# Applied Algorithms

# CSCI-B505 / INFO-I500

## Lecture 19.

## Randomized Algorithms - II

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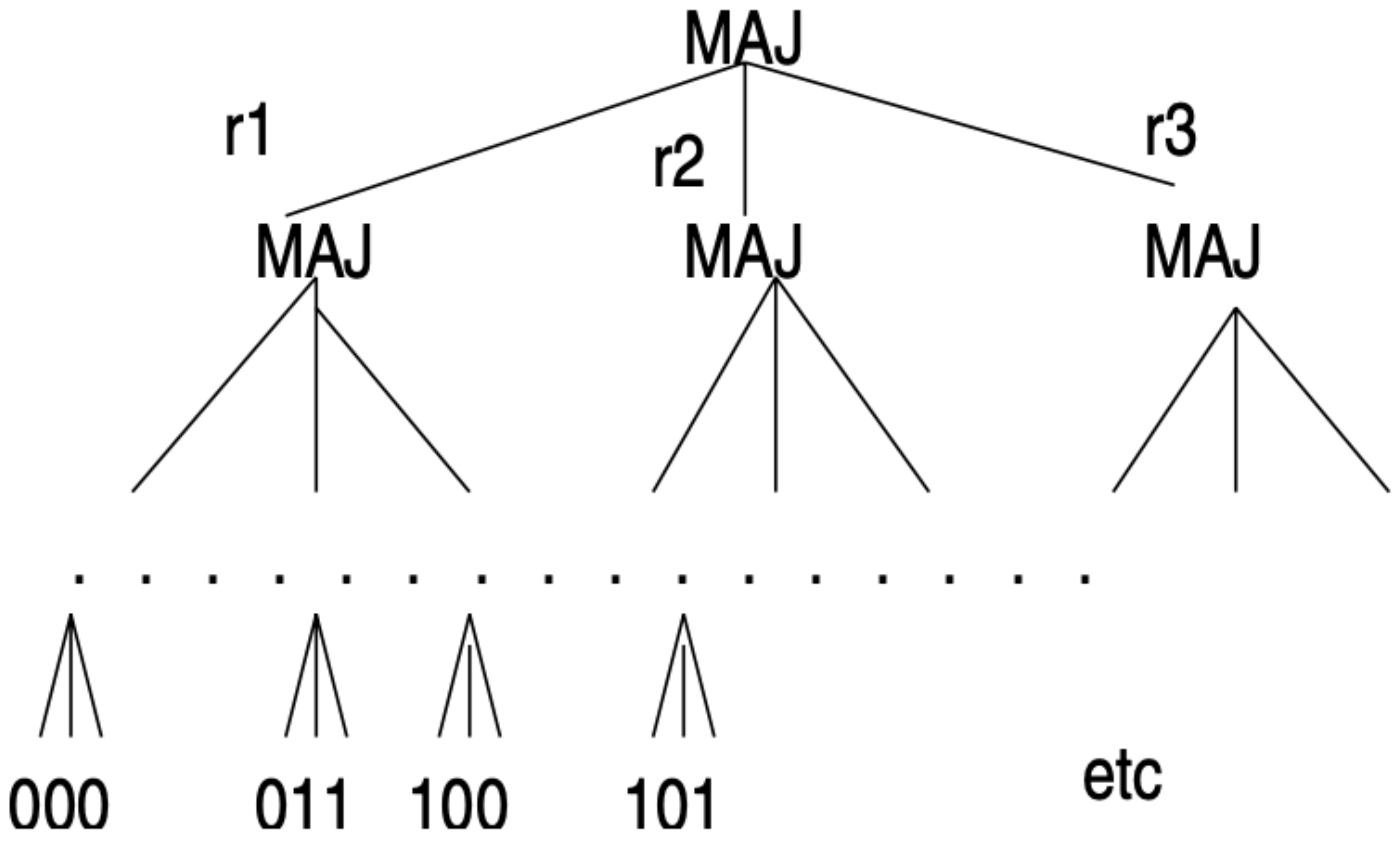
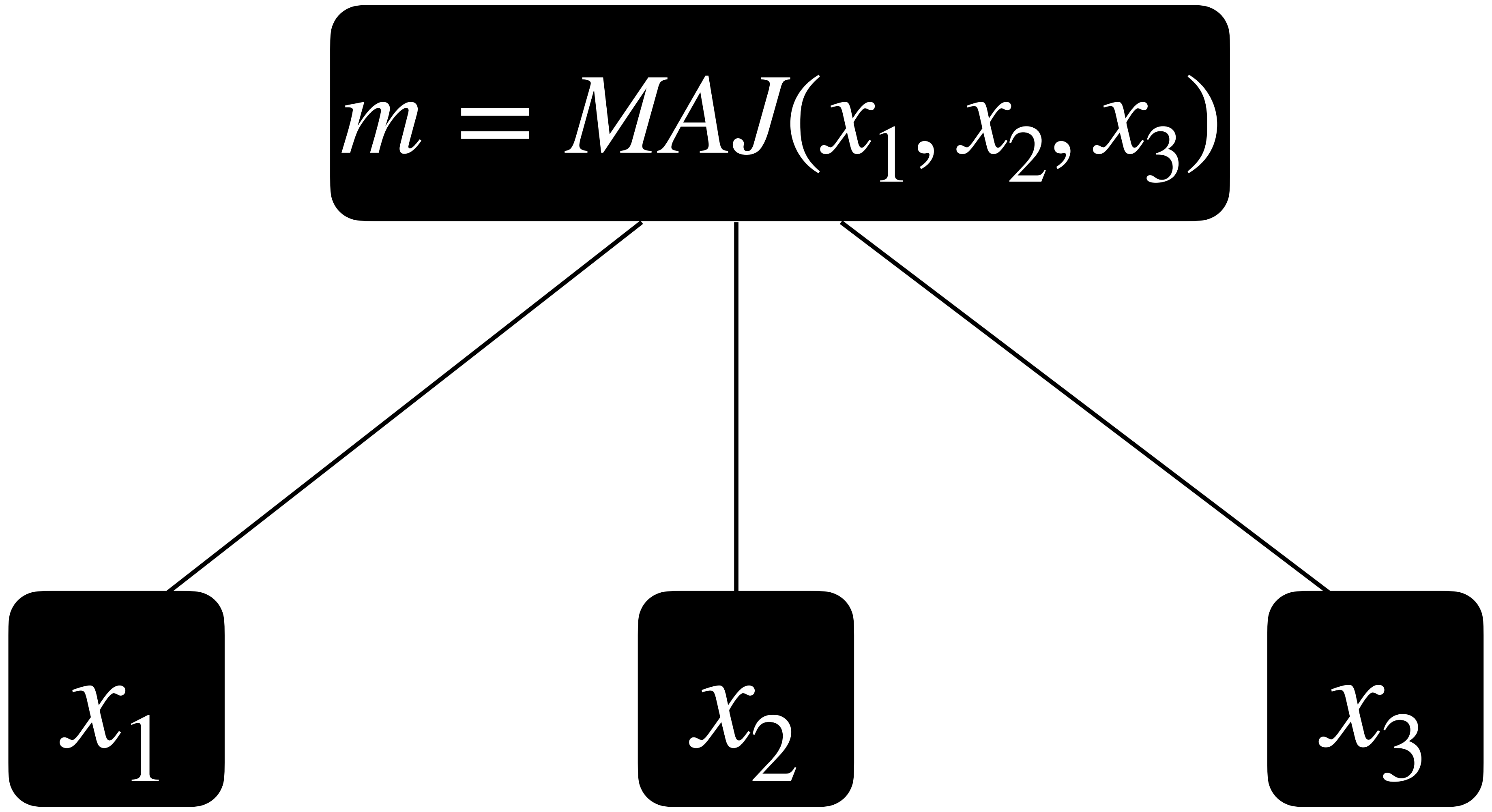
- Majority Tree Evaluation
- Approximate Median Revisited
- Matrix Multiplication Verification
- Randomized 3SAT

# Majority Tree



- A ternary tree of depth n
- Leaf nodes are 0 or 1
- Inner nodes are **MAJ** gates that output the value of the majority value of its children, e.g. if children are 0, 0, and 1 then the output is 0, which will be fed to the upper level.
- We can evaluate the result by traversing all nodes that is $O(3^n)$
- Can randomization help to reduce the number of visited nodes and still provide the correct result?
- Since we need the correct result, it should be a **Las Vegas** type randomization.

# Majority Tree



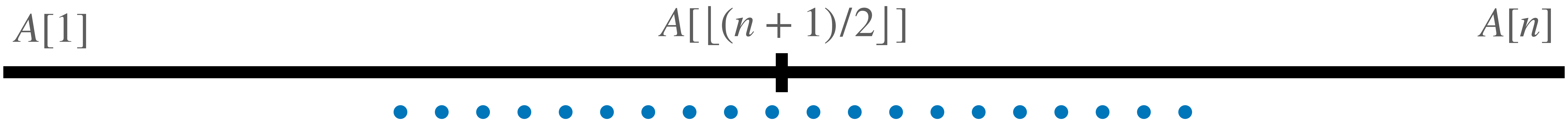
- Assume we perform a breads-first bottom-up traversal! A DFS would be more advantageous though
- On the average we visit (8/3 = 2.66) children instead of 3 to decide the majority, and thus, the expected complexity reduces to $O(2.66^n)$

$$\frac{2}{3} \cdot 3 + \frac{1}{3} \cdot 2 = \frac{8}{3}$$

- Chose **randomly** 2 from $x_1, x_2, x_3$.
- There are 3 different ways of that selection.
- Regardless of the values of the $x_i$, at least one of those selections removes the necessity of visiting the third one !
- Then, with such a randomization with at least 1/3 probability, no need to visit the third.

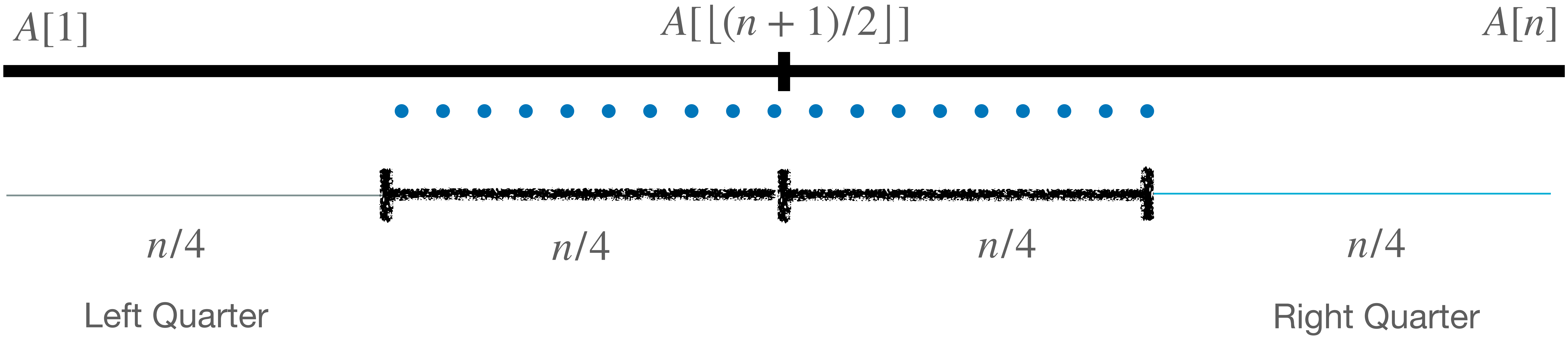
| $x_1x_2x_3$ | $x_1x_2$ | $x_1x_3$ | $x_2x_3$ |
|---|---|---|---|
| **000** | 00 | 00 | 00 |
| **001** | 00 | 01 | 01 |
| **010** | 01 | 00 | 10 |
| **011** | 01 | 01 | 11 |
| **100** | 10 | 10 | 00 |
| **101** | 10 | 11 | 01 |
| **110** | 11 | 10 | 10 |
| **111** | 11 | 11 | 11 |

# Approximate Median Revisited



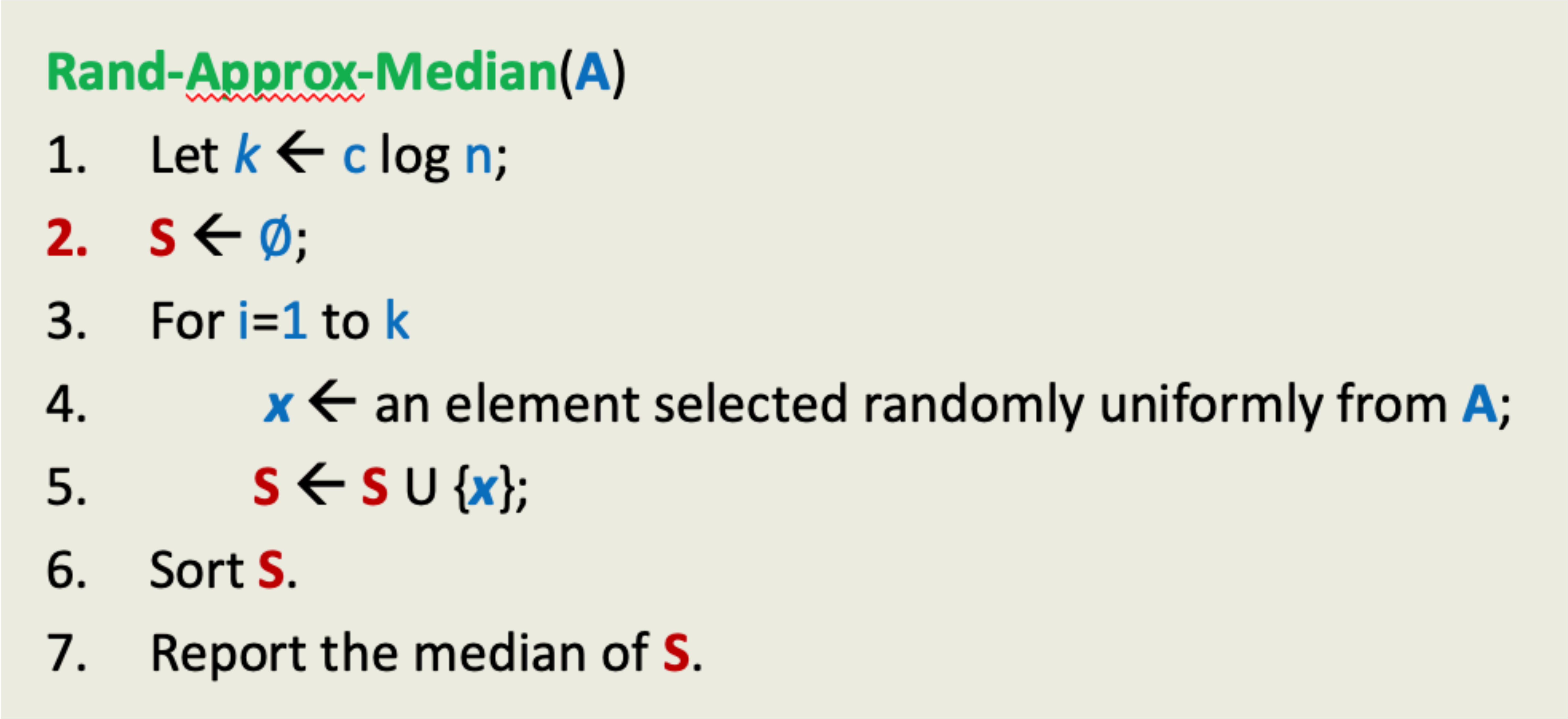
- For the $\gamma$-approximate median, where $0 < \gamma \leq 1/2$, we had
    - **Monte Carlo** algorithm that runs in $O(c \log n)$ with a failure prob $(1-2\gamma)^c$
    - **Las Vegas** algorithm that returns a correct answer expectedly in $O(n/\gamma)$ time
- Now we assume $\gamma = 1/4$, and try to come up with a Monte Carlo type algorithm that runs in $O(\log n \log \log n)$ time with a failure probability of $2n^{-c/5}$ over $n$ numbers

# Approximate Median Revisited



- Assume we randomly select $k$ integers, sort them and return the median.
- When is that computed not in the shaded region and thus, erroneous ?
    - That happens if at least $k/2$ are chosen from the left quarter (or the right quarter).
    - The probability that $k/2$ are from the left quarter is $(1/2)^{k/5}$ (with some detailed probability calculations)
    - So, if we chose $k = c \log n$, it is $n^{-c/5}$.
    - Also considering the right quarter, the total failure probability is $2n^{-c/5}$

# Approximate Median Revisited



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Rand-Approx-Median(A)
Let $k$ ← c log n;
S ← Ø;
For i=1 to k
    $x$ ← an element selected randomly uniformly from A;
    S ← S U {$x$};
Sort S.
Report the median of S.
```

- Complexity is $O(\log n \log \log n)$
- Assume $c = 10$, then the failure probability is $2n^{-c/5} = 2n^{-2}$
- Particularly efficient for large $n$

# Matrix Multiplication Verification

## Freivald’s Algorithm

$$\begin{bmatrix} \mathrm{A} \end{bmatrix}_{n \times n} \begin{bmatrix} \mathrm{B} \end{bmatrix}_{n \times n} \overset{?}{=} \begin{bmatrix} \mathrm{C} \end{bmatrix}_{n \times n}$$

- We would like to verify whether $A \times B = C$, where all are $n \times n$ matrices.
- We can repeat the operation, and then check, $O(n^3)$ time
- Freivald’s randomized algorithm performs verification in $O(kn^2)$ time with a failure probability of $2^{-k}$. (*Yes, a Monte Carlo type*)

# Matrix Multiplication Verification

## Freivald’s Algorithm

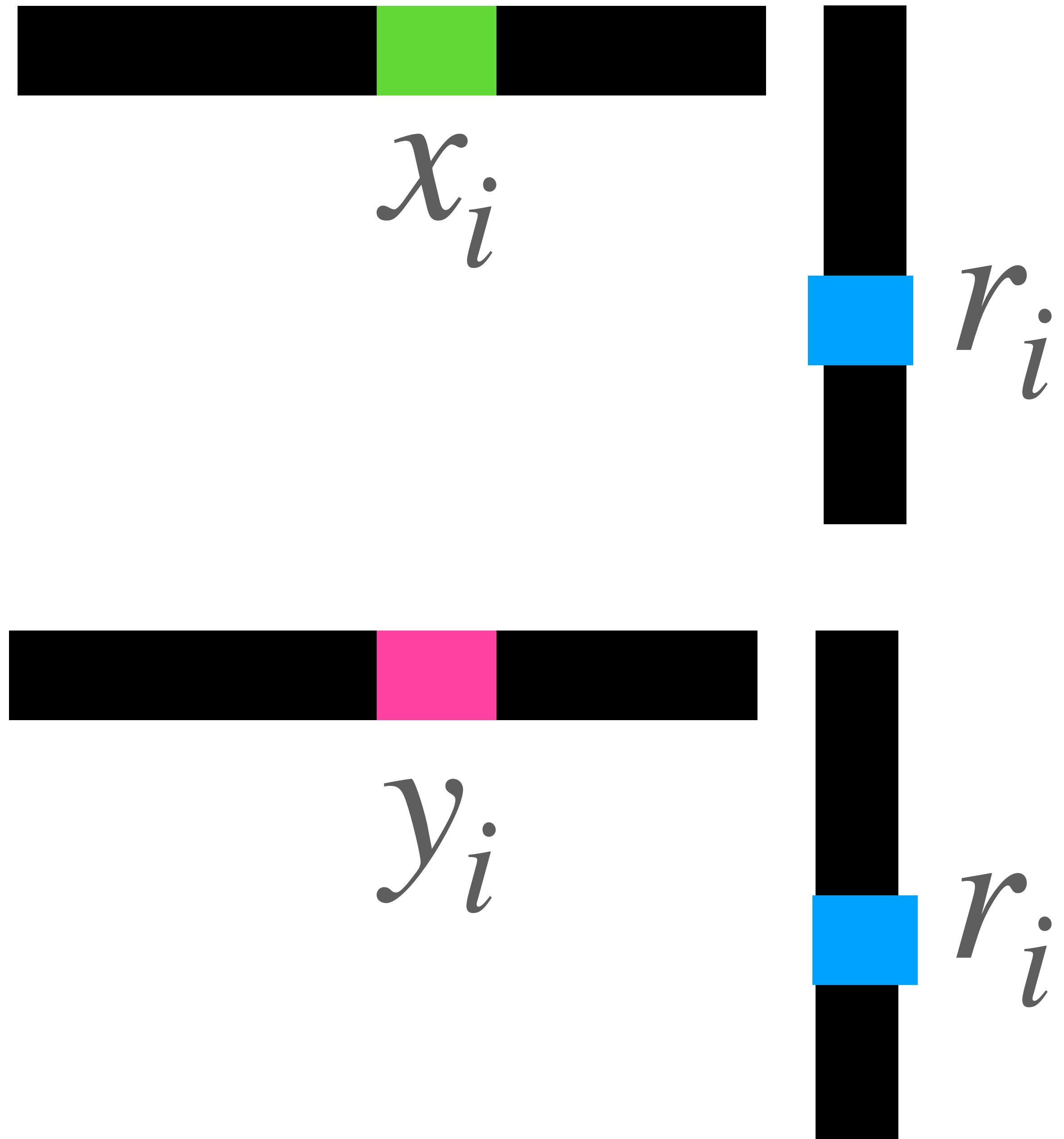
$$\underset{n \times n}{\begin{bmatrix} \quad A \quad \end{bmatrix}} \text{x} \underset{n \times n}{\begin{bmatrix} \quad B \quad \end{bmatrix}} \text{x} \underset{n \times 1}{\begin{bmatrix} r \end{bmatrix}} \overset{?}{=} \underset{n \times n}{\begin{bmatrix} \quad C \quad \end{bmatrix}} \text{x} \underset{n \times 1}{\begin{bmatrix} r \end{bmatrix}}$$

- Pick a random vector $r = [r_1 \ r_2 \ r_3 \ldots r_n]$, where $r_i \in \{0,1\}$.
- If $A \times B \times r = C \times r$, then output YES, otherwise NO.
- If NO, then $A \times B = C$ is not correct, but otherwise, there is a probability that the multiplication is wrong, but the test generated  a false true.
- The time complexity is $O(n^2)$.

# Matrix Multiplication Verification

## Freivald’s Algorithm

- **What is the probability of that failure in Freivald’s method ?**
- Let $x$ and $y$ be two vectors and $r$ is a vector of randomly selected 0s and 1s.
- If $x \neq y$, it is still possible that $x^t r = y^t r$.
- If $x \neq y$, then there is at least one dimension, where $x_i \neq y_i$



$x_i r_i + \alpha = y_i r_i + \beta$ , where $\alpha$ and $\beta$ are the sum of the remaining dimensions

| | $\alpha = \beta$ | $\alpha \neq \beta$ |
|---|---|---|
| $r_i = 0$ | | X |
| $r_i = 1$ | X | |

X marks the cases that reveal $x \neq y$.

**With 1/2 probability, the verification might be wrong.**

# Matrix Multiplication Verification

## Freivald’s Algorithm

- **What is the probability of that failure in Freivald’s method ?**
- Assume $D = AB$, and we check whether $Dr = Cr$.
- The previous analysis showed that it may hide the truth with 1/2 probability.
- Therefore, if we apply it $k$ times, the failure probability is $2^{-k}$ and the time complexity becomes $O(k \cdot n^2)$.
- Notice that this is better than $O(n^3)$.

# Randomized Approximation for MAX 3-SAT

- 3-SAT: Given $k$ clauses $C_1, C_2, \ldots, C_k$, where each clause is the disjunction of three variables from $X = \{x_1, x_2, \ldots, x_n\}$, does there a truth assignment that satisfies all clauses?

- The core problem in NP-completeness.

- MAX 3-SAT: Satisfy as many clauses as possible.

**Interestingly, a random assignment of truth values to $X$ is expected to satisfy 7/8 of all $k$ clauses !**

# Randomized Approximation for MAX 3-SAT

- $(x_i \vee x_j \vee x_k)$ turns false only when $x_i = x_j = x_k = 0$.
- Since we randomly selected the truth values for variables, each variable is 0 or 1 with probability 1/2
- Therefore, per each clause, the probability that it becomes false is 1/8, which means any random assignment satisfies the clause with 7/8 probability.
- Assume $Z_i = 1$ if clause $C_i$ is satisfied, where $E[Z_i] = 7/8$.
- $E[Z_1 + Z_2 + \ldots + Z_k] = E[Z_1] + E[Z_2] + \ldots + E[Z_k] = k \cdot \frac{7}{8}$

This implies there should be a truth assignment for every SAT formula that satisfies 7/8 of the clauses !