Applied Algorithms

CSCI-B505 / INFO-I500

Lecture 6.

Amortized Analysis - 1

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- Amortized Analysis
  - Aggregate Method
    - Accounting Method
  - Potential Method

Amortized Analysis ?

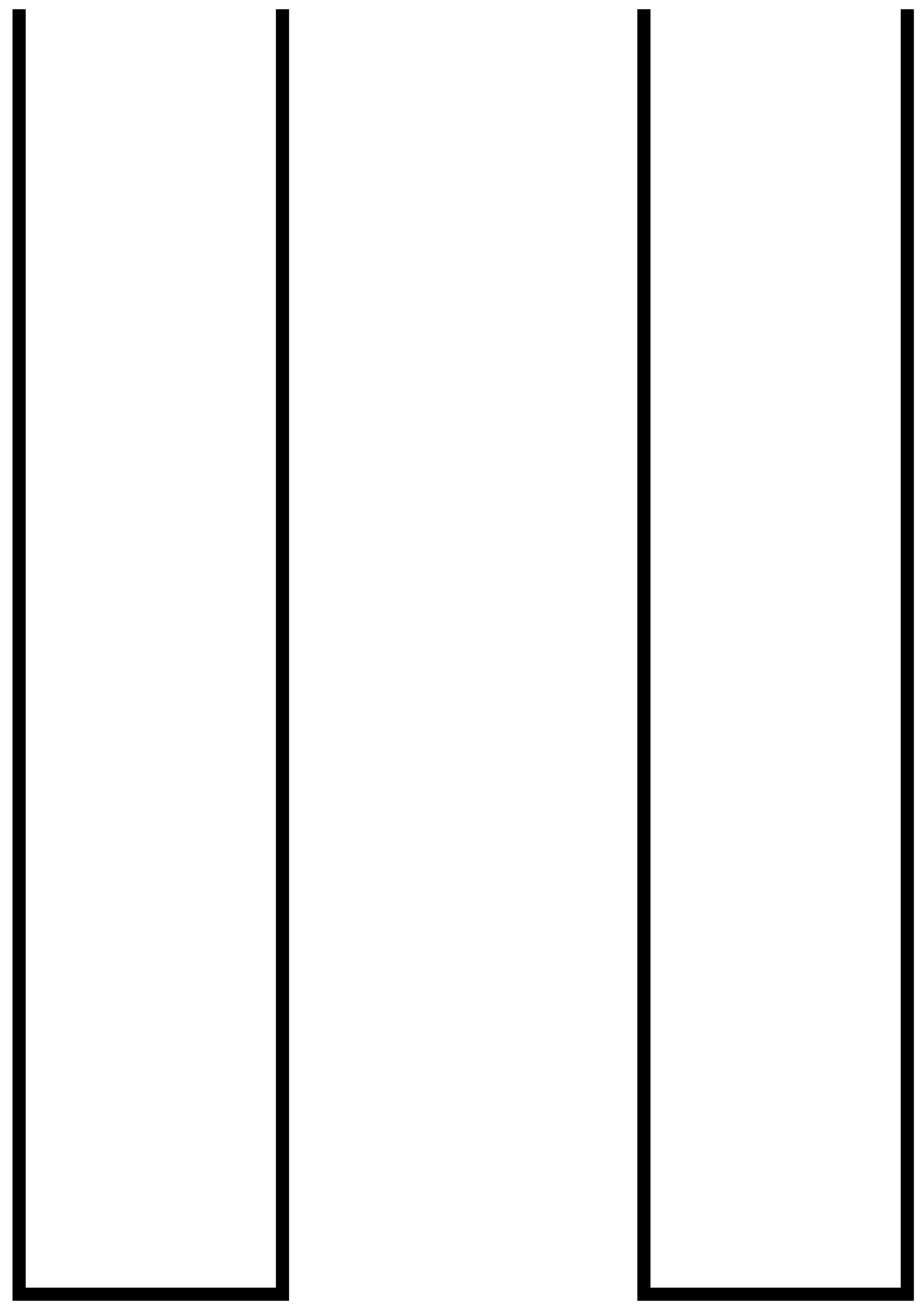
Amortize:

- *gradually write off the initial cost of (an asset) over a period*
- *reduce or pay off (a debt) with regular payments*

- In an algorithm, there may be cheap operations and expensive operations.
- Regular worst-case analysis assumes the expensive operations always dominate the execution.
- However, there can be a **deterministic** limit on the number of times *expensive* happens.

Let’s see on an example…

Queue Implementation with Two Stacks



Stack-1 Stack-2

Implement a queue by using two stacks.

Enqueue(x):
  Push x into stack-1.

Dequeue():
  If stack-2 is empty, then
    Pop everything from stack-1 and push into stack-2;
  Pop from stack-2

Enqueue(7)	Enqueue(2)	Enqueue(9)	Dequeu()	Enqueue(8)
Dequeu()	Dequeu()	Enqueue(1)	Enqueue(2)	Dequeu()

Queue Implementation with Two Stacks

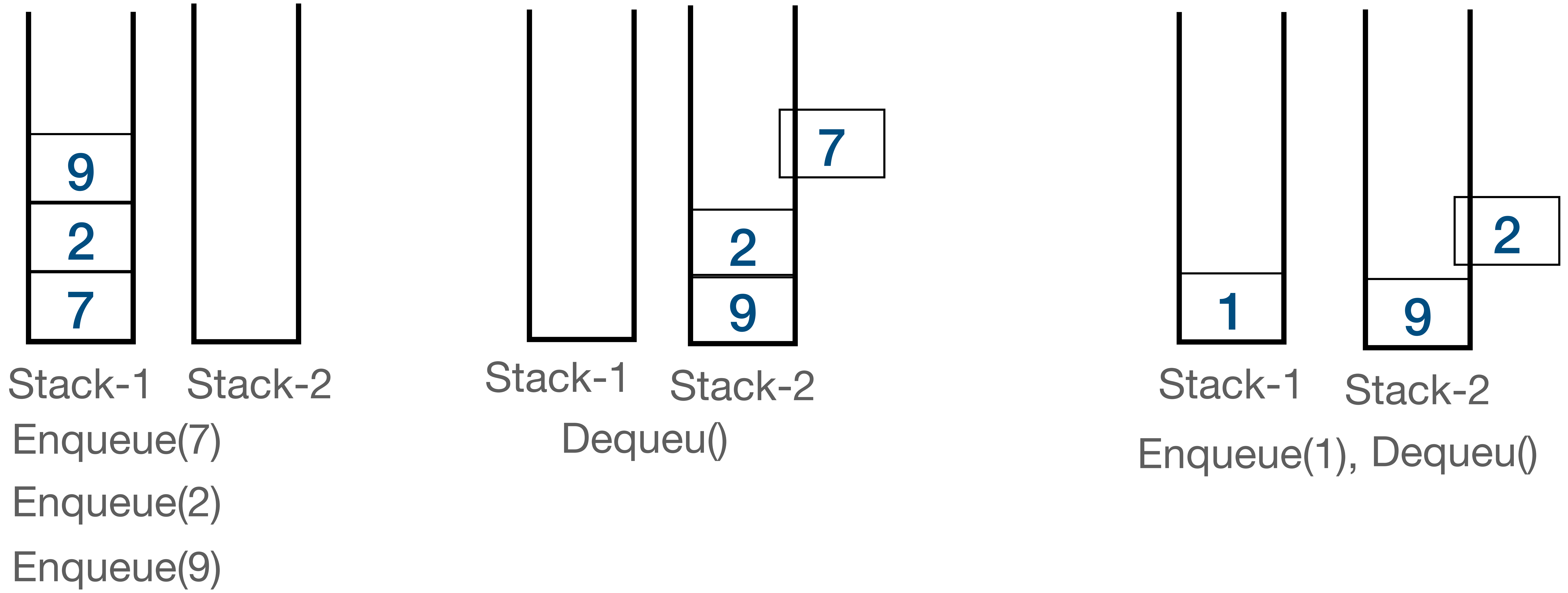
Assume n insert or fetch operations will be executed.
What will be the time-complexity on this implementation?

```
for (i=1 to n){
     operation = randomSelect(enqueue, dequeue);
     execute the operation;}
```

- Enqueue() is cheap, worst case $O(1)$-time.
- Dequeue() is expensive, worst-case $O(n)$-time.
- If I always do the expensive operation then worst-case complexity becomes $O(n^2)$!?

Is this correct ?

Queue Implementation with Two Stacks



Once an expensive ‘dequeue’ happens,
some of the following ‘dequeue’s will always be cheap.

Queue Implementation with Two Stacks

Once an expensive dequeue happens, the following ones are always cheap.

- How many times an item is inserted into the stack-1 and stack-2 ?
- How many times it is popped from stack-1 and stack-2 ?
- For n enqueue/dequeue, **at most** $4n$ push/pop are achieved
- $\frac{4n}{n} = 4 \in O(1)$ **per each enqueue/dequeu operation.**

This is **NOT average-case** analysis,
but a **worst-case** analysis.

Binary Counter

Assume we have a k-bit binary counter, and the cost of incrementing this counter is defined as being equal to the number of bits flipped. What is the cost of incrementing this counter n times?

A[5]	A[4]	A[3]	A[2]	A[1]	A[0]	COST
0	0	0	0	0	0	
0	0	0	0	0	**1**	1
0	0	0	0	**1**	**0**	2
0	0	0	0	1	**1**	1
0	0	0	**1**	**0**	**0**	3
0	0	0	1	0	**1**	1

```
INCREMENT(A)
  i = 0
  while i < A.length and A[i] == 1
        A[i] = 0
        i = i + 1
  if i < A.length
        A[i] = 1
```

Binary Counter

Regular worst-case analysis:

- At most how many bits can be flipped ?
  - All of the k bits, e.g., 011111 →100000
- Thus, if we consider n increments then it makes $O(n \cdot k)$

A[5]	A[4]	A[3]	A[2]	A[1]	A[0]	COST
0	0	0	0	0	0	
0	0	0	0	0	**1**	**1**
0	0	0	0	**1**	**0**	**2**
0	0	0	0	1	**1**	**1**
0	0	0	**1**	**0**	**0**	**3**
0	0	0	1	0	**1**	**1**
⋯	⋯	⋯	⋯	⋯	⋯	

INCREMENT(A)

$i = 0$
while $i < A.length$ and $A[i] == 1$
    $A[i] = 0$
    $i = i + 1$
if $i < A.length$
    $A[i] = 1$

However, it is not possible to have consecutive increments with k-bits flip! So…

Binary Counter

A[5]	A[4]	A[3]	A[2]	A[1]	A[0]	COST
0	0	0	0	0	0	
0	0	0	0	0	**1**	**1**
0	0	0	0	**1**	**0**	**2**
0	0	0	0	1	**1**	**1**
0	0	0	**1**	**0**	**0**	**3**
0	0	0	1	0	**1**	**1**
0	0	0	1	**1**	**0**	**2**

A[0] flips at each increment

A[1] flips once at each 2 increments

A[2] flips once at each 4 increments

……

A[k-1] flips once at each 2^{k-1} increments

So total cost of n increment operations is

$$n + \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \ldots + \left\lfloor \frac{n}{2^{k-1}} \right\rfloor < n \sum_{i=0}^{\infty} \frac{1}{2^i} = 2n$$

n increment operations cost less than 2n flips.

Thus, each increment costs 2, which makes $O(1)$ time per increment.

Aggregate Method

- Compute the total cost of n operations.
- Divide this cost by n to compute the cost of one operation.
- This is the **aggregate** method of amortized analysis.
- We have two alternative approaches, ***accounting*** and ***potential***

Accounting Method

- Again we assume n operations will be achieved.
- We compute an **amortized cost** per operation
- Before each operation, we deposit in an account the **amortized cost** of that operation.
- Each operation drops exactly the regular **required** amount from the account, where the excess amount from cheap operations are expected to **amortize** the expensive ones.
- If there appears a case that there is not enough money in the account (**bankruptcy**), then the operation can not be performed. Thus, it should be strictly avoided.
- **What should we assume the amortized cost to avoid the bankruptcy ?**

Accounting Method

$$\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} \hat{c}_i$$

- The regular cost of the i^{th} operation is c_i
- The amortized cost of i^{th} operation is $\hat{c}_i$
- **For any $k = 1 \ldots n$, the total regular cost should never exceed the total amortized cost.**

Accounting Method

Queue with Two Stacks

	Stack operations	Stack-1	Stack-2	Actual Cost	Deposit	Remaining Balance
Enqueue(7)	1-push	7		1	\$2	\$1
Enqueue(2)	1-push	7.2		1	\$2	\$2
Enqueue(9)	1-push	7,2,9		1	\$2	\$3
Dequeue()	3-pop, 3-push, 1-pop		9.2	7	\$0	**BANKRUPTCY** !
Dequeue()	1-pop		2	1		
Enqueue(8)	1-push	8	2	1		
Dequeue()	1-pop	8		1		
Enqueue(7)	1-push	8.7		1		
Dequeue()	2-pop, 2-push, 1-pop		2.7	5		

Assume the amortized cost for enqueue is **\$2** , and dequeue is **free** !

Not a good choice ! We may face a bankruptcy

Accounting Method

Queue with Two Stacks

	Stack operations	Stack-1	Stack-2	Actual Cost	Deposit	Remaining Balance
Enqueue(7)	1-push	7		1	$4	$3
Enqueue(2)	1-push	7.2		1	$4	$6
Enqueue(9)	1-push	7,2,9		1	$4	$9
Dequeue()	3-pop, 3-push, 1-pop		9.2	7	**$0**	$2
Dequeue()	1-pop		2	1	**$0**	$1
Enqueue(8)	1-push	8	2	1	$4	$4
Dequeue()	1-pop	8		1	**$0**	$3
Enqueue(7)	1-push	8.7		1	$4	$6
Dequeue()	2-pop, 2-push, 1-pop		7	5	**$0**	$1

If we the amortized cost for enqueue is **$4** , and dequeue is free, then it seems no bankruptcy !
We know it takes no more than 4 stack operations per each item in the queue. So, we pay $4 dollars at the enqueue phase, and use the remaining $3 during the later dequeue operations.

Accounting Method

Binary Counter

```
INCREMENT(A)
i = 0
while i < A.length and A[i] == 1
    A[i] = 0
    i = i + 1
if i < A.length
    A[i] = 1
```

$1 \rightarrow 0$ flipping, FREE, no cost

$0 \rightarrow 1$ flipping, **$? amortized cost**

- There are two different flip operations as $1 \rightarrow 0$ and $0 \rightarrow 1$.
- At each increment **some number of** ones flip into 0, and **one** zero at the end flips to 1.
- Assume $1 \rightarrow 0$ is free, so no worries on '*some number of*'
- What should be the amortized cost of $0 \rightarrow 1$ to accommodate this?

Potential Method

$$\sum_{i=1}^{i=n} \hat{c}_i \quad = \quad \sum_{i=1}^{i=n} c_i + \phi(D_i) - \phi(D_{i-1}) \quad = \quad \phi(D_n) - \phi(D_0) + \sum_{i=0}^{i=n} c_i$$

- Again we consider n operations, but the focus is on the used data structure D.
- Function $\phi(D)$ that defines the potential energy of the data structure D.
- The amortized cost $\hat{c}_i = c_i + \phi(D_i) - \phi(D_{i-1})$, where c_i is the actual real cost.
- If $\phi(D_n) \geq \phi(D_0)$ can be maintained, then the amortized cost is fine since potential never goes negative.
- **The issue is to propose such a ϕ function.**

Potential Method

Queue with two stacks

Assume $\phi = 2 \cdot x$, where x is the number of items in stack-1.

After enqueue operation the potential increases by $2 = \phi(k+1) - \phi(k)$.

Therefore, the amortized cost is $\hat{c}_E = c_E + \phi(k+1) - \phi(k) = 1 + 2 = 3$.

How about amortized cost of $\hat{c}_D$. Two cases:

1) Stack-2 is not empty. Then, no change in the potential and the amortized cost is equal to actual cost of 1.
2) Stack-2 is empty, and stack-1 has k elements. Then the potential difference is $-2k$. The actual cost is $2k + 1$ (why?). So, amortized cost is $-2k + 2k + 1 = 1$.

Potential Method

Binary counter

ϕ **is the number of set bits (equal to 1) in the counter.**

After an increment, assume t_i bits are flipped from 1 to 0. Then the **actual cost** is $t_i + 1$.

(Think about a case like 011011 -> 011100, $t_i = ?$)

$$\phi(D_i) - \phi(D_{i-1}) = [\phi(D_{i-1}) - t_i + 1] - \phi(D_{i-1}) = 1 - t_i\ .$$

Then the amortized cost is $t_i + 1 + 1 - t_i = 2$.

Reading assignment

- Read chapter 14 Amortized Analysis from Cormen.
- Read the paper 'Amortized Computational Complexity' by Tarjan, which dates back to 1985, on a very nice review of what amortized analysis is. Here is the link https://www.cs.princeton.edu/courses/archive/spr06/cos423/Handouts/Amortized.pdf
- Yet another paper I suggest you to look at is Amortized Efficiency of List Update and Paging Rules available at https://scholar.google.com/scholar?output=instlink&q=info:gElaOowSipkJ:scholar.google.com/&hl=en&as_sdt=0,15&scillfp=3816853723830843295&oi=lle
- We will study some further examples in the next lecture.