y)}

{(4,w), (3,x), (8,x), (10, y)}

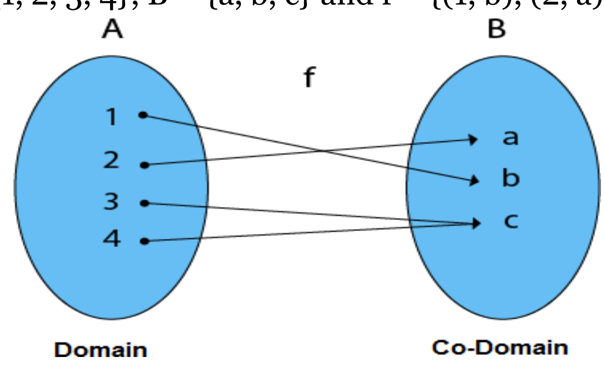
Answer: Thus, {(4,w), (3,x), (10,z), (8, y)} represent a one to one function.

Onto Functions:

- A function in which every element of Co-Domain Set has one pre-image.
- It is also known as surjective function.
- If a function is a Surjective Function, as every element of B is the image of some A.

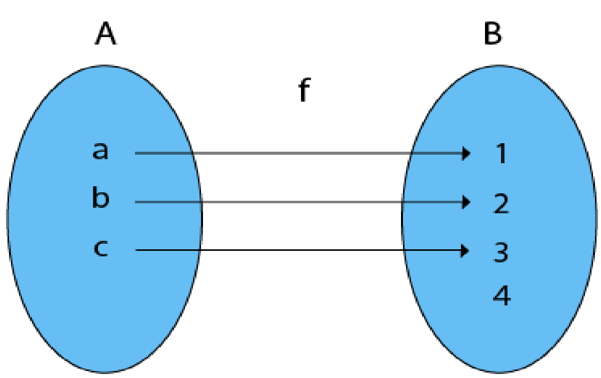
Example:

Consider, A = {1, 2, 3, 4}, B = {a, b, c} and f = {(1, b), (2, a), (3, c), (4, c)}.



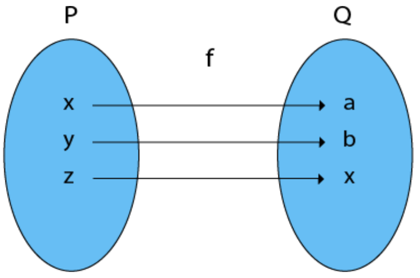
Into Functions :

- A function in which there must be an element of co-domain Y does not have a pre-image in domain X.



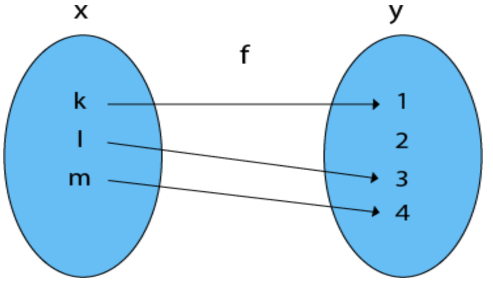
One-to-One Onto Functions:

- A function which is both injective (one to - one) and surjective (onto) is called bijective (One-to-One Onto) Function.
- It is also known as Bijective function.



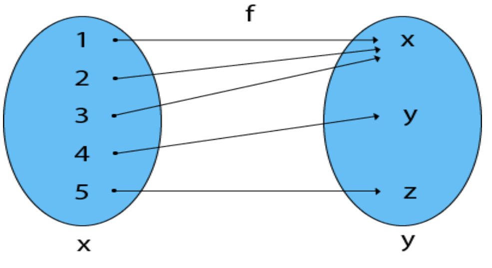
One-One Into Functions:

- Let f: X → Y.
- The function f is called one-one into function if different elements of X have different unique images of Y.



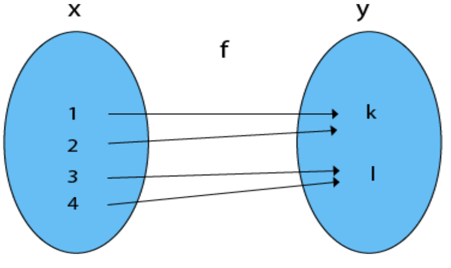
Many-One Functions:

- Let f: X → Y.
- The function f is said to be many-one functions if there exist two or more than two different elements in X having the same image in Y.



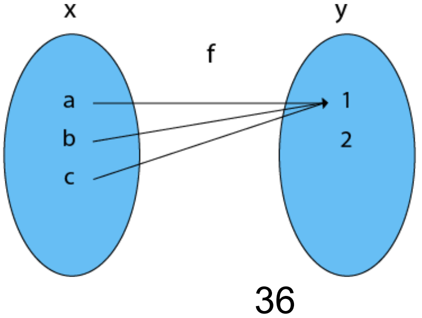
Many-One Onto Functions:

- Let f: X → Y.
- The function f is called many-one onto function if and only if is both many one and onto.



Many-One Into Functions:

- Let f: X → Y.
- The function f is called the many-one function if and only if is both many one and into function.



Example 2: Determine if $g(x) = -3x^3 - 1$ is a one to one function using the algebraic approach.

Solution:

In order for function to be a one to one function, $g(x_1) = g(x_2)$ if and only if $x_1 = x_2$.

Let us start solving now: $g(x_1) = -3x_1^3 - 1$

$g(x_2) = -3x_2^3 - 1$

We will start with $g(x_1) = g(x_2)$. Then

$-3x_1^3 - 1 = -3x_2^3 - 1$

$-3x_1^3 = -3x_2^3$ $(x_1)^3 = (x_2)^3$

Removing the cube roots from both sides of the equation will lead us to $x_1 = x_2$.

Answer: Hence, $g(x) = -3x^3 - 1$ is a one to one function.

Unit - 05

Recurrence Relation

- Definition – Examples (Fibonacci, Factorial etc.),
- Linear recurrence relations with constants coefficients –
- Homogenous solutions –
- Particular solutions –
- Total solutions – Problems.

Recurrence Relation:

- An equation that next term express in terms of one or more of the previous term of the sequence.
- A recurrence relation is an equation that recursively defines a sequence.

Example :

Concider a sequence of real numbers

$$\{ a_0, a_1, a_3, a_4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ a_n \ldots\ldots\ldots \}$$

- A formula which relates $\mathbf{a_n}$ with one or more preceeding terms is called recurrence relation.
- recurrence relation also related to Fibonacci series.

Exp :

$\{0, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\}$

$$\mathbf{F_n = F_{n-1} + F_{n-2}}$$

Where,

$n \geq 2$ and $F_1 = 0$

and $F_2 = 1$

Recurrence Relations of the form :

$$\mathbf{C_0a_n + C_1a_{n-1} + C_2a_{n-2} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + C_ka_{n-k} = f(n)}$$

Where,

$C_0, C_1, C_2 \ldots\ldots C_n$ are constant and F(n) is a function.

Order : The difference between highest subscript & lowest subscript of the relation.

In the above equation,

Order = Highest subscript – Lowest subscript

$= n - (n - k)$

$= k$

So, the given equation is k^{th} order.

Unit - 05

Recurrence Relation

Recurrence Relations

Linear recurrence relation

f(n) = 0

Non-Linear recurrence relation

f(n) ≠ 0

Note : If f(n) = 0, then the equation (i) is called homogeniouslinear recurrence relation otherwise equation (i) is said to be non-homogenious recurrence relation.

Solution of recurrence relation :

1. Sum rule
2. Substitution method
3. Method of characteristics root

Sum rule :

Q1. T(p) = T(p-1) + 5 Find T(3) when T(0) = 2.

Q2. $a_n = a_{n-1} + a_{n-2}$ (n>2) then find a_5 where, a_1=2 and a_2=3

Substitution method :

Q. Find the Solution of recurrence relation $a_n = n \cdot a_{n-1}$ where $n>1$ and $a_0=1$.

Soln:

$$a_n = n \cdot a_{n-1}$$

put $n=1 \Rightarrow a_1 = 1 \cdot a_{1-1} = 1 \cdot a_0 = 1 \cdot 1 = 1$ $\{a_0 = 1\}$

$n=2 \Rightarrow a_2 = 2 \cdot a_{2-1} = 2 \cdot a_1 = 2 \cdot 1$

$n=3 \Rightarrow a_3 = 3 \cdot a_{3-1} = 3 \cdot a_2 = 3 \cdot 2 \cdot 1$

$\vdots$

$a_n = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$

$\therefore \boxed{a_n = \underline{|n}}$

Q. Find the Solution of recurrence relation $a_n = a_{n-1} + n$ where $a_0 = 2$ and $n \geq 1$.

Soln:

$$a_n = a_{n-1} + n$$

put $n=1 \Rightarrow a_1 = a_0 + 1$

$n=2 \Rightarrow a_2 = a_1 + 2$
$= a_0 + 1 + 2$

$n=3 \Rightarrow a_3 = a_2 + 3 = a_0 + 1 + 2 + 3$

$\vdots$

$n=n \quad a_n = a_0 + 1 + 2 + 3 + \cdots n$

$= a_0 + (1 + 2 + 3 + \cdots n)$

$= a_0 + \frac{n(n+1)}{2}$

$\boxed{a_n = 2 + \frac{n(n+1)}{2}}$ $\{a_0 = 2 \text{ Given}\}$

Method of characteristics root

Step-1: Find the order of given recurrence relation

$$a_n \longrightarrow x^2$$
$$a_{n-1} \longrightarrow x$$
$$a_{n-2} \longrightarrow 1$$

Step-2: Generate the characteristics equation, simplify it & find the roots.

Step-3: Solution

$$a_n = A(\text{root}1)^n + B(\text{root}2)^n$$

where, A and B are arbitrary Constant.

Q. Solve the following recurrence relation:

(i) $a_n = 6a_{n-1} - 8a_{n-2}$ (ii) $a_n = 4a_{n-1} - 4a_{n-2}$

Sol^n: (i) $a_n = 6a_{n-1} - 8a_{n-2}$

Step-1: order = highest subscript − lowest subscript

$$= n-(n-2) = \cancel{n}-\cancel{n}+2$$
$$= 2$$

Step-2:

$$a_n = 6a_{n-1} - 8a_{n-2}$$
$$x^2 = 6x - 8$$
$$\therefore x^2 - 6x + 8 = 0$$
$$x^2 - 4x - 2x + 8 = 0$$
$$x(x-4) - 2(x-4) = 0$$
$$(x-2)(x-4) = 0$$
$$\boxed{x = 2, 4}$$

Step-3:

$$a_n = A(2)^n + B(4)^n$$

OR,

$$a_n = A(4)^n + B(2)^n$$

Ans

Unit – 06 : Graph Theory

- Introduction –
- Definition of a graph –
- sub graph –
- Isomorphism-
- Walk, Paths and circuits –
- connectedness and components –
- Euler graphs.

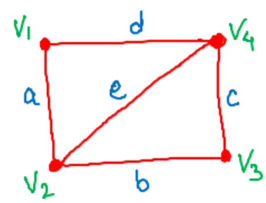
Graph Theory :

- Graph Theory is the study of points and lines.
- It is a pictorial representation that represents the Mathematical truth.
- Graph theory is the study of relationship between the vertices (nodes) and edges (lines).
- It was introduced by the famous Swiss mathematician named **Leonhard Euler**.

What is Graph ?

- A graph is a set of vertices and edges.
- It is a pictorial representation of any data in an organised manner.
- Vertices and edges are known as nodes & lines respectively.
- Each edge is associated with two endpoints called nodes.
- A graph G is denoted as G(V, E).

V = Set of vertices(nodes) E = Set of Edges(lines)



$V=\{v_1, v_2, v_3, v_4\}$

$E=\{a, b, c, d, e\}$

Note : A graph with no edges is called null graph or empty graph.

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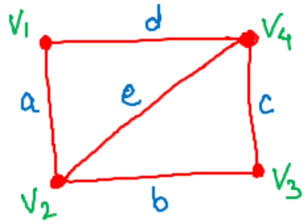

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Note : A graph with no edges is called null graph or empty graph.

$E=\{a, b, c, d, e\}$

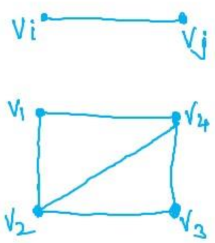
Unit – 06 : Graph Theory

Types of graph :

Undirected graph

In a undirected graph each edge is repersented by a set of two vertices such as $\{v_i, v_j\}$.

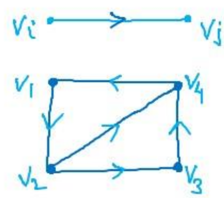
(v_i, v_j) = An edge between v_i & v_j.



Directed graph

In a directed graph each edge is repersented by $\{v_i, v_j\}$.

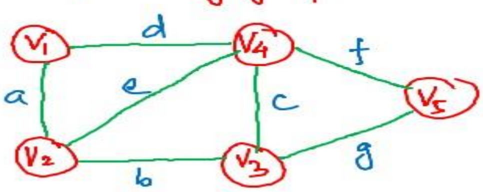
(v_i, v_j) = An edge from v_i to v_j.



Sub-Graph :

A graph G'=(V', E') is a subgraph of another graph G=(V, E) iff V(G') ⊆ V(G), and E(G') ⊆ E(G).

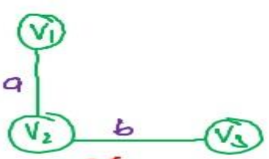
In the following graph G(V,E)



$V(G) = \{v_1, v_2, v_3, v_4, v_5\}$

$E(G) = \{a, b, c, d, e, f, g\}$

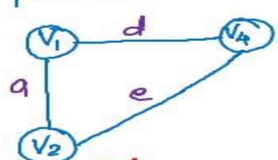
Some possible sub-graph is :—



G_1'

$V(G_1') = \{v_1, v_2, v_3\}$

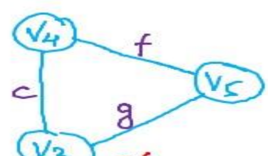
$E(G_1') = \{a, b\}$



G_2'

$V(G_2') = \{v_1, v_2, v_4\}$

$E(G_2') = \{a, d, e\}$



G_3'

$V(G_3') = \{v_3, v_4, v_5\}$

$E(G_3') = \{c, g, f\}$

Isomorphic Graph

Two graphs G_1 and G_2 are said to be isomorphic if they must hold the following Conditions —

<I> No. of vertices in G_1 = No. of vertices in G_2.

<II> No. of Edges in G_1 = No. of Edges in G_2.

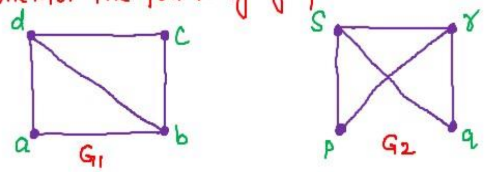
<III> Degree sequence of G_1 and G_2 are same.

<IV> Complement of G_1 and G_2 are also same.

$$G_1 \cong G_2$$

Read as isomorphic

Q. Find whether the following graphs are isomorphic or not?



Sol^n :-

Step-1: $|V(G_1)| = |V(G_2)| = 4$

Step-2: $|E(G_1)| = |E(G_2)| = 5$

Step-3: Degree sequence are also same

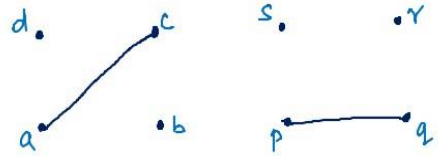
$G_1 \rightarrow \{3,3,2,2\}$

$G_2 \rightarrow \{3,3,2,2\}$

Step-4: Taking Complement of G_1 & G_2, we have

Complement

G_1' G_2'

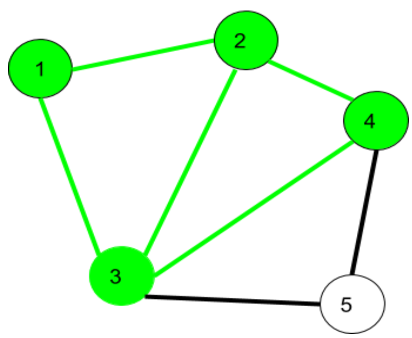


$\therefore \overline{G_1} \cong \overline{G_2}$

$\therefore$ Graph G_1 and G_2 are isomorphic graphs.

Walk :

- A walk is a sequence of vertices and edges of a graph.
- If we traverse a graph then we get a walk.
- In walk Vertices and Edges can be repeated.



Here,

1->2->3->4->2->1->3 is a walk.

Walk can be open or closed.

Open walk :

A walk is said to be an open walk if the starting and ending vertices are different i.e. the origin vertex and terminal vertex are different.

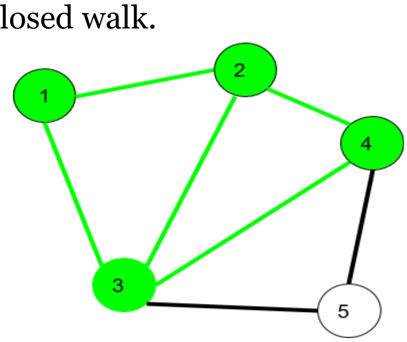
Closed walk :

A Walk is said to be a closed walk if the starting and ending vertices are identical i.e. if a walk starts and ends at the same vertex, then it is said to be a closed walk.

In the above diagram:

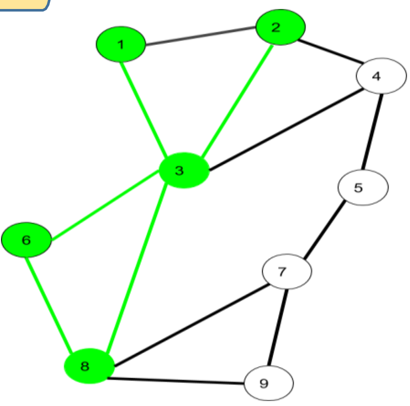
1->2->3->4->5->3 is an open walk.

1->2->3->4->5->3->1 is a closed walk.



Trail :

- Trail is an open walk in which no edge is repeated.
- Vertex can be repeated.

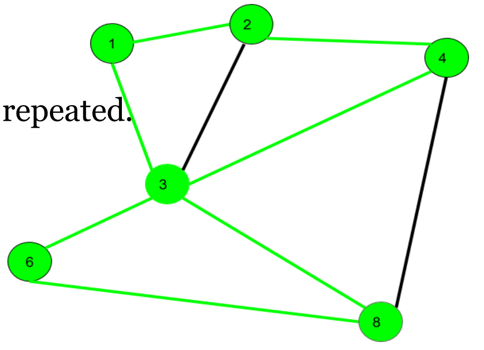


Here 1->3->8->6->3->2 is trail

Also 1->3->8->6->3->2->1 will be a closed trail

Circuit :

- Traversing a graph such that not an edge is repeated but vertex can be repeated and it is closed also is called circuit.
- It is a known as closed trail.
- In circuit Vertex can be repeated but Edge can not be repeated.



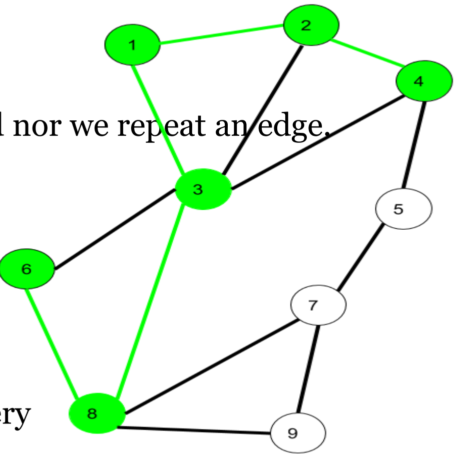
Here 1->2->4->3->6->8->3->1 is a circuit.

Circuit is a closed trail.

These can have repeated vertices only.

Path :

- In path both Vertex and Edge can not be repeated.
- It is a trail in which neither vertices nor edges are repeated.
- If we traverse a graph such that we do not repeat a vertex and nor we repeat an edge.
- As path is also a trail, thus it is also an open walk.



Here,

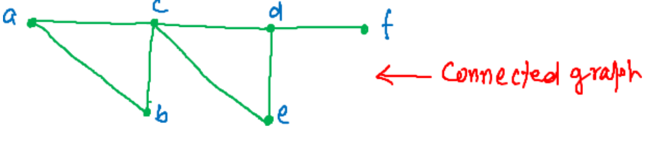
6->8->3->1->2->4 is a Path

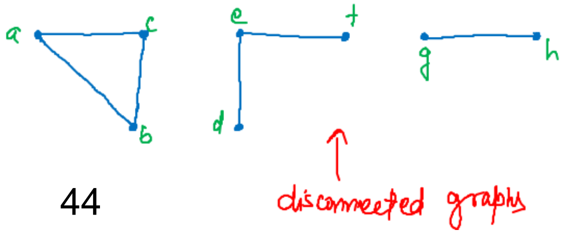
Connected graph :

- A graph is said to be connected if there exist a path between every pair of vertices.

Connectedness/Disconnected graph :

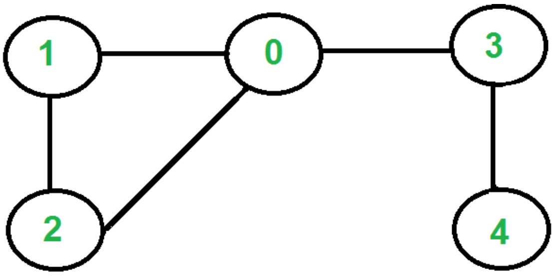
- A graph which is not connected is called disconnected graph.
- It has two or more connected components.



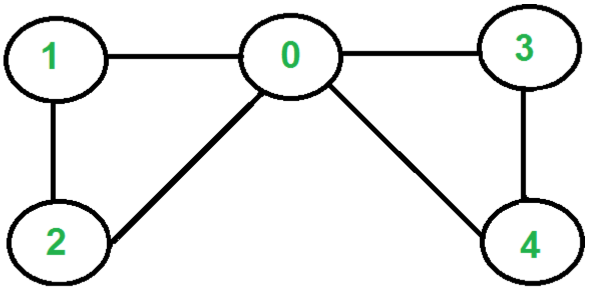


Euler graph :

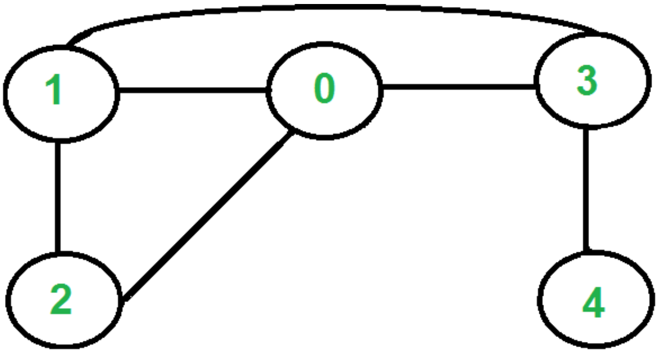
- Any graph is called Euler graph if must exist a Euler path and Euler circuit in a graph.
  1. Eulerian Path is a path in graph that visits every edge exactly once.
  2. Eulerian Circuit is an Eulerian Path which starts and ends on the same vertex.



The graph has Eulerian Paths, for example "4 3 0 1 2 0", but no Eulerian Cycle. Note that there are two vertices with odd degree (4 and 0)



The graph has Eulerian Cycles, for example "2 1 0 3 4 0 2" Note that all vertices have even degree



The graph is not Eulerian. Note that there are four vertices with odd degree (0, 1, 3 and 4)

Unit-07 : Counting:

- Introduction –
- Basic counting principles,
- Factorial Notation,
- Binomial coefficients,
- Permutations,
- Combinations,
- The pigeonhole principle,

What is Counting ?

- To understand the number of all possible outcomes for a series of events is called counting.
- The different ways in which 10 lettered PAN numbers can be generated in such a way that the first five letters are capital alphabets and the next four are digits and the last is again a capital letter.
- This can be calculated by using the mathematical theory of Counting.
- The fundamental counting rule, permutation rule and the combination rule is used by Counting.

What are the rules of Sum and Product?

- In order to decompose the difficult counting problems into simple problems
  - Rule of Sum(Sum rule) and
  - Rule of Product(Product rule).

Basic Counting Principles

Sum Rule Principle:

- Assume some event E can occur in m ways and a second event F can occur in n ways, and suppose both events cannot occur simultaneously.
- Then E or F can occur in (m + n) ways.

Example:

If 8 male professor and 5 female professor teaching DMS then the student can choose professor in 8+5=13 ways.

Product Rule Principle:

- Suppose there is an event E which can occur in m ways and, independent of this event, there is a second event F which can occur in n ways.

- Then combinations of E and F can occur in (m x n) ways.

Example: In class, there are 4 boys and 10 girls if a boy and a girl have to be chosen for the class monitor, the students can choose class monitor in 4 x 10 = 40 ways.

Factorial Function:

- The product of the first n natural number is called factorial n.
- It is denoted by n!, read "n Factorial."
- The Factorial n can also be written as n!

$$n! = n\,(n-1)\,(n-2)\,(n-3)\ldots\ldots 1.$$

$$0! = 1 \quad \text{and} \quad 1! = 1.$$

Example1: Find the value of 5!

Solution:

$$5! = 5 \times (5-1)\,(5-2)\,(5-3)\,(5-4)$$
$$= 5 \times 4 \times 3 \times 2 \times 1 = 120$$

Example2: Find the value of $\frac{10!}{8!}$.

Solution: $\frac{10!}{8!} = \frac{10 \times 9 \times 8!}{8!} = 10 \times 9 = 90$

Binomial Coefficients:

- Binomial Coefficient is represented by n_{C_r} where r and n are positive integer with r ≤ n is defined as follows:

$$\binom{n}{r} = \frac{n\,(n-1)(n-2)\ldots\ldots\ldots\ldots(n-r+1)}{1 * 2 * 3 \ldots\ldots\ldots\ldots(r-1)\,r}$$

Example:

$$8_{C_2} = \frac{8!}{2!\,6!} = \frac{8 \times 7 \times 6}{2 \times 6} = 28.$$

What are Permutations?

- A permutation is an arrangement of some elements in which order matters.
- In other words a Permutation is an ordered Combination of elements.

Examples

- From a set S ={x, y, z} by taking two at a time, all permutations are –

  xy, yx, xz, zx, yz, zy.

- A permutation of three digit numbers from the set of numbers S={1,2,3} can be as –

  123, 132, 213, 231, 312, 321.

Number of Permutations

- The number of permutations of 'n' different things taken 'r' at a time is denoted $^{n}P_{r}$ by

$$n_{P_r} = \frac{n!}{(n-r)!}$$

where **n! = 1.2.3....(n−1).n**

Example: $4 \text{ x } {}^{n}p_{3} = {}^{n+1}P_{3}$

Solution:

$$4 \text{ x } \frac{n!}{(n-3)!} = \frac{(n+1)!}{(n+1-3)!}$$

$$\frac{4 \text{ x } n!}{(n-3)!} = \frac{(n+1) \text{x } n!}{(n-2)(n-3)!}$$

$$4\,(n-2) = (n+1)$$

$$4n - 8 = n+1$$

$$3n = 9$$

$$n = 3.$$

Some important formula :

- Number of permutations of n distinct elements taking n elements at a time = $^{n}P_{n}$
- If there are n elements in which a1 are of some kind, a2 are of another kind; a3 are alike of third kind and so on and a_r are of r^{th} kind,

  Where, (a1 + a2 + ...+ ar) = n

  then, number of permutations of these n objects is

$$\frac{n!}{a1!\, a2! \ldots\ldots\ldots ar!}$$

Problems on permutations.

Problem 1 – In how many ways a bunch of 6 different cards can be permutated?

Solution – As 6 cards are taken at a time from a deck of 6 cards, the permutation will be

$$^{n}P_{n} = 6\,! = 720$$

Problem 2 – In how many different ways the letters of the word 'READER' can be arranged?

Solution – There are 6 letters word (2 E, 1 A, 1D and 2R.) in the word 'READER'.The permutation will be

= 6!/[(2!)(1!)(1!)(2!)]

= 180.

What are Combinations?

- The selection of some of the given elements by not considering the order is known as a combination.
- The number of all combinations of n things, taken r at a time is –

$$^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

Problem 1 : Find the number of subsets of the set {1,2,3,4,5,6} having 3 elements.

Solution : The cardinality of the set is 6. As the ordering does not matter, the number of subsets is

$$^{6}C_{3} = 20$$

Problem : Out of 6 men and 5 women in a room, in how many ways can be 3 men and 2 women chosen?

Solution

3 men from 6 men can be chosen in

$^{6}C_{3}$ways

and 2 women from 5 can be chosen in

$^{5}C_{2}$ways

Hence, the total number of ways is $^{6}C_{3} \times {}^{5}C_{2} = 20 \times 10 = 200$

The Pigeonhole Principle

- If n pigeonholes are occupied by n+1 or more pigeons, then at least one pigeonhole is occupied by greater than one pigeon.
- Generalized pigeonhole principle is: -
  - If n pigeonholes are occupied by **kn+1** or more pigeons,
  - where k is a positive integer, then at least one pigeonhole is occupied by **k+1** or more pigeons.

Q1: Find the minimum number of students in a class to be sure that three of them are born in the same month.

Solution:

Here,

$n = 12$ months are the Pigeonholes

$k + 1 = 3$

$k = 2$

$kn + 1 = 2x12 + 1 = \mathbf{25}$

Q2: Find the minimum number of teachers in a school to be sure that four of them are born in the same month.

Solution:

Here,

$n = 12$ months are the Pigeonholes

$k + 1 = 4$

$k = 3$

$kn + 1 = 3x12 + 1 = \mathbf{37}$

Q3: A box contain 10 blue balls, 20 red balls, 8 green balls, 15 yellow balls and 25 white balls. How many balls must be choose to ensure that we have 12 balls of the same colour.

Solution :

$n = 5$

$k + 1 = 12$

$k = 11$

$\mathbf{kn + 1 = 11 \times 5 + 1 = 56}$

Unit-08 : Probability Theory :

1. Introduction,
2. Sample space and Events,
3. Finite probability spaces,
4. Conditional probability,
5. Independent Events,
6. Independent Repeated Trials,
7. Binomial Distribution,
8. Random variables.

Unpredictable: Unknown in advance.

Random experiment:

- Unpredictable outcomes of an experiments is known as random experiments.
- Example:
    - Tossing a coin
    - Rolling a dice
    - Drawing a card

Sample Space:

- The collection of all possible outcomes of an experiment is known as sample space.
- It is denoted by S.

Events:

- The outcomes of an experiments are known as events.
- Mathematically events is the subset of sample space.

Probability:

- The probability of an events defined as the ratio between the favorable number of cases to the total number of outcomes of an experiment.

$$\boxed{P(E) = \frac{m}{n}}$$

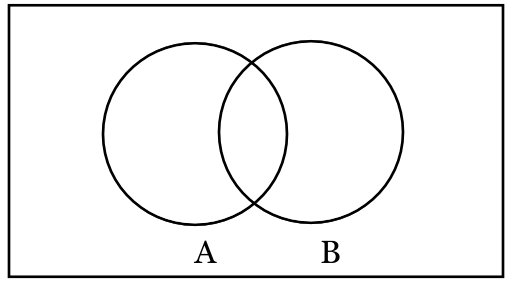
Where, m = Favorable number of cases

n = total number of outcomes = sample space

Rules of Probability:

- P (Sample Space) = 1
- $0 \leq P(E) \leq 1$
- For impossible events P(E) = 0.

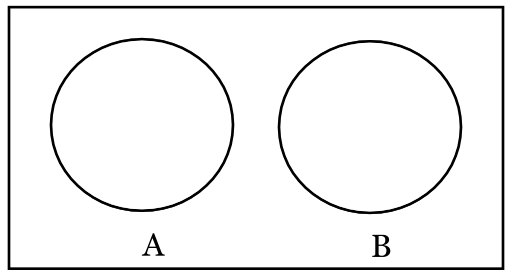
Dependent Events :



$$\boxed{\mathbf{P(A \cap B) = P(A) + P(B)}}$$

Mutually Exclusive :

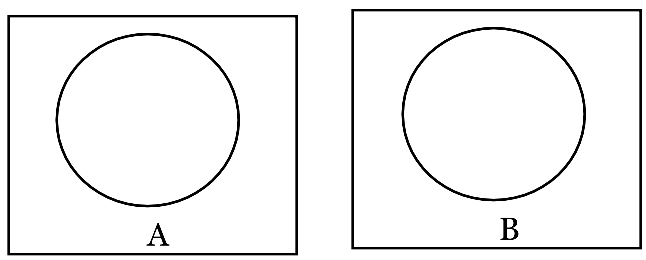
- Occurence of one events does not depends upon the occurrence of other events in the same sample space then those events are known as mutually exclusive events.



$$\boxed{\mathbf{P(A \cap B) = 0}}$$

Independent Events :

- Occurence of one events does not depends upon the occurrence of other events in the different sample space then those events are known as independent events.



$$\boxed{\mathbf{P(A \cap B) = P(A) \,.\, P(B)}}$$

Addition Theorem :

- **If A and B are two events :**

$$P(A \text{ or } B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- **If A and B are two mutually exclusive events :**

$$P(A \cup B) = P(A) + P(B)$$

Conditional Probability/Multiplication Theorem :

$$P(A \cap B) = P(A) \,.\, P(B/A)$$

$$= P(B) \,.\, P(A/B)$$

Therefore,

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Q. If $P(A) = 7/13$, $P(B) = 9/13$ and $P(A \cap B) = 4/13$, evaluate $P(A|B)$.

Solution:

$$P(A|B) = P(A \cap B)/P(B)$$
$$= (4/13)/(9/13)$$
$$= 4/9.$$

Q. If $P(A) = 1/3$, $P(B) = 1/4$ and $P(A \cup B) = 5/12$, Find $P(A/B)$ and $P(B/A)$?

Q. If A and B are two events and $P(A) = 3/8$, $P(B) = 1/2$, $P(A \cap B) = 1/4$ then the value of $P(A^c \cap B^c) = K/8$, find the value of K. [$P(A^c \cap B^c) = 1 - P(A \cup B)$]

Probability formula with the complementary rule :

- Whenever an event is the complement of another event.
- If A is an event, then

$$P(\text{not } A) = 1 - P(A)$$

or

$$P(A') = 1 - P(A).$$
$$P(A) + P(A') = 1.$$

Tossing a Coin :

- A single coin on tossing has two outcomes, a head, and a tail.
- Total number of possible outcomes = 2.
- Sample Space = {H, T}; H: Head, T: Tail

P(H) = Number of heads/Total outcomes = 1/2P(T)=

Number of Tails/ Total outcomes = 1/2

Tossing Two Coins

- In the process of tossing two coins, we have a total of four outcomes.
- Total number of outcomes = 4.
- Sample Space = {(H, H), (H, T), (T, H), (T, T)}

Tossing Three Coins

- The number of total outcomes on tossing three coins simultaneously is equal to $2^3 = 8$.
- Total number of outcomes = $2^3 = 8$

Sample Space = **{**(H, H, H),
(H, H, T),
(H, T, H),
(T, H, H),
(T, T, H),
(T, H, T),
(H, T, T),
(T, T, T)**}**

Q. A coin is tossed twice at random. What is the probability of getting

(i) at least one head

(ii) the same face?

Solution:

The possible outcomes are HH, HT, TH, TT.

n(S) = **4.**

(i) Number of favourable outcomes for event E = No. of outcomes having at least one head

n(E) = **3** (HH, HT, TH).

P(E) = **3/4**

(ii) Number of favourable outcomes for event E = No. of outcomes having the same face

n(E) = 2 (HH, TT).

P(Same face) = **2/4** = **½**

Q. Three coins are tossed at a time find the probability of getting

a. at most one head

b. at least one tail

Solution :

n(S) = $2^3 = 8$

n(E) = 4 (HTT, THT, TTH, TTT)

(a) P(E) = 4/8 = **1/2**

(b) $n(E) = 7$(THH, HTH, HHT, TTH, HTT, THT, TTT)

$P(E) = 7/8 =$ **7/8**

Q : What is the probability of a die showing a number 3 or number 5 ?

Solution :

Let,

P(3) is the probability of getting a number 3

$P(3) = 1/6$

P(5) is the probability of getting a number 5

$P(5) = 1/6$

So,

$P(3 \text{ or } 5) = P(3) + P(5)$

$P(3 \text{ or } 5) = (1/6) + (1/6) = 2/6$

$P(3 \text{ or } 5) =$ **1/3**

Q : Find the probability of getting a number less than 5 when a dice is rolled by using the probability formula.

Solution

Given: Sample space = {1,2,3,4,5,6} Getting

a number less than 5 = {1,2,3,4}Therefore,

$n(S) = 6$

$n(A) = 4$

$P(A) = n(A)/n(S)$

$p(A) = 4/6 = 2/3$

Random Variable :

- Connecting the outcomes of an experiments with the real value is known as random variable.
- A random variable is a rule that assigns a numerical value to each outcome in a sample space.
- Random variables may be either discrete or continuous.
- A random variable is said to be discrete if it assumes only specified values in an interval, otherwise, it is continuous.

Example :

S = {HH, HT, TH, TT}

$= \{ 2, \quad 1, \quad 1, \quad 0 \}$ If H=1 and T=0.

Random Variable :

Discrete Random Variable

Discrete Distribution

1. **Binomial Distribution**
2. **Poison Distribution**

Continuous Random Variable

ContinuousDistribution

1. **Exponential Distribution**
2. **Uniform Distribution**
3. **Normal Distribution**

Mean & Variance :

- If x is a random variable then,

Mean = E(x)

$$\sum_{x=0}^{n} x.\, p(x)$$

Variance =V(x)

$$E(x^2) - (E(x))^2$$

$$= \sum x2.p(x) - (x.\, p(x))^2$$

$$\textbf{Variance + Mean}^2 = \textbf{E}(x^2)$$

Q. Find the Mean and Variance for a single dice when it is thrown ?

x	1	2	3	4	5	6
P(x)	1/6	1/6	1/6	1/6	1/6	1/6

Solution :

Mean = E(x)

$$\sum^{n}$$

$x=0$ x. p(x)

$= 1.p(1) + 2.p(2) + 3.p(3) + 4.p(4) + 5.p(5) + 6.p(6)$

$= 1/6[1 + 2+ 3 + 4 + 5 + 6]$

$= 7/2$

Variance = V(x)

$= \sum x2.p(x) - (x.p(x))^2$

$\sum_{x=1}^{6} x2.p(x) = 1^2.p(1) + 2^2.p(2) + 3^3.p(3) + 4^4.p(4) + 5^5.p(5) + 6^6.p(6)$

$= 1/6[1 + 4 + 9 + 16 + 25 + 36] = 91/6$

Variance $= \sum x2.p(x) - (x.p(x))^2 = 91/6 - (7/2)^2$

$= 35/12$

For a dice Mean & Variance :

$$\boxed{\begin{aligned} \text{Mean} &= \frac{7n}{2} \\ \text{Variance} &= \frac{35n}{12} \end{aligned}}$$

Q. 3 dice are thrown , find the Mean and Variance for the some of the numbers on them.

Solution :

$n = 3$

Mean $= \frac{7n}{2} = \frac{7x3}{2} = \frac{21}{2}$

Variance $= \frac{35n}{12} = \frac{35x3}{12} = \frac{35}{4}$

Binomial Distribution Formula

- If x is said to be binomial random variable then it allow the value from 0 to n with the parameters n and p.

$$\mathbf{B(x: n, p) = {}^{n}C_{x}\, p^{x}\, (q)^{n-x}}$$

Where,

x = Random variable

n = Number of trials/repetation.

p = Number of success

q =Number of failure.

Q. Find the probability of getting 9 exactly 2 in 3 times with a pair of dice.

Solution :

$n = 3$

$x = 2$

$P(Sum=9) = 4/36 = 1/9$

$p = 1/9$

$q = 1-1/9 = 8/9$

$B(x, n, p) = {}^{n}C_{x}\, p^{x}\, (q)^{n-x}$

$= {}^{3}C_{2}\, (1/9)^{2}\, (8/9)^{3-2}$

$= 8/243$

Binomial Distribution Mean and Variance

For a binomial distribution, the mean, variance and standard deviation for the given number of success are represented using the formulas

- Mean, $\mu = np$
- Variance, $\sigma^2 = npq$
- Standard Deviation $\sigma = \sqrt{(npq)}$

Where p is the probability of success q is the probability of failure, where q = 1-p