Applied Algorithms

CSCI-B505 / INFO-I500

Lecture 17.

Graph Data Structures and Algorithms - I

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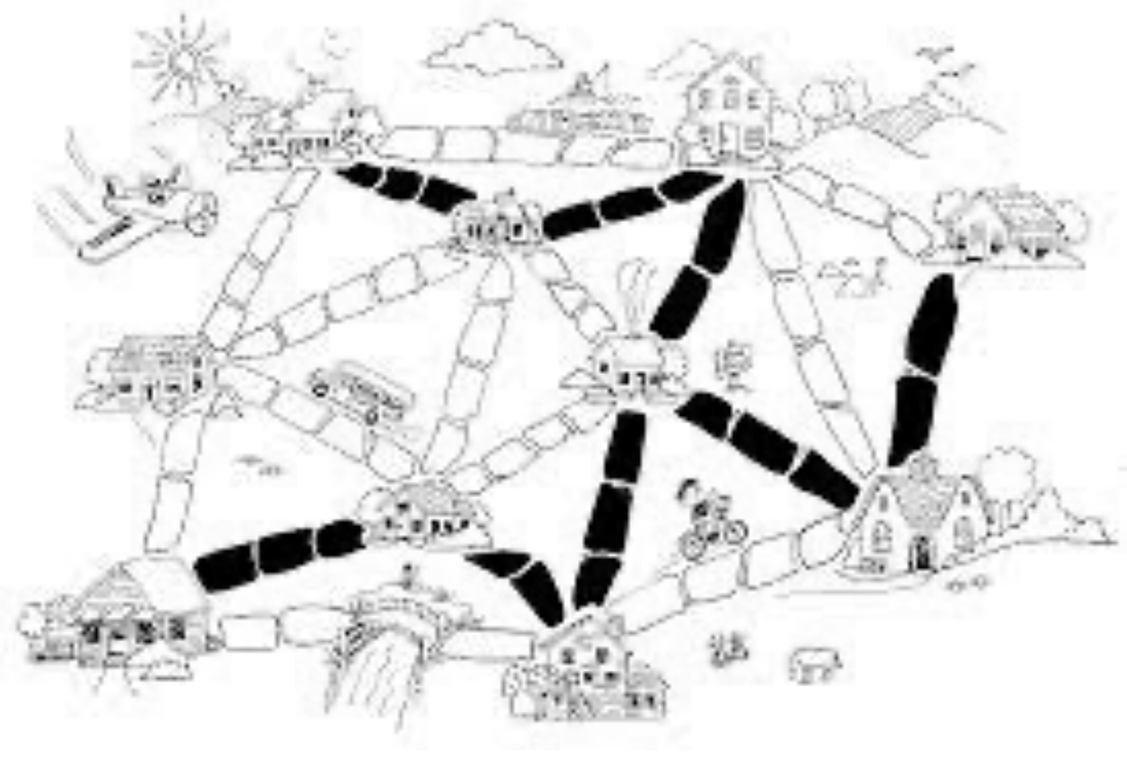
- Graphs
- How to represent graphs
- Graph Traversals

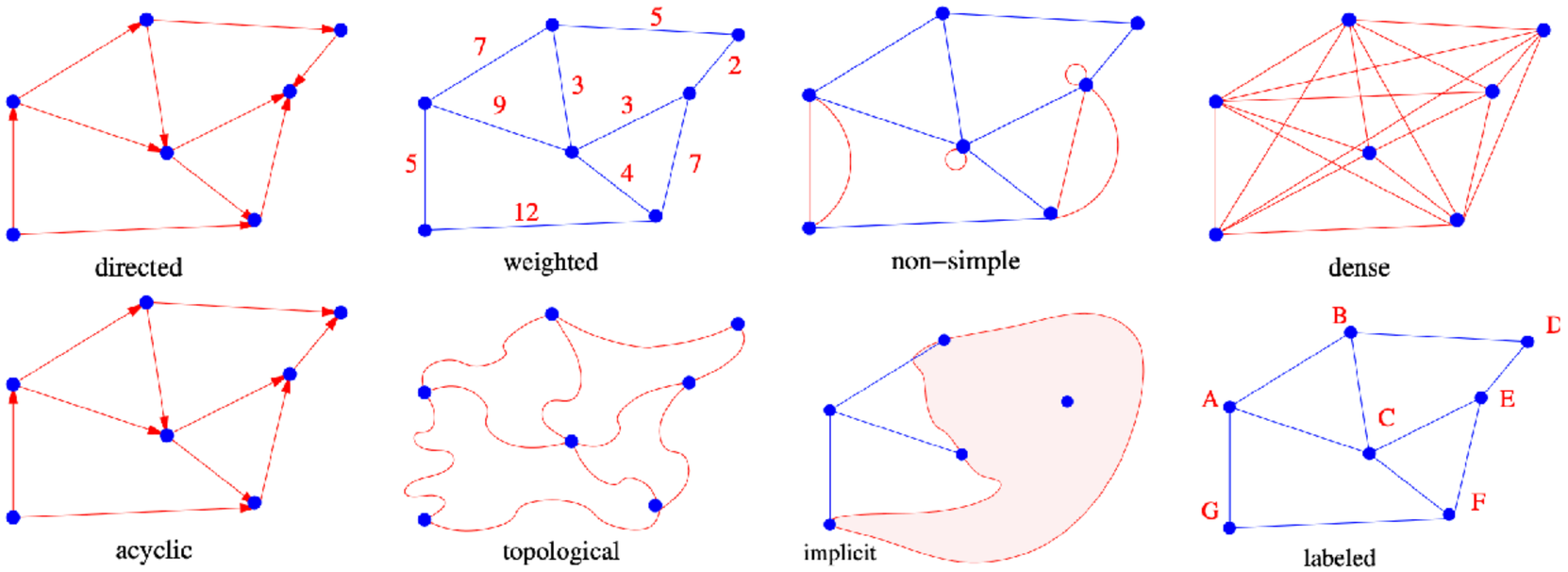
Graphs

- Graphs encode many practical real life scenarios
- Major usage: Modeling our problem with a graph representation and investigating which graph algorithms can help us to solve it.
- Notice that it is very hard to come up with a novel graph algorithm



Some terminology

- G(V,E) represents a graph.



Graph Algorithms - General View

Polynomial Time

- Connected components
- Minimum spanning tree
- Shortest path
- Eulerian cycle
- Edge/vertex connectivity
- Transitive closure
- Network flow
- Planarity testing
- Graph drawing
- …..

NP-Hard

- Clique
- Independent set/vertex cover
- Traveling salesman
- Hamiltonian cycle
- Vertex/edge coloring
- Graph partitioning
- Graph isomorphism
- ….

Representing Graphs

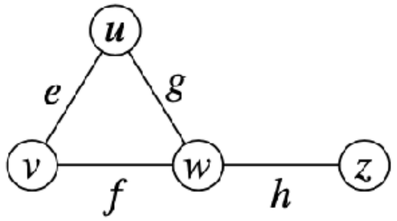
- Different data structures
- Important notice: The complexity of graph algorithms strongly depend on the data structure used to represent the graph.

Operation	Edge List	Adj. List	Adj. Map	Adj. Matrix
vertex_count()	$O(1)$	$O(1)$	$O(1)$	$O(1)$
edge_count()	$O(1)$	$O(1)$	$O(1)$	$O(1)$
vertices()	$O(n)$	$O(n)$	$O(n)$	$O(n)$
edges()	$O(m)$	$O(m)$	$O(m)$	$O(m)$
get_edge(u,v)	$O(m)$	$O(\min(d_u, d_v))$	$O(1)$ exp.	$O(1)$
degree(v)	$O(m)$	$O(1)$	$O(1)$	$O(n)$
incident_edges(v)	$O(m)$	$O(d_v)$	$O(d_v)$	$O(n)$
insert_vertex(x)	$O(1)$	$O(1)$	$O(1)$	$O(n^2)$
remove_vertex(v)	$O(m)$	$O(d_v)$	$O(d_v)$	$O(n^2)$
insert_edge(u,v,x)	$O(1)$	$O(1)$	$O(1)$ exp.	$O(1)$
remove_edge(e)	$O(1)$	$O(1)$	$O(1)$ exp.	$O(1)$

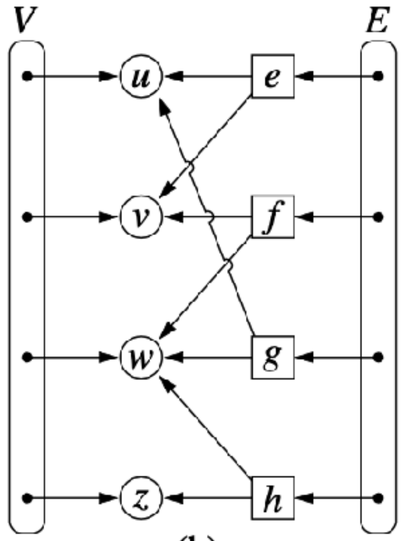
Table 14.1: A summary of the running times for the methods of the graph ADT, using the graph representations discussed in this section. We let n denote the number of vertices, m the number of edges, and d_v the degree of vertex v. Note that the adjacency matrix uses $O(n^2)$ space, while all other structures use $O(n+m)$ space.

- Edge list
- Adjacency list/map
- Adjacency matrix

Edge List



(a)



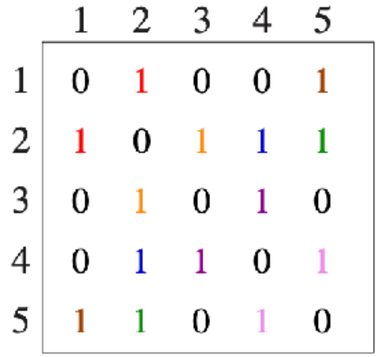
(b)

Operation	Running Time
vertex_count( ), edge_count( )	$O(1)$
vertices( )	$O(n)$
edges( )	$O(m)$
get_edge(u,v), degree(v), incident_edges(v)	$O(m)$
insert_vertex(x), insert_edge(u,v,x), remove_edge(e)	$O(1)$
remove_vertex(v)	$O(m)$

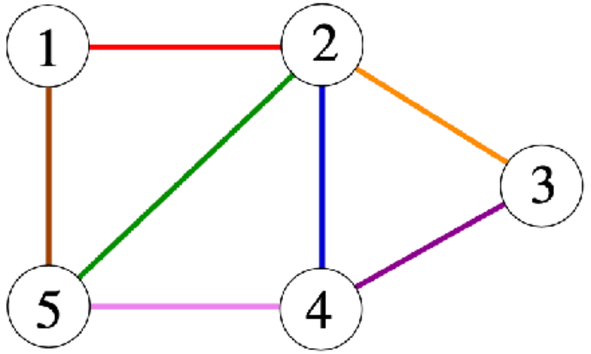
Adjacency Matrix / List

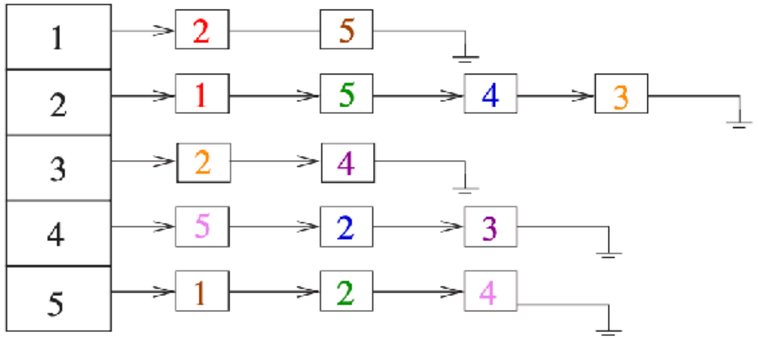
Comparison	Winner
Faster to test if (x, y) is in graph?	adjacency matrices
Faster to find the degree of a vertex?	adjacency lists
Less memory on sparse graphs?	adjacency lists $(m+n)$ vs. (n^2)
Less memory on dense graphs?	adjacency matrices (a small win)
Edge insertion or deletion?	adjacency matrices $O(1)$ vs. $O(d)$
Faster to traverse the graph?	adjacency lists $\Theta(m+n)$ vs. $\Theta(n^2)$
Better for most problems?	adjacency lists

?



	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0





Adjacency Matrix

Operation	Edge List	Adj. List	Adj. Map	Adj. Matrix
vertex_count()	$O(1)$	$O(1)$	$O(1)$	$O(1)$
edge_count()	$O(1)$	$O(1)$	$O(1)$	$O(1)$
vertices()	$O(n)$	$O(n)$	$O(n)$	$O(n)$
edges()	$O(m)$	$O(m)$	$O(m)$	$O(m)$
get_edge(u,v)	$O(m)$	$O(\min(d_u, d_v))$	$O(1)$ exp.	$O(1)$
degree(v)	$O(m)$	$O(1)$	$O(1)$	$O(n)$
incident_edges(v)	$O(m)$	$O(d_v)$	$O(d_v)$	$O(n)$
insert_vertex(x)	$O(1)$	$O(1)$	$O(1)$	$O(n^2)$
remove_vertex(v)	$O(m)$	$O(d_v)$	$O(d_v)$	$O(n^2)$
insert_edge(u,v,x)	$O(1)$	$O(1)$	$O(1)$ exp.	$O(1)$
remove_edge(e)	$O(1)$	$O(1)$	$O(1)$ exp.	$O(1)$

← R/S Dictionaries, O(1)

Adjacency List

Operation	Running Time
vertex_count(), edge_count()	$O(1)$
vertices()	$O(n)$
edges()	$O(m)$
get_edge(u,v)	$O(\min(\deg(u), \deg(v)))$
degree(v)	$O(1)$
incident_edges(v)	$O(\deg(v))$
insert_vertex(x), insert edge(u,v,x)	$O(1)$
remove_edge(e)	$O(1)$
remove_vertex(v)	$O(\deg(v))$

Graph Traversal

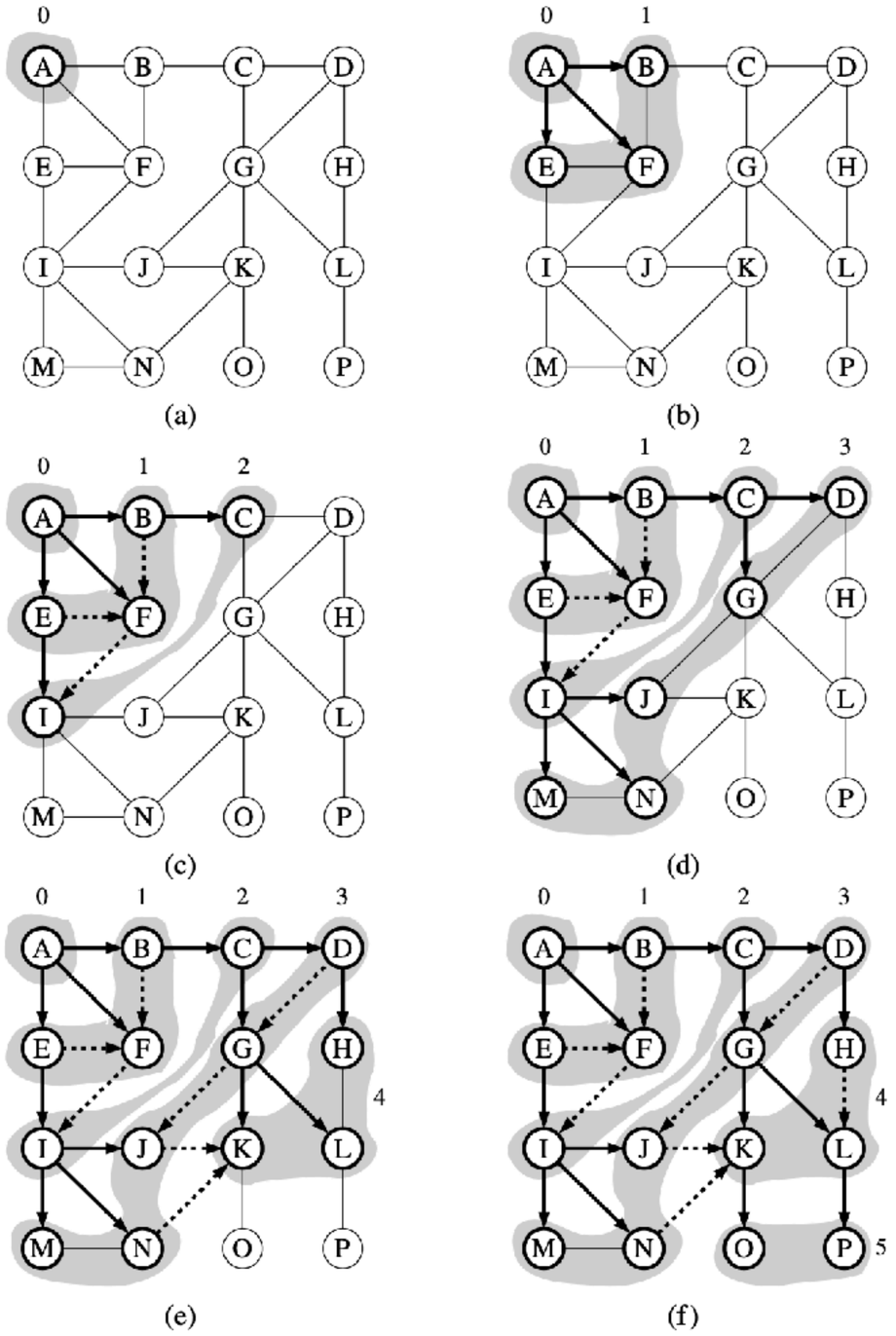
- Visit every vertex in a graph
- A key operation in many graph algorithms, e.g., connected components, reachability, shortest-path, spanning-tree, cycle-detection etc…
- We need a systematic way to avoid again and again visiting the same vertex or get stuck somewhere in the graph
- So, keep track of visited vertices
- Breadth-first or depth-first

Breadth-First Graph Traversal

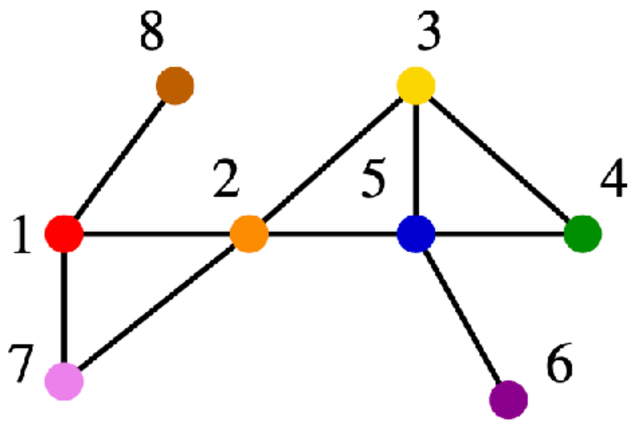
```
1  def BFS(g, s, discovered):
2    """Perform BFS of the undiscovered portion of Graph g starting at Vertex s.
3
4    discovered is a dictionary mapping each vertex to the edge that was used to
5    discover it during the BFS (s should be mapped to None prior to the call).
6    Newly discovered vertices will be added to the dictionary as a result.
7    """
8    level = [s]                          # first level includes only s
9    while len(level) > 0:
10     next_level = [ ]                   # prepare to gather newly found vertices
11     for u in level:
12       for e in g.incident_edges(u):    # for every outgoing edge from u
13         v = e.opposite(u)
14         if v not in discovered:        # v is an unvisited vertex
15           discovered[v] = e            # e is the tree edge that discovered v
16           next_level.append(v)         # v will be further considered in next pass
17     level = next_level                 # relabel 'next' level to become current
```

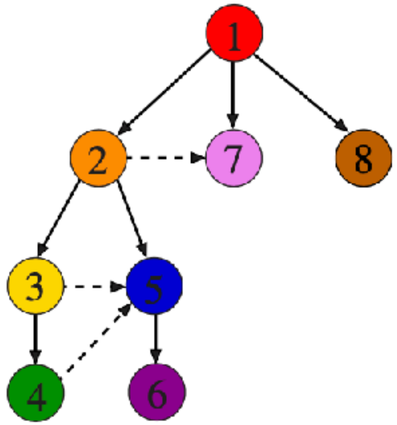
$O(n + m)$-time with adjacency list data structure

Breadth-first search procedure has some interesting properties….



Breadth-First Graph Traversal





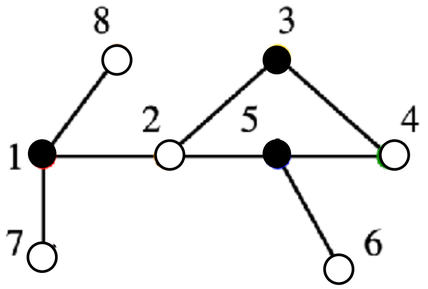
Proposition 14.16: *Let G be an undirected or directed graph on which a BFS traversal starting at vertex s has been performed. Then*

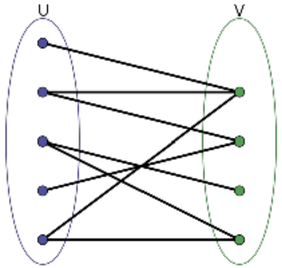
- *The traversal visits all vertices of G that are reachable from s.*
- *For each vertex v at level i, the path of the BFS tree T between s and v has i edges, and any other path of G from s to v has at least i edges.*
- *If (u, v) is an edge that is not in the BFS tree, then the level number of v can be at most 1 greater than the level number of u.*

"a path in a breadth- first search tree rooted at vertex s to any other vertex v is guaranteed to be the **shortest** such path from s to v **in terms of the number of edges** "

BFS can also be used in

- detecting the connected components of a graph
- two-coloring verification of the graph (is it a bipartite graph)

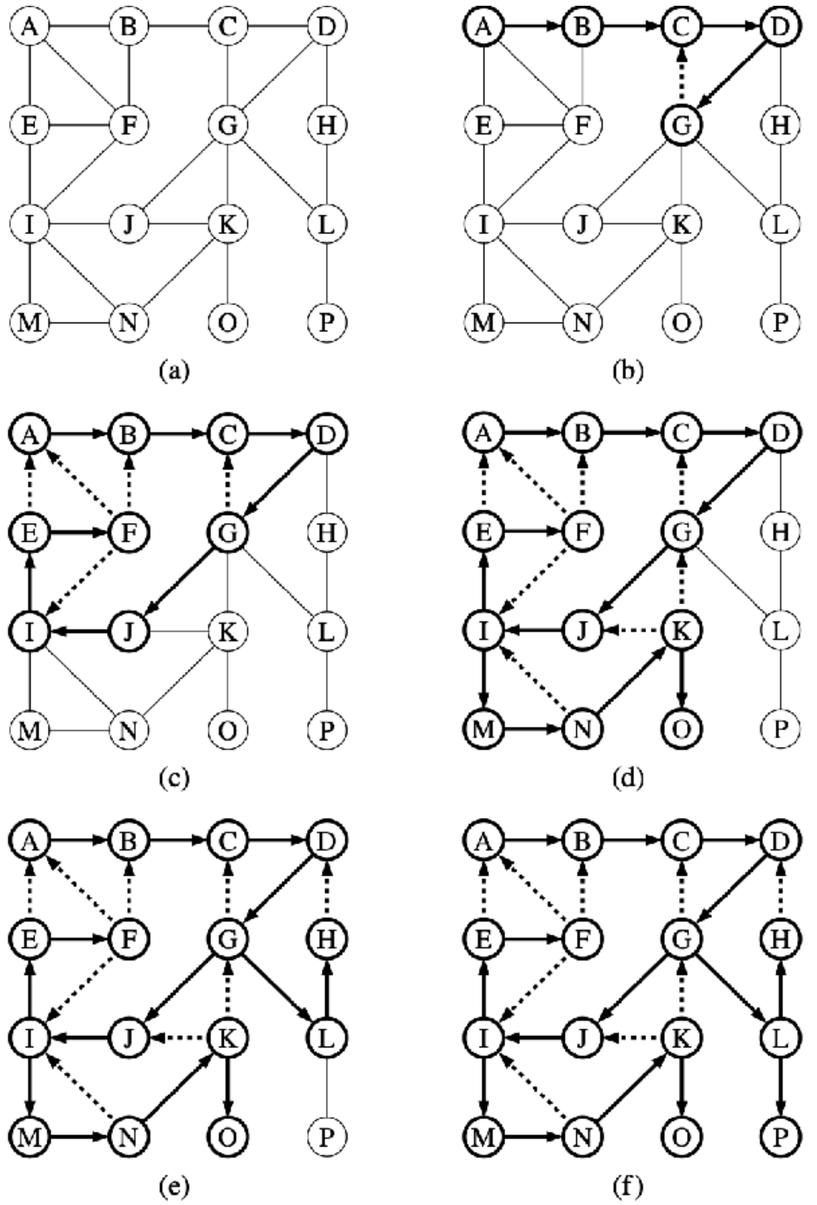




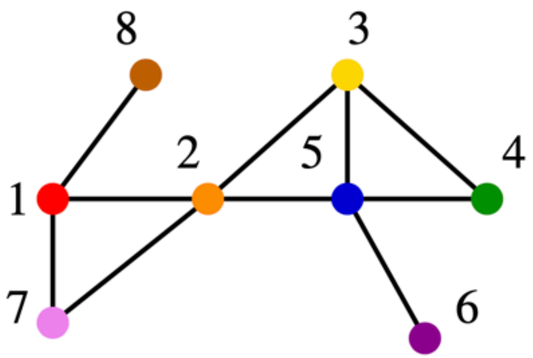
Depth-First Graph Traversal

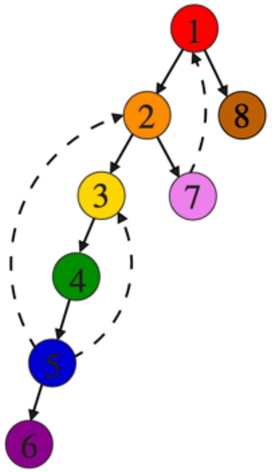
Algorithm DFS(G,u): {We assume u has already been marked as visited}
Input: A graph G and a vertex u of G
Output: A collection of vertices reachable from u, with their discovery edges
for each outgoing edge e = (u,v) of u **do**
if vertex v has not been visited **then**
Mark vertex v as visited (via edge e).
Recursively call DFS(G,v).

$O(n + m)$-time with adjacency list data structure



Depth-First Graph Traversal





Proposition 14.14: *Let G be an undirected graph with n vertices and m edges. A DFS traversal of G can be performed in $O(n+m)$ time, and can be used to solve the following problems in $O(n+m)$ time:*

- *Computing a path between two given vertices of G, if one exists.*
- *Testing whether G is connected.*
- *Computing a spanning tree of G, if G is connected.*
- *Computing the connected components of G.*
- *Computing a cycle in G, or reporting that G has no cycles.*

Proposition 14.15: *Let $\vec{G}$ be a directed graph with n vertices and m edges. A DFS traversal of $\vec{G}$ can be performed in $O(n+m)$ time, and can be used to solve the following problems in $O(n+m)$ time:*

- *Computing a directed path between two given vertices of $\vec{G}$, if one exists.*
- *Computing the set of vertices of $\vec{G}$ that are reachable from a given vertex s.*
- *Testing whether $\vec{G}$ is strongly connected.*
- *Computing a directed cycle in $\vec{G}$, or reporting that $\vec{G}$ is acyclic.*
- *Computing the* **transitive closure** *of $\vec{G}$ (see Section 14.4).*

DFS can also be used in

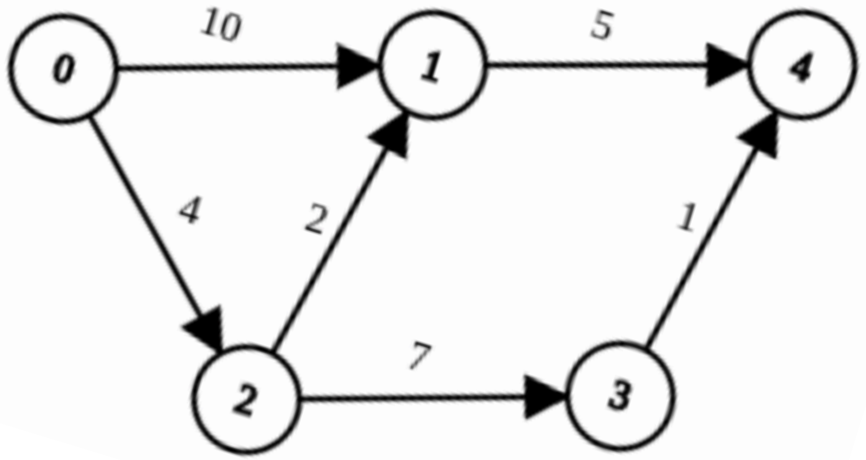
- Testing for connectivity
  - Undirected graphs: Start from a random vertex and see all vertices are reached or not
  - Directed graphs: Run DFS from a random point, reverse all directions of the graph and run DFS again. If all are visited on both cases, it is strongly connected.
- Computing all connected components: Run DFS from a random point. If unvisited vertices are left at the end, repeat until all are visited.
- Detecting cycles: Through the creation of the DFS tree…

Reading assignment

- Skiena chapter 7.

- Goodrich et al. 14.1, 14.2, 14.3

$D_0 = 0$ $D_1 = \infty$ $D_4 = \infty$



$D_2 = \infty$ $D_3 = \infty$