CS & IT ENGINEERING

Algorithms

Introduction to Algorithms and Analysis

Lecture No.- 05

By- Dr. Khaleel Khan Sir

Recap of Previous Lecture

Topic Asymptotic Notations

Topic Big Oh, Omega, Theta Notations

Topic

Topic

Topic

Topics to be Covered

Topics

- Asymptotic Notations
- Little Oh, Little Omega
- Properties of ASN
- Problem Solving

Topic: Asymptotic Notations

U.B 1) Big-oh: $f(n)$ is $O(g(n)) \implies f(n) \le c \cdot g(n)$

Whenever $n \ge n_0$;

L.B 2) Big-omega (Ω): $f(n)$ is $\Omega(g(n))$ iff

$f(n) \ge c \cdot g(n)$,

$n \ge n_0$

Tight Bound 3) Theta (Θ): $f(n)$ is $\Theta(g(n))$ iff

$f(n)$ is $O(\underline{g(n)})$ &

$f(n)$ is $\Omega(\underline{g(n)})$

Topic: Asymptotic Notations

$\Longrightarrow$ Smaller functions are in the order of Bigger function

$\Longrightarrow$ Bigger fn's are in the omega of Smaller fn

Ex: $n^2 < n^3$

$n^2 < 2^n$

$\Longrightarrow$ If the rate of Growth of Both the functions are equal, then they are theta of each other;

Topic: Asymptotic Notations

1) $f(n) = 2^{\log_2 n} = O(n)$ ✓

$= n^{\log_2 2} = n^1 =$

2) $f(n) = 2^{\sqrt{n} \cdot \log_2 n} = O(n^{\sqrt{n}})$

$= 2^{\log_2 n^{\sqrt{n}}}$

$= n^{\sqrt{n}(\log_2 2)} = n^{\sqrt{n}}$

Is

3) $f(n) = 2^{2n} = (2^2)^n$

$\neq O(2^n)$

2^n is $O(4^n)$ ✓

4^n is $\Omega(2^n)$ ✓

$4^n \neq O(2^n)$

Topic: Asymptotic Notations

Is $2^{n+1} = O(2^n)$ ✓

$$\underset{}{2\cdot 2^n} \le \underbrace{4}_{c}\cdot \underbrace{2^n}_{g}$$

$$\boxed{n = 2^{\log_2 n}}$$

$$f(n) = n^{1/\log_2 n} = \quad \text{(Value)}$$

$$= \left(2^{\cancel{\log_2 n}}\right)^{\frac{1}{\cancel{\log n}}} = 2^1 = 2 \ (c)$$

$$= O(1)$$

$$f(n) = n^{1/\log n} = x$$

$$= \log n^{1/\log n} = \log_2 x$$

$$= \frac{1}{\cancel{\log n}} \cdot \cancel{\log n} = \log_2 x$$

$\log_2 x = 1$

$\Rightarrow x = 2^1$

$O(1)$

f is $O(g)$ ✓

$f(n) = \log_2 n < g(n) = \sqrt{n}$

$a < b$

$\log a < \log b$ ✓

$a > b$

$\log a > \log b$ ✓

$a = b$

$\log a = \log b$ ✗

$\log_2 \log_2 n$

$\log \sqrt{n}$

$\log n^{1/2}$

$\log \log n < \frac{1}{2} \cdot \log(n)$

n^2	2^n
4	4
9	8

Topic: Asymptotic Notations

$f(n) = \frac{1}{n}$ (decr)

$g(n) = \log n$ (inc)

$n = 8$

$\frac{1}{8} < \log_2 8 = 3$

Topic: Asymptotic Notations

$f(n) = n^2 < ; \quad g(n) = n^3$

$\boxed{\underset{=}{2} \cdot \cancel{\log n}}$ $\quad \boxed{\underset{=}{3} \cdot \cancel{\log n}}$

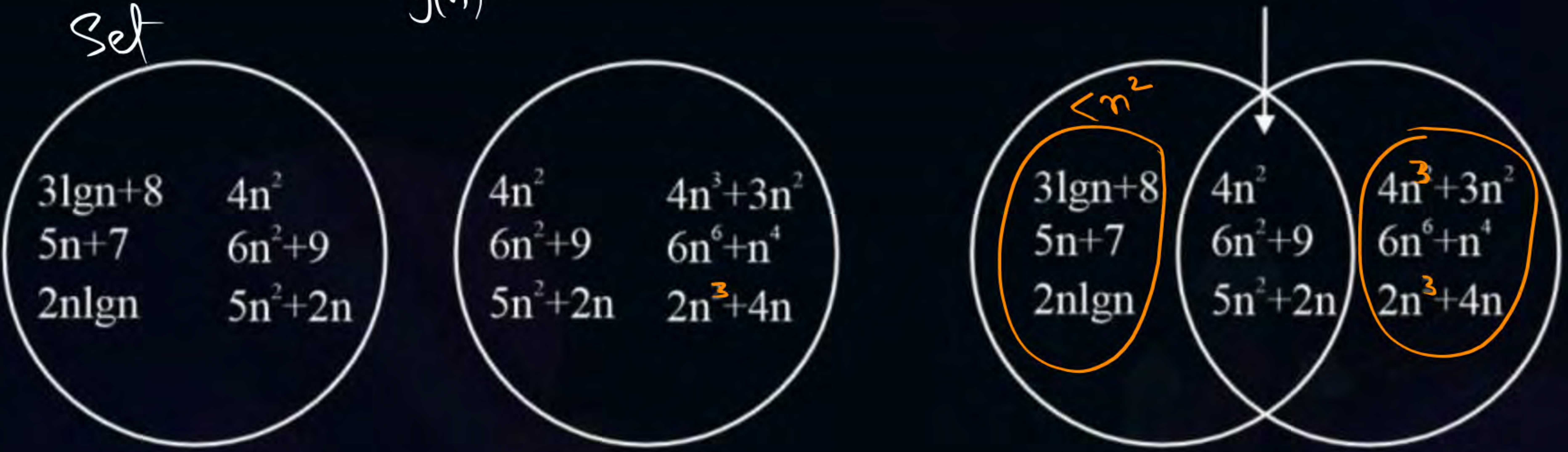
Value $\quad$ Value

$2 < 3$

Topic : Time Complexity

$O(n^3)$
$g(n)$

$\theta(n^2)$



(a) $O(n^2)$ (b) $\Omega(n^2)$ (b) $\theta(n^2) = O(n^2) \cap \Omega(n^2)$

Note:- The set $O(n^2)$, $\Omega(n^2)$, $\Theta(n^2)$. Some exemplary member are shown.

Topic : Exponentials

For all real $a > 0$, m, n

$a^0 = 1$

$a^1 = a$

$a^{-1} = 1/a$

$(a^m)^n = a^{mn}$

$(a^m)^n = (a^n)^m$

$a^m \cdot a^n = a^{m+n}$

Topic: Analysis of Algorithms

$\log x^y = y \log x$

$\log xy = \log x + \log y$

$\log \log n = \log(\log n)$

$a^{\log_b^x} = x^{\log_b^a}$

$\log n = \log_{10}^n$

$\log^k n = (\log n)^k$

$\boxed{\log \frac{x}{y} = \log x - \log y}$

$\log_b^x = \frac{\log_a^x}{\log_a^b} = \frac{\log_a x}{\log_a b}$

$a^{\log_c b} = b^{\log_c a}$

$\log^{10} n = (\log n)^{10}$

$\log_2$

$\boxed{a = b^{\log_b a}}$ $a^{\log_b b} = a^1$

$\log_c(ab) = \log_c a + \log_c b$

$\log_b a^n = n \cdot \log_b a$

$\log_b a = \frac{\log_c a}{\log_c b}$

$\log_b {}^{1/a} = -\log_b a$ | $\log_b \frac{1}{a}$

$\boxed{\log_b a = \frac{1}{\log_a b}}$

$a^{\log_b c} = c^{\log_b a}$

Slide 13

Topic : Geometric Sum Formula

1. The geometric sum formula for finite terms is given as:

if r = 1, $S_n = n*a$

$\sum_{i=0}^{n} x^i$ [(n+1) Terms]

$r = \frac{1}{2} < 1$

$\boxed{\text{if } |r| < 1, \quad S_n = \frac{a(1-r^n)}{1-r}}$

if |r| > 1, $S_n = \frac{a(r^n-1)}{r-1}$

$$\boxed{\sum_{i=1}^{n} \frac{1}{2}^i = \frac{\cancel{\frac{1}{2}}\left(1-\frac{1}{2^n}\right)}{1-\cancel{\frac{1}{2}}} = 1 - \frac{1}{2^n}}$$

$$\boxed{\sum_{i=1}^{n} 2^i = \frac{2(2^n-1)}{2-1} = 2^{n+1} - 2}$$

Where

- a is the first term
- r is the common ratio
- n is the number of terms

Topic : Geometric Sum Formula

$$\boxed{\sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \quad \underline{\underline{r<1}}}$$

2. The geometric sum formula for infinite terms is given as:
   if $|r| < 1$, $\quad S_\infty = \dfrac{a}{1-r}$

   If $|r| > 1$, the series does not converge and it has no sum.

Airthmetic series

$$\sum_{k=1}^{n} k = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

Geometric series

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 \ldots + x^n = \frac{x^{n+1} - 1}{x - 1} (x \neq 1)$$

Harmonic series

$$\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \ldots + \frac{1}{n} \approx \log n$$

$$\sum_{k=1}^{n} 1/k \sim \int_{1}^{n} 1/k = [\log k]_{1}^{n} = \log n$$

Dominance Relation

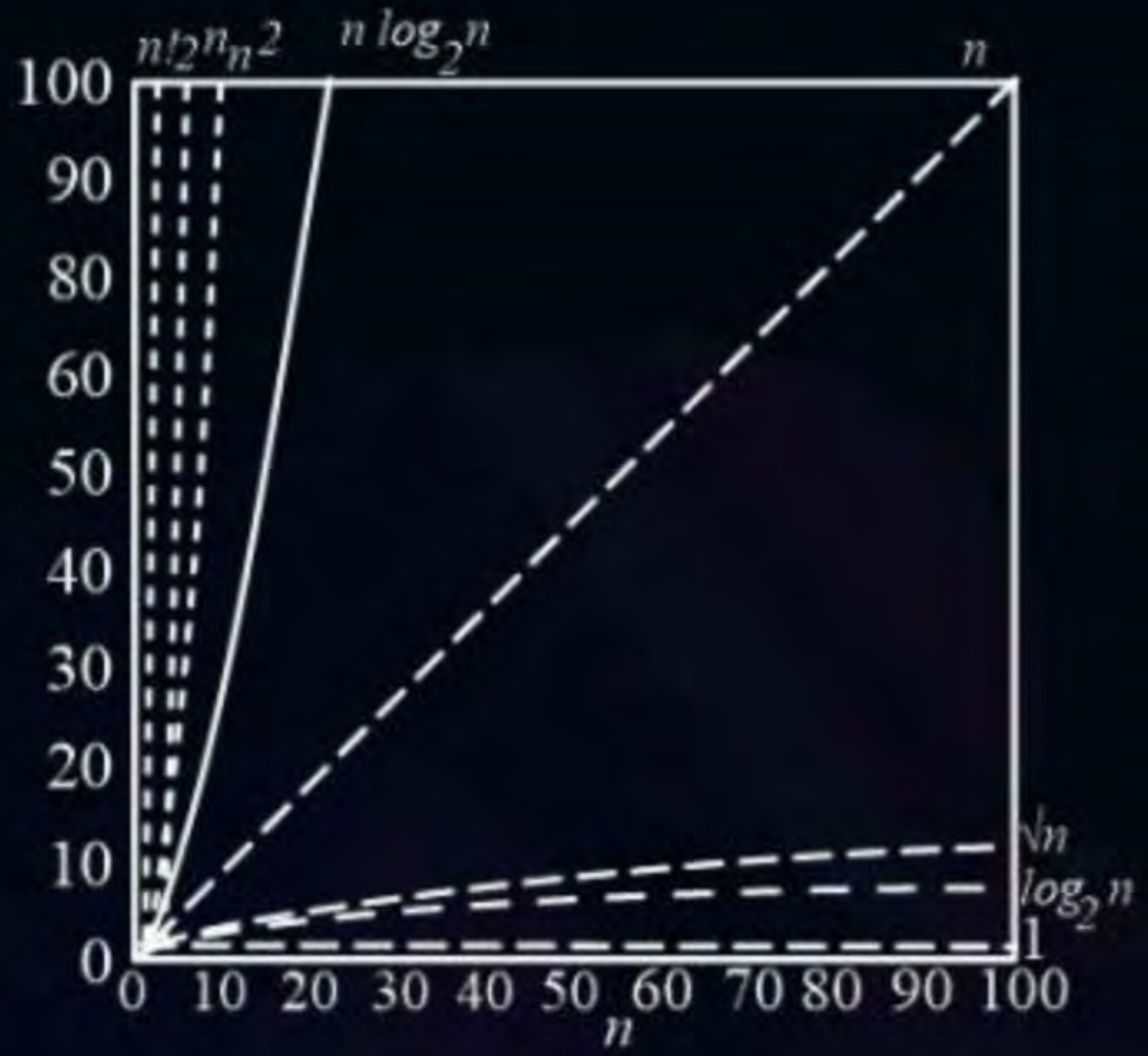
Constants < Logarithms < Poly < Exponential

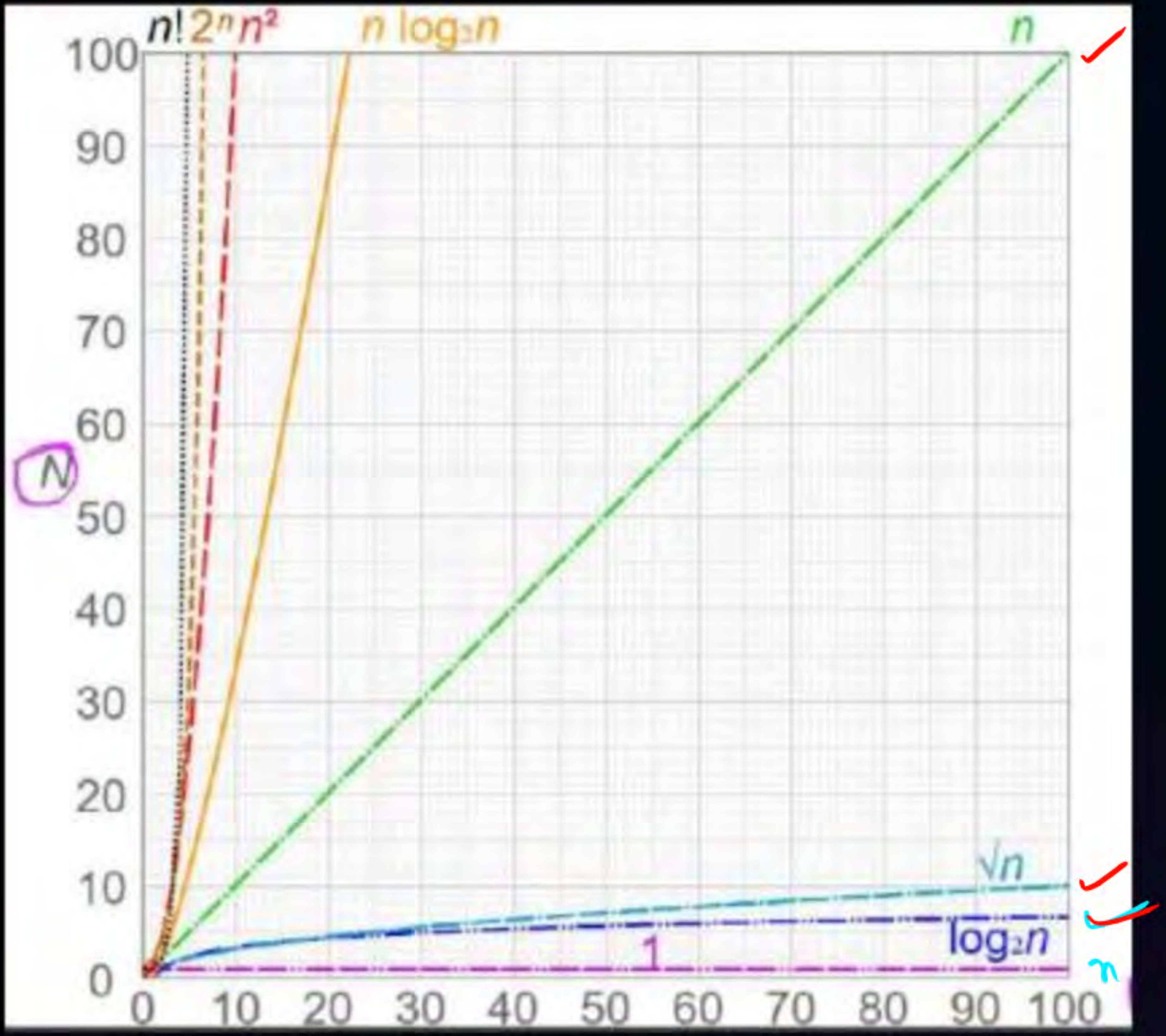
$$C < \log n < \sqrt{n} < n < n \cdot \log n < n^2 < n^3 \cdots$$

$$< 2^n < 4^n < n^n$$

$$< n^{n^n}$$

Topic: Analysis of Algorithms





1) $f(n) = \sum_{i=1}^{n} 1 = n = O(n)$ ✓

2) $f(n) = \sum_{i=1}^{n} n = n \cdot \sum_{i=1}^{n} 1 = n \cdot n = O(n^2)$

3) $f(n) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} = O(n^2)$

4) $f(n) = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

$= O(n^3)$

5) $f(n) = \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2 = O(n^4)$

6) $f(n) = \sum_{i=1}^{n} 2^i = \left(2^{n+1} - 2\right) = O(2^n)$

7) $f(n) = \sum_{i=1}^{n} 1/2^i = \left(1 - \frac{1}{2^n}\right) = O(1)$

8) $\boxed{f(n) = \sum_{i=1}^{n} i \cdot 2^i = (n-1) \cdot 2^{n+1} + 2}$

$\left[n \cdot 2^{n+1} - 2^{n+1} + 2\right]$

$\Downarrow$

$O(n \cdot 2^n)$ ✓

9) $f(n) = \prod_{i=1}^{n} 1 = O(1)$ ✓

10) $f(n) = \prod_{i=1}^{n} i = (1 \cdot 2 \cdot 3 \cdot \quad n) = \boxed{n! = n \cdot (n-1)(n-2) \cdot \cdot \cdot \cdot 1}$

$$\boxed{n! = n(n-1)(n-2) \cdot \cdot 1 < n \cdot n \cdot n \cdot \cdot \cdot \cdot n}$$

$n! < 1 \cdot n^n \Rightarrow O(n^n)$ loose Bound

$n!$ vs n^n

11) $f(n) = \sum_{i=1}^{n} \log i$

Stirlings Approximation

$$n! \sim \sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n \quad < \quad n^n$$

Vs

Is $n! = \Omega(n^n)$?

$$\left[n(n-1)(n-2)\cdots 1\right] \times \left[n\cdot n \cdots n\right]$$

$$\boxed{n! \neq \Omega(n^n)}$$

$$\sqrt{2\pi}\,\sqrt{n}\cdot\frac{\cancel{n^n}}{e^n} \qquad \cancel{n^n}$$

$$\boxed{c\cdot\sqrt{n} \quad < \quad e^n}$$

Is $n!\ \Omega(n)$ ✓ (loose

$n!\ \Omega(n^2)$ ✓

$$f(n) = \sum_{i=1}^{n} \log i = \log 1 + \log 2 + \cdots + \log n = \log(1\cdot 2\cdot 3 \cdots n)$$
$$= \log(n!)$$

$(n^2 + n^3 - n)$

$\log(n!) = O(\quad)$

$= \Omega(\quad)$

$n!$ is $O(n^n)$

$\log a\cdot b = \log a + \log b$

1) LHS RHS

$$n! \sim \sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n$$

$$\log n! = \log\left(\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n\right)$$
$$= \left[\log\sqrt{2\pi} + \frac{1}{2}\log n + n\cdot\log n + \log \frac{1}{e^n}\right]$$
$$= \left(\frac{1}{2}\log 2\pi\right)$$
$$= \left[C + \frac{1}{2}\log n + n\log n - n\cdot\log e\right]$$

$O(n\cdot\log n)$ $\Omega(n\cdot\log n)$

$$\boxed{\log n! = \Theta(n\log n)}$$

2) $\log n! = \sum_{i=1}^{n} \log i \sim \int_1^n \log x \cdot dx = \left[x(\log x - 1) + c\right]_1^n$

$$\sum_{i=1}^{n} \log i = \left[n \cdot \log n - n + c\right]$$

$O(n \cdot \log n)$ $\quad \Omega(n \cdot \log n)$

$$\boxed{\log n! = \Theta(n \cdot \log n)}$$

3) $\sum_{i=1}^{n} \log i = \log 1 + \log 2 + \cdots + \log n$

1) $\log n + \log n-1 + \log n-2 + \cdots + \log 1$

U.B

$< (\log n + \log n + \log n + \cdots + \log n)$

$< n \cdot \log n$

$O(n \cdot \log n)$

2) $\log n + \log n-1 + \cdots + 1 > (\log n/2 + \log n/2 + \log n/2$

L.B

$(n/2)^{+} \cdots$

$$\boxed{\log n! > \frac{n}{2} \cdot \log n/2}$$

$\Omega(n \cdot \log n)$

$\therefore \log n! = \Theta(n \log n)$

$$n! < n^n \Rightarrow O(n^n)$$

$$\log n! = \Theta(n \log n)$$

$$f(n) = \sum_{i=1}^{n} i^{1/2} = O(\quad)$$

THANK - YOU