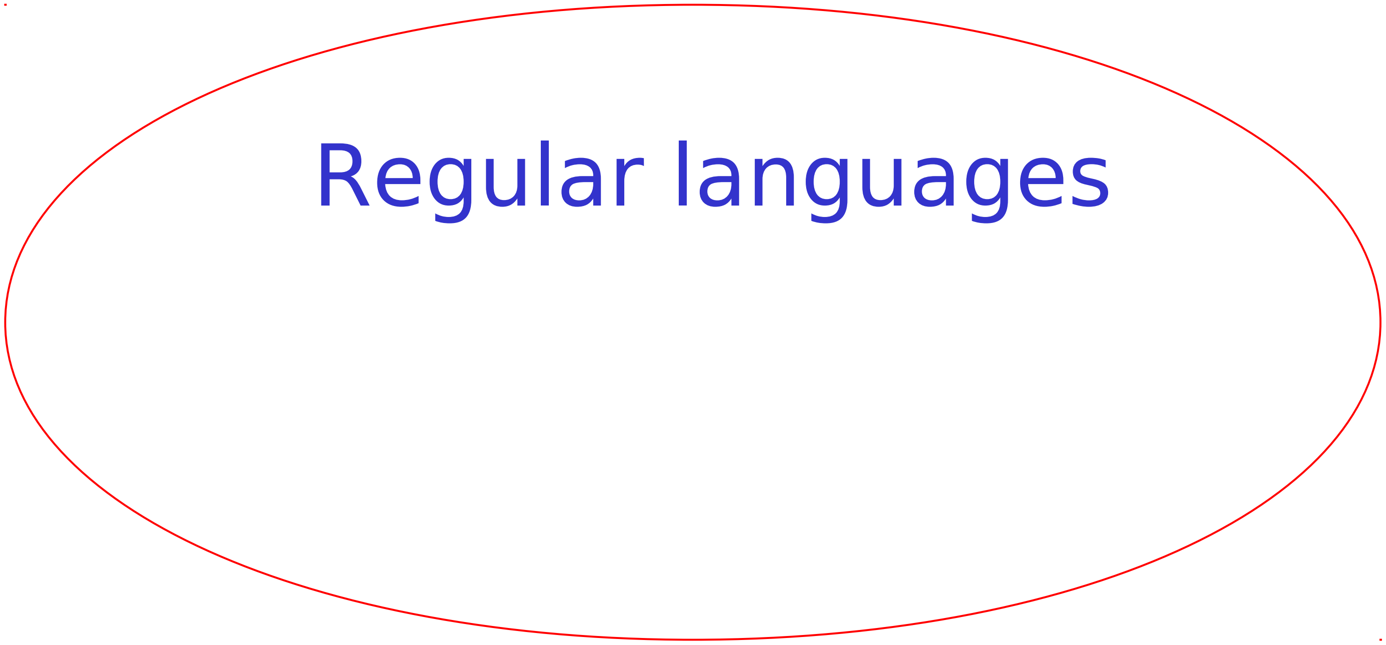
Pumping Lemma Theorem and Examples

The Pumping Lemma:

- Given a infinite regular language L
- there exists an integer m (critical length)
- for any string $w \in L$ with length $|w| \geq m$
- we can write $w = x\, y\, z$
- with $|x\, y| \leq m$ and $|y| \geq 1$
- such that: $x\, y^i\, z \in L \qquad i = 0, 1, 2, \ldots$

Non-regular languages

$$L = \{vv^R : v \in \Sigma^*\}$$



Regular languages

Theorem: The language

$$L = \{vv^R : v \in \Sigma^*\} \qquad \Sigma = \{a, b\}$$

is not regular

Proof: Use the Pumping Lemma

$$L = \{vv^R : v \in \Sigma^*\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$$L = \{vv^R : v \in \Sigma^*\}$$

Let m be the critical length for L

Pick a string w such that: $w \in L$

and length $|w| \geq m$

We pick $w = a^m b^m b^m a^m$

From the Pumping Lemma:

we can write: $w = a^m b^m b^m a^m = x\ y\ z$

with lengths: $|x\,y| \leq m, \quad |y| \geq 1$

$$w = xyz = \overbrace{\underbrace{a\ldots a}_{x}\underbrace{a\ldots a}_{y}\ldots a}^{m}\underbrace{\overbrace{b\ldots b}^{m}\overbrace{b\ldots b}^{m}\overbrace{a\ldots a}^{m}}_{z}$$

Thus: $y = a^k, \quad 1 \leq k \leq m$

$x\,y\,z = a^m b^m b^m a^m$ $\quad y = a^k, \quad 1 \le k \le m$

From the Pumping Lemma: $x\,y^i\,z \in L$

$i = 0, 1, 2, \ldots$

Thus: $x\,y^2\,z \in L$

$x\,y\,z = a^m b^m b^m a^m$ $y = a^k, \quad 1 \le k \le m$

From the Pumping Lemma: $x\,y^2\,z \in L$

$$xy^2z = \overbrace{\underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...a}_{y}}^{m+k}\underbrace{...a\overbrace{b...b}^{m}\overbrace{b...b}^{m}\overbrace{a...a}^{m}}_{z} \in L$$

Thus: $a^{m+k}b^m b^m a^m \in L$

$$a^{m+k}b^m b^m a^m \in L \qquad k \geq 1$$

BUT: $L = \{vv^R : v \in \Sigma^*\}$

$$\Downarrow$$

$$a^{m+k}b^m b^m a^m \notin L$$

CONTRADICTION!!!

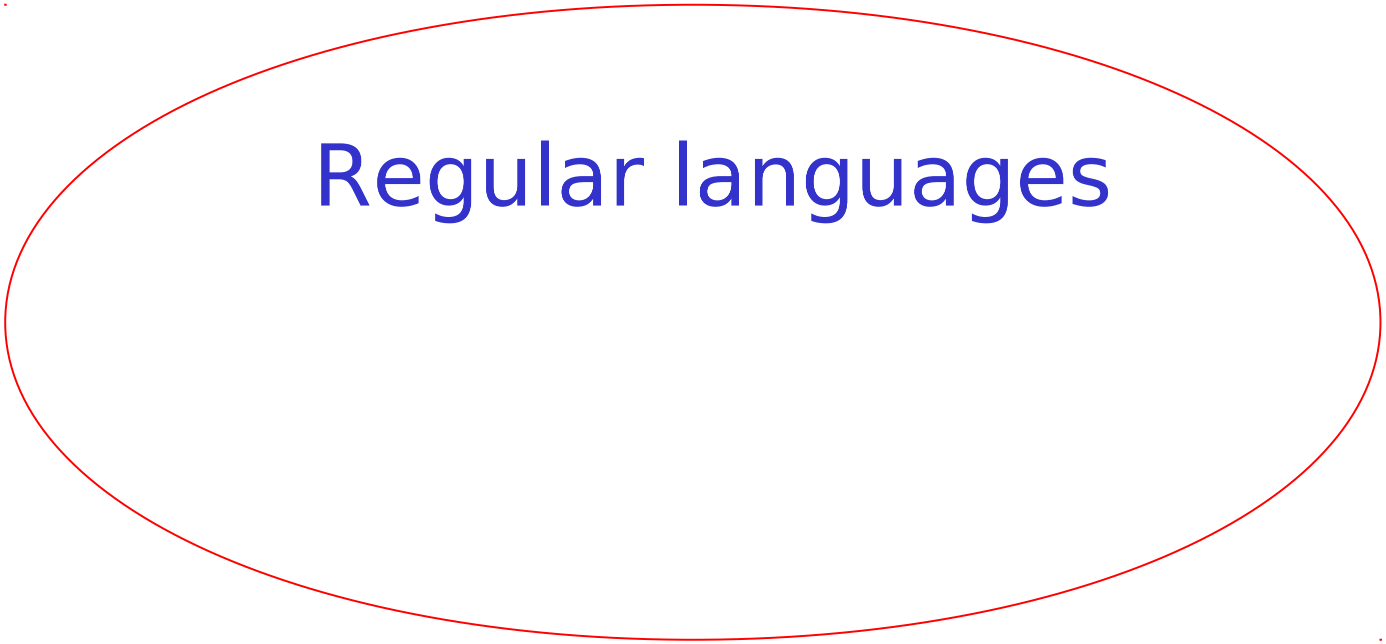
Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

END OF PROOF

Non-regular languages

$$L = \{a^n b^l c^{n+l} : \ n, l \geq 0\}$$



Regular languages

Theorem: The language

$$L = \{a^n b^l c^{n+l} : n, l \geq 0\}$$

is not regular

Proof: Use the Pumping Lemma

$$L = \{a^n b^l c^{n+l} : n,l \geq 0\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$$L = \{a^n b^l c^{n+l} : n,l \geq 0\}$$

Let m be the critical length of L

Pick a string w such that: $w \in L$ and

length $|w| \geq m$

We pick $w = a^m b^m c^{2m}$

From the Pumping Lemma:

We can write $w = a^m b^m c^{2m} = x\ y\ z$

With lengths $|x\ y| \leq m, \quad |y| \geq 1$

$$w = xyz = \overbrace{\underbrace{a\ldots a}_{x}\underbrace{a\ldots a}_{y}\rlap{$\underbrace{}_{z}$}a\ldots a}^{m}\overbrace{b\ldots b}^{m}\overbrace{c\ldots cc\ldots c}^{2m}$$

Thus: $y = a^k, \quad 1 \leq k \leq m$

$x\,y\,z = a^m b^m c^{2m}$ $y = a^k, \quad 1 \le k \le m$

From the Pumping Lemma: $x\,y^i\,z \in L$

$i = 0, 1, 2, \ldots$

Thus: $x\,y^0\,z = xz \in L$

$x\,y\,z = a^m b^m c^{2m}$ $\qquad$ $y = a^k, \quad 1 \le k \le m$

From the Pumping Lemma: $\quad xz \in L$

$$xz = \overbrace{\underbrace{a\ldots a}_{x}a\ldots a}^{m-k}\underbrace{\overbrace{b\ldots b}^{m}\overbrace{c\ldots cc\ldots c}^{2m}}_{z} \in L$$

Thus: $\quad a^{m-k} b^m c^{2m} \in L$

$$a^{m-k} b^m c^{2m} \in L \qquad k \geq 1$$

BUT: $L = \{a^n b^l c^{n+l} : n, l \geq 0\}$

$$\Downarrow$$

$$a^{m-k} b^m c^{2m} \notin L$$

CONTRADICTION!!!

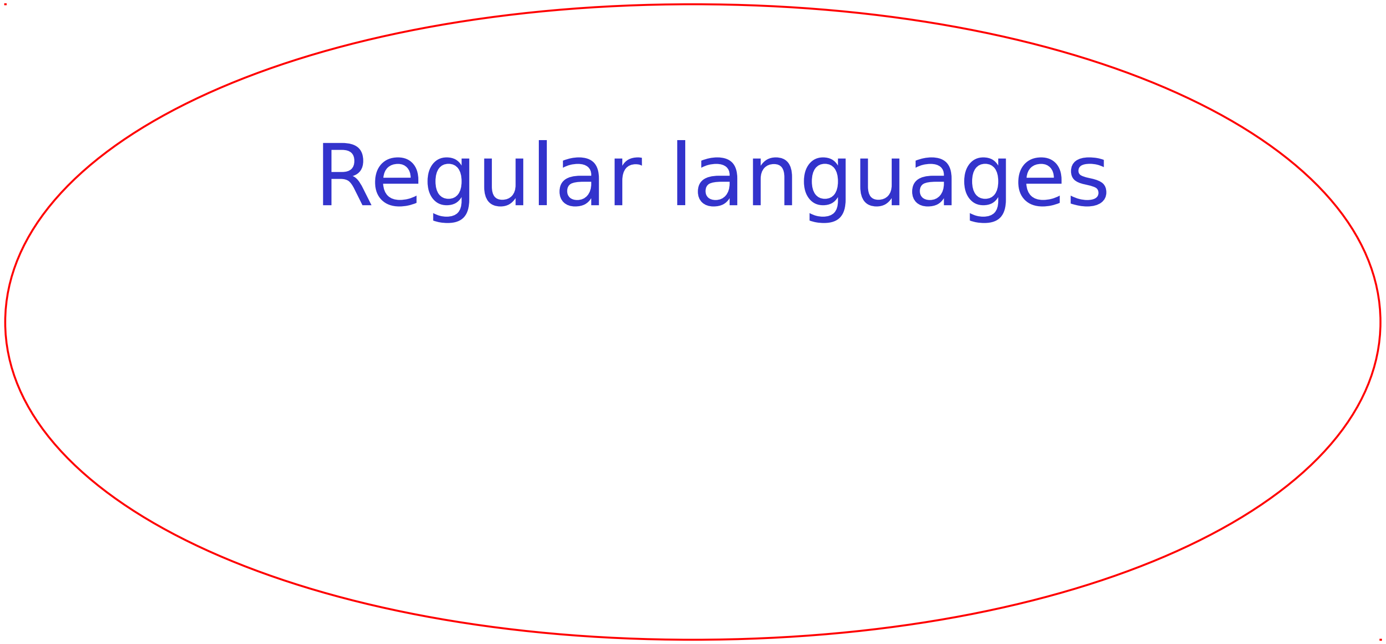
Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

END OF PROOF

Non-regular languages

$$L = \{a^{n!} : n \geq 0\}$$



Regular languages

Theorem: The language $L = \{a^{n!} : n \geq 0\}$

is not regular

$$n! = 1 \cdot 2 \cdots (n - 1) \cdot n$$

Proof: Use the Pumping Lemma

$$L = \{a^{n!} : n \geq 0\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$$L = \{a^{n!} : n \geq 0\}$$

Let m be the critical length of L

Pick a string w such that: $w \in L$

length $|w| \geq m$

We pick $w = a^{m!}$

From the Pumping Lemma:

We can write $w = a^{m!} = x\ y\ z$

With lengths $|\,x\ y\,| \leq m, \quad |\,y\,| \geq 1$

$$w = xyz = a^{m!} = \mathrlap{\overbrace{}^{m}}\underbrace{a\ldots a}_{x}\underbrace{a\ldots a}_{y}\underbrace{a\ldots a\overbrace{a\ldots aa\ldots a}^{m!-\,m}}_{z}$$

Thus: $y = a^{k}, \quad 1 \leq k \leq m$

$$x\,y\,z = a^{m!}$$

$$y = a^k, \quad 1 \le k \le m$$

From the Pumping Lemma: $$x\,y^i\,z \in L$$

$$i = 0, 1, 2, \ldots$$

Thus: $$x\,y^2\,z \in L$$

$x\,y\,z = a^{m!}$ $\qquad$ $y = a^{k}, \quad 1 \le k \le m$

From the Pumping Lemma: $x\,y^{2}\,z \in L$

$$xy^{2}z = \mathrlap{\overbrace{}^{m+k}\overbrace{}^{m!-m}}\underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...a}_{y}\underbrace{a...aa...aa...a}_{z} \in L$$

Thus: $a^{m!+k} \in L$

$$a^{m!+k} \in L \qquad\qquad 1 \leq k \leq m$$

Since: $L = \{a^{n!} : n \geq 0\}$

⇓

There must exist p such that:

$$m! + k = p!$$

However: $m!+k \leq m!+m$ for $m>1$

$$\leq m!+m!$$

$$< m!m + m!$$

$$= m!(m+1)$$

$$= (m+1)!$$

$$\Downarrow$$

$$m!+k < (m+1)!$$

$$\Downarrow$$

$m!+k \neq p!$ for any p

$$a^{m!+k} \in L \qquad 1 \leq k \leq m$$

BUT: $L = \{a^{n!} : n \geq 0\}$

⇓

$$a^{m!+k} \notin L$$

CONTRADICTION!!!

Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

END OF PROOF