HOMEWORK 5

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Instructions:

- Please submit your answers in a single pdf file and your code in a zip file. pdf preferably made using latex. No need to submit latex code.
- Submit code for programming exercises. Though we provide a base code with python (jupyter notebook, you can import it in colab and choose GPU as the runtime environment), you can use any programming language you like as long as you use the same model and dataset.

1 Implementation: GAN (55 pts)

In this part, you are expected to implement GAN with MNIST dataset. We have provided a base jupyter notebook (gan-base.ipynb) for you to start with, which provides a model setup and training configurations to train GAN with MNIST dataset.

(a) Implement training loop and report learning curves and generated images in epochs 1, 50, and 100. Note that drawing learning curves and visualization of images are already implemented in provided jupyter notebook. (20 pts)

Procedure 1 Training GAN, modified from Goodfellow et al. (2014)

```
Input: $m$: real data batch size, $n_z$: fake data batch size
Output: Discriminator $D$, Generator $G$
for number of training iterations do
    # Training discriminator
    Sample minibatch of $n_z$ noise samples $\{\mathbf{z}^{(1)}, \mathbf{z}^{(2)}, \cdots, \mathbf{z}^{(n_z)}\}$ from noise prior $p_g(\mathbf{z})$
    Sample minibatch of $\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \cdots, \mathbf{x}^{(m)}\}$
    Update the discriminator by ascending its stochastic gradient:
        $$\nabla_{\theta_d}\Big(\frac{1}{m}\sum_{i=1}^{m}\log D(\mathbf{x}^{(i)}) + \frac{1}{n_z}\sum_{i=1}^{n_z}\log(1 - D(G(\mathbf{z}^{(i)})))\Big)$$
    # Training generator
    Sample minibatch of $n_z$ noise samples $\{\mathbf{z}^{(1)}, \mathbf{z}^{(2)}, \cdots, \mathbf{z}^{(n_z)}\}$ from noise prior $p_g(\mathbf{z})$
    Update the generator by ascending its stochastic gradient:
        $$\nabla_{\theta_g}\frac{1}{n_z}\sum_{i=1}^{n_z}\log D(G(\mathbf{z}^{(i)}))\Big)$$
end for
# The gradient-based updates can use any standard gradient-based learning rule. In the base code, we are using Adam optimizer (Kingma and Ba, 2014)
```

The expected results are as follows.

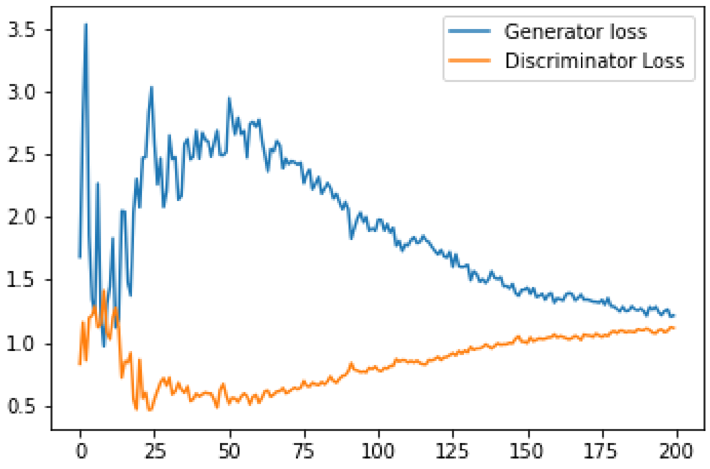
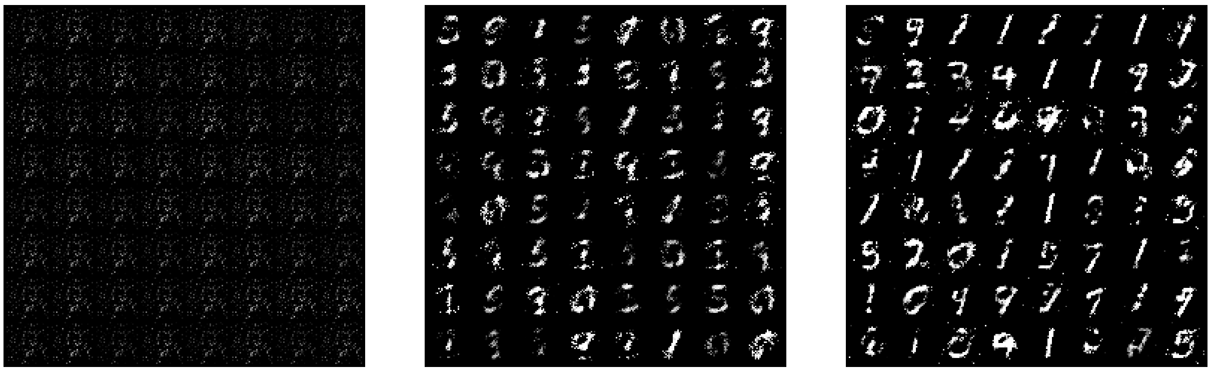


Figure 1: Learning curve



(a) epoch 1 (b) epoch 50 (c) epoch 100

Figure 2: Generated images by G

Ans: After training the GAN using the above steps, the following learning curve and images at epoch 1, epoch 50 and epoch 100 were obtained.

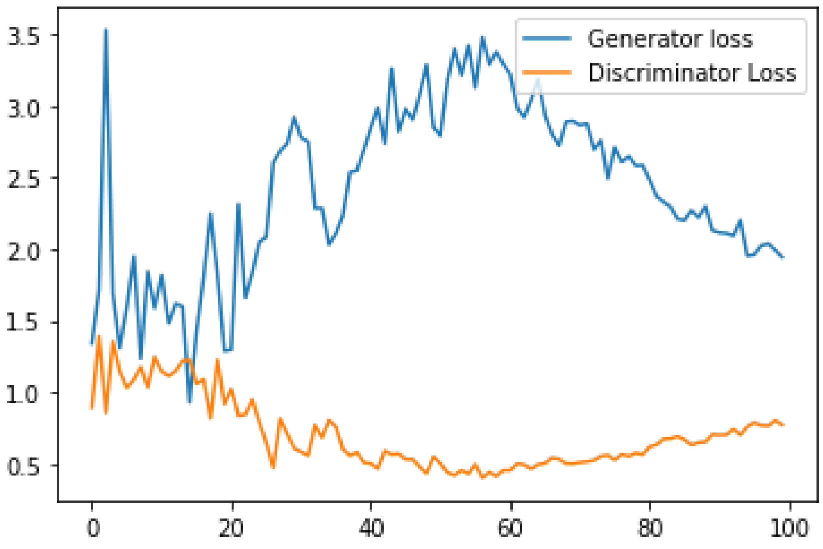


Figure 3: Learning curve

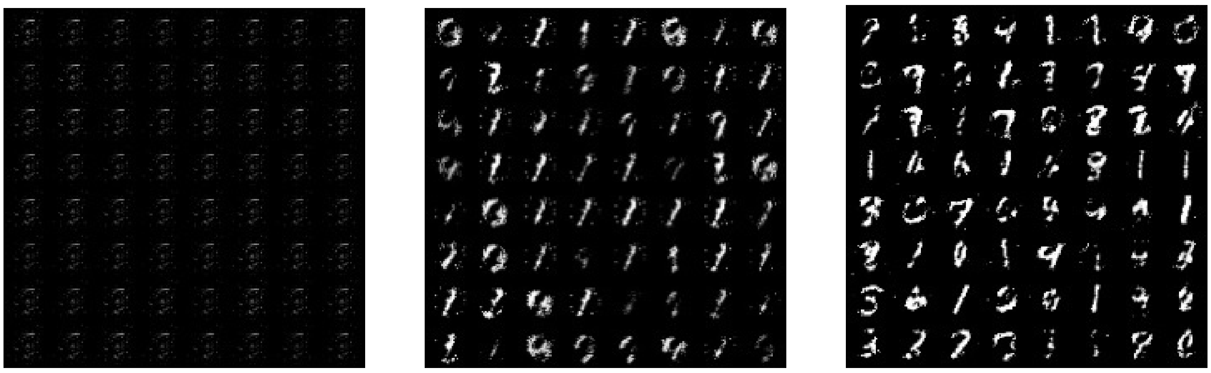


Figure 4: Generated images by G

(b) Replace the generator update rule as the original one in the slide, "Update the generator by descending its stochastic gradient:"

$$\nabla_{\theta_g} \frac{1}{n_z} \sum_{i=1}^{n_z} \log(1 - D(G(\mathbf{z}^{(i)})))\Big) \; ,$$

and report learning curves and generated images in epochs 1, 50, and 100. Compare the result with (a). Note that it may not work. If training does not work, explain why it does not work. (10 pts)

Ans: After updating the generator by descending the gradient, the generator loss and discriminator loss were 0.05609245 and 0.08100681 respectively in epoch 1. In the next epoch, the loss of the generator and discriminator were 0 until 100 epochs. The training has clearly not worked even though the loss has converged since this particular change to the generator leads to a vanishing gradient and the discriminator becomes optimal (D is close to D*). i.e. $\frac{1}{n_z}\sum_{i=1}^{n_z} \log(1 - D(G(\mathbf{z}^{(i)})))\Big)$ tends to 0. The discriminator rejects samples from the generator and becomes too strong that the generator cannot keep up.
The images at epoch 1, epoch 50, epoch 100, and learning curve look like the following:

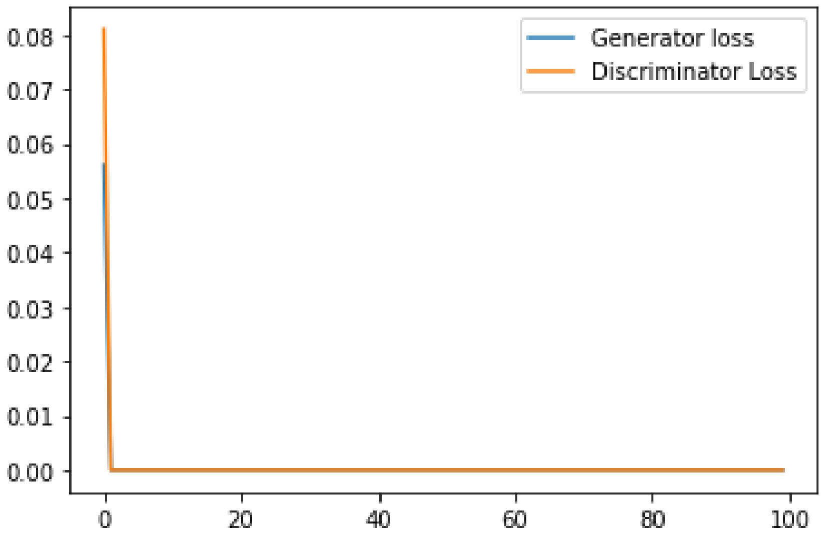
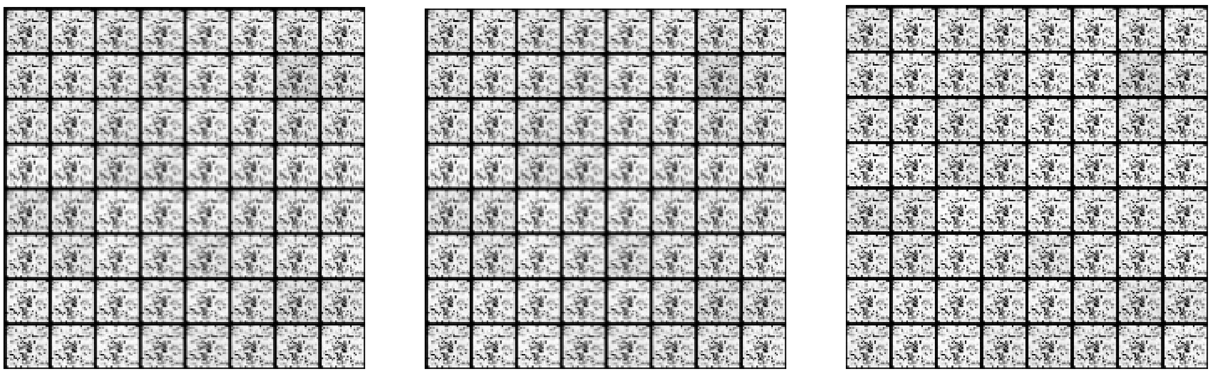


Figure 5: Learning curve



(a) epoch 1 (b) epoch 50 (c) epoch 100

Figure 6: Generated images by G

(c) Except for the method that we used in (a), how can we improve training for GAN? Implement that and report learning curves and generated images in epochs 1, 50, and 100. (10 pts)

Ans: Based on suggestions provided in the original paper and other data science blogs, the learning rate was modified to 0.0001 for the generator and the discriminator and additionally the number of steps of k in discriminator was modified to 5 to improve the training of the GAN. Although the losses of the generator and discriminator converged earlier than in part (a), the generator started producing class 1 images mostly. The learning curve and images at epoch 1, epoch 50 and epoch 100 are the following:

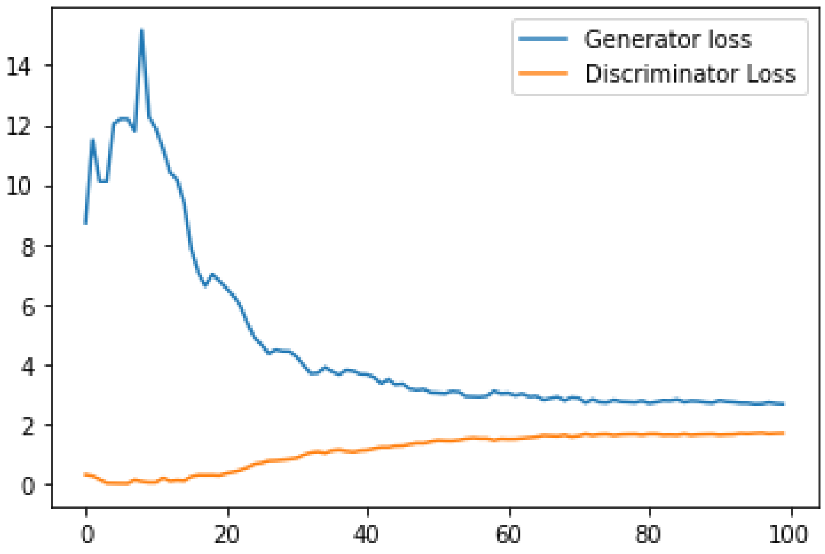
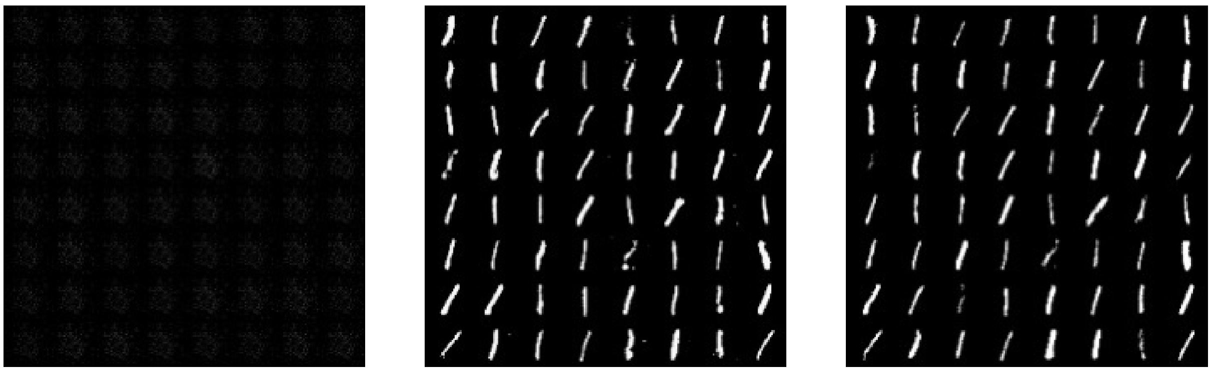


Figure 7: Learning curve



(a) epoch 1 (b) epoch 50 (c) epoch 100

Figure 8: Generated images by G

2 Ridge regression [20 pts]

Derive the closed-form solution in matrix form for the ridge regression problem:

$$\min_{\boldsymbol{\beta}} \left(\frac{1}{n} \sum_{i=1}^{n} (\mathbf{z}_i^\top \boldsymbol{\beta} - y_i)^2 \right) + \lambda \|\boldsymbol{\beta}\|_{\mathbf{A}}^2$$

where

$$\|\boldsymbol{\beta}\|_{\mathbf{A}}^2 := \boldsymbol{\beta}^\top \mathbf{A} \boldsymbol{\beta}$$

and

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This $\mathbf{A}$ matrix has the effect of NOT regularizing the bias β_0, which is standard practice in ridge regression. Note: Derive the closed-form solution, do not blindly copy lecture notes.

Let $f(\boldsymbol{\beta}) = \min_{\boldsymbol{\beta}} \left(\frac{1}{n} \sum_{i=1}^{n} (\mathbf{z}_i^\top \boldsymbol{\beta} - y_i)^2 \right) + \lambda \|\boldsymbol{\beta}\|_{\mathbf{A}}^2$

Rewriting the problem as follows:

$f(\boldsymbol{\beta}) = \frac{1}{n}||Z\boldsymbol{\beta} - y||^2 + \lambda\|\boldsymbol{\beta}\|^2_{\mathbf{A}}$

$f(\boldsymbol{\beta}) = \frac{1}{n}[(Z\boldsymbol{\beta} - y)^\top(Z\boldsymbol{\beta} - y)] + \lambda\boldsymbol{\beta}^\top\mathbf{A}\boldsymbol{\beta}$

$f(\boldsymbol{\beta}) = \frac{1}{n}(\boldsymbol{\beta}^\top Z^\top Z\beta - Z^\top\boldsymbol{\beta}^\top y - y^\top Z\boldsymbol{\beta} + y^\top y) + \lambda\boldsymbol{\beta}^\top\mathbf{A}\boldsymbol{\beta}$

To find the closed form min solution we differentiate $f(\boldsymbol{\beta})$ with respect to $\boldsymbol{\beta}$ and set the gradient to 0

$\frac{1}{n}[2\boldsymbol{\beta}Z^\top Z - y^\top Z - Z^\top y] + 2\lambda\boldsymbol{\beta}\mathbf{A} = 0$

$\frac{2}{n}Z^\top Z\boldsymbol{\beta} + 2\lambda\mathbf{A}\boldsymbol{\beta} = \frac{2}{n}Z^\top y$

$\boldsymbol{\beta}(Z^\top Z + \lambda\mathbf{A}n) = Z^\top y$

The closed form solution can be written as:

$\boldsymbol{\beta} = (Z^\top Z + \lambda\mathbf{A}n)^{-1}Z^T y$

3 Review the change of variable in probability density function [25 pts]

In Flow based generative model, we have seen $p_\theta(x) = p(f_\theta(x))|\frac{\partial f_\theta(x)}{\partial x}|$. As a hands-on (fixed parameter) example, consider the following setting.

Let X and Y be independent, standard normal random variables. Consider the transformation $U = X + Y$ and $V = X - Y$. In the notation used above, $U = g_1(X, Y)$ where $g_1(X, Y)$ where $g_1(x, y) = x + y$ and $V = g_2(X, Y)$ where $g_2(x, y) = x - y$. The joint pdf of X and Y is $f_{X,Y} = (2\pi)^{-1}exp(-x^2/2)exp(-y^2/2), -\infty < x < \infty, -\infty < y < \infty$. Then, we can determine u, v values by x, y, i.e. $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

(a) (5 pts) Compute Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

(5 pts)

$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

From the above matrix we can get the following equations:

u = x + y

v = x - y

x = (u + v)/2 and y = (u - v)/2

$\frac{\partial x}{\partial u} = \frac{1}{2}; \frac{\partial x}{\partial v} = \frac{1}{2}; \frac{\partial y}{\partial u} = \frac{1}{2}; \frac{\partial y}{\partial v} = \frac{-1}{2}$

$$J = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{bmatrix}$$

(b) (Forward) Show that the joint pdf of U, V is

$$f_{U,V}(u, v) = \left(\frac{1}{\sqrt{2\pi}\sqrt{2}}exp(-u^2/4)\right)\left(\frac{1}{\sqrt{2\pi}\sqrt{2}}exp(-v^2/4)\right)$$

(10 pts)

(Hint: $f_{U,V}(u,v) = f_{X,Y}(?,?)|det(J)|)$

We know that:

x = (u + v)/2 and y = (u - v)/2

We can express $f_{U,V}(u,v) = f_{X,Y}((u+v)/2, (u-v)/2)|det(J)|$ as:

$(2\pi)^{-1}exp(-x^2/2)exp(-y^2/2) = \frac{1}{2\pi}exp(-\frac{1}{2}((u+v)/2)^2)exp(-\frac{1}{2}((u-v)/2)^2) * |-\frac{1}{2}|$

$=\frac{1}{4\pi}exp(\frac{-1}{2}[\frac{u^2+v^2+2uv}{4}])exp(\frac{-1}{2}[\frac{u^2+v^2-2uv}{4}])$

$f_{U,V}(u,v) = \left(\frac{1}{\sqrt{2\pi}\sqrt{2}}exp(-u^2/4)\right)\left(\frac{1}{\sqrt{2\pi}\sqrt{2}}exp(-v^2/4)\right)$

(c) (Inverse) Check whether the following equation holds or not.

$$f_{X,Y}(x,y) = f_{U,V}(x+y, x-y)|det(J)^{-1}|$$

(10 pts)

We know that :

$f_{X,Y} = (2\pi)^{-1}exp(-x^2/2)exp(-y^2/2)$

From the previous part we now know that:

$$f_{U,V}(u,v) = \left(\frac{1}{\sqrt{2\pi}\sqrt{2}}exp(-u^2/4)\right)\left(\frac{1}{\sqrt{2\pi}\sqrt{2}}exp(-v^2/4)\right)$$

Therefore we can write

$$f_{U,V}(x+y, x-y)|det(J)^{-1}| = \left(\frac{1}{\sqrt{2\pi}\sqrt{2}}exp(-(x+y)^2/4)\right)\left(\frac{1}{\sqrt{2\pi}\sqrt{2}}exp(-(x-y)^2/4)\right)|-2|$$

$$= \left(\frac{1}{2\pi}exp(\frac{-x^2-y^2-2xy}{4})exp(\frac{-x^2-y^2+2xy}{4})\right)$$

$$= \left(\frac{1}{2\pi}exp(-2x^2/4)exp(-2y^2/4)\right) = (2\pi)^{-1}exp(-(x^2/2))exp(-(y^2/2))$$

Therefore the equation

$$f_{X,Y}(x,y) = f_{U,V}(x+y, x-y)|det(J)^{-1}|$$

holds.

References

Goodfellow, I., Pouget-Abadie, J., Mirza, M., Xu, B., Warde-Farley, D., Ozair, S., Courville, A., and Bengio, Y. (2014). Generative adversarial nets. *Advances in neural information processing systems*, 27.

Kingma, D. P. and Ba, J. (2014). Adam: A method for stochastic optimization. *arXiv preprint arXiv:1412.6980*.