Computer Vision Minor 1

Singhal, Madhur
2015CS10235

September 12, 2017

1 Declaration of Originality

2 Question 5

We assume that the Optical flow can be given by the following equation, ignoring the higher order terms.

$$\begin{bmatrix} u \\ v \end{bmatrix} \approx \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} + \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

This expression assumes affine motion of the object (or the camera). We now use the standard optical flow equation and substitute the approximation for u and v.

$$uI_x + vI_y + I_t = 0$$

$$\left(u_0 + u_x x + v_y y\right) I_x + \left(v_0 + v_x x + v_y y\right) I_y + I_t = 0$$

$$\begin{bmatrix} I_x & xI_x & yI_y & I_y & xI_y & yI_y \end{bmatrix} \begin{bmatrix} u_0 \\ u_x \\ u_y \\ v_0 \\ v_x \\ v_y \end{bmatrix} = -I_t$$

Now, we assume that that the velocities u_0, v_0 and the gradients u_x, u_y, v_x and v_y are constant over a $5x5$ patch of the image around any point (x, y). Then the above equation is valid for all those points and we can collect all the 25 equations formed into a matrix.

$$\begin{bmatrix} I_{x_1} & x_1 I_{x_1} & y_1 I_{y_1} & I_{y_1} & x_1 I_{y_1} & y_1 I_{y_1} \\ I_{x_2} & x_2 I_{x_2} & y_2 I_{y_2} & I_{y_2} & x_2 I_{y_2} & y_2 I_{y_2} \\ & & & .... & & \end{bmatrix} \begin{bmatrix} u_0 \\ u_x \\ u_y \\ v_0 \\ v_x \\ v_y \end{bmatrix} = - \begin{bmatrix} I_{t_1} \\ I_{t_2} \\ \cdots \end{bmatrix}$$

We can write the above equation as

$$AX = B$$

Since the row rank is larger than the number of columns, we will use the pseudo-inverse to find the solution that minimizes the least squares error. This is because the least square solution is the projection of B onto the space which is orthogonal to the range space of A, but the null space of $A^\top$ is the space orthogonal to range space of A. Thus the pseudo-inverse is just the matrix which projects B onto $A^\top$.

Thus the solution is given by

$$X = \left(A^\top A\right)^{-1} A^\top B$$

3 Question 6

Let us first express the velocity of a 3D point sin terms of the given parameters.

$$V = -U - \Omega \times P$$

$$\begin{bmatrix} \dfrac{\partial X}{\partial t} \\ \dfrac{\partial Y}{\partial t} \\ \dfrac{\partial Z}{\partial t} \end{bmatrix} = - \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} - \begin{bmatrix} Z\Omega_2 - Y\Omega_3 \\ X\Omega_3 - Z\Omega_1 \\ Y\Omega_1 - X\Omega_2 \end{bmatrix}$$

Now we express the 2D image coordinates assuming the pinhole camera model.

$$x = \frac{fX}{Z}$$

$$y = \frac{fY}{Z}$$

Differentiate the 2D image coordinates with respect to time.

$$u = \frac{fZ\frac{\partial X}{\partial t} - fX\frac{\partial Z}{\partial t}}{Z^2}$$

$$v = \frac{fZ\frac{\partial Y}{\partial t} - fY\frac{\partial Z}{\partial t}}{Z^2}$$

Now substitute the expression for derivatives of a 3D coordinate.

$$u = \frac{fZ\left(-U_1 - Z\Omega_2 + Y\Omega_3\right) - fX\left(-U_3 - Y\Omega_1 + X\Omega_2\right)}{Z^2}$$

$$v = \frac{fZ\left(-U_2 - X\Omega_3 + Z\Omega_1\right) - fY\left(-U_3 - Y\Omega_1 + X\Omega_2\right)}{Z^2}$$

Simplifying and substituting the pinhole model expression into the above equation we will get the desired result.

$$u = -\frac{fU_1}{Z} - f\Omega_2 + y\Omega_3 - x\left(-\frac{U_3}{Z} - \frac{y\Omega_1}{f} + \frac{x\Omega_2}{f}\right)$$

$$v = -\frac{fU_2}{Z} - x\Omega_3 + f\Omega_1 - y\left(-\frac{U_3}{Z} - \frac{y\Omega_1}{f} + \frac{x\Omega_2}{f}\right)$$

4 Question 7

5 Question 8

The decomposition of the velocity tensor gradient is given as the following.

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \frac{div(v)}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{curl(v)}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \frac{def(v)}{2}\begin{bmatrix} \cos(2\mu) & \sin(2\mu) \\ \sin(2\mu) & -\cos(2\mu) \end{bmatrix}$$

Simplifying we get four equations.

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} \frac{div(v)}{2} + \frac{def(v)\cos(2\mu)}{2} & -\frac{curl(v)}{2} + \frac{def(v)\sin(2\mu)}{2} \\ \frac{curl(v)}{2} + \frac{def(v)\sin(2\mu)}{2} & \frac{div(v)}{2} - \frac{def(v)\cos(2\mu)}{2} \end{bmatrix}$$

Now we can show all of the desired relations by looking at the appropriate linear combinations of these four equations.

$$u_x + v_y = \frac{div(v)}{2} + \frac{def(v)\cos(2\mu)}{2} + \frac{div(v)}{2} - \frac{def(v)\cos(2\mu)}{2} = div(v)$$

$$-(u_y - v_x) = v_x - u_y = \frac{curl(v)}{2} + \frac{def(v)\sin(2\mu)}{2} - \left(-\frac{curl(v)}{2} + \frac{def(v)\sin(2\mu)}{2}\right) = curl(v)$$

$$u_x - v_y = \frac{div(v)}{2} + \frac{def(v)\cos(2\mu)}{2} - \left(\frac{div(v)}{2} - \frac{def(v)\cos(2\mu)}{2}\right) = def(v)\cos(2\mu)$$

$$u_y + v_x = -\frac{curl(v)}{2} + \frac{def(v)\sin(2\mu)}{2} + \frac{curl(v)}{2} + \frac{def(v)\sin(2\mu)}{2} = def(v)\sin(2\mu)$$

Hence all the relations given have been verified.