CS & IT ENGINEERING

Algorithms

Introduction to Algorithms and Analysis

Lecture No.- 06

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Recap of Previous Lecture

Topic Problem Solving with Big Notations

Topic

Topic

Topic

Topic

Topics to be Covered

Topics

- Small Notations (o ; ω)
- Properties of Asymptotic Notations
- Problem Solving

Topic: Asymptotic Notations

$$f(n) = \sum_{i=1}^{n} \sqrt{i} = O(\quad) = \left[\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n}\right]$$

$$\int_{i=1}^{n} i^{1/2}\, di = \left[\frac{i^{3/2}}{3/2}\right]_1^n$$

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\boxed{f(n) = n^{3/2} * c}$$

$$= O\left(n^{3/2}\right) = O\left(n^{1.5}\right) = O\left(n\sqrt{n}\right) \checkmark$$

$$\sum_{x=1}^{n} 1/x \sim \int 1/x\, dx$$

$$\left[\log x\right]_1^n$$

$$= O(\log n)$$

Topic: Asymptotic Notations

Small/Little Notations :

→ The bounds provided by Big-Notations (O, Ω), may or may not be tight;

$f(n) = n$
- $O(n)$: tight Bound
- $O(n^2)$: Loose Bound
- $\Omega(n)$: Tight
- $\Omega(1)$: Loose

→ The bounds provided by Little/Small Notations is $\boxed{\text{always NOT Asymptotically tight}}$; (loose Bound)

Topic: Asymptotic Notations

4) Small-oh (o): proper Upper Bound

$f(n)$ is $o(g(n))$ iff for all $c > 0$

$\boxed{f(n) < c \cdot g(n)}$, whenever $(n > n_0)$

$n_0 > 0$

1) $f(n) = n$
- $O(n^2)$
- $O(n)$ ✓
- $o(n^2)$ ✓
- $o(n^3)$ ✓
- $o(n)$ ✗
- $o(n \cdot \log n)$

2) $f(n) = n^2+n+1$
- $O(n^2)$
- $o(n^3)$
- $o(n^2 \log n)$
- $o(n^2 \sqrt{n})$

Topic: Asymptotic Notations

5) Small omega (ω): Proper Lower Bound

$f(n)$ is $\omega(g(n))$, iff for all $c > 0$

$\boxed{f(n) > c \cdot g(n)}$, whenever

$n > n_0$

$(n_0 > 0)$

1) $f(n) = n$

- $\Omega(n)$ ✓
- $\Omega(1)$ ✓
- $\Omega(\log n)$ ✓
- $\Omega(n^2)$ ✗
- $\omega(n)$ ✗
- $\omega(\log n)$ ✓
- $\omega(1)$ ✓
- $\omega(\sqrt{n})$ ✓
- $\omega(1/n)$ ✓

Topic: Asymptotic Notations

Properties of ASN :

I. Analogy b/w Real No's & A.S.N

Let a, b : real No's & f, g : +ve functions

I. If $f(n)$ is $O(g(n)) \Longleftrightarrow a <= b$

II. If $f(n)$ is $\Omega(g(n)) \Longleftrightarrow a >= b$

III. If $f(n)$ is $\Theta(g(n)) \Longleftrightarrow a = b$

IV. If $f(n)$ is $o(g(n)) \Longleftrightarrow a < b$

V. If $f(n)$ is $\omega(g(n)) \Longleftrightarrow a > b$

Topic: Analysis of Algorithms

$\log x^y = y \log x$

$\log xy = \log x + \log y$

$\log \log n = \log(\text{long})$

$a^{\log_b^x} = x^{\log_b^a}$

$\log n = \log_{10}^n$

$\log^k n = (\log n)^k$

$\log \frac{x}{y} = \log x - \log y$

$\log_b^x = \frac{\log_a^x}{\log_a^b}$

$a = b^{\log_b a}$

$\log_c(ab) = \log_c a + \log_c b$

$\log_b a^n = n \cdot \log_b a$

$\log_b a = \frac{\log_c a}{\log_c b}$

$\log_b{}^{1/a} = -\log_b a$

$\log_b a = \frac{1}{\log_a b}$

$a^{\log_b c} = c^{\log_b a}$

Topic : Geometric Sum Formula

1. The geometric sum formula for finite terms is given as:

if $r = 1$, $S_n = n*a$

if $|r| < 1$, $S_n = \frac{a(1 - r^n)}{1-r}$

if $|r| > 1$, $S_n = \frac{a(r^n - 1)}{r - 1}$

Where

- a is the first term
- r is the common ratio
- n is the number of terms

Topic: Analysis of Algorithms

Airthmetic series

$$\sum_{k=1}^{n} k = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

Geometric series

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 \ldots . + x^n = \frac{x^{n+1} - 1}{x - 1} (x \neq 1)$$

Harmonic series

$$\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \ldots . + \frac{1}{n} \approx \log n$$

PYQ

$f(n) = \sum_{i=1}^{n} i^3 = x$, choices for x

I. $\theta(n^4)$ II. $\theta(n^5)$

III. $O(n^5)$ IV. $\Omega(n^3)$

a) I, II, III

b) II, III, IV

c) I, II, III, IV

d) I, III, IV

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$= \frac{n^2(n+1)^2}{4}$$

$$= \frac{n^2\left[n^2+2n+1\right]}{4}$$

$$\boxed{f(n) = \frac{n^4+2n^3+n^2}{4}}$$

$O(n^5)$ ✓

$\Omega(n^5)$ ✗

$O(n^4)$ $\Omega(n^4)$

$\Theta(n^4)$

Topic : General Properties of Big Oh Notation

$\cancel{10}n < \cancel{100} \cdot n^2$

Let $d(n)$, $e(n)$, $f(n)$, and $g(n)$ be functions mapping nonnegative integers to nonnegative reals. Then

$d(n) = n - O(n)$

$= 10 \cdot n - O(n)$

1. If $d(n)$ is $O(f(n))$, then $\boxed{ad(n) \text{ is } O(f(n))}$, for any constant $a > 0$.
2. If $d(n)$ is $O(f(n))$ and $e(n)$ is $O(g(n))$, then $d(n) + e(n)$ is $\boxed{O(f(n) + g(n))}$.
3. If $d(n)$ is $O(f(n))$ and $e(n)$ is $O(g(n))$, then $d(n)e(n)$ is $\boxed{O(f(n)g(n))}$.
4. If $d(n)$ is $O(f(n))$ and $f(n)$ is $O(g(n))$, then $d(n)$ is $O(g(n))$. Transitive
5. If $f(n)$ is a polynomial of degree d (that is, $f(n) = (a_0 + a_1 n + \ldots + a_d n^d)$ then $f(n)$ is $O(n^d)$.

   $1 + n + n^2 = O(n^2)$
6. n^x is $O(a^n)$ for any fixed $x > 0$ and $a > 1$.
7. $\log n^x$ is $O(\log n)$ for any fixed $x > 0$.
8. $\log^x n$ is $O(n^y)$ for any fixed constants $x > 0$ and $y > 0$.

$(n+k)^m \begin{cases} O(n^m) \\ \Omega(n^m) = \Theta(n^m) \end{cases}$ $(k, m) > 0$

$d(n) = n^2 \Longrightarrow O(n^2)$ ⎫
$e(n) = n^3 \Longrightarrow O(n^3)$ ⎭ Max

$n^2 + n^3 \Longrightarrow O(n^3) = O(\text{Max}(f(n), g(n)))$

$O(n^2 + n^3)$

$d(n) * e(n) = n^2 * n^3 = O(n^5)$

$= O(f(n) * g(n))$

$d(n) = n - O(n)$

$\neq$

$f(n) = n^3 - O(n^3)$

$x > 0$
$a > 1$

$$n^x = O(a^n)$$

Polynomial Exponential

$$\log n^x = O(\log n) \qquad x > 0$$

$$\Rightarrow (x \cdot \log n) =$$

$$\Rightarrow \log^x n = (\log n)^x$$

$$= O(n^y)$$

Log Poly

Const's < Log < Poly < Expo

Discrete Properties of ASN

$n = O(n^2)$ ✓
$n^2 = \Omega(n)$ ✓

$a \le b$
$f(n) = n^2$
$O(n^2)$ ✓

Properties \ ASN	O	Ω	Θ	o	ω
Reflexive	✓	✓	✓	✗	✗
Symmetric	✗	✗	✓	✗	✗
Transitive	✓	✓	✓	✓	✓
Transpose Symmetry	if $f(n)$ is $O(g(n))$ then $g(n)$ is $\Omega(f(n))$		✗	if $f(n)$ is $o(g(n))$ then $g(n)$ is $\omega(f(n))$	

if $a \le b \Rightarrow b \ge a$

$a \le b$;
$b \le c$
$\Rightarrow a \le c$

n in $O(n^2)$
n^2 in $O(n^3)$
$\therefore n$ in $O(n^3)$

Tricotomy Property : [For any Two real no's (a, b)

I. $a < b$
II. $a > b$
III. $a = b$

If 'f' & 'g' are +ve fn's

???

I. $f <_A g \Longrightarrow O, o$
II. $f >_A g \Longrightarrow \Omega, \omega$
III. $f =_A g \Longrightarrow \Theta$

ASN/ASF's does not Satisfy Tricotomy Property

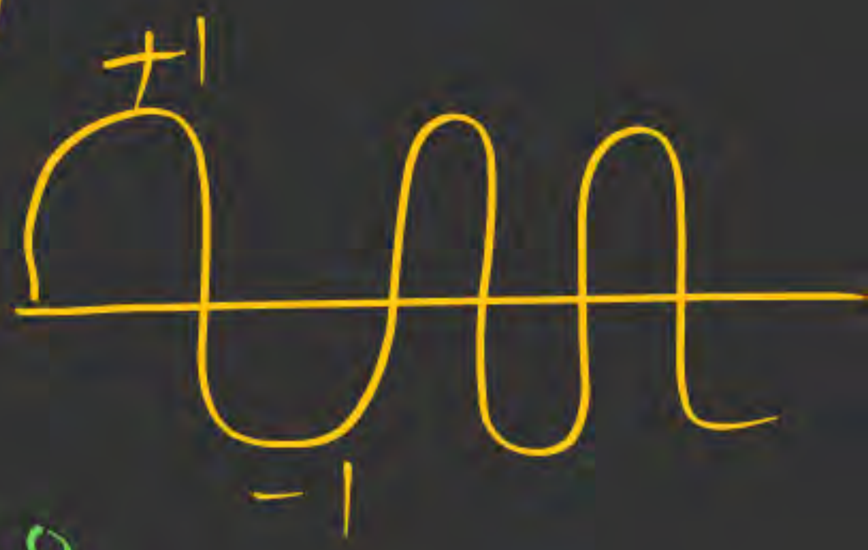
Does ASN/ASFn's obey Tricotomy Property always? (Y/N ✓)

✓ 1) $f(n) = n$; $g(n) = n^2$ $(f \leq c \cdot g)$
$\llcorner O, o$

✓ 2) $f(n) = \log n$; $g(n) = 1/n$ $f \geq c \cdot g$
Ω, ω

3) $f(n) = n^2+10$; $g(n) = 10n^2+5$

$$O(n^2) = O(n^2) \Rightarrow \Theta(n^2)$$



4) $f(n) = n$, $g(n) = n^{1+\sin n}$, $n > 0$

$$= n^{1-1} = n^0 \quad (f > g)$$

$$= n \qquad\qquad = n^2 \quad (f < g)$$

$\mathrm{Min}(\sin x) = -1$

$\mathrm{Max}(\sin x) = +1$

$[-1, +1]$

(These 2-fn's never converge)

Topic : Asymptotic Notations & Apriori Analysis

State True / False

$a \cdot d(n) = O(f(n))$

1. $100n.\log n = O(n.\log n)$ T
2. $2^{n+1} = O(2^n)$: $2 \cdot 2^n = O(2^n)$: T
3. $2^{2n} = O(2^n)$ $\Rightarrow (2^2)^n = 4^n$: F
4. $0<x<y$ then $n^x = O(n^y)$: T $2 < 3$ $\hookrightarrow n^2 < n^3$ ✓
5. $(n+k)^m \neq \theta(n^m)$ $(k, m) > 0$: F
6. $\sqrt{\log n} = O(\log\log n)$: F
7. $\log(n)$ is $\Omega(1/n)$: T
8. 2^{n^2} is $O(n!)$: F
9. n^2 is $O(2^{2\log n})$: T $2^{\log_2 n^2} = n^2$
10. $a^n \neq O(n^x)$, $a>1$, $x>0$: T $n = 1024$
11. $2^{\log_2 n^2}$ is $O(n^2)$: T

$\log(\sqrt{\log_2 n}) > \log(\log_2 \log_2 n)$

$\frac{1}{2}\log\log n > \log\log\log n$

$2^{\log_2 n^2} = (n^2)^{\log_2 2} = n^2$

$$2^{n^2} > O(n!)$$

$$\log\left(2^{n^2}\right) \text{ vs } \log\left(n^n\right)$$

$$n^2 \cdot \log_2 2 \qquad n \cdot \log_2 n$$

$$n^2 > n \log n$$

$0 < x < y \qquad (x < y)$

$n^x < n^y$. ✓

$n^3 < n^5$: ✓

H/W :

Let $f(n)$ & $g(n)$ be +ve fn's

if $f(n)$ is $O(g(n))$ then

Is $f(n)$ always $O\left((f(n))^2\right)$?

Is $2^{f(n)} = O\left(2^{g(n)}\right)$

$\log_2 n$ (Inc) $\quad$ $1/n$ (Dec)

$\log_2 8 \quad 1/8$

$3 > 0.125$

THANK - YOU